ON ACCUMULATION POINTS OF THE SET OF PV-NUMBERS

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1. A real algebraic integer α (>1) is said to be a PV-number if any conjugate ($\neq \alpha$) of α are in the unit circle $|z|<1$. Let S be the set of all PV-numbers and S′ the set of all accumulation points of S. Since S is closed ([3]), we have $S' \subset S$. Dufrenoy and Pisot found the least element of S′ and proved that the elements of S′ which are less than 1.8 are only two : $\eta_1=(1+\sqrt{5})/2=1.618\cdots$, zero of $1+z-z^2$, and $\eta_2=1.754\cdots$, zero of $1-z+2z^2-z^3$ ([1]). In this paper we shall show that the number of the elements of S' which are less than 1.84 is at most five, and give the unique expansions of $A(z)/Q(z)$ corresponding to these elements.

2. Let θ be a PV-number. Let P(z) be the irreducible polynomial of θ :

$$P(z)=p_0+p_1z+\cdots+p_{s-1}z^{s-1}+\varepsilon z^s \ (p_0>0,\ \varepsilon=\pm1),$$

and $Q(z)$ the reciprocal polynomial :

$$Q(z)=\varepsilon z^s P(\frac{1}{z}).$$

For a number $\theta \in S$, θ belongs to S′ if and only if there exists a polynomial $A(z) \in \mathbf{Z}[z]$ such that

$$|A(z)| \leq |P(z)| \quad \text{on} \quad |z|=1,$$

where the equality holds only at finite points on $|z|=1$ ([3] Th. 1).

Assume that $\theta \in S'$ and let $P(z)$, $Q(z)$ and $A(z)$ be the polynomials corresponding to θ. We can assume that $A(0)>0$. Then it is easy to see that $A(z)/Q(z)$ has a single pole of order 1 at $1/\theta$ in $|z| \leq 1$. Therefore we have the power series expansion of $A(z)/Q(z)$ at zero :

$$\frac{A(z)}{Q(z)} = v_0 + v_1 z + \cdots + v_n z^n + \cdots$$

We briefly denote this expansion by $(v_0, v_1, \cdots, v_n, \cdots)$.

The following was proved in [2]. For any natural number n, there exist polynomials $E_n(z)$ and $E_n^*(z)$ ($\in \mathbf{Q}(z)$, of degree n) such that, if we put $D_n(z) = -z^n E_n(\frac{1}{z})$ and $D_n^*(z) = z^n E_n^*(\frac{1}{z})$, then the expansion of $\frac{D_n(z)}{E_n(z)}$ at zero is $(v_0, v_1, \cdots, v_{n-1}, w_n, \cdots)$ and that of $\frac{D_n^*(z)}{E_n^*(z)}$ is $(v_0, v_1, \cdots, v_{n-1}, w_n^*, \cdots)$, where w_n and w_n^* are uniquely determined. Then the following relations holds.

(1) $w_n + 1 \leq v_n \leq w_n^* - 1$, except the case $(1, 1, 1, \cdots)$. $(n \geq 3)$.

(2) $D_{n+2}(z) = (1+z)D_{n+1}(z) - z\dfrac{v_{n+1} - w_{n+1}}{v_n - w_n} D_n(z)$ $(n \geq 1)$.

(3) $w_{n+1}^* - w_{n+1} = 2\dfrac{D_{n+1}(1)}{D_n(1)}(v_n - w_n)$ $(n \geq 1)$.

(4) $w_{n+1}^* - w_{n+1} = 4\dfrac{(w_n^* - v_n)(v_n - w_n)}{(w_n^* - w_n)}$ $(n \geq 3)$.

If $\theta < 1.84$, then

(5) $v_{n+1} - w_{n+1} < \dfrac{2.84}{1.84} \dfrac{D_{n+1}(1.84)}{D_n(1.84)} (v_n - w_n)$ $(n \geq 1)$.

The sequence $w_n^* - w_n$ is monotone decreasing, i. e.,

(6) $w_{n+1}^* - w_{n+1} \leq w_n^* - w_n$.

Hence, if $w_n^* - w_n < 3$ then $v_n, v_{n+1}, v_{n+2}, \cdots$ are determined from (1) without ambiguity.

3. Let θ be any number of S$'$ which is less than 1.84. We shall give an expansion $(v_0, v_1, \cdots, v_n, \cdots)$ of $A(z)/Q(z)$ correspond-

ing to θ.

We have $v_0 = 1$, $D_1(z) = 1-z$ $w_1 = 0$ and $v_1 = 1$ or 2, in a similar way to [1]. Therefore we have only to consider two cases, i. e., the case (1, 1, . . .) and the case (1, 2, . . .).

(I) The case (1, 2, . . .).

As it is hnown ([2] p77, p79)

$$(7)\qquad D_2(z) = v_0 + \frac{v_1}{1+v_0} z - z^2, \qquad w_2 = v_0^2 - 1 + \frac{v_1^2}{1+v_0},$$

we have

$$D_2(z) = 1+z-z^2, \quad w_2 = 2.$$

Hence, from (5) it follows $v_2 - w_2 < 2.005\ldots$, so $v_2 \leq 4$. On the other hand, since $w_2 \leq v_2$ ([1] p56), we obtain

$$v_2 = 2 \text{ or } v_2 = 3 \text{ or } v_2 = 4.$$

The case $v_2 = 2$ corresponds to η_1 ; if $v_2 = 3$ there does not exist θ which correspond to (1, 2, 3, . . .) ([1]).

For the case (1, 2, 4, . . .), we have $D_3(z) = 1+z+z^2 - z^3$ from (2). As $E_3(z) = 1 - (z+z^2+z^3)$, we have

$$\frac{D_3(z)}{E_3(z)} = 1 + 2z + 4z^2 + 6z^3 + \ldots,$$

so it follows $w_3 = 6$. On the other hand, the formula (5) implies $v_3 - w_3 < 0.02\ldots$, hence $v_3 \leq 6$. This contradicts to (1). So there is no element in S$'$ corresponding to (1, 2, 4, . . .).

(II) The case (1, 1, . . .).

From (7), we have

$$D_2(z) = 1 + \frac{1}{2} z - z^2, \quad w_2 = \frac{1}{2}.$$

From (5) it follows that $v_2 - w_2 < 2.69\ldots$, so we have $v_2 \leq 3$, hence

$$v_2 = 1 \quad \text{or} \quad v_2 = 2 \quad \text{or} \quad v_2 = 3\,.$$

The expantions (1, 1, 1, . . .) and (1, 1, 3, . . .) correspond to η_1 and η_2, respectively (〔1〕). Therefore we have only to deal with the case (1, 1, 2, . . .).

In this case, we obtain $D_3(z) = 1 + z^2 - z^3$ from (2). As $E_3(z) = 1 - z - z^3$ and

$$\frac{D_3(z)}{E_3(z)} = 1 + z + 2z^2 + 2z^3 + \ldots,$$

we have $w_3 = 2$. It follows $v_3 - w_3 < 2.91\ldots$ from (5), so $v_3 \leq 4$. From (1) we have

$$v_3 = 3 \quad \text{or} \quad v_3 = 4.$$

For the case $v_3 = 3$, i. e., (1, 1, 2, 3, . . .), we *only obtain* (1, 1, 2, 3, 5, 8, . . .), (1, 1, 2, 3, 5, 9, 16, . . .) and (1, 1, 2, 3, 6,. . .) which correspond to η_1, η_2 and η_3, respectively, where $\eta_3 = 1.839\ldots$, zero of $1 + z + z^2 - z^3$ (〔1〕).

Therefore it is enough to deal with the case (1, 1, 2, 4, . . .). From (2), we have

$$D_4(z) = 1 - \frac{1}{3}z + \frac{1}{3}z^2 + \frac{4}{3}z^3 - z^4, \quad w_4 = \frac{17}{3}.$$

From (3) it follows that $w_4^* - w_4 = \frac{16}{3}$, so $w_4^* = 11$. Hence we obtain $7 \leq v_4 \leq 10$ by (1). As $v_4 - w_4 < 2.73\ldots$ from (5), we have

$$v_4 = 7 \quad \text{or} \quad v_4 = 8$$

If $v_4 = 8$, then we calculate $D_5(z)$ by (2), from which we have $w_5 = 13.5$ and $w_5^* = 18.75 : 15 \leq v_5 \leq 17$. But we have $v_5 = 15$ using the formula (5). Next we calculate $D_6(z)$ by (2), from which we have $w_6 = 26.71\ldots$ and $w_6^* = 30.99\ldots : 28 \leq v_6 \leq 29$. This contradicts (5), hence $v_4 = 7$.

From (2) and (4), we have

$$D_5(z) = 1 + z^3 + z^4 - z^5, \quad w_5 = 11, \quad w_5^* = 15.$$

So it follows $v_5 = 12$ or $v_5 = 13$ or $v_5 = 14$ by (1). If $v_5 = 12$, then it corresponds to η_2 ([1]). If $v_5 = 14$, we have $v_6 = 27$ by (1), which contradicts (5).

Therefore we have only to deal with the case (1, 1, 2, 7, 13, . .). Since

$$D_6(z) = 1 - \frac{1}{2}z + \frac{1}{2}z^2 + \frac{1}{2}z^3 + \frac{3}{2}z^5 - z^6,$$

$$w_6 = 22, \quad w_6^* = 26,$$

it follows $23 \leq v_6 \leq 25$ from (1). On the other hand (5) implies $v_6 = 23$ or $v_6 = 24$. The number θ corresponding to the case $v_6 = 23$ does not exist (cf. [1]). Hence we have $v_6 = 24$. Next we have

$$D_7(z) = 1 - \frac{1}{2}z + z^3 - \frac{1}{2}z^4 + \frac{1}{2}z^5 + \frac{3}{2}z^6 - z^7,$$

$$w_7 = 42.5, \quad w_7^* = 46.5,$$

so it follows $v_7 = 44$ or $v_7 = 45$. If $v_7 = 45$, then we have $v_8 = 83$ and $v_9 = 154$ which contradicts (5), so $v_7 = 44$.

For the expansion (l, 1, 2, 4, 7, 13, 24, 44, . . .) we have

$$D_8(z) = 1 - \frac{1}{4}z - \frac{1}{8}z^2 + \frac{5}{8}z^3 + \frac{1}{8}z^4 + \frac{7}{8}z^6 + \frac{5}{4}z^7 - z^8,$$

$$w_8 = 78.875, \quad w_8^* = 82.625,$$

which gives $v_8 = 80$ or $v_8 = 81$ by (1).

If $v_8 = 81$, in a similar way, we have a contradiction.

If $v_8 = 80$, then we calculate $D_9(z)$, from which we have $w_9 = 144.5$, while (3) gives $w_9^* = 147.65$. Hence it follws $v_9 = 146$. Furthermore we calculate $D_{10}(z)$, from which $w_{10} = 265$ and $w_{10}^* = 268.14. . .$, so we have $v_{10} = 266$ or $v_{10} = 267$.

For $v_{10} = 266$, it follows $w_{11}^* - w_{11} > 2.72. . .$

from (4), then v_{11}, v_{12}, . . . are uniquely determined by (6).

For $v_{10} = 267$, it follows $w_{11}^{*} - w_{11} < 2.90...$ from (4), then v_{11}, v_{12}, . . . are uniquely determined.

Therefore we obtain that the unique expansions of $A(z)/Q(z)$ corresponding to the numbers of S′ which are less than 1.84 are at most the following :

① (1, 2, 2, . . .),

② (1, 1, 1, . . .),

③ (1, 1, 3, . . .),

④ (1, 1, 2, 3, 6, . . .),

⑤ (1, 1, 2, 3, 5, 8, . . .),

⑥ (1, 1, 2, 3, 5, 9, . . .),

⑦ (1, 1, 2, 4, 7, 13, 24, 44, 80, 146, 266, . . .),

⑧ (1, 1, 2, 4, 7, 13, 24, 44, 80, 146, 267, . . .),

where ①, ② and ⑤ correspond to η_1, ③ and ⑥ to η_2, and ④ to η_3.

Finally, HITAC M-280H in Computer Centre University of Tokyo was used in order to determine w_n from $D_n(z)$. The program and its application to the case (1, 1, 2, 4, 7, 13, 24, 44, . . .) will be shown. 'DN(Z)', 'C(Z)', 'E(Z), 'W(Z)' and 'W8' in the list indicate $D_n(z)$, $1 - E_n(z)$, $1/E_n(z)$, $D_n(z)/E_n(z)$ and w_8, respectively. In this case,

$$n = 8,$$

$$D_n(z) = 1 - \frac{1}{4}z - \frac{1}{8}z^2 + \frac{5}{8}z^3 + \frac{1}{8}z^4 + \frac{7}{8}z^6 + \frac{5}{4}z^7 - z^8,$$

$$1 - E_n(z) = \frac{5}{4}z + \frac{7}{8}z^2 + \frac{1}{8}z^4 + \frac{5}{8}z^5 - \frac{1}{8}z^6 - \frac{1}{4}z^7 + z^8,$$

$$\frac{1}{E_n(z)} = 1 + \frac{5}{4}z + \frac{39}{16}z^2 + \frac{265}{64}z^3 + \frac{1903}{256}z^4 + \dots + \frac{5298671}{65536}z^8 + \dots,$$

$$\frac{D_n(z)}{E_n(z)} = 1 + z + 2z^2 + 4z^3 + 7z^4 + 13z^5 + 24z^6 + 44z^7 + \frac{631}{8}z^8 + \dots,$$

$$w_8 = \frac{631}{8}.$$

```
C       PV-NUMBER
        INTEGER*8 A(20),B(20),C(20),D(20),E(20),
        F(20)
        INTEGER*8 CC(20),DD(20),AA(20),BB(20)
        INTEGER*8 C1(20),D1(20)
        READ(5,1000) N
        READ(5,1100) (A(I),B(I),I=1,N+1)
        WRITE(6,2000) N
        WRITE(6,2500)
        WRITE(6,3000)(A(I),B(I),I=1,N+1)
        C(1)=0
        D(1)=1
        DO 10 I=2,N+1
        C(I)=A(N-I+2)
        D(I)=B(N-I+2)
     10 CONTINUE
        WRITE(6,1111)
        WRITE(6,3000)(C(I),D(I),I=1,N+1)
        E(1)=1
        F(1)=1
        DO 20 I=2,N+1
        E(I)=0
        F(I)=1
     20 CONTINUE
        CALL ADDPLY(N,E,F,C,D)
        DO 25 I=1,N+1
        C1(I)=C(I)
        D1(I)=D(I)
     25 CONTINUE
        DO 30 JJ=2,N
        CALL PRDPLY(N,C1,D1,C,D,CC,DD)
        CALL ADDPLY(N,E,F,CC,DD)
        DO 30 I=1,N+1
        C1(I)=CC(I)
        D1(I)=DD(I)
     30 CONTINUE
        WRITE(6,1200)
        WRITE(6,3000)(E(I),F(I),I=1,N+1)
        CALL PRDPLY(N,A,B,E,F,AA,BB)
        WRITE(6,1300)
        WRITE(6,3000)(AA(I),BB(I),I=1,N+1)
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      WRITE(6,5000) N,AA(N+1),BB(N+1)
 1000 FORMAT(I3)
 1100 FORMAT(16I5)
 1111 FORMAT(1H0,'C(Z)=')
 1200 FORMAT(1H0,'E(Z)=')
 1300 FORMAT(1H0,'W(Z)=')
 2000 FORMAT(1H0,'N =',I3)
 2500 FORMAT(1H0,'DN(Z)=')
 3000 FORMAT(1H ,I30,'/',I30)
 5000 FORMAT(1H0,'W',I2,' ='/1H ,I30,'/',I30)
      STOP
      END
      SUBROUTINE PRDPLY(N,A,B,C,D,E,F)
      INTEGER*8 A(20),B(20),C(20),D(20),E(20),
      F(20)
      INTEGER*8 EE,FF
      INTEGER*8 LA,LB,LC,LD,LE,LF
      DO 20 K=1,N+1
      EE=0
      FF=1
      DO 30 J=1,K
      LA=A(J)
      LB=B(J)
      LC=C(K-J+1)
      LD=D(K-J+1)
      CALL PRODQ(LA,LB,LC,LD,LE,LF)
      CALL ADDQ(EE,FF,LE,LF)
   30 CONTINUE
      E(K)=EE
      F(K)=FF
   20 CONTINUE
      RETURN
      END
      SUBROUTINE PRDCTP(M,N,A,B,C,D,E,F)
      INTEGER*8 A(20),B(20),C(20),D(20),E(20),
      F(20)
      INTEGER*8 EE,FF
      INTEGER*8 LA,LB,LC,LD,LE,LF
      DO 10 I=M+2,M+N+1
      A(I)=0
      B(I)=1
   10 CONTINUE
      DO 15 I=N+2,M+N+1
      C(I)=0
      D(I)=1
   15 CONTINUE
      DO 22 I=1,M+1
      CALL TSUBUN(A(I),B(I))
   22 CONTINUE
      DO 33 I=1,N+1
      CALL TSUBUN(C(I),D(I))
   33 CONTINUE
      DO 20 K=1,M+N+1
```

```
      EE=0
      FF=1
      DO 30 J=1,K
      LA=A(J)
      LB=B(J)
      LC=C(K-J+1)
      LD=D(K-J+1)
      CALL PRODQ(LA,LB,LC,LD,LE,LF)
      CALL ADDQ(EE,FF,LE,LF)
   30 CONTINUE
      E(K)=EE
      F(K)=FF
   20 CONTINUE
      RETURN
      END
      SUBROUTINE ADDPLY(N,A,B,C,D)
      INTEGER*8 A(20),B(20),C(20),D(20),E(20),
      F(20)
      DO 30 I=1,N+1
      CALL ADDQ(A(I),B(I),C(I),D(I))
   30 CONTINUE
      RETURN
      END
      SUBROUTINE PRODQ(LA,LB,LC,LD,LE,LF)
      INTEGER*8 LA,LB,LC,LD,LE,LF
      IF(LA.EQ.0.OR.LC.EQ.0) GOTO 9
      CALL TSUBUN(LA,LB)
      CALL TSUBUN(LC,LD)
      CALL TSUBUN(LA,LD)
      CALL TSUBUN(LC,LB)
      LE=LA*LC
      LF=LB*LD
      RETURN
    9 LE=0
      LF=1
      RETURN
      END
      SUBROUTINE ADDQ(EE,FF,LE,LF)
      INTEGER*8 EE,FF,LE,LF,LCOMM
      CALL EUCLID(FF,LF,LCOMM)
      EE=LF/LCOMM*EE+FF/LCOMM*LE
      FF=FF/LCOMM*LF
      CALL TSUBUN(EE,FF)
      RETURN
      END
      SUBROUTINE TSUBUN(LA,LB)
      INTEGER*8 LA,LB,LC
      IF(LA.EQ.0) GOTO 99
      CALL EUCLID(LA,LB,LC)
      LA=LA/LC
      LB=LB/LC
      RETURN
   99 LB=1
```

```
      RETURN
      END
      SUBROUTINE EUCLID(A,B,C)
      INTEGER*8 A,B,C
      INTEGER*8 LA,LB,LQ,LR
      LA=ABS(A)
      LB=ABS(B)
    1 LQ=LA/LB
      LR=LA-LB*LQ
      IF(LR.EQ.0)  GOTO 9
      LA=LB
      LB=LR
      GO TO 1
    9 C=LB
      RETURN
      END
```

```
N =  8

DN(Z)=
                 1/           1
                -1/           4
                -1/           8
                 5/           8
                 1/           8
                 0/           1
                 7/           8
                 5/           4
                -1/           1

C(Z)=
                 0/           1
                 5/           4
                 7/           8
                 0/           1
                 1/           8
                 5/           8
                -1/           8
                -1/           4
                 1/           1

E(Z)=
                 1/           1
                 5/           4
                39/          16
               265/          64
              1903/         256
             14025/        1024
            100703/        4096
            726649/       16384
           5298671/       65536
```

```
W(Z)=
                         1/                1
                         1/                1
                         2/                1
                         4/                1
                         7/                1
                        13/                1
                        24/                1
                        44/                1
                       631/                8

W 8 =
                       631/                8
```

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