Polluted and Sustainable Growth Paths

Hans W. Gottinger

Abstract

The contribution of this paper is to extend turnpike versions of optimal growth to include pollution. We introduce a pollution abatement technology which has the characteristics of a constant rate of progress toward pollution reduction. The requirements on this rate of advance in order to assure a consumption happy future are derived. Two major economic policy conclusions are drawn.

Key-words. Economic Growth, Technology, Pollution Abatement, Sustainability.

1. INTRODUCTION

The contribution of this paper is to extend turnpike versions of optimal growth, such as those by Cass (1966) and Shell (1967), to include pollution and technologies for pollution abatement. Such problems have received renewed interest through the recent work of Ploeg and Withagen (1991), Barrett (1992), Tahvonen and Kuuluvainen (1993), Beltratti, Chichilnisky and Heal (1993) and Eismont (1994).

The underlying model can be described as an aggregate one sector model of the Solow (1956) type. After introducing definitions and notation in Section 2, in Section 3 we will state the optimal growth problem under pollution for an economy in which technological progress is focused on pollution abatement. Pollution abatement has the characteristics of a constant rate of progress towards pollution reduction. Sections 4 and 5 describe the optimal paths with and without a critical pollution level. Section 6 summarizes the results, and the requirements on the rate of progress to assure a sustainable consumption future are derived.

2. DEFINITIONS AND NOTATION

Labour L, is provided by a population, N which is assumed to be growing at some constant rate ρ. If the labour force is a constant proportion of the total population, then

$$L=\rho N \qquad \frac{\dot{N}}{N}=\rho \qquad \frac{\dot{L}}{L}=\rho$$

That is, the growth rate in the labour force is equal to that of total population. Capital stock, K, is assumed to grow at a rate dependent upon the investment, I, and the rate of capital depreciation μ. Thus,

$$\dot{K}=I-\mu K$$

The investment variable is one of the two controls which this model economy has at its disposal to steer its course toward attainment of its specified objectives. The model essentially assumes that there is a social planner which will determine the proper investment at the proper time to reach this economy's goals. The social planner will make the necessary plans over a given planning time interval. (Gottinger, 1992).

The three factors of production are combined together for production according to a Cobb-Douglas function, given by

$$Y=AL^{\Psi} K^{\alpha} E^{\beta}.$$

with partial elasticities Ψ, α, β and the state of technology A.
The following neoclassical features of the production function are also assumed:

(i) constant returns to scale, i.e. $\Psi+\alpha+\beta=1$
(ii) positive α, β
(iii) diminishing marginal rate of substitution between factors
(iv) all available factors are employed.

Pollution generation is assumed to be a linear function of the amount of energy produced. Also, there exists a dissipation of pollution such that the net production rate of pollution is given by

$$\dot{P}=\theta E_0-\sigma P$$

where θ and σ are constants.

Since pollution is a stock that has very definite adverse effects upon the quality of life, it cannot be allowed to increase indefinitely. Therefore, it can be assumed there is a critical or maximum level of pollution which the social planner will not allow to be exceeded. We shall examine the effects upon a growth when the critical level is reached. The stock of pollution at the critical level will be designated as P_c.

Finally, the social planner must decide upon an objective for the economy. It will be assumed that the desirable direction of growth for the economy is toward maximizing a discounted net consumption over time. The term net is used here to indicate that gross consumption designated by C above only encompasses that obtained directly from the output of production Y. The harmful effects of pollution will be a cost that degrades consumption so that net consumption will be designated by

$$C-\Sigma P$$

where Σ is a constant. But, this net consumption will be discounted in time so that the objective to be maximized by the social planner is

$$\int_{O}^{T}(C-\Sigma P)e^{-\delta t}\,dt,\quad \delta \text{ being the discount rate.}$$

In order to simplify the handling of the model, it is convenient to eliminate the labour variable by defining all other variables in terms of labour.

Thus,

$$y=(Y/L)=A(K/L)^{\alpha}(E_0/L)^{\beta}=Ak^{\alpha}E^{\beta} \tag{1}$$

$$\begin{aligned}\dot{k}=(\dot{K}/\dot{L})=(\dot{K}/L)-(K/L^2)\dot{L}=(I/L)=(\mu K/L)\\ -(K/L)(\dot{L}/L)\\ =1-(\mu+\rho)k\end{aligned} \tag{2}$$

$$\begin{aligned}\dot{p}=(\dot{P}/\dot{L})=(\dot{P}/L)-(P/L^2)\dot{L}=(E_0/L)-(\sigma P/L)\\ -(P/L)(\dot{L}/L)\\ =E-(\sigma+\rho)p\end{aligned} \tag{3}$$

$$y=c+i+E$$

where as indicated

$y = (Y/L)$

$k = (K/L)$

$p = (P/L)$

$i = (I/L)$

$E = (E_0/L)$

A further specification of our model is that capital investment can be expressed as

$i = sy$

where s is the capital output coefficient. This, in effect, has transformed our control variable from investment i to the capital output coefficient s.

Let $y = c + sy + E$,

where $c = (C/L)$

thus $c = (1-s)y - E$ (4)

The pollution constraint is given by

$$P_c \geq P = pL_0e^{\rho t} \quad (5)$$

Thus, equations (1) – (5) represent the economic model under consideration.

The objective can be redefined in per capita terms as maximizing the per capita discounted net consumption over a specified time period, therefore

$$\max_{s,E} \int_{O}^{T} (c - \Sigma p)e^{-\delta t}dt \quad (6)$$

3. GENERAL PROBLEM FORMULATION: POLLUTED TURNPIKE AND POLLUTION ABATEMENT TECHNOLOGY

The simplified case that will be analyzed is one in which the production technology is assumd constant, but there exists research and development toward pollution abatement. The problem can be concisely stated as determining the growth path of

s(t), E(t), k(t), p(t)

in order to maximize

$$\int_{O}^{T} (c(t) - \Sigma p(t))e^{-\delta t}\, dt$$

subject to equations

$$\begin{aligned} &c(t) = (1-s(t))y(t) - E(t) \\ &y(t) = Ak^{\alpha}(t)E^{\beta}(t) \\ &\dot{k}(t) = s(t)y(t) = (\mu+\rho)k(t) \\ &\dot{p}(t) = e^{-\pi t}\theta\, E(t) - (\sigma+\rho)p(t) \end{aligned} \tag{7}$$

and inequalities

$$\begin{aligned} &p(t)L_0e^{\rho t} \leq P_c \\ &s(t) + E(t)/y(t) \leq 1 \\ &s(t) \geq 0 \\ &E(t) \geq 0 \end{aligned}$$

where $e^{-\pi t}$ is the pollution abatement technology. π is the rate, at which progress in pollution abatement advances. This is a model in which abatement progress advances at a constant rate similar to a neutral technology advance.

Given that a solution exists for this problem, then there exists auxiliary variables $\Psi_p(t)$, $\Psi_k(t)$ which are defined by

$$\dot{\Psi}_p(t) = \Psi_p(t)(\sigma+\rho) + \Sigma e^{-\delta t} - \lambda(t) L_0 e^{\rho t,} \quad (8)$$

$$\dot{\Psi}_k(t) = -\Psi_k(t)(\alpha\bar{s}(t)\bar{y}(t)/\bar{k}(t) - (\mu+\rho)) \quad (9)$$

$$+\xi(t)(\bar{E}(t)\alpha/\bar{k}(t)\bar{y}(t))$$

$$+\Psi_0(t)(\bar{s})(t-1)(\alpha\bar{y}(t)/\bar{k}(t))e^{-\delta t}$$

$$\dot{\Psi}_0(t) = 0 . \quad (10)$$

The bar above the control and phase variables designates values along the optimal path.

$\Psi_0(t)$, $\Psi_p(t)$ and $\Psi_k(t)$ are continuous. The multiplier function $\xi(t)$ is zero when $\tilde{\chi}(k(t),s(t),E(t),t) < 0$ and when $\xi(t) \leq 0$, $\tilde{\chi}(\bar{k}(t),\bar{s}(t),\bar{E}(t),t) < 0$. As before, the pollution and capital level at t = T will not be specified so that the transversality conditions (iii) in Appendix A yield that

$$\Psi_p(T) = 0$$

$$\Psi_k(T) = 0$$

$$\Psi_0(T) = v_0$$

Since $\Psi_0(t)$ is a constant, and assumed not zero, we may set Ψ_0 equal to one without altering any necessary conditions of the theorem in Appendix A.

The H function is defined as

$$H(\Psi_k, \Psi_p, k, p, s, E, t) =$$

$$s(\Psi_k - e^{-\delta t})Ak^\alpha E^\beta + \Psi_\rho(e^{-\pi t}\theta E - (\sigma+\rho)p) \\ -\Psi_k((\mu+\rho)k) + (Ak^\alpha E^\beta - E - \Sigma p)e^{-\delta t} \\ -\lambda(L_0 e^{\rho t}(e^{-\pi t}\theta E - (\sigma+\rho)p)) \tag{11}$$

Condition (ii) of Appendix A requires that at each instant of time

$$H(\Psi_k(t), \Psi_p(t), \bar{k}(t), \bar{p}(t), \bar{s}(t), \bar{E}(t), t) \\ = \max_{\substack{s \geq 0 \\ E \geq 0}} H(\Psi_k(t), \Psi_p(t), \bar{k}(t), \bar{p}(t), s, E, t), \\ s + (E/A\bar{k}^\alpha(t)\bar{E}^\beta) \leq 1$$

To perform the maximization of H, consider first examining the optimal program for s with the energy program E(t) given. H may be written as

$$H(\Psi_k(t), \Psi_p(t), \bar{k}(t), \bar{p}(t), s, \bar{E}(t), t) = \\ s(\Psi_k(t) - e^{-\delta t})A\bar{k}^\alpha(t)\bar{E}^\beta(t) \\ + (A\bar{k}^\alpha(t)\bar{E}^\beta(t) - \Sigma\bar{p}(t) - \bar{E}(t))e^{-\delta t} \\ - \Psi_k(t)(\mu+\rho)\bar{k}(t) + \Psi_p(t)(\theta\bar{E}(t) - (\sigma+\rho)\bar{p}(t)) \\ + \lambda(t)(L_0 e^{\rho t}(e^{-\pi t}\theta\bar{E}(t) - (\sigma+\rho)\bar{p}(t)))$$

Performing the maximization of H with respect to s we again find that since the s control appears linearly, the H–function will take on a maximum depending upon the value of the coefficient $(\Psi_k(t) - e^{-\delta t})$. If

$$\Psi_k(t) > e^{-\delta t} \text{ then } \bar{s}(t) = 1 - (\bar{E}(t)/\bar{y}(t)) \tag{12}$$

$$\Psi_k(t) < e^{-\delta t} \text{ then } \bar{s}(t) = 0 \tag{13}$$

The first situation (12) is a no–consumption path while the second is a no investment path. The optimal growth path of most interest is to us the one which corresponds to the singular arc relative to the control s. The singular arc occurs when $\Psi_k(t)=e^{-\delta t}$. Along the singular arc, the shadow price of capital will equal the discount factor. The necessary conditions along this singular arc, well known in optimal control (see, e.g. Bryson and Ho (1969),

provide that

$$(\partial\bar{y}(t)/\partial\bar{k}(t))=\bar{y}_k(t)=\mu+\rho+\delta \tag{14}$$

$$(\dot{\bar{y}}(t)/\bar{y}(t))=(\dot{\bar{k}}(t)/\bar{k}(t)) \tag{15}$$

Suppose initially, that the critical pollution level has not been reached, but eventually with time it is attained. The general relationship for the energy share of output is obtained by satisfying the maximum condition, this is

$$\frac{\bar{E}(t)}{\bar{y}(t)}=\frac{\beta}{1-(\Psi_\rho e^{-\pi t}\,\theta-\lambda L_0 e^{(\rho-\pi)t})e^{\delta t}} \tag{16}$$

This expression is used to determine the energy level as long as the critical pollution level has not been attained. This equation differs from the non-technological case by the exponential term involving pollution abatement. With time, the term in the denominator will approach one faster than without technological progress in pollution abatement.

The share of output for investment is derived again to be

$$\bar{s}(t)=\left(\frac{\alpha}{\rho+\delta+\mu}\right)\{\mu+\rho-\left(\frac{\beta}{\alpha-1}\right)\left(\frac{\dot{\bar{E}}(t)}{\bar{E}(t)}\right)\}. \tag{17}$$

Since $\alpha<1$, the term in the parenthesis is positive when energy growth is positive.

4. OPTIMAL PATH WITH NO CRITICAL POLLUTION LEVEL

The first case to consider is when the pollution level is never to reach the critical level, the equation (8) can be solved so that

$$\Psi_p(t)=\frac{\Sigma e^{-\delta t}}{\sigma+\rho+\delta}(e^{(\sigma+\rho+\delta)(t-T)}-1)$$

Substituting this into expression (16)

$$\frac{\bar{E}(t)}{\bar{y}(t)}=\frac{\beta}{1-\left(\frac{\Sigma\theta e^{-(\delta+\pi)t}}{\sigma+\rho+\delta}\right)(e^{(\sigma+\rho+\delta)(t-T)}-1)} \tag{18}$$

This expression indicated the same general property. That is, the share of output going to energy initially begins at a level less than β.

$$\frac{\bar{E}(t)}{\bar{y}(t)}=\frac{\beta}{1+\frac{\Sigma\theta(1-e^{-(\sigma+\rho+\delta)T})}{\sigma+\rho+\delta}} \tag{19}$$

From this level, the energy share will increase with time to the value of β, the partial elasticity of production for energy. However, the path along which the energy share variable moves is different due to the pollution abatement technology. This energy share path is closer to the value of β along the path. This will be shown diagrammatically later in Figure 2.

The equation for the energy share (18) can be written as

$$\bar{E}(t)=(\beta/g(t))\bar{y}(t)$$

where

$$g(t)=1-\frac{\Sigma\theta e^{-(\delta+\pi)t}}{\sigma+\rho+\delta}(e^{(\sigma+\rho+\delta)(t-T)}-1)$$

Taking a time derivative,

$$(\dot{\bar{E}}(t)/\bar{E}(t)) = (\dot{\bar{y}}(t)/\bar{y}(t)) - (\dot{\bar{g}}(t)/\bar{g}(t))$$

using

$$(\dot{\bar{y}}(t)/\bar{y}(t)) = (\beta/(1-\alpha))(\dot{\bar{E}}(t)/\bar{E}(t))$$
$$(\dot{\bar{E}}(t)/\bar{E}(t)) = -((1-\alpha)/(1-\alpha-\beta))(\dot{\bar{g}}(t)/\bar{g}(t))$$

where

$$\frac{\dot{g}(t)}{g(t)} = \frac{-((\sigma+\rho+\pi)e^{(\sigma+\rho+\delta)(t-T)} + (\delta+\pi))\theta\Sigma e^{-(\delta+\pi)t}}{\sigma+\rho+\delta-\Sigma\theta e^{(\sigma+\rho+\delta)(t-T)}e^{-(\delta+\pi)t} + \Sigma\theta e^{-(\delta+\pi)t}} \quad (20)$$

The substitution of this expression into equation (16) gives us an explicit function of time for the investment share of output.

Using the above expressions, we may extract some qualitative information which may be displayed diagrammatically. Let us do so; in particular, we present a comparison of the case with and without the pollution abatement technology. Figure 1 depicts the reduction in pollution per capita with time as pollution abatement. This figure, of course, only presents one possible comparison in that the initial level of pollution will play a major role in how these paths will move. However, the figure does show the divergence between the time paths with and without pollution abatement which is the major point to be brought out by the figure. The figure shown assumes that the values of E and the initial pollution level are such that there is initially a positive growth in the pollution level. With a long enough time period, the case with pollution abatement will eventually cause the pollution per capita level to decrease in time and approach zero.

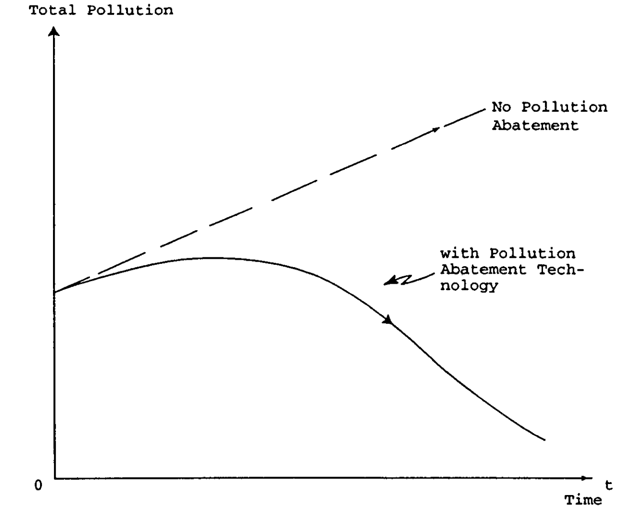

Figure 1: Total Pollution Time Path

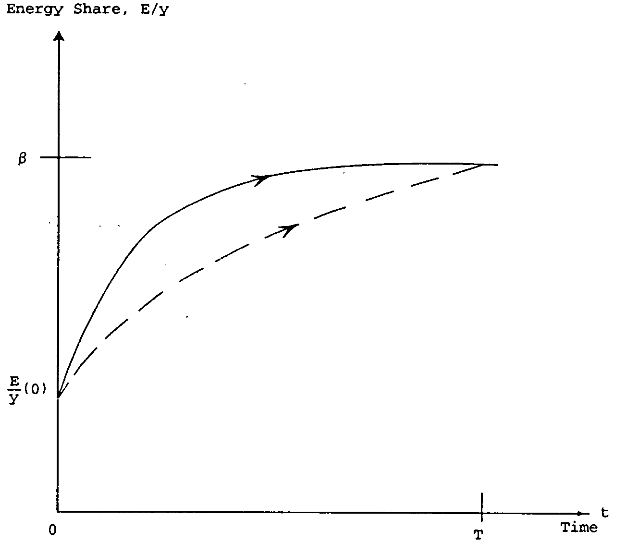

Figure 2: Energy Share Time Path

Figure 2 presents the comparison of the path of the energy share with time. Again the dashed line illustrates the case where there exists no pollution abatement advance. This path begins at time zero with a value equal to equation (18). With time, the path will move toward β and will be equal to β at the time horizon. For the case where there are advances made toward the reduction of pollution generated by energy, the path is shown by the solid curve. The initial and final points for this curve are exactly the same as in the dashed line. However, the economy with the pollution abatement advance will experience a higher share of output to energy between the initial and final points. That is, the energy share path arches closer to β in the case where pollution abatement progress exists.

Figure 3 presents a control plane diagram which depicts the growth path of the two control variables s and E/y in time. For the case where pollution abatement exists,

$$\bar{s}(0)=\frac{\alpha}{\mu+\rho+\delta}\{\mu+\rho+\left(\frac{\beta}{1-\alpha-\beta}\right)$$

$$\left[\frac{(\sigma+\rho-\pi)\Sigma\theta e^{-(\sigma+\rho+\delta)T}+\Sigma\theta(\delta+\pi)}{\sigma+\rho+\delta-\Sigma\theta e^{-(\sigma+\rho+\delta)}T+\Sigma\theta}\right]\} \tag{21}$$

which is greater or less than β depending upon the value of the constant parameters. Suppose it is positive. The energy share is given at t=0 by equation (20). At the final time T, the energy share is β and the investment share is

$$\bar{s}(T)=\left(\frac{\alpha}{\mu+\rho+\delta}\right)\left\{\mu+\rho+\frac{\beta\Sigma\theta e^{-(\delta+\pi)T}}{1-\alpha-\beta}\right\} \tag{22}$$

For the case where pollution abatement technology is a constant, $\pi = 0$,

$$\bar{s}_0(0) = \left(\frac{\alpha}{\mu+\rho+\delta}\right)\left\{\mu+\rho+\left(\frac{\beta}{1-\alpha-\beta}\right)\right.$$

$$\left.\left[\frac{\Sigma\theta e^{-(\sigma+\rho+\delta)T}}{1+(\theta\Sigma/(\alpha+\rho+\delta))(1-e^{-(\sigma+\rho+\delta)T})}\right]\right\}. \quad (23)$$

and

$$\bar{s}(T) = \left(\frac{\alpha}{\mu+\rho+\delta}\right)\{\mu+\rho+(\beta\theta\Sigma/(1-\alpha-\beta))\} \quad (24)$$

Thus, the starting points differ for the two cases. The values of the energy share in both cases are the same but the investment share of output in the pollution abatement progress model is higher than without pollution abatement. This is seen by examining equations (21) and (23). However, the reverse is true at the planning horizon. Equations (22) and (24) show that

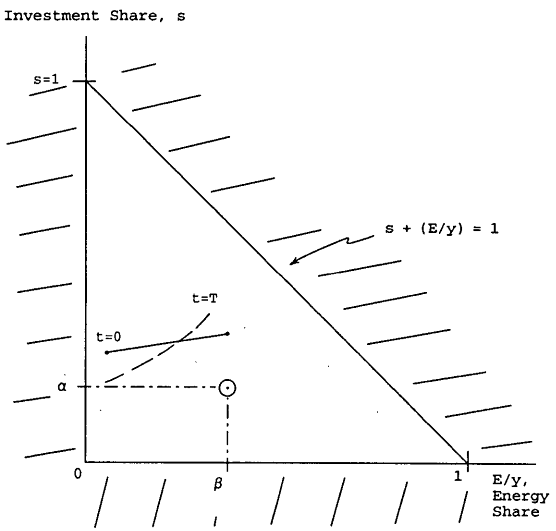

Figure 3: Output Allocation Paths

at the time T, the output share to investment is greater for the case where there is no progress in the abatement pollution through technical change. These properties of the growth paths are shown in Figure 1, the two paths representing the constant pollution technology case. Thus, the change in investment share of output is less when there is pollution abatement advance.

5. OPTIMAL PATH WHEN A CRITICAL POLLUTION LEVEL IS REACHED

Suppose the critical pollution level P_c is reached along the optimal growth path, then

$$P_c = \bar{p}(t) L_0 e^{\rho t}$$

and

$$\bar{p}(t) = (P_c / L_0) e^{-\rho t}$$

or

$$(\dot{\bar{p}}(t) / \bar{p}(t)) = -\rho$$

Using the fact that

$$\dot{\bar{p}}(t) = e^{-\pi t} \theta E(t) - (\sigma + \rho) \bar{p}(t)$$

we can derive

$$\bar{E}(t) = \bar{E}_c \, e^{(\pi - \rho)(t - t_c)} \tag{25}$$

where

$$\bar{E}c = \frac{\sigma P_c}{\theta L_0}$$

t_c = time at which the critical pollution level is reached.

The energy per capita is an exponentially changing function of time where the rate of change is

$$(\dot{\bar{E}}(t)/\bar{E}(t))=\pi-\rho$$

The investment share of output is given then by

$$\bar{s}(t)=\left(\frac{\alpha}{\mu+\rho+\delta}\right)\{\mu+\rho+\frac{\beta}{1-\alpha}(\pi-\rho)\} \tag{26}$$

This is a constant which depends upon the parameters of the problem. The output growth is given by

$$(\dot{\bar{y}}(t)/\bar{y}(t))=\left(\frac{\beta(\pi-\rho)}{1-\alpha}\right)$$

and

$$y(t)=\bar{y}_c\, e^{\left(\frac{\beta(\pi-\rho)}{1-\alpha}\right)(t-t_c)} \tag{27}$$

where $\bar{y}_c=$

$$\left[\frac{\alpha A^{(1/\alpha)}}{\rho+\delta+\mu}\right]^{(\alpha/(1-\alpha))} \bar{E}_c^{\beta} \text{ (output level at point of critical pollution)}$$

The expression for capital is given by

$$\bar{k}(t)=\left(\frac{\alpha\bar{y}(t)}{\rho+\delta+\mu}\right)$$

The net consumption per capita is then given by

$$C_n(t)=(1-s)\bar{y}_c\, e^{\left(\frac{\beta(\pi-\rho)}{1-\alpha}\right)(t-t_c)}-\bar{E}_c\, e^{(\pi-\rho)(t-t_c)}-\Sigma\bar{p}_c\, e^{-\rho(t-t_c)} \tag{28}$$

where s is constant and the subscript c denotes the value of the variable when the critical pollution level is reached. We note dire results if there is no pollution abatement progress, i.e. $\pi = 0$. If this happens then net consumption will decrease exponentially with time and future generations of our model economy can expect a life of low consumption.

Suppose our economy does progress at a constant rate of pollution reduction through technological advance. But suppose the rate of progress is not as great as the rate of population growth, that is

$$\pi < \rho$$

As long as this rate of progress in reducing pollution is less than the growth of population, the net consumption per capita will decrease exponentially with time. The only thing that such a rate of progress buys is time.

If our economy was able to provide progress in abatement of pollution at a rate equal to the rate of population growth then we can expect a level of net consumption which is growing exponentially with time.The gross consumption will at least be as good as existed at the time when the critical pollution level was reached. The first and second terms of equation (28) are constant and the pollution per capita term is decreasing. Since the pollution level is held constant, the cost of this pollution will be spread over more people with time due to the constant growth of the population. Whether or not this consumption level is satisfactory depends upon the parameters in equations (25) and (27). The key is (σ/θ), the ratio of the pollution dissipation rate and the pollution generation rate.

Of course, things will become even better when the rate of technological advance in pollution mitigation is greater than the rate of population growth.

This is seen in equation (25). For $\pi > \rho$, the equation indicates that the energy will increase exponentially with time. If the time difference between T and t_c is small, then we would expect that there might only be a short period in which the economy would expect to experience a deprived level of consumption. The energy level would eventually rise and accordingly so would the total output level, leading to more net consumption. This would occur only over the short period between t_c and T.

On the other hand, if the time difference between t_c and T is large, that is, the critical pollution level was reached much ahead of the planning horizon, then energy according to equation (25) could take on arbitrarily large values. If this is the case, then we could expect the energy level to reach that which corresponds to not being at the critical pollution level. This would then mean the optimal path would move off the pollution constraint and could ignore it the rest of the way. This is possible in this case, due to the fact, that pollution abatement technology will allow the total energy to increase without a corresponding increase in total pollution.

This latter case is of particular interest to our economy and some of the possible characteristics of this case can be illustrated by our phase and control plane diagrams. Figure 4 shows the time path of total pollution. The path begins below the critical level, moves to and along the critical level between times t_c and t_o. During this time, the pollution technology is improving the pollution problem until t_o is reached at which time the economy can move off the critical path.

Figure 5 presents the energy per capita path with time. As long as the

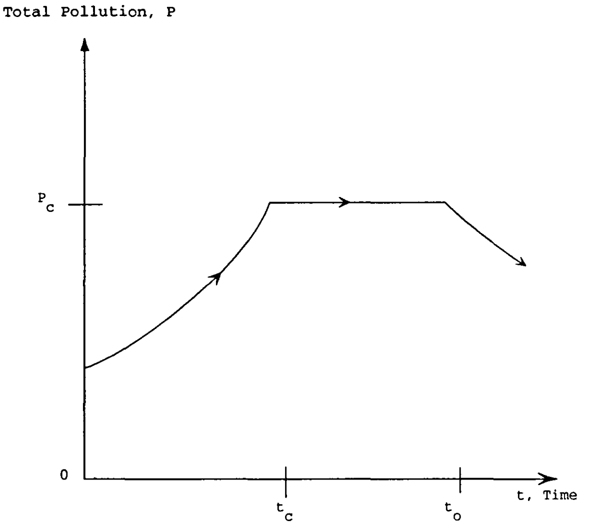

Figure 4: Total Pollution History

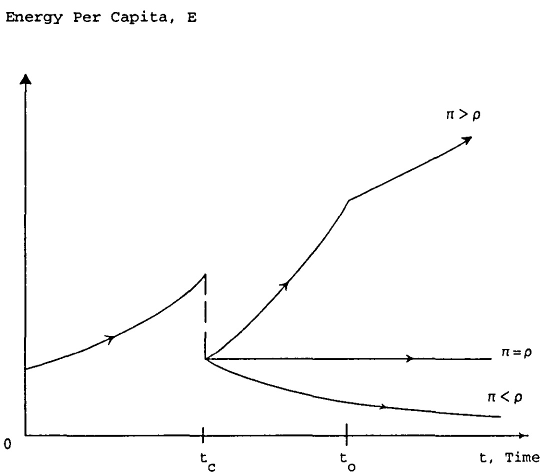

Figure 5: Energy Per Capita History

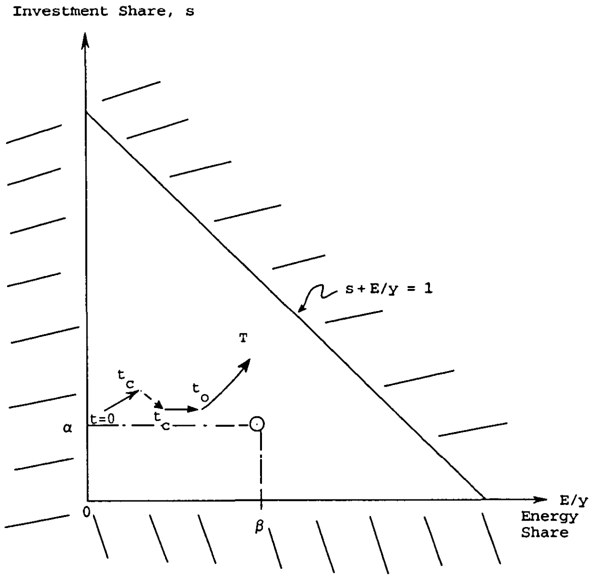

Figure 6: Output Allocation Path

critical pollution level is not yet reached, the energy per capita will increase as shown until t_c is reached. At this point, the critical pollution level was reached and there had to be an adjustment downward in energy per capita produced. This is shown by the dashed line at t_c. But due to the pollution abatement advance being larger than the population growth, the energy level will increase with time even though growth is along the critical pollution level. When time t_o is reached, the energy path will continue to increase in an environment in which pollution will be decreasing.

Figure 6 illustrates the control path. Between times $t=0$ and t_c, the path is moving away from the origin as both s and E/y are increasing. At t_c, the values of s and E/y will be discontinuous and the path moves in the direction shown between the two points t_c. During the time at which the growth path is along the critical pollution, the controls will move from t_c and t_o. Along this path, s is constant and E/y is increasing. When t_o is reached, the energy share will move toward β with the path shown between t_o and $t=T$.

6. SUMMARY AND CONCLUSIONS

The results of this paper can be summarized.

Given pollution abatement technological advance, the model economy can expect only relatively short periods of deprived net consumption if the rate of pollution abatement progress π is greater than the pollution growth rate. If $\pi=\rho$, then the net consumption level near the end of the planning horizon can expect to reach some maximum level of consumption set by the constants of the problem. This may or may not be a satisfactory level. If $\pi<\rho$, then the Malthusian results will develop.

The technical change factor analyzed is one directed at pollution abatement. This type of progress assumes that with time, advances are made toward the reduction of pollution generated. This type of progress will at least buy our economy some time with respect to the time when subsistence may set in. To insure a happy consumption for the future in this model, the following condition must exist:

$$\pi>\rho.$$

π is the rate of pollution abatement progress. The condition requires the pollution abatement progress rate to be greater than the population growth. If this requirement is met, then the economy can expect to move off the critical pollution level at some time in the future. The economy will eventually move toward pollution free growth.

A situation of most concern would be where the advances in pollution abatement were not enough to set off the population growth, that is, $\pi<\rho$. In this case the neutral technical advance would here compensate for this deficiency. This requires that $y> | \beta(\pi-\rho) |$.

Note that the rate of neutral technical advance does not have to compensate for the entire difference between pollution abatement advance and population growth rate since β is less than one. The compensation is less than the difference by a factor equal to the partial elasticity of production for capital. A second possible situation would be if the neutral technical advance was not sufficient to sustain a desired level of consumption, i.e.

$$\lambda<\beta\rho$$

then the pollution abatement advance must take up the slack. The requirement on this advance is that

$$\pi > \left| \frac{(\lambda-\beta\rho)}{\beta} \right|$$

Since β is less than unity, the pollution abatement advance must do more than just make up for the deficiency of the neutral progress.

The general requirement for a guaranteed future of satisfactory consumption in the combined technology case is

$$\lambda+\beta(\pi-\rho)>0$$

Because of the technical progress multiplier, small changes in the rate of neutral technical progress will provide much larger changes in the output growth. This can be to the economy's advantage if the neutral technical advance rate is as easy to improve as the pollution abatement advance rate.

However, the multiplier will be to the economy's detriment if the neutral progress rate is more difficult to improve.

There are two distinct policy implications that result from the analysis. The first is that:

> *An economy which takes into account the cost of pollution due to production should appropriately shift its use of the factors in production away from the polluting factor to the non–polluting factors. Here, the polluting factor is energy and the non–polluting factor is capital.*

This would be an expected result. For the particular economy of interest, the production process will become more capital intensive than it would be without considering pollution. An example of this policy in practice might be the emphasis of mass rapid transit rather than the automobile for commuter transportation. The mass rapid transit system would be more highly capital oriented than the automobile. Huge capital development is needed for the rapid transit system but its overall energy use and associated output is much lower than that of the automobile.

A second policy implication is that regarding priorities for research and development. This implies that:

Research and development priorities should emphasize progress in increased productivity through efficiency in production rather than progress in direct development of methods to reduce pollution.

This policy is implied by the fact that a multiplier effect exists with productivity advances. Such a multiplier does not exist with advances in the pollution reduction. The policy implies that the economy will reap more benefits by increasing its ability to produce more goods with a given quantity of factors for production, while holding pollution at a given level, than by using up its research resources to reduce pollution directly.

That is, since productivity is increasing, an economy can afford to reduce its use of a given polluting factor without a corresponding reduction in its desired consumption level. This fact is emphasized by the technical change multiplier effect. On the other hand, if advances are made only with respect to reduction of pollution, the economy's output per capita will always be limited.

APPENDIX A

A theorem of necessary conditions for a maximum in problems with both bounded controls and bounded state variables

The general problem is based on a theorem by L.W. Neustadt (1975) and involves a maximization problem where two inequality constraints exist. One inequality constraint is a function of a phase variable and time, the other is a function of the two control variables and one phase variable.

Necessary conditions for a problem involving both of these types of constraints has not yet appeared in the literature. The theorem is stated in general terms.

Suppose the control process is described by

$$\dot{x}(t) = f(x(t), u(t), t) \tag{A 1}$$

where x is a differentiable $(n+1)$–dimensional real vector function t and u is an r–dimensional real piecewise continuous vector function of time, and $0 \leq t \leq T$. The function f is continuously differentiable in x and continuous in u,x and t. The vector x has components $(x^0, x^1, \ldots x^n)$ and f has the components $(f^0, f^1, \ldots f^n)$.

The function to be maximized is

$$J = x^0(T) \tag{A 2}$$

There are two scalar constraints which must be satisfied at each instant of time. These are

$$\tilde{\chi}(x(t), t) \leq 0$$
$$\tilde{\tilde{\chi}}(x(t), u(t), t) \leq 0$$

where $\tilde{\chi}$ and $\tilde{\tilde{\chi}}$ are given scalar valued functions. Also the following must be satisfied at the initial and final times:

$$\chi(x(0), x(T)) = 0$$

where χ is a given m–dimensional vector–valued function χ must be once differentiable, $\tilde{\chi}$ twice differentiable and $\tilde{\tilde{\chi}}$ once differentiable.

THEOREM

Suppose $\bar{\chi}(t), \bar{u}(t)$ maximizes

$$J=x^0(T)$$

Subject to

$$\dot{x}(t)=f(x(t), u(t), t) \qquad 0\leq t\leq T$$

$$\tilde{\chi}(x(t), t)\leq 0 \qquad 0\leq t\leq T$$

$$\tilde{\tilde{\chi}}(x(t), u(t), t)\leq 0 \qquad 0\leq t< T$$

$$\chi(x(0), x(T))=0$$

$$u^i(t)\geq 0 \text{ for all } t,\ i$$

Then there exists an m-dimensional row vector

$$v=(v^1,\dots,v^m),$$

a number

$$v_o\geq 0,$$

scalar, piecewise continuous multiplier functions

$$\lambda(t), \xi(t)$$

and an auxiliary $(n+1)$-dimensional row vector function

$$\Psi(t)$$

which is differentiable such that

(i) $$\dot{\Psi}(t)=-\Psi(t)f_x(\bar{x}(t), \bar{u}(t), t) -\xi(t)\tilde{\tilde{\chi}}_x(\bar{x}(t), \bar{u}(t), t) +\lambda(t)Q_x(\bar{x}(t), t)$$

where $Q(x,t)=\tilde{\chi}_x(x, t)f(x,\bar{u}(t), t)+\tilde{\chi}_t(x,t)$.

f_x is the Jacobian matrix of partial derivatives of components of f with respect to the components of x, $\tilde{\chi}_x$, $\tilde{\tilde{\chi}}_x$ and Q_x are row vectors obtained by taking the partial derivatives of $\tilde{\chi}$, $\tilde{\tilde{\chi}}$ and Q with respect to the components of x.

$\tilde{\chi}_t$ is the partial derivative of $\tilde{\chi}$ with respect to t.

(ii)
$$\{\Psi(t) - \lambda(t)\tilde{\chi}_x(\bar{x}(t), t)\} f(\bar{x}(t), u(t), t) = \sup_{v \varepsilon \omega(\bar{x}(t), t)} \{\Psi(t) - \lambda(t)\tilde{\Psi}_x(\bar{x}(t), t)\} f(\bar{x}(t), v, t)$$

where

$$\omega(x,t) = \{v: v_1 \geq 0 \text{ for all } i, \tilde{\chi}(x, v, t) < 0\}$$

(iii) The following transversality conditions are satisfied:

$$\Psi(0) - \lambda(0)\tilde{x}_x(\bar{x}(0), 0) = -v\chi_{x_1}(\bar{x}(0), \bar{x}(T))$$
$$\Psi(T) = v\chi_{x_2}(\bar{x}(0), \bar{x}(T)) + (v^o, 0, \ldots 0)$$

where x_{x_1} and x_{x_2} are the Jacobian matrices obtained by taking the partial derivatives of x components with respect to the components of the first and second arguments of x which we denote as x_1 and x_2 respectively.

(iv) The following inequality holds:

$$\{\{\Psi(t) - \lambda(t)\tilde{\chi}_x(\bar{x}(t), t)\} f_u(\bar{x}(t), \bar{u}(t), t) + \xi(t)\tilde{\chi}_u(\bar{x}(t), \bar{u}(t), t\}\{v - \bar{u}(t)\} \leq 0$$

for all v such that $v^i \geq 0$ for aii i and where $\tilde{\tilde{\chi}}_u$ has the obvious meaning.

(v) The multiplier function $\lambda(t)$ is

a. non–increasing

b. constant on every interval on which $\tilde{\chi}(\bar{x}(t)t) < 0$

c. equal to zero at T

d. continuous from the right on the open interval $(0, T)$

(vi) The multiplier function $\xi(t)$ is

a. non-positive

b. zero for all t in which $\tilde{\chi}(x(t), u(t), t) < 0$

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