ON THE BREAK INTERVAL SEQUENCES OF EQUITABLE ROUND-ROBIN TOURNAMENTS

YUTAKA SUEYOSHI, RYUICHI HARASAWA and AICHI KUDO

Graduate School of Engineering
Nagasaki University
1-14 Bunkyo-machi
Nagasaki 852-8521, Japan
e-mail: {sueyoshi, harasawa, kudo}@cis.nagasaki-u.ac.jp

Abstract

We study the mathematical structure of equitable round-robin tournaments with home-away assignments, and give some necessary conditions for the feasibility of home-away tables, by using their friend-enemy tables and break interval sequences. We examine the relation of these conditions and enumerate the feasible break interval sequences. By our method, making schedules of equitable round-robin tournaments can be reduced to determining some sequences of positive integers satisfying certain inequalities.

1. Introduction

Making schedules of sports competitions such as professional football leagues, college basketball conference, $\cdots$ etc. is often too much time-consuming. There are some papers studying the mathematical structure of sports scheduling problems [1, 2, 3, 4, 5, 6, 12, 14], most of which are taking graph theoretical approach. For more recent papers using integer programming (IP), constraint programming (CP), metaheuristic approaches and combinations thereof, see the literature in [7, 8, 13].

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In [11], Miyashiro and Matsui studied the feasibility of home-away tables (HATs for short) of equitable round-robin tournaments by a different approach. Using a simple necessary condition for the feasibility of an equitable HAT, and solving some integer programming problems by computer search, they determined all the feasible HATs satisfying both the opening and the closing conditions, up to 20 teams.

Furthemore, in [9, 10], Miyashiro, Iwasaki and Matsui gave stronger necessary conditions for the feasibility of a HAT in round-robin tournaments. They conjecture that their conditions are also sufficient for round-robin tournaments with a minimum number of breaks, equivalently, for equitable round-robin tournaments, and showed that the proposed conditions are sufficient if the number of teams is less than or equal to 26, by computational experiment.

In this paper, we propose another approach which uses the *friend-enemy tables* and the *break interval sequences* of equitable round-robin tournaments, and obtain some necessary conditions for their feasibility, which are variants of Miyashiro-Iwasaki-Matsui's conditions. We examine the relation of these conditions, and enumerate the feasible break interval sequences and the corresponding HATs. For example, we can determine all the feasible HATs of equitable round-robin tournaments satisfying both the opening and the closing conditions, up to 26 teams.

2. Equitable Round-Robin Tournaments

In this section, we recall some definitions and basic facts from [1, 2, 9, 10, 11]. We consider a round-robin tournament consisiting of $2n$ teams ($n \geq 1$) and $2n - 1$ slots. In a round-robin tournament, each team must play one game against every other team. In each slot, each team plays one game, either at home or away. Table 1 is a *schedule* for a round-robin tournament consisiting of 8 teams, in which the rows are indexed by teams, the columns are indexed by slots, the entries of each row show the opponents of the team at different slots, and the home games are underlined.

Table 1 Schedule of 8 teams

	1	2	3	4	5	6	7
1	$\underline{7}$	8	2	$\underline{3}$	4	$\underline{5}$	6
2	$\underline{8}$	7	$\underline{1}$	$\underline{4}$	3	$\underline{6}$	5
3	$\underline{5}$	6	$\underline{7}$	1	$\underline{2}$	$\underline{4}$	8
4	$\underline{6}$	5	$\underline{8}$	2	$\underline{1}$	3	7
5	3	$\underline{4}$	$\underline{6}$	7	$\underline{8}$	1	$\underline{2}$
6	4	$\underline{3}$	5	8	$\underline{7}$	2	$\underline{1}$
7	1	$\underline{2}$	3	$\underline{5}$	6	8	$\underline{4}$
8	2	$\underline{1}$	4	$\underline{6}$	5	$\underline{7}$	$\underline{3}$

A home-away table (HAT) is the table which shows that each team plays whether a home-game or an away-game on each slot. Table 2 is the HAT corresponding to Table 1, where a home-game is denoted by 'H', and an away-game by 'A'.

Table 2 HAT corresponding to Table 1

	1	2	3	4	5	6	7
1	H	A	A	H	A	H	A
2	H	A	H	H	A	H	A
3	H	A	H	A	H	H	A
4	H	A	H	A	H	A	A
5	A	H	H	A	H	A	H
6	A	H	A	A	H	A	H
7	A	H	A	H	A	A	H
8	A	H	A	H	A	H	H

A HAT is called *feasible*, if there exists a schedule corresponding to it. A feasible HAT satisfies the following conditions (*consistency*):

- The rows of the HAT are mutually distinct.
- On each slot, the numbers of 'H's and 'A's coincide.

In the following, we assume that all HATs satisfy the consistency, and investigate the feasibility of HATs satisfying some extra conditions.

A row of a HAT is called a *home-away pattern* (HA-pattern for short). If an HA-pattern has consecutive 'H's or 'A's, then we say the HA-pattern has a *break*. In Table 2, every team has exactly one break. For example, teams 1 and 5 have a break on slot 3, teams 2 and 6 have a break on slot 4, and so on. Since there are only two HA-patterns with no breaks, "$HAHA\cdots AH$" and "$AHAH\cdots HA$", we have the following proposition.

Proposition 1 (de Werra [1, 2]). *For every HAT consisting of* $2n$ *teams, the number of breaks is greater than or equal to* $2n-2$.

Though a HAT with a minimum number of breaks may be optical, we rather consider a HAT in which every team has exactly one break, from the point of view of fairness. Such a HAT is called an *equitable* HAT, and the corresponding round-robin tournament is also called *equitable* [2]. For example, Table 1 is a schedule of an equitable round-robin tournament and Table 2 is the corresponding equitable HAT.

Remark 1. Since a HAT with a minimum number of breaks is equivalent to an equitable HAT by a cyclic rotation of the slots [9, 10], it makes no difference for the feasibility whether we consider a HAT with a minimum number of breaks or an equitable HAT.

If an HA-pattern x' is obtained by changing symbols 'H' for 'A' and 'A' for 'H' from an HA-pattern x, then we say x' is the *complement* of x and vice versa. In Table 2, the HA-patterns of teams $5, 6, 7$ and 8 are the complements of the HA-patterns of teams $1, 2, 3$ and 4, respectively.

In an equitable HAT, every slot (column) has at most two breaks, because otherwise at least two rows coincide, which contradicts the consistency.

Since every team (row) has exactly one break and each slot (column) has the same number of 'H's and 'A's, the complement of each row must be contained in an equitable HAT. Therefore we have the following proposition.

Proposition 2 (de Werra [1, 2]). *If an equitable HAT contains an HA-pattern* x*, then it also contains the complement of* x.

We say a round-robin tournament (or its HAT) satisfies the *opening* (resp.

closing) *condition* if it has no break on slot 2 (resp. on slot $2n-1$) [11]. For example, the HAT described in Table 2 satisfies the opening condition, but not the closing condition.

3. Break Interval Sequences

To determine whether a given equitable HAT is feasible or not, we introduce some simple necessary conditions which generarize the *triple break constraint* in [11]. First, we change the HAT of an equitable round-robin tournament to a so-called *friend-enemy table* (FET for short), which is also called a 0-1 *expression* in [10]. For example, the FET corresponding to Table 2 is the following Table 3.

Table 3 FET corresponding to Table 2

	1	2	3	4	5	6	7
1	◯	◯	◯	◯	◯	◯	◯
2	◯	◯	×	◯	◯	◯	◯
3	◯	◯	×	×	×	◯	◯
4	◯	◯	×	×	×	×	◯
5	×	×	×	×	×	×	×
6	×	×	◯	×	×	×	×
7	×	×	◯	◯	◯	×	×
8	×	×	◯	◯	◯	◯	×

On each slot of an FET, the teams with the same symbol (◯ or ×) cannot play a game with each other. We see easily from Table 3 that team 1 must play a game with team 2 on slot 3, team 3 must play a game with team 4 on slot 6, team 1 must play games with teams 3 and 4 on slots 4 and 5, and so on. In this way, we can easily make a schedule correspondig to Table 3, as described in Table 1.

We call a row of an FET a *friend-enemy pattern* (FE-pattern for short). In Table 3 obtained from Table 2, team 1 has a break on slot 3, where the friend teams $2, 3$ and 4 of team 1 go over to the enemy, and teams $2,\ 3$ and 4 have a break on slots $4,\ 6$ and 7 respectively, where they take sides with team 1 again. Since the FE-patterns of teams $5, 6, 7$ and 8 are the complements of the ones of teams $1, 2, 3$ and 4 respectively, the relations among teams $5, 6, 7$ and 8 are the same as the relations among teams $1, 2, 3$ and 4.

Conversely, if an FET consisting of $2n$ teams satisfies the following three conditions, with respect to a sequence $\{s_i\}_{1 \le i \le n}$ of positive integers satisfying $2 \le s_1 < s_2 < \cdots < s_n \le 2n - 1$, then the FET corresponds to an equitable HAT which has breaks on slots s_i $(1 \le i \le n)$:

1. The FE-pattern of team 1 (resp. $n + 1$) is "◯ ◯ $\cdots$ ◯" (resp. "$\times \times \cdots \times$").
2. The friend teams 2, 3, $\cdots$, n (resp. $n + 2$, $n + 3$, $\cdots$, $2n$) of team 1 (resp. $n + 1$) go over to the enemy on slot s_1.
3. Teams 2, 3, $\cdots$, n (resp. $n + 2$, $n + 3$, $\cdots$, $2n$) take sides with team 1 (resp. $n + 1$) again on slots s_2, s_3, $\cdots$, s_n, respectively.

We call an FET *equitable* and denote it by $(s_1, s_2, \cdots, s_n)$ for short, if it satisfies the three conditions above. For example, the FET in Table 3 can be denoted by (3, 4, 6, 7). Put $r_i = s_{i+1} - s_i$ for $1 \le i \le n - 1$, then r_i's are the intervals of successive breaks. In the following, we denote the sequence $\{r_i\}_{1 \le i \le n-1}$ by $\langle r_1, r_2, \cdots, r_{n-1} \rangle$ and call it the *break interval sequence* (BIS for short) of $(s_1, s_2, \cdots, s_n)$. For example, the BIS of (3, 4, 6, 7) is $\langle 1, 2, 1 \rangle$.

Remark 2. The sequence $(s_1, s_2, \cdots, s_n)$ associated with an equitable FET coincides with the *break sequence* considered in [2] for the corresponding HAT. If we add the term $r_n = s_1 - s_n + (2n-1)$ to the BIS $\langle r_1, r_2, \cdots, r_{n-1} \rangle$, then $\{r_i\}_{1 \le i \le n}$ coincides with the sequence considered in [5] for a HAT with a minimum number of breaks.

For a given equitable FET $(s_1, s_2, \cdots, s_n)$, the n teams 1, 2, $\cdots$, n (resp. $n+1$, $n+2$, $\cdots$, $2n$) must play games with each other in $s_n - s_1$ consecutive slots s_1, $s_1 + 1$, $\cdots$, $s_n - 1$. The total number of games among n teams is equal to ${}_nC_2$. On the other hand, we can have only one game among them on each slot $s \in \{s_1, \cdots, s_2 - 1\}$, and at most two games on each slot $s \in \{s_2, \cdots, s_3 - 1\}$, and so on. Therefore, we obtain a necessary condition for the feasibility of an FET, as follows.

Proposition 3 (*global condition*). *If an equitable FET* $(s_1,\ s_2,\ \cdots,\ s_n)$ *is feasible, then its BIS* $\langle r_1,\ r_2,\ \cdots,\ r_{n-1}\rangle$ *must satisfy the following condition*:

$$\begin{cases} r_1 + 2r_2 + \cdots + \frac{n}{2} r_{\frac{n}{2}} + \cdots + 2r_{n-2} + r_{n-1} \geq {}_nC_2 & (if\ n\ is\ even), \\ r_1 + 2r_2 + \cdots + \frac{n-1}{2}(r_{\frac{n-1}{2}} + r_{\frac{n+1}{2}}) + \cdots + 2r_{n-2} + r_{n-1} \geq {}_nC_2 & (if\ n\ is\ odd). \end{cases}$$

For example, the BIS $\langle 1,\ 2,\ 1\rangle$ satisfies the global condition, because

$$1 + 2 \cdot 2 + 1 = 6 \geq {}_4C_2.$$

Similarly, we obtain *local conditions* as follows. For each i $(1 \leq i \leq n-2)$, three teams i, $i+1$ and $i+2$ must play games with each other in $s_{i+2} - s_i$ consecutive slots s_i, $s_i + 1$, $\cdots$, s_{i+1}, $\cdots$, $s_{i+2} - 1$, so that a feasible FET must satisfy the *three-team condition*

$$r_i + r_{i+1} \geq {}_3C_2 = 3.$$

This is equivalent to that there are no consecutive 1's in a feasible BIS $\langle r_1,\ r_2,\ \cdots,\ r_{n-1}\rangle$. In other words, the breaks do not appear in three consecutive slots in a feasible FET. This condition is also called a *triple break constraint* in [11]. In general, we have the following proposition.

Proposition 4 (*m-team conditions*). *If an equitable FET* $(s_1,\ s_2,\ \cdots,\ s_n)$ $(n \geq 3)$ *is feasible, then its BIS* $\langle r_1,\ r_2,\ \cdots,\ r_{n-1}\rangle$ *must satisfy the following conditions*:

$$\begin{cases} r_i + 2r_{i+1} + \cdots + \frac{m}{2} r_{i+\frac{m-2}{2}} + \cdots + 2r_{i+m-3} + r_{i+m-2} \geq {}_mC_2 & (if\ m\ is\ even), \\ r_i + 2r_{i+1} + \cdots + \frac{m-1}{2}(r_{i+\frac{m-3}{2}} + r_{i+\frac{m-1}{2}}) + \cdots & \\ \qquad\qquad + 2r_{i+m-3} + r_{i+m-2} \geq {}_mC_2 & (if\ m\ is\ odd) \end{cases}$$

for each m $(3 \leq m \leq n)$ *and* i $(1 \leq i \leq n-m+1)$. *In particular, the n-team condition coincides with the global condition.*

Remark 3. The m-team condition for teams $i,\ i+1, \cdots, i+m-1$ coincides with Miyashiro-Iwasaki-Matsui's necessary condition [9, 10] for $T = \{i,\ i+1,\ \cdots,\ i+m-1\}$:

$$\sum_{s=1}^{2n-1} \min\{A(T,s),\ H(T,s)\} \geq \frac{|T|(|T|-1)}{2} = {}_mC_2,$$

where $A(T,s)$ (resp. $H(T,s)$) denotes the number of 'A's (resp. 'H's) in the slot s among T. They conjecture that the above conditions for all i, m with $1 \leq i \leq n$ and $3 \leq m \leq n$ are also sufficient for the feasibility of a HAT with a minimum number of breaks, equivalently, of an equitable HAT.

If all the three-team conditions are satisfied, then the four-team conditions are automatically satisfied, because

$$r_i + 2r_{i+1} + r_{i+2} = (r_i + r_{i+1}) + (r_{i+1} + r_{i+2}) \geq {}_3C_2 + {}_3C_2 = 6 = {}_4C_2.$$

On the other hand, the five-team conditions

$$r_i + 2r_{i+1} + 2r_{i+2} + r_{i+3} \geq {}_5C_2 = 10$$

for i ($1 \leq i \leq n-4$) can be used to reduce the number of candidates of feasible FETs. For example, the sequences $\langle 1,\ 2,\ 1,\ 2\rangle$ and $\langle 2,\ 1,\ 2,\ 1\rangle$ satisfy all the three-team conditions, but not the five-team condition, so that they never appear in a feasible FET. Conversely, if a sequence $\langle r_i,\ r_{i+1},\ r_{i+2},\ r_{i+3}\rangle$ satisfies all the three-team conditions and is equal to neither $\langle 1,\ 2,\ 1,\ 2\rangle$ nor $\langle 2,\ 1,\ 2,\ 1\rangle$, then it satisfies the five-team condition. Thus, we have the following theorem.

Theorem 1. *Let a sequence $\langle r_i,\ r_{i+1},\ r_{i+2},\ r_{i+3}\rangle$ satisfy all the three-team conditions, then it satisfies the five-team condition if and only if it is equal to neither $\langle 1,\ 2,\ 1,\ 2\rangle$ nor $\langle 2,\ 1,\ 2,\ 1\rangle$.*

Example. Let $(s_1,\ s_2,\ s_3,\ s_4,\ s_5)$ be an equitable FET satisfying the opening condition. Then we have $s_1 \geq 3$ and $s_5 \leq 9$, so that $r_1 + r_2 + r_3 + r_4 = s_5 - s_1 \leq 6$. Therefore, if $(s_1,\ s_2,\ s_3,\ s_4,\ s_5)$ is feasible, we must have $\langle r_1,\ r_2,\ r_3,\ r_4\rangle = \langle 1,\ 2,\ 2,\ 1\rangle$ by Theorem 1. Hence the FET is equal to (3, 4, 6, 8, 9) (Table 4) and the corresponding HAT is Table 5.

Table 4 FET $(3, 4, 6, 8, 9)$

	1	2	3	4	5	6	7	8	9
1	◯	◯	◯	◯	◯	◯	◯	◯	◯
2	◯	◯	×	◯	◯	◯	◯	◯	◯
3	◯	◯	×	×	×	◯	◯	◯	◯
4	◯	◯	×	×	×	×	×	◯	◯
5	◯	◯	×	×	×	×	×	×	◯
6	×	×	×	×	×	×	×	×	×
7	×	×	◯	×	×	×	×	×	×
8	×	×	◯	◯	◯	×	×	×	×
9	×	×	◯	◯	◯	◯	◯	×	×
10	×	×	◯	◯	◯	◯	◯	◯	×

Table 5 HAT corresponding to $(3, 4, 6, 8, 9)$

	1	2	3	4	5	6	7	8	9
1	H	A	A	H	A	H	A	H	A
2	H	A	H	H	A	H	A	H	A
3	H	A	H	A	H	H	A	H	A
4	H	A	H	A	H	A	H	H	A
5	H	A	H	A	H	A	H	A	A
6	A	H	H	A	H	A	H	A	H
7	A	H	A	A	H	A	H	A	H
8	A	H	A	H	A	A	H	A	H
9	A	H	A	H	A	H	A	A	H
10	A	H	A	H	A	H	A	H	H

Using these tables, we can easily make a schedule for the FET $(3, 4, 6, 8, 9)$ as in Table 6.

Table 6 Schedule for $(3, 4, 6, 8, 9)$ $\cdots$ $\langle 1,\ 2,\ 2,\ 1\rangle$

	1	2	3	4	5	6	7	8	9
1	8	9	2	3	4	5	6	7	10
2	9	10	1	5	3	7	4	8	6
3	10	7	8	1	2	4	5	6	9
4	6	8	10	9	1	3	2	5	7
5	7	6	9	2	10	1	3	4	8
6	4	5	7	8	9	10	1	3	2
7	5	3	6	10	8	2	9	1	4
8	1	4	3	6	7	9	10	2	5
9	2	1	5	4	6	8	7	10	3
10	3	2	4	7	5	6	8	9	1

Similarly, we can prove the following theorems.

Theorem 2. *If a sequence $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+4}\rangle$ satisfies all the three-team and the five-team conditions, then it also satisfies the six-team condition.*

Proof. It follows from the three-team and the five-team conditions that

$$\begin{aligned} r_i + 2r_{i+1} + 3r_{i+2} + 2r_{i+3} + r_{i+4} &\geq {}_5C_2 + (r_{i+2} + r_{i+3} + r_{i+4}) \\ &\geq 10 + 4 = 14. \end{aligned}$$

If the sequence $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+4}\rangle$ does not satisfy the six-team condition, then the equalities must hold. Hence we have $\langle r_{i+2},\ r_{i+3},\ r_{i+4}\rangle = \langle 1,\ 2,\ 1\rangle$ by the three-team conditions. Similarly, we have $\langle r_i,\ r_{i+1},\ r_{i+2}\rangle = \langle 1,\ 2,\ 1\rangle$, and therefore $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+4}\rangle = \langle 1,\ 2,\ 1,\ 2,\ 1\rangle$, which does not satisfy the five-team conditions by Theorem 1. This contradicts our assumption. Hence $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+4}\rangle$ must satisfy the six-team condition. □

Theorem 3. *If a sequence $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+5}\rangle$ satisfies all the three-team and the five-team conditions, then it satisfies the seven-team condition if and only if it is equal to none of the following sequences*: $\langle 1,\ 2,\ 2,\ 1,\ 2,\ 2\rangle$, $\langle 2,\ 1,\ 2,\ 2,\ 1,\ 2\rangle$, $\langle 2,\ 2,\ 1,\ 2,\ 2,\ 1\rangle$.

Proof. If a sequence $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+5}\rangle$ satisfies all the three-team and the five-team conditions, then we have

$$\begin{aligned} & r_i + 2r_{i+1} + 3r_{i+2} + 3r_{i+3} + 2r_{i+4} + r_{i+5} \\ = \ & (r_i + 2r_{i+1} + 2r_{i+2} + r_{i+3}) + (r_{i+2} + 2r_{i+3} + 2r_{i+4} + r_{i+5}) \\ \geq \ & {}_5C_2 + {}_5C_2 = 20. \end{aligned}$$

If $\langle r_i, r_{i+1}, \cdots, r_{i+5}\rangle$ does not satisfy the seven-team condition, then the equality must hold. Hence the sequences $\langle r_i, r_{i+1}, r_{i+2}, r_{i+3}\rangle$ and $\langle r_{i+2}, r_{i+3}, r_{i+4}, r_{i+5}\rangle$ must be equal to one of the following sequences, respectively:

$$\langle 1, 2, 1, 3\rangle, \quad \langle 1, 2, 2, 1\rangle, \quad \langle 2, 1, 2, 2\rangle, \quad \langle 2, 2, 1, 2\rangle, \quad \langle 3, 1, 2, 1\rangle.$$

Joining these sequences and checking the five-team and the seven-team conditions, we see that $\langle r_i, r_{i+1}, \cdots, r_{i+5}\rangle$ must be equal to one of the following sequences: $\langle 1, 2, 2, 1, 2, 2\rangle$, $\langle 2, 1, 2, 2, 1, 2\rangle$, $\langle 2, 2, 1, 2, 2, 1\rangle$. The converse is straightforward. □

Theorem 4. *If a sequence $\langle r_i, r_{i+1}, \cdots, r_{i+6}\rangle$ satisfies all the j-team conditions for $j = 3$, 5 and 7, then it also satisfies the eight-team condition.*

Proof. It follows from the j-team conditions for $j = 3$, 5 and 7 that

$$\begin{aligned} & r_i + 2r_{i+1} + 3r_{i+2} + 4r_{i+3} + 3r_{i+4} + 2r_{i+5} + r_{i+6} \\ \geq \ & {}_7C_2 + (r_{i+3} + r_{i+4} + r_{i+5} + r_{i+6}) \geq 21 + 6 = 27. \end{aligned}$$

If the sequence $\langle r_i, r_{i+1}, \cdots, r_{i+6}\rangle$ does not satisfy the eight-team condition, then we must have $\langle r_{i+3}, r_{i+4}, r_{i+5}, r_{i+6}\rangle = \langle 1, 2, 2, 1\rangle$ by Theorem 1. Similarly, we have $\langle r_i, r_{i+1}, r_{i+2}, r_{i+3}\rangle = \langle 1, 2, 2, 1\rangle$, and therefore $\langle r_i, r_{i+1}, \cdots, r_{i+6}\rangle = \langle 1, 2, 2, 1, 2, 2, 1\rangle$, which does not satisfy the seven-team conditions by Theorem 3. This contradicts our assumption, and therefore $\langle r_i, r_{i+1}, \cdots, r_{i+6}\rangle$ must satisfy the eight-team condition. □

Theorem 5. *If a sequence $\langle r_i, r_{i+1}, \cdots, r_{i+7}\rangle$ satisfies all the j-team conditions for $j = 3$, 5 and 7, then it satisfies the nine-team condition if and only if it is equal to none of the following sequences:*

$$\begin{aligned} &\langle 1, 2, 1, 3, 1, 2, 1, 3\rangle, \ \langle 1, 2, 1, 3, 1, 2, 2, 1\rangle, \ \langle 1, 2, 2, 1, 3, 1, 2, 1\rangle, \\ &\langle 1, 2, 2, 2, 1, 2, 2, 2\rangle, \ \langle 1, 3, 1, 2, 1, 3, 1, 2\rangle, \ \langle 2, 1, 2, 2, 1, 3, 1, 2\rangle, \\ &\langle 2, 1, 2, 2, 2, 1, 2, 2\rangle, \ \langle 2, 1, 3, 1, 2, 1, 3, 1\rangle, \ \langle 2, 1, 3, 1, 2, 2, 1, 2\rangle, \\ &\langle 2, 2, 1, 2, 2, 2, 1, 2\rangle, \ \langle 2, 2, 2, 1, 2, 2, 2, 1\rangle, \ \langle 3, 1, 2, 1, 3, 1, 2, 1\rangle. \end{aligned}$$

Proof. If a sequence $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+7}\rangle$ satisfies all the j-team conditions for $j = 3$, 5 and 7, then we have

$$\begin{aligned} & r_i + 2r_{i+1} + 3r_{i+2} + 4r_{i+3} + 4r_{i+4} + 3r_{i+5} + 2r_{i+6} + r_{i+7} \\ \geq\ & {}_8C_2 + (r_{i+4} + r_{i+5} + r_{i+6} + r_{i+7}) \geq 28 + 6 = 34. \end{aligned}$$

If $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+7}\rangle$ does not satisfy the nine-team condition, then $r_{i+4} + r_{i+5} + r_{i+6} + r_{i+7}$ must be equal to either 6 or 7. Hence the sequence $\langle r_{i+4},\ r_{i+5},\ r_{i+6},\ r_{i+7}\rangle$ is equal to one of the following sequences:

$$\begin{aligned} &\langle 1,\ 2,\ 1,\ 3\rangle(20, 15),\ \langle 1,\ 2,\ 2,\ 1\rangle(15, 15),\ \langle 1,\ 2,\ 2,\ 2\rangle(19, 16), \\ &\langle 1,\ 2,\ 3,\ 1\rangle(18, 17),\ \langle 1,\ 3,\ 1,\ 2\rangle(18, 17),\ \langle 1,\ 3,\ 2,\ 1\rangle(17, 18), \\ &\langle 2,\ 1,\ 2,\ 2\rangle(18, 17),\ \langle 2,\ 1,\ 3,\ 1\rangle(17, 18),\ \langle 2,\ 2,\ 1,\ 2\rangle(17, 18), \\ &\langle 2,\ 2,\ 2,\ 1\rangle(16, 19),\ \langle 3,\ 1,\ 2,\ 1\rangle(15, 20). \end{aligned}$$

Similarly, $\langle r_i,\ r_{i+1},\ r_{i+2},\ r_{i+3}\rangle$ is also equal to one of the above sequences (in parentheses, we described $r_i + 2r_{i+1} + 3r_{i+2} + 4r_{i+3}$ and $4r_{i+4} + 3r_{i+5} + 2r_{i+6} + r_{i+7}$, respectively). Joining these sequences and checking the j-team conditions for $j = 3$, 5, 7 and 9, we see that $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+7}\rangle$ must be equal to one of the sequences described in the theorem. The converse is straightforward. □

Theorem 6. *If a sequence* $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+8}\rangle$ *satisfies all the* j*-team conditions for* $j = 3$, 5, 7 *and* 9, *then it satisfies the ten-team condition if and only if it is equal to neither* $\langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2,\ 1\rangle$ *nor* $\langle 1,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle$.

Proof. If a sequence $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+8}\rangle$ satisfies all the j-team conditions for $j = 3$, 5, 7 and 9, then we have

$$\begin{aligned} & r_i + 2r_{i+1} + 3r_{i+2} + 4r_{i+3} + 5r_{i+4} + 4r_{i+5} + 3r_{i+6} + 2r_{i+7} + r_{i+8} \\ \geq\ & {}_9C_2 + (r_{i+4} + \cdots + r_{i+8}) \geq 36 + 8 = 44. \end{aligned}$$

If $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+8}\rangle$ does not satisfy the ten-team condition, then the equalities must hold. Hence the sequence $\langle r_{i+4},\ r_{i+5},\ r_{i+6},\ r_{i+7},\ r_{i+8}\rangle$ is equal to one of the following sequences:

$\langle 1,\ 2,\ 1,\ 3,\ 1\rangle(25, 23)$, $\langle 1,\ 2,\ 2,\ 1,\ 2\rangle(25, 23)$, $\langle 1,\ 2,\ 2,\ 2,\ 1\rangle(24, 24)$,
$\langle 1,\ 3,\ 1,\ 2,\ 1\rangle(23, 25)$, $\langle 2,\ 1,\ 2,\ 2,\ 1\rangle(23, 25)$.

Similarly, $\langle r_i,\ r_{i+1},\ r_{i+2},\ r_{i+3},\ r_{i+4}\rangle$ is also equal to one of the above sequences (in parentheses, we described $r_i+2r_{i+1}+\cdots+5r_{i+4}$ and $5r_{i+4}+4r_{i+5}+\cdots+r_{i+8}$, respectively). Joining these sequences and checking the j-team conditions for $j = 5$, 7, 9 and 10, we see that the sequence $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+8}\rangle$ must be equal to either $\langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2,\ 1\rangle$ or $\langle 1,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle$. The converse is straightforward. $\square$

4. Construction of Equitable Tournaments

In this section, using the necessary conditions introduced in §3, we construct equitable round-robin tournaments satisfying both the opening and the closing conditions.

Let $(s_1,\ s_2,\ \cdots,\ s_n)$ be a feasible (and equitable) FET satisfying the opening and the closing conditions, and let $\langle r_1,\ r_2,\ \cdots,\ r_{n-1}\rangle$ be its BIS. Since $(s_1,\ s_2,\ \cdots,\ s_n)$ satisfies the opening and the closing conditions, we have $s_1 \geq 3$ and $s_n \leq 2n-2$, so that $r_1 + r_2 + \cdots + r_{n-1} = s_n - s_1 \leq 2n-5$. On the other hand, we have $r_i + r_{i+1} \geq 3$ for any i $(1 \leq i \leq n-2)$ by the three-team conditions, so that

$$r_1 + r_2 + \cdots + r_{n-1} \geq \left[\frac{3}{2}(n-1)\right] = \begin{cases} \frac{3}{2}(n-1) - \frac{1}{2} & (n:\ \text{even}), \\ \frac{3}{2}(n-1) & (n:\ \text{odd}). \end{cases}$$

Therefore, we have $\frac{1}{2}n \geq 3$ (resp. $\frac{1}{2}n \geq \frac{7}{2}$), hence $2n \geq 12$ if n is even (resp. $2n \geq 14$ if n is odd).

(i) The case $2n = 12$: Since $r_1+r_2+\cdots+r_5 \leq 7$, we have $\langle r_1,\ r_2,\ r_3,\ r_4,\ r_5\rangle = \langle 1,\ 2,\ 1,\ 2,\ 1\rangle$ by the three-team conditions, which does not satisfy the five-team conditions by Theorem 1. Therefore, there is no feasible (and equitable) FET with 12 teams satisfying both the opening and the closing conditions.

(ii) The case $2n = 14$: Since $r_1 + r_2 + \cdots + r_6 \leq 9$, the sequence $\langle r_1,\ r_2,\ \cdots,\ r_6\rangle$ contains three 1's and three 2's by the three-team conditions.

However, by Theorem 1, there is no BIS consisting of three 1's and three 2's which satisfies all the three-team and the five-team conditions. Therefore, there is no feasible (and equitable) FET with 14 teams satisfying both the opening and the closing conditions.

(iii) The case $2n = 16$: Since $r_1 + r_2 + \cdots + r_7 \leq 11$, $r_1 + r_2 + r_3 \geq 4$ and $r_5 + r_6 + r_7 \geq 4$, we have $r_4 \leq 3$. If $r_4 = 1$, then we have $\langle r_1, r_2, \cdots, r_7 \rangle = \langle 1, 2, 2, 1, 2, 2, 1 \rangle$ by Theorem 1, which does not satisfy the seven-team conditions by Theorem 3. If $r_4 = 2$, then we have $\langle r_1, r_2, \cdots, r_7 \rangle = \langle 1, 2, 1, 2, \cdots \rangle$ or $\langle \cdots, 2, 1, 2, 1 \rangle$, neither of which satisfies the five-team conditions by Theorem 1. If $r_4 = 3$, then we have $\langle r_1, r_2, \cdots, r_7 \rangle = \langle 1, 2, 1, 3, 1, 2, 1 \rangle$, which satisfies all the local conditions. The corresponding FET is (3, 4, 6, 7, 10, 11, 13, 14). In this case, the global condition holds exactly:

$$1 + 2 \cdot 2 + 3 \cdot 1 + 4 \cdot 3 + 3 \cdot 1 + 2 \cdot 2 + 1 = 28 = {}_8C_2.$$

Table 7 is a part of the FET $(3, 4, 6, 7, 10, 11, 13, 14)$ for eight teams $\{1, 2, \cdots, 8\}$.

Table 7 Part of FET $(3, 4, 6, 7, 10, 11, 13, 14)$

	3	4	5	6	7	8	9	10	11	12	13
1	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
2	×	◯	◯	◯	◯	◯	◯	◯	◯	◯	◯
3	×	×	×	◯	◯	◯	◯	◯	◯	◯	◯
4	×	×	×	×	◯	◯	◯	◯	◯	◯	◯
5	×	×	×	×	×	×	×	◯	◯	◯	◯
6	×	×	×	×	×	×	×	×	◯	◯	◯
7	×	×	×	×	×	×	×	×	×	×	◯
8	×	×	×	×	×	×	×	×	×	×	×

Using Table 7, we can easily make a schedule for these eight teams, as in Table 8.

Table 8 Part of Schedule for $(3, 4, 6, 7, 10, 11, 13, 14)$

	3	4	5	6	7	8	9	10	11	12	13
1	2	$\underline{3}$	4	$\underline{5}$	6	$\underline{7}$	8				
2	$\underline{1}$	$\underline{4}$	3	$\underline{6}$	5	$\underline{8}$	7				
3		1	$\underline{2}$	$\underline{4}$	8	$\underline{5}$	6	$\underline{7}$			
4		2	$\underline{1}$	3	7	$\underline{6}$	5	$\underline{8}$			
5				1	$\underline{2}$	3	$\underline{4}$	$\underline{6}$	7	$\underline{8}$	
6				2	$\underline{1}$	4	$\underline{3}$	5	8	$\underline{7}$	
7					$\underline{4}$	1	$\underline{2}$	3	$\underline{5}$	6	8
8					$\underline{3}$	2	$\underline{1}$	4	$\underline{6}$	5	$\underline{7}$

Similarly, we can make a schedule for the remaining eight teams. Using Table 8 and its counterpart for $\{9, 10, \cdots, 16\}$, we can easily make a schedule for $(3, 4, 6, 7, 10, 11, 13, 14)$, as in Table 9. Thus, $(3, 4, 6, 7, 10, 11, 13, 14)$ is the only feasible (and equitable) FET consisting of 16 teams satisfying both the opening and the closing conditions.

Table 9 Schedule for $(3, 4, 6, 7, 10, 11, 13, 14)$ $\cdots$ $\langle 1, 2, 1, 3, 1, 2, 1 \rangle$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	$\underline{13}$	14	2	$\underline{3}$	4	$\underline{5}$	6	$\underline{7}$	8	$\underline{9}$	10	$\underline{11}$	12	$\underline{15}$	16
2	$\underline{14}$	13	$\underline{1}$	$\underline{4}$	3	$\underline{6}$	5	$\underline{8}$	7	$\underline{10}$	9	$\underline{12}$	11	$\underline{16}$	15
3	$\underline{15}$	16	$\underline{11}$	1	$\underline{2}$	$\underline{4}$	8	$\underline{5}$	6	$\underline{7}$	12	$\underline{9}$	10	$\underline{13}$	14
4	$\underline{16}$	15	$\underline{12}$	2	$\underline{1}$	3	7	$\underline{6}$	5	$\underline{8}$	11	$\underline{10}$	9	$\underline{14}$	13
5	$\underline{9}$	10	$\underline{15}$	16	$\underline{13}$	1	$\underline{2}$	3	$\underline{4}$	$\underline{6}$	7	$\underline{8}$	14	$\underline{11}$	12
6	$\underline{10}$	9	$\underline{16}$	15	$\underline{14}$	2	$\underline{1}$	4	$\underline{3}$	5	8	$\underline{7}$	13	$\underline{12}$	11
7	$\underline{11}$	12	$\underline{13}$	14	$\underline{15}$	16	$\underline{4}$	1	$\underline{2}$	3	$\underline{5}$	6	8	$\underline{9}$	10
8	$\underline{12}$	11	$\underline{14}$	13	$\underline{16}$	15	$\underline{3}$	2	$\underline{1}$	4	$\underline{6}$	5	$\underline{7}$	$\underline{10}$	9
9	5	$\underline{6}$	$\underline{10}$	11	$\underline{12}$	13	$\underline{14}$	15	$\underline{16}$	1	$\underline{2}$	3	$\underline{4}$	7	$\underline{8}$
10	6	$\underline{5}$	9	12	$\underline{11}$	14	$\underline{13}$	16	$\underline{15}$	2	$\underline{1}$	4	$\underline{3}$	8	$\underline{7}$
11	7	$\underline{8}$	3	$\underline{9}$	10	12	$\underline{16}$	13	$\underline{14}$	15	$\underline{4}$	1	$\underline{2}$	5	$\underline{6}$
12	8	$\underline{7}$	4	$\underline{10}$	9	$\underline{11}$	$\underline{15}$	14	$\underline{13}$	16	$\underline{3}$	2	$\underline{1}$	6	$\underline{5}$
13	1	$\underline{2}$	7	$\underline{8}$	6	$\underline{9}$	10	$\underline{11}$	12	14	$\underline{15}$	16	$\underline{5}$	3	$\underline{4}$
14	2	$\underline{1}$	8	$\underline{7}$	5	$\underline{10}$	9	$\underline{12}$	11	$\underline{13}$	$\underline{16}$	15	$\underline{6}$	4	$\underline{3}$
15	3	$\underline{4}$	5	$\underline{6}$	8	$\underline{7}$	12	$\underline{9}$	10	$\underline{11}$	13	$\underline{14}$	$\underline{16}$	1	$\underline{2}$
16	4	$\underline{3}$	6	$\underline{5}$	7	$\underline{8}$	11	$\underline{10}$	9	$\underline{12}$	14	$\underline{13}$	15	2	$\underline{1}$

(iv) The case $2n = 18$: Since $r_1 + r_2 + \cdots + r_8 \leq 13$, $r_1 + \cdots + r_4 \geq 6$ and $r_5 + \cdots + r_8 \geq 6$, we have $\langle r_1, r_2, \cdots, r_8 \rangle = \langle 1, 2, 2, 1, 3, 1, 2, 1 \rangle$ or $\langle 1, 2, 1, 3, 1, 2, 2, 1 \rangle$ by Theorems 1 and 3. However, these sequences do not satisy the nine-team condition by Theorem 5. Therefore, there is no feasible (and equitable) FET with 18 teams satisfying both the opening and the closing conditions.

(v) The case $2n = 20$: Since $r_1 + r_2 + \cdots + r_9 \leq 15$, $r_1 + \cdots + r_4 \geq 6$ and $r_6 + \cdots + r_9 \geq 6$, we have $r_5 \leq 3$. If $r_5 = 1$, then we have $r_1 + r_2 + r_3 + r_4 = r_6 + r_7 + r_8 + r_9 = 7$ and $\langle r_1, r_2, \cdots, r_9 \rangle = \langle 1, 2, 1, 3, 1, 3, 1, 2, 1 \rangle$ by Theorems 1, 3, 5 and 6. This BIS satisfies all the local conditions and the corresponding FET is (3, 4, 6, 7, 10, 11, 14, 15, 17, 18). In fact, this FET is feasible. In a similar way as in the case $2n = 16$, we can make its schedule. Table 10 is a part of a schedule.

Table 10 Part of Schedule for $(3, 4, 6, 7, 10, 11, 14, 15, 17, 18)$ $\cdots$ $\langle 1, 2, 1, 3, 1, 3, 1, 2, 1 \rangle$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	$\underline{17}$	18	2	$\underline{3}$	4	$\underline{5}$	6	$\underline{7}$	8	$\underline{9}$	10	$\underline{11}$	12	$\underline{13}$	14	$\underline{15}$	16	$\underline{19}$	20
2	$\underline{18}$	17	$\underline{1}$	$\underline{4}$	3	$\underline{6}$	8	$\underline{9}$	5	$\underline{10}$	14	$\underline{7}$	11	$\underline{12}$	13	$\underline{16}$	15	$\underline{20}$	19
3	$\underline{19}$	20	$\underline{17}$	1	$\underline{2}$	$\underline{4}$	5	$\underline{6}$	10	$\underline{7}$	8	$\underline{13}$	9	$\underline{11}$	12	$\underline{14}$	18	$\underline{15}$	16
4	$\underline{20}$	19	$\underline{18}$	2	$\underline{1}$	3	7	$\underline{5}$	6	$\underline{8}$	12	$\underline{9}$	10	$\underline{14}$	11	$\underline{13}$	17	$\underline{16}$	15
5	$\underline{13}$	14	$\underline{20}$	15	$\underline{19}$	1	$\underline{3}$	4	$\underline{2}$	$\underline{6}$	7	$\underline{10}$	8	$\underline{9}$	16	$\underline{11}$	12	$\underline{17}$	18
6	$\underline{14}$	13	$\underline{19}$	16	$\underline{20}$	2	$\underline{1}$	3	$\underline{4}$	5	9	$\underline{8}$	7	$\underline{10}$	15	$\underline{12}$	11	$\underline{18}$	17
7	$\underline{15}$	16	$\underline{13}$	20	$\underline{18}$	19	$\underline{4}$	1	$\underline{17}$	3	$\underline{5}$	2	$\underline{6}$	$\underline{8}$	10	$\underline{9}$	14	$\underline{11}$	12
8	$\underline{16}$	15	$\underline{14}$	19	$\underline{17}$	20	$\underline{2}$	18	$\underline{1}$	4	$\underline{3}$	6	$\underline{5}$	7	9	$\underline{10}$	13	$\underline{12}$	11
9	$\underline{11}$	12	$\underline{16}$	18	$\underline{15}$	17	$\underline{20}$	2	$\underline{19}$	1	$\underline{6}$	4	$\underline{3}$	5	$\underline{8}$	7	10	$\underline{13}$	14
10	$\underline{12}$	11	$\underline{15}$	17	$\underline{16}$	18	$\underline{19}$	20	$\underline{3}$	2	$\underline{1}$	5	$\underline{4}$	6	$\underline{7}$	8	$\underline{9}$	$\underline{14}$	13

If $r_5 = 2$ or 3, then we have $\langle r_1, r_2, \cdots, r_9 \rangle = \langle 1, 2, 2, 1, 2, 3, 1, 2, 1 \rangle$, $\langle 1, 2, 1, 3, 2, 1, 2, 2, 1 \rangle$ or $\langle 1, 2, 2, 1, 3, 1, 2, 2, 1 \rangle$ by Theorems 1, 3, 5 and 6. These BIS's satisfy all the local conditions and the corresponding FETs are (3, 4, 6, 8, 9, 11, 14, 15, 17, 18), (3, 4, 6, 7, 10, 12, 13, 15, 17, 18) or (3, 4, 6, 8, 9, 12, 13, 15, 17, 18), respectively. In fact, they are also feasible and we can make their schedules in a similar way as above.

Thus we have 4 feasible (and equitable) FETs with 20 teams satisfying both the opening and the closing conditions. For all of these feasible FETs, the global conditions hold exactly: $r_1 + 2r_2 + \cdots + 5r_5 + \cdots + r_9 = 45 = {}_{10}C_2$.

(vi) The case $2n = 22$: Since $r_1 + r_2 + \cdots + r_{10} \leq 17$, and since $r_1 + \cdots + r_5 \geq 8$ and $r_6 + \cdots + r_{10} \geq 8$ by the three-team and the five-team conditions, we have $r_1 + \cdots + r_5 = 8$ or $r_6 + \cdots + r_{10} = 8$. If $r_1 + \cdots + r_5 = 8$, then we have $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5 \leq 25$, where the equality holds if and only if $\langle r_1, \cdots, r_5 \rangle = \langle 1, 2, 1, 3, 1 \rangle$ or $\langle 1, 2, 2, 1, 2 \rangle$. Therefore, we must have $r_6 + \cdots + r_{10} = 9$ and $5r_6 + 4r_7 + 3r_8 + 2r_9 + r_{10} \geq 30$ by the global condition. Hence we have $\langle r_6, \cdots, r_{10} \rangle = \langle 2, 3, 1, 2, 1 \rangle$ or $\langle 3, 1, 2, 2, 1 \rangle$, both of which satisfy $5r_6 + 4r_7 + 3r_8 + 2r_9 + r_{10} = 30$. Therefore, we have $\langle r_1, r_2, \cdots, r_{10} \rangle = \langle 1, 2, 1, 3, 1, 2, 3, 1, 2, 1 \rangle$, $\langle 1, 2, 1, 3, 1, 3, 1, 2, 2, 1 \rangle$ or $\langle 1, 2, 2, 1, 2, 3, 1, 2, 2, 1 \rangle$ by Theorem 3. These BIS's satisfy all the local conditions and the corresponding FETs are (3, 4, 6, 7, 10, 11, 13, 16, 17, 19, 20), (3, 4, 6, 7, 10, 11, 14, 15, 17, 19, 20) or (3, 4, 6, 8, 9, 11, 14, 15, 17, 19, 20), respectively. In fact, these FETs are feasible. For example, Table 11 is a part of a schedule for (3, 4, 6, 8, 9, 11, 14, 15, 17, 19, 20).

Table 11 Part of Schedule for (3, 4, 6, 8, 9, 11, 14, 15, 17, 19, 20) $\cdots$ $\langle 1, 2, 2, 1, 2, 3, 1, 2, 2, 1 \rangle$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	$\underline{19}$	20	2	$\underline{3}$	4	$\underline{5}$	6	$\underline{7}$	8	$\underline{9}$	10	$\underline{11}$	12	$\underline{13}$	14	$\underline{15}$	16	$\underline{17}$	18	$\underline{21}$	22
2	$\underline{20}$	18	$\underline{1}$	$\underline{5}$	3	$\underline{6}$	4	$\underline{8}$	7	$\underline{10}$	13	$\underline{9}$	11	$\underline{14}$	12	$\underline{16}$	15	$\underline{19}$	17	$\underline{22}$	21
3	$\underline{21}$	22	$\underline{19}$	1	$\underline{2}$	$\underline{4}$	5	$\underline{6}$	9	$\underline{7}$	11	$\underline{8}$	10	$\underline{12}$	13	$\underline{14}$	17	$\underline{15}$	16	$\underline{18}$	20
4	$\underline{22}$	19	$\underline{21}$	20	$\underline{1}$	3	$\underline{2}$	$\underline{5}$	11	$\underline{6}$	8	$\underline{7}$	9	$\underline{10}$	15	$\underline{12}$	13	$\underline{16}$	14	$\underline{17}$	18
5	$\underline{18}$	21	$\underline{22}$	2	$\underline{17}$	1	$\underline{3}$	4	6	$\underline{8}$	9	$\underline{10}$	7	$\underline{11}$	16	$\underline{13}$	12	$\underline{14}$	15	$\underline{20}$	19
6	$\underline{15}$	14	$\underline{20}$	22	$\underline{21}$	2	$\underline{1}$	3	$\underline{5}$	4	7	$\underline{17}$	8	$\underline{9}$	10	$\underline{11}$	18	$\underline{12}$	13	$\underline{19}$	16
7	$\underline{16}$	15	$\underline{18}$	21	$\underline{22}$	20	$\underline{19}$	1	$\underline{2}$	3	$\underline{6}$	4	$\underline{5}$	$\underline{8}$	11	$\underline{9}$	14	$\underline{10}$	12	$\underline{13}$	17
8	$\underline{17}$	16	$\underline{14}$	18	$\underline{20}$	22	$\underline{21}$	2	$\underline{1}$	5	$\underline{4}$	3	$\underline{6}$	7	9	$\underline{10}$	11	$\underline{13}$	19	$\underline{15}$	12
9	$\underline{14}$	17	$\underline{15}$	19	$\underline{16}$	21	$\underline{18}$	22	$\underline{3}$	1	$\underline{5}$	2	$\underline{4}$	6	$\underline{8}$	7	10	$\underline{11}$	20	$\underline{12}$	13
10	$\underline{13}$	12	$\underline{17}$	15	$\underline{19}$	18	$\underline{22}$	20	$\underline{21}$	2	$\underline{1}$	5	$\underline{3}$	4	$\underline{6}$	8	$\underline{9}$	7	11	$\underline{16}$	14
11	$\underline{12}$	13	$\underline{16}$	17	$\underline{18}$	19	$\underline{20}$	21	$\underline{4}$	22	$\underline{3}$	1	$\underline{2}$	5	$\underline{7}$	6	$\underline{8}$	9	$\underline{10}$	$\underline{14}$	15

The case $r_6 + \cdots + r_{10} = 8$ can be treated similarly.

Thus we have 6 feasible (and equitable) FETs with 22 teams satisfying both the opening and the closing conditions. For all of these feasible FETs, the global conditions hold exactly: $r_1 + 2r_2 + \cdots + 5r_5 + 5r_6 + \cdots + r_{10} = 55 = {}_{11}C_2$.

(vii) The case $2n = 24$: Since $r_1 + r_2 + \cdots + r_{11} \leq 19$, and since $r_1 + \cdots + r_5 \geq 8$ and $r_7 + \cdots + r_{11} \geq 8$ by the three-team and the five-team conditions, we have $r_6 \leq 3$. If $r_6 = 1$, then we must have $r_1 + \cdots + r_5 = r_7 + \cdots + r_{11} = 9$ by the three-team and the five-team conditions. Therefore, we have $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5 \leq 30$, where the equality holds if and only if $\langle r_1, \cdots, r_5 \rangle = \langle 1, 2, 1, 3, 2 \rangle$ or $\langle 1, 2, 2, 1, 3 \rangle$. Similarly, we have $5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11} \leq 30$, where the equality holds if and only if $\langle r_7, \cdots, r_{11} \rangle = \langle 2, 3, 1, 2, 1 \rangle$ or $\langle 3, 1, 2, 2, 1 \rangle$. By the global condition, we must have $\langle r_1, r_2, \cdots, r_{11} \rangle = \langle 1, 2, 1, 3, 2, 1, 2, 3, 1, 2, 1 \rangle$, $\langle 1, 2, 1, 3, 2, 1, 3, 1, 2, 2, 1 \rangle$, $\langle 1, 2, 2, 1, 3, 1, 2, 3, 1, 2, 1 \rangle$ or $\langle 1, 2, 2, 1, 3, 1, 3, 1, 2, 2, 1 \rangle$.

These BIS's satisfy all the local conditions and the corresponding FETs are (3, 4, 6, 7, 10, 12, 13, 15, 18, 19, 21, 22), (3, 4, 6, 7, 10, 12, 13, 16, 17, 19, 21, 22), (3, 4, 6, 8, 9, 12, 13, 15, 18, 19, 21, 22) or (3, 4, 6, 8, 9, 12, 13, 16, 17, 19, 21, 22), respectively. In fact, these FETs are feasible. For example, Table 12 is a part of a schedule for (3, 4, 6, 7, 10, 12, 13, 15, 18, 19, 21, 22).

Table 12 Part of Schedule for $(3, 4, 6, 7, 10, 12, 13, 15, 18, 19, 21, 22)$ $\cdots$ $\langle 1, 2, 1, 3, 2, 1, 2, 3, 1, 2, 1\rangle$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
1	$\underline{21}$	22	2	$\underline{3}$	4	$\underline{5}$	6	$\underline{7}$	8	$\underline{9}$	10	$\underline{11}$	12	$\underline{13}$	14	$\underline{15}$	16	$\underline{17}$	18	$\underline{19}$	20	$\underline{23}$	24
2	$\underline{22}$	21	$\underline{1}$	$\underline{4}$	3	$\underline{6}$	5	$\underline{8}$	7	$\underline{10}$	9	$\underline{12}$	11	$\underline{14}$	13	$\underline{16}$	18	$\underline{15}$	17	$\underline{20}$	19	$\underline{24}$	23
3	$\underline{23}$	24	$\underline{16}$	1	$\underline{2}$	$\underline{4}$	7	$\underline{5}$	11	$\underline{8}$	6	$\underline{9}$	15	$\underline{12}$	10	$\underline{13}$	17	$\underline{14}$	19	$\underline{18}$	21	$\underline{20}$	22
4	$\underline{24}$	23	$\underline{15}$	2	$\underline{1}$	3	12	$\underline{6}$	5	$\underline{7}$	8	$\underline{10}$	16	$\underline{11}$	19	$\underline{14}$	13	$\underline{9}$	20	$\underline{17}$	18	$\underline{22}$	21
5	$\underline{17}$	18	$\underline{24}$	23	$\underline{21}$	1	$\underline{2}$	3	$\underline{4}$	$\underline{6}$	7	$\underline{8}$	9	$\underline{10}$	12	$\underline{11}$	14	$\underline{13}$	15	$\underline{16}$	22	$\underline{19}$	20
6	$\underline{13}$	20	$\underline{23}$	24	$\underline{22}$	2	$\underline{1}$	4	$\underline{18}$	5	$\underline{3}$	$\underline{7}$	8	$\underline{9}$	11	$\underline{12}$	15	$\underline{10}$	16	$\underline{14}$	17	$\underline{21}$	19
7	$\underline{20}$	19	$\underline{22}$	21	$\underline{24}$	23	$\underline{3}$	1	$\underline{2}$	4	$\underline{5}$	6	10	$\underline{8}$	16	$\underline{9}$	11	$\underline{12}$	13	$\underline{15}$	14	$\underline{17}$	18
8	$\underline{19}$	16	$\underline{21}$	22	$\underline{23}$	24	$\underline{20}$	2	$\underline{1}$	3	$\underline{4}$	5	$\underline{6}$	7	9	$\underline{10}$	12	$\underline{11}$	14	$\underline{13}$	15	$\underline{18}$	17
9	$\underline{18}$	14	$\underline{20}$	19	$\underline{17}$	22	$\underline{23}$	24	$\underline{21}$	1	$\underline{2}$	3	$\underline{5}$	6	$\underline{8}$	7	$\underline{10}$	$\underline{4}$	12	$\underline{11}$	13	$\underline{15}$	16
10	$\underline{14}$	17	$\underline{19}$	20	$\underline{18}$	21	$\underline{22}$	23	$\underline{24}$	2	$\underline{1}$	4	$\underline{7}$	5	$\underline{3}$	8	$\underline{9}$	6	11	$\underline{12}$	16	$\underline{13}$	15
11	$\underline{15}$	13	$\underline{18}$	17	$\underline{20}$	19	$\underline{21}$	22	$\underline{3}$	24	$\underline{23}$	1	$\underline{2}$	4	$\underline{6}$	5	$\underline{7}$	8	$\underline{10}$	9	12	$\underline{16}$	14
12	$\underline{16}$	15	$\underline{17}$	18	$\underline{19}$	20	$\underline{4}$	21	$\underline{22}$	23	$\underline{24}$	2	$\underline{1}$	3	$\underline{5}$	6	$\underline{8}$	7	$\underline{9}$	10	$\underline{11}$	$\underline{14}$	13

If $r_6 = 2$ and $r_1 + \cdots + r_5 = 8$, then we have $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5 \leq 25$, so that we must have $r_7 + \cdots + r_{11} = 9$ and $5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11} \geq 29$ by the global condition. Since $5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11} \leq 30$, we must have $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5 = 24$ or 25, and $5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11} = 29$ or 30. Therefore, $\langle r_1, \cdots, r_5\rangle$ is equal to either $\langle 1, 2, 1, 3, 1\rangle(25)$ or $\langle 1, 2, 2, 2, 1\rangle(24)$ by Theorems 1 and 3 (in parentheses, we described $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5$). Similarly, $\langle r_7, \cdots, r_{11}\rangle$ is equal to one of the following sequences (in parentheses, we described $5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11}$): $\langle 2, 3, 1, 2, 1\rangle(30)$, $\langle 1, 4, 1, 2, 1\rangle(29)$, $\langle 3, 1, 2, 2, 1\rangle(30)$, $\langle 2, 2, 2, 2, 1\rangle(29)$. Joining these sequences and checking the global condition, we obtain 6 BIS's $\langle r_1, r_2, \cdots, r_{11}\rangle$ as follows (in parentheses, we described $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5 + 6r_6 + 5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11}$):

$$\langle 1, 2, 1, 3, 1, 2, 2, 3, 1, 2, 1\rangle(67),\ \langle 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1\rangle(66),$$
$$\langle 1, 2, 1, 3, 1, 2, 3, 1, 2, 2, 1\rangle(67),\ \langle 1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1\rangle(66),$$
$$\langle 1, 2, 2, 2, 1, 2, 2, 3, 1, 2, 1\rangle(66),\ \langle 1, 2, 2, 2, 1, 2, 3, 1, 2, 2, 1\rangle(66).$$

These sequences satisfy all the local conditions and the corresponding FETs are feasible. The case $r_7 + \cdots + r_{11} = 8$ can be treated similarly, and we obtain 6 feasible BIS's.

If $r_6 = 3$, then we have $r_1 + \cdots + r_5 = r_7 + \cdots + r_{11} = 8$. Since $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5 \leq 25$ and $5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11} \leq 25$, we must have $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5 \geq 23$, and $5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11} \geq 23$ by the global condition. Therefore, $\langle r_1, \cdots, r_5 \rangle$ and $\langle r_7, \cdots, r_{11} \rangle$ are equal to one of the following sequences, respectively (in parentheses, we described $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5$ and $5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11}$, respectively):

$$\langle 1, 2, 1, 3, 1 \rangle (25, 23), \ \langle 1, 2, 2, 1, 2 \rangle (25, 23), \ \langle 1, 2, 2, 2, 1 \rangle (24, 24),$$
$$\langle 1, 3, 1, 2, 1 \rangle (23, 25), \ \langle 2, 1, 2, 2, 1 \rangle (23, 25).$$

Joining these sequences and checking the global condition, we obtain 17 BIS's $\langle r_1, r_2, \cdots, r_{11} \rangle$ as follows (in parentheses, we described $r_1 + 2r_2 + 3r_3 + 4r_4 + 5r_5 + 6r_6 + 5r_7 + 4r_8 + 3r_9 + 2r_{10} + r_{11}$):

$$\langle 1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 1 \rangle (66), \ \langle 1, 2, 1, 3, 1, 3, 1, 2, 2, 1, 2 \rangle (66),$$
$$\langle 1, 2, 1, 3, 1, 3, 1, 2, 2, 2, 1 \rangle (67), \ \langle 1, 2, 1, 3, 1, 3, 1, 3, 1, 2, 1 \rangle (68),$$
$$\langle 1, 2, 1, 3, 1, 3, 2, 1, 2, 2, 1 \rangle (68), \ \langle 1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 1 \rangle (66),$$
$$\langle 1, 2, 2, 1, 2, 3, 1, 2, 2, 1, 2 \rangle (66), \ \langle 1, 2, 2, 1, 2, 3, 1, 2, 2, 2, 1 \rangle (67),$$
$$\langle 1, 2, 2, 1, 2, 3, 1, 3, 1, 2, 1 \rangle (68), \ \langle 1, 2, 2, 1, 2, 3, 2, 1, 2, 2, 1 \rangle (68),$$
$$\langle 1, 2, 2, 2, 1, 3, 1, 2, 2, 2, 1 \rangle (66), \ \langle 1, 2, 2, 2, 1, 3, 1, 3, 1, 2, 1 \rangle (67),$$
$$\langle 1, 2, 2, 2, 1, 3, 2, 1, 2, 2, 1 \rangle (67), \ \langle 1, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1 \rangle (66),$$
$$\langle 1, 3, 1, 2, 1, 3, 2, 1, 2, 2, 1 \rangle (66), \ \langle 2, 1, 2, 2, 1, 3, 1, 3, 1, 2, 1 \rangle (66),$$
$$\langle 2, 1, 2, 2, 1, 3, 2, 1, 2, 2, 1 \rangle (66).$$

These sequences satisfy all the local conditions and the corresponding FETs are feasible.

Thus we have 33 feasible (and equitable) FETs with 24 teams satisfying both the opening and the closing conditions.

(viii) The case $2n = 26$: Since $r_1 + r_2 + \cdots + r_{12} \leq 21$, and since $r_1 + \cdots + r_6 \geq 10$ and $r_7 + \cdots + r_{12} \geq 10$ by the three-team and the five-team conditions, we have $r_1 + \cdots + r_6 = 10$ or $r_7 + \cdots + r_{12} = 10$. If $r_1 + \cdots + r_6 = 10$, then we have $r_1 + 2r_2 + \cdots + 6r_6 \leq 37$, where the equality holds if and only if $\langle r_1, \cdots, r_6 \rangle = \langle 1, 2, 1, 3, 1, 2 \rangle$. Therefore, we must have $r_7 + \cdots + r_{12} = 11$ and $6r_7 + 5r_8 + \cdots + r_{12} \geq 41$ by the global condition. In this case, we have $6r_7 + 5r_8 + \cdots + r_{12} \leq 43$, where the equality holds if and only if $\langle r_7, \cdots, r_{12} \rangle = \langle 3, 1, 3, 1, 2, 1 \rangle$ or $\langle 3, 2, 1, 2, 2, 1 \rangle$, so that we must have $r_1 + 2r_2 + \cdots + 6r_6 \geq 35$ by the global condition. Hence $\langle r_1, \cdots, r_6 \rangle$ is equal to one of the following sequences (in parentheses, we described $r_1 + 2r_2 + \cdots + 6r_6$):

$$\langle 1,\ 2,\ 1,\ 3,\ 1,\ 2\rangle(37),\ \langle 1,\ 2,\ 1,\ 3,\ 2,\ 1\rangle(36),\ \langle 1,\ 2,\ 2,\ 1,\ 3,\ 1\rangle(36),$$
$$\langle 1,\ 2,\ 2,\ 2,\ 1,\ 2\rangle(36),\ \langle 1,\ 2,\ 2,\ 2,\ 2,\ 1\rangle(35).$$

Similarly, $\langle r_7, \cdots, r_{12}\rangle$ is equal to one of the following sequences (in parentheses, we described $6r_7 + 5r_8 + \cdots + r_{12}$):

$$\langle 3,\ 1,\ 3,\ 1,\ 2,\ 1\rangle(43),\ \langle 2,\ 2,\ 3,\ 1,\ 2,\ 1\rangle(42),\ \langle 1,\ 3,\ 3,\ 1,\ 2,\ 1\rangle(41),$$
$$\langle 2,\ 1,\ 4,\ 1,\ 2,\ 1\rangle(41),\ \langle 3,\ 2,\ 1,\ 2,\ 2,\ 1\rangle(43),\ \langle 2,\ 3,\ 1,\ 2,\ 2,\ 1\rangle(42),$$
$$\langle 1,\ 4,\ 1,\ 2,\ 2,\ 1\rangle(41),\ \langle 3,\ 1,\ 2,\ 2,\ 2,\ 1\rangle(42),\ \langle 2,\ 2,\ 2,\ 2,\ 2,\ 1\rangle(41),$$
$$\langle 3,\ 1,\ 2,\ 1,\ 3,\ 1\rangle(41),\ \langle 3,\ 1,\ 2,\ 2,\ 1,\ 2\rangle(41).$$

Joining these sequences, we obtain 28 BIS's which satisfy the global condition. Among them, only the sequences $\langle 1, 2, 1, 3, 1, 2, 1, 3, 3, 1, 2, 1\rangle$, $\langle 1, 2, 1, 3, 1, 2, 2, 1, 4, 1, 2, 1\rangle$ and $\langle 1, 2, 2, 2, 1, 2, 2, 2, 3, 1, 2, 1\rangle$ do not satisfy the nine-team conditions by Theorem 5. All of others satisfy all the local conditions and the corresponding FETs are feasible. For example, Table 13 is a part of a schedule for (3, 4, 6, 7, 10, 11, 13, 16, 17, 19, 21, 22, 24), whose BIS is $\langle 1, 2, 1, 3, 1, 2, 3, 1, 2, 2, 1, 2\rangle$.

Table 13 Part of Schedule for $(3, 4, 6, 7, 10, 11, 13, 16, 17, 19, 21, 22, 24)$ $\cdots$ $\langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 3,\ 1,\ 2,\ 2,\ 1,\ 2\rangle$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
1	23	24	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	25	26
2	24	22	1	4	3	6	5	9	7	11	13	10	15	8	12	14	17	16	19	18	21	20	23	26	25
3	26	23	25	1	2	4	7	5	6	10	12	9	13	11	8	16	15	14	17	20	22	19	24	18	21
4	21	25	26	2	1	3	8	6	5	7	9	12	10	17	13	11	14	15	20	16	19	23	18	22	24
5	25	26	16	24	23	1	2	3	4	6	7	8	11	12	9	10	13	18	15	14	17	22	20	21	19
6	20	21	24	25	26	2	1	4	3	5	8	7	9	10	11	13	19	12	16	17	18	14	15	23	22
7	22	19	17	26	18	20	3	1	2	4	5	6	8	9	10	12	11	13	14	15	16	25	21	24	23
8	16	20	22	23	24	25	4	26	1	21	6	5	7	2	3	9	10	11	12	13	14	15	17	19	18
9	19	18	23	21	25	26	24	2	22	1	4	3	6	7	5	8	12	10	13	11	15	16	14	17	20
10	18	14	21	22	19	24	26	25	23	3	1	2	4	6	7	5	8	9	11	12	13	17	16	20	15
11	17	16	20	18	22	23	25	21	26	2	24	1	5	3	6	4	7	8	10	9	12	13	19	15	14
12	15	17	19	20	21	22	23	24	25	26	3	4	1	5	2	7	9	6	8	10	11	18	13	14	16
13	14	15	18	19	20	21	22	23	24	25	2	26	3	1	4	6	5	7	9	8	10	11	12	16	17

The case $r_7 + \cdots + r_{12} = 10$ can be treated similarly.

Thus we have 50 feasible (and equitable) FETs with 26 teams satisfying both the opening and the closing conditions.

Finally, we summarize the results in Table 14, where we show all the feasible (and equitable) BIS's and the corresponding FETs for $2n \leq 26$.

Table 14 Feasible BIS's and the corresponding FETs for $2n \le 26$

$2n$	feasible BIS's	corresponding FETs
16	⟨1, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 14)
20	⟨1, 2, 1, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 15, 17, 18)
	⟨1, 2, 2, 1, 2, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 15, 17, 18)
	⟨1, 2, 1, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 12, 13, 15, 17, 18)
	⟨1, 2, 2, 1, 3, 1, 2, 2, 1⟩	(3, 4, 6, 8, 9, 12, 13, 15, 17, 18)
22	⟨1, 2, 1, 3, 1, 2, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 16, 17, 19, 20)
	⟨1, 2, 1, 3, 1, 3, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 15, 17, 19, 20)
	⟨1, 2, 2, 1, 2, 3, 1, 2, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 15, 17, 19, 20)
	⟨1, 2, 1, 3, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 12, 13, 16, 17, 19, 20)
	⟨1, 2, 2, 1, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 12, 13, 16, 17, 19, 20)
	⟨1, 2, 2, 1, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 8, 9, 12, 14, 15, 17, 19, 20)
24	⟨1, 2, 1, 3, 2, 1, 2, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 12, 13, 15, 18, 19, 21, 22)
	⟨1, 2, 1, 3, 2, 1, 3, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 12, 13, 16, 17, 19, 21, 22)
	⟨1, 2, 2, 1, 3, 1, 2, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 12, 13, 15, 18, 19, 21, 22)
	⟨1, 2, 2, 1, 3, 1, 3, 1, 2, 2, 1⟩	(3, 4, 6, 8, 9, 12, 13, 16, 17, 19, 21, 22)
	⟨1, 2, 1, 3, 1, 2, 2, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 15, 18, 19, 21, 22)
	⟨1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 14, 18, 19, 21, 22)
	⟨1, 2, 1, 3, 1, 2, 3, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 16, 17, 19, 21, 22)
	⟨1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 15, 17, 19, 21, 22)
	⟨1, 2, 2, 2, 1, 2, 2, 3, 1, 2, 1⟩	(3, 4, 6, 8, 10, 11, 13, 15, 18, 19, 21, 22)
	⟨1, 2, 2, 2, 1, 2, 3, 1, 2, 2, 1⟩	(3, 4, 6, 8, 10, 11, 13, 16, 17, 19, 21, 22)
	⟨1, 2, 1, 3, 2, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 12, 14, 15, 18, 19, 21, 22)
	⟨1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 11, 12, 14, 15, 18, 19, 21, 22)
	⟨1, 2, 2, 1, 3, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 12, 14, 15, 18, 19, 21, 22)
	⟨1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 10, 12, 14, 15, 18, 19, 21, 22)
	⟨1, 2, 1, 3, 2, 2, 1, 2, 2, 2, 1⟩	(3, 4, 6, 7, 10, 12, 14, 15, 17, 19, 21, 22)
	⟨1, 2, 2, 1, 3, 2, 1, 2, 2, 2, 1⟩	(3, 4, 6, 8, 9, 12, 14, 15, 17, 19, 21, 22)
	⟨1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 1⟩	(3, 4, 6, 7, 10, 11, 14, 15, 17, 18, 21, 22)
	⟨1, 2, 1, 3, 1, 3, 1, 2, 2, 1, 2⟩	(3, 4, 6, 7, 10, 11, 14, 15, 17, 19, 20, 22)
	⟨1, 2, 1, 3, 1, 3, 1, 2, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 15, 17, 19, 21, 22)
	⟨1, 2, 1, 3, 1, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 15, 18, 19, 21, 22)
	⟨1, 2, 1, 3, 1, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 16, 17, 19, 21, 22)
	⟨1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 1⟩	(3, 4, 6, 8, 9, 11, 14, 15, 17, 18, 21, 22)
	⟨1, 2, 2, 1, 2, 3, 1, 2, 2, 1, 2⟩	(3, 4, 6, 8, 9, 11, 14, 15, 17, 19, 20, 22)
	⟨1, 2, 2, 1, 2, 3, 1, 2, 2, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 15, 17, 19, 21, 22)
	⟨1, 2, 2, 1, 2, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 15, 18, 19, 21, 22)
	⟨1, 2, 2, 1, 2, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 16, 17, 19, 21, 22)
	⟨1, 2, 2, 2, 1, 3, 1, 2, 2, 2, 1⟩	(3, 4, 6, 8, 10, 11, 14, 15, 17, 19, 21, 22)
	⟨1, 2, 2, 2, 1, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 10, 11, 14, 15, 18, 19, 21, 22)

24	⟨1, 2, 2, 2, 1, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 8, 10, 11, 14, 16, 17, 19, 21, 22)
	⟨1, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1⟩	(3, 4, 7, 8, 10, 11, 14, 15, 18, 19, 21, 22)
	⟨1, 3, 1, 2, 1, 3, 2, 1, 2, 2, 1⟩	(3, 4, 7, 8, 10, 11, 14, 16, 17, 19, 21, 22)
	⟨2, 1, 2, 2, 1, 3, 1, 3, 1, 2, 1⟩	(3, 5, 6, 8, 10, 11, 14, 15, 18, 19, 21, 22)
	⟨2, 1, 2, 2, 1, 3, 2, 1, 2, 2, 1⟩	(3, 5, 6, 8, 10, 11, 14, 16, 17, 19, 21, 22)
26	⟨1, 2, 1, 3, 1, 2, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 2, 2, 2, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 15, 17, 20, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 2, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 16, 18, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 2, 2, 3, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 15, 18, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 14, 18, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 2, 3, 1, 2, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 16, 17, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 13, 15, 17, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 2, 3, 1, 2, 1, 3, 1⟩	(3, 4, 6, 7, 10, 11, 13, 16, 17, 19, 20, 23, 24)
	⟨1, 2, 1, 3, 1, 2, 3, 1, 2, 2, 1, 2⟩	(3, 4, 6, 7, 10, 11, 13, 16, 17, 19, 21, 22, 24)
	⟨1, 2, 1, 3, 2, 1, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 12, 13, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 1, 3, 2, 1, 2, 2, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 12, 13, 15, 17, 20, 21, 23, 24)
	⟨1, 2, 1, 3, 2, 1, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 12, 13, 16, 18, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 2, 1, 2, 3, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 12, 13, 15, 18, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 2, 1, 3, 1, 2, 2, 2, 1⟩	(3, 4, 6, 7, 10, 12, 13, 16, 17, 19, 21, 23, 24)
	⟨1, 2, 2, 1, 3, 1, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 12, 13, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 1, 3, 1, 2, 2, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 12, 13, 15, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 1, 3, 1, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 8, 9, 12, 13, 16, 18, 19, 21, 23, 24)
	⟨1, 2, 2, 1, 3, 1, 2, 3, 1, 2, 2, 1⟩	(3, 4, 6, 8, 9, 12, 13, 15, 18, 19, 21, 23, 24)
	⟨1, 2, 2, 1, 3, 1, 3, 1, 2, 2, 2, 1⟩	(3, 4, 6, 8, 9, 12, 13, 16, 17, 19, 21, 23, 24)
	⟨1, 2, 2, 2, 1, 2, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 10, 11, 13, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 2, 1, 2, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 8, 10, 11, 13, 16, 18, 19, 21, 23, 24)
	⟨1, 2, 2, 2, 1, 2, 2, 3, 1, 2, 2, 1⟩	(3, 4, 6, 8, 10, 11, 13, 15, 18, 19, 21, 23, 24)
	⟨1, 2, 2, 2, 1, 2, 3, 1, 2, 2, 2, 1⟩	(3, 4, 6, 8, 10, 11, 13, 16, 17, 19, 21, 23, 24)
	⟨1, 2, 2, 2, 2, 1, 3, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 10, 12, 13, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 2, 2, 1, 3, 2, 1, 2, 2, 1⟩	(3, 4, 6, 8, 10, 12, 13, 16, 18, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 3, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 1, 3, 2, 2, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 12, 14, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 1, 2, 3, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 1, 3, 2, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 12, 14, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 13, 14, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 2, 1, 3, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 10, 11, 14, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1⟩	(3, 4, 6, 8, 10, 12, 14, 16, 17, 20, 21, 23, 24)
	⟨1, 3, 1, 2, 1, 3, 2, 1, 3, 1, 2, 1⟩	(3, 4, 7, 8, 10, 11, 14, 16, 17, 20, 21, 23, 24)
	⟨2, 1, 2, 2, 1, 3, 2, 1, 3, 1, 2, 1⟩	(3, 5, 6, 8, 10, 11, 14, 16, 17, 20, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 3, 1, 2, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 15, 17, 20, 21, 23, 24)
	⟨1, 2, 1, 3, 2, 2, 1, 2, 3, 1, 2, 1⟩	(3, 4, 6, 7, 10, 12, 14, 15, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 1, 2, 3, 1, 2, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 15, 17, 20, 21, 23, 24)

	⟨1, 2, 2, 1, 3, 2, 1, 2, 3, 1, 2, 1⟩	(3, 4, 6, 8, 9, 12, 14, 15, 17, 20, 21, 23, 24)
	⟨1, 2, 2, 2, 1, 3, 1, 2, 3, 1, 2, 1⟩	(3, 4, 6, 8, 10, 11, 14, 15, 17, 20, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 3, 1, 3, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 15, 18, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 2, 2, 1, 3, 1, 2, 2, 1⟩	(3, 4, 6, 7, 10, 12, 14, 15, 18, 19, 21, 23, 24)
	⟨1, 2, 2, 1, 2, 3, 1, 3, 1, 2, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 15, 18, 19, 21, 23, 24)
	⟨1, 2, 2, 1, 3, 2, 1, 3, 1, 2, 2, 1⟩	(3, 4, 6, 8, 9, 12, 14, 15, 18, 19, 21, 23, 24)
26	⟨1, 2, 2, 2, 1, 3, 1, 3, 1, 2, 2, 1⟩	(3, 4, 6, 8, 10, 11, 14, 15, 18, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 3, 2, 1, 2, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 16, 17, 19, 21, 23, 24)
	⟨1, 2, 2, 1, 2, 3, 2, 1, 2, 2, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 16, 17, 19, 21, 23, 24)
	⟨1, 2, 2, 1, 3, 2, 2, 1, 2, 2, 2, 1⟩	(3, 4, 6, 8, 9, 12, 14, 16, 17, 19, 21, 23, 24)
	⟨1, 2, 2, 2, 1, 3, 2, 1, 2, 2, 2, 1⟩	(3, 4, 6, 8, 10, 11, 14, 16, 17, 19, 21, 23, 24)
	⟨1, 2, 1, 3, 1, 3, 1, 2, 2, 2, 2, 1⟩	(3, 4, 6, 7, 10, 11, 14, 15, 17, 19, 21, 23, 24)
	⟨1, 2, 2, 1, 2, 3, 1, 2, 2, 2, 2, 1⟩	(3, 4, 6, 8, 9, 11, 14, 15, 17, 19, 21, 23, 24)

Remark 4. From the above discussions, the local conditions seem to be sufficient for the feasibility of equitable round-robin tournaments satisfying both the opening and the closing conditions. However, we can construct a counterexample for this. For example, in the case $2n = 36$, the BIS ⟨1, 2, 1, 3, 1, 2, 1, 5, 1, 2, 1, 3, 1, 3, 1, 2, 1⟩ for the FET (3, 4, 6, 7, 10, 11, 13, 14, 19, 20, 22, 23, 26, 27, 30, 31, 33, 34) satisfies all the local conditions, but is infeasible. To see this, for any FET $(s_1, s_2, \cdots, s_n)$ we unite the slots $2n-1$ and 1 cyclically as in [9, 10] and consider the local conditions for n teams $\{i, i{+}1, \cdots, i{+}n{-}1\}$ and $2n{-}1$ consecutive slots $\{s_i, s_i{+}1, \cdots, 2n{-}1, 1, \cdots, s_i-1\}$. The corresponding BIS is $\langle r_i, \cdots, r_{n-1}, r_n, r_1, \cdots, r_{i-2}\rangle$, where $r_1 + \cdots + r_{n-1} + r_n = 2n-1$. In particular, for the above FET and 18 teams $\{9, 10, \cdots, 26\}$, the corresponding BIS is ⟨1, 2, 1, 3, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1⟩, which does not satisfy the global condition and therefore is infeasible.

5. Conclusion

We studied the mathematical structure of equitable round-robin tournaments with home-away assignments, and proposed an approach which uses friend-enemy tables and break interval sequences. We obtained some necessary conditions for the feasibility of equitable home-away tables, by using their break interval sequences. By checking some inequalities for the break interval sequences and using the corresponding friend-enemy tables, we determined all the feasible home-away tables of equitable round-robin tournaments satisfying both the opening and the closing conditions, up to 26 teams.

It is still an open problem whether Miyashiro-Iwasaki-Matsui's necessary conditions are also sufficient for the feasibility of home-away tables of equitable round-robin tournaments, or equivalently, of round-robin tournaments with a minimum number of breaks.

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