# Nonlinear Response of an Annular Sector Plate Subjected to In-Plane Dynamic Moment

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The nonlinear dynamic instability of an annular sector plate subjected to equal and opposite time-varying moments is examined. The equation of motion describing a large deflection of the annular sector plate based upon Berger's approximate equation is analyzed by the Galerkin method.
The resulting equations for time variables are integrated by using the Runge-Kutta-Gill method. Numerical results are presented for various boundary conditions, damping forces, and static moments.

## I. INTRODUCTION

Out-of-plane vibrations of a thin plate may be observed under in-plane periodic forces by reason of parametric excitation.[1] Since the parametric resonance may induce fatigue cracks or acoustic radiations, it is important to clarify the conditions under which the unstable motions occur.

The dynamic stability of an annular sector plate such as a web plate of an arch rib or corner member of a rigid-frame is examined in this study.

The dynamic stability which is determined by the small deflection theory has been presented.[2] The amplitudes of unstable regions become infinite under assumptions of the small deflection theory. However, the amplitudes are bounded because of the stretching of the middle plane of the plate. From this fact, the amplitudes of unstable motions must be estimated by the large deflection theory of the plate.

The purpose of the present paper is to present an analytical approach to the investigation of the nonlinear response of an annular sector plate subjected to an in-plane dynamic moment. The equation of motion describing a large deflection of the plate based upon Berger's approximate equations[3] is analyzed by the Galerkin method. The resulting equations for time variables are integrated by using the Runge-Kutta-Gill method.

Numerical results are presented for different boundary conditions, damping forces, and static moment.

## II. DIFFERENTIAL EQUATION AND BOUNDARY CONDITION

Figure 1 shows an annular sector plate with an opening angle $\alpha$, outer radius a and inner radius b.

The polar coordinates $(r, \theta)$ are taken in the neutral surface of the plate. Equal and opposite moments M, which consist of the static moment $M_0$ and sinusoidally time-varying moment $M_t \cos\Omega t$, act along the radial edges. In-plane forces $N_r, N_\theta$, and $N_{r\theta}$ due to the moment M are given as follows[4]:

$$N_r = \frac{-4\,(M_0 + M_t \cos\Omega t)}{N}\left(\frac{a^2 b^2}{r^2}\ln\frac{a}{b} + a^2 \ln\frac{r}{a} + b^2 \ln\frac{b}{a}\right) \qquad (1.\,a)$$

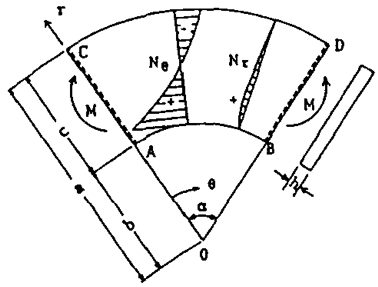


Fig. 1. Geometry and coordinates of an annular sector plate.

$$N_\theta = \frac{-4\,(M_0 + M_t \cos\Omega t)}{N}\left(-\frac{a^2 b^2}{r^2}\ln\frac{a}{b} + a^2\ln\frac{r}{a} + b^2\ln\frac{b}{r} + a^2 - b^2\right) \qquad (1.\text{b})$$

$$N_{r\theta} = 0 \qquad (1.\text{c})$$

where $N = (a^2 - b^2)^2 - 4a^2 b^2 (\ln(a/b))^2$, $M = M_0 + M_t\cos\Omega t$, $N_r$, $N_\theta$ and $N_{r\theta}$ are functions of independent variable r, and $\Omega$ and $M_t$ are the forcing circular frequency and amplitude of the sinusoidally time varying moment.

When the transverse inertia term is added to Berger's equations,[3] the basic equations for large-amplitude free vibrations of an annular sector plate subjected to an in-plane moment can be written as

$$L(w) = \rho h\frac{\partial^2 w}{\partial t^2} + D\nabla^4 w - \frac{1}{r}\frac{\partial}{\partial r}\left(rN_r\frac{\partial w}{\partial r}\right) - N_\theta\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} - N_f\nabla^2 w = 0 \qquad (2)$$

$$\frac{N_f h^2}{12D} = \frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 \qquad (3)$$

where w is the plate deflection, t is the time, h is the plate thickness, $\rho$ is the mass density, $D = Eh^3/12(1-\nu^2)$ is the bending stiffness, E is Young's modulus, $\nu$ is Poisson's ratio, u and v are the in-plane displacements in the r and $\theta$ directions, and

$\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2$ is the biharmonic operator in the polar coordinates and $N_f$ is a constant.

The following two boundary conditions for bending are considered in the present analysis:

case I: simply supported along all edges; i.e.,

$$w = M_\theta = 0\ (\theta = 0, \alpha),\ w = M_r = 0\ \ (r = b, a); \qquad (4.\text{a})$$

case II: simply supported along the loaded edges and clamped along the other edges

$$w = M_\theta = 0\ (\theta = 0, \alpha),\ w = \frac{\partial w}{\partial r} = 0\ (r = b, a), \qquad (4.\text{b})$$

where $M_\theta$ is the bending moment in the radial direction and $M_r$ is the bending moment in the angular direction.

With regard to in-plane boundary conditions, all edges are immovable. Since it is difficult to satisfy the in-plane constraints exactly, the average in-plane constraint boundary conditions are employed:

$$\int u r d\theta = 0\ (r = b, a),\quad \int v dr = 0\ (\theta = 0, \alpha). \qquad (4.\text{c})$$

Since $N_f$ is independent of r and $\theta$ in the equations, we can multiply Eq. (3) by $rdrd\theta$ and integrate over the area of the annular sector plate as shown in Fig. 1 to find

$$\frac{N_f h^2}{12D}\alpha(a^2-b^2)=\iint\left(\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\right)rdrd\theta+\frac{1}{2}\iint\left\{\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial\theta}\right)^2\right\}rdrd\theta. \quad (5)$$

We find, from Eq. (4. c),

$$\frac{N_f h^2}{12D}\alpha(a^2-b^2)=\frac{1}{2}\iint\left\{\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial\theta}\right)^2\right\}rdrd\theta. \quad (6)$$

Since the edges are either simply supported or clamped, the right-hand side of Eq. (6) can be integrated by parts and is simplified as follows:

$$\frac{N_f h^2}{12D}\alpha(a^2-b^2)=-\frac{1}{2}\iint w\nabla^2 w rdrd\theta. \quad (7)$$

## III. METHOD OF SOLUTION

Taking these boundary conditions into account, we assume the solution of Eq. (1) by

$$w=h\Sigma T_{in}(t)W_{in}(r,\theta) \quad (8)$$

where $T_{in}$ is an unknown function of the time variable and $W_{in}$ is an eigen-function associated with free vibration satisfying the geometric boundary condition of the plate, defined as

$$W_{in}=R_{in}(r)\sin\alpha_n\theta. \quad (9)$$

$R_{in}=A_{in}J_{\alpha n}(k_{in}\xi)+B_{in}Y_{\alpha n}(k_{in}\xi)+C_{in}I_{\alpha n}(k_{in}\xi)+D_{in}K_{\alpha n}(k_{in}\xi)$
in which $A_{in}, B_{in}, C_{in}$ and $D_{in}$ are constants of integration dependent on the boundary conditions, $J_{\alpha n}$ and $Y_{\alpha n}$ are the Bessel function, $I_{\alpha n}$ and $K_{\alpha n}$ are the modified Bessel function, $k_{in}=a^4\sqrt{\rho h\omega_{in}{}^2/D}$, $\xi=r/a$, $\omega_{in}$ is the radian frequency for the linear case. $\alpha_n=n\pi/\alpha$, and n=1, 2, …is an integer.

Substituting Eqs. (8) and (7) into Eq. (2) and applying the Galerkin method, one has

$$\iint LW_{jn}\xi d\xi d\theta=0 \quad (10)$$

where j=1, 2, ….

Performing integrations, one has

$$\ddot{T}_{jn}+p^2{}_{jn}T_{jn}+(\bar{M}_0+\bar{M}_t\cos\bar{\omega}\tau)\Sigma E_{ijn}T_{in}+\Sigma F_{ijn}T_{in}\Sigma\Sigma G_{lmn}T_{ln}T_{mn}=0 \quad (11)$$

where $p_{jn}, E_{ijn}, F_{ijn}$ and $G_{lmn}$ are constants dependent on vibration mode(Appendix). The following nondimensional quantities have been introduced in the above equation[3]:

$$\bar{M}_0=\frac{M_0}{M_{cr}},\quad \bar{M}_t=\frac{M_t}{M_{cr}},\quad \bar{\omega}=\frac{\Omega}{\Omega'_1}\ \text{and}\ \tau=\omega'_1 t \quad (12)$$

where $\Omega'_1$ is the lowest natural radian frequency, $M_{cr}=\lambda_{cr}D$ is the buckling moment and $\lambda_{cr}$ is the eigen-value of buckling which is determined by boundary conditions and geometrical parameters of the plate.[4]

Geometrical parameters in the present analysis are the opening angle $\alpha$ and the radius ratio $\beta(=b/a)$. The aspect ratio of the annular sector plate may be defined by the rectangular plate analogy as $\mu=\ell/c$, in which $\ell=(a+b)\alpha/2$ is the mean arc length and c is the radial edge length.

## IV. METHOD OF TIME RESPONSE ANALYSIS

The dynamic unstable regions of the present problem consist of simple parametric and sum type combination resonances.[3]

Combination resonance is the vibration of the two-degrees-of-freedom system. The two-degrees-of-freedom approach is adopted to obtain a time response. Time variables are numerically integrated using the Runge-Kutta-Gill method. The purpose of the present analysis is to determine the amplitudes of unstable motions which occur under the assumptions of the small deflection theory.

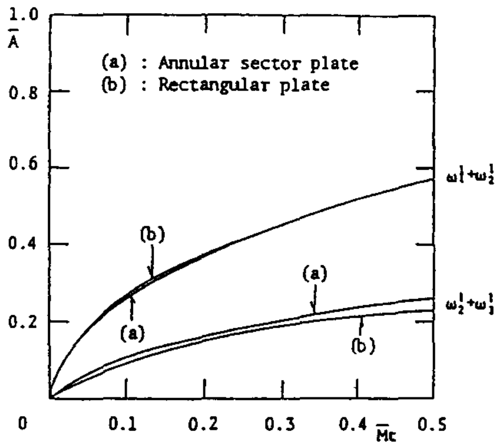


Fig. 2. Accuracy of the present solution.

Therefore , the initial conditions for the time variables are $T_{ai}=T_{aj}=0.01$ and $\dot{T}_{ai}=\dot{T}_{aj}=0.00$ to satisfy the small amplitude vibration. Poisson's ratio $\nu$ of the plate is taken as 0. 3.

## V. NUMERICAL RESULTS

### V-1. *Accuracy of solution*

Consider the annular sector plate with opening angle $\alpha=\pi/18$ and $\beta=0.839$. The aspect ratio of the plate is unity($\mu$=1. 0). The boundary conditions are simply supported on all edges(case I).

The amplitudes of combination resonances, $\omega^1{}_1+\omega^1{}_2$ and $\omega^1{}_2+\omega^1{}_3$ for central frequencies, are shown in Fig. 2. In this figure, the abscissa $\bar{M}_t$ shows the nondimensional moment, and the ordinate $\bar{A}$ indicates the maximum amplitude which is nondimensionalized by the plate thickness h. The amplitudes of the unstable motions for the square plate which are obtained by using Karman's theory are also shown. From the comparison of these results, it will be seen that in spite of approximate Berger's basic equation, the present solution is

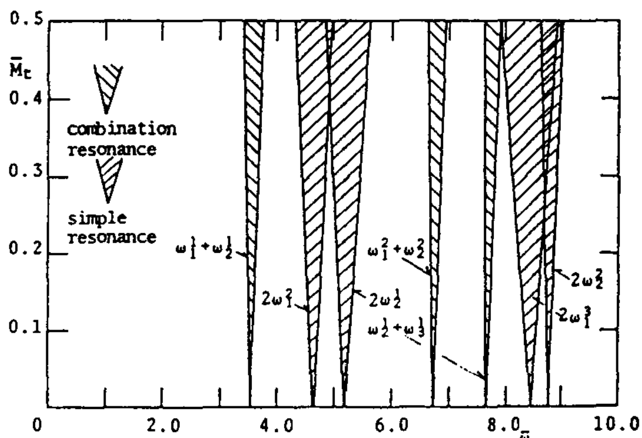


Fig. 3. Unstable regions for an annular sector plate subject to moment $\bar{M}_t$ : case I, $\alpha$=60° and $\mu$=1. 0.

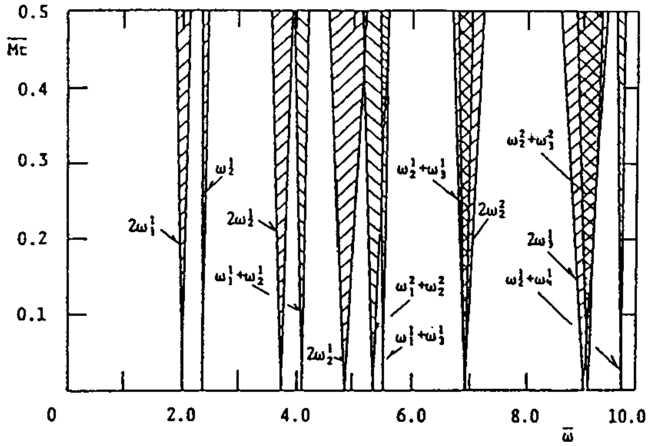


Fig. 4. Unstable regions for an annular sector plate subject to moment $\bar{M}_t$ : case II, $\alpha$=60° and $\mu$=1.0.

in reasonable agreement with that of the more reliable basic equation, i.e., Karman's equation.

*V-2. Effect of boundary conditions*

Figures. 3 and 4 show unstable regions of an annular sector plate with no static moment($\bar{M}_0$=0.0) for case I and case II which are obtained by linear analysis.[3] In these figures, the ordinate $\bar{M}_t$ denotes the amplitude of the periodic moment normalized to the buckling moment, while the abscissa $\bar{\omega}$ is the exciting nondimensional frequency. Further, the cross-hatched portions represent the regions of various types of instability such as both simple parametric resonances($2\omega^n_i/k$) and combination resonances of the sum type($(\omega^n_i+\omega^n_j)/k$), which contain the secondary unstable region(k=2) as well as the primary unstable region(k=1).

The widths of primary unstable regions of the simple resonances are broader than those of the combination resonance. When two frequencies have adjacent half-wave numbers in the radial direction and the same half-wave number in the

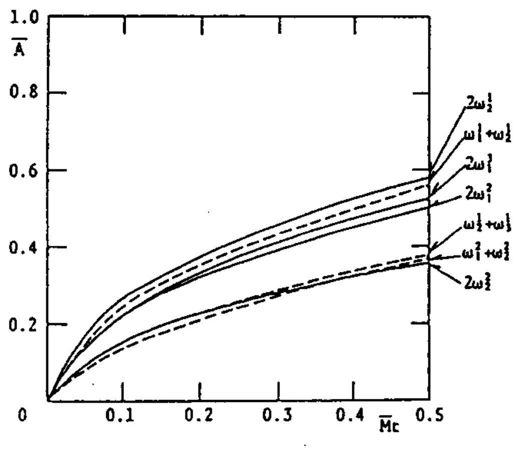


Fig. 5. Amplitudes of unstable motions: case I, $\alpha$=60°, $\mu$=1.0.

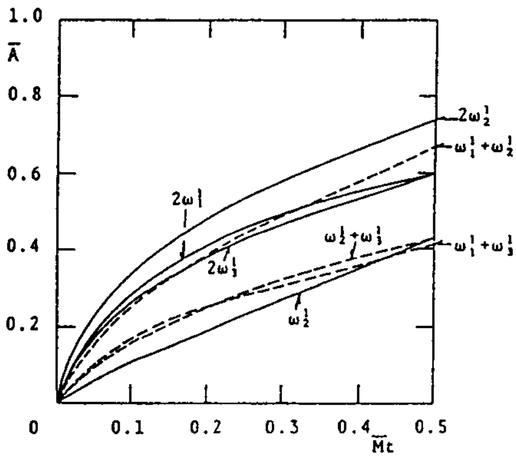


Fig. 6. Amplitudes of unstable motions: case II, $\alpha$=60°, $\mu$=1.0. (for n=1)

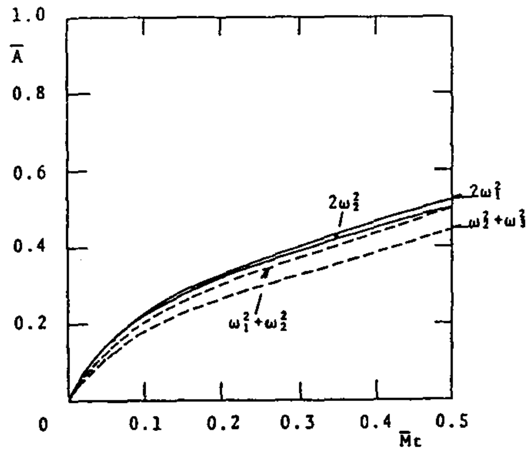


Fig. 7. Amplitudes of unstable motions: case II, $\alpha$=60° and $\mu$=1.0 (for n=2).

angular direction, unstable regions of the combination resonances such as $\omega^n{}_i+\omega^n{}_{i+1}$ are obtained in the present problem. The amplitudes of these unstable regions obtained by linear analysis grow indefinitely. However, nonlinear time responses of the unstable regions are bounded because of the nonlinear terms effect, which is caused by in-plane stretching forces due to the deflection of the plate.

The amplitudes for each central frequency $2\omega^n{}_i/k$ and $(\omega^n{}_i+\omega^n{}_j)/k$ of the unstable motions are shown in Figs. 5, 6 and 7. In these figures, the abscissa $\bar{M}_t$ shows the nondimensional moment and the ordinate $\bar{A}$ indicates the amplitude which is non-dimensionalized by the plate thickness. The amplitude of the simple resonance is greater than that of the combination resonance for each boundary condition. This fact is quite at variance with that of the rectangular plate in which the amplitudes of combination resonances are greater. The amplitudes are not directly dependent on the widths of the unstable regions and for case II are greater than for case I.

### V-3. *Effect of damping*

Amplitudes of the simple resonance $2\omega^1{}_2$ and the combination resonance

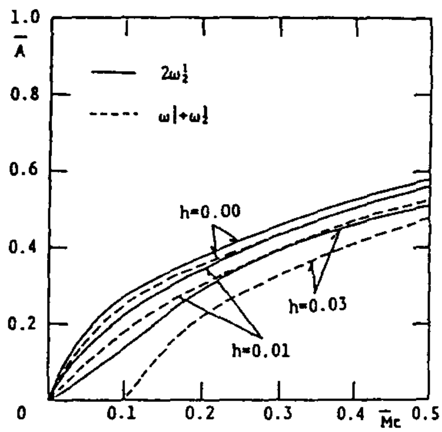


Fig. 8. Effect of damping on the amplitudes of unstable motions: case I, $\alpha$=60°, $\mu$=1.0.

$\omega^1_i+\omega^1_j$ for various magnitudes of damping constant($h^n_i=h^n_j=h$) are shown in Fig. 8. The effect of damping decreases the amplitudes of unstable motions, and this tendency is conspicuous where the exciting moment $\bar{M}_t$ is small. The unstable motion does not occur if the damping effect is greater than the divergent (negative damping) effect of the parametric instability. The effect of damping becomes smaller as the moment $\bar{M}_t$ increases.

*V-4. Effect of static moment*

Figure 9 shows the amplitudes of unstable motions for the simple resonance $2\omega^1_2$ and the combination resonance $\omega^1_1+\omega^1_2$ of an annular sector plate subjected to static moments $\bar{M}_0=0.0$ and $0.3$. The static moment $\bar{M}_0$ has an influence upon the amplitudes of unstable motions.

The effect of the static moment is to increase the amplitude of the combination resonance $\omega^1_1+\omega^1_2$ and decrease the amplitude of the simple resonance $2\omega^1_2$.

## VI. CONCLUSIONS

The present paper shows the nonlinear dynamic instability of an annular sector plate subjected to an in-plane dynamic moment. The conclusions are as follows:

(1) Amplitudes of the out-of-plane vibrations of an annular sector plate subjected to an in-plane dynamic moment can be satisfactorily estimated using Berger's approximate equation.

(2) Amplitudes of simple parametric resonances are greater than those of combination resonances for the present case. This is quite different from the response of the rectangular plate, in which the amplitudes of combination resonances are greater.

(3) Damping decreases the amplitudes of unstable motions. This effect is conspicuous where the parametric excitation moment is small.

(4) The static moment influences the amplitudes of unstable motions.

## APPENDIX

$$A_{jn}=\int R_{jn}^2\,\xi d\xi$$

$$E_{ijn}=-\frac{4\lambda_{cr}}{Nk_{11}}\int\left(\xi f_1\frac{dR_{in}}{d\xi}-\frac{\alpha_n^2}{\xi^2}f_2R_{in}\right)R_{jn}\,\xi d\xi\frac{1}{A_{jn}}$$

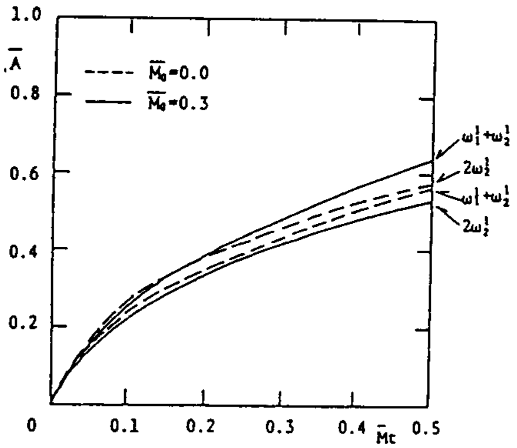


Fig. 9. Effect of static moment on the amplitudes of unstable motions: case I, $\alpha=60°$ and $\mu=1.0$.

$$F_{ijn}=\frac{6}{1-\beta^2}\frac{\lambda_{cr}}{k_{11}{}^4}\int S_{in}R_{jn}\xi d\xi\frac{1}{A_{jn}}$$

$$G_{lmn}=\int R_{ln}S_{mn}\xi d\xi$$

$$p_{jn}=\left(\frac{k_{jn}}{k_{11}}\right)^2$$

$$S_{jn}=k_{jn}{}^2\{-A_{jn}J_{\alpha n}(k_{jn}\xi)-B_{jn}Y_{\alpha n}(k_{jn}\xi)+C_{jn}I_{\alpha n}(k_{jn}\xi)+D_{jn}K_{\alpha n}(k_{jn}\xi)\}$$

$$N=(1-\beta^2)^2-4\beta^2\ln^2(1/\beta)$$

$$f_1(\xi)=\frac{\beta^2}{\xi^2}\ln\frac{1}{\beta}+\ln\xi+\beta^2\ln\frac{\beta}{\xi}$$

$$f_2(\xi)=-\frac{\beta^2}{\xi^2}\ln\frac{1}{\beta}+\ln\xi+\beta^2\ln\frac{\beta}{\xi}+1-\beta^2$$

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