ON EQUITABLE ROUND-ROBIN TOURNAMENTS WITH MAXIMAL BREAK INTERVAL GREATER THAN OR EQUAL TO 5

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Abstract

In our earlier paper, we studied the mathematical structure of equitable round-robin tournaments with home-away assignments, and gave some necessary conditions for their feasibility in terms of friend-enemy tables and break interval sequences. We also enumerated all the feasible home-away tables of such tournaments satisfying both the opening and the closing conditions, up to 26 teams.

In this paper, we study the maximal break interval of such tournaments. From this point of view, the tournaments satisfying both the opening and the closing conditions correspond to the case where the maximal break interval is greater than or equal to 4. The aim of this paper is to examine the case where the maximal break interval is greater than or equal to 5. We enumerate all the feasible cyclic break interval sequences of such tournaments, up to 42 teams.

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1. Introduction

Making a fair schedule of a sports league with home-away assignments is important for a tournament organizer, but such scheduling is often a too much time-consuming task. For the study of the mathematical structure of the scheduling problem of *equitable* round-robin tournaments, there are several approaches [1, 2, 3, 4, 5, 6, 9, 10, 11, 13, 14, 15].

De Werra [2, 3] initiated a graph theoretical appproach in 1980s, and represented the scheduling problem of round-robin tournaments with home-away assignments as finding an *oriented coloring* of a complete graph K_{2n} with $2n$ vertices, namely as finding a decomposition of the edge set of K_{2n} into 1-*factors* with orientation $\{\overrightarrow{F}_1, \overrightarrow{F}_2, \cdots, \overrightarrow{F}_{2n-1}\}$. He obtained a *canonical* 1-*factorization*, which gives a *canonical schedule* of an equitable round-robin tournament consisting of $2n$ teams.

Miyashiro, Iwasaki and Matsui [9, 10, 11] studied the feasibility of home-away tables (HATs for short) of round-robin tournaments and gave some necessary conditions for the feasibility of a HAT. They also conjectured that their conditions are also sufficient for the feasibility of HATs corresponding to equitable round-robin tournaments, and showed that the proposed conditions are sufficient if the number of teams is less than or equal to 26, by computational experiments.

In recent years, there are many works in the context of operations research using integer programming (IP), constraint programming (CP), metaheuristic approaches and combinations thereof (see the literature in [1, 7, 8, 12, 14]).

In our earlier paper [13], we proposed another approach which uses the *friend-enemy tables* (*FETs* for short) and *break interval sequences* (*BIS's* for short) of equitable round-robin tournaments, and obtained some necessary conditions for their feasibility which are variants of Miyashiro-Iwasaki-Matsui's conditions. We also enumerated all the feasible HATs (or FETs) of equitable round-robin tournaments satisfying both the *opening* and the *closing conditions*, up to 26 teams. Thanks to these conditions, the number of candidates of feasible HATs can be reduced remarkably.

In this paper, we consider *cyclic* break interval sequences and study their *maximal break interval*. From this point of view, the equitable round-robin tournaments satisfying both the opening and the closing conditions correspond to the case where the maximal break interval is greater than or equal to 4, and de Werra's canonical schedule corresponds to the case where the maximal break interval is equal to 2.

The aim of this paper is to examine the case where the maximal break interval is greater than or equal to 5. We enumerate all the feasible cyclic break interval sequences with maximal break interval greater than or equal to 5, up to 42 teams. Our results partially answer the question proposed by Zeng and Mizuno [14, Section 2], where double round-robin tournaments with a minimum number of breaks and large separation of games for all pairs of teams are considered.

2. Cyclic Break Interval Sequences

In this section, we recall some definitions and basic facts from [2, 3, 9, 10, 11, 13]. We consider a round-robin tournament consisiting of $2n$ teams ($n \geq 1$) and $2n-1$ slots. In a round-robin tournament with home-away assignments, each team must play one game against every other team, and in each slot, each team plays one game, either at home or away. Table 1 is a *schedule* for a round-robin tournament consisiting of 8 teams, in which the rows are indexed by teams, the columns are indexed by slots, the entries of each row show the opponents of the team at different slots, and the home games are underlined.

Table 1. Schedule of 8 teams

	1	2	3	4	5	6	7
1	$\underline{7}$	8	2	$\underline{3}$	4	$\underline{5}$	6
2	$\underline{8}$	7	$\underline{1}$	$\underline{4}$	3	$\underline{6}$	5
3	$\underline{5}$	6	$\underline{7}$	1	$\underline{2}$	$\underline{4}$	8
4	$\underline{6}$	5	$\underline{8}$	2	$\underline{1}$	3	7
5	3	$\underline{4}$	$\underline{6}$	7	$\underline{8}$	1	$\underline{2}$
6	4	$\underline{3}$	5	8	$\underline{7}$	2	$\underline{1}$
7	1	$\underline{2}$	3	$\underline{5}$	6	8	$\underline{4}$
8	2	$\underline{1}$	4	$\underline{6}$	5	$\underline{7}$	$\underline{3}$

Table 2. HAT corresponding to Table 1

	1	2	3	4	5	6	7
1	H	A	$\underline{\text{A}}$	H	A	H	A
2	H	A	H	$\underline{\text{H}}$	A	H	A
3	H	A	H	A	H	$\underline{\text{H}}$	A
4	H	A	H	A	H	A	$\underline{\text{A}}$
5	A	H	$\underline{\text{H}}$	A	H	A	H
6	A	H	A	$\underline{\text{A}}$	H	A	H
7	A	H	A	H	A	$\underline{\text{A}}$	H
8	A	H	A	H	A	H	$\underline{\text{H}}$

A home-away table (HAT) is the table which shows whether each team plays a home-game or an away-game on each slot. Table 2 is the HAT corresponding to Table 1, where a home-game is denoted by 'H', and an away-game by 'A'.

A row of a HAT is called a *home-away pattern* (*HA-pattern* for short). If an HA-pattern has consecutive 'H's or 'A's, then we say the HA-pattern has a *break*. In Table 2, every team has exactly one break (underlined).

A HAT is called *feasible*, if there exists a schedule corresponding to it. A feasible HAT satisfies the following conditions (*consistency*):

- The HA-patterns of the HAT are mutually distinct.
- On each slot, the numbers of 'H's and 'A's coincide.

Since there are only two HA-patterns with no breaks, the minimum number of breaks is equal to $2n-2$ [2, 3]. On the other hand, if every team has exactly one break, then we call such a HAT and the corresponding round-robin tournament *equitable*. For example, Table 1 is a schedule of an equitable round-robin tournament and Table 2 is the corresponding equitable HAT.

Since a HAT with a minimum number of breaks is equivalent to an equitable HAT by a cyclic rotation of the slots [9, 10], it makes no difference for the feasibility which one we consider. In the following, we do not distinguish between them.

If an HA-pattern x' is obtained by changing symbols 'H' for 'A' and 'A' for 'H' from an HA-pattern x, then we say x' is the *complement* of x and vice versa. In Table 2, the HA-patterns of teams $5, 6, 7$ and 8 are the complements of the HA-patterns of teams $1, 2, 3$ and 4, respectively.

In a feasible and equitable HAT, every slot (column) has at most two breaks. Since every team (row) has exactly one break and each slot (column) has the same number of 'H's and 'A's, the complement of each row must be contained in a feasible and equitable HAT [2, 3].

To determine whether a given equitable HAT is feasible or not, we introduced some simple necessary conditions [13].

First, we change the HAT of an equitable round-robin tournament to a *friend-enemy table* (*FET* for short). For example, the FET corresponding to Table 2 is the following Table 3.

Table 3. FET corresponding to Table 2

	1	2	3	4	5	6	7
1	◯	◯	◯	◯	◯	◯	◯
2	◯	◯	×	◯	◯	◯	◯
3	◯	◯	×	×	×	◯	◯
4	◯	◯	×	×	×	×	◯
5	×	×	×	×	×	×	×
6	×	×	◯	×	×	×	×
7	×	×	◯	◯	◯	×	×
8	×	×	◯	◯	◯	◯	×

On each slot of an FET, the teams with the same symbol (◯ or ×) cannot play a game with each other. Using Table 3, we can easily make a schedule correspondig to Table 2, as described in Table 1.

We call a row of an FET a *friend-enemy pattern* (*FE-pattern* for short). In Table 3, team 1 has a break on slot 3, where the friend teams $2, 3$ and 4 of team 1 go over to the enemy, and teams 2, 3 and 4 have a break on slots 4, 6 and 7, respectively, where they take sides with team 1 again. The relations among teams $5, 6, 7$ and 8 are the same as those among teams $1, 2, 3$ and 4.

Conversely, if an FET consisting of $2n$ teams satisfies the following three conditions, with respect to a sequence $\{s_i\}_{1\leq i\leq n}$ of positive integers satisfying $1 \leq s_1 < s_2 < \cdots < s_n \leq 2n-1$, then the FET corresponds to an equitable HAT which has breaks on slots s_i $(1 \leq i \leq n)$:

1. The FE-pattern of team 1 (resp. $n+1$) is "◯ ◯ $\cdots$ ◯" (resp. "× × $\cdots$ ×").
2. The friend teams 2, 3, $\cdots$, n (resp. $n+2$, $n+3$, $\cdots$, $2n$) of team 1 (resp. $n+1$) go over to the enemy on slot s_1.
3. Teams 2, 3, $\cdots$, n (resp. $n+2$, $n+3$, $\cdots$, $2n$) take sides with team 1 (resp. $n+1$) again on slots s_2, s_3, $\cdots$, s_n, respectively.

We call such an FET *equitable* and denote it by $(s_1, s_2, \cdots, s_n)$ for short. Put $r_i = s_{i+1} - s_i$ $(1 \leq i \leq n-1)$ and $r_n = s_1 - s_n + (2n-1)$, then r_i's are the intervals of successive breaks. In the following, we denote the sequences $\{r_i\}_{1\leq i\leq n-1}$ and $\{r_i\}_{1\leq i\leq n}$ by $\langle r_1,\, r_2,\, \cdots,\, r_{n-1}\rangle$ and $[r_1,\, r_2,\, \cdots,\, r_n]$, respectively, and call them the *break interval sequence* (*BIS* for short) and the *cyclic break interval sequence* (*cyclic BIS* for short) of $(s_1,\, s_2,\, \cdots,\, s_n)$. For example, the FET in Table 3 can be denoted by (3, 4, 6, 7), and its BIS and cyclic BIS are $\langle 1,\, 2,\, 1\rangle$ and [1, 2, 1, 3], respectively. In the following, we also write $\langle r_j,\, \cdots,\, r_k\rangle$ for a subsequence $\{r_i\}_{j\leq i\leq k}$ of a cyclic BIS $[r_1,\, r_2,\, \cdots,\, r_n]$.

An FET and its cyclic BIS are called *feasible*, if there exists a schedule corresponding to them. For a given feasible FET $(s_1,\, s_2,\, \cdots,\, s_n)$, the m teams $i,\ i+1,\ \cdots,\ i+m-1$ $(1 \leq i \leq n,\ 2 \leq m \leq n)$ must play games with each other in $s_{i+m-1} - s_i$ consecutive slots $s_i,\ s_i+1,\ \cdots,\ s_{i+m-1}-1$ (suffixes of slots are considered mod n and slots are considered mod $2n-1$). The total number of games among m teams is equal to ${}_mC_2$. On the other hand, we can have only one game among them on each slot $s \in \{s_i,\, \cdots,\, s_{i+1}-1\}$, and at most two games on each slot $s \in \{s_{i+1},\, \cdots,\, s_{i+2}-1\}$, and so on. Therefore, the sequence $\langle r_i,\, r_{i+1},\, \cdots,\, r_{i+m-2}\rangle$ must satisfy the following condition [13] :

$$\begin{cases} r_i + 2r_{i+1} + \cdots + \frac{m}{2} r_{i+\frac{m-2}{2}} + \cdots + 2r_{i+m-3} + r_{i+m-2} \geq {}_mC_2 & \text{(if } m \text{ is even)}, \\ r_i + 2r_{i+1} + \cdots + \frac{m-1}{2}(r_{i+\frac{m-3}{2}} + r_{i+\frac{m-1}{2}}) + \cdots & \\ \qquad\qquad + 2r_{i+m-3} + r_{i+m-2} \geq {}_mC_2 & \text{(if } m \text{ is odd)}. \end{cases}$$

We call this condition the *m-team condition* for teams $i,\ i{+}1,\ \cdots,\ i{+}m{-}1$ (or a *local condition*). In pariticular, we call an n-team condition a *global condition.* If a cyclic BIS $[r_1,\, r_2,\, \cdots,\, r_n]$ satisfies the m-team conditions for all i $(1 \leq i \leq n)$ and m $(2 \leq m \leq n)$, then we say $[r_1,\, r_2,\, \cdots,\, r_n]$ is *probable.* Obviously, a feasible cyclic BIS is probable. Conversely, Miyashiro, Iwasaki and Matsui conjecture that a probable cyclic BIS is feasible [9, 10].

We say a round-robin tournament (or its HAT) satisfies the *opening* (resp. *closing*) *condition* if it has no break on slot 2 (resp. on slot $2n-1$) [11]. For example, the HAT described in Table 2 satisfies the opening condition, but not the closing condition.

For a cyclic BIS $[r_1,\, r_2,\, \cdots,\, r_n]$, we call the value $\max\limits_{1\leq i\leq n} r_i$ its *maximal break interval.* If an equitable round-robin tournament satisfies the opening and the closing conditions, then the corresponding FET $(s_1,\, s_2,\, \cdots,\, s_n)$ satisfies $s_1 \geq 3$ and $s_n \leq 2n-2$. Therefore, its cyclic BIS $[r_1,\, r_2,\, \cdots,\, r_n]$

satisfies $\max_{1\le i\le n} r_i \ge r_n = s_1 - s_n + (2n-1) \ge 4$.

Remark 1. De Werra's canonical schedule for $2n$ teams can be constructed as follows [2, 3]. On each slot i $(1 \le i \le 2n-1)$, team $2n$ plays a game with team i at home if i is odd and away if i is even. For each k $(1 \le k \le n-1)$, team $i-k$ plays a game with team $i+k$ on slot i $(1 \le i \le 2n-1)$ at home if k is odd and away if k is even. Here, we consider the numbers $i-k$ and $i+k$ modulo $2n-1$. We show the schedule in Table 4. In the canonical schedule, teams 1 and $2n$ have a break on slot 1 and teams $2j+1$ and $2j$ have a break on slot $2j+1$ for each j $(1 \le j \le n-1)$. Therefore, its FET is $(1,\,3,\,5,\,\cdots,\,2n-1)$ and the corresponding cyclic BIS is $[2,\,2,\,\cdots,\,2,\,1]$. In particular, its maximal break interval is equal to 2. This is the minimum value of all the maximal break intervals.

Table 4. De Werra's canonical schedule

	1	2	3	4	$\cdots$	$\cdots$	$2n-3$	$2n-2$	$2n-1$
1	$2n$	$\underline{3}$	5	$\underline{7}$	$\cdots$	$\cdots$	$2n-6$	$\underline{2n-4}$	$2n-2$
2	$2n-1$	$\underline{2n}$	$\underline{4}$	6	$\cdots$	$\cdots$	$\underline{2n-7}$	$2n-5$	$\underline{2n-3}$
3	$\underline{2n-2}$	1	$2n$	$\underline{5}$	$\cdots$	$\cdots$	$2n-8$	$\underline{2n-6}$	$2n-4$
4	$2n-3$	$\underline{2n-1}$	2	$\underline{2n}$	$\cdots$	$\cdots$	$\underline{2n-9}$	$2n-7$	$\underline{2n-5}$
5	$\underline{2n-4}$	$2n-2$	$\underline{1}$	3	$\cdots$	$\cdots$	$2n-10$	$\underline{2n-8}$	$2n-6$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\vdots$
$2n-4$	5	$\underline{7}$	9	$\underline{11}$	$\cdots$	$\cdots$	$\underline{2n-2}$	1	$\underline{3}$
$2n-3$	$\underline{4}$	6	$\underline{8}$	10	$\cdots$	$\cdots$	$2n$	$\underline{2n-1}$	2
$2n-2$	3	$\underline{5}$	7	$\underline{9}$	$\cdots$	$\cdots$	$2n-4$	$\underline{2n}$	$\underline{1}$
$2n-1$	$\underline{2}$	4	$\underline{6}$	8	$\cdots$	$\cdots$	$\underline{2n-5}$	$2n-3$	$2n$
$2n$	$\underline{1}$	2	$\underline{3}$	4	$\cdots$	$\cdots$	$\underline{2n-3}$	$2n-2$	$\underline{2n-1}$

Remark 2. In [14], Zeng and Mizuno considers construction of double round-robin tournaments with a minimum number of breaks and large separation of games for all pairs of teams. If $(s_1,\,s_2,\,\cdots,\,s_n)$ is a feasible FET with $s_1 = 1$ and $s_n \le 2n-2k-1$ (i.e., $\max_{1\le i\le n} r_i \ge r_n \ge 2k+1$), then we can make a schedule of such a double round-robin tournament with $2k$-separation. Namely, if $(\overrightarrow{F}_1,\,\overrightarrow{F}_2,\,\cdots,\,\overrightarrow{F}_{2n-1})$ is a schedule for $(s_1,\,s_2,\,\cdots,\,s_n)$ and $\overleftarrow{F}_i$ denotes the same edge set with reversed orientation,

then $(\overrightarrow{F}_1, \overrightarrow{F}_2, \cdots, \overrightarrow{F}_{2n-1}, \overleftarrow{F}_{2n-2k-1}, \cdots, \overleftarrow{F}_{2n-1}, \overleftarrow{F}_{2n-2k-2}, \cdots, \overleftarrow{F}_1)$ is a schedule of a double round-robin tournament with minimum breaks ($4n-4$ breaks) and $2k$-separation. In this schedule, two teams (teams 1 and $n+1$ in a standard notation of FET) have no breaks and the other $2n-2$ teams have two breaks each. Zeng and Mizuno found such schedules with 2-separation (i.e., $k=1$) for $4 \leq n \leq 34$ by computational experiments, and posed the problem of finding schedules of above form with more than 2-separation (i.e., $k > 1$). Our results in this paper will show that there exist such schedules with 4-separation (i.e., $k=2$) for $n = 16$, 20 and 21, and there is none with more than 2-separation for $n \leq 15$ and $17 \leq n \leq 19$. Recently, Zeng and Mizuno [15] investigted this problem further and obtained some interesting results by using the method developed in [6] and computational experiments.

3. Preliminary Results

In this section, we recall some facts from [13] and prove some lemmas which will be used in Section 4.

In [13], we proved the following propositions.

Proposition 1 ([13]). *Let a sequence $\langle r_i, r_{i+1}, r_{i+2}\rangle$ satisfy all the 3-team conditions for teams i, $i+1$, $i+2$ and $i+3$. Then it also satisfies the 4-team condition.*

Proposition 2 ([13, Theorem 1]). *Let a sequence $\langle r_i, r_{i+1}, r_{i+2}, r_{i+3}\rangle$ satisfy all the 3-team conditions for teams i, $i+1$, $i+2$, $i+3$ and $i+4$. Then it satisfies the 5-team condition if and only if it is equal to neither $\langle 1, 2, 1, 2\rangle$ nor $\langle 2, 1, 2, 1\rangle$.*

Proposition 3 ([13, Theorem 2]). *Let a sequence $\langle r_i, r_{i+1}, \cdots, r_{i+4}\rangle$ satisfy all the 3-team and 5-team conditions for teams $i, i+1, \cdots, i+5$. Then it also satisfies the 6-team condition.*

Proposition 4 ([13, Theorem 3]). *Let a sequence $\langle r_i, r_{i+1}, \cdots, r_{i+5}\rangle$ satisfy all the 3-team and 5-team conditions for teams $i, i+1, \cdots, i+6$. Then it satisfies the 7-team condition if and only if it is equal to none of the following sequences: $\langle 1, 2, 2, 1, 2, 2\rangle$, $\langle 2, 1, 2, 2, 1, 2\rangle$, $\langle 2, 2, 1, 2, 2, 1\rangle$.*

Proposition 5 ([13, Theorem 4]. *Let a sequence* $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+6}\rangle$ *satisfy all the* j*-team conditions* ($j = 3,\ 5$ *and* 7) *for teams* $i,\ i+1,\ \cdots,\ i+7$. *Then it also satisfies the* 8*-team condition.*

Proposition 6 ([13, Theorem 5]. *Let a sequence* $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+7}\rangle$ *satisfy all the* j*-team conditions* ($j = 3,\ 5$ *and* 7) *for teams* $i,\ i+1,\ \cdots,\ i+8$. *Then it satisfies the* 9*-team condition if and only if it is equal to none of the following sequences*:

$$\langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1,\ 3\rangle,\ \langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 1\rangle,\ \langle 1,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,$$
$$\langle 1,\ 2,\ 2,\ 2,\ 1,\ 2,\ 2,\ 2\rangle,\ \langle 1,\ 3,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2\rangle,\ \langle 2,\ 1,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2\rangle,$$
$$\langle 2,\ 1,\ 2,\ 2,\ 2,\ 1,\ 2,\ 2\rangle,\ \langle 2,\ 1,\ 3,\ 1,\ 2,\ 1,\ 3,\ 1\rangle,\ \langle 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 1,\ 2\rangle,$$
$$\langle 2,\ 2,\ 1,\ 2,\ 2,\ 2,\ 1,\ 2\rangle,\ \langle 2,\ 2,\ 2,\ 1,\ 2,\ 2,\ 2,\ 1\rangle,\ \langle 3,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle.$$

Proposition 7 ([13, Theorem 6]. *Let a sequence* $\langle r_i,\ r_{i+1},\ \cdots,\ r_{i+8}\rangle$ *satisfy all the* j*-team conditions* ($j = 3,\ 5,\ 7$ *and* 9) *for teams* $i,\ i+1,\ \cdots,\ i+9$. *Then it satisfies the* 10*-team condition if and only if it is equal to neither* $\langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2,\ 1\rangle$ *nor* $\langle 1,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle$.

Next, we prove some lemmas.

Lemma 1. *Let a sequence* $\langle r_i,\ r_{i+1}\rangle$ *satisfy the* 3*-team condition for teams* $i,\ i+1$ *and* $i+2$. *Then we have* $r_i + r_{i+1} \geq 3$, *i.e., two* 1*'s are not adjacent in a probable cyclic BIS. Further, if* $r_i + r_{i+1} = 3$, *then* $\langle r_i,\ r_{i+1}\rangle = \langle 1,\ 2\rangle(5)$ *or* $\langle 2,\ 1\rangle(4)$ (*in parentheses, we described* $r_i + 2r_{i+1}$).

Proof. The lemma easily follows from the assumption. □

Lemma 2. *Let a sequence* $\langle r_i,\ r_{i+1},\ r_{i+2}\rangle$ *satisfy all the* 3*-team conditions for teams* $i,\ i+1,\ i+2$ *and* $i+3$. *Then we have* $r_i + r_{i+1} + r_{i+2} \geq 4$. *Further, we have the following*:

(i) *If* $r_i + r_{i+1} + r_{i+2} = 4$, *then* $\langle r_i,\ r_{i+1},\ r_{i+2}\rangle = \langle 1,\ 2,\ 1\rangle(8)$ (*in parentheses, we described* $r_i + 2r_{i+1} + 3r_{i+2}$).

(ii) *If* $r_i + r_{i+1} + r_{i+2} = 5$, *then* $\langle r_i,\ r_{i+1},\ r_{i+2}\rangle$ *is equal to one of the following sequences* (*in parentheses, we described* $r_i + 2r_{i+1} + 3r_{i+2}$):

$$\langle 1,\ 2,\ 2\rangle(11),\quad \langle 1,\ 3,\ 1\rangle(10),\quad \langle 2,\ 1,\ 2\rangle(10),\quad \langle 2,\ 2,\ 1\rangle(9).$$

Proof. Since two 1's are not adjacent in $\langle r_i, r_{i+1}, r_{i+2}\rangle$ by Lemma 1, we have $r_i + r_{i+1} + r_{i+2} \geq 4$. The rest of the proof is straightforward. $\square$

Lemma 3. *Let a sequence* $\langle r_i, r_{i+1}, r_{i+2}, r_{i+3}\rangle$ *satisfy all the* 3*-team and* 5*-team conditions for teams* $i, i+1, i+2, i+3$ *and* $i+4$. *Then we have* $r_i + r_{i+1} + r_{i+2} + r_{i+3} \geq 6$. *Further, we have the following*:

(i) *If* $r_i+r_{i+1}+r_{i+2}+r_{i+3} = 6$, *then* $\langle r_i, r_{i+1}, r_{i+2}, r_{i+3}\rangle = \langle 1, 2, 2, 1\rangle(15)$ (*in parentheses, we described* $r_i + 2r_{i+1} + 3r_{i+2} + 4r_{i+3}$).

(ii) *If* $r_i+r_{i+1}+r_{i+2}+r_{i+3} = 7$, *then* $\langle r_i, r_{i+1}, r_{i+2}, r_{i+3}\rangle$ *is equal to one of the following sequences* (*in parentheses, we described* $r_i+2r_{i+1}+3r_{i+2}+4r_{i+3}$):

$$\langle 1, 2, 1, 3\rangle(20),\ \langle 1, 2, 2, 2\rangle(19),\ \langle 1, 2, 3, 1\rangle(18),\ \langle 1, 3, 1, 2\rangle(18),$$
$$\langle 1, 3, 2, 1\rangle(17),\ \langle 2, 1, 2, 2\rangle(18),\ \langle 2, 1, 3, 1\rangle(17),\ \langle 2, 2, 1, 2\rangle(17),$$
$$\langle 2, 2, 2, 1\rangle(16),\ \langle 3, 1, 2, 1\rangle(15).$$

Proof. Since $r_i + r_{i+1} \geq 3$ and $r_{i+2} + r_{i+3} \geq 3$ by Lemma 1, we have $r_i + r_{i+1} + r_{i+2} + r_{i+3} \geq 6$. The assertions (i) and (ii) follow by enumeration and Proposition 2. $\square$

Lemma 4. *Let a sequence* $\langle r_i, r_{i+1}, \cdots, r_{i+4}\rangle$ *satisfy all the* 3*-team and* 5*-team conditions for teams* $i, i+1, \cdots, i+5$. *Then we have* $r_i + r_{i+1} + \cdots + r_{i+4} \geq 8$. *Further, if* $r_i + r_{i+1} + \cdots + r_{i+4} = 8$, *then* $\langle r_i, r_{i+1}, \cdots, r_{i+4}\rangle$ *is equal to one of the following sequences* (*in parentheses, we described* $r_i + 2r_{i+1} + \cdots + 5r_{i+4}$):

$$\langle 1, 2, 1, 3, 1\rangle(25),\ \langle 1, 2, 2, 1, 2\rangle(25),\ \langle 1, 2, 2, 2, 1\rangle(24),$$
$$\langle 1, 3, 1, 2, 1\rangle(23),\ \langle 2, 1, 2, 2, 1\rangle(23).$$

Proof. If $r_i + r_{i+1} + \cdots + r_{i+4} \leq 7$, then we have $\langle r_i, r_{i+1}, \cdots, r_{i+4}\rangle = \langle 1, 2, 1, 2, 1\rangle$ by Lemma 1, which contradicts Proposition 2. Therefore, we must have $r_i + r_{i+1} + \cdots + r_{i+4} \geq 8$. The rest of the assertion follows by enumeration and Proposition 2. $\square$

Lemma 5. *Let a sequence* $\langle r_i, r_{i+1}, \cdots, r_{i+5}\rangle$ *satisfy all the* j*-team conditions* ($j = 3, 5$ *and* 7) *for teams* $i, i+1, \cdots, i+6$. *Then we have* $r_i + r_{i+1} + \cdots + r_{i+5} \geq 10$. *Further, we have the following*:

(i) *If* $r_i+r_{i+1}+\cdots+r_{i+5} = 10$, *then* $\langle r_i, r_{i+1}, \cdots, r_{i+5}\rangle$ *is equal to one of the following sequences* (*in parentheses, we described* $r_i+2r_{i+1}+\cdots+6r_{i+5}$):

$\langle 1, 2, 1, 3, 1, 2\rangle(37)$, $\langle 1, 2, 1, 3, 2, 1\rangle(36)$, $\langle 1, 2, 2, 1, 3, 1\rangle(36)$,
$\langle 1, 2, 2, 2, 1, 2\rangle(36)$, $\langle 1, 2, 2, 2, 2, 1\rangle(35)$, $\langle 1, 2, 3, 1, 2, 1\rangle(34)$,
$\langle 1, 3, 1, 2, 2, 1\rangle(34)$, $\langle 2, 1, 2, 2, 2, 1\rangle(34)$, $\langle 2, 1, 3, 1, 2, 1\rangle(33)$.

(ii) *If* $l := r_i + r_{i+1} + \cdots + r_{i+5} \geq 11$, *then we have* $r_i + 2r_{i+1} + \cdots + 6r_{i+5} \leq 25 + 6(l-8)$, *and the equality holds if and only if* $\langle r_i, r_{i+1}, \cdots, r_{i+5}\rangle = \langle 1, 2, 1, 3, 1, l-8\rangle$ *or* $\langle 1, 2, 2, 1, 2, l-8\rangle$.

Proof. If $r_i + r_{i+1} + \cdots + r_{i+5} \leq 9$, then we have $r_i + r_{i+1} = r_{i+2} + r_{i+3} = r_{i+4} + r_{i+5} = 3$ by Lemma 1. Therefore, $\langle r_i, r_{i+1}, r_{i+2}, r_{i+3}\rangle = \langle r_{i+2}, r_{i+3}, r_{i+4}, r_{i+5}\rangle = \langle 1, 2, 2, 1\rangle$ by Lemma 3 (i), which is a contradiction. Hence, we must have $r_i + r_{i+1} + \cdots + r_{i+5} \geq 10$.

(i) The assertion follows by enumeration and Propositions 2 and 4.

(ii) We have $r_i + r_{i+1} + \cdots + r_{i+4} \geq 8$ by Lemma 4, and the equality holds if and only if $\langle r_i, r_{i+1}, \cdots, r_{i+4}\rangle$ is equal to one of the five sequences described in Lemma 4. Among them, the value $r_i + 2r_{i+1} + \cdots + 5r_{i+4}$ becomes maximum when $\langle r_i, r_{i+1}, \cdots, r_{i+4}\rangle = \langle 1, 2, 1, 3, 1\rangle$ or $\langle 1, 2, 2, 1, 2\rangle$. Therefore, the sequences $\langle r_i, r_{i+1}, \cdots, r_{i+4}, r_{i+5}\rangle = \langle 1, 2, 1, 3, 1, l-8\rangle$ and $\langle 1, 2, 2, 1, 2, l-8\rangle$ with $l \geq 11$ satisfy all the j-team conditions ($j = 3$, 5 and 7) by Propositions 2 and 4, and the equality

$$r_i + 2r_{i+1} + \cdots + 5r_{i+4} + 6r_{i+5} = 25 + 6(l-8)$$

holds. On the other hand, for any sequence $\langle r_i, r_{i+1}, \cdots, r_{i+4}, r_{i+5}\rangle$ with $l' := r_i + r_{i+1} + \cdots + r_{i+4} > 8$, we have

$$\begin{aligned} r_i + 2r_{i+1} + \cdots + 5r_{i+4} + 6r_{i+5} &\leq 25 + 5(l'-8) + 6(l-l') \\ &= 25 + 6(l-8) - (l'-8) < 25 + 6(l-8), \end{aligned}$$

which completes the proof. □

Lemma 6. *Let a sequence* $\langle r_i, r_{i+1}, \cdots, r_{i+6}\rangle$ *satisfy all the* j*-team conditions* ($j = 3$, 5 *and* 7) *for teams* $i, i+1, \cdots, i+7$. *Then we have* $r_i + r_{i+1} + \cdots + r_{i+6} \geq 11$. *Further, if* $r_i + r_{i+1} + \cdots + r_{i+6} = 11$, *then* $\langle r_i, r_{i+1}, \cdots, r_{i+6}\rangle = \langle 1, 2, 1, 3, 1, 2, 1\rangle(44)$ (*in parentheses, we described* $r_i + 2r_{i+1} + \cdots + 7r_{i+6}$).

Proof. Since $r_i + r_{i+1} + \cdots + r_{i+5} \geq 10$ by Lemma 5, we have $r_i + r_{i+1} + \cdots + r_{i+6} \geq 11$. By Lemma 5 (i), Propositions 2, 4 and Lemma 1, the equality holds if and only if $\langle r_i, r_{i+1}, \cdots, r_{i+6}\rangle = \langle 1, 2, 1, 3, 1, 2, 1\rangle$. □

Lemma 7. *Let a sequence* $\langle r_i, r_{i+1}, \cdots, r_{i+7}\rangle$ *satisfy all the* j*-team conditions* ($j = 3, 5, 7$ *and* 9) *for teams* $i, i+1, \cdots, i+8$. *Then we have* $r_i + r_{i+1} + \cdots + r_{i+7} \geq 14$. *Further, we have the following*:

(i) *If* $r_i + r_{i+1} + \cdots + r_{i+7} = 14$, *then* $\langle r_i, r_{i+1}, \cdots, r_{i+7}\rangle$ *is equal to one of the following sequences* (*in parentheses, we described* $r_i + 2r_{i+1} + \cdots + 8r_{i+7}$) :

$\langle 1, 2, 1, 3, 1, 2, 2, 2\rangle(67)$, $\langle 1, 2, 1, 3, 1, 2, 3, 1\rangle(66)$, $\langle 1, 2, 1, 3, 1, 3, 1, 2\rangle(66)$,
$\langle 1, 2, 1, 3, 1, 3, 2, 1\rangle(65)$, $\langle 1, 2, 1, 3, 2, 1, 2, 2\rangle(66)$, $\langle 1, 2, 1, 3, 2, 1, 3, 1\rangle(65)$,
$\langle 1, 2, 1, 3, 2, 2, 1, 2\rangle(65)$, $\langle 1, 2, 1, 3, 2, 2, 2, 1\rangle(64)$, $\langle 1, 2, 1, 3, 3, 1, 2, 1\rangle(63)$,
$\langle 1, 2, 1, 4, 1, 2, 2, 1\rangle(63)$, $\langle 1, 2, 2, 1, 2, 3, 1, 2\rangle(66)$, $\langle 1, 2, 2, 1, 2, 3, 2, 1\rangle(65)$,
$\langle 1, 2, 2, 1, 3, 1, 2, 2\rangle(66)$, $\langle 1, 2, 2, 1, 3, 1, 3, 1\rangle(65)$, $\langle 1, 2, 2, 1, 3, 2, 1, 2\rangle(65)$,
$\langle 1, 2, 2, 1, 3, 2, 2, 1\rangle(64)$, $\langle 1, 2, 2, 1, 4, 1, 2, 1\rangle(63)$, $\langle 1, 2, 2, 2, 1, 2, 3, 1\rangle(65)$,
$\langle 1, 2, 2, 2, 1, 3, 1, 2\rangle(65)$, $\langle 1, 2, 2, 2, 1, 3, 2, 1\rangle(64)$, $\langle 1, 2, 2, 2, 2, 1, 2, 2\rangle(65)$,
$\langle 1, 2, 2, 2, 2, 1, 3, 1\rangle(64)$, $\langle 1, 2, 2, 2, 2, 2, 1, 2\rangle(64)$, $\langle 1, 2, 2, 2, 2, 2, 2, 1\rangle(63)$,
$\langle 1, 2, 2, 2, 3, 1, 2, 1\rangle(62)$, $\langle 1, 2, 2, 3, 1, 2, 2, 1\rangle(62)$, $\langle 1, 2, 3, 1, 2, 1, 3, 1\rangle(63)$,
$\langle 1, 2, 3, 1, 2, 2, 1, 2\rangle(63)$, $\langle 1, 2, 3, 1, 2, 2, 2, 1\rangle(62)$, $\langle 1, 2, 3, 1, 3, 1, 2, 1\rangle(61)$,
$\langle 1, 2, 3, 2, 1, 2, 2, 1\rangle(61)$, $\langle 1, 3, 1, 2, 1, 3, 2, 1\rangle(63)$, $\langle 1, 3, 1, 2, 2, 1, 3, 1\rangle(63)$,
$\langle 1, 3, 1, 2, 2, 2, 1, 2\rangle(63)$, $\langle 1, 3, 1, 2, 2, 2, 2, 1\rangle(62)$, $\langle 1, 3, 1, 2, 3, 1, 2, 1\rangle(61)$,
$\langle 1, 3, 1, 3, 1, 2, 2, 1\rangle(61)$, $\langle 1, 3, 2, 1, 2, 2, 2, 1\rangle(61)$, $\langle 1, 3, 2, 1, 3, 1, 2, 1\rangle(60)$,
$\langle 2, 1, 2, 2, 1, 3, 2, 1\rangle(63)$, $\langle 2, 1, 2, 2, 2, 1, 3, 1\rangle(63)$, $\langle 2, 1, 2, 2, 2, 2, 1, 2\rangle(63)$,
$\langle 2, 1, 2, 2, 2, 2, 2, 1\rangle(62)$, $\langle 2, 1, 2, 2, 3, 1, 2, 1\rangle(61)$, $\langle 2, 1, 2, 3, 1, 2, 2, 1\rangle(61)$,
$\langle 2, 1, 3, 1, 2, 2, 2, 1\rangle(61)$, $\langle 2, 1, 3, 1, 3, 1, 2, 1\rangle(60)$, $\langle 2, 1, 3, 2, 1, 2, 2, 1\rangle(60)$,
$\langle 2, 2, 1, 2, 2, 2, 2, 1\rangle(61)$, $\langle 2, 2, 1, 2, 3, 1, 2, 1\rangle(60)$, $\langle 2, 2, 1, 3, 1, 2, 2, 1\rangle(60)$,
$\langle 2, 2, 2, 1, 3, 1, 2, 1\rangle(59)$.

(ii) *If* $l := r_i + r_{i+1} + \cdots + r_{i+7} \geq 15$, *then we have* $r_i + 2r_{i+1} + \cdots + 8r_{i+7} \leq 44 + 8(l - 11)$, *and the equality holds if and only if* $\langle r_i, r_{i+1}, \cdots, r_{i+7}\rangle = \langle 1, 2, 1, 3, 1, 2, 1, l - 11\rangle$.

Proof. If $r_i + r_{i+1} + \cdots + r_{i+7} \leq 13$, then we have $\langle r_i, r_{i+1}, r_{i+2}, r_{i+3}\rangle = \langle 1, 2, 2, 1\rangle$ and $r_{i+4} + r_{i+5} + r_{i+6} + r_{i+7} = 7$, or $\langle r_{i+4}, r_{i+5}, r_{i+6}, r_{i+7}\rangle =$

$\langle 1, 2, 2, 1\rangle$ and $r_i + r_{i+1} + r_{i+2} + r_{i+3} = 7$ by Lemmas 1 and 3. However, by Lemmas 1, 3 and Propositions 2, 4 and 6, all combinations of these sequences are impossible. Hence, we must have $r_i + r_{i+1} + \cdots + r_{i+7} \geq 14$.

(i) The assertion follows by enumeration and Propositions 2, 4 and 6.

(ii) By Lemma 6, we have $r_i + r_{i+1} + \cdots + r_{i+6} \geq 11$ and the equality holds if and only if $\langle r_i, r_{i+1}, \cdots, r_{i+6}\rangle = \langle 1, 2, 1, 3, 1, 2, 1\rangle$. Therefore, the sequence $\langle r_i, r_{i+1}, \cdots, r_{i+6}, r_{i+7}\rangle = \langle 1, 2, 1, 3, 1, 2, 1, l-11\rangle$ with $l \geq 15$ satisfies all the j-team conditions ($j = 3$, 5, 7 and 9) by Propositions 2, 4 and 6, and the equality

$$r_i + 2r_{i+1} + \cdots + 7r_{i+6} + 8r_{i+7} = 44 + 8(l-11)$$

holds. On the other hand, for any sequence $\langle r_i, r_{i+1}, \cdots, r_{i+6}, r_{i+7}\rangle$ with $l' := r_i + r_{i+1} + \cdots + r_{i+6} > 11$, we have

$$\begin{aligned} r_i + 2r_{i+1} + \cdots + 7r_{i+6} + 8r_{i+7} &\leq 44 + 7(l'-11) + 8(l-l') \\ &= 44 + 8(l-11) - (l'-11) < 44 + 8(l-11), \end{aligned}$$

which completes the proof. □

4. Maximal Break Interval Greater than or Equal to 5

In this section, using the results in Section 3, we enumerate all the feasible cyclic BIS's with maximal break interval ≥ 5, up to 42 teams.

Let $(s_1, s_2, \cdots, s_n)$ be a feasible and equitable FET with maximal break interval ≥ 5. Let $[r_1, r_2, \cdots, r_{n-1}, r_n]$ be its cyclic BIS. We assume that $r_n = \max_{1\leq i\leq n} r_i \geq 5$ for simplicity. Then, we have

$$r_1 + r_2 + \cdots + r_{n-1} = s_n - s_1 = (2n-1) - r_n \leq 2n - 6.$$

If $n \leq 13$, then we have $\max_{1\leq i\leq n} r_i \leq 4$ by the classification in [13]. Therefore, we examine the cases $14 \leq n \leq 21$ in the following.

(1) The case $n = 14$: Since $r_1 + r_2 + \cdots + r_{13} \leq 28 - 6 = 22$, and $r_1 + \cdots + r_6 \geq 10$ and $r_8 + \cdots + r_{13} \geq 10$ by Lemma 5, we have $r_7 \leq 2$. On the other hand, by the global condition, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + \cdots + 2r_{12} + r_{13} \geq {}_{14}C_2 = \frac{14 \cdot 13}{2} = 91.$$

(1-i) The case $r_7 = 1$: In this case, we may assume that $r_1 + \cdots + r_6 = 10$ and $r_8 + \cdots + r_{13} \leq 11$ by symmetry. Therefore, by Lemma 5, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + \cdots + 2r_{12} + r_{13} \leq 37 + 7 + 43 = 87 < 91,$$

which contradicts the global condition.

(1-ii) The case $r_7 = 2$: In this case, we have $r_1 + \cdots + r_6 = r_8 + \cdots + r_{13} = 10$. Therefore, by Lemma 5, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + \cdots + 2r_{12} + r_{13} \leq 37 + 14 + 37 = 88 < 91,$$

which contradicts the global condition.

Hence, there is no feasible cyclic BIS with $n = 14$ and $\max_{1 \leq i \leq 14} r_i \geq 5$.

(2) The case $n = 15$: Since $r_1 + r_2 + \cdots + r_{14} \leq 30 - 6 = 24$, and $r_1 + \cdots + r_6 \geq 10$ and $r_9 + \cdots + r_{14} \geq 10$ by Lemma 5, we have $3 \leq r_7 + r_8 \leq 4$. On the other hand, by the global condition, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + 7r_8 + \cdots + 2r_{13} + r_{14} \geq {}_{15}C_2 = \frac{15 \cdot 14}{2} = 105.$$

(2-i) The case $r_7 + r_8 = 3$: In this case, we may assume that $r_1 + \cdots + r_6 = 10$ and $r_9 + \cdots + r_{14} \leq 11$ by symmetry. Therefore, by Lemma 5, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + 7r_8 + \cdots + 2r_{13} + r_{14} \leq 37 + 21 + 43 = 101 < 105,$$

which contradicts the global condition.

(2-ii) The case $r_7 + r_8 = 4$: In this case, we have $r_1 + \cdots + r_6 = r_9 + \cdots + r_{14} = 10$. Therefore, by Lemma 5, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + 7r_8 + \cdots + 2r_{13} + r_{14} \leq 37 + 28 + 37 = 102 < 105,$$

which contradicts the global condition.

Hence, there is no feasible cyclic BIS with $n = 15$ and $\max_{1 \le i \le 15} r_i \ge 5$.

(3) The case $n = 16$: Since $r_1 + r_2 + \cdots + r_{15} \le 32 - 6 = 26$, and $r_1 + \cdots + r_6 \ge 10$ and $r_{10} + \cdots + r_{15} \ge 10$ by Lemma 5, we have $4 \le r_7 + r_8 + r_9 \le 6$. On the other hand, by the global condition, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + 8r_8 + 7r_9 + \cdots + 2r_{14} + r_{15} \ge {}_{16}C_2 = \frac{16 \cdot 15}{2} = 120.$$

(3-i) The case $r_7 + r_8 + r_9 = 4$: In this case, by Lemma 2 we have $\langle r_7, r_8, r_9 \rangle = \langle 1, 2, 1 \rangle$ and $7r_7 + 8r_8 + 7r_9 = 30$.

(3-i-a) If $r_1 + \cdots + r_6 = 10$ and $r_{10} + \cdots + r_{15} \le 12$, then by Lemma 5, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + 8r_8 + 7r_9 + \cdots + 2r_{14} + r_{15} \le 37 + 30 + 49 = 116 < 120,$$

which contradicts the global condition.

(3-i-b) If $r_1 + \cdots + r_6 = 11$ and $r_{10} + \cdots + r_{15} \le 11$, then by Lemma 5, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + 8r_8 + 7r_9 + \cdots + 2r_{14} + r_{15} \le 43 + 30 + 43 = 116 < 120,$$

which contradicts the global condition.

(3-i-c) If $r_1 + \cdots + r_6 = 12$ and $r_{10} + \cdots + r_{15} = 10$, then we have a contradiction in the same way as the case (3-i-a).

(3-ii) The case $r_7 + r_8 + r_9 = 5$: In this case, the value $7r_7 + 8r_8 + 7r_9$ becomes maximum when $\langle r_7, r_8, r_9 \rangle = \langle 1, 3, 1 \rangle$, and the maximum value is $7+24+7 = 38$. We may assume that $r_1 + \cdots + r_6 = 10$ and $r_{10} + \cdots + r_{15} \le 11$ by symmetry. Therefore, by Lemma 5, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + 8r_8 + 7r_9 + \cdots + 2r_{14} + r_{15} \le 37 + 38 + 43 = 118 < 120,$$

which contradicts the global condition.

(3-iii) The case $r_7 + r_8 + r_9 = 6$: In this case, the value $7r_7 + 8r_8 + 7r_9$ becomes maximum when $\langle r_7, r_8, r_9 \rangle = \langle 1, 4, 1 \rangle$, and the maximum value is $7 + 32 + 7 = 46$. Since $r_1 + \cdots + r_6 = r_{10} + \cdots + r_{15} = 10$, by Lemma 5, we have

$$r_1 + 2r_2 + \cdots + 7r_7 + 8r_8 + 7r_9 + \cdots + 2r_{14} + r_{15} \leq 37 + 46 + 37 = 120.$$

The equality holds if and only if $\langle r_1, r_2, \cdots, r_{15}\rangle = \langle 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1\rangle$. In fact, the cyclic BIS [1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5] satisfies all the local conditions, and therefore it is probable. The corresponding FET is (1, 2, 4, 5, 8, 9, 11, 12, 16, 17, 19, 20, 23, 24, 26, 27). We can make its schedule in a similar way as in [13]. First, we make schedules for two subsets of teams $\{1, 2, \cdots, 16\}$ and $\{17, 18, \cdots, 32\}$, and then make a schedule between these two subsets. Table 5 in Appendix is the upper half of a schedule. Thus, we have obtained a unique feasible cyclic BIS with $n = 16$ and $\max_{1\leq i\leq 16} r_i \geq 5$, and its maximal break interval is exactly equal to 5.

(4) The case $n = 17$: Since $r_1 + r_2 + \cdots + r_{16} \leq 34 - 6 = 28$, and $r_1 + \cdots + r_8 \geq 14$ and $r_9 + \cdots + r_{16} \geq 14$ by Lemma 7, we have $r_1 + \cdots + r_8 = r_9 + \cdots + r_{16} = 14$. Therefore, by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 8r_8 + 8r_9 + \cdots + 2r_{15} + r_{16} \leq 67 + 67 = 134,$$

which contradicts the global condition:

$$r_1 + 2r_2 + \cdots + 8r_8 + 8r_9 + \cdots + 2r_{15} + r_{16} \geq {}_{17}C_2 = \frac{17 \cdot 16}{2} = 136.$$

Hence, there is no feasible cyclic BIS with $n = 17$ and $\max_{1\leq i\leq 17} r_i \geq 5$.

(5) The case $n = 18$: Since $r_1 + r_2 + \cdots + r_{17} \leq 36 - 6 = 30$, and $r_1 + \cdots + r_8 \geq 14$ and $r_{10} + \cdots + r_{17} \geq 14$ by Lemma 7, we have $r_9 \leq 2$. On the other hand, by the global condition, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + \cdots + 2r_{16} + r_{17} \geq {}_{18}C_2 = \frac{18 \cdot 17}{2} = 153.$$

(5-i) The case $r_9 = 1$: In this case, we may assume that $r_1 + \cdots + r_8 = 14$ and $r_{10} + \cdots + r_{17} \leq 15$ by symmetry. Therefore, by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + \cdots + 2r_{16} + r_{17} \leq 67 + 9 + 76 = 152 < 153,$$

which contradicts the global condition.

(5-ii) The case $r_9 = 2$: In this case, we have $r_1 + \cdots + r_8 = r_{10} + \cdots + r_{17} = 14$. Therefore, by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + \cdots + 2r_{16} + r_{17} \leq 67 + 18 + 67 = 152 < 153,$$

which contradicts the global condition.

Hence, there is no feasible cyclic BIS with $n = 18$ and $\max\limits_{1 \leq i \leq 18} r_i \geq 5$.

(6) The case $n = 19$: Since $r_1 + r_2 + \cdots + r_{18} \leq 38 - 6 = 32$, and $r_1 + \cdots + r_8 \geq 14$ and $r_{11} + \cdots + r_{18} \geq 14$ by Lemma 7, we have $3 \leq r_9 + r_{10} \leq 4$. On the other hand, by the global condition, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + 9r_{10} + \cdots + 2r_{17} + r_{18} \geq {}_{19}C_2 = \frac{19 \cdot 18}{2} = 171.$$

(6-i) The case $r_9 + r_{10} = 3$: In this case, we may assume that $r_1 + \cdots + r_8 = 14$ and $r_{11} + \cdots + r_{18} \leq 15$ by symmetry. Therefore, by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + 9r_{10} + \cdots + 2r_{17} + r_{18} \leq 67 + 27 + 76 = 170 < 171,$$

which contradicts the global condition.

(6-ii) The case $r_9 + r_{10} = 4$: In this case, we have $r_1 + \cdots + r_8 = r_{11} + \cdots + r_{18} = 14$. Therefore, by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + 9r_{10} + \cdots + 2r_{17} + r_{18} \leq 67 + 36 + 67 = 170 < 171,$$

which contradicts the global condition.

Hence, there is no feasible cyclic BIS with $n = 19$ and $\max\limits_{1 \leq i \leq 19} r_i \geq 5$.

(7) The case $n = 20$: Since $r_1 + r_2 + \cdots + r_{19} \leq 40 - 6 = 34$, and $r_1 + \cdots + r_8 \geq 14$ and $r_{12} + \cdots + r_{19} \geq 14$ by Lemma 7, we have $4 \leq r_9 + r_{10} + r_{11} \leq 6$. On the other hand, by the global condition, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + 10r_{10} + 9r_{11} + \cdots + 2r_{18} + r_{19} \geq {}_{20}C_2 = \frac{20 \cdot 19}{2} = 190.$$

(7-i) The case $r_9 + r_{10} + r_{11} = 4$: In this case, by Lemma 2, we have $\langle r_9, r_{10}, r_{11} \rangle = \langle 1, 2, 1 \rangle$ and $9r_9 + 10r_{10} + 9r_{11} = 38$.

(7-i-a) If $r_1 + \cdots + r_8 = 14$ and $r_{12} + \cdots + r_{19} \leq 16$, then by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + 10r_{10} + 9r_{11} + \cdots + 2r_{18} + r_{19} \leq 67 + 38 + 84 = 189 < 190,$$

which contradicts the global condition.

(7-i-b) If $r_1 + \cdots + r_8 = 15$ and $r_{12} + \cdots + r_{19} \leq 15$, then by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + 10r_{10} + 9r_{11} + \cdots + 2r_{18} + r_{19} \leq 76 + 38 + 76 = 190.$$

The equality holds if and only if $\langle r_1, r_2, \cdots, r_{19} \rangle = \langle 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1 \rangle$. In fact, the cyclic BIS [1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5] satisfies all the local conditions, and therefore it is probable.

(7-i-c) If $r_1 + \cdots + r_8 = 16$ and $r_{12} + \cdots + r_{19} = 14$, then we have a contradiction in the same way as the case (7-i-a).

(7-ii) The case $r_9 + r_{10} + r_{11} = 5$: In this case, $\langle r_9, r_{10}, r_{11} \rangle$ is equal to $\langle 1, 3, 1 \rangle(48)$, $\langle 1, 2, 2 \rangle(47)$, $\langle 2, 2, 1 \rangle(47)$ or $\langle 2, 1, 2 \rangle(46)$ (in parentheses, we described $9r_9 + 10r_{10} + 9r_{11}$). We may assume that $r_1 + \cdots + r_8 = 14$ and $r_{12} + \cdots + r_{19} \leq 15$ by symmetry. Therefore, by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 9r_9 + 10r_{10} + 9r_{11} + \cdots + 2r_{18} + r_{19} \leq 67 + 48 + 76 = 191 \ (> 190).$$

This implies that $9r_9 + 10r_{10} + 9r_{11} \geq 47$.

(7-ii-a) If $\langle r_9, r_{10}, r_{11} \rangle = \langle 1, 3, 1 \rangle$, then we have $r_8 \geq 2$ and $r_{12} \geq 2$. Hence, $r_1 + 2r_2 + \cdots + 8r_8 \leq 66$ and $8r_{12} + \cdots + 2r_{18} + r_{19} = 76$ by Lemma 7 and Proposition 7. Therefore, the equalities must hold, and $\langle r_1, r_2, \cdots, r_{19} \rangle$ is equal to one of the following sequences:

$$\begin{aligned}
&\langle 1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 1, 4, 1, 2, 1, 3, 1, 2, 1 \rangle, \\
&\langle \underline{1, 2, 1, 3, 2, 1, 2, 2, 1, 3}, 1, 4, 1, 2, 1, 3, 1, 2, 1 \rangle, \\
&\langle 1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 1, 4, 1, 2, 1, 3, 1, 2, 1 \rangle, \\
&\langle \underline{1, 2, 2, 1, 3, 1, 2, 2, 1, 3}, 1, 4, 1, 2, 1, 3, 1, 2, 1 \rangle.
\end{aligned}$$

Among them, the underlined sequences do not satisy the 11-team condition. On the other hand, the cyclic BIS's [1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5], [1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5] and the reversed ones satisfy all the local conditions, and therefore they are probable.

(7-ii-b) If $\langle r_9, r_{10}, r_{11}\rangle = \langle 1, 2, 2\rangle$ or $\langle 2, 2, 1\rangle$, then $\langle r_1, \cdots, r_8\rangle = \langle 1, 2, 1, 3, 1, 2, 2, 2\rangle$ and $\langle r_{12}, \cdots, r_{19}\rangle = \langle 4, 1, 2, 1, 3, 1, 2, 1\rangle$ by Lemma 7. Therefore, by Proposition 7, we have $r_9 \geq 2$ and $\langle r_1, r_2, \cdots, r_{19}\rangle = \langle 1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1\rangle$. In fact, the cyclic BIS [1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5] and the reversed one satisfy all the local conditions, and therefore they are probable.

(7-iii) The case $r_9 + r_{10} + r_{11} = 6$: In this case, $\langle r_9, r_{10}, r_{11}\rangle$ is equal to $\langle 1, 4, 1\rangle(58)$, $\langle 1, 3, 2\rangle(57)$, $\langle 2, 3, 1\rangle(57)$, $\langle 1, 2, 3\rangle(56)$, $\langle 2, 2, 2\rangle(56)$, $\langle 3, 2, 1\rangle(56)$, $\langle 2, 1, 3\rangle(55)$ or $\langle 3, 1, 2\rangle(55)$ (in parentheses, we described $9r_9 + 10r_{10} + 9r_{11}$). Since $r_1 + \cdots + r_8 = r_{12} + \cdots + r_{19} = 14$, by Lemma 7, we have

$$r_1+2r_2+\cdots+9r_9+10r_{10}+9r_{11}+\cdots+2r_{18}+r_{19} \leq 67+58+67 = 192\ (> 190).$$

This implies that $9r_9 + 10r_{10} + 9r_{11} \geq 56$.

(7-iii-a) If $\langle r_9, r_{10}, r_{11}\rangle = \langle 1, 4, 1\rangle$, then we have $r_8 \geq 2$ and $r_{12} \geq 2$. Hence, $r_1 + 2r_2 + \cdots + 8r_8 \leq 66$ and $8r_{12} + \cdots + 2r_{18} + r_{19} \leq 66$ by Lemma 7 and Proposition 7. Therefore, the equalities must hold, and $\langle r_1, r_2, \cdots, r_{19}\rangle$ is equal to one of the following sequences or the reversed ones:

$$\begin{aligned}
&\langle 1, 2, 1, 3, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 3, 1, 2, 1\rangle,\\
&\langle 1, 2, 1, 3, 1, 3, 1, 2, 1, 4, 1, 2, 2, 1, 2, 3, 1, 2, 1\rangle,\\
&\langle 1, 2, 1, 3, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 2, 1, 2, 2, 1\rangle,\\
&\langle 1, 2, 1, 3, 1, 3, 1, 2, 1, 4, 1, 2, 2, 1, 3, 1, 2, 2, 1\rangle,\\
&\langle 1, 2, 1, 3, 2, 1, 2, 2, 1, 4, 1, 2, 2, 1, 2, 3, 1, 2, 1\rangle,\\
&\langle 1, 2, 1, 3, 2, 1, 2, 2, 1, 4, 1, 2, 1, 3, 2, 1, 2, 2, 1\rangle,\\
&\langle 1, 2, 1, 3, 2, 1, 2, 2, 1, 4, 1, 2, 2, 1, 3, 1, 2, 2, 1\rangle,\\
&\langle 1, 2, 2, 1, 2, 3, 1, 2, 1, 4, 1, 2, 1, 3, 2, 1, 2, 2, 1\rangle,\\
&\langle 1, 2, 2, 1, 2, 3, 1, 2, 1, 4, 1, 2, 2, 1, 3, 1, 2, 2, 1\rangle,\\
&\langle 1, 2, 2, 1, 3, 1, 2, 2, 1, 4, 1, 2, 2, 1, 3, 1, 2, 2, 1\rangle.
\end{aligned}$$

If we add the term $r_{20} = 5$ to the sequences above, then the obtained cyclic BIS's and the reversed ones satisfy all the local conditions, and therefore they are probable.

(7-iii-b) If $\langle r_9, r_{10}, r_{11}\rangle = \langle 1, 3, 2\rangle$ or $\langle 2, 3, 1\rangle$, then we may assume that $\langle r_9, r_{10}, r_{11}\rangle = \langle 1, 3, 2\rangle$ by symmetry. In this case, we have $r_8 \geq 2$. Hence, $r_1 + 2r_2 + \cdots + 8r_8 \leq 66$ and $8r_{12} + \cdots + 2r_{18} + r_{19} \leq 67$ by Lemma 7 and Proposition 7. Therefore, the equalities must hold, and $\langle r_1, r_2, \cdots, r_{19}\rangle$ is equal to one of the following sequences:

$$\langle 1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 2, 2, 2, 2, 1, 3, 1, 2, 1\rangle,$$
$$\langle \underline{1, 2, 1, 3, 2, 1, 2, 2, 1, 3}, 2, 2, 2, 2, 1, 3, 1, 2, 1\rangle,$$
$$\langle 1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 2, 2, 2, 2, 1, 3, 1, 2, 1\rangle,$$
$$\langle \underline{1, 2, 2, 1, 3, 1, 2, 2, 1, 3}, 2, 2, 2, 2, 1, 3, 1, 2, 1\rangle.$$

Among them, the underlined sequences do not satisy the 11-team condition. On the other hand, the cyclic BIS's [1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5], [1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5] and the reversed ones satisfy all the local conditions, and therefore they are probable.

(7-iii-c) If $\langle r_9, r_{10}, r_{11}\rangle = \langle 1, 2, 3\rangle$, $\langle 3, 2, 1\rangle$ or $\langle 2, 2, 2\rangle$, then we have $r_1 + 2r_2 + \cdots + 8r_8 = 8r_{12} + \cdots + 2r_{18} + r_{19} = 67$. Hence, $\langle r_1, \cdots, r_8\rangle = \langle 1, 2, 1, 3, 1, 2, 2, 2\rangle$ and $\langle r_{12}, \cdots, r_{19}\rangle = \langle 2, 2, 2, 1, 3, 1, 2, 1\rangle$ by Lemma 7. Therefore, we have $r_9 \geq 2$ and $r_{11} \geq 2$ by Proposition 7, namely, $\langle r_9, r_{10}, r_{11}\rangle = \langle 2, 2, 2\rangle$. Hence, $\langle r_1, r_2, \cdots, r_{19}\rangle = \langle 1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1\rangle$. In fact, the cyclic BIS [1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5] satisfies all the local conditions, and therefore it is probable.

Hence, there exist 28 probable cyclic BIS's with $n = 20$ and $\max_{1\leq i\leq 20} r_i \geq 5$, and their maximal break intervals are exactly equal to 5. We can make their schedules in a similar way as in the case $n = 16$. For example, Table 6 in Appendix is the upper half of a schedule for the FET (1, 2, 4, 5, 8, 9, 11, 12, 16, 17, 19, 20, 24, 25, 27, 28, 31, 32, 34, 35) corresponding to the cyclic BIS [1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5] in (7-i-b). Thus, we have obtained 28 feasible cyclic BIS's with $n = 20$ and $\max_{1\leq i\leq 20} r_i \geq 5$.

(8) The case $n = 21$: Since $r_1 + r_2 + \cdots + r_{20} \leq 42 - 6 = 36$, and $r_1 + \cdots + r_8 \geq 14$ and $r_{13} + \cdots + r_{30} \geq 14$ by Lemma 7, we have $6 \leq r_9 + r_{10} + r_{11} + r_{12} \leq 8$. On the other hand, by the global condition, we have

$$r_1 + 2r_2 + \cdots + 10r_{10} + 10r_{11} + \cdots + 2r_{19} + r_{20} \geq {}_{21}C_2 = \frac{21 \cdot 20}{2} = 210.$$

(8-i) The case $r_9 + r_{10} + r_{11} + r_{12} = 6$: In this case, by Lemma 3, we have $\langle r_9, r_{10}, r_{11}, r_{12} \rangle = \langle 1, 2, 2, 1 \rangle$ and $9r_9 + 10r_{10} + 10r_{11} + 9r_{12} = 58$.

(8-i-a) If $r_1 + \cdots + r_8 = 14$ and $r_{13} + \cdots + r_{20} \leq 16$, then by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 10r_{10} + 10r_{11} + \cdots + 2r_{19} + r_{20} \leq 67 + 58 + 84 = 209 < 210,$$

which contradicts the global condition.

(8-i-b) If $r_1 + \cdots + r_8 = 15$ and $r_{13} + \cdots + r_{20} \leq 15$, then by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 10r_{10} + 10r_{11} + \cdots + 2r_{19} + r_{20} \leq 76 + 58 + 76 = 210.$$

The equality holds if and only if $\langle r_1, r_2, \cdots, r_{20} \rangle = \langle 1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1 \rangle$. In fact, the cyclic BIS [1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5] satisfies all the local conditions, and therefore it is probable.

(8-i-c) If $r_1 + \cdots + r_8 = 16$ and $r_{13} + \cdots + r_{20} = 14$, then we have a contradiction in the same way as the case (8-i-a).

(8-ii) The case $r_9 + r_{10} + r_{11} + r_{12} = 7$: In this case, by Lemma 3, $\langle r_9, r_{10}, r_{11}, r_{12} \rangle$ is equal to $\langle 1, 2, 3, 1 \rangle(68)$, $\langle 1, 3, 2, 1 \rangle(68)$, $\langle 1, 2, 2, 2 \rangle(67)$, $\langle 2, 2, 2, 1 \rangle(67)$, $\langle 1, 3, 1, 2 \rangle(67)$, $\langle 2, 1, 3, 1 \rangle(67)$, $\langle 1, 2, 1, 3 \rangle(66)$, $\langle 3, 1, 2, 1 \rangle(66)$, $\langle 2, 1, 2, 2 \rangle(66)$ or $\langle 2, 2, 1, 2 \rangle(66)$ (in parentheses, we described $9r_9 + 10r_{10} + 10r_{11} + 9r_{12}$). We may assume that $r_1 + \cdots + r_8 = 14$ and $r_{13} + \cdots + r_{20} \leq 15$ by symmetry. Therefore, by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 10r_{10} + 10r_{11} + \cdots + 2r_{19} + r_{20} \leq 67 + 68 + 76 = 211 \ (> 210).$$

This implies that $9r_9 + 10r_{10} + 10r_{11} + 9r_{12} \geq 67$.

(8-ii-a) If $\langle r_9, r_{10}, r_{11}, r_{12} \rangle = \langle 1, 2, 3, 1 \rangle$ or $\langle 1, 3, 2, 1 \rangle$, then we have $r_8 \geq 2$ and $r_{13} \geq 2$. Hence, $r_1 + 2r_2 + \cdots + 8r_8 \leq 66$ and $8r_{13} + \cdots + 2r_{19} + r_{20} = 76$ by Lemma 7 and Proposition 7. Therefore, the equalities must hold, and $\langle r_1, r_2, \cdots, r_{20} \rangle$ is equal to one of the following sequences:

$$
\begin{aligned}
&\langle 1,\ 2,\ 1,\ 3,\ 1,\ 3,\ \underline{1,\ 2,\ 1,\ 2},\ 3,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
&\langle 1,\ 2,\ 1,\ 3,\ \underline{2,\ 1,\ 2,\ 2,\ 1,\ 2},\ 3,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
&\langle 1,\ 2,\ 2,\ 1,\ 2,\ 3,\ \underline{1,\ 2,\ 1,\ 2},\ 3,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
&\langle 1,\ 2,\ \underline{2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 1,\ 2},\ 3,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
&\langle \underline{1,\ 2,\ 1,\ 3,\ 1,\ 3,\ 1,\ 2,\ 1,\ 3,\ 2,\ 1},\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
&\langle \underline{1,\ 2,\ 1,\ 3,\ 2,\ 1,\ 2,\ 2,\ 1,\ 3},\ 2,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
&\langle \underline{1,\ 2,\ 2,\ 1,\ 2,\ 3,\ 1,\ 2,\ 1,\ 3,\ 2,\ 1},\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
&\langle \underline{1,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 1,\ 3},\ 2,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle.
\end{aligned}
$$

For all of them, the underlined sequences do not satisy one of the 5-team, 7-team, 9-team, 11-team or 13-team conditions.

(8-ii-b) If $\langle r_9,\ r_{10},\ r_{11},\ r_{12}\rangle = \langle 1,\ 2,\ 2,\ 2\rangle$, $\langle 2,\ 2,\ 2,\ 1\rangle$, $\langle 1,\ 3,\ 1,\ 2\rangle$ or $\langle 2,\ 1,\ 3,\ 1\rangle$, then $\langle r_1,\ \cdots,\ r_8\rangle = \langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2\rangle$ and $\langle r_{13},\ \cdots,\ r_{20}\rangle = \langle 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle$ by Lemma 7. Therofore, by Proposition 7, we have $r_9 \geq 2$ and $\langle r_1,\ r_2,\ \cdots,\ r_{20}\rangle$ is equal to one of the following sequences:

$$
\begin{aligned}
&\langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
&\langle \underline{1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2,\ 2,\ 1},\ 3,\ 1,\ 4,\ 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle.
\end{aligned}
$$

Among them, the underlined sequence does not satisy the 11-team condition. On the other hand, the cyclic BIS [1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5] and the reversed one satisfy all the local conditions, and therefore they are probable.

(8-iii) The case $r_9 + r_{10} + r_{11} + r_{12} = 8$: In this case, $\langle r_9,\ r_{10},\ r_{11},\ r_{12}\rangle$ is equal to one of the following sequences (in parentheses, we described $9r_9 + 10r_{10} + 10r_{11} + 9r_{12}$):

$$
\begin{aligned}
&\langle 1,\ 2,\ 4,\ 1\rangle(78),\ \langle 1,\ 3,\ 3,\ 1\rangle(78),\ \langle 1,\ 4,\ 2,\ 1\rangle(78),\ \langle 1,\ 2,\ 3,\ 2\rangle(77),\\
&\langle 1,\ 3,\ 2,\ 2\rangle(77),\ \langle 1,\ 4,\ 1,\ 2\rangle(77),\ \langle 2,\ 1,\ 4,\ 1\rangle(77),\ \langle 2,\ 2,\ 3,\ 1\rangle(77),\\
&\langle 2,\ 3,\ 2,\ 1\rangle(77),\ \langle 1,\ 2,\ 2,\ 3\rangle(76),\ \langle 1,\ 3,\ 1,\ 3\rangle(76),\ \langle 2,\ 1,\ 3,\ 2\rangle(76),\\
&\langle 2,\ 2,\ 2,\ 2\rangle(76),\ \langle 2,\ 3,\ 1,\ 2\rangle(76),\ \langle 3,\ 1,\ 3,\ 1\rangle(76),\ \langle 3,\ 2,\ 2,\ 1\rangle(76),\\
&\langle 1,\ 2,\ 1,\ 4\rangle(75),\ \langle 2,\ 2,\ 1,\ 3\rangle(75),\ \langle 3,\ 2,\ 1,\ 2\rangle(75),\ \langle 2,\ 1,\ 2,\ 3\rangle(75),\\
&\langle 3,\ 1,\ 2,\ 2\rangle(75),\ \langle 4,\ 1,\ 2,\ 1\rangle(75).
\end{aligned}
$$

Since $r_1 + \cdots + r_8 = r_{13} + \cdots + r_{20} = 14$, by Lemma 7, we have

$$r_1 + 2r_2 + \cdots + 10r_{10} + 10r_{11} + \cdots + 2r_{19} + r_{20} \le 67 + 78 + 67 = 212 \ (> 210).$$

This implies that $9r_9 + 10r_{10} + 10r_{11} + 9r_{12} \ge 76$.

(8-iii-a) If $\langle r_9, r_{10}, r_{11}, r_{12}\rangle = \langle 1, 2, 4, 1\rangle$, $\langle 1, 3, 3, 1\rangle$ or $\langle 1, 4, 2, 1\rangle$, then we have $r_8 \ge 2$ and $r_{13} \ge 2$. Hence, $r_1 + 2r_2 + \cdots + 8r_8 \le 66$ and $8r_{13} + \cdots + 2r_{19} + r_{20} \le 66$ by Lemma 7 and Proposition 7. Therefore, the equalities must hold, and $\langle r_1, r_2, \cdots, r_{20}\rangle$ is equal to one of the following sequences or the reversed ones:

$$\begin{gathered}
\langle 1, 2, 1, 3, 1, 3, \underline{1, 2, 1, 2}, 4, 1, \cdots\rangle, \\
\langle 1, 2, 1, 3, \underline{2, 1, 2, 2, 1, 2}, 4, 1, \cdots\rangle, \\
\langle 1, 2, 2, 1, 2, 3, \underline{1, 2, 1, 2}, 4, 1, \cdots\rangle, \\
\langle 1, 2, \underline{2, 1, 3, 1, 2, 2, 1, 2}, 4, 1, \cdots\rangle, \\
\langle 1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 3, 1, \cdots\rangle, \\
\langle \underline{1, 2, 1, 3, 2, 1, 2, 2, 1, 3}, 3, 1, \cdots\rangle, \\
\langle 1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 3, 1, \cdots\rangle, \\
\langle \underline{1, 2, 2, 1, 3, 1, 2, 2, 1, 3}, 3, 1, \cdots\rangle,
\end{gathered}$$

Among them, the underlined sequences do not satisy one of the 5-team, 7-team, 9-team or 11-team conditions. On the other hand, the following cyclic BIS's satisfy all the local conditions, and therefore they are probable:

$$\begin{gathered}
[1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5], \\
[1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 3, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5], \\
[1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5], \\
[1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 3, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5].
\end{gathered}$$

(8-iii-b) If $\langle r_9, r_{10}, r_{11}, r_{12}\rangle = \langle 1, 2, 3, 2\rangle$, $\langle 1, 3, 2, 2\rangle$, $\langle 1, 4, 1, 2\rangle$, $\langle 2, 1, 4, 1\rangle$, $\langle 2, 2, 3, 1\rangle$ or $\langle 2, 3, 2, 1\rangle$, then we may assume that $r_9 = 1$ by symmetry. In this case, we have $r_8 \ge 2$. Hence, $r_1 + 2r_2 + \cdots + 8r_8 \le 66$ and $8r_{13} + \cdots + 2r_{19} + r_{20} \le 67$ by Lemma 7 and Proposition 7. Therefore, the equalities must hold, and $\langle r_1, r_2, \cdots, r_{20}\rangle$ is equal to one of the following sequences:

$$
\begin{array}{l}
\langle 1,\ 2,\ 1,\ 3,\ 1,\ 3,\ \underline{1,\ 2,\ 1,\ 2},\ 3,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle 1,\ 2,\ 1,\ 3,\ \underline{2,\ 1,\ 2,\ 2,\ 1,\ 2},\ 3,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle 1,\ 2,\ 2,\ 1,\ 2,\ 3,\ \underline{1,\ 2,\ 1,\ 2},\ 3,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle 1,\ 2,\ \underline{2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 1,\ 2},\ 3,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle 1,\ 2,\ 1,\ 3,\ 1,\ 3,\ 1,\ 2,\ 1,\ 3,\ 2,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle \underline{1,\ 2,\ 1,\ 3,\ 2,\ 1,\ 2,\ 2,\ 1,\ 3},\ 2,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle 1,\ 2,\ 2,\ 1,\ 2,\ 3,\ 1,\ 2,\ 1,\ 3,\ 2,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle \underline{1,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 1,\ 3},\ 2,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle 1,\ 2,\ 1,\ 3,\ 1,\ 3,\ 1,\ 2,\ 1,\ 4,\ \underline{1,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1}\rangle,\\
\langle 1,\ 2,\ 1,\ 3,\ 2,\ 1,\ 2,\ 2,\ 1,\ 4,\ \underline{1,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1}\rangle,\\
\langle 1,\ 2,\ 2,\ 1,\ 2,\ 3,\ 1,\ 2,\ 1,\ 4,\ \underline{1,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1}\rangle,\\
\langle 1,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 1,\ 4,\ \underline{1,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1}\rangle.
\end{array}
$$

Among them, the underlined sequences do not satisy one of the 5-team, 7-team, 9-team or 11-team conditions. On the other hand, the cyclic BIS's [1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5], [1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5] and the reversed ones satisfy all the local conditions, and therefore they are probable.

(8-iii-c) If $\langle r_9,\ r_{10},\ r_{11},\ r_{12}\rangle = \langle 1,\ 2,\ 2,\ 3\rangle$, $\langle 1,\ 3,\ 1,\ 3\rangle$, $\langle 2,\ 1,\ 3,\ 2\rangle$, $\langle 2,\ 2,\ 2,\ 2\rangle$, $\langle 2,\ 3,\ 1,\ 2\rangle$, $\langle 3,\ 1,\ 3,\ 1\rangle$ or $\langle 3,\ 2,\ 2,\ 1\rangle$, then we have $r_1+2r_2+\cdots+8r_8 = 8r_{13}+\cdots+2r_{19}+r_{20} = 67$. Hence, $\langle r_1,\ \cdots,\ r_8\rangle = \langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2\rangle$ and $\langle r_{13},\ \cdots,\ r_{20}\rangle = \langle 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle$ by Lemma 7. Therefore, we have $r_9 \geq 2$ and $r_{12} \geq 2$ by Proposition 7, namely, $\langle r_9,\ r_{10},\ r_{11},\ r_{12}\rangle = \langle 2,\ 1,\ 3,\ 2\rangle$, $\langle 2,\ 2,\ 2,\ 2\rangle$ or $\langle 2,\ 3,\ 1,\ 2\rangle$. Hence, $\langle r_1,\ r_2,\ \cdots,\ r_{20}\rangle$ is equal to one of the following sequences:

$$
\begin{array}{l}
\langle \underline{1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2,\ 2,\ 1},\ 3,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1\rangle,\\
\langle 1,\ 2,\ 1,\ 3,\ 1,\ 2,\ 2,\ 2,\ 2,\ 3,\ \underline{1,\ 2,\ 2,\ 2,\ 2,\ 1,\ 3,\ 1,\ 2,\ 1}\rangle.
\end{array}
$$

Among them, the underlined sequences do not satisy the 11-team condition. On the other hand, the cyclic BIS [1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5] satisfies all the local conditions, and therefore it is probable.

Hence, there exist 12 probable cyclic BIS's with $n = 21$ and $\max_{1 \le i \le 21} r_i \ge 5$, and their maximal break intervals are exactly equal to 5. We can make their schedules in a similar way as in the case $n = 16$. For example, Table 7 in Appendix is the upper half of a schedule for the FET (1, 2, 4, 5, 8, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 30, 33, 34, 36, 37) corresponding to the cyclic BIS [1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5] in (8-iii-c). Thus, we have obtained 12 feasible cyclic BIS's with $n = 21$ and $\max_{1 \le i \le 21} r_i \ge 5$.

We summarize the results in Table 8 in Appendix, where we show all the feasible and equitable cyclic BIS's with $\max_{1 \le i \le n} r_i \ge 5$ for $n \le 21$.

5. Conclusion

We studied the maximal interval of the cyclic break interval sequences of equitable round-robin tournaments with $2n$ teams. We enumerated all the feasible cyclic break interval sequences with maximal break interval greater than or equal to 5 for $n \le 21$. We proved that there exist such cyclic break interval sequences for $n = 16$, 20 and 21, and there is none for $n \le 15$ and $17 \le n \le 19$. We also showed some schedules for such tournamnents.

It is of interest to know how the maximal break intervals of feasible and equitable round-robin tournaments with $2n$ teams increases when n increases. We will treat this problem in a forthcoming paper.

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Table 5. Schedule for $(1,2,4,5,8,9,11,12,16,17,19,20,23,24,26,27)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15
3	19	1	2	4	7	5	6	8	11	9	10	15	16	13	14
4	20	2	1	3	8	6	5	7	12	10	9	16	15	14	13
5	31	32	29	1	2	3	4	6	7	8	13	9	10	11	12
6	32	31	30	2	1	4	3	5	8	7	14	10	9	12	11
7	29	30	31	23	3	1	2	4	5	6	8	11	12	9	10
8	30	29	32	24	4	2	1	3	6	5	7	12	11	10	9
9	27	28	25	32	31	30	29	1	2	3	4	5	6	7	8
10	28	27	26	31	32	29	30	2	1	4	3	6	5	8	7
11	25	26	27	30	29	32	31	28	3	1	2	7	8	5	6
12	26	25	28	29	30	31	32	27	4	2	1	8	7	6	5
13	23	21	22	28	27	25	26	32	31	29	5	1	2	3	4
14	24	22	21	27	28	26	25	31	32	30	6	2	1	4	3
15	21	24	23	26	25	28	27	30	29	32	31	3	4	1	2
16	22	23	24	25	26	27	28	29	30	31	32	4	3	2	1

	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
1	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
2	18	17	20	19	22	21	24	23	26	25	28	27	30	29	32	31
3	12	20	17	18	23	24	21	22	27	28	25	26	31	32	29	30
4	11	19	18	17	24	23	22	21	28	27	26	25	32	31	30	29
5	14	15	16	21	17	18	19	20	22	23	24	30	25	26	27	28
6	13	16	15	22	18	17	20	19	21	24	23	29	26	25	28	27
7	15	13	14	16	19	20	17	18	24	21	22	32	27	28	25	26
8	16	14	13	15	20	19	18	17	23	22	21	31	28	27	26	25
9	10	11	12	13	14	15	16	26	17	18	19	20	21	22	23	24
10	9	12	11	14	13	16	15	25	18	17	20	19	22	21	24	23
11	4	9	10	12	15	13	14	16	19	20	17	18	23	24	21	22
12	3	10	9	11	16	14	13	15	20	19	18	17	24	23	22	21
13	6	7	8	9	10	11	12	14	15	16	30	24	17	18	19	20
14	5	8	7	10	9	12	11	13	16	15	29	23	18	17	20	19
15	7	5	6	8	11	9	10	12	13	14	16	22	19	20	17	18
16	8	6	5	7	12	10	9	11	14	13	15	21	20	19	18	17

Table 6. Schedule for $(1, 2, 4, 5, 8, 9, 11, 12, 16, 17, 19, 20, 24, 25, 27, 28, 31, 32, 34, 35)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	2	$\underline{3}$	4	$\underline{5}$	6	$\underline{7}$	8	$\underline{9}$	10	$\underline{11}$	12	$\underline{13}$	14	$\underline{15}$	16	$\underline{17}$	19	$\underline{20}$	18
2	$\underline{1}$	$\underline{4}$	3	$\underline{6}$	5	$\underline{8}$	7	$\underline{10}$	9	$\underline{12}$	11	$\underline{14}$	13	$\underline{16}$	15	$\underline{18}$	20	$\underline{19}$	17
3	$\underline{28}$	1	$\underline{2}$	$\underline{4}$	7	$\underline{5}$	6	$\underline{8}$	11	$\underline{9}$	10	$\underline{12}$	17	$\underline{14}$	13	$\underline{19}$	15	$\underline{16}$	20
4	$\underline{27}$	2	$\underline{1}$	3	8	$\underline{6}$	5	$\underline{7}$	12	$\underline{10}$	14	$\underline{11}$	18	$\underline{13}$	9	$\underline{20}$	16	$\underline{15}$	19
5	$\underline{39}$	36	$\underline{35}$	1	$\underline{2}$	3	$\underline{4}$	$\underline{6}$	8	$\underline{7}$	13	$\underline{15}$	11	$\underline{9}$	10	$\underline{12}$	17	$\underline{18}$	14
6	$\underline{40}$	35	$\underline{36}$	2	$\underline{1}$	4	$\underline{3}$	5	7	$\underline{8}$	9	$\underline{10}$	12	$\underline{11}$	14	$\underline{16}$	18	$\underline{17}$	15
7	$\underline{38}$	37	$\underline{40}$	39	$\underline{3}$	1	$\underline{2}$	4	$\underline{6}$	5	8	$\underline{9}$	10	$\underline{12}$	11	$\underline{15}$	14	$\underline{13}$	27
8	$\underline{37}$	38	$\underline{39}$	40	$\underline{4}$	2	$\underline{1}$	3	$\underline{5}$	6	$\underline{7}$	$\underline{16}$	9	$\underline{10}$	12	$\underline{11}$	13	$\underline{14}$	28
9	$\underline{34}$	33	$\underline{38}$	37	$\underline{40}$	39	$\underline{35}$	1	$\underline{2}$	3	$\underline{6}$	7	$\underline{8}$	5	$\underline{4}$	$\underline{10}$	12	$\underline{11}$	16
10	$\underline{33}$	34	$\underline{37}$	38	$\underline{39}$	40	$\underline{36}$	2	$\underline{1}$	4	$\underline{3}$	6	$\underline{7}$	8	$\underline{5}$	9	11	$\underline{12}$	13
11	$\underline{35}$	30	$\underline{29}$	33	$\underline{38}$	37	$\underline{40}$	39	$\underline{3}$	1	$\underline{2}$	4	$\underline{5}$	6	$\underline{7}$	8	$\underline{10}$	9	12
12	$\underline{36}$	29	$\underline{30}$	34	$\underline{37}$	38	$\underline{39}$	40	$\underline{4}$	2	$\underline{1}$	3	$\underline{6}$	7	$\underline{8}$	5	$\underline{9}$	10	$\underline{11}$
13	$\underline{32}$	31	$\underline{25}$	36	$\underline{34}$	35	$\underline{38}$	37	$\underline{40}$	39	$\underline{5}$	1	$\underline{2}$	4	$\underline{3}$	33	$\underline{8}$	7	$\underline{10}$
14	$\underline{31}$	32	$\underline{26}$	35	$\underline{33}$	36	$\underline{37}$	38	$\underline{39}$	40	$\underline{4}$	2	$\underline{1}$	3	$\underline{6}$	34	$\underline{7}$	8	$\underline{5}$
15	$\underline{29}$	40	$\underline{32}$	28	$\underline{36}$	34	$\underline{33}$	31	$\underline{38}$	37	$\underline{35}$	5	$\underline{39}$	1	$\underline{2}$	7	$\underline{3}$	4	$\underline{6}$
16	$\underline{30}$	39	$\underline{31}$	27	$\underline{35}$	33	$\underline{34}$	32	$\underline{37}$	38	$\underline{36}$	8	$\underline{40}$	2	$\underline{1}$	6	$\underline{4}$	3	$\underline{9}$
17	$\underline{24}$	25	$\underline{34}$	32	$\underline{31}$	30	$\underline{29}$	36	$\underline{35}$	33	$\underline{40}$	39	$\underline{3}$	38	$\underline{37}$	1	$\underline{5}$	6	$\underline{2}$
18	$\underline{23}$	26	$\underline{33}$	31	$\underline{32}$	29	$\underline{30}$	35	$\underline{36}$	34	$\underline{39}$	40	$\underline{4}$	37	$\underline{38}$	2	$\underline{6}$	5	$\underline{1}$
19	$\underline{25}$	28	$\underline{27}$	30	$\underline{29}$	32	$\underline{31}$	34	$\underline{33}$	36	$\underline{38}$	37	$\underline{35}$	40	$\underline{39}$	3	$\underline{1}$	2	$\underline{4}$
20	$\underline{26}$	27	$\underline{28}$	29	$\underline{30}$	31	$\underline{32}$	33	$\underline{34}$	35	$\underline{37}$	38	$\underline{36}$	39	$\underline{40}$	4	$\underline{2}$	1	$\underline{3}$

	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39
1	$\underline{21}$	22	$\underline{25}$	23	$\underline{24}$	26	$\underline{27}$	28	$\underline{29}$	30	$\underline{31}$	32	$\underline{33}$	34	$\underline{35}$	36	$\underline{37}$	38	$\underline{39}$	40
2	$\underline{22}$	21	$\underline{26}$	24	$\underline{23}$	25	$\underline{28}$	27	$\underline{30}$	29	$\underline{32}$	31	$\underline{34}$	33	$\underline{36}$	35	$\underline{38}$	37	$\underline{40}$	39
3	$\underline{23}$	24	$\underline{18}$	21	$\underline{22}$	27	$\underline{25}$	26	$\underline{31}$	32	$\underline{29}$	30	$\underline{35}$	36	$\underline{33}$	39	$\underline{34}$	40	$\underline{37}$	38
4	$\underline{24}$	23	$\underline{17}$	22	$\underline{21}$	28	$\underline{26}$	25	$\underline{32}$	31	$\underline{30}$	29	$\underline{36}$	35	$\underline{34}$	40	$\underline{33}$	39	$\underline{38}$	37
5	$\underline{20}$	19	$\underline{21}$	16	$\underline{25}$	23	$\underline{24}$	29	$\underline{27}$	28	$\underline{22}$	33	$\underline{31}$	26	$\underline{37}$	38	$\underline{40}$	34	$\underline{30}$	32
6	$\underline{19}$	20	$\underline{22}$	13	$\underline{26}$	24	$\underline{23}$	30	$\underline{28}$	27	$\underline{21}$	34	$\underline{32}$	25	$\underline{38}$	37	$\underline{39}$	33	$\underline{29}$	31
7	$\underline{18}$	17	$\underline{20}$	19	$\underline{16}$	21	$\underline{22}$	23	$\underline{24}$	25	$\underline{26}$	28	$\underline{29}$	30	$\underline{31}$	32	$\underline{35}$	36	$\underline{33}$	34
8	$\underline{15}$	18	$\underline{19}$	20	$\underline{17}$	22	$\underline{21}$	24	$\underline{23}$	26	$\underline{25}$	27	$\underline{30}$	29	$\underline{32}$	31	$\underline{36}$	35	$\underline{34}$	33
9	$\underline{13}$	14	$\underline{15}$	18	$\underline{20}$	19	$\underline{17}$	21	$\underline{22}$	23	$\underline{24}$	25	$\underline{26}$	27	$\underline{30}$	28	$\underline{31}$	32	$\underline{36}$	29
10	$\underline{14}$	15	$\underline{16}$	17	$\underline{19}$	20	$\underline{18}$	22	$\underline{21}$	24	$\underline{23}$	26	$\underline{25}$	28	$\underline{29}$	27	$\underline{32}$	31	$\underline{35}$	30
11	$\underline{16}$	13	$\underline{14}$	15	$\underline{18}$	17	$\underline{20}$	19	$\underline{25}$	21	$\underline{27}$	23	$\underline{24}$	31	$\underline{28}$	34	$\underline{26}$	22	$\underline{32}$	36
12	$\underline{17}$	16	$\underline{13}$	14	$\underline{15}$	18	$\underline{19}$	20	$\underline{26}$	22	$\underline{28}$	24	$\underline{23}$	32	$\underline{27}$	33	$\underline{25}$	21	$\underline{31}$	35
13	9	$\underline{11}$	12	$\underline{6}$	$\underline{14}$	15	$\underline{16}$	18	$\underline{20}$	19	$\underline{17}$	21	$\underline{22}$	23	$\underline{26}$	29	$\underline{30}$	27	$\underline{28}$	24
14	10	$\underline{9}$	11	$\underline{12}$	13	16	$\underline{15}$	17	$\underline{19}$	20	$\underline{18}$	22	$\underline{21}$	24	$\underline{25}$	30	$\underline{29}$	28	$\underline{27}$	23
15	8	$\underline{10}$	9	$\underline{11}$	12	$\underline{13}$	14	16	$\underline{18}$	17	$\underline{20}$	19	$\underline{27}$	21	$\underline{22}$	24	$\underline{23}$	30	$\underline{25}$	26
16	11	$\underline{12}$	10	$\underline{5}$	7	$\underline{14}$	13	$\underline{15}$	$\underline{17}$	18	$\underline{19}$	20	$\underline{28}$	22	$\underline{21}$	23	$\underline{24}$	29	$\underline{26}$	25
17	12	$\underline{7}$	4	$\underline{10}$	8	$\underline{11}$	9	$\underline{14}$	16	$\underline{15}$	13	18	$\underline{20}$	19	$\underline{23}$	21	$\underline{27}$	26	$\underline{22}$	28
18	7	$\underline{8}$	3	$\underline{9}$	11	$\underline{12}$	10	$\underline{13}$	15	$\underline{16}$	14	$\underline{17}$	$\underline{19}$	20	$\underline{24}$	22	$\underline{28}$	25	$\underline{21}$	27
19	6	$\underline{5}$	8	$\underline{7}$	10	$\underline{9}$	12	$\underline{11}$	14	$\underline{13}$	16	$\underline{15}$	18	$\underline{17}$	$\underline{20}$	26	$\underline{21}$	24	$\underline{23}$	22
20	5	$\underline{6}$	7	$\underline{8}$	9	$\underline{10}$	11	$\underline{12}$	13	$\underline{14}$	15	$\underline{16}$	17	$\underline{18}$	19	25	$\underline{22}$	23	$\underline{24}$	21

Table 7. Schedule for $(1, 2, 4, 5, 8, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 30, 33, 34, 36, 37)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
2	1	4	3	6	5	8	7	10	11	9	13	12	15	14	17	16	19	21	18	20
3	41	1	2	4	7	5	6	8	9	10	11	14	13	12	15	18	20	17	21	19
4	40	2	1	3	9	6	5	7	12	13	10	8	11	16	14	15	21	20	25	17
5	38	41	42	1	2	3	4	6	8	7	14	11	16	9	12	10	15	13	17	18
6	39	40	41	2	1	4	3	5	7	8	9	10	12	11	13	14	17	15	19	16
7	37	39	40	42	3	1	2	4	6	5	8	9	10	17	11	12	13	16	14	15
8	36	37	38	41	42	2	1	3	5	6	7	4	9	10	19	11	14	18	16	12
9	35	38	37	39	4	42	34	1	3	2	6	7	8	5	10	13	11	12	15	30
10	34	42	39	38	40	41	37	2	1	3	4	6	7	8	9	5	12	11	13	14
11	33	35	36	37	41	40	42	39	2	1	3	5	4	6	7	8	9	10	12	13
12	32	36	35	40	37	39	41	42	4	38	1	2	6	3	5	7	10	9	11	8
13	31	33	34	35	36	38	40	41	42	4	2	1	3	39	6	9	7	5	10	11
14	30	34	33	36	39	37	38	40	41	42	5	3	1	2	4	6	8	35	7	10
15	29	31	32	34	38	35	39	36	37	41	42	40	2	1	3	4	5	6	9	7
16	24	32	31	33	34	36	35	38	39	40	41	42	5	4	1	2	37	7	8	6
17	28	30	26	32	33	34	36	37	38	39	40	41	42	7	2	1	6	3	5	4
18	26	29	27	31	32	33	30	35	36	37	38	39	40	41	42	3	1	8	2	5
19	25	26	28	29	35	30	31	32	34	33	39	36	41	42	8	40	2	1	6	3
20	42	27	30	28	29	31	32	33	40	35	36	37	38	34	39	41	3	4	1	2
21	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	4	2	3	1

	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
1	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
2	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	22	42	38	39	40	41
3	16	22	27	23	29	24	31	25	32	33	35	30	36	37	38	39	26	42	34	28	40
4	19	18	22	24	23	26	27	28	29	34	31	32	33	38	35	36	37	41	42	39	30
5	21	20	19	22	24	23	25	27	28	30	29	31	32	34	33	40	39	37	26	36	35
6	20	21	23	18	22	25	24	26	27	29	30	28	31	33	32	38	35	34	37	42	36
7	28	19	21	20	18	22	23	24	26	27	25	29	30	32	34	35	41	36	31	33	38
8	17	13	20	21	25	15	26	23	24	28	27	22	29	31	30	34	40	35	33	32	39
9	18	16	17	14	21	19	20	22	25	26	28	27	24	29	31	33	36	40	41	23	32
10	15	17	18	19	20	21	22	16	23	25	26	24	28	30	29	32	31	33	36	35	27
11	14	15	16	17	19	20	21	18	22	24	23	26	27	28	25	31	30	32	29	38	34
12	13	14	15	16	17	18	19	20	21	23	24	25	26	27	28	30	33	31	22	34	29
13	12	8	14	15	16	17	18	19	20	22	21	23	25	26	27	29	24	30	32	37	28
14	11	12	13	9	15	16	17	21	19	20	22	18	23	25	26	28	32	27	24	29	31
15	10	11	12	13	14	8	16	17	18	19	20	21	22	24	23	27	28	25	30	26	33
16	3	9	11	12	13	14	15	10	17	21	18	19	20	23	22	26	29	28	27	30	25
17	8	10	9	11	12	13	14	15	16	18	19	20	21	22	24	25	27	29	35	31	23
18	9	4	10	6	7	12	13	11	15	17	16	14	19	21	20	24	34	23	28	25	22
19	4	7	5	10	11	9	12	13	14	15	17	16	18	20	21	23	22	24	38	27	37
20	6	5	8	7	10	11	9	12	13	14	15	17	16	19	18	21	23	26	25	22	24
21	5	6	7	8	9	10	11	14	12	16	13	15	17	18	19	20	25	22	23	24	26

Table 8. Feasible cyclic BIS's with maximal break interval ≥ 5 for $n \leq 21$

n	case	feasible cyclic BIS's
16	(3-iii)	[1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5]
20	(7-i-b)	[1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5]
	(7-ii-a)	[1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 2, 1, 4, 1, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5]
		[1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 2, 1, 4, 1, 3, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5]
	(7-ii-b)	[1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5]
	(7-iii-a)	[1, 2, 1, 3, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 3, 1, 2, 1, 4, 1, 2, 2, 1, 2, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 2, 1, 2, 2, 1, 4, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 3, 1, 2, 1, 4, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5]
		[1, 2, 2, 1, 2, 3, 1, 2, 1, 4, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 3, 1, 2, 1, 4, 1, 2, 2, 1, 3, 1, 2, 2, 1, 5]
		[1, 2, 2, 1, 3, 1, 2, 2, 1, 4, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 2, 1, 2, 2, 1, 4, 1, 2, 2, 1, 2, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 2, 1, 2, 2, 1, 4, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5]
		[1, 2, 2, 1, 2, 3, 1, 2, 1, 4, 1, 2, 2, 1, 2, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 2, 1, 2, 2, 1, 4, 1, 2, 2, 1, 3, 1, 2, 2, 1, 5]
		[1, 2, 2, 1, 3, 1, 2, 2, 1, 4, 1, 2, 2, 1, 2, 3, 1, 2, 1, 5]
		[1, 2, 2, 1, 2, 3, 1, 2, 1, 4, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5]
		[1, 2, 2, 1, 2, 3, 1, 2, 1, 4, 1, 2, 2, 1, 3, 1, 2, 2, 1, 5]
		[1, 2, 2, 1, 3, 1, 2, 2, 1, 4, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5]
		[1, 2, 2, 1, 3, 1, 2, 2, 1, 4, 1, 2, 2, 1, 3, 1, 2, 2, 1, 5]
	(7-iii-b)	[1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 2, 2, 2, 2, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5]
		[1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 2, 2, 2, 2, 3, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5]
	(7-iii-c)	[1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5]
21	(8-i-b)	[1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5]
	(8-ii-b)	[1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 2, 1, 4, 1, 2, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 2, 1, 4, 1, 2, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5]
	(8-iii-a)	[1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 3, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5]
		[1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5]
		[1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 3, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5]
	(8-iii-b)	[1, 2, 1, 3, 1, 3, 1, 2, 1, 3, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 3, 1, 2, 1, 3, 1, 3, 1, 2, 1, 5]
		[1, 2, 2, 1, 2, 3, 1, 2, 1, 3, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5]
		[1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 3, 1, 2, 1, 3, 2, 1, 2, 2, 1, 5]
	(8-iii-c)	[1, 2, 1, 3, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 1, 3, 1, 2, 1, 5]