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Algebraic $\mathfrak{L}_q$-norms and complexity-like properties of Jacobi polynomials: Degree and parameter asymptotics

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Abstract

The Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ conform the canonical family of hypergeometric orthogonal polynomials (HOPs) with the two-parameter weight function $(1-x)^\alpha(1+x)^\beta, \alpha,\beta > -1$, on the interval $[-1,+1]$. The spreading of its associated probability density (i.e., the Rakhmanov density) over the support interval has been quantified, beyond the dispersion measures (moments around the origin, variance), by the algebraic $\mathfrak{L}_q$-norms (Shannon and Rényi entropies) and the monotonic complexity-like measures of Cramér–Rao, Fisher–Shannon, and LMC (López-Ruiz, Mancini, and Calbet) types. These quantities, however, have been often determined in an analytically highbrow, non-handy way; specially when the degree or the parameters (α,β) are large. In this work, we determine in a simple, compact form the leading term of the entropic and complexity-like properties of the Jacobi polynomials in the two extreme situations: ($n \to \infty$; fixed α,β) and ($\alpha \to \infty$; fixed n,β). These two asymptotics are relevant per se and because they control the physical entropy and complexity measures of the high energy (Rydberg) and high dimensional (pseudoclassical) states of many exactly, conditional exactly, and quasi-exactly solvable quantum- mechanical potentials which model numerous atomic and molecular systems.

1 | INTRODUCTION

The formalization of the intuitive notion of simplicity/complexity of probability distributions is a formidable task, not yet well established in spite of a huge number of efforts in many scientific and technological areas from quantum chemistry [1–4] and quantum technologies [5–9] to applied mathematics and approximation theory [10–13]. To a large extent, this is because of the great diversity of configurational shapes from perfect order to maximal randomness (or perfect disorder) which have the bi- and multi-parametric probability distributions associated to the quantum one- and many-body systems and the special functions of applied mathematics and mathematical physics [14].

This enormous amount of geometrical forms cannot be captured by a single quantity, but it requires a number of measures of intrinsic and extrinsic characters. The latter ones refer to algorithmic and computational complexities [15, 16] which are closely related to the time required for a computer to solve a given problem, so that they depend on the chosen computer. The former ones refer to statistical measures of complexity, extracted from the density functional theory of electronic systems [17], which quantify the degree of structure or pattern of one- and many-electron systems in terms of the single electron density. The main quantities of this type are the Cramér–Rao [2, 10], Fisher–Shannon [18–20], and LMC (Lópezruiz, Mancini and Calbet) [21, 22] complexities and modifications of them [23–33]. Here we will use the basic density-dependent complexity measures (Cramér–Rao, Fisher–Shannon and LMC) recently introduced in electronic structure (see, e.g., References [29, 34–36] and the reviews [37, 38]), which are of intrinsic character in the sense that they do not depend on the context but on the probability density of the system under consideration. These quantities are given by the product of two spreading measures of dispersion (variance) and entropic (Fisher information, Shannon entropy, Rényi entropy) types, so that each measures two configurational shapes of the system in a simultaneous manner.

The hypergeometric polynomials, $\{p_n(x)\}$, orthogonal with respect to the weight function $h(x)$ on the support interval Λ, are known to often control the quantum-mechanical wavefunctions of the bound states in numerous quantum systems [14, 39–43]. The Hermite, Laguerre and Jacobi polynomials are the three canonical families of real HOPs [44–47]. Recently the entropy- and complexity-like properties of these polynomials, which determine their spreading on the support interval, have begun to be investigated by means of the entropy- and complexity-like measures [11, 48, 49] of the associated Rakhmanov density $\rho_n(x) = p_n^2(x)h(x)$. This normalized-to-unity probability density function governs the $(n \to +\infty)$-asymptotics of the ratio of two polynomials with consecutive orders [50], and characterizes the Born's probability density of the bound stationary states of a great deal of quantum-mechanical potentials which model numerous atomic and molecular systems [12, 14, 41, 51–53]. The numerical evaluation of the integral functionals corresponding to the dispersion, entropic and complexity measures of the HOPs by means of the standard quadratures is not convenient, because the highly oscillatory nature of the integrand renders Gaussian quadrature ineffective as the number of quadrature points grows linearly with n and the evaluation of high-degree polynomials are subject to round-off errors. Indeed, since all the zeros of p_n belong to the interval of orthogonality, the increasing number of integrable singularities spoil any attempt to achieve reasonable accuracy even for rather small n [54, 55].

The entropic and complexity-like measures of the three canonical families of the HOPs have been analytically calculated in terms of the degree and the parameters which characterize their weight function. The resulting analytical expressions for the Cramér–Rao complexity of Hermite and Laguerre polynomials are simple and compact [11], but for the rest of complexity measures of HOPs this is not at all true because the involved entropic components have a somewhat highbrow, non-handy form, being mostly useful in an algorithmic sense *only* (see the review Dehesa et al. [48]). This is especially true for the Fisher–Shannon and LMC complexities of HOPs. However, recently, the Fisher–Shannon and LMC measures of the Hermite $H_n(x)$, Laguerre $L_n^{(\alpha)}(x)$ and Gegenbauer polynomials $C_n^{(\alpha)}(x)$ have been determined [49] when $n \to \infty$ and when $\alpha \to \infty$, obtaining simple and transparent expressions.

The goal of the present work is the analytical evaluation of the Cramér–Rao, Fisher–Shannon and LMC complexities of the whole family of Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$, with $\alpha,\beta > -1$, in the extreme situations ($n \to \infty$; fixed α,β) and ($\alpha \to \infty$; fixed n,β). These polynomials [14, 46, 56–59] are known to be orthonormal with respect to the weight function $h_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$ as

$$\int_{-1}^{+1} \hat{P}_n^{(\alpha,\beta)}(x)\hat{P}_m^{(\alpha,\beta)}(x)h_{\alpha,\beta}(x)dx = \delta_{mn}. \tag{1}$$

Then, the associated Rakhmanov probability density $\rho_n(x)$ is given by

$$\rho_n(x) = \left[\hat{P}_n^{(\alpha,\beta)}(x)\right]^2 h_{\alpha,\beta}(x). \tag{2}$$

Moreover we will denote the orthogonal Jacobi polynomials $P_n^{(\alpha,\beta)}(x) = \hat{P}_n^{(\alpha,\beta)}(x)(\kappa_n)^{1/2}$, with the normalization constant given (see, e.g., Olver et al. [46]) by

$$\kappa_n = \int_{-1}^{+1} \left|P_n^{(\alpha,\beta)}(x)\right|^2 h_{\alpha,\beta}(x)dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}. \tag{3}$$

The special case $\alpha = \beta = \lambda - \frac{1}{2}$ corresponds to the ultraspherical or Gegenbauer polynomials $C_n^{(\lambda)}(x), \lambda > -\frac{1}{2}$, with slightly different normalization, and the case $\alpha = \beta = 0$ corresponds to the Legendre polynomials (see, e.g., Olver et al. [46]).

The structure of this paper is as follows. In Sections 2–4 we obtain the asymptotic behavior for the Cramér–Rao, Fisher–Shannon and LMC complexities of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ when $(n \to \infty; \text{fixed}\, \alpha,\beta)$ and when $(\alpha \to \infty; \text{fixed}\, n,\beta)$ in a simple, compact and transparent form, respectively. Then, some concluding remarks and a few open related issues are pointed out.

2 | CRAMÉR–RAO COMPLEXITY OF JACOBI POLYNOMIALS

The Cramér–Rao complexity of the Jacobi polynomials is given by the corresponding quantity of its associated Rakhmanov density (2), which quantifies the combined balance of the pointwise probability concentration over its support interval jointly with the spreading of the probability around the centroid. It is defined [2, 10, 60] by

$$\mathcal{C}_{\mathrm{CR}}\left[\hat{P}_n^{(\alpha,\beta)}\right] = F\left[\hat{P}_n^{(\alpha,\beta)}\right] \times V\left[\hat{P}_n^{(\alpha,\beta)}\right], \tag{4}$$

where $F\left[\hat{P}_n^{(\alpha,\beta)}\right]$ and $V\left[\hat{P}_n^{(\alpha,\beta)}\right]$ are the Fisher information [61, 62] and the variance of the Rakhmanov density (2), which are defined as

$$F\left[\hat{P}_n^{(\alpha,\beta)}\right] = \int_{-1}^{+1} \frac{\left[\rho_n'(x)\right]^2}{\rho_n(x)} dx, \quad \text{and} \quad V\left[\hat{P}_n^{(\alpha,\beta)}\right] = \langle x^2 \rangle - \langle x \rangle^2,$$

respectively, with the expectation value $\langle x^k \rangle = \int_{-1}^{+1} x^k \rho_n(x) dx$ for $k = 1,2$. In this section we give the explicit expressions of these three spreading quantities and we find in a simple compact form the values of the Cramér–Rao complexity of the Jacobi polynomials in the two following asymptotical regimes: ($n \to \infty$; fixed α,β) and ($\alpha \to \infty$; fixed n,β).

The particularly elegant algebraic properties of the Jacobi polynomials (see, e.g., References [14, 46, 59]) have allowed to encounter the following expression

$$F\left[\hat{P}_n^{(\alpha,\beta)}\right] = \begin{cases} 2n(n+1)(2n+1), & \alpha,\beta = 0, \\ \dfrac{2n+\beta+1}{4}\left[\dfrac{n^2}{\beta+1} + n + (4n+1)(n+\beta+1) + \dfrac{(n+1)^2}{\beta-1}\right], & \alpha = 0, \beta > 1, \\ \dfrac{2n+\alpha+\beta+1}{4(n+\alpha+\beta-1)}\left[n(n+\alpha+\beta-1)\left(\dfrac{n+\alpha}{\beta+1} + 2 + \dfrac{n+\beta}{\alpha+1}\right)\right. & \\ \left. +(n+1)(n+\alpha+\beta)\left(\dfrac{n+\alpha}{\beta-1} + 2 + \dfrac{n+\beta}{\alpha-1}\right)\right], & \alpha,\beta > 1, \end{cases} \tag{5}$$

(and ∞ otherwise) for the Fisher information [63, 64], and

$$\begin{aligned} V\left[\hat{P}_n^{(\alpha,\beta)}\right] = & \frac{4(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)} \\ & + \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}, \end{aligned} \tag{6}$$

for the variance [10] of Jacobi polynomials. Then, from Equations (4)–(6) one has [11] the following values for the Cramér–Rao complexity of the Jacobi polynomials

$$C_{\mathrm{CR}}\left[\hat{P}_n^{(\alpha,\beta)}\right] = \begin{cases} 2n(n+1)\left[\dfrac{(n+1)^2}{2n+3} + \dfrac{n^2}{2n-1}\right], & \alpha = \beta = 0, \\ \left[\dfrac{(n+1)^2(n+\beta+1)^2}{(2n+\beta+2)^2(2n+\beta+3)} + \dfrac{n^2(n+\beta)^2}{(2n+\beta-1)(2n+\beta)^2}\right] & \\ \times\left[\dfrac{n^2}{\beta+1} + n + (4n+1)(n+\beta+1) + \dfrac{(n+1)^2}{\beta-1}\right], & \alpha = 0, \beta > 1, \\ \left[\dfrac{(n+1)(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)} + \dfrac{n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2}\right] & \\ \times\dfrac{1}{n+\alpha+\beta-1}\left[n(n+\alpha+\beta-1)\left(\dfrac{n+\alpha}{\beta+1} + 2 + \dfrac{n+\beta}{\alpha+1}\right)\right. & \\ \left. +(n+1)(n+\alpha+\beta)\left(\dfrac{n+\alpha}{\beta-1} + 2 + \dfrac{n+\beta}{\alpha-1}\right)\right], & \alpha > 1, \beta > 1, \end{cases}$$

From this expression we easily obtain the high-degree asymptotics ($n \to \infty$; fixed α,β)

$$C_{\rm CR}\left[\hat{P}_n^{(\alpha,\beta)}\right]=\begin{cases}2n^3+\mathcal{O}(n^2), & \alpha=\beta=0,\\ \frac{1}{2}\left(2+\frac{\beta}{\beta^2-1}\right)n^3+\mathcal{O}(n^2), & \alpha=0,\beta>1,\\ \frac{1}{2}\left(\frac{\alpha}{\alpha^2-1}+\frac{\beta}{\beta^2-1}\right)n^3+\mathcal{O}(n^2), & \alpha>1,\beta>1,\end{cases}\tag{7}$$

and the high-parameter asymptotics ($\alpha\to\infty$; fixed n,β)

$$C_{\rm CR}\left[\hat{P}_n^{(\alpha,\beta)}\right]=\frac{(1+\beta+2n\beta)(1+\beta+2n(1+n+\beta))}{\beta^2-1},\quad \beta>1,\quad \alpha\to\infty,\tag{8}$$

for the Cramér–Rao complexity of the Jacobi polynomials. Summarizing, we first observe that in the limit $n\to\infty$ the Cramér–Rao complexity of the Jacobi polynomials follow a qualitative n^3-law, similarly to the corresponding quantity of Laguerre polynomials [11], despite the fact that the respective weight functions are different. This is because the two factors (variance and Fisher information) of the Cramér–Rao complexity of Laguerre polynomials have a linear (Fisher information) and quadratic (variance) dependence on n, while for the Jacobi polynomials the Fisher information has a cubic dependence on the degree and the variance is constant. Moreover, in the limit ($\alpha\to\infty$; fixed n,β) the Cramér–Rao complexity of Laguerre and Jacobi polynomials has also a mutual similar behavior, having a constant leading term that depends on n,β for the Jacobi polynomials and n for the Laguerre polynomials. In this limit the Fisher information and variance have a direct and inverse quadratic dependence on α for the Jacobi polynomials, while for the Laguerre polynomials the Fisher information and the variance have an inverse and direct linear dependence on α, respectively.

3 | FISHER–SHANNON COMPLEXITY OF JACOBI POLYNOMIALS

This statistical quantity is given by the Fisher–Shannon complexity of the Rakhmanov density (2) of the Jacobi polynomials, which is defined [19, 20] as

$$\mathcal{C}_{\rm FS}\left[\hat{P}_n^{(\alpha,\beta)}\right]=F\left[\hat{P}_n^{(\alpha,\beta)}\right]\times\frac{1}{2\pi e}e^{2S\left[\hat{P}_n^{(\alpha,\beta)}\right]}=\frac{1}{2\pi e}F\left[\hat{P}_n^{(\alpha,\beta)}\right]\times\left(\mathcal{L}_S\left[\hat{P}_n^{(\alpha,\beta)}\right]\right)^2,\tag{9}$$

where the symbols $F\left[\hat{P}_n^{(\alpha,\beta)}\right]$ and $\mathcal{L}_S[p_n]=e^{S[p_n]}$ denote the Fisher information and the Shannon entropic power or Shannon spreading length of the polynomial $\hat{P}_n^{(\alpha,\beta)}(x)$, respectively. Note that $\mathcal{C}_{\rm FS}\left[\hat{P}_n^{(\alpha,\beta)}\right]$ measures the gradient content of the Rakhmanov probability density $\rho_n(x)$ associated to the polynomial $\hat{P}_n^{(\alpha,\beta)}(x)$ and its total extent along the support interval $[-1,+1]$ simultaneously.

The explicit expression of the Fisher–Shannon complexity of the Jacobi polynomials for generic values (n,α,β) is unknown up until now. This is so because, although the Fisher information has been given by Equation (5), the Shannon entropy is not known in spite of many efforts. However, there are two extreme situations in which the value of this quantity can be analytically evaluated; namely, when ($\alpha\to\infty$; fixed n,β) and when

TABLE 1 First order asymptotics for the Shannon spreading length $\mathcal{L}_S$ and the Fisher–Shannon complexity $\mathcal{C}_{\rm FS}$ measures of the orthonormal Jacobi polynomials $\widehat{P}_n^{(\alpha,\beta)}(x)$, when $n\to\infty$ and $\alpha\to\infty$

Measure of $\widehat{P}_n^{(\alpha,\beta)}(x)$	$n\to\infty$		$\alpha\to\infty$
$\mathcal{L}_S\left[\widehat{P}_n^{(\alpha,\beta)}\right]$	$\frac{\pi}{e}$		$\frac{1}{\alpha}$
$\mathcal{C}_{\rm FS}\left[\widehat{P}_n^{(\alpha,\beta)}\right]$	$\frac{2\pi}{e^3}n^3$	$\alpha,\beta=0$	$\frac{(1+\beta+2n\beta)}{8\pi e(\beta^2-1)}$
	$\frac{\pi}{4e^3}\left(4+\frac{1}{\beta-1}+\frac{1}{\beta+1}\right)n^3$	$\alpha=0,\beta>1$	
	$\frac{\pi(\alpha+\beta)(\alpha\beta-1)}{2e^3(\alpha^2-1)(\beta^2-1)}n^3$	$\alpha,\beta>1$	
	∞	otherwise	

$(n\to\infty;\text{fixed}\,\alpha,\beta)$. The goal of this section is to obtain these two parameter and degree asymptotics in a compact way for the Shannon spreading length, and then for the Fisher–Shannon complexity (Equation 9) of the orthonormal Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$. The main results are briefly summarized in Table 1.

First, we realize that the Shannon-like integral functional of the orthonormal Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$ is given by

$$S\left[\hat{P}_n^{(\alpha,\beta)}\right]=-\int_{-1}^{+1}\left[\hat{P}_n^{(\alpha,\beta)}(x)\right]^2 h_{\alpha,\beta}(x)\log\left\{\left[\hat{P}_n^{(\alpha,\beta)}(x)\right]^2 h_{\alpha,\beta}(x)\right\}dx=E\left[\hat{P}_n^{(\alpha,\beta)}\right]+I\left[\hat{P}_n^{(\alpha,\beta)}\right], \tag{10}$$

with the functional [65].

$$\begin{aligned}I\left[\hat{P}_n^{(\alpha,\beta)}\right]&=-\int_{-1}^{+1}\left[\hat{P}_n^{(\alpha,\beta)}(x)\right]^2 h_{\alpha,\beta}(x)\log h_{\alpha,\beta}(x)\\&=(\alpha+\beta)\left(\frac{1}{2n+\alpha+\beta+1}+2\psi(2n+\alpha+\beta+1)-\psi(n+\alpha+\beta+1)-\log(2)\right)\\&\quad-(\alpha\psi(n+\alpha+1)+\beta\psi(n+\beta+1))\end{aligned} \tag{11}$$

and the Shannon entropy $E\left[\hat{P}_n^{(\alpha,\beta)}\right]$, which is defined by

$$E\left[\hat{P}_n^{(\alpha,\beta)}\right]=-\int_{-1}^{+1}\left[\hat{P}_n^{(\alpha,\beta)}(x)\right]^2 h_{\alpha,\beta}(x)\log\left[\hat{P}_n^{(\alpha,\beta)}(x)\right]^2dx. \tag{12}$$

The analytical determination of this entropic measure is a formidable task. Indeed, it has been calculated for integer values of the polynomial parameters in a somewhat highbrow manner *only*. However, we find below that they can be expressed in a simple and compact way for the two extreme situations mentioned above.

3.1 | Asymptotics $n\to\infty$

The Shannon entropy of the Jacobi polynomials has been shown to have the following degree asymptotics

$$E\left(\hat{P}_n^{(\alpha,\beta)}\right)=\log(\pi)-1-(\alpha+\beta)\log(2)+\mathcal{O}\left(n^{-1}\right),\quad n\to\infty \tag{13}$$

for fixed (α,β) [66, 67]. Moreover, from Equation (11) and the known asymptotical behavior [46] of the involved gamma $\Gamma(x)$ and digamma $\psi(x)$ functions, we find that for fixed (α,β) the integral functional $I\left[\hat{P}_n^{(\alpha,\beta)}\right]$ fulfills the asymptotics

$$I\left[\hat{P}_n^{(\alpha,\beta)}\right]=(\alpha+\beta)\log(2)+\mathcal{O}\left(n^{-1}\right),\quad n\to\infty \tag{14}$$

Therefore, from Equations (10), (13), and (14) we find that the asymptotics for the Shannon-like functional of the Jacobi polynomials is

$$S\left[\hat{P}_n^{(\alpha,\beta)}\right]=\log(\pi)-1+\mathcal{O}\left(n^{-1}\right),\quad n\to\infty \tag{15}$$

so that the Shannon spreading length of the Jacobi polynomials has the behavior

$$\mathcal{L}_S\left[\hat{P}_n^{(\alpha,\beta)}\right]\sim\frac{\pi}{e},\quad n\to\infty \tag{16}$$

On the other hand, from expression (5) we can obtain the following $(n\to\infty;\text{fixed}\,\alpha,\beta)$ asymptotics for the Fisher information of the Jacobi polynomials:

$$F\left[\hat{P}_n^{(\alpha,\beta)}\right]=\begin{cases}4n^3+\mathcal{O}(n^2), & \alpha,\beta=0,\\ \frac{1}{2}\left(4+\frac{1}{\beta-1}+\frac{1}{\beta+1}\right)n^3+\mathcal{O}(n^2), & \alpha=0,\beta>1,\\ \frac{(\alpha+\beta)(\alpha\beta-1)}{(\alpha^2-1)(\beta^2-1)}n^3+\mathcal{O}(n^2), & \alpha,\beta>1,\\ \infty, & \text{otherwise.}\end{cases} \tag{17}$$

Finally, taking into account (9), (16), and (17), we have that the Fisher–Shannon complexity of the Jacobi polynomials has the following asymptotics ($n\to\infty$; fixed α,β) behavior

$$\mathcal{C}_{\text{FS}}\left[\hat{P}_n^{(\alpha,\beta)}\right]=\begin{cases}\frac{2\pi}{e^3}n^3+\mathcal{O}(n^2), & \alpha,\beta=0,\\ \frac{\pi}{4e^3}\left(4+\frac{1}{\beta-1}+\frac{1}{\beta+1}\right)n^3+\mathcal{O}(n^2), & \alpha=0,\beta>1,\\ \frac{\pi(\alpha+\beta)(\alpha\beta-1)}{2e^3(\alpha^2-1)(\beta^2-1)}n^3+\mathcal{O}(n^2), & \alpha,\beta>1,\\ \infty, & \text{otherwise.}\end{cases} \tag{18}$$

which extends and includes the corresponding asymptotical quantity recently obtained for the subfamily of Gegenbauer polynomials [48] to the whole Jacobi family of orthogonal polynomials. Moreover, we observe that in the limit $n\to\infty$ the Fisher–Shannon complexity of the Jacobi polynomials behaves qualitatively similar to the corresponding quantity of Laguerre polynomials [49], following a n^3-law, despite the fact that the weight functions are very different in each case; this is because for the Laguerre polynomials both the Fisher information and the Shannon spreading length have a linear dependence on n, while for the Jacobi polynomials the Fisher information has a cubic dependence on the degree and the Shannon spreading length is constant.

Interestingly, for the quantum systems with a solvable quantum-mechanical potential with bound-states wavefunctions controlled by Jacobi polynomials (e.g., some supersymmetric quantum systems) (see, e.g., References [12, 40–42]), the Fisher–Shannon measure (18) allows one to find the quantum-classical limit of the physical Fisher–Shannon complexity; they correspond to the high-energy or Rydberg states since for such a limit $n\to\infty$, the wavelengths of particles are small in comparison with the characteristic dimensions of the system and the wavefunctions of the quasi-classical state.

3.2 | Asymptotics $\alpha\to\infty$

Now we determine the Fisher–Shannon complexity $\mathcal{C}_{\text{FS}}\left[\hat{P}_n^{(\alpha,\beta)}\right]$, given by Equation (9), in the limit $\alpha\to\infty$ with fixed degree n,β. We start by evaluating the Shannon entropy (12) of the orthogonal Jacobi polynomials in this limit by means of the relation

$$E\left[P_n^{(\alpha,\beta)}\right]=2\frac{d}{dp}\left[\mathcal{N}_p\left[P_n^{(\alpha,\beta)}\right]\right]_{p=2}, \tag{19}$$

where the symbol $\mathcal{N}_p$ denotes the norm of the orthogonal Jacobi polynomials defined as

$$\mathcal{N}_p\left[P_n^{(\alpha,\beta)}\right]=\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta\left|P_n^{(\alpha,\beta)}(x)\right|^p dx. \tag{20}$$

This quantity can be analytically estimated for $\alpha\to\infty$ by taking into account the known relation [46]

$$\lim_{\alpha\to\infty}\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)}=\left(\frac{1+x}{2}\right)^n, \quad \text{with} \quad P_n^{(\alpha,\beta)}(1)=\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}. \tag{21}$$

Then, from Equations (20) and (21) we have the asymptotics

$$\begin{aligned}\mathcal{N}_p\left[P_n^{(\alpha,\beta)}\right] &\sim P_n^{(\alpha,\beta)}(1)2^{-np}\left(\frac{{}_2F_1(1,-\alpha,2+np+\beta;-1)}{1+np+\beta}+\frac{{}_2F_1(1,-np-\beta,2+\alpha;-1)}{1+\alpha}\right)\\ &= P_n^{(\alpha,\beta)}(1)2^{-np}(1+np+\beta)^{-1}(1+\alpha)^{-1}\times\\ &\quad((1+\alpha)_2F_1(1,-\alpha,2+np+\beta;-1)+(1+np+\beta)_2F_1(1,-np-\beta,2+\alpha;-1)),\end{aligned} \tag{22}$$

where ${}_2F_1(a,b,c;x)$ denotes the Gaussian hypergeometric function [46]. Then, taking into account the known relation between the hypergeometric functions

$$(1-a)_2F_1(1,a,2-b,-1)+(1-b)_2F_1(1,b,2-a,-1)=\frac{2^{1-a-b}\Gamma(2-a)\Gamma(2-b)}{\Gamma(2-a-b)}, \tag{23}$$

with the parameters $a=-\alpha$ and $b=-np-\beta$, the asymptotical behavior (22) simplifies as

$$\mathcal{N}_p\left[P_n^{(\alpha,\beta)}\right]\sim\frac{\Gamma(\alpha+n+1)}{n!}\frac{\Gamma(1+np+\beta)}{\Gamma(2+\alpha+np+\beta)}2^{1+\alpha+\beta}. \tag{24}$$

Thus, according to Equations (19) and (24), one has that the Shannon entropy of the orthogonal Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in the current limit is given as

$$\begin{aligned}E\left[P_n^{(\alpha,\beta)}\right] &\sim 2^{2+\alpha+\beta}\frac{\Gamma(1+n+\alpha)\Gamma(1+2n+\beta)}{\Gamma(n)\Gamma(2+2n+\alpha+\beta)}(\psi(1+2n+\beta)-\psi(2+2n+\alpha+\beta))\\ &= 2^{2+\alpha+\beta}\alpha^{-n-\beta-1}\left(\frac{\Gamma(1+2n+\beta)}{\Gamma(n)}(\psi(1+2n+\beta)-\log(\alpha))+\mathcal{O}\left(\alpha^{-2}\right)\right),\quad \alpha\to\infty,\end{aligned}$$

so that we can express the Shannon entropy of the orthonormal Jacobi polynomials as

$$\begin{aligned}E\left[\hat{P}_n^{(\alpha,\beta)}\right] &= \frac{1}{\kappa_n^P}E\left[P_n^{(\alpha,\beta)}\right]+\log(\kappa_n)=2\alpha^{-n}\left(\frac{n\Gamma(1+2n+\beta)}{\Gamma(1+n+\beta)}(\psi(1+2n+\beta)-\log(\alpha))+\mathcal{O}\left(\alpha^{-2}\right)\right)\\ &\quad+(1+\alpha+\beta)\log(2)+\log\left(\frac{\Gamma(1+n+\beta)}{n!}\right)-(1+\beta)\log(\alpha).\end{aligned} \tag{25}$$

Moreover, from Equation (11), we have the following asymptotics for the auxiliary functional $I\left[P_n^{(\alpha,\beta)}\right]$

$$I\left[\hat{P}_n^{(\alpha,\beta)}\right]=-\alpha\log(2)+1+2n+\beta-\beta\log(2)+\beta\log(\alpha)-\beta\psi(1+n+\beta)+\mathcal{O}\left(\alpha^{-1}\right),\quad \alpha\to\infty. \tag{26}$$

A similar result follows for $\beta\to\infty$ by exchanging $\alpha\leftrightarrow\beta$. Then, according to Equations (10) and (25), and (26), we find the following asymptotics for the Shannon-like integral functional of the Jacobi polynomials

$$S\left[\hat{P}_n^{(\alpha,\beta)}\right]\sim-\log(\alpha)+\mathcal{O}(1),\quad \alpha\to\infty, \tag{27}$$

so that the Shannon entropy power or spreading length of Jacobi polynomials behaves as

$$\mathcal{L}_S\left[\hat{P}_n^{(\alpha,\beta)}\right]\sim\frac{1}{\alpha},\quad \alpha\to\infty. \tag{28}$$

On the other hand, the asymptotics ($\alpha\to\infty$, fixed n,β) for the Fisher information (5) of the Jacobi polynomials $F[\hat{P}_n^{(\alpha,\beta)}]$ turns out to be

$$F\left[\hat{P}_n^{(\alpha,\beta)}\right]=\frac{(1+\beta+2n\beta)}{4(\beta^2-1)}\alpha^2+\mathcal{O}(\alpha),\quad \alpha\to\infty. \tag{29}$$

Finally, the substitution of the last two quantities into Equation (9) gives rise to the following ($\alpha \to \infty$, fixed n,β)-asymptotics for the Fisher–Shannon complexity of the orthonormal Jacobi polynomials:

$$\mathcal{C}_{\mathrm{FS}}\left[\hat{P}_n^{(\alpha,\beta)}\right] \sim \frac{(1+\beta+2n\beta)}{8\pi e(\beta^2-1)} + \mathcal{O}(\alpha^{-1}), \quad \alpha \to \infty. \tag{30}$$

The corresponding result for the polynomials with ($\alpha \to \infty; \beta \to \infty$; fixed n) remains to be found; we have not been able to find it because the involved asymptotical behavior of the second-order entropic moment $W_2\left[\hat{P}_n^{(\alpha,\beta)}\right]$ is a non-trivial task.

Finally, for multidimensional quantum systems with stationary states controlled by Jacobi polynomials (e.g., some supersymmetric quantum systems) (see, e.g., References [12, 40–42]), the Fisher–Shannon measure (30) and its asymptotical extension for ($\alpha \to \infty; \beta \to \infty$; fixed n) allow us to find the pseudo-classical limit of the physical Fisher–Shannon complexity; they correspond to the high-dimensional or pseudoclassical states. The latter is because the wavefunctions of such extreme states involve polynomials orthogonal with respect to a Jacobi weight function where both parameters α and β are directly proportional to the system's dimensionality.

4 | LMC COMPLEXITY OF JACOBI POLYNOMIALS

In this section we investigate the LMC complexity of the orthonormal Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$, which is defined as

$$\mathcal{C}_{\mathrm{LMC}}\left[\hat{P}_n^{(\alpha,\beta)}\right] = W_2\left[\hat{P}_n^{(\alpha,\beta)}\right] \times \mathcal{L}_S\left[\hat{P}_n^{(\alpha,\beta)}\right], \tag{31}$$

where the second-order entropic moment $W_2\left[\hat{P}_n^{(\alpha,\beta)}\right]$, which measures the disequilibrium or deviation from uniformity, is given by

$$W_2\left[\hat{P}_n^{(\alpha,\beta)}\right] = \int_{-1}^{+1}\left(\left[\hat{P}_n^{(\alpha,\beta)}(x)\right]^2 h_{\alpha,\beta}(x)\right)^2 dx = \int_{-1}^{+1}(1-x)^{2\alpha}(1+x)^{2\beta}\left[\hat{P}_n^{(\alpha,\beta)}(x)\right]^4 dx. \tag{32}$$

This statistical complexity quantifies the combined balance of the disequilibrium and the total extent of the polynomials along its weight function. The explicit expression of this measure in terms of the degree n and the parameters (α,β) has not yet been determined in a handy way, because neither $W_2\left[\hat{P}_n^{(\alpha,\beta)}\right]$ nor the spreading length $\mathcal{L}_S\left[\hat{P}_n^{(\alpha,\beta)}\right]$ are analytically known. In this section we obtain simple and compact analytical expressions for $\mathcal{C}_{\mathrm{LMC}}\left[\hat{P}_n^{(\alpha,\beta)}\right]$ in the two extreme situations, ($n \to \infty$; fixed α,β) and ($\alpha \to \infty$; fixed n,β). They are briefly summarized in Table 2.

4.1 | Asymptotics $n \to \infty$

To determine the ($n \to \infty$; fixed α,β)-asymptotics of the LMC complexity $\mathcal{C}_{\mathrm{LMC}}\left[\hat{P}_n^{(\alpha,\beta)}\right]$ we start by calculating the disequilibrium $W_2\left[\hat{P}_n^{(\alpha,\beta)}\right]$ in the limit $n \to \infty$ by means of Theorem 3 of Aptekarev et al. [68], obtaining

TABLE 2 First order asymptotics for the disequilibrium W_2 and the LMC complexity $\mathcal{C}_{\mathrm{LMC}}$ measures of the orthonormal Jacobi polynomials $\widehat{P}_n^{(\alpha,\beta)}(x)$, when $n \to \infty$ and $\alpha \to \infty$

Measure of $\widehat{P}_n^{(\alpha,\beta)}(x)$	$n \to \infty$		$\alpha \to \infty$
$W_2\left[\widehat{P}_n^{(\alpha,\beta)}\right]$	$\frac{2^{\alpha+\beta-2}3}{\pi^2}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$	$\beta > 0$	$\frac{\Gamma(1+4n+2\beta)}{2^{2(1+2n+\beta)}n!^2\Gamma(1+n+\beta)}\alpha$
	$\log(n)$	$\beta = 0$	
	$n^{-2\beta}$	$-1 < \beta < 0$	
$\mathcal{C}_{\mathrm{LMC}}\left[\widehat{P}_n^{(\alpha,\beta)}\right]$	$\frac{2^{\alpha+\beta-2}3}{\pi e}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$	$\beta > 0$	$\frac{\Gamma(1+4n+2\beta)}{2^{2(1+2n+\beta)}n!^2\Gamma(1+n+\beta)}$
	$\frac{\pi}{e}\log(n)$	$\beta = 0$	
	$\frac{\pi}{e}n^{-2\beta}$	$-1 < \beta < 0$	

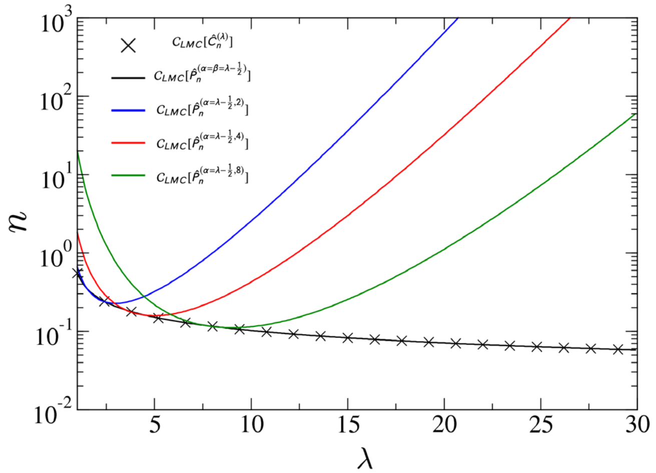

FIGURE 1 Comparison of the LMC complexity $\mathcal{C}_{\text{LMC}}$ measures of the orthonormal Gegenbauer polynomials $\widehat{C}_n^{(\lambda)}(x)$ and the orthonormal Jacobi polynomials $\widehat{P}_n^{(\alpha,\beta)}(x)$, with ($\alpha=\lambda-2$ and $\beta=\lambda-2,2,4,8$), for various values of λ and $n\to\infty$

$$W_2\left[\hat{P}_n^{(\alpha,\beta)}\right]\sim\begin{cases}\dfrac{2^{\alpha+\beta-2}3}{\pi^2}\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, & \beta>0,\\ \log(n), & \beta=0,\\ n^{-2\beta}, & -1<\beta<0,\end{cases}\tag{33}$$

and then we keep in mind the corresponding asymptotics (16) for the Shannon spreading length $\mathcal{L}_S\left[\hat{P}_n^{(\alpha,\beta)}\right]$. The substitution of the asymptotical values of these two entropic quantities into Equation (31) gives rise to the following asymptotical behavior ($n\to\infty$) of the LMC complexity of the orthonormal Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$:

$$\mathcal{C}_{\text{LMC}}\left[\hat{P}_n^{(\alpha,\beta)}\right]\sim\begin{cases}\dfrac{2^{\alpha+\beta-2}3}{\pi e}\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, & \beta>0,\\ \dfrac{\pi}{e}\log(n), & \beta=0,\\ \dfrac{\pi}{e}n^{-2\beta}, & -1<\beta<0,\end{cases}\tag{34}$$

which extends and includes the corresponding asymptotical quantity recently obtained for the subfamily of Gegenbauer polynomials [48] to the whole Jacobi family of orthogonal polynomials. See also Figure 1, where the LMC complexity $\mathcal{C}_{\text{LMC}}$ measures of the orthonormal Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$, with ($\alpha=\lambda-2$ and $\beta=\lambda-2,2,4,8$), and the orthonormal Gegenbauer polynomials $\hat{C}_n^{(\lambda)}(x)$ are compared for various values of λ and $n\to\infty$. Note that both Jacobi and Gegenbauer complexities match when $\beta=\lambda-2$ as one would expect, what is a partial checking of our results. On the other hand, we observe that in the limit $n\to\infty$ the LMC complexity of the Jacobi polynomials behaves very different to the corresponding quantity of Laguerre polynomials [48], as one expects because of their weight functions are so distinct, except in the special case $\beta=0$. Indeed, for $\beta=0$ there happens the following phenomenon: the disequilibrium has a logarithmic dependence on n while the Shannon spreading length is constant so that the total balance for the LMC complexity of the Jacobi polynomials obeys the $\log n$-law as also occurs for the LMC complexity of Laguerre polynomials [48].

Here again, for the quantum systems with a solvable quantum-mechanical potential with bound-states wavefunctions controlled by Jacobi polynomials (e.g., some supersymmetric quantum systems) (see, e.g., References [12, 40–42]), the LMC measure (18) allows one to find the quantum-classical limit of the physical LMC complexity; they correspond to the high-energy or Rydberg states.

4.2 | Asymptotics $\alpha\to\infty$

Here we determine the LMC complexity $\mathcal{C}_{\text{LMC}}\left[\hat{P}_n^{(\alpha,\beta)}\right]$, given by Equation (31), in the limit ($\alpha\to\infty$; fixed n,β). First we realize that the Shannon spreading length $\mathcal{L}_S\left[\hat{P}_n^{(\alpha,\beta)}\right]$, has been already obtained in Equation (28). Then, we calculate the second-order entropic moment $W_2\left[\hat{P}_n^{(\alpha,\beta)}\right]$, given by Equation (32). We use the limiting relation (21) into (32), obtaining the value

$$W_2\left[P_n^{(\alpha,\beta)}\right] \sim P_n^{(\alpha,\beta)}(1)^4 2^{-4n}\Gamma(1+2\alpha)\Gamma(1+4n+2\beta)\times \left(\frac{1}{\Gamma(1+2\alpha)^2}\widetilde{F}_1(1,-2\alpha,2+4n+2\beta,-1)+\frac{1}{\Gamma(1+4n+2\beta)^2}\widetilde{F}_1(1,-4n-2\beta,2+2\alpha,-1)\right) = \frac{\Gamma(\alpha+n+1)^4}{(n!)^4\Gamma(\alpha+1)^4}\frac{\Gamma(1+4n+2\beta)\Gamma(1+2\alpha)}{\Gamma(2+2\alpha+4n+2\beta)}2^{1+2\alpha+2\beta} \tag{35}$$

for the orthogonal Jacobi polynomials. Now, the corresponding asymptotics for the second-order entropic power of the orthonormal Jacobi polynomials is given by

$$W_2\left[\hat{P}_n^{(\alpha,\beta)}\right] = \frac{1}{(\kappa_n)^2}W_2\left[P_n^{(\alpha,\beta)}\right] \sim \frac{\Gamma(1+4n+2\beta)}{2^{2(1+2n+\beta)}(n!)^2\Gamma(1+n+\beta)}\alpha, \quad \alpha\to\infty. \tag{36}$$

Finally, the combination of Equations (31), (28), and (36) lead to the asymptotical behavior

$$\mathcal{C}_{\mathrm{LMC}}\left[\hat{P}_n^{(\alpha,\beta)}\right] = \frac{\Gamma(1+4n+2\beta)}{2^{2(1+2n+\beta)}(n!)^2\Gamma(1+n+\beta)}, \quad \alpha\to\infty \tag{37}$$

for the LMC complexity of the (orthonormal) Jacobi polynomials with ($\alpha\to\infty$; fixed n,β). It remains to find the corresponding result for the polynomials with ($\alpha\to\infty;\beta\to\infty$; fixed n), which we have not been able to find because the involved asymptotical behavior of the second-order entropic moment $W_2\left[\hat{P}_n^{(\alpha,\beta)}\right]$ is a non-trivial task.

Finally, for multidimensional quantum systems with bound states controlled by Jacobi polynomials (e.g., some supersymmetric quantum systems) (see, e.g., References [12, 40–42]), the LMC measure (37) and its asymptotical extension when ($\alpha\to\infty;\beta\to\infty$; fixed n) allow us to find the pseudo-classical limit of the physical LMC complexity; they correspond to the high-dimensional or pseudoclassical states. This is because the wavefunctions of such extreme states involve polynomials orthogonal with respect to a Jacobi weight function where both parameters α and β are directly proportional to the space dimensionality of the system.

5 | CONCLUSIONS AND OPEN PROBLEMS

In this work we have determined the Cramér–Rao, Fisher–Shannon and LMC complexity-like measures of the Jacobi polynomials $\hat{P}_n^{(\alpha,\beta)}(x)$, with $\alpha,\beta > -1$, in the extreme situations ($n\to\infty$; fixed α,β) and ($\alpha\to\infty$; fixed n,β). They are given by the leading term of the degree and parameter asymptotics of the corresponding statistical properties of the associated probability density (Rakhmanov's density), respectively. Each of these complexity quantifiers capture in a simultaneous way two polynomial's configurational facets of dispersion (variance) and entropic (Fisher, Shannon) types. Briefly, in the limit ($n\to\infty$; fixed α,β) we have found that both Cramér–Rao and Fisher–Shannon complexities follow a qualitatively similar n^3-law behavior for all β (see Table 1), but the LMC complexity has a different asymptotical n-behavior depending on β (see Table 2). Moreover, in the limit ($\alpha\to\infty$; fixed n,β) we have found that the Fisher information and the variance follow a direct and inverse quadratic dependence on α, respectively, while the second-order entropic moment and the Shannon entropy power follow a direct and inverse linear dependence on α. The combination of the two dispersion/entropic factors involved for the Cramér–Rao, Fisher–Shannon and LMC complexities lead to a constant leading term for all the complexity measures. These results can potentially have a strong impact on the calculations of quantum chemical properties of one- and many-electron systems whose wavefunctions are controlled by Jacobi polynomials; basically, this is because the entropy and complexity measures quantify the different facets of the internal disorder of the system which are manifest in the great diversity of configurational shapes of the electron probability density. Moreover, the usefulness of these results in quantum chemistry and physics is due to the fact that for the high energy (Rydberg) and high-dimensional states they can predict without any further calculation the values for the three measures of complexities of the corresponding systems. This is because such results are expressed directly from first principles; that is to say in terms of the principal quantum number of the states and the dimensionality of the system.

Finally, a number of open related problems can be highlighted. First, the extension of these results to the varying Jacobi polynomials [69–71] (i.e., when the parameters depend on the polynomial degree) as well as to the exceptional Jacobi polynomials [42, 72, 73], which are very useful to standard and supersymmetric quantum mechanics [41]. Second, the determination of the general statistical complexity measures [74] of Fisher-Rényi [25, 28, 31, 32, 75] and LMC-Rényi [24, 26, 28, 53, 76] types for the standard and varying Jacobi polynomials; this includes the calculation of the Rényi entropy of such polynomials. These open issues are not only interesting per se but also because of their chemical and physical applications, especially for the extreme quantum states of highly excited Rydberg and high dimensional types of numerous atomic and molecular systems whose bound states are described by wavefunctions controlled by these polynomials.

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AUTHOR CONTRIBUTIONS

Jesus Sanchez-Dehesa: Conceptualization; data curation; funding acquisition; investigation; resources; supervision; validation. **Nahual Sobrino:** Data curation; formal analysis; investigation; methodology; software; validation.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study are openly available in https://arxiv.org/abs/2110.11441.

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