

CAN ONE IDENTIFY TWO UNITAL JB*-ALGEBRAS BY THE METRIC SPACES DETERMINED BY THEIR SETS OF UNITARIES?

MARÍA CUETO-AVELLANEDA, ANTONIO M. PERALTA

ABSTRACT. Let M and N be two unital JB*-algebras and let $\mathcal{U}(M)$ and $\mathcal{U}(N)$ denote the sets of all unitaries in M and N , respectively. We prove that the following statements are equivalent:

- (a) M and N are isometrically isomorphic as (complex) Banach spaces;
- (b) M and N are isometrically isomorphic as real Banach spaces;
- (c) There exists a surjective isometry $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$.

We actually establish a more general statement asserting that, under some mild extra conditions, for each surjective isometry $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ we can find a surjective real linear isometry $\Psi : M \rightarrow N$ which coincides with Δ on the subset $e^{iM_{sa}}$. If we assume that M and N are JBW*-algebras, then every surjective isometry $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ admits a (unique) extension to a surjective real linear isometry from M onto N . This is an extension of the Hatori–Molnár theorem to the setting of JB*-algebras.

1. INTRODUCTION

Every surjective isometry between two real normed spaces X and Y is an affine mapping by the Mazur–Ulam theorem. It seems then natural to ask whether the existence of a surjective isometry between two proper subsets of X and Y can be employed to identify metrically both spaces. By a result of P. Mankiewicz (see [34]) every surjective isometry between convex bodies in two arbitrary normed spaces can be uniquely extended to an affine function between the spaces. The so-called Tingley’s problem, which ask if a surjective isometry between the unit spheres of two normed spaces can be also extended to a surjective linear isometry between the spaces, came out in the eighties (cf. [42]). To the best of our knowledge, Tingley’s problem remains open even for two dimensional spaces (see [5] where it is solved for non-strictly convex two dimensional spaces). A full machinery has been developed in the different partial positive solutions to Tingley’s problem in the case of classical Banach spaces, C*- and operator algebras and JB*-triples (see, for example the references [2, 5, 8, 10, 11, 15, 16, 18, 19, 20, 30, 32, 35, 36, 38, 39, 43] and the surveys [47, 37]).

The question at this stage is whether in Tingley’s problem the unit spheres can be reduced to strictly smaller subsets. Even in the most favorable case of a finite dimensional normed space X , we cannot always conclude that every surjective isometry on the set of extreme points of the closed unit ball of X can be extended

2010 *Mathematics Subject Classification*. Primary 47B49, 46B03, 46B20, 46A22, 46H70 Secondary 46B04, 46L05, 17C65 .

Key words and phrases. Isometry; Jordan *-isomorphism, Unitary set, JB*-algebra, JBW*-algebra, extension of isometries.

to a surjective real linear isometry on X (see [10, Remark 3.15]). So, the sets of extreme points is not enough to determine a surjective real linear isometry. The existence of an additional structure on X provides new candidates, this is the case of unital C^* -algebras. In a unital C^* -algebra A , the set $\mathcal{U}(A)$ of all unitary elements in A is, in general, strictly contained in the set of all extreme points of the closed unit ball of A . The symbol A_{sa} will stand for the set of self-adjoint elements in A . We recall that an element u in A is called *unitary* if $uu^* = \mathbf{1}_A = u^*u$, that is, u is invertible with inverse u^* . The set of all unitaries in A will be denoted by $\mathcal{U}(A)$. It is well known that $\mathcal{U}(A)$ is contained in the unit sphere of A and it is a subgroup of A which is also self-adjoint (i.e., u^* and uv lie in $\mathcal{U}(A)$ for all $u, v \in \mathcal{U}(A)$). However, the set $\mathcal{U}(A)$ is no longer stable under Jordan products of the form $a \circ b := \frac{1}{2}(ab + ba)$. Namely, let $u, v \in \mathcal{U}(A)$ the element $w = u \circ v$ is a unitary if and only if $\mathbf{1}_A = ww^* = w^*w$, that is,

$$\mathbf{1}_A = \frac{1}{4}(uv + vu)(v^*u^* + u^*v^*) = \frac{1}{4}(2 \cdot \mathbf{1}_A + uvu^*v^* + vuv^*u^*),$$

equivalently, $\mathbf{1}_A = \frac{uvu^*v^* + vuv^*u^*}{2}$ and thus $uvu^*v^* = vuv^*u^* = \mathbf{1}_A$, because $\mathbf{1}_A$ is an extreme point of the closed unit ball of A . In particular $uv = vu$. That is $u \circ v \in \mathcal{U}(A)$ if and only if u and v commute. Despite the instability of unitaries under Jordan products, expressions of the form uvu lie in $\mathcal{U}(A)$ for all $u, v \in \mathcal{U}(A)$, and they can be even expressed in terms of the Jordan product because $uvu = 2(u \circ v) \circ u - u^2 \circ v$.

O. Hatori and L. Molnár proved in [27, Theorem 1], that for each surjective isometry $\Delta : \mathcal{U}(A) \rightarrow \mathcal{U}(B)$, where A and B are unital C^* -algebras, the identity $\Delta(e^{iA_{sa}}) = e^{iB_{sa}}$ holds, and there is a central projection $p \in B$ and a Jordan $*$ -isomorphism $J : A \rightarrow B$ satisfying

$$\Delta(e^{ix}) = \Delta(1)(pJ(e^{ix}) + (1-p)J(e^{ix})^*), \quad (x \in A_{sa}).$$

In particular A and B are Jordan $*$ -isomorphic. Actually, every surjective isometry between the unitary groups of two von Neumann algebras admits an extension to a surjective real linear isometry between these algebras (see [27, Corollary 3]). These influencing results have played an important role in some of the recent advances on Tingley's problem and in several other problems.

Let us take a look at some historical precedents. S. Sakai proved in [41] that if M and N are AW^* -factors, $\mathcal{U}(M)$, $\mathcal{U}(N)$ their respective unitary groups, and ρ a uniformly continuous group isomorphism from $\mathcal{U}(M)$ into $\mathcal{U}(N)$, then there is a unique map f from M onto N which is either a linear or conjugate linear $*$ -isomorphism and which agrees with ρ on $\mathcal{U}(M)$. In the case of W^* -factors not of type I_{2n} the continuity assumption was shown to be superfluous by H.A. Dye in [14, Theorem 2]. In the results by Hatori and Molnár, the mapping Δ is merely a distance preserving bijection between the unitary groups of two unital C^* -algebras or two von Neumann algebras.

The proofs of the Hatori-Molnár theorems are based, among other things, on a study on isometries and maps compatible with inverted Jordan triple products on groups by O. Hatori, G. Hirasawa, T. Miura, L. Molnár [25]. Despite of the attractive terminology, the study of the surjective isometries between the sets of unitaries of two unital JB^* -algebras has not been considered. There are different difficulties which are inherent to the Jordan setting. As we commented above, the set of unitary elements in a unital C^* -algebra is not stable under Jordan products.

Motivated by the pioneering works of I. Kaplansky, JB*-algebras were introduced as a Jordan generalization of C*-algebras (see subsection 1.1 for the detailed definitions). For example every Jordan self-adjoint subalgebra of a C*-algebra is a JB*-algebra (these JB*-algebras are called JC*-algebras), but there exists exceptional JB*-algebras which cannot be represented as JC*-algebras.

Unitaries in unital C*-algebras and JB*-algebras have been intensively studied. They constitute the central notion in the Russo–Dye theorem [40] and its JB*-algebra-analogue in the Wright–Youngson–Russo–Dye theorem [45], which are milestone results in the field of functional analysis. The interest remains very active, for example, we recently obtained a metric characterization of unitary elements in a unital JB*-algebra (cf. [12]).

By the Gelfand–Naimark theorem, every unitary u in a unital C*-algebra A can be viewed as a unitary element in the algebra $B(H)$, of all bounded linear operators on a complex Hilbert space H , in such a way that u itself is a unitary on H . Consequently, one-parameter unitary groups in A are under the hypotheses of some well known results like Stone’s one-parameter theorem. However, unitary elements in a unital JB*-algebra M cannot always be regarded as unitaries on some complex Hilbert space H . The lacking of a suitable Jordan version of Stone’s one-parameter theorem for JB*-algebras leads us to establish an appropriate result for uniformly continuous one-parameter groups of unitaries in an arbitrary unital JB*-algebra in Theorem 3.1.

Let M and N denote two arbitrary unital JB*-algebras whose sets of unitaries are denoted by $\mathcal{U}(M)$ and $\mathcal{U}(N)$, respectively. In our first main result (Theorem 3.4) we prove that for each surjective isometry $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ satisfying one of the following statements:

- (1) $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$;
- (2) there exists a unitary ω_0 in N such that $U_{\omega_0}(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$,

there exists a unitary ω in N satisfying

$$\Delta(e^{iM_{sa}}) = U_{\omega^*}(e^{iN_{sa}}),$$

Furthermore, we can find a central projection $p \in N$, and a Jordan *-isomorphism $\Phi : M \rightarrow N$ such that

$$\begin{aligned} \Delta(e^{ih}) &= U_{\omega^*}(p \circ \Phi(e^{ih})) + U_{\omega^*}((\mathbf{1}_N - p) \circ \Phi(e^{ih})^*) \\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi((e^{ih})^*)), \end{aligned}$$

for all $h \in M_{sa}$. Consequently, the restriction $\Delta|_{e^{iM_{sa}}}$ admits a (unique) extension to a surjective real linear isometry from M onto N . We remark that Δ is merely a distance preserving bijection.

Among the consequences of the previous result we prove that the following statements are equivalent for any two unital JB*-algebras M and N :

- (a) M and N are isometrically isomorphic as (complex) Banach spaces;
- (b) M and N are isometrically isomorphic as real Banach spaces;
- (c) There exists a surjective isometry $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$

(see Corollary 3.8).

Finally, in Theorem 3.9 we prove that any surjective isometry between the sets of unitaries of any two JBW*-algebras admits a (unique) extension to a surjective real linear isometry between these algebras.

Our proofs, which are completely independent from the results for C^* -algebras, undoubtedly benefit from results in JB^* -triple theory. This beautiful subject (meaning the theory of JB^* - and JBW^* -triples) makes simpler and more accessible our arguments.

1.1. Definitions and background. A complex (respectively, real) *Jordan algebra* M is a (non-necessarily associative) algebra over the complex (respectively, real) field whose product is abelian and satisfies the so-called *Jordan identity*: $(a \circ b) \circ a^2 = a \circ (b \circ a^2)$ ($a, b \in M$). A *normed Jordan algebra* is a Jordan algebra M equipped with a norm, $\|\cdot\|$, satisfying $\|a \circ b\| \leq \|a\| \|b\|$ ($a, b \in M$). A *Jordan Banach algebra* is a normed Jordan algebra whose norm is complete. Every real or complex associative Banach algebra is a real Jordan Banach algebra with respect to the product $a \circ b := \frac{1}{2}(ab + ba)$.

Let M be a Jordan Banach algebra. Given $a, b \in M$, we shall write $U_{a,b} : M \rightarrow M$ for the bounded linear operator defined by

$$U_{a,b}(x) = (a \circ x) \circ b + (b \circ x) \circ a - (a \circ b) \circ x,$$

for all $x \in M$. The mapping $U_{a,a}$ will be simply denoted by U_a . One of the fundamental identities in Jordan algebras assures that

$$(1) \quad U_a U_b U_a = U_{U_a(b)}, \text{ for all } a, b \text{ in a Jordan algebra } M$$

(see [24, 2.4.18]). The multiplication operator by an element $a \in M$ will be denoted by M_a , that is, $M_a(b) = a \circ b$ ($b \in M$).

Henceforth, the powers of an element a in a Jordan algebra M will be denoted as follows:

$$a^1 = a; \quad a^{n+1} = a \circ a^n, \quad n \geq 1.$$

If M is unital, we set $a^0 = \mathbf{1}_M$. An algebra \mathcal{B} is called *power associative* if the subalgebras generated by single elements of \mathcal{B} are associative. In the case of a Jordan algebra M this is equivalent to say that the identity $a^m \circ a^n = a^{m+n}$, holds for all $a \in M$, $m, n \in \mathbb{N}$. It is known that any Jordan algebra is power associative ([24, Lemma 2.4.5]).

By analogy with the associative case, if M is a unital Jordan Banach algebra, the closed subalgebra generated by an element $x \in M$ and the unit is always associative, and hence we can always consider the elements of the form e^x in M , defined by

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ (cf. [6, § 1.1.29]). Let us suppose that M is a unital Jordan Banach subalgebra of an associative unital Banach algebra, and let x be an element in M . Take a and b in M such that $a = e^{itx}$ and $b = e^{isx}$, where $t, s \in \mathbb{R}$. From now on, it will be useful to keep in mind that

$$\begin{aligned} a \circ b &= e^{itx} \circ e^{isx} = \frac{1}{2}(e^{itx} e^{isx} + e^{isx} e^{itx}) \\ &= \frac{1}{2}(e^{i(t+s)x} + e^{i(t+s)x}) = e^{i(t+s)x} = e^{itx} e^{isx} = ab. \end{aligned}$$

An element a in a unital Jordan Banach algebra M is called *invertible* whenever there exists $b \in M$ satisfying $a \circ b = 1$ and $a^2 \circ b = a$. The element b is unique and it will be denoted by a^{-1} (cf. [24, 3.2.9] and [6, Definition 4.1.2]). We know from [6, Theorem 4.1.3] that an element $a \in M$ is invertible if and only if U_a is a bijective mapping, and in such a case $U_a^{-1} = U_{a^{-1}}$.

A *JB*-algebra* is a complex Jordan Banach algebra M equipped with an algebra involution $*$ satisfying $\|\{a, a, a\}\| = \|a\|^3$, $a \in M$ (where $\{a, a, a\} = U_a(a^*) = 2(a \circ a^*) \circ a - a^2 \circ a^*$). We know from a result by M.A. Youngson that the involution of every JB*-algebra is an isometry (cf. [48, Lemma 4]).

A *JB-algebra* is a real Jordan Banach algebra J in which the norm satisfies the following two axioms for all $a, b \in J$:

- (i) $\|a^2\| = \|a\|^2$;
- (ii) $\|a^2\| \leq \|a^2 + b^2\|$.

The hermitian part, M_{sa} , of a JB*-algebra, M , is always a JB-algebra. A celebrated theorem due to J.D.M. Wright asserts that, conversely, the complexification of every JB-algebra is a JB*-algebra (see [44]). We refer to the monographs [24] and [6] for the basic notions and results in the theory of JB- and JB*-algebras.

Every C*-algebra A is a JB*-algebra when equipped with its natural Jordan product $a \circ b = \frac{1}{2}(ab + ba)$ and the original norm and involution. Norm-closed Jordan *-subalgebras of C*-algebras are called *JC*-algebras*. JC*-algebras which are also dual Banach spaces are called *JW*-algebras*. Any JW*-algebra is a weak*-closed Jordan *-subalgebra of a von Neumann algebra.

We recall that an element u in a unital JB*-algebra M is a *unitary* if it is invertible and its inverse coincides with u^* . As in the associative setting, we shall denote by $\mathcal{U}(M)$ the set of all unitary elements in M . Let us observe that if a unital C*-algebra is regarded as a JB*-algebra both notions of unitaries coincide. An element s in a unital JB-algebra J is called a *symmetry* if $s^2 = \mathbf{1}_J$. If M is a JB*-algebra, the symmetries in M are defined as the symmetries in its self-adjoint part M_{sa} .

A celebrated result in the theory of JB*-algebras is the so-called Shirshov-Cohn theorem, which affirms that the JB*-subalgebra of a JB*-algebra generated by two self-adjoint elements (and the unit element) is a JC*-algebra, that is, a JB*-subalgebra of some $B(H)$ (cf. [24, Theorem 7.2.5] and [44, Corollary 2.2]).

Two elements a, b in a Jordan algebra M are said to *operator commute* if

$$(a \circ c) \circ b = a \circ (c \circ b),$$

for all $c \in M$ (cf. [24, 4.2.4]). By the *centre* of M (denoted by $Z(M)$) we mean the set of all elements of M which operator commute with any other element in M . Any element in the center is called *central*.

A JB*-algebra may admit two different Jordan products compatible with the same norm. However, when JB*-algebras are regarded as JB*-triples, any surjective linear isometry between them is a triple isomorphism (see [33, Proposition 5.5]). This fact produces a certain uniqueness of the triple product (see next section for more details). We recall the definition of JB*-triples.

A JB*-triple is a complex Banach space E equipped with a continuous triple product $\{., ., .\} : E \times E \times E \rightarrow E$, $(a, b, c) \mapsto \{a, b, c\}$, which is bilinear and symmetric in (a, c) and conjugate linear in b , and satisfies the following axioms for all $a, b, x, y \in E$:

- (a) $L(a, b)L(x, y) = L(x, y)L(a, b) + L(L(a, b)x, y) - L(x, L(b, a)y)$, where $L(a, b) : E \rightarrow E$ is the operator defined by $L(a, b)x = \{a, b, x\}$;
- (b) $L(a, a)$ is a hermitian operator with non-negative spectrum;
- (c) $\|\{a, a, a\}\| = \|a\|^3$.

The definition presented here dates back to 1983 and it was introduced by W. Kaup in [33].

Examples of JB*-triples include all C*-algebras and JB*-algebras with the triple products of the form

$$(2) \quad \{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x),$$

and

$$(3) \quad \{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*,$$

respectively.

The triple product of every JB*-triple is a non-expansive mapping, that is,

$$(4) \quad \|\{a, b, c\}\| \leq \|a\|\|b\|\|c\| \text{ for all } a, b, c \text{ (see [22, Corollary 3]).}$$

Let E be a JB*-triple. Each element e in E satisfying $\{e, e, e\} = e$ is called a *tripotent*. Each tripotent $e \in E$, determines a decomposition of E , known as the *Peirce decomposition* associated with e , in the form

$$E = E_2(e) \oplus E_1(e) \oplus E_0(e),$$

where $E_j(e) = \{x \in E : \{e, e, x\} = \frac{j}{2}x\}$ for each $j = 0, 1, 2$.

Triple products among elements in Peirce subspaces satisfy the following *Peirce arithmetic*:

$$\begin{aligned} \{E_i(e), E_j(e), E_k(e)\} &\subseteq E_{i-j+k}(e) \text{ if } i - j + k \in \{0, 1, 2\}, \\ \{E_i(e), E_j(e), E_k(e)\} &= \{0\} \text{ if } i - j + k \notin \{0, 1, 2\}, \end{aligned}$$

and $\{E_2(e), E_0(e), E\} = \{E_0(e), E_2(e), E\} = 0$. Consequently, each Peirce subspace $E_j(e)$ is a JB*-subtriple of E .

The projection $P_k(e)$ of E onto $E_k(e)$ is called the *Peirce k -projection*. It is known that Peirce projections are contractive (cf. [21, Corollary 1.2]) and determined by the following identities $P_2(e) = Q(e)^2$, $P_1(e) = 2(L(e, e) - Q(e)^2)$, and $P_0(e) = \text{Id}_E - 2L(e, e) + Q(e)^2$, where $Q(e) : E \rightarrow E$ is the conjugate or real linear map defined by $Q(e)(x) = \{e, x, e\}$.

It is worth remarking that if e is a tripotent in a JB*-triple E , the Peirce 2-subspace $E_2(e)$ is a unital JB*-algebra with unit e , product $x \circ_e y := \{x, e, y\}$ and involution $x^{*e} := \{e, x, e\}$, respectively (cf. [6, Theorem 4.1.55]).

Following standard notation, a tripotent e in in a JB*-triple E is called *unitary* if $E_2(e) = E$.

Remark 1.1. *The reader should be warned that if a unital JB*-algebra M is regarded as a JB*-triple we have two, a priori, different uses of the word “unitary”. However, there is no conflict between these two notions because unitary elements in a unital JB*-algebra M are precisely the unitary tripotents in M when the latter is regarded as a JB*-triple (cf. [3, Proposition 4.3] or [6, Theorem 4.2.24, Definition 4.2.25 and Fact 4.2.26]).*

2. UNITARIES IN JB^* -ALGEBRAS AND INVERTED JORDAN TRIPLE PRODUCTS

Unitary elements in JB^* -algebras have been intensively studied for many geometric reasons. As in the setting of C^* -algebras, they play a protagonist role in the Wright–Youngson extension of the Russo–Dye theorem for JB^* -algebras [46] (see also [6, Corollary 3.4.7 and Fact 4.2.39]). Different applications can be found on the study of surjective isometries between JB - and JB^* -algebras (see [46, 29] and [6, Proposition 4.2.44]).

The definition of unitary in a JB^* -algebra and its natural connection with the notion of unitary (tripotent) in the setting of JB^* -triples has been recalled at the introduction. We shall next revisit some basic properties with the aim of clarifying and make accessible our subsequent arguments.

The first result, which has been almost outlined in the introduction, has been borrowed from [6].

Lemma 2.1. [6, Lemma 4.2.41, Theorem 4.2.28, Corollary 3.4.32], [46], [29] *Let M be a unital JB^* -algebra, and let u be a unitary element in M . Then the following statements hold:*

- (a) *The Banach space of M becomes a unital JB^* -algebra with unit u for the (Jordan) product defined by $x \circ_u y := U_{x,y}(u^*) = \{x, u, y\}$ and the involution $*_u$ defined by $x^{*u} := U_u(x^*) = \{u, x, u\}$. (This JB^* -algebra $M(u) = (M, \circ_u, *_u)$ is called the u -isotope of M .)*
- (b) *The unitary elements of the JB^* -algebras M and $(M, \circ_u, *_u)$ are the same, and they also coincide with the unitary tripotents of M when the latter is regarded as a JB^* -triple.*
- (c) *The triple product of M satisfies*

$$\begin{aligned} \{x, y, z\} &= (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^* \\ &= (x \circ_u y^{*u}) \circ_u z + (z \circ_u y^{*u}) \circ_u x - (x \circ_u z) \circ_u y^{*u}, \end{aligned}$$

for all $x, y, z \in M$. Actually, the previous identities hold when \circ is replaced with any Jordan product on M making the latter a JB^ -algebra with the same norm.*

- (d) *The mapping $U_u : M \rightarrow M$ is a surjective isometry and hence a triple isomorphism. Consequently, $U_u(\mathcal{U}(M)) = \mathcal{U}(M)$. Furthermore, the operator $U_u : (M, \circ_u, *_u) \rightarrow (M, \circ_u, *_u)$ is a Jordan $*$ -isomorphism.*

Proof. Statements (a) and (b) can be found in [6, Lemma 4.2.41] (see also [3, Proposition 4.3]). Moreover, Theorem 4.2.28 (vii) in [6] assures that the mapping U_u is a surjective linear isometry. The remaining statements are consequences of the fact that a linear bijection between JB^* -triples is an isometry if and only if it is a triple isomorphism (cf. [33, Proposition 5.5] and [7, Theorem 5.6.57]). Furthermore each unital triple isomorphism between unital JB^* -algebras must be a Jordan $*$ -isomorphism. \square

Let u be a unitary element in a unital C^* -algebra A . It is known that $\|1 - u\| < 2$ implies that $u = e^{ih}$ for some $h \in A_s$ (see [31, Exercise 4.6.6]). In our next lemma we combine this fact with the Shirshov–Cohn theorem.

Lemma 2.2. *Let u, v be two unitaries in a unital JB^* -algebra M . Let us suppose that $\|u - v\| = \eta < 2$. Then the following statements hold:*

- (a) *There exists a self-adjoint element h in the u -isotope JB^* -algebra $M(u) = (M, \circ_u, *_u)$ such that $v = e^{ih}$, where the exponential is computed in the JB^* -algebra $M(u)$.*
- (b) *There exists a unitary w in M satisfying $U_w(u^*) = v$.*
- Moreover, if $\|u - v\| = \eta = |1 - e^{it_0}| = \sqrt{2}\sqrt{1 - \cos(t_0)}$ for some $t_0 \in (-\pi, \pi)$, we can further assume that $\|w - u\|, \|w - v\| \leq \sqrt{2}\sqrt{1 - \cos(\frac{t_0}{2})}$.

Proof. We consider the unital JB^* -algebra $M(u) = (M, \circ_u, *_u)$. Let \mathcal{C} denote the JB^* -subalgebra of $M(u)$ generated by v and its unit –i.e. u –. Let us observe that the product and involution on \mathcal{C} are precisely $\circ_u|_{\mathcal{C} \times \mathcal{C}}$ and $*_u|_{\mathcal{C}}$, respectively. Since v is unitary in $M(u)$ (cf. Lemma 2.1(b)), the JB^* -subalgebra \mathcal{C} must be isometrically Jordan $*$ -isomorphic to a unital commutative C^* -algebra, that is, to some $C(\Omega)$ for an appropriate compact Hausdorff space Ω , and under this identification, u corresponds to the unit (cf. [24, 3.2.4. The spectral theorem] or [6, Proposition 3.4.2 and Theorem 4.1.3(v)]).

Since $\|u - v\| < 2$, v is a unitary in \mathcal{C} and u is the unit element of the C^* -algebra $\mathcal{C} \equiv C(\Omega)$, we can find a self-adjoint element $h \in \mathcal{C}_{sa}$ such that $v = e^{ih}$ (see [31, Exercise 4.6.6]), where the exponential is, of course, computed with respect to the structure of \mathcal{C} , that is with the product and involution of $M(u)$. This finishes the proof of (a).

By setting $w = e^{i\frac{h}{2}}$ we get a unitary element in \mathcal{C} satisfying $w \cdot u^{*u} \cdot w = w \cdot u \cdot w = v$ (let us observe that the involution of \mathcal{C} is precisely the restriction of $*_u$ to \mathcal{C}). Let U_a^u denote the U operator on the unital JB^* -algebra $M(u)$ associated with the element a . Since \mathcal{C} is a unital JB^* -subalgebra of $M(u)$, we deduce that

$$v = w \cdot u^{*u} \cdot w = U_w^u(u^{*u}) = \{w, u, w\} = U_w(u^*),$$

and clearly w is a unitary in M because it is a unitary in $M(u)$ (cf. Lemma 2.1(b) and (c)). We have therefore concluded the proof of (b).

The final statement is a clear consequence of the identification of \mathcal{C} with $C(\Omega)$ in which u corresponds to the unit and v and w with e^{ih} and $e^{i\frac{h}{2}}$, respectively. \square

The next lemma can be deduced by similar arguments to those given in the previous result.

Lemma 2.3. *Let u and w be unitary elements in a unital JB^* -algebra M . Suppose $U_w(u^*) = u$ and $\|u - w\| < 2$. Then $w = u$.*

Proof. Let $M(w) = (M, \circ_w, *_w)$ and let \mathcal{C} denote the JB^* -subalgebra of $M(w)$ generated by u and its unit. By applying the identification of \mathcal{C} with an appropriate $C(\Omega)$ space as the one given in the proof of the previous lemma, we can identify w with the unit of $C(\Omega)$ and u with a unitary in this commutative unital C^* -algebra with $u^{*w} = U_w(u^*) = u$ (self-adjoint in \mathcal{C}) and $\|u - w\| < 2$, which implies that $w = u$. \square

We shall gather next some results on isometries between metric groups due to O. Hatori, G. Hirasawa, T. Miura and L Molnár [25]. The conclusions in the just quoted paper provided the tools applied in the study of surjective isometries between the unitary groups of unital C^* -algebras in subsequent references [26] and [27].

Henceforth, let \mathcal{G} be a group and let (X, d) be a non-trivial metric space such that X is a subset of \mathcal{G} and

$$(5) \quad yx^{-1}y \in X \text{ for all } x, y \in X$$

(note that we are not assuming that X is a subgroup of \mathcal{G}).

Definition 2.4. *Let us fix a, b in X . We shall say that condition $B(a, b)$ holds for (X, d) if the following properties hold:*

(B.1) *For all $x, y \in X$ we have $d(bx^{-1}b, by^{-1}b) = d(x, y)$.*

(B.2) *There exists a constant $K > 1$ satisfying*

$$d(bx^{-1}b, x) \geq Kd(x, b),$$

$$\text{for all } x \in L_{a,b} = \{x \in X : d(a, x) = d(ba^{-1}b, x) = d(a, b)\}.$$

Definition 2.5. *Let us fix $a, b \in X$. We shall say that condition $C_1(a, b)$ holds for (X, d) if the following properties hold:*

(C.1) *For every $x \in X$ we have $ax^{-1}b, bx^{-1}a \in X$;*

(C.2) *$d(ax^{-1}b, ay^{-1}b) = d(x, y)$, for all $x, y \in X$.*

We shall say that condition $C_2(a, b)$ holds for (X, d) if there exists $c \in X$ such that $ca^{-1}c = b$ and $d(cx^{-1}c, cy^{-1}c) = d(x, y)$ for all $x, y \in X$.

An element $x \in X$ is called *2-divisible* if there exists $y \in X$ such that $y^2 = x$. X is called *2-divisible* if every element in X is 2-divisible. Furthermore, X is called *2-torsion free* if it contains the unit of \mathcal{G} and the condition $x^2 = 1$ with $x \in X$ implies $x = 1$.

We shall need the following result taken from [25].

Theorem 2.6. [25, Theorem 2.4] *Let (X, d_X) and (Y, d_Y) be two metric spaces. Pick two points $a, c \in X$. Suppose that $\varphi : X \rightarrow X$ is a distance preserving map such that $\varphi(c) = c$ and $\varphi \circ \varphi$ is the identity map on X . Let*

$$L_{a,c} = \{x \in X : d_X(a, x) = d_X(\varphi(a), x) = d_X(a, c)\}.$$

Suppose that there exists a constant $K > 1$ such that $d_X(\varphi(x), x) \geq Kd_X(x, c)$ holds for every $x \in L_{a,c}$. If δ is a bijective distance preserving map from X onto Y , and ψ is a bijective distance preserving map from Y onto itself such that $\psi(\Delta(a)) = \Delta(\varphi(a))$ and $\psi(\Delta(\varphi(a))) = \Delta(a)$, then we have $\psi(\Delta(c)) = \Delta(c)$. \square

Conditions $B(a, b)$, $C_1(a, b)$ and $C_2(a, b)$ are perfectly applied in [25] (and subsequently in [26]) to establish a generalization of the Mazur-Ulam theorem for commutative groups [25, Corollary 5.1], and to present a metric characterization of normed real-linear spaces among commutative metric groups [25, Corollary 5.4]. Despite of the tempting title of [25] for the audience on Jordan structures—i.e. “*Isometries and maps compatible with inverted Jordan triple products on groups*”—, the results in the just quoted reference have not been applied in a proper Jordan setting yet. There are so many handicaps reducing its potential applicability. We are aimed to present a first application in this paper.

For the discussion in this paragraph, let \mathcal{A} be a unital JC*-algebra which will be regarded as a JB*-subalgebra of some $B(H)$. Let us observe that the unit of \mathcal{A} must be a projection $\mathbf{1}_{\mathcal{A}}$ in $B(H)$, and thus by replacing H with $\mathbf{1}_{\mathcal{A}}(H)$, we can always assume that \mathcal{A} and $B(H)$ share the same unit. We shall denote the product of $B(H)$ by mere juxtaposition. The set $\mathcal{U}(\mathcal{A})$ of all unitaries in \mathcal{A} is

not in general a subgroup of $\mathcal{U}(B(H))$ –the latter is not even stable under Jordan products–, however $U_u(v) = uvu$, and u^* lie in $\mathcal{U}(\mathcal{A})$ for all $u, v \in \mathcal{U}(\mathcal{A})$ (cf. Lemma 2.1). The set $\mathcal{U}(B(H))$ is a group for its usual product and will be equipped with the distance provided by the operator norm. Conditions of the type $C_1(a, b)$ do not hold for $(\mathcal{U}(\mathcal{A}), \|\cdot\|)$ because products of the form $ax^{-1}b$ do not necessarily lie in $\mathcal{U}(\mathcal{A})$ for all $a, b, x \in \mathcal{U}(\mathcal{A})$. The set $\mathcal{U}(\mathcal{A})$ is not 2-torsion free since $-1 \in \mathcal{U}(\mathcal{A})$. Furthermore, the identity $yx^{-1}y = y^2x^{-1}$ does not necessarily hold for $x, y \in \mathcal{U}(\mathcal{A})$. We have therefore justified that [25, Corollaries 3.9, 3.10 and 3.11] cannot be applied in the Jordan setting, even under more favorable hypothesis of working with a JC^* -algebra.

Let $\text{Iso}(Z)$ denote the group of all surjective linear isometries on a Banach space Z . Hidden within the proof of [26, Theorem 6], it is shown that for a complex Banach spaces Z , condition $B(a, b)$ is satisfied for elements a, b in $\text{Iso}(Z)$ which are at distance strictly smaller than $\frac{1}{2}$. Let us concretize the exact statement.

Lemma 2.7. [26, Proof of Theorem 6] *Let Z be a complex Banach space, let u, v be two elements in $\text{Iso}(Z)$ with $\|u - v\| < \frac{1}{2}$. Then for $K = 2 - 2\|u - v\| > 1$, the inequality*

$$\|vw^{-1}v - w\| \geq K\|w - v\|$$

holds for every w in the set

$$L_{u,v} = \{w \in \text{Iso}(Z) : \|u - w\| = \|vu^{-1}v - w\| = \|u - v\|\}.$$

The next result is a consequence of the previous lemma in the case in which M is a JC^* -algebra by just regarding M as a unital Jordan $*$ -subalgebra of some $B(H)$ with the same unit (we can always see $\mathcal{U}(M)$ inside $\text{Iso}(H)$). The existence of exceptional JB^* -algebras which cannot be embedded as Jordan $*$ -subalgebras of $B(H)$ (see [24, Corollary 2.8.5], [6, Example 3.1.56]), forces us to develop a new argument.

Lemma 2.8. *Let u, v be two elements in $\mathcal{U}(M)$, where M is a unital JB^* -algebra. Suppose $\|u - v\| < 1/2$. Then the Jordan version of condition $B(u, v)$ holds for $\mathcal{U}(M)$, that is,*

- (a) *For all $x, y \in \mathcal{U}(M)$ we have $\|U_v(x^{-1}) - U_v(y^{-1})\| = \|x^* - y^*\| = \|x - y\|$.*
- (b) *The constant $K = 2 - 2\|u - v\| > 1$ satisfies that*

$$\|U_v(w^*) - w\| = \|U_v(w^{-1}) - w\| \geq K\|w - v\|,$$

for all w in the set

$$L_{u,v} = \{w \in \mathcal{U}(M) : \|u - w\| = \|U_v(u^{-1}) - w\| = \|U_v(u^*) - w\| = \|u - v\|\}.$$

Proof. Statement (a) is clear from Lemma 2.1(d) and the fact that the involution on M is an isometry.

Let us consider the u -isotope JB^* -algebra $M(u) = (M, \circ_u, *_u)$ of M . The U -operator on $M(u)$ will be denoted by U^u . We fix an element $w \in L_{u,v}$. Since $\|u - w\| = \|u - v\| < \frac{1}{2}$, we deduce from Lemma 2.2 the existence of two self-adjoint elements $h_1, h_2 \in M(u)$ such that $v = e^{ih_1}$ and $w = e^{ih_2}$. Let \mathcal{B} denote the JB^* -subalgebra of $M(u)$ generated by u, h_1, h_2 . The Shirshov-Cohn theorem assures the existence of a complex Hilbert space H such that \mathcal{B} is a JB^* -subalgebra of $B(H)$ and both share the same unit u (the product of $B(H)$ will be denoted by mere

juxtaposition and the involution by \sharp). Obviously $u, v, w \in \mathcal{U}(\mathcal{B}) \subseteq \mathcal{U}(B(H)) \subseteq \text{Iso}(H)$ with $\|u - w\|_{\mathcal{B}} = \|u - w\|_M = \|u - v\|_M = \|u - v\|_{\mathcal{B}}$. Let us compute

$$\begin{aligned} U_v^u(u) &= 2(v \circ_u u) \circ_u v - (v \circ_u v) \circ_u u \\ &= 2v \circ_u v - v \circ_u v = v \circ_u v = \{v, u, v\} = U_v(u^*), \end{aligned}$$

and

$$\begin{aligned} \|vu^{-1}v - w\|_{B(H)} &= \|vu^{-1}v - w\|_{\mathcal{B}} = \|U_v^u(u^{\sharp}) - w\|_{\mathcal{B}} = \|U_v^u(u^{*u}) - w\|_{\mathcal{B}} \\ &= \|U_v^u(u) - w\|_{\mathcal{B}} = \|U_v(u^*) - w\|_{\mathcal{B}} = \|U_v(u^*) - w\|_M. \end{aligned}$$

Lemma 2.7 proves that for $K = 2 - 2\|u - v\| > 1$ we have

$$\|U_v^u(w^{*u}) - w\|_{\mathcal{B}} = \|U_v^u(w^{-1}) - w\|_{\mathcal{B}} = \|vw^{-1}v - w\|_{B(H)} \geq K\|w - v\|_{\mathcal{B}} = K\|w - v\|.$$

On the other hand, by the uniqueness of the triple product (see [33, Proposition 5.5] or Lemma 2.1(c)) we have

$$U_v^u(w^{*u}) = \{v, w, v\}_{\mathcal{B}} = \{v, w, v\}_{M(u)} = \{v, w, v\}_M = U_v(w^*).$$

All together gives

$$\begin{aligned} \|U_v(w^{-1}) - w\| &= \|U_v(w^*) - w\|_M = \|U_v(w^*) - w\|_{\mathcal{B}} \\ &= \|U_v^u(w^{*u}) - w\|_{\mathcal{B}} \geq K\|w - v\|, \end{aligned}$$

which completes the proof. \square

We can now establish a key result for our goals.

Theorem 2.9. *Let $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ be a surjective isometry, where M and N are unital JB^* -algebras. Suppose $u, v \in \mathcal{U}(M)$ with $\|u - v\| < \frac{1}{2}$. Then the following statements are true:*

- (1) *The Jordan version of condition $B(u, v)$ holds for $\mathcal{U}(M)$;*
- (2) *The Jordan version of condition $C_2(\Delta(u), \Delta(U_v(u^*)))$ holds for $\mathcal{U}(N)$;*
- (3) *The identity $\Delta(U_v(u^*)) = \Delta(U_v(u^{-1})) = U_{\Delta(v)}(\Delta(u)^*) = U_{\Delta(v)}(\Delta(u)^{-1})$ holds.*

Proof. Statement (1) follows from Lemma 2.8.

(2) By hypotheses $\|\Delta(u) - \Delta(v)\| = \|u - v\| < \frac{1}{2}$. Let \mathcal{B} denote the JB^* -subalgebra of the u -isotope $M(u)$ generated by u and v . We shall denote by U^u the U -operator in $M(u)$. Again, by the Shirshov-Cohn theorem we can find a complex Hilbert space H such that \mathcal{B} is a JB^* -subalgebra of $B(H)$ and u is the unit of $B(H)$. The product of $B(H)$ will be denoted by mere juxtaposition and the involution by \sharp . In this case we have

$$\begin{aligned} \|\Delta(u) - \Delta(U_v(u^*))\| &= \|u - U_v(u^*)\| = \|u - \{v, u, v\}\| = \|u - \{v, u, v\}_{\mathcal{B}}\|_{\mathcal{B}} \\ &= \|u - U_v^u(u^{*u})\|_{\mathcal{B}} = \|u - U_v^u(u^{*u})\|_{B(H)} = \|u - vu^{\sharp}v\|_{B(H)} \\ &= \|u - vv\|_{B(H)} = \|v^{\sharp} - v\|_{B(H)} \leq \|v^{\sharp} - u\|_{B(H)} + \|u - v\|_{B(H)} \\ &= 2\|u - v\|_{B(H)} = 2\|u - v\|_{\mathcal{B}} = 2\|u - v\| < 1 \end{aligned}$$

(cf. Lemma 2.1(c)).

Lemma 2.2(b) assures the existence of $w \in \mathcal{U}(N)$ satisfying

$$(6) \quad U_w(\Delta(u)^*) = \Delta(U_v(u^*)), \text{ and } \|w - \Delta(v)\| < 1 \text{ (or smaller).}$$

This shows that the Jordan version of $C_2(\Delta(u), \Delta(U_u(v^*)))$ holds for $\mathcal{U}(N)$ because the remaining requirement, i.e.,

$$\|U_w(x^{-1}) - U_w(y^{-1})\| = \|x^{-1} - y^{-1}\| = \|x^* - y^*\| = \|x - y\|$$

holds for all $x, y \in \mathcal{U}(N)$ (cf. Lemma 2.1(d)).

(3) Let $w \in \mathcal{U}(N)$ be the element found in the proof of (2). We define a couple of mappings $\varphi : \mathcal{U}(M) \rightarrow \mathcal{U}(M)$ and $\psi : \mathcal{U}(N) \rightarrow \mathcal{U}(N)$ given by $\varphi(x) := U_v(x^{-1}) = U_v(x^*)$ and $\psi(y) := U_w(y^{-1}) = U_w(y^*)$, respectively. Clearly, φ and ψ are distance preserving bijections (cf. Lemma 2.1(d)).

It is clear that $\varphi(v) = v$ and $\varphi \circ \varphi$ is the identity mapping on $\mathcal{U}(M)$. Furthermore, by (6) we have

$$\begin{aligned} \psi(\Delta(u)) &= U_w(\Delta(u)^*) = \Delta(U_v(u^*)) = \Delta(\varphi(u)) \\ \psi(\Delta(\varphi(u))) &= U_w(\Delta(U_v(u^*)^*)) = U_{w^*}(\Delta(U_v(u^*)))^* = \Delta(u)^{**} = \Delta(u). \end{aligned}$$

Since by (1) the Jordan version of condition $B(u, v)$ holds for $\mathcal{U}(M)$, and by (2) the Jordan version of $C_2(\Delta(u), \Delta(U_v(u^*)))$ holds for $\mathcal{U}(N)$, we can see that all the hypotheses of Theorem 2.6 [25, Theorem 2.4] are satisfied. We deduce from the just quoted theorem that $U_w(\Delta(v)^*) = \Delta(v)$, and since $\|w - \Delta(v)\| < 2$ (cf. (6)), Lemma 2.3 guarantees that $w = \Delta(v)$ and hence $\Delta(U_v(u^*)) = U_{\Delta(v)}(\Delta(u)^*)$ as desired (see (6)). \square

3. SURJECTIVE ISOMETRIES BETWEEN SETS OF UNITARIES

In this section we shall try to find a precise description of the surjective isometries between the sets of unitaries in two unital JB^* -algebras. Our first goal is to find conditions under which any such surjective isometry can be extended to a surjective real linear isometry between these JB^* -algebras.

We recall that a *one-parameter group* of bounded linear operators on a Banach space Z is a mapping $\mathbb{R} \rightarrow B(Z)$, $t \mapsto E(t)$ satisfying $E(0) = I$ and $E(t+s) = E(s)E(t)$, for all $s, t \in \mathbb{R}$. A one-parameter group $\{E(t) : t \in \mathbb{R}\}$ is uniformly continuous at the origin if $\lim_{t \rightarrow 0} \|E(t) - I\| = 0$. It is known that being uniformly continuous at zero is equivalent to the existence of a bounded linear operator $R \in B(Z)$ such that $E(t) = e^{tR}$ for all $t \in \mathbb{R}$, where the exponential is computed in the Banach algebra $B(Z)$ (see, for example, [4, Proposition 3.1.1]). A one-parameter group on $\{E(t) : t \in \mathbb{R}\}$ on a complex Hilbert space H is called *strongly continuous* if for each ξ in H the mapping $t \mapsto E(t)(\xi)$ is continuous ([9, Definition 5.3, Chapter X]). A *one-parameter unitary group* on H is a one-parameter group on H such that $E(t)$ is a unitary element for each $t \in \mathbb{R}$.

The celebrated Stone's one-parameter theorem affirms that for each strongly continuous one-parameter unitary group $\{E(t) : t \in \mathbb{R}\}$ on a complex Hilbert space H there exists a self-adjoint operator $h \in B(H)$ such that $E(t) = e^{ith}$, for every $t \in \mathbb{R}$ ([9, 5.6, Chapter X]).

We recall that a *triple derivation* on a JB^* -triple E is a linear mapping $\delta : E \rightarrow E$ satisfying a ternary version of Leibniz' rule

$$\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}, \quad (a, b, c \in E).$$

We shall apply that every triple derivation is automatically continuous (see [1, Corollary 2.2]). If $\delta : M \rightarrow M$ is a triple derivation on a unital JB^* -algebra, it is known that $\delta(\mathbf{1}_M)^* = -\delta(\mathbf{1}_M)$, that is, $i\delta(\mathbf{1}_M) \in M_{sa}$ (cf. [28, Proof of Lemma 1]).

The existence of exceptional JB*-algebra which cannot be represented inside a $B(H)$ space, adds extra difficulties to apply Stone's one-parameter theorem. The study of uniformly continuous one-parameter groups of surjective isometries (i.e. triple isomorphisms), Jordan *-isomorphisms and orthogonality preserving operators on JB*-algebras has been recently initiated in [23]. We complement the results in the just quoted reference with the next result, which is a Jordan version of Stone's theorem for uniformly continuous unitary one-parameter groups.

Theorem 3.1. *Let M be a unital JB*-algebra. Suppose $\{u(t) : t \in \mathbb{R}\}$ is a family in $\mathcal{U}(M)$ satisfying $u(0) = \mathbf{1}$, and $U_{u(t)}(u(s)) = u(2t + s)$, for all $t, s \in \mathbb{R}$. We also assume that the mapping $t \mapsto u(t)$ is continuous. Then there exists $h \in M_{sa}$ such that $u(t) = e^{ith}$ for all $t \in \mathbb{R}$.*

Proof. We shall first prove that

$$(7) \quad u(s + t) = u(t) \circ u(s) \text{ for all } t, s \in \mathbb{R}.$$

Fix a real t , it follows from the hypothesis that

$$u(t)^2 = U_{u(t)}(u(0)) = u(2t), \text{ and } u(t)^3 = U_{u(t)}(u(t)) = u(3t).$$

Arguing by induction on n , it can be established that $u(t)^n = u(nt)$ for all $n \in \mathbb{N}$, $t \in \mathbb{R}$. Indeed, for any integer $n \geq 4$, by the induction hypothesis

$$u(t)^n = U_{u(t)}(u(t)^{n-2}) = U_{u(t)}(u((n-2)t)) = u(nt).$$

The identity $U_{u(t)}U_{u(-t)}U_{u(t)} = U_{u(t)}$ (see (1)) together with the fact that $U_{u(t)}$ is invertible in $B(M)$ proves that $U_{u(t)}^* = U_{u(t)}^{-1} = U_{u(-t)}$ and thus $u(t)^* = u(t)^{-1} = u(-t)$ for all $t \in \mathbb{R}$ (cf. [6, Theorem 4.1.3]).

Given an integer $n \leq 0$ we observe that $u(t)^n = (u(t)^{-n})^{-1} = (u(t)^{-n})^*$ for every $t \in \mathbb{R}$, and thus

$$u(t)^n = (u(t)^{-n})^* = u((-n)t)^* = u(nt).$$

We have therefore shown that

$$(8) \quad u(t)^n = u(nt), \text{ for all } t \in \mathbb{R} \text{ and } n \in \mathbb{Z}.$$

By the continuity of the mapping $t \mapsto u(t)$, in order to prove (7) it suffices to show that the identity $u(r + r') = u(r) \circ u(r')$ holds for any rational numbers r and r' . Therefore, let us take $r = n/m$ and $r' = n'/m'$, with $n, n' \in \mathbb{Z}$ and $m, m' \in \mathbb{N}$. By a couple of applications of (8) and the power associativity of M (see [24, Lemma 2.4.5]) we have

$$\begin{aligned} u(r + r') &= u\left(\frac{nm' + mn'}{mm'}\right) = u\left(\frac{1}{mm'}\right)^{nm' + mn'} = u\left(\frac{1}{mm'}\right)^{nm'} \circ u\left(\frac{1}{mm'}\right)^{mn'} \\ &= u\left(\frac{nm'}{mm'}\right) \circ u\left(\frac{mn'}{mm'}\right) = u(r) \circ u(r'), \end{aligned}$$

as desired.

Let us define a mapping $\Phi : \mathbb{R} \rightarrow \text{Iso}(M)$, $t \mapsto \Phi(t) = U_{u(t)}$. Clearly, Φ is continuous with $\Phi(0) = Id_M$. By applying the fundamental identity (1) one sees that

$$\Phi(t)\Phi(s)\Phi(t) = U_{u(t)}U_{u(s)}U_{u(t)} = U_{U_{u(t)}(u(s))} = U_{u(2t+s)} = \Phi(2t + s),$$

for all $s, t \in \mathbb{R}$. It then follows that $\Phi(t)^2 = \Phi(t)\Phi(0)\Phi(t) = \Phi(2t)$, $\Phi(3t) = \Phi(t)^3$, and by induction on $n(\geq 3)$, $\Phi(nt) = \Phi(2t + (n-2)t) = \Phi(t)\Phi((n-2)t)\Phi(t) = \Phi(t)\Phi(t)^{n-2}\Phi(t) = \Phi(t)^n$ for all $t \in \mathbb{R}$, $n \in \mathbb{N}$. Therefore $\Phi(nt) = \Phi(t)^n$ for all $t \in \mathbb{R}$, $n \in \mathbb{Z}$.

Since, for each real t , $\Phi(t)\Phi(-t)\Phi(t) = \Phi(t)$ and $\Phi(t) \in \text{Iso}(M)$ we can deduce that $\Phi(-t) = \Phi(t)^{-1}$ for all $t \in \mathbb{R}$. It follows that, for a negative integer n and each real t , we have

$$\Phi(nt) = \Phi((-n)(-t)) = \Phi(-t)^{-n} = \Phi(t)^n,$$

an identity which then holds for all $n \in \mathbb{Z}$.

We claim that $t \mapsto \Phi(t)$ is a one-parameter group of surjective isometries on M . Let us fix two rational numbers $\frac{n}{m}, \frac{n'}{m'}$ with $m, m' \in \mathbb{N}$, $n, n' \in \mathbb{Z}$. It follows from the above properties that

$$\begin{aligned} \Phi\left(\frac{n}{m} + \frac{n'}{m'}\right) &= \Phi\left(\frac{nm' + n'm}{mm'}\right) = \Phi\left(\frac{1}{mm'}\right)^{nm' + n'm} \\ &= \Phi\left(\frac{1}{mm'}\right)^{nm'} \Phi\left(\frac{1}{mm'}\right)^{n'm} = \Phi\left(\frac{n}{m}\right) \Phi\left(\frac{n'}{m'}\right). \end{aligned}$$

It follows from the continuity of Φ that

$$\Phi(t+s) = \Phi(t)\Phi(s), \text{ for all } t, s \in \mathbb{R},$$

that is, $\Phi(t)$ is a uniformly continuous one-parameter group of surjective isometries on M . By [23, Lemma 3.1] there exists a triple derivation $\delta : M \rightarrow M$ satisfying $\Phi(t) = e^{t\delta} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \delta^n$ for all $t \in \mathbb{R}$, where the exponential is computed in $B(M)$.

Let us observe that $w(t) := \Phi(t)(\mathbf{1}) = U_{u(t)}(\mathbf{1}) = u(t)^2 = U_{u(t)}(u(0)) = u(2t)$ for all $t \in \mathbb{R}$, and hence by (7)

$$\begin{aligned} w(t+s) &= \Phi(t+s)(\mathbf{1}) = u(2(t+s)) = u(2t) \circ u(2s) \\ &= \Phi(t)(\mathbf{1}) \circ \Phi(s)(\mathbf{1}) = w(t) \circ w(s), \end{aligned}$$

for all $s, t \in \mathbb{R}$. We shall next show that

$$\delta(\mathbf{1})^n = \delta^n(\mathbf{1}), \text{ for all natural } n.$$

Since $w(t+s) = w(t) \circ w(s)$, by taking derivatives in t at $t=0$ we get

$$\sum_{n=1}^{\infty} \frac{s^{n-1}}{(n-1)!} \delta^n(\mathbf{1}) = \frac{\partial}{\partial t|_{t=0}} w(t+s) = \frac{\partial}{\partial t|_{t=0}} w(t) \circ w(s) = \delta(\mathbf{1}) \circ w(s),$$

for all $s \in \mathbb{R}$. Taking a new derivative in s at $s=0$ we have

$$\begin{aligned} \delta^2(\mathbf{1}) &= \frac{\partial}{\partial s|_{s=0}} \frac{\partial}{\partial t|_{t=0}} w(t+s) = \delta(\mathbf{1}) \circ \frac{\partial}{\partial s|_{s=0}} w(s) \\ &= \delta(\mathbf{1}) \circ (\delta(\mathbf{1}) \circ w(0)) = M_{\delta(\mathbf{1})}^2(w(0)) = \delta(\mathbf{1})^2. \end{aligned}$$

Similarly,

$$\delta^n(\mathbf{1}) = \frac{\partial^{n-1}}{\partial s^{n-1}|_{s=0}} \frac{\partial}{\partial t|_{t=0}} w(t+s) = M_{\delta(\mathbf{1})}^n(w(0)) = \delta(\mathbf{1})^n,$$

which gives the desired statement.

It follows from the above identities that

$$u(t) = \Phi\left(\frac{t}{2}\right) = e^{\frac{t}{2}\delta}(\mathbf{1}) = \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} \delta^n(\mathbf{1}) = \sum_{n=0}^{\infty} \frac{t^n}{2^n n!} \delta(\mathbf{1})^n = e^{t\frac{\delta(\mathbf{1})}{2}}, \text{ for all } t \in \mathbb{R},$$

where, as we commented before this proposition, $\delta(\mathbf{1})^* = -\delta(\mathbf{1})$ in M (cf. [28, Proof of Lemma 1]). \square

We continue by enunciating a variant of an argument which has been applied in several cases before.

Remark 3.2. *Suppose u is a unitary element in a unital JB^* -algebra M such that $\|\mathbf{1}-u\| < 2$. By Lemma 2.2(a) we can find a self-adjoint element $h \in M_{sa}$ satisfying $u = e^{ih}$. Let us consider the unitary $\omega = e^{-i\frac{h}{2}} \in \mathcal{U}(M)$ and the mapping $U_\omega : M \rightarrow M$. Let us observe that $U_\omega(u) = \mathbf{1}$ (just apply that u and ω operator commute by definition). Since ω is unitary in M , and hence $U_\omega^{-1} = U_{\omega^*}$ (cf. [6, Theorem 4.1.3]) we can conclude from (4) that $U_\omega : M(u) = (M, \circ_u, *_u) \rightarrow M$ is a unital surjective isometry (see also Lemma 2.1(d) or [6, Theorem 4.2.28] for a direct argument). We can therefore conclude from Theorem 6 in [46] that $U_\omega : M(u) = (M, \circ_u, *_u) \rightarrow M$ is a Jordan $*$ -isomorphism.*

As in the case of unital C^* -algebras (cf. the discussion preceding Proposition 4.4.10 in [31]), not each unitary element of a unital JB^* -algebra M is of the form e^{ih} for some $h \in M_{sa}$. However, if we assume that M is a JBW^* -algebra the conclusion is different. Let u be a unitary element in a JBW^* -algebra M . Let \mathcal{W} denote the JBW^* -subalgebra of M generated by u, u^* and the unit of M . Clearly \mathcal{W} is an associative JBW^* -algebra (cf. [24, Theorem 3.2.2, Remark 3.2.3 and Theorem 4.4.16]), and we can therefore assume that \mathcal{W} is a commutative von Neumann algebra. Theorem 5.2.5 in [31] implies the existence of an element $h \in \mathcal{W}_{sa} \subseteq M_{sa}$ such that $e^{ih} = u$ (in \mathcal{W} and also in M). We therefore have

$$(9) \quad \mathcal{U}(M) = \{e^{ih} : h \in M_{sa}\}$$

for all JBW^* -algebra M .

The next lemma, which is a Jordan version of [26, Lemma 7], will be required later.

Lemma 3.3. *Let M and N be two unital JB^* -algebras. Let $\{u_k : 0 \leq k \leq 2^n\}$ be a subset of $\mathcal{U}(M)$ and let $\Phi : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ be a mapping such that $U_{u_{k+1}}(u_k^*) = u_{k+2}$ and*

$$\Phi(U_{u_{k+1}}(u_k^*)) = U_{\Phi(u_{k+1})}(\Phi(u_k)^*),$$

for all $0 \leq k \leq 2^n - 2$. Then $U_{u_{2^n-1}}(u_0^*) = u_{2^n}$ and

$$\Phi(U_{u_{2^n-1}}(u_0^*)) = U_{\Phi(u_{2^n-1})}(\Phi(u_0)^*).$$

Proof. We shall argue by induction on n . The statement is clear for $n = 1$. Suppose that our statement is true for every family with $2^n + 1$ elements satisfying the conditions above. Let $\{w_k : 0 \leq k \leq 2^{n+1}\}$ be a subset of $\mathcal{U}(M)$ such that $U_{w_{k+1}}(w_k^*) = w_{k+2}$ and

$$\Phi(U_{w_{k+1}}(w_k^*)) = U_{\Phi(w_{k+1})}(\Phi(w_k)^*),$$

for all $0 \leq k \leq 2^{n+1} - 2$. Set $u_k = w_{2k}$. We shall next show that we can apply the induction hypothesis to the family $\{u_k : 0 \leq k \leq 2^n\} \subseteq \mathcal{U}(M)$. Fix $0 \leq k \leq 2^n - 2$,

$$\begin{aligned} u_{k+2} &= w_{2k+4} = U_{w_{2k+3}}(w_{2k+2}^*) = U_{U_{w_{2k+2}}(w_{2k+1}^*)}(w_{2k+2}^*) \\ &= U_{w_{2k+2}} U_{w_{2k+1}^*} U_{w_{2k+2}}(w_{2k+2}^*) = U_{w_{2k+2}} U_{w_{2k+1}^*}(w_{2k+2}^*) \\ &= U_{w_{2k+2}} U_{w_{2k+1}^*} U_{w_{2k+1}}(w_{2k}^*) = U_{w_{2k+2}}(w_{2k}^*) = U_{u_{k+1}}(u_k^*), \end{aligned}$$

where in the fourth equality we applied the identity (1).

On the other hand, the previous identities and the induction hypothesis also give

$$\begin{aligned} \Phi(U_{u_{k+1}}(u_k^*)) &= \Phi(U_{w_{2k+2}}(w_{2k}^*)) = \Phi(U_{w_{2k+3}}(w_{2k+2}^*)) = U_{\Phi(w_{2k+3})}(\Phi(w_{2k+2}^*)) \\ &= U_{\Phi(U_{w_{2k+2}}(w_{2k+1}^*))}(\Phi(w_{2k+2}^*)) = U_{U_{\Phi(w_{2k+2})}(\Phi(w_{2k+1}^*))}(\Phi(w_{2k+2}^*)) \\ &= U_{\Phi(w_{2k+2})} U_{\Phi(w_{2k+1}^*)} U_{\Phi(w_{2k+2})}(\Phi(w_{2k+2}^*)) \\ &= U_{\Phi(w_{2k+2})} U_{\Phi(w_{2k+1}^*)}(\Phi(w_{2k+2}^*)) \\ &= U_{\Phi(w_{2k+2})} U_{\Phi(w_{2k+1}^*)}(\Phi(U_{w_{2k+1}}(w_{2k}^*))) \\ &= U_{\Phi(w_{2k+2})} U_{\Phi(w_{2k+1}^*)} U_{\Phi(w_{2k+1})}(\Phi(w_{2k}^*)) \\ &= U_{\Phi(w_{2k+2})}(\Phi(w_{2k}^*)) = U_{\Phi(u_{k+1})}(\Phi(u_k)^*), \end{aligned}$$

where in the sixth equality we applied the identity (1).

It follows from the induction hypothesis that

$$U_{w_{2^n}}(w_0^*) = U_{u_{2^{n-1}}}(u_0^*) = u_{2^n} = w_{2^{n+1}}$$

and

$$\Phi(U_{w_{2^n}}(w_0^*)) = \Phi(U_{u_{2^{n-1}}}(u_0^*)) = U_{\Phi(u_{2^{n-1}})}(\Phi(u_0)^*) = U_{\Phi(w_{2^n})}(\Phi(w_0)^*),$$

which finishes the induction argument. \square

We are now ready to establish our first main result, which asserts that, under some mild conditions, for each surjective isometry Δ between the unitary sets of two unital JB^* -algebras M and N we can find a surjective real linear isometry $\Psi : M \rightarrow N$ which coincides with Δ on the subset $e^{iM_{sa}}$.

Theorem 3.4. *Let $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ be a surjective isometry, where M and N are two unital JB^* -algebras. Suppose that one of the following holds:*

- (1) $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$;
- (2) *There exists a unitary ω_0 in N such that $U_{\omega_0}(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$.*

Then there exists a unitary ω in N satisfying

$$\Delta(e^{iM_{sa}}) = U_{\omega^*}(e^{iN_{sa}}).$$

Furthermore, there exists a central projection $p \in N$ and a Jordan $$ -isomorphism $\Phi : M \rightarrow N$ such that*

$$\begin{aligned} \Delta(e^{ih}) &= U_{\omega^*}(p \circ \Phi(e^{ih})) + U_{\omega^*}((\mathbf{1}_N - p) \circ \Phi(e^{ih})^*) \\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi((e^{ih})^*)), \end{aligned}$$

for all $h \in M_{sa}$. Consequently, the restriction $\Delta|_{e^{iM_{sa}}}$ admits a (unique) extension to a surjective real linear isometry from M onto N .

Proof. If Δ satisfies (1), by Remark 3.2 (see also Lemma 2.2(a)) there exists a unitary ω in N such that $U_\omega(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$, and U_ω is an isometric Jordan $*$ -isomorphism from the $\Delta(\mathbf{1})$ -isotope $N(\Delta(\mathbf{1}))$ onto N . Since $\mathcal{U}(N, \circ_{\Delta(\mathbf{1}_M)}, *_{\Delta(\mathbf{1}_M)}) = \mathcal{U}(N)$ (see Lemma 2.1) we have $U_\omega(\mathcal{U}(N)) = \mathcal{U}(N)$. In case that (2) holds we take $\omega = \omega_0$.

The surjective isometry $\Delta_0 : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$, $\Delta_0(u) = U_\omega(\Delta(u))$, satisfies $\Delta_0(\mathbf{1}_M) = \mathbf{1}_N$, that is, Δ_0 is a unital surjective isometry between the unitary sets of M and N .

Fix $h \in M_{sa}$. We consider the continuous mapping $\mathbb{R} \rightarrow \mathcal{U}(M)$, $E_h(t) = e^{ith}$. Let u and v be two arbitrary unitary elements in $E_h(\mathbb{R})$ (that is, $u = e^{ith}$ and $v = e^{ish}$, for some $t, s \in \mathbb{R}$). Choose now a positive integer m such that $e^{\frac{\|i(s-t)h\|_M}{2^m}} - 1 < \frac{1}{2}$. It follows from this assumption that

$$(10) \quad \left\| e^{\frac{i(s-t)h}{2^m}} - \mathbf{1}_M \right\|_M \leq e^{\frac{\|i(s-t)h\|_M}{2^m}} - 1 < \frac{1}{2}.$$

Let us define the family $\{u_k : 0 \leq k \leq 2^{m+1}\}$, where

$$u_k = u \circ e^{\frac{ik(s-t)h}{2^m}}, \quad 0 \leq k \leq 2^{m+1}.$$

Since u and $e^{\frac{ik(s-t)h}{2^m}}$ operator commute in M , the element u_k is a unitary in M for every $k \in \{0, \dots, 2^{m+1}\}$. Clearly, $u_0 = u$, $u_{2^m} = v$ and $u_{2^{m+1}} = U_v(u^*)$.

Any two elements in the family $\{u_k\}_{k=0}^{2^{m+1}}$ operator commute, thus it is not hard to see that for any $0 \leq k \leq 2^{m+1} - 2$,

$$U_{u_{k+1}}(u_k^{-1}) = U_{u_{k+1}}(u_k^*) = u_{k+2}.$$

On the other hand, by our election of m in (10), we have

$$\begin{aligned} \|u_{k+1} - u_k\|_M &= \left\| u \circ e^{\frac{i(k+1)(s-t)h}{2^m}} - u \circ e^{\frac{ik(s-t)h}{2^m}} \right\|_M \\ &\leq \|u\|_M \left\| e^{\frac{ik(s-t)h}{2^m}} \circ e^{\frac{i(s-t)h}{2^m}} - e^{\frac{ik(s-t)h}{2^m}} \right\|_M \\ &= \left\| e^{\frac{ik(s-t)h}{2^m}} \circ \left(e^{\frac{i(s-t)h}{2^m}} - \mathbf{1}_M \right) \right\|_M \\ &\leq \left\| e^{\frac{ik(s-t)h}{2^m}} \right\|_M \left\| e^{\frac{i(s-t)h}{2^m}} - \mathbf{1}_M \right\|_M \leq e^{\frac{\|i(s-t)h\|_M}{2^m}} - 1 < \frac{1}{2}, \end{aligned}$$

for any $0 \leq k \leq 2^{m+1} - 1$. Theorem 2.9(3) affirms that the identity

$$\Delta(U_{u_{k+1}}(u_k^{-1})) = \Delta(U_{u_{k+1}}(u_k^*)) = U_{\Delta(u_{k+1})}(\Delta(u_k)^*) = U_{\Delta(u_{k+1})}(\Delta(u_k)^{-1}),$$

holds for every $0 \leq k \leq 2^{m+1} - 1$.

Lemma 3.3 applied to Δ and the family $\{u_k : 0 \leq k \leq 2^{m+1}\}$ proves that $U_{u_{2^m}}(u_0^*) = u_{2^{m+1}} = U_v(u^*)$ and

$$(11) \quad \Delta(U_v(u^*)) = \Delta(U_{u_{2^m}}(u_0^*)) = U_{\Delta(u_{2^m})}(\Delta(u_0)^*) = U_{\Delta(v)}(\Delta(u)^*),$$

for any u, v arbitrary elements in the one-parameter unitary group $\{E_h(t) : t \in \mathbb{R}\}$. By similar arguments applied to Δ_0 , or by applying U_ω at the previous identity, we deduce that

$$(12) \quad U_{\Delta_0(v)}(\Delta_0(u)^*) = \Delta_0(U_v(u^*)),$$

for every u, v in $\{E_h(t) : t \in \mathbb{R}\}$. Taking $v = \mathbf{1} = E_h(0)$ and u arbitrary in (12) and having in mind that $\Delta_0(\mathbf{1}) = \mathbf{1}$ we get

$$(13) \quad \Delta_0(u)^* = \Delta_0(u^*), \text{ for all } u \in \{E_h(t) : t \in \mathbb{R}\}.$$

Furthermore, for any two $u, v \in \{E_h(t) : t \in \mathbb{R}\}$ their adjointed u^*, v^* also lie in the set $\{E_h(t) : t \in \mathbb{R}\}$ and thus by (12) and (13) we derive

$$U_{\Delta_0(v)}(\Delta_0(u)) = U_{\Delta_0(v)}(\Delta_0(u^*))^* = \Delta_0(U_v(u^{**})) = \Delta_0(U_v(u)),$$

for every u, v in $\{E_h(t) : t \in \mathbb{R}\}$, that is,

$$U_{\Delta_0(E_h(t))}(\Delta_0(E_h(s))) = \Delta_0(U_{E_h(t)}(E_h(s))) = \Delta_0(E_h(2t + s)),$$

for all $t, s \in \mathbb{R}$.

By applying Theorem 3.1 to $\{\Delta_0(E_h(t)) : t \in \mathbb{R}\}$ we deduce the existence (as well as the uniqueness) of a self-adjoint element y in N such that $\Delta_0(E_h(t)) = e^{ity} \in \mathcal{U}(N)$ for every $t \in \mathbb{R}$.

We can therefore define a mapping $f : M_{sa} \rightarrow N_{sa}$ as the one which maps h into y , that is, $f(h) = y$ (where y is the unique element in N_{sa} such that $\Delta_0(E_h(t)) = e^{ity} \in \mathcal{U}(N)$ for every $t \in \mathbb{R}$). Thus, f satisfies

$$(14) \quad \Delta_0(e^{ith}) = e^{itf(h)},$$

for each $t \in \mathbb{R}$, and each $h \in M_{sa}$. We shall show that f is actually a surjective isometry.

Let us first observe that the injectivity of Δ_0 implies that f also is injective. On the other hand, replacing Δ_0 by Δ_0^{-1} in the previous arguments, we can deduce the existence of an injective mapping $g : N_{sa} \rightarrow M_{sa}$ such that

$$(15) \quad \Delta_0^{-1}(e^{ity}) = e^{itg(y)},$$

for any $y \in N_{sa}$, and any $t \in \mathbb{R}$. Therefore, by combining the properties of f and g , we derive that $y = f(g(y))$ ($y \in N_{sa}$), and hence f is surjective.

The bijectivity of f allows us to assure that Δ_0 maps any one-parameter unitary group $\{e^{ith}\}_{t \in \mathbb{R}}$ in $\mathcal{U}(M)$ (for any self-adjoint h in M) onto the one-parameter unitary group $\{e^{ity}\}_{t \in \mathbb{R}}$ in $\mathcal{U}(N)$, with $y \in N_{sa}$. Consequently, for any $y \in N_{sa}$, and any real number t ,

$$U_{\omega^*}(e^{ity}) = U_{\omega^*}(\Delta_0(e^{ith})) = U_{\omega^*}U_{\omega}(\Delta(e^{ith})) = \Delta(e^{ith}).$$

That proves the first statement, namely,

$$\Delta(e^{iM_{sa}}) = U_{\omega^*}(e^{iN_{sa}}).$$

Our next goal is to prove that f is an isometry. To this end, given $h, h' \in M_{sa}$ and a real number t , let us compute the following limits (with respect to the norm topology) as $t \rightarrow 0$:

$$\frac{e^{ith} - e^{ith'}}{t} = \frac{e^{ith} - \mathbf{1}_N}{t} - \frac{e^{ith'} - \mathbf{1}_N}{t} \longrightarrow ih - ih',$$

and

$$\frac{e^{itf(h)} - e^{itf(h')}}{t} = \frac{e^{itf(h)} - \mathbf{1}_N}{t} - \frac{e^{itf(h')} - \mathbf{1}_N}{t} \longrightarrow if(h) - if(h').$$

On the other hand, by (14), we have

$$\|e^{itf(h)} - e^{itf(h')}\|_N = \|\Delta_0(e^{ith}) - \Delta_0(e^{ith'})\|_N = \|e^{ith} - e^{ith'}\|_N.$$

Therefore, by uniqueness of the limits above, $\|f(h) - f(h')\|_N = \|h - h'\|_N$. The arbitrariness of h and h' in M_{sa} gives the expected conclusion, that is, $f : M_{sa} \rightarrow N_{sa}$ is a surjective isometry. Moreover, since $\Delta_0(\mathbf{1}_M) = \mathbf{1}_N$, we deduce that $f(0) = 0$, and hence the Mazur–Ulam theorem implies that f is a surjective real linear isometry from M_{sa} onto N_{sa} .

It is known since the times of Kaplansky that the self-adjoint part of any JB*-algebra is a JB-algebra. Thus, $f : M_{sa} \rightarrow N_{sa}$ can be regarded as a linear surjective isometry between JB-algebras. Theorem 1.4 and Corollary 1.11 in [29] guarantee the existence of a central symmetry $f(\mathbf{1}_M)$ in M_{sa} and a unital surjective linear isometry $\Phi : M \rightarrow N$ such that

$$(16) \quad f(h) = f(\mathbf{1}_M) \circ \Phi(h),$$

for every $h \in M_{sa}$. Therefore Φ is a unital triple isomorphism between unital JB*-algebras, and hence Φ must be a Jordan *-isomorphism (cf. [33, Proposition 5.5]).

By construction there exists a central projection p in N such that $f(\mathbf{1}_M) = 2p - \mathbf{1}_N = p - (\mathbf{1}_M - p)$, where p and $(\mathbf{1}_N - p)$ clearly are orthogonal projections in N , and for any $n > 0$,

$$(2p - \mathbf{1}_N)^n = (p - (\mathbf{1}_N - p))^n = p + (-1)^n(\mathbf{1}_N - p).$$

Finally, we shall describe Δ_0 in terms of p and Φ . To achieve this goal, we shall employ the equalities obtained in (14), (16), and the definition of the exponential in a Jordan algebra. According to this, given an arbitrary $h \in M_{sa}$, we have

$$\Delta_0(e^{ih}) = e^{if(h)} = e^{if(\mathbf{1}_M) \circ \Phi(h)} = e^{i(2p - \mathbf{1}_N) \circ \Phi(h)} = \sum_{n=0}^{\infty} \frac{(i(2p - \mathbf{1}_N) \circ \Phi(h))^n}{n!}.$$

Since $f(\mathbf{1}_N)$ (and hence p) is central, it operator commutes with any element in N . Additionally, Φ is a Jordan *-isomorphism, and we can thus conclude that

$$\begin{aligned} \Delta_0(e^{ih}) &= \sum_{n=0}^{\infty} \frac{i^n (2p - \mathbf{1}_N)^n \circ \Phi(h)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n (p + (-1)^n(\mathbf{1}_N - p)) \circ \Phi(h)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{i^n p \circ \Phi(h)^n}{n!} + \sum_{n=0}^{\infty} \frac{i^n (-1)^n (\mathbf{1}_N - p) \circ \Phi(h)^n}{n!} \\ &= p \circ \sum_{n=0}^{\infty} \frac{i^n \Phi(h)^n}{n!} + (\mathbf{1}_N - p) \circ \sum_{n=0}^{\infty} \frac{i^n (-1)^n \Phi(h)^n}{n!} \\ &= p \circ \Phi \left(\sum_{n=0}^{\infty} \frac{i^n h^n}{n!} \right) + (\mathbf{1}_N - p) \circ \Phi \left(\sum_{n=0}^{\infty} \frac{i^n (-1)^n h^n}{n!} \right) \\ &= p \circ \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ \Phi(e^{-ih}) = p \circ \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ \Phi(e^{ih})^*. \end{aligned}$$

The arbitrariness of the self-adjoint element h in M gives the following statement

$$\Delta_0(e^{ih}) = U_{\omega}(\Delta(e^{ih})) = p \circ \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ \Phi(e^{ih})^*, \quad (h \in M_{sa}),$$

and consequently,

$$\begin{aligned}
(17) \quad \Delta(e^{ih}) &= U_{\omega^*}(\Delta(e^{ih})) = U_{\omega^*}(p \circ \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ \Phi(e^{ih})^*) \\
&= U_{\omega^*}(P_2(p)\Phi(e^{ih})) + U_{\omega^*}(P_2(\mathbf{1}_N - p)\Phi((e^{ih})^*)) \\
&= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi((e^{ih})^*))
\end{aligned}$$

because U_{ω^*} is a triple isomorphism. \square

It should be remarked that the idea of employing one-parameter unitary groups was already employed by O. Hatori and L. Molnár in [27], where they were motivated by previous results on uniformly continuous group isomorphisms of unitary groups in AW*-factors due to Sakai (see [41]). In our proof this idea is combined with the Jordan version of Stone's one-parameter theorem developed in Theorem 3.1.

Corollary 3.5. *Let $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ be a surjective isometry, where M and N are two unital JB*-algebras. Then there exist a central projection p in the $\Delta(\mathbf{1}_M)$ -isotope $N(\Delta(\mathbf{1}_M))$ and an isometric Jordan *-isomorphism $\Phi : M \rightarrow N(\Delta(\mathbf{1}_M))$ such that*

$$\begin{aligned}
\Delta(e^{ih}) &= p \circ_{\Delta(\mathbf{1}_M)} \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ_{\Delta(\mathbf{1}_M)} \Phi(e^{ih})^{*\Delta(\mathbf{1}_M)} \\
&= p \circ_{\Delta(\mathbf{1}_M)} \Phi(e^{ih}) + (\mathbf{1}_N - p) \circ_{\Delta(\mathbf{1}_M)} \Phi((e^{ih})^*),
\end{aligned}$$

for all $h \in M_{sa}$.

Proof. The desired statement follows from Theorem 3.4 by just observing that $\Delta(\mathbf{1}_M)$ is the unit of the $\Delta(\mathbf{1}_M)$ -isotope $N(\Delta(\mathbf{1}_M))$, and $\mathcal{U}(N) = \mathcal{U}(N(\Delta(\mathbf{1}_M)))$ (cf. Lemma 2.1(b)). \square

Remark 3.6. *Let $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ be a surjective isometry, where M and N are two unital JB*-algebras. We have shown in the proof of Theorem 3.4 (see also Remark 3.2) that the assumption $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$ implies the existence of a unitary ω in N such that $U_{\omega}(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$. So, condition (1) seems stronger, but (2) is all what is needed in the proof of this theorem.*

Remark 3.7. *From the point of view of JB*-triples, the conclusion of the previous Theorem 3.4 can be also stated in the following terms: There exist two orthogonal tripotents u_1 and u_2 in M and two orthogonal tripotents \tilde{u}_1 and \tilde{u}_2 in N , a linear surjective isometry (i.e. triple isomorphism) $\Psi_1 : M_2(u_1) \rightarrow N_2(\tilde{u}_1)$ and a conjugate linear surjective isometry (i.e. triple isomorphism) $\Psi_2 : M_2(u_2) \rightarrow N_2(\tilde{u}_2)$ such that $M = M_2(u_1) \oplus^{\infty} M_2(u_2)$, $N = N_2(\tilde{u}_1) \oplus^{\infty} N_2(\tilde{u}_2)$, and the surjective real linear isometry $\Psi = \Psi_1 + \Psi_2 : M_2(u_1) \oplus^{\infty} M_2(u_2) \rightarrow N_2(\tilde{u}_1) \oplus^{\infty} N_2(\tilde{u}_2)$ restricted to $e^{iM_{sa}}$ coincides with Δ .*

To see this conclusion, we resume the proof of Theorem 3.4 from its final paragraph. Set $\tilde{\Psi} = U_{\omega^*}\Phi$, $\tilde{u}_1 = U_{\omega^*}(p)$, $\tilde{u}_2 = U_{\omega^*}(\mathbf{1}_N - p)$, $u_1 = \tilde{\Psi}^{-1}(\tilde{u}_1)$ and $u_2 = \tilde{\Psi}^{-1}(\tilde{u}_2)$. Since $N = N_2(p) \oplus^{\ell^{\infty}} N_2(\mathbf{1}_N - p)$, U_{ω^*} and $\tilde{\Psi}$ are triple isomorphisms, we deduce that $N = N_2(\tilde{u}_1) \oplus^{\infty} N_2(\tilde{u}_2)$ and $M = M_2(u_1) \oplus^{\infty} M_2(u_2)$. The identity in (17) actually proves that

$$\begin{aligned}
\Delta(e^{ih}) &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(e^{ih})) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi((e^{ih})^*)) \\
&= P_2(\tilde{u}_1)\tilde{\Psi}P_2(u_1)(e^{ih}) + P_2(\tilde{u}_2)\tilde{\Psi}P_2(u_2)((e^{ih})^*),
\end{aligned}$$

for all $h \in M_{sa}$. It can be easily checked that the maps $\Psi_1(x) = P_2(\tilde{u}_1)\tilde{\Psi}P_2(u_1)(x)$ and $\Psi_2(x) = P_2(\tilde{u}_2)\tilde{\Psi}P_2(u_2)(x^*)$ ($x \in M$) give the desired statement.

The next corollary asserts that the Banach spaces underlying two unital JB*-algebras are isometrically isomorphic if and only if the metric spaces determined by the unitary sets of these algebras are isometric.

Corollary 3.8. *Two unital JB*-algebras M and N are Jordan *-isomorphic if and only if there exists a surjective isometry $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ satisfying one of the following*

- (1) $\|\mathbf{1}_N - \Delta(\mathbf{1}_M)\| < 2$;
- (2) *There exists a unitary ω in N such that $U_\omega(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$.*

Furthermore, the following statements are equivalent for any two unital JB*-algebras M and N :

- (a) *M and N are isometrically isomorphic as (complex) Banach spaces;*
- (b) *M and N are isometrically isomorphic as real Banach spaces;*
- (c) *There exists a surjective isometry $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$.*

Proof. The first equivalence follows from Theorem 3.4.

We deal next with the second statement. The implication (a) \Rightarrow (b) is clear. (b) \Rightarrow (c) Suppose we can find a surjective real linear isometry $\Phi : M \rightarrow N$. It follows from [13, Corollary 3.2] (or even from [17, Corollary 3.4]) that Φ is a triple homomorphism, that is, Φ preserves triple products of the form $\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$. In particular Φ maps unitaries in M to unitaries in N , and hence $\Phi(\mathcal{U}(M)) = \mathcal{U}(N)$. Therefore $\Delta = \Phi|_{\mathcal{U}(M)} : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ is a surjective isometry.

The implication (c) \Rightarrow (a) is given by Corollary 3.5. \square

As we have seen in Remark 3.2(9) for each JBW*-algebra M the set of all unitaries in M is precisely the set $e^{iM_{sa}}$. We are now ready to establish the main result of this paper in which we relax some of the hypotheses in Theorem 3.4 at the cost of considering surjective isometries between the unitary sets of two JBW*-algebras.

Theorem 3.9. *Let $\Delta : \mathcal{U}(M) \rightarrow \mathcal{U}(N)$ be a surjective isometry, where M and N are two JBW*-algebras. Then there exist a unitary ω in N , a central projection $p \in N$, and a Jordan *-isomorphism $\Phi : M \rightarrow N$ such that*

$$\begin{aligned} \Delta(u) &= U_{\omega^*}(p \circ \Phi(u)) + U_{\omega^*}((\mathbf{1}_N - p) \circ \Phi(u)^*) \\ &= P_2(U_{\omega^*}(p))U_{\omega^*}(\Phi(u)) + P_2(U_{\omega^*}(\mathbf{1}_N - p))U_{\omega^*}(\Phi(u)^*), \end{aligned}$$

for all $u \in \mathcal{U}(M)$. Consequently, Δ admits a (unique) extension to a surjective real linear isometry from M onto N .

Proof. We only need to appeal to the identities $\mathcal{U}(M) = e^{iM_{sa}}$ and $\mathcal{U}(N) = e^{iN_{sa}}$, and to the arguments in Remark 3.2 to find a unitary $\omega \in N$ such that $U_\omega(\Delta(\mathbf{1}_M)) = \mathbf{1}_N$. The rest is a consequence of Theorem 3.4. \square

Acknowledgements Authors partially supported by the Spanish Ministry of Science, Innovation and Universities (MICINN) and European Regional Development Fund project no. PGC2018-093332-B-I00, Programa Operativo FEDER 2014-2020 and Consejería de Economía y Conocimiento de la Junta de Andalucía grant number A-FQM-242-UGR18, and Junta de Andalucía grant FQM375.

REFERENCES

- [1] T.J. Barton, Y. Friedman, Bounded derivations of JB^* -triples, *Quart. J. Math. Oxford Ser.* **41** (2), 255–268 (1990).
- [2] J. Becerra Guerrero, M. Cueto-Avellaneda, F.J. Fernández-Polo, A.M. Peralta, On the extension of isometries between the unit spheres of a JBW^* -triple and a Banach space, to appear in *J. Inst. Math. Jussieu*. doi:10.1017/S1474748019000173 arXiv: 1808.01460
- [3] R. Braun, W. Kaup, H. Upmeyer, A holomorphic characterisation of Jordan- C^* -algebras, *Math. Z.* **161**, 277–290 (1978).
- [4] O. Bratteli, D.W. Robinson, *Operator algebras and quantum statistical mechanics. 1. C^* - and W^* -algebras, symmetry groups, decomposition of states*. Second edition. Texts and Monographs in Physics. Springer-Verlag, New York, 1987.
- [5] J. Cabello-Sánchez, A reflection on Tingley’s problem and some applications, *J. Math. Anal. Appl.* **476** (2), 319–336 (2019).
- [6] M. Cabrera García, A. Rodríguez Palacios, *Non-associative normed algebras. Vol. 1*, vol. 154 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2014. The Vidav-Palmer and Gelfand-Naimark theorems.
- [7] M. Cabrera García, A. Rodríguez Palacios, *Non-associative normed algebras. Vol. 2*, vol. 167 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2018. Representation theory and the Zel’manov approach.
- [8] L. Cheng, Y. Dong, On a generalized Mazur–Ulam question: extension of isometries between unit spheres of Banach spaces, *J. Math. Anal. Appl.* **377**, 464–470 (2011).
- [9] J.B. Conway, *A Course in Functional Analysis*, Springer-Verlag, 1990.
- [10] M. Cueto-Avellaneda, A.M. Peralta, The Mazur–Ulam property for commutative von Neumann algebras, *Linear and Multilinear Algebra* **68**, no. 2, 337–362 (2020).
- [11] M. Cueto-Avellaneda, A.M. Peralta, On the Mazur–Ulam property for the space of Hilbert-space-valued continuous functions, *J. Math. Anal. Appl.* **479**, no. 1, 875–902 (2019).
- [12] M. Cueto-Avellaneda, A.M. Peralta, Metric characterisation of unitaries in JB^* -algebras, to appear in *Mediterranean Journal of Mathematics*, arXiv:1907.04738.
- [13] T. Dang, Real isometries between JB^* -triples, *Proc. Amer. Math. Soc.* **114**, 971–980 (1992).
- [14] H.A. Dye, On the geometry of projections in certain operator algebras, *Ann. of Math. (2)* **61**, 73–89 (1955).
- [15] F.J. Fernández-Polo, J.J. Garcés, A.M. Peralta, I. Villanueva, Tingley’s problem for spaces of trace class operators, *Linear Algebra Appl.* **529**, 294–323 (2017).
- [16] F.J. Fernández-Polo, E. Jordá, A.M. Peralta, Tingley’s problem for p -Schatten von Neumann classes, to appear in *J. Spectr. Theory*. arXiv:1803.00763
- [17] F.J. Fernández-Polo, J. Martínez, A.M. Peralta, Surjective isometries between real JB^* -triples, *Math. Proc. Cambridge Phil. Soc.*, **137** 709–723 (2004).
- [18] F.J. Fernández-Polo, A.M. Peralta, Tingley’s problem through the facial structure of an atomic JBW^* -triple, *J. Math. Anal. Appl.* **455**, 750–760 (2017).
- [19] F.J. Fernández-Polo, A.M. Peralta, On the extension of isometries between the unit spheres of von Neumann algebras, *J. Math. Anal. Appl.* **466**, 127–143 (2018).
- [20] F.J. Fernández-Polo, A.M. Peralta, Low rank compact operators and Tingley’s problem, *Adv. Math.* **338**, 1–40 (2018).
- [21] Y. Friedman, B. Russo, Structure of the predual of a JBW^* -triple, *J. Reine Angew. Math.*, **356**, 67–89 (1985).
- [22] Y. Friedman, B. Russo, The Gelfand–Naimark theorem for JB^* -triples, *Duke Math. J.*, **53**, 139–148 (1986).
- [23] J.J. Garcés, A.M. Peralta, One-parameter groups of orthogonality preservers, preprint 2020, arXiv:2004.04155v1.
- [24] H. Hanche-Olsen, E. Størmer, *Jordan Operator Algebras*, Pitman, London, 1984.

- [25] O. Hatori, G. Hirasawa, T. Miura, L. Molnár, Isometries and maps compatible with inverted Jordan triple products on groups, *Tokyo J. Math.* **35**, no. 2, 385–410 (2012).
- [26] O. Hatori, L. Molnár, Isometries of the unitary group, *Proc. Amer. Math. Soc.* **140**, no. 6, 2127–2140 (2012).
- [27] O. Hatori, L. Molnár, Isometries of the unitary groups and Thompson isometries of the spaces of invertible positive elements in C^* -algebras, *J. Math. Anal. Appl.* **409**, 158–167 (2014).
- [28] T. Ho, J. Martinez-Moreno, A.M. Peralta, B. Russo, Derivations on real and complex JB^* -triples, *J. London Math. Soc.* (2) **65**, no. 1, 85–102 (2002).
- [29] J.M. Isidro, A. Rodríguez-Palacios, Isometries of JB -algebras, *Manuscripta Math.* **86**, 337–348 (1995).
- [30] A. Jiménez-Vargas, A. Morales-Campoy, A.M. Peralta, M.I. Ramírez, The Mazur–Ulam property for the space of complex null sequences, *Linear and Multilinear Algebra* **67**, no. 4, 799–816 (2019).
- [31] R.V. Kadison, J.R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. I*, vol. 100 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983. Elementary theory.
- [32] O.F.K. Kalenda, A.M. Peralta, Extension of isometries from the unit sphere of a rank-2 Cartan factor, preprint 2019. arXiv:1907.00575
- [33] W. Kaup, A Riemann mapping theorem for bounded symmetric domains in complex Banach spaces, *Math. Z.* **183**, 503–529 (1983).
- [34] P. Mankiewicz, On extension of isometries in normed linear spaces, *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys.* **20**, 367–371 (1972).
- [35] M. Mori, Tingley’s problem through the facial structure of operator algebras, *J. Math. Anal. Appl.* **466**, no. 2, 1281–1298 (2018).
- [36] M. Mori, N. Ozawa, Mankiewicz’s theorem and the Mazur–Ulam property for C^* -algebras, *Studia Math.* **250**, no. 3, 265–281 (2020).
- [37] A.M. Peralta, A survey on Tingley’s problem for operator algebras, *Acta Sci. Math. (Szeged)* **84**, 81–123 (2018).
- [38] A.M. Peralta, Extending surjective isometries defined on the unit sphere of $\ell_\infty(\Gamma)$, *Rev. Mat. Complut.* **32**, no. 1, 99–114 (2019).
- [39] A.M. Peralta, R. Tanaka, A solution to Tingley’s problem for isometries between the unit spheres of compact C^* -algebras and JB^* -triples, *Sci. China Math.* **62**, no. 3, 553–568 (2019).
- [40] B. Russo, H.A. Dye, A note on unitary operators in C^* -algebras, *Duke Math. J.* **33**, 413–416 (1966).
- [41] S. Sakai, On the group isomorphism of unitary groups in AW^* -algebras, *Tohoku Math. J.* **7**, 87–95 (1955).
- [42] D. Tingley, Isometries of the unit sphere, *Geom. Dedicata* **22**, 371–378 (1987).
- [43] R. Wang, X. Huang, The Mazur–Ulam property for two-dimensional somewhere-flat spaces, *Linear Algebra Appl.* **562**, 55–62 (2019).
- [44] J.D.M. Wright, Jordan C^* -algebras, *Michigan Math. J.* **24** 291–302 (1977).
- [45] J.D.M. Wright, M.A. Youngson, A Russo–Dye theorem for Jordan C^* -algebras. In *Functional analysis: surveys and recent results (Proc. Conf., Paderborn, 1976)*, pp. 279–282. North-Holland Math. Studies, Vol. 27; Notas de Mat., No. 63, North-Holland, Amsterdam, 1977.
- [46] J.D.M. Wright, M.A. Youngson, On isometries of Jordan algebras, *J. London Math. Soc.* **17**, 339–44 (1978).
- [47] X. Yang, X. Zhao, On the extension problems of isometric and nonexpansive mappings. In: *Mathematics without boundaries*. Edited by Themistocles M. Rassias and Panos M. Pardalos. 725–748, Springer, New York, 2014.
- [48] M.A. Youngson, A Vidav theorem for Banach Jordan algebras, *Math. Proc. Cambridge Philos. Soc.* **84**, 2, 263–272 (1978).

(M. Cueto-Avellaneda, A.M. Peralta) DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, FACULTAD DE CIENCIAS, UNIVERSIDAD DE GRANADA, 18071 GRANADA, SPAIN.

E-mail address: mcueto@ugr.es, aperalta@ugr.es