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ON BOUNDARY NULL CONTROLLABILITY OF STRONGLY DEGENERATE HYPERBOLIC SYSTEMS ON STAR-SHAPED PLANAR NETWORK

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Abstract. In this paper we discuss the problem of boundary exact null controllability for weakly and strongly degenerate linear wave equation defined on star-shaped planar network. The network is represented by a singular measure in a bounded planar domain. The novelty of this article lies in the degeneration of the leading coefficient representative of the material properties at the common node of network. We discuss the existence of weak and strong solutions to the degenerate hyperbolic problem and establish the corresponding controllability properties.

Key words: Degenerate hyperbolic equation, weighted Sobolev spaces, planar network, singular measures, boundary controllability..

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1. Introduction

In this paper we focus on the study of boundary exact null controllability for the simplified 1-d hyperbolic model of a multi-body structure consisting of a finite number of flexible strings distributed along a star-shaped network with a defect at the common node. The practical relevance of such systems is obvious. Observation and control of systems exhibiting damage is a major desire for a control theory aiming at sustainability. See [12], where a quasistatic evolution of damage has been considered in the context of optimal control. Coupled systems and in particular flexible multi-body systems suffer from defects at the coupling interfaces. On the other side, the concept of degeneration may be used as a constructive element that comes into play when properties of meta-materials are in the focus, like in cloaking, see e.g. [11].

This article is the first attempt to investigate this class of problems from a mathematical control perspective in the context of partial differential equations on metric graphs.

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Wave equations on metric graphs were the subject of extensive investigations of many mathematicians since the early 80's (see, e.g., [7, 16, 28] and references therein). In spite of the fact that there is an extensive literature on this topic as far as well-posedness, controllability, optimal control, stabilizability and other related problems as domain decomposition are concerned, problems related to material deterioration or damage have received little attention so far.

While parabolic controllability problems with degeneration have been discussed in detail in e.g. [2], wave equations with degeneration in the leading coefficients are much less explored, see however [3, 10] for problems in which damage occurs at the boundary of the domain, and [15] where the 'damaged' point is internal. Very recently, in-span damage has been considered for the 1-d parabolic equation in [4]. In this work, we embark on degeneration at junction points for 1-d wave equations. Junctions are seen as building blocks for general networks. Degeneration occurring in general networks — also for more general structural elements — is then the subject of forthcoming publications.

To be more specific, the main issue in this article concerns the question how the defect at the node $x = 0$ affects the solution of the system (1.4)–(1.6) and its controllability properties. We also refer to the recent article [14], where the same authors use the representation of the metric graph by singular measures and provide a qualitative analysis of the boundary observability problem. In contrast to [14], however, in this paper we, mainly focus on the controllability issues and on a complete well-posedness analysis of the degenerate initial boundary value problem, based on the method of transposition. Furthermore, we provide more general assumptions on the admissible weight function $a(\cdot)$ and the initial state (y_0, y_1).

Of course, in particular 1-d boundary value problems for degenerate elliptic and parabolic equations per se have received a lot of attention in the recent years (see, for instance, [5, 6, 21–23]). Also, sensitivity results for systems and also for optimal control problems with respect to defects in the domain belong to this cycle of ideas.

Typically, one differentiates between with weak and strong degeneration, characterized by the properties of the coefficient $a(x)$ in the neighbourhood of the defect, e.g. by the parameter η_a in (1.7). While the effect of weak degeneration ($\eta_a \in [0, 1)$) preserves the controllability properties in principle, the effect of strong degeneration ($\eta_a \in [1, 2)$) on the controllability properties is, in fact, quite significant, as, loosely speaking, quadratic degeneration leads to a lack of controllability or observability with the corresponding control or observation times exhibiting blow up as η_a tends towards 2.

While there are many topics to be explored in this context, in this article we provide a qualitative analysis of system (1.4)–(1.6), prove a boundary observability result, and find out how the degree of degeneracy in the principle coefficient $a(x)$ affects the controllability time and the solutions of system (1.4)–(1.6).

Let Ω be an open bounded set in $\mathbb{R}^2$ such that 0 is an interior point and Ω has a sufficiently smooth boundary $\partial\Omega$. Let $I_1, I_2, \ldots, I_N$ be a collection of segments starting at the origin and directed along the vectors $v_1, v_2, \ldots, v_N$. Hereinafter we suppose that

$$\frac{v_i}{|v_i|} \neq \frac{v_j}{|v_j|} \quad \text{for} \ \ i \neq j, \quad I_i \subset \Omega \ \forall\, i = 1, \ldots, N,$$

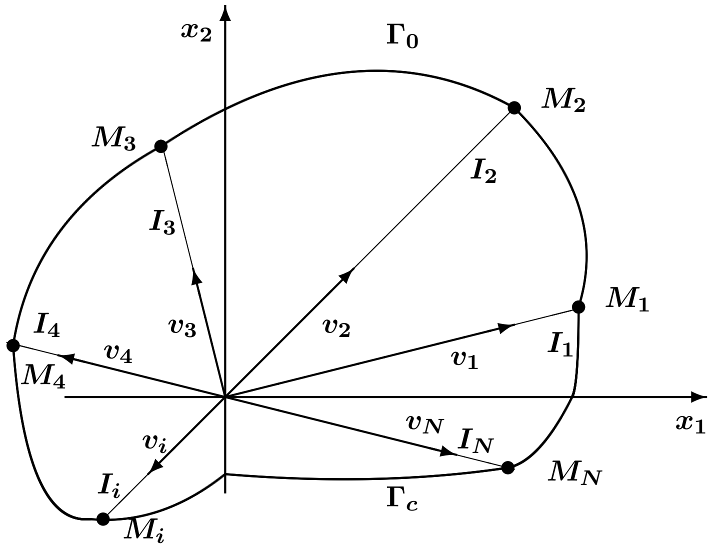
and all the end points of these segments $K = \{M_i, \ i = 1, \ldots, N\}$ belong to the boundary, $K \subset \partial\Omega$ (see the picture below).

The length of the segment I_i is denoted by ℓ_i. In the sequel, we call the object

$$G = \big\langle \{I_1, \ldots, I_N\}, K \cup \{0\} \big\rangle$$

a star-shaped planar network. On each interval (edge of the network) we choose an orientation in accordance to the direction of the vectors v_i. As a result, I_i can be parametrized as a function of its length by mean of the function $z_i : [0, \ell_i] \to I_i$, i.e.,

$$z_i(\xi) = \xi \frac{v_i}{|v_i|_{\mathbb{R}^2}}, \quad \forall\, \xi \in [0, \ell_i], \quad |z_i(\xi)| = \sqrt{z_{i,1}^2 + z_{i,2}^2} = \xi, \quad \text{and} \quad z_i(\ell_i) = M_i.$$



Picture 1. Example of a star-shaped planar network

Let $a : \Omega \to \mathbb{R}$ be a given function. We say that $a : \Omega \to \mathbb{R}$ is an admissible weight function if:

(i) $a \in C(\overline{\Omega}) \cap C^1(\overline{\Omega} \setminus \{0\})$;

(ii) $a(0) = 0$ and $a(x) > 0$ for each $x \in \overline{\Omega} \setminus \{0\}$;

(iii) There exists an open convex neighbourhood $\mathcal{U} \subset \Omega$ of the origin in $\mathbb{R}^2$ such that

$$(\nabla a(x), x)_{\mathbb{R}^2} > 0, \quad \forall\, x \in (I_i \cap \mathcal{U}) \setminus \{0\}, \tag{1.1}$$

$$\eta_{i,a} := \sup_{x \in I_i \cap \mathcal{U}} \frac{(\nabla a(x), x)_{\mathbb{R}^2}}{a(x)} = \lim_{\substack{x \to 0 \\ x \in I_i \cap \mathcal{U}}} \frac{(\nabla a(x), x)_{\mathbb{R}^2}}{a(x)} < 2, \tag{1.2}$$

$$a\left(\cdot \frac{v_i}{|v_i|}\right) \in C^{[\eta_{i,a}]}\left([0, \ell_i]\right) \tag{1.3}$$

for each $i \in \{1, \ldots, N\}$, where $[\cdot]$ stands for the integer part.

For further convenience, we denote the points where the intervals I_i intersect the boundary of the neighbourhood $\mathcal{U}$ by L_i for each $i = 1, \ldots, N$. So, $I_i \cap \partial\mathcal{U} = \{L_i\}$ and, hence, L_i can be represented as follows $L_i = \ell_i^* \frac{v_i}{|v_i|}$ for some $\ell_i^* \in (0, \ell_i]$. Moreover, if $a : \Omega \to \mathbb{R}$ is an admissible weight function, then relation (1.1) implies that, for each $i \in \{1, \ldots, N\}$, the mapping $x \mapsto a\,(x)$ is monotonically increasing along the interval $(I_i \cap \mathcal{U}) \setminus \{0\}$.

In the sequel, we will use the notation $|v_i|$ and (a, b) instead of $|v_i|_{\mathbb{R}^2}$ and $(a, b)_{\mathbb{R}^2}$, respectively.

With each segment I_i we associate a singular measure μ_i concentrated on it, where it is assumed that μ_i is uniformly distributed on I_i and coincides with Lebesgue measure $\mathcal{L}^1$. Setting $d\mu = \sum_{i=1}^N d\mu_i$, we see that μ is a singular measure with respect to the Lebesgue measure $\mathcal{L}^2$, and $\mu\left(\Omega \setminus \cup_{i=1}^N I_i\right) = 0$. Therefore, any functions $f = f(x)$ and $g = g(x)$ taking the same values on the planar network G coincide as elements of $L^1(\Omega, d\mu)$, provided they have finite norms in this Lebesgue space.

The purpose of this article is to discuss the well-posedness and boundary controllability of the following Cauchy-Dirichlet problem

$$u_{tt} - \operatorname{div}^\mu(a \nabla^\mu u) = 0 \quad \text{in } (0, \infty) \times \Omega, \tag{1.4}$$

$$u(t, M_i) = f_i(t) \quad \text{for a.a. } t \in (0, \infty) \text{ and } i = 1, \ldots, N, \tag{1.5}$$

$$u(0, x) = y_0(x), \quad u_t(0, x) = y_1(x) \quad \text{for } \mu\text{-a.a. } x \in \Omega, \tag{1.6}$$

where (y_0, y_1) is a given initial state, $a : \Omega \to \mathbb{R}$ is a weight function with properties (i)–(iii), and $f_i \in L^2_{loc}(0, \infty)$, $i = 1, \ldots, N$, are control functions. Here, $\nabla^\mu u$ stands for some μ-gradient of the function u, and div^μ is the divergence operator with respect to the singular measure μ. The precise definition of these notions are given in Section 3.

So, we deal with a hyperbolic system on the star-shaped planar network with boundary sources, f_i, acting on the system as controls through the Dirichlet boundary conditions at the nodes $x = M_i$. In contrast to the standard case that is widely studied in the literature (see, for instance, [7, 16, 18]), we assume that the planar network has a defect at the common node $x = 0$. Loosely speaking, since $a(0) = 0$, it means that all considered strings lose their elasticity property at $x = 0$.

Arguing in the spirit of paper [3], the degeneracy of the hyperbolic system (1.4) at the central node $x = 0$ is proposed to be measured by the parameter η_a which is defined as

$$\eta_a = \max_{i\in\{1,\dots,N\}} \eta_{i,a} = \max_{i\in\{1,\dots,N\}} \left[\sup_{x\in I_i\cap\mathcal{U}} \frac{(\nabla a(x), x)}{a(x)} \right] \tag{1.7}$$

$$= \max_{i\in\{1,\dots,N\}} \left[\lim_{\xi\to+0} \frac{\xi}{a\big(\xi\frac{v_i}{|v_i|}\big)} \frac{d}{d\xi} a\big(\xi\frac{v_i}{|v_i|}\big) \right],$$

where $\frac{(\nabla a(x), v_i)_{\mathbb{R}^2}}{|v_i|_{\mathbb{R}^2}} = \frac{da}{dv_i}$ is the directional derivative of $a = a(x)$ along the vector v_i.

We say that (1.4) is weakly degenerate if $\eta_a \in [0, 1)$ and this system degenerates strongly if $\eta_a \in [1, 2)$. We show that the boundary observability and null controllability properties no longer hold true if $\eta_a \geqslant 2$. In particular, we are faced with a blow-up of the observability time when η_a converges to 2 from below.

With that in mind, in Section 2 we introduce a special classes of weighted Sobolev spaces that are strictly associated with the original initial-boundary value problem. It allows us to study in details some special properties of their elements in the regions which are in close vicinity to the 'damaged' point. In Section 3, we mainly focus on the description of the Sobolev spaces with respect to the singular measure μ. We study their main properties and show how these are related to the geometry of the star-shaped network. Section 4 is devoted to well-posedness of the controlled system. In Section 5, we derive the observability inequality which guarantees that the total energy of the weak solutions of degenerate wave equations on a planar network can be 'observed' from the boundary measurement at the nodes $\{x = M_i\}_{i=1}^N$. The last Section 6, we discuss the questions of exact and null boundary controllability of the original degenerate system and the lack of these properties for the strong degeneration case.

2. Preliminaries

We begin with some auxiliary results which are the direct consequence of properties (i)–(iii) of the weight function $a(\cdot)$.

Proposition 2.1. Let $a : \Omega \to \mathbb{R}$ be an admissible weight function. Then

$$\frac{1}{a} \in L^1(\Omega, d\mu) \quad \text{provided } \eta_{i,a} \in [0, 1) \text{ for all } i = 1, \dots, N; \tag{2.1}$$

$$\frac{1}{a} \notin L^1(\Omega, d\mu) \quad \text{if } \exists\, i_0 \in \{1, \dots, N\} \text{ such that } \eta_{i,a} \in [1, 2); \tag{2.2}$$

$$a\left(\xi\frac{v_i}{|v_i|}\right) \geqslant a(L_i)\left(\frac{\xi}{\ell_i^*}\right)^{\eta_{i,a}}, \quad \forall\, \xi \in [0, \ell_i^*],\ \forall\, i = 1, \dots, N. \tag{2.3}$$

Proof. In view of property (iii), for a fixed $i \in \{1, \dots, N\}$, we have

$$(\nabla a(x), x) = |x|\frac{(\nabla a(x), v_i)}{|v_i|} = |x|\frac{da(x)}{dv_i} \leqslant \eta_{i,a} a(x), \quad \forall\, x \in I_i \cap \mathcal{U}. \tag{2.4}$$

Since, for each $x \in I_i \cap \mathcal{U}$, there is $\xi \in [0, \ell_i^*]$ such that $x = \xi \frac{v_i}{|v_i|}$, it follows from (2.4) that

$$\xi \frac{da(x)}{d\xi} \leqslant \eta_{i,a} a(x), \quad x = \xi \frac{v_i}{|v_i|}, \quad \forall\, \xi \in [0, \ell_i^*]$$

Then, after integration of this inequality over $[\xi, \ell_i^*]$, in view of property (iii), we obtain

$$\frac{1}{a\left(\xi \frac{v_i}{|v_i|}\right)} \leqslant \frac{1}{a\left(L_i\right)} \left(\frac{\ell_i^*}{\xi}\right)^{\eta_{i,a}}, \quad \forall\, \xi \in [0, \ell_i^*]. \tag{2.5}$$

where $L_i = \ell_i^* \frac{v_i}{|v_i|}$. From this (2.3) follows. Moreover, taking into account that

$$\left\|\frac{1}{a}\right\|_{L^1(\Omega, d\mu)} = \sum_{i=1}^{N} \int_0^{\ell_i} \left[a\left(\xi \frac{v_i}{|v_i|}\right)\right]^{-1} d\xi,$$

it follows from (2.5) that $\frac{1}{a} \in L^1(\Omega, d\mu)$ provided $\max_{1 \leqslant i \leqslant N} \eta_{i,a} \in [0, 1)$.

It remains to prove property (2.2). Assume that, for a given $i \in \{1, \dots, N\}$, we have $\eta_{i,a} \in [1, 2)$. Using the fact that $a\left(\cdot \frac{v_i}{|v_i|}\right) \in C^1\left([0, \ell_i]\right)$ (see (1.3)), and $a : \Omega \to \mathbb{R}$ satisfies monotonicity condition (1.1) and relation (2.3), and

$$\eta_{i,a} := \sup_{x \in I_i \cap \mathcal{U}} \frac{|x| \left(\nabla a(x), v_i\right)}{a(x)|v_i|} = \sup_{\xi \in [0, \ell_i^*]} \frac{\xi \frac{d}{d\xi} a\left(\xi \frac{v_i}{|v_i|}\right)}{a\left(\xi \frac{v_i}{|v_i|}\right)} = \lim_{\xi \to +0} \xi \frac{\frac{d}{d\xi} a\left(\xi \frac{v_i}{|v_i|}\right)}{a\left(\xi \frac{v_i}{|v_i|}\right)} \geqslant 1,$$

we get

$$\xi \frac{\|\frac{d}{d\xi} a\left(\cdot \frac{v_i}{|v_i|}\right)\|_{C([0,\ell_i])} + 1}{a\left(\xi \frac{v_i}{|v_i|}\right)} > 1 \quad \text{for } \xi \in (0, \ell_i] \text{ small enough.}$$

Hence, for the given range of ξ, we have $\frac{1}{a\left(\xi \frac{v_i}{|v_i|}\right)} \geqslant \frac{\text{const}}{\xi}$. Hence, $1/a \notin L^1(\Omega, d\mu_i)$. The proof is complete. □

We now introduce some weighted Sobolev spaces that are naturally associated with the restrictions of the weight function $a : \overline{\Omega} \to \mathbb{R}$ along the intervals I_i, $i = 1, \dots, N$, and with the corresponding degenerate elliptic operators (see, for instance, [3,21]). For each $i \in \{1, \dots, N\}$ and a smooth u, we define the functional $\|\cdot\|_{a,I_i}$ as follows

$$\|u\|_{a,I_i} = \left(\int_0^{\ell_i} \left[u^2(s) + a\left(s \frac{v_i}{|v_i|}\right) |u'(s)|^2\right] ds\right)^{1/2}.$$

Let $W^{1,2}(0, \ell_i)$ be the standard Sobolev space. We denote by $H_a^1(0, \ell_i)$, $H_{a,0}^1(0, \ell_i)$, and $W_a^1(0, \ell_i)$ the spaces which are defined as follows:

$H_a^1(0,\ell_i)$ is the closure of the set $\varphi \in C_0^\infty(\mathbb{R})$ with respect to the $\|\cdot\|_{a,I_i}$-norm;

$H_{a,0}^1(0,\ell_i)$ is the closure of the set $\{\varphi \in C^\infty(\mathbb{R}) \ : \ \varphi(\ell_i)=0\}$ with respect to the $\|\cdot\|_{a,I_i}$-norm;

$W_a^1(0,\ell_i)$ is the space of functions $u \in L^2(0,\ell_i)$ with distributional derivatives u' that satisfy $u' \in L^2(0,\ell_i; a\left(s\frac{v_i}{|v_i|}\right) ds) \cap L^1(0,\ell_i)$, where

$$L^2(0,\ell_i; a\left(s\frac{v_i}{|v_i|}\right) ds) = \left\{ v:(0,\ell_i)\to\mathbb{R} \ : \ \begin{array}{c} v \ \text{ is measurable and} \\ \displaystyle\int_0^{\ell_i} a\left(s\frac{v_i}{|v_i|}\right) v^2(s)\,ds < +\infty \end{array} \right\}.$$

First note that in view of the inclusion $\{\varphi \in C^\infty(\mathbb{R}) \ : \ \varphi(\ell_i)=0\} \subset C^\infty([0,\ell_i])$, we have: $H_{a,0}^1(0,\ell_i) \subset H_a^1(0,\ell_i)$. Moreover, due to compactness of the embedding

$$H_a^1(\varepsilon,\ell_i) \hookrightarrow C^{0,1}([\varepsilon,\ell_i]), \qquad \text{for all } \varepsilon \in (0,\ell_i),$$

we see that, if $u \in H_a^1(0,\ell_i)$, then $u(\cdot)$ is an absolutely continuous function in $(0,\ell_i]$. So, the condition $u(\ell_i)=0$ is consistent for all $u \in H_{a,0}^1(0,\ell_i)$. Therefore, $H_{a,0}^1(0,\ell_i)$ can be equivalently defined as the closed subspace of $H_a^1(0,\ell_i)$ such that

$$H_{a,0}^1(0,\ell_i) := \left\{ u \in H_a^1(0,\ell_i) \ : \ u(\ell_i)=0 \right\}.$$

It is easy to see that $H_a^1(0,\ell_i)$ is a Hilbert space with the scalar product

$$\langle \varphi,\psi\rangle_{H_a^1(0,\ell_i)} = \int_0^{\ell_i} \left[a\left(s\frac{v_i}{|v_i|}\right) \varphi'(s)\psi'(s) + \varphi(s)\psi(s) \right] ds, \quad \forall\, \varphi,\psi \in H_a^1(0,\ell_i)$$

and associated norm

$$\|\varphi\|_{H_a^1(0,\ell_i)} = \left(\int_0^{\ell_i} \left[a\left(s\frac{v_i}{|v_i|}\right) |\varphi'(s)|^2 + |\varphi(s)|^2 \right] ds \right)^{\frac{1}{2}}, \quad \forall\, \varphi \in H_a^1(0,\ell_i).$$

Let us show that $H_{a,0}^1(0,\ell_i)$ is a Banach space with respect to the norm

$$\|\varphi\|_{H_{a,0}^1(0,\ell_i)} = \left(\int_0^{\ell_i} a\left(s\frac{v_i}{|v_i|}\right) |\varphi'(s)|^2\,ds \right)^{1/2}. \tag{2.6}$$

The fact that (2.6) defines an equivalent norm on $H_{a,0}^1(0,\ell_i)$ is a simple consequence of the following version of Friedrichs's inequality.

Proposition 2.2. Let $a:\Omega\to\mathbb{R}$ be a given weight function. Then

$$\|\varphi\|_{L^2(0,\ell_i)} \leqslant C_{i,a} \|\varphi\|_{H_{a,0}^1(0,\ell_i)}, \quad \forall\, \varphi \in H_{a,0}^1(0,\ell_i), \quad \forall\, i=1,\dots,N, \tag{2.7}$$

where

$$C_{i,a}^2 = \min\left\{ \frac{(\ell_i^*)^2}{a(L_i)\,(2-\eta_{i,a})} + \frac{\ell_i^2 - (\ell_i^*)^2}{2 \min\limits_{x\in[L_i,M_i]} a(x)}, 4\max\left\{ \frac{(\ell_i^*)^{\eta_{i,a}}}{a(L_i)}, \frac{\ell_i^2}{\min\limits_{x\in[L_i,M_i]} a(x)} \right\}\right\}. \tag{2.8}$$

Proof. Let φ be an arbitrary element of $H^1_{a,0}(0,\ell_i)$. Arguing as in [3], we will prove two different bounds for $\|u\|^2_{L^2(0,\ell_i)}$ in terms of $\|u\|_{H^1_{a,0}(0,\ell_i)}$. Then the conclusion (2.8) will follow by taking the minimum of the two corresponding constants.

Fixing an arbitrary $s \in (0,\ell_i)$, we begin with the following chain of relations

$$\begin{aligned}|\varphi(s)| &= \Big| \int_s^{\ell_i} \varphi'(\xi)\,d\xi \Big| = \Big| \int_s^{\ell_i} \sqrt{a\Big(\xi\frac{v_i}{|v_i|}\Big)}\varphi'(\xi)\frac{1}{\sqrt{a\Big(\xi\frac{v_i}{|v_i|}\Big)}}\,d\xi \Big| \\ &\leqslant \|\varphi\|_{H^1_{a,0}(0,\ell_i)} \Big(\int_s^{\ell_i} \frac{d\xi}{a\Big(\xi\frac{v_i}{|v_i|}\Big)} \Big)^{\frac12}.\end{aligned}$$

From this and estimate (2.3), by Fubini's theorem, we obtain the first bound that was mentioned above

$$\begin{aligned}\|u\|^2_{L^2(0,\ell_i)} &\leqslant \|u\|^2_{H^1_{a,0}(0,\ell_i)} \int_0^{\ell_i}\int_s^{\ell_i} \frac{d\xi}{a\left(\xi\frac{v_i}{|v_i|}\right)}\,ds \\ &= \|u\|^2_{H^1_{a,0}(0,\ell_i)} \int_0^{\ell_i}\int_0^{\xi} ds \frac{1}{a\left(\xi\frac{v_i}{|v_i|}\right)}\,d\xi \\ &= \|u\|^2_{H^1_{a,0}(0,\ell_i)} \left(\int_0^{\ell_i^*} \frac{\xi}{a\left(\xi\frac{v_i}{|v_i|}\right)}\,d\xi + \int_{\ell_i^*}^{\ell_i} \frac{\xi}{a\left(\xi\frac{v_i}{|v_i|}\right)}\,d\xi \right) \\ &\overset{\text{by (2.3)}}{\leqslant} \|u\|^2_{H^1_a(0,\ell_i)} \left[\frac{1}{a(L_i)} \int_0^{\ell_i^*} \xi^{1-\eta_{i,a}} (\ell_i^*)^{\eta_{i,a}}\,ds + \frac{\ell_i^2-(\ell_i^*)^2}{2\min\limits_{x\in[L_i,M_i]} a(x)} \right] \\ &= \left[\frac{(\ell_i^*)^2}{a(L_i)\,(2-\eta_{i,a})} + \frac{\ell_i^2-(\ell_i^*)^2}{2\min\limits_{x\in[L_i,M_i]} a(x)} \right] \|u\|^2_{H^1_{a,0}(0,\ell_i)}. \end{aligned} \tag{2.9}$$

Further, as an alternative proof, we adapt a reasoning here that can be used to prove Hardy's inequality. With that in mind, we observe that, for all $s \in [0,\ell_i]$, the following transformation is valid

$$\int_s^{\ell_i} \xi\varphi'(\xi)\varphi(\xi)\,d\xi = \frac12 \int_s^{\ell_i} \xi\frac{d}{d\xi}\varphi^2(\xi)\,d\xi = -\frac12 s\varphi^2(s) - \frac12\int_s^{\ell_i} \varphi^2(\xi)\,d\xi. \tag{2.10}$$

Hence,

$$0 \leqslant \int_s^{\ell_i} \left[\xi\varphi'(\xi) + \frac{1}{2}\varphi(\xi)\right]^2 d\xi = \int_s^{\ell_i} \left[\xi^2 \left[\varphi'(\xi)\right]^2 + \frac{1}{4}\varphi^2(\xi) + \xi\varphi'(\xi)\varphi(\xi)\right] d\xi$$
$$\overset{\text{by (2.10)}}{=} \int_s^{\ell_i} \left[\xi^2 \left[\varphi'(\xi)\right]^2 - \frac{1}{4}\varphi^2(\xi)\right] d\xi - \frac{1}{2}s\varphi^2(s).$$

From this, we deduce that

$$\int_s^{\ell_i} \varphi^2(\xi)\, d\xi \leqslant 4\int_s^{\ell_i^*} \xi^2 \left[\varphi'(\xi)\right]^2 d\xi + 4\int_{\ell_i^*}^{\ell_i} \xi^2 \left[\varphi'(\xi)\right]^2 d\xi$$
$$\leqslant 4\int_s^{\ell_i^*} \xi^{\eta_{i,a}} \left[\varphi'(\xi)\right]^2 d\xi + 4\ell_i^2 \int_{\ell_i^*}^{\ell_i} \left[\varphi'(\xi)\right]^2 d\xi$$
$$\overset{\text{by (2.3)}}{\leqslant} 4\max\left\{\frac{(\ell_i^*)^{\eta_{i,a}}}{a(L_i)}, \frac{\ell_i^2}{\min\limits_{x\in[L_i,M_i]} a(x)}\right\} \int_s^{\ell_i} a\left(\xi\frac{v_i}{|v_i|}\right) \left[\varphi'(\xi)\right]^2 d\xi.$$

Passing to the limit in the last relation as $s \downarrow 0$, we obtain

$$\|u\|^2_{L^2(0,\ell_i)} \leqslant 4\max\left\{\frac{(\ell_i^*)^{\eta_{i,a}}}{a(L_i)}, \frac{\ell_i^2}{\min\limits_{x\in[L_i,M_i]} a(x)}\right\} \|u\|^2_{H^1_{a,0}(0,\ell_i)}. \tag{2.11}$$

The conclusion follows from (2.9) and (2.11). $\square$

3. On Sobolev Spaces with Respect to a Singular Measure

In this section we focus on a short description of Sobolev spaces defined with respect to the singular measure $\mu = \sum_{i=1}^N \mu_i$ which is associated with the star-shaped planar network G. For more details, we refer to our recent work [14]. Let $a : \Omega \to \mathbb{R}$ be a given weight function with properties (i)–(iii). Let $L^2(\Omega, a\, d\mu)$ be the weighted Lebesgue space of μ-measurable functions for which the following norm is finite

$$\|f\|_{L^2(\Omega, a\, d\mu)} = \left(\int_\Omega |f|^2 a\, d\mu\right)^{\frac{1}{2}}.$$

Let $C_0^\infty(\Omega)$ be the set of test functions. Let

$$V_{pot}(\Omega, a\, d\mu) := \mathrm{cl}_{\|\cdot\|_{L^2(\Omega,a\,d\mu)^2}} \left\{\nabla\psi \,:\, \psi \in C_c^\infty(\mathbb{R}^2)\right\}$$

be the so-called space of potential vectors. Following the standard procedure (see [24–26]), we define the Sobolev space $W^{1,2}_{a,0}(\Omega, d\mu)$ as follows.

Definition 3.1. We say that a function $u \in L^2(\Omega, d\mu)$ belongs to the Sobolev space $W^{1,2}_{a,0}(\Omega, d\mu)$ if there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that

$$u_k \to u \text{ in } L^2(\Omega, d\mu) \quad \text{and} \quad \nabla u_k \to z \text{ in } L^2(\Omega, a\, d\mu)^2 \quad \text{as } k \to \infty. \tag{3.1}$$

The set of the limits z is denoted by $\Gamma^\mu(u)$, and its elements are called μ-gradients of the function u.

It is clear that $\Gamma^\mu(u)$ is a closed affine subspace of $L^2(\Omega, a\, d\mu)^2$, and if $u \in W^{1,2}_{a,0}(\Omega, d\mu)$ and $z \in \Gamma^\mu(u)$ then $z \in V_{pot}(\Omega, a\, d\mu)$. In order to specify the structure of elements of $W^{1,2}_{a,0}(\Omega, d\mu)$, we give the following result (for the proof, we refer to [14]).

Proposition 3.1. If a Borel function u belongs to the Sobolev space $W^{1,2}_{a,0}(\Omega, d\mu)$ then its restriction $u_i = u|_{I_i}$ on each segment I_i is an $H^1_{a,0}$-function of a single variable. Namely,

$$u_i \in H^1_{a,0}(0, \ell_i), \ i = 1, \dots, N, \tag{3.2}$$

$$\frac{du_i}{d\xi} = \left(z, \frac{v_i}{|v_i|}\right)\Big|_{x=\xi \frac{v_i}{|v_i|}} \text{ for a.a. } \xi \in I_i, \text{ i=1,...,N}, \quad \forall\, z \in \Gamma^\mu(u), \tag{3.3}$$

where $\frac{du_i}{d\xi}$ stands for the weak derivative of $u_i = u\left(\xi \frac{v_i}{|v_i|}\right)$.

As follows from (3.3), the set of all gradients $\Gamma^\mu(u)$ of any function $u \in W^{1,2}_{a,0}(\Omega, d\mu)$ is not a singleton. So, a typical function $u \in W^{1,2}_{a,0}(\Omega, d\mu)$ can have many gradients. Moreover, it is clear that $\Gamma^\mu(u)$ has the structure $\Gamma^\mu(u) = \nabla^\mu u + \Gamma^\mu(0)$, where $\nabla^\mu u$ is some gradient with properties

$$\left(\nabla^\mu u, \frac{v_i}{|v_i|}\right)\Big|_{x=\xi \frac{v_i}{|v_i|}} = \frac{d}{d\xi} u\big(\xi \frac{v_i}{|v_i|}\big), \quad \forall\, i = 1, \dots, N,$$

and $\Gamma^\mu(0)$ is the set of μ-gradients of 0.

We say that a vector-valued function $b \in L^2(\Omega, a\, d\mu)^2$ and a Borel function $h \in L^2(\Omega, d\mu)$ are connected by the relation $h = \operatorname{div}^\mu(ab)$ if

$$\int_\Omega (b, \nabla\varphi)\; a\, d\mu = -\int_\Omega h\, \varphi\, d\mu, \quad \forall\, \varphi \in C_0^\infty(\Omega). \tag{3.4}$$

The following result reveals some extra properties of functions $u \in W^{1,2}_{a,0}(\Omega, d\mu)$ and their restriction onto the planar network G.

Theorem 3.1 ([14]). *Let $a : \Omega \to \mathbb{R}$ be a given weight function with properties (i)–(iii). Let u be an arbitrary element of $W^{1,2}_{a,0}(\Omega, d\mu)$. Then the following trans-*

mission (or Kirchhhoff) conditions at the origin hold true

$$\sum_{i=1}^{N} \lim_{\xi_i \searrow 0} \Big[u\big(\xi_i \frac{v_i}{|v_i|}\big) a\Big(\xi_i \frac{v_i}{|v_i|}\Big) \frac{dw\big(\xi_i \frac{v_i}{|v_i|}\big)}{d\xi} \Big] = 0, \quad \forall\, w \in W^{1,2}_{a,0}(\Omega, d\mu) \text{ such that } \operatorname{div}^{\mu}(a\nabla^{\mu} w) \in L^2(\Omega, d\mu), \tag{3.5}$$

$$\sum_{i=1}^{N} \lim_{\xi_i \searrow 0} \Big[a\Big(\xi_i \frac{v_i}{|v_i|}\Big) \frac{du\big(\xi_i \frac{v_i}{|v_i|}\big)}{d\xi} \Big] = 0 \quad \textit{provided } \operatorname{div}^{\mu}(a\nabla^{\mu} u) \in L^2(\Omega, d\mu). \tag{3.6}$$

Moreover, if $1 \leqslant \eta_{i,a} < 2$ *for all* $i = 1, \dots, N$, *then*

$$\sum_{i=1}^{N} \lim_{\xi_i \searrow 0} \Big[u\big(\xi_i \frac{v_i}{|v_i|}\big) a\Big(\xi_i \frac{v_i}{|v_i|}\Big) \Big] = 0, \quad \forall\, u \in W^{1,2}_{a,0}(\Omega, d\mu). \tag{3.7}$$

If $u \in W^{1,2}_{a,0}(\Omega, d\mu)$ *and* $\eta_{i,a} \in [0, 1)$ *for all* $i = 1, \dots, N$, *then the restrictions* u_i *to each segment* I_i *belong to the corresponding spaces* $H^1_{a_i,0}(0, \ell_i)$ *and the values of the restricted functions at the origin coincide for all segments.*

4. On Well-Possedness of Cauchy-Dirichlet Problems for Degenerate Hyperbolic Equation

Let Ω be an open bounded subset of $\mathbb{R}^2$ with sufficiently regular boundary $\partial\Omega$. We also assume that 0 is an interior point of Ω.

Let $G = \big\langle \{I_1, \dots, I_N\}, K \cup \{0\} \big\rangle$ be a given 'star-shaped' planar network with the set of end-points K satisfying condition $K \subset \partial\Omega$. Let $\mu = \mu_1 + \dots + \mu_N$ be a singular Borel measure ($\sum_{i=1}^{N} \mu_i (\Omega \setminus I_i) = 0$) associated with the planar network G.

Our main interest in this section is to study the following Cauchy-Dirichlet problem

$$u_{tt} - \operatorname{div}^{\mu}(a\nabla^{\mu} u) = 0 \quad \text{in } (0, \infty) \times \Omega, \tag{4.1}$$

$$u(t, M_i) = 0 \quad \text{for a.a. } t \in (0, \infty) \text{ and } i = 1, \dots, N, \tag{4.2}$$

$$u(0, x) = y_0(x), \quad u_t(0, x) = y_1(x) \quad \text{for } \mu\text{-a.a. } x \in \Omega, \tag{4.3}$$

where $y_0 \in W^{1,2}_{a,0}(\Omega, d\mu)$, $y_1 \in L^2(\Omega, d\mu)$ are given distributions, and $a : \Omega \to \mathbb{R}$ is a weight function with properties (i)–(iii).

Definition 4.1. We say that a pair $(u, \nabla^{\mu} u)$, where $u \in W^{1,2}_{a,0}(\Omega, d\mu)$ and $\nabla^{\mu} u$ is its some gradient, satisfies equation (4.1) in $L^2(\Omega, d\mu)$ and boundary condition (4.2) if for any $z \in W^{1,2}_{a,0}(\Omega, d\mu)$ and any gradient $\nabla^{\mu} z$ of z, we have

$$\int_{\Omega} u_{tt} z \, d\mu + \int_{\Omega} (\nabla^{\mu} u, \nabla^{\mu} z)\, a \, d\mu = 0. \tag{4.4}$$

Remark 4.1. It is worth to notice that such interpretation of relation (4.1)–(4.2) is quite natural. Indeed, the left-hand side of (4.4) is the inner product in $L^2(\Omega, d\mu) \times L^2(\Omega, a\, d\mu)^2$. Hence, by the Riesz representation theorem, there exists a pair $(y, \nabla^\mu u) \in L^2(\Omega, d\mu) \times L^2(\Omega, a\, d\mu)^2$ such that

$$\int_\Omega yz\, d\mu + \int_\Omega (\nabla^\mu u, \nabla^\mu z)\, a\, d\mu = 0 \quad \forall\, z \in W^{1,2}_{a,0}(\Omega, d\mu),\ \forall\, \nabla^\mu z \in \Gamma^\mu(z),$$

Taking for a test function in (4.4) the pair $(0, w)$, where $w \in \Gamma^\mu(0)$, we see that the vector $\nabla^\mu u$ is orthogonal to $\Gamma^\mu(0)$. Using the Riesz theorem again, we conclude that there is a unique element $\nabla^\mu u$ of the set of μ-gradients of u satisfying this orthogonality condition. In view of this, we can expect that the solution of Cauchy-Dirichlet problem (4.1)–(4.3), if it exists, is unique. Since the gradient $\nabla^\mu u$ chosen by a solution u to the problem (4.1)–(4.2) satisfies condition $\nabla^\mu u \perp \Gamma^\mu(0)$, it follows that $\nabla^\mu u$ is the tangential gradient for $u \in W^{1,2}_{a,0}(\Omega, d\mu)$ (see [13, Section 2.8]). Moreover, as follows from Theorem 3.1, the orthogonality condition $\nabla^\mu u \perp \Gamma^\mu(0)$ is equivalent to the Kirchhoff relation (3.6).

In order to give the notions of weak and strong (or classical) solutions for problem (4.1)–(4.3), we introduce two Hilbert spaces $X = \prod_{i=1}^N H^1_{a,0}(0, \ell_i)$ and $Y = \prod_{i=1}^N L^2(0, \ell_i)$ with the scalar products

$$(\mathbf{u}, \mathbf{w})_X = \sum_{i=1}^N (u_i, w_i)_{H^1_{a_i,0}(0,\ell_i)} \overset{\text{by (2.7)}}{=} \sum_{i=1}^N \int_0^{\ell_i} u_i' w_i' a_i\, d\xi, \quad \forall\, \mathbf{u}, \mathbf{w} \in X,$$
$$(\mathbf{u}, \mathbf{w})_Y = \sum_{i=1}^N (u_i, w_i)_{L^2(0,\ell_i)} = \sum_{i=1}^N \int_0^{\ell_i} u_i w_i\, d\xi, \quad \forall\, \mathbf{u}, \mathbf{w} \in Y,$$

respectively. Here, $a_i(\xi) = a\big(\xi \frac{v_i}{|v_i|}\big)$, $\xi \in [0, \ell_i]$, for all $i = 1, \ldots, N$.

Arguing in the spirit of the classical wave equation, we define the operator $\mathcal{A} : D(\mathcal{A}) \subset W^{1,2}_{a,0}(\Omega, d\mu) \times L^2(\Omega, d\mu) \to W^{1,2}_{a,0}(\Omega, d\mu) \times L^2(\Omega, d\mu)$, associated with the problem (4.1)–(4.3), as follows

$$D(\mathcal{A}) = \Big\{(u, w) \in W^{1,2}_{a,0}(\Omega, d\mu) \times W^{1,2}_{a,0}(\Omega, d\mu) \ : \ \operatorname{div}^\mu(a \nabla^\mu u) \in L^2(\Omega, d\mu)\Big\}, \tag{4.5}$$

$$\mathcal{A}(u, w) = (w, \operatorname{div}^\mu(a \nabla^\mu u)), \quad \forall\, (u, w) \in D(\mathcal{A}). \tag{4.6}$$

For arbitrary elements $u \in W^{1,2}_{a,0}(\Omega, d\mu)$ and $w \in L^2(\Omega, d\mu)$, we set

$$\mathbf{u} = [u_1, \ldots, u_N]^t = \left[u\big(\xi_1 \frac{v_1}{|v_1|}\big), \ldots, u\big(\xi_N \frac{v_N}{|v_N|}\big)\right]^t,$$
$$\mathbf{w} = [w_1, \ldots, w_N]^t = \left[w\big(\xi_1 \frac{v_1}{|v_1|}\big), \ldots, w\big(\xi_N \frac{v_N}{|v_N|}\big)\right]^t.$$

Then Proposition 3.1 implies that $\mathbf{u} \in X$ and $\mathbf{w} \in Y$, i.e., the mapping

$$W^{1,2}_{a,0}(\Omega, d\mu) \times L^2(\Omega, d\mu) \ni \begin{bmatrix} u \\ w \end{bmatrix} \overset{\Phi}{\mapsto} \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \in X \times Y \tag{4.7}$$

is well defined. Here, by Φ-transformation we mean the restriction operator onto the planar network G. If, in addition, the distribution u in (4.7) is such that $\operatorname{div}^\mu(a\nabla^\mu u) \in L^2(\Omega, d\mu)$ then the components of vector-valued function $\mathbf{u} \in X$ satisfy transmission conditions (3.5)–(3.6).

Taking into account Theorem 3.1, we see that

$$\Big(\mathcal{A} : D(\mathcal{A}) \to W^{1,2}_{a,0}(\Omega, d\mu) \times L^2(\Omega, d\mu)\Big) \overset{\Phi}{\mapsto} \Big(\mathcal{B} : D(\mathcal{B}) \subset X\times Y \to R(\mathcal{B}) \subset X\times Y\Big),$$

where

$$D(\mathcal{B}) = \left\{ \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \ : \ \mathbf{u} \in \prod_{i=1}^N H^2_a(0,\ell_i), \ \mathbf{w} \in \prod_{i=1}^N H^1_{a,0}(0,\ell_i) \right\}, \tag{4.8}$$

$$R(\mathcal{B}) = \left\{ \begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} \ : \ \begin{array}{l} \mathbf{u} \in \prod_{i=1}^N H^1_{a,0}(0,\ell_i), \ \mathbf{w} \in \prod_{i=1}^N L^2(0,\ell_i), \\ \sum_{i=1}^N \lim_{\xi_i \searrow 0} \left[a_i(\xi_i) \frac{du_i(\xi_i)}{d\xi} \right] = 0, \\ \sum_{i=1}^N \lim_{\xi_i \searrow 0} \left[u_i(\xi_i) a_i(\xi_i) \frac{d\varphi\left(\xi_i \frac{v_i}{|v_i|}\right)}{d\xi} \right] = 0, \\ \forall\, \varphi \in W^{1,2}_{a,0}(\Omega, d\mu) \text{ such that } \operatorname{div}^\mu(a\nabla^\mu\varphi) \in L^2(\Omega, d\mu) \end{array} \right\}, \tag{4.9}$$

$$\mathcal{B}\begin{bmatrix} \mathbf{u} \\ \mathbf{w} \end{bmatrix} = \left[w_1, \dots, w_N, \frac{d}{d\xi}\Big(a_1 \frac{du_1}{d\xi}\Big), \dots, \frac{d}{d\xi}\Big(a_N \frac{du_N}{d\xi}\Big) \right]^t, \tag{4.10}$$

$a_i = a\left(\xi \frac{v_i}{|v_i|}\right)$, and $H^2_a(0,\ell_i) = \left\{ z \in H^1_{a,0}(0,\ell_i) \ : \ a_i z' \in H^1(0,\ell_i) \right\}$ for $i = 1, \dots, N$.

Arguing as in [9, Section II.2] and in [14], it is easy to show that $D(\mathcal{B})$ is a dense subset of $X \times Y$ and $\mathcal{B} : D(\mathcal{B}) \subset X \times Y \to X \times Y$ is the generator of a contraction semi-group in $X \times Y$.

For the further convenience, let us denote this semi-group by $e^{\mathcal{B}t}$. Then for any $y_0 \in W^{1,2}_{a,0}(\Omega, d\mu)$ and $y_1 \in L^2(\Omega, d\mu)$, the representation $U(t) = e^{\mathcal{B}t}\Phi \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$ gives the so-called mild solution of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt} U(t) = \mathcal{B}U(t), \quad U(t) = \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix} \\ U(0) = \Phi \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \end{cases} \tag{4.11}$$

supplied by the transmission conditions

$$\sum_{i=1}^N \lim_{\xi_i \searrow 0} \left[a_i(\xi_i) \frac{\partial u_i(t,\xi_i)}{\partial \xi} \right] = 0, \tag{4.12}$$

$$\sum_{i=1}^N \lim_{\xi_i \searrow 0} \left[u_i(t,\xi_i) a_i(\xi_i) \frac{d\varphi\left(\xi_i \frac{v_i}{|v_i|}\right)}{d\xi} \right] = 0, \tag{4.13}$$

$$\forall\, \varphi \in W^{1,2}_{a,0}(\Omega, d\mu) \text{ such that } \operatorname{div}^\mu(a\nabla^\mu\varphi) \in L^2(\Omega, d\mu).$$

When $\Phi \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in D(\mathcal{B})$, the above solution is classical in the sense that

$$U \in C^1([0,\infty); X \times Y) \cap C([0,\infty); D(\mathcal{B}))$$

and the equation holds on $[0,\infty)$ (see [1, Section 4.6] for the details).

Thus, in view of the above consideration, we arrive at the following conclusion.

Theorem 4.1. *For given $y_0 \in W^{1,2}_{a,0}(\Omega, d\mu)$ and $y_1 \in L^2(\Omega, d\mu)$, Cauchy-Dirichlet problem* (4.1)–(4.3) *admits a unique weak solution, i.e., there exists a unique pair $(u, \nabla^\mu u)$ such that*

$$u \in C^1([0,\infty); L^2(\Omega, d\mu)) \cap C([0,\infty); W^{1,2}_{a,0}(\Omega, d\mu)), \tag{4.14}$$

$\nabla^\mu u \in \Gamma^\mu(u)$, the pair $(u, \nabla^\mu u)$ is related by integral equality (4.4)*, and $\Phi(u(t)) = \mathbf{u}(t)$ for all $t \geqslant 0$, where $\begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix} = e^{\mathcal{B}t}\Phi \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$. Moreover, if y_0 and y_1 are such that $\Phi \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in D(\mathcal{B})$ then the function u is the strong solution of problem* (4.1)–(4.3) *meaning that*

$$\Phi(u(t)) = \mathbf{u}(t)\ \forall\, t \geqslant 0, \qquad \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{w}(t) \end{bmatrix} = e^{\mathcal{B}t}\Phi \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}, \quad \mathbf{u} = [u_1, \dots, u_N]^t, \tag{4.15}$$

$$u_i \in C^2([0,\infty); L^2(0,\ell_i)) \cap C^1([0,\infty); H^1_{a_i,0}(0,\ell_i)) \cap C([0,\infty); H^2_{a_i}(0,\ell_i)), \tag{4.16}$$

and equations

$$\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial \xi}\left(a\big(\xi \frac{v_i}{|v_i|}\big) \frac{\partial u_i}{\partial \xi} \right) = 0 \quad \text{in } (0,\infty) \times (0,\ell_i) \quad \text{for } i = 1, \dots, N$$

satisfied for all $t \geqslant 0$ and a.a. $\xi \in (0, \ell_i)$.

5. The problem of boundary observability

Let $(u, \nabla^\mu u)$ be a weak solution of Cauchy-Dirichlet problem (4.1)–(4.3) (see Theorem 4.1). We define the energy of this solution as follows (see [3, 7] for comparison)

$$\begin{aligned} E_u(y_0, y_1, t) &= \frac{1}{2} \int_\Omega \left[|u_t|^2 + a |\nabla^\mu u|^2 \right] d\mu \\ &= \frac{1}{2} \sum_{i=1}^N \int_0^{\ell_i} \left[\left| u_t\big(t, \xi \frac{v_i}{|v_i|}\big) \right|^2 + a\big(\xi \frac{v_i}{|v_i|}\big) \left| \frac{d}{d\xi} u\big(t, \xi \frac{v_i}{|v_i|}\big) \right|^2 \right] d\xi. \end{aligned} \tag{5.1}$$

Then property (4.14) implies that $E(t)$ is a continuous function. To begin with, let us show that the energy $E_u(y_0, y_1, t)$ of any weak solution u does not dissipate in time.

Proposition 5.1. Let $a : \Omega \to \mathbb{R}$ be a given weight function with properties (i)–(iii), and let u be a weak solution of (4.1)–(4.2). Then

$$E_u(y_0, y_1, t) = E_u(y_0, y_1, 0), \quad \forall\, t \geqslant 0. \tag{5.2}$$

Proof. Suppose, first, that u is a strong solution of (4.1)–(4.2). Then this function satisfies transmission conditions (3.5)–(3.6), and $u_t(t, \cdot) \in W^{1,2}_{a,0}(\Omega, d\mu)$. Hence, we can consider u_t as a test function in integral identity (4.4) (see Definition 4.1). As a result, we obtain

$$0 \stackrel{\text{by (4.4)}}{=} \int_\Omega u_{tt} u_t \, d\mu + \int_\Omega (\nabla^\mu u, \nabla^\mu u_t)\, a\, d\mu = \frac{d}{dt} E_u(y_0, y_1, t).$$

Thus, the energy of the strong solution u is constant. The same conclusion can be extended to any weak solution by approximation arguments. To do so, it is enough to make use of the approximation property (3.1) and utilize it for the test functions in integral identity (4.4). $\square$

We recall that the problem of boundary observability of the system (4.1)–(4.3) can be formulated as follows: (4.1)–(4.3) is said to be observable (via the normal derivative at the nodes M_i) in time $T > 0$ if there exists a constant $C > 0$ such that for any $y_0 \in W^{1,2}_{a,0}(\Omega, d\mu)$ and $y_1 \in L^2(\Omega, d\mu)$ the corresponding weak solution of (4.1)–(4.3) satisfies inequality

$$\sum_{i=1}^{N} \int_0^T \left| \frac{\partial u(t, M_i)}{\partial v_i} \right|^2 dt \geqslant C E_u(y_0, y_1, 0). \tag{5.3}$$

Inequality (5.3), when it holds, guarantees that the total energy of solutions can be 'observed' from the boundary measurement at the nodes $\{x = M_i\}_{i=1}^N$. Any constant satisfying (5.3) is called an observability constant for (4.1)–(4.3) in time T. The supremum of all observability constants for (4.1)–(4.3) is denoted by C_T. Equivalently, (4.1)–(4.3) is observable if

$$C_T = \inf_{(y_0,y_1) \neq (0,0)} \left[E_u^{-1}(y_0, y_1, 0) \sum_{i=1}^{N} \int_0^T \left| \frac{\partial u(t, M_i)}{\partial v_i} \right|^2 dt \right] > 0.$$

The inverse $c_T = 1/C_T$ is sometimes called the cost of observability.

We begin with a few technical results.

Lemma 5.1. *For any weak solution u of (4.1)–(4.3) we have that $\frac{\partial u(t,M_i)}{\partial v_i} \in L^2(0,T)$ for all $T > 0$ and $i = 1, \ldots, N$. Moreover, for each index $i = 1, \ldots, N$,* *the following relations hold true*

$$\ell_i a(M_i)\int_0^T \left|\frac{\partial u(t,M_i)}{\partial v_i}\right|^2 dt \leqslant D\,E_u(y_0,y_1,0), \tag{5.4}$$

$$\ell_i a(M_i)\int_0^T \left|\frac{\partial u(t,M_i)}{\partial v_i}\right|^2 dt = 2\Big[\int_0^{\ell_i}\xi\frac{\partial u(t,\xi\frac{v_i}{|v_i|})}{\partial\xi}\frac{\partial u(t,\xi\frac{v_i}{|v_i|})}{\partial t}\,d\xi\Big]_{t=0}^{t=T}$$
$$+\int_0^T\int_0^{\ell_i}\Big[\Big(\frac{\partial u(t,\xi\frac{v_i}{|v_i|})}{\partial t}\Big)^2+\Big(1-\frac{\xi\frac{d}{d\xi}a(\xi\frac{v_i}{|v_i|})}{a(\xi\frac{v_i}{|v_i|})}\Big)a\big(\xi\frac{v_i}{|v_i|}\big)\Big(\frac{\partial u(t,\xi\frac{v_i}{|v_i|})}{\partial\xi}\Big)^2\Big]\,d\xi dt, \tag{5.5}$$

where

$$D = 4\max\left\{1,\frac{(\ell_i^*)^2}{a(L_i)},\frac{(\ell_i)^2}{\min\limits_{x\in[L_i,M_i]}a(x)}\right\}+2T\left(1+\max\left\{2,\max_{\xi\in[\ell_i^*,\ell_i]}\frac{\xi|a_i'(\xi)|}{a_i(\xi)}\right\}\right).$$

Proof. Assume that y_0 and y_1 are such that $\Phi\begin{bmatrix}y_0\\y_1\end{bmatrix}\in D(\mathcal{B})$. Then the function u, given by formula (4.15), is the strong solution of problem (4.1)–(4.3). Hence, its restriction $\mathbf{u}$ on the planar network G satisfies transmission conditions (4.12)–(4.13) and the relations (see (4.11))

$$\ddot{u}_i-\big(a_iu_i'\big)'=0,\quad \text{in } (0,T)\times(0,\ell_i),\quad \forall\, i=1,\dots,N, \tag{5.6}$$

where $a_i=a\left(\xi\frac{v_i}{|v_i|}\right)$, and we write down $\dot{u}_i$ instead $\frac{\partial u_i}{\partial t}$, $\ddot{u}_i$ instead $\frac{\partial^2u_i}{\partial t^2}$, and u_i' instead $\frac{\partial u_i}{\partial\xi}$. Following in many aspects [3, Lemma 3.2], we multiply equations (5.6) by $\frac{\xi}{\ell_i}u_i'$. Integrating over $(0,T)\times(0,\ell_i)$, we obtain

$$0=\int_0^T\int_0^{\ell_i}\xi u_i'\left[\ddot{u}_i-\big(a_iu_i'\big)'\right]d\xi dt=\left[\int_0^{\ell_i}\xi u_i'\dot{u}_i\,d\xi\right]_{t=0}^{t=T}-\int_0^T\int_0^{\ell_i}\xi\dot{u}_i'\dot{u}_i\,d\xi dt$$
$$-\int_0^T\int_0^{\ell_i}\xi a_i'(u_i')^2\,d\xi dt-\int_0^T\int_0^{\ell_i}\xi a_iu_i'u_i''\,d\xi dt=\left[\int_0^{\ell_i}\xi u_i'\dot{u}_i\,d\xi\right]_{t=0}^{t=T}$$
$$-\int_0^T\int_0^{\ell_i}\xi a_i'(u_i')^2\,d\xi dt-\int_0^T\int_0^{\ell_i}\left[\xi\left(\frac{\dot{u}_i^2}{2}\right)'+\xi a_i\left(\frac{(u_i')^2}{2}\right)'\right]d\xi dt. \tag{5.7}$$

Utilizing conditions (4.2) and continuity property of the function

$$z(\xi)=\left\{\begin{matrix}\xi a_i(\xi)(u_i'(\xi))^2,\ \xi>0\\ 0,\ \xi=0\end{matrix}\right\}\quad \text{for } u_i\in H_a^2(0,\ell_i)$$

(see [3, Proposition 2.5]), we can rewrite the last term in (5.7) as follows

$$\int_0^T \int_0^{\ell_i} \xi \left(\frac{\dot{u}_i^2}{2}\right)' d\xi dt = -\frac{1}{2}\int_0^T \int_0^{\ell_i} \dot{u}_i^2 \, d\xi dt + \frac{1}{2}\int_0^T \left[\xi \dot{u}_i^2\right]_{\xi=0}^{\xi=\ell_i} dt$$
$$= -\frac{1}{2}\int_0^T \int_0^{\ell_i} \dot{u}_i^2 \, d\xi dt, \tag{5.8}$$

$$\int_0^T \int_0^{\ell_i} \xi a_i \left(\frac{(u_i')^2}{2}\right)' d\xi dt = -\frac{1}{2}\int_0^T \int_0^{\ell_i} (\xi a_i)' (u_i')^2 \, d\xi dt + \frac{1}{2}\int_0^T \left[\xi a_i (u_i')^2\right]_{\xi=0}^{\xi=\ell_i} dt$$
$$= -\frac{1}{2}\int_0^T \int_0^{\ell_i} \left(1 + \frac{\xi a_i'}{a_i}\right) a_i (u_i')^2 \, d\xi dt$$
$$+ \frac{\ell_i}{2} a_i(\ell_i) \int_0^T (u_i'(t,\ell_i))^2 \, dt. \tag{5.9}$$

In order to deduce identity (5.5) it remains to insert (5.8) and (5.9) into (5.7) and take into account that

$$a_i(\ell_i) = a\Big(\frac{\ell_i v_i}{|v_i|}\Big) = a(M_i), \;\; u_i'(t,\ell_i) = \frac{\partial u\big(t, \xi \frac{v_i}{|v_i|}\big)}{\partial \xi}\Big|_{\xi=\ell_i} = \frac{\partial u(t, M_i)}{\partial v_i}$$

As for the estimate (5.4), we observe that

$$\Big(1 - \frac{\xi a_i'}{a_i)}\Big) \overset{\text{by (1.2)}}{\leqslant} \left(1 + \max\left\{2, \max_{\xi\in[\ell_i^*,\ell_i]} \frac{\xi |a_i'(\xi)|}{a_i(\xi)}\right\}\right),$$

$$\left|\int_0^{\ell_i} \xi u_i' \dot{u}_i \, d\xi\right| \leqslant \frac{1}{2}\int_0^{\ell_i} \left[(\dot{u}_i)^2 + \frac{\xi^2}{a_i} a_i \left(u_i'\right)^2\right] d\xi$$
$$\overset{\text{by (2.3)}}{\leqslant} \frac{1}{2}\int_0^{\ell_i^*} \left[(\dot{u}_i)^2 + \left(\frac{\xi}{\ell_i^*}\right)^{2-\eta_{i,a}} \frac{(\ell_i^*)^2}{a(L_i)} a_i \left(u_i'\right)^2\right] d\xi$$
$$+ \frac{1}{2}\int_{\ell_i^*}^{\ell_i} \left[(\dot{u}_i)^2 + \frac{\xi^2}{a_i} a_i \left(u_i'\right)^2\right] d\xi$$
$$\leqslant \max\left\{1, \frac{(\ell_i^*)^2}{a(L_i)}, \frac{(\ell_i)^2}{\min\limits_{x\in[L_i,M_i]} a(x)}\right\} E_u(y_0,y_1,0).$$

As a result, we have

$$2\left|\left[\int_0^{\ell_i} \xi u_i' \dot{u}_i \, d\xi\right]_{t=0}^{t=T}\right| \leqslant 4\max\left\{1, \frac{(\ell_i^*)^2}{a(L_i)}, \frac{(\ell_i)^2}{\min\limits_{x\in[L_i,M_i]} a(x)}\right\} E_u(y_0,y_1,0), \tag{5.10}$$

$$\int_0^T \int_0^{\ell_i} \left[\dot{u}_i^2 + \left(1 - \frac{\xi a_i'}{a_i}\right) a_i (u_i')^2\right] d\xi dt$$
$$\leqslant 2T \left(1 + \max\left\{2, \max_{\xi\in[\ell_i^*,\ell_i]} \frac{\xi |a_i'(\xi)|}{a_i(\xi)}\right\}\right) E_u(y_0,y_1,0). \tag{5.11}$$

From this and (5.5) estimate (5.4) follows.

In order to extend (5.5) and (5.4) to the case of weak solutions associated with the initial data $y_0 \in W^{1,2}_{a,0}(\Omega, d\mu)$ and $y_1 \in L^2(\Omega, d\mu)$, it suffices to approximate such data by $\{y_{0,n}\}_{n\in\mathbb{N}}$ and $\{y_{1,n}\}_{n\in\mathbb{N}}$ such that $\Phi \begin{bmatrix} y_{0,n} \\ y_{1,n} \end{bmatrix} \in D(\mathcal{B})$ and using (5.4) to show that the normal derivatives $\left\{\frac{\partial u_n(t,M_i)}{\partial v_i}\right\}_{n\in\mathbb{N}}$ of the corresponding classical solutions give Cauchy sequences in $L^2(0,T)$ for each $i = 1,\dots,N$. $\square$

Lemma 5.2. *For any weak solution $u(t,x)$ of the problem* (4.1)–(4.3)*, we have*

$$\sum_{i=1}^N \int_0^T \int_0^{\ell_i} \left[a_i \left(\frac{\partial u_i}{\partial \xi}\right)^2 - \left(\frac{\partial u_i}{\partial t}\right)^2 \right] d\xi dt + \sum_{i=1}^N \left[\int_0^{\ell_i} u_i \frac{\partial u_i}{\partial t}\, d\xi \right]_{t=0}^{t=T} = 0, \quad \forall T > 0. \tag{5.12}$$

Proof. Let u be a strong solution of problem (4.1)–(4.3). Then, multiplying equation (5.6) by u_i and integrating over $(0,T)\times(0,\ell_i)$, we obtain

$$\begin{aligned} 0 = \int_0^T \int_0^{\ell_i} u_i \left(\ddot{u}_i - \left(a_i u_i'\right)' \right) d\xi dt &= \int_0^T \int_0^{\ell_i} \left[a_i \left(u_i'\right)^2 - \left(\dot{u}_i\right)^2 \right] d\xi dt \\ &+ \left[\int_0^{\ell_i} u_i \dot{u}_i \, d\xi \right]_{t=0}^{t=T} - \left[\int_0^T u_i a_i u_i' \, dt \right]_{\xi=0}^{\xi=\ell_i} \end{aligned} \tag{5.13}$$

Since $u = u(t,x)$ vanishes at nodes M_i, $i = 1,\dots,N$, and it satisfies transmission condition (4.12)–(4.13), it follows that

$$\sum_{i=1}^N \left[\int_0^T u_i a_i u_i' \, dt \right]_{\xi=0}^{\xi=\ell_i} = 0.$$

Summing up relation (5.13) for all $i = 1,\dots,N$, we arrive at the announced equality (5.12). It remains to notice that an approximation argument allows to extend this conclusion to any weak solutions. $\square$

Remark 5.1. In fact, if we deal with a strong solution u of the problem (4.1)–(4.3), then the main assertion of Lemma 5.2 can be enhanced. Namely, assume that the initial data (y_0, y_1) in (4.3) are sufficiently regular, i.e. $\Phi \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in D(\mathcal{B})$. Then it follows from (4.16) that $u_i = u_i(t,\xi)$ and $a_i(\xi)u_i'(t,\xi)$ are continuous functions on $[0,T]\times[0,\ell_i]$ for each $i = 1,\dots,N$. Hence, there exist constants L_1 and L_2, independent of i, such that

$$\lim_{\xi \searrow 0} u_i(t,\xi) = L_1, \quad \lim_{\xi \searrow 0} a_i(\xi) u_i'(t,\xi) = L_2.$$

Then transmission conditions (4.12)–(4.13) imply that $L_1 L_2 = 0$. As a result, the last term in (5.13) is equal to zero. So, instead of the announced equality (5.12), we have the following one

$$\int_0^T \int_0^{\ell_i} \left[a_i \left(\frac{\partial u_i}{\partial \xi}\right)^2 - \left(\frac{\partial u_i}{\partial t}\right)^2 \right] d\xi dt + \left[\int_0^{\ell_i} u_i \frac{\partial u_i}{\partial t}\, d\xi \right]_{t=0}^{t=T} = 0, \quad \forall T > 0$$

for each $i = 1, \dots, N$.

We are now in a position to prove the main result of this section.

Theorem 5.1. *Let $a : \Omega \to \mathbb{R}$ be a given weight function satisfying properties (i)–(iii). We assume that*

$$\frac{d \ln a\left(\xi \frac{v_i}{|v_i|}\right)}{d\xi} \leqslant \frac{d \ln \xi^{\eta_{i,a}}}{d\xi}, \quad \forall \xi \in [\ell_i^*, \ell_i], \quad \forall i = 1, \dots, N. \tag{5.14}$$

Let u be a mild solution of (4.1)–(4.3). *Then, for every $T > 0$, the estimate*

$$\sum_{i=1}^N \ell_i a(M_i) \int_0^T \left| \frac{\partial u(t, M_i)}{\partial v_i} \right|^2 dt \geqslant C^* E_u(y_0, y_1, 0), \tag{5.15}$$

holds true with

$$\begin{aligned} C^* = {} & (2 - \max\{\eta_{1,a}, \dots, \eta_{N,a}\})\, T \\ & - 4 \sum_{i=1}^N \max \left\{ 1, \frac{(\ell_i^*)^2}{a(L_i)}, \frac{(\ell_i)^2}{\min\limits_{x \in [L_i, M_i]} a(x)} \right\} - 2 \max\{\eta_{1,a}, \dots, \eta_{N,a}\} \sum_{i=1}^N C_{i,a}. \end{aligned} \tag{5.16}$$

Here, the constants $C_{i,a}$ are given by relations (2.8).

Proof. Since the case of weak solutions can be recovered by an approximation arguments, we restrict ourself by assumptions that u is a classical solution of the problem (4.1)–(4.2). Then summing up (5.5) for $i = 1, \dots, N$ and adding to its right hand side the left side of (5.12) multiplied by $\frac{1}{2} \max\{\eta_{1,a}, \dots, \eta_{N,a}\}$, we obtain

$$\begin{aligned} & \sum_{i=1}^N \ell_i a_i(\ell_i) \int_0^T (u_i'(t, \ell_i))^2\, dt \\ & \qquad = 2 \sum_{i=1}^N \Big[\int_0^{\ell_i} \xi u_i' \dot{u}_i\, d\xi \Big]_{t=0}^{t=T} + \frac{\max\{\eta_{1,a}, \dots, \eta_{N,a}\}}{2} \sum_{i=1}^N \Big[\int_0^{\ell_i} u_i \dot{u}_i\, d\xi \Big]_{t=0}^{t=T} \\ & \qquad + \sum_{i=1}^N \int_0^T \int_0^{\ell_i} \left(1 - \frac{\max\{\eta_{1,a}, \dots, \eta_{N,a}\}}{2} \right) \dot{u}_i^2\, d\xi dt \\ & \qquad + \sum_{i=1}^N \int_0^T \int_0^{\ell_i} \left(1 + \frac{\max\{\eta_{1,a}, \dots, \eta_{N,a}\}}{2} - \frac{\xi a_i'}{a_i} \right) a_i \left(u_i'\right)^2\, d\xi dt \\ & \qquad = J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Since

$$-\frac{\xi a_i'(\xi)}{a_i(\xi)} \overset{\text{by } (1.2)}{\geqslant} - \max\{\eta_{1,a}, \dots, \eta_{N,a}\}, \quad \forall \xi \in [0, \ell_i^*], \quad \forall i = 1, \dots, N,$$

$$-\frac{\xi a_i'(\xi)}{a_i(\xi)} \overset{\text{by } (5.14)}{\geqslant} -\eta_{i,a} \quad \forall \xi \in [\ell_i^*, \ell_i],, \quad \forall i = 1, \dots, N,$$

it follows that

$$J_3 + J_4 \geqslant (2 - \max\{\eta_{1,a}, \dots, \eta_{N,a}\}) \int_0^T E_u(y_0, y_1, t)\, dt$$
$$= (2 - \max\{\eta_{1,a}, \dots, \eta_{N,a}\})\, T E_u(y_0, y_1, 0). \tag{5.17}$$

Taking into account that

$$J_1 = 2\sum_{i=1}^N \Big[\int_0^{\ell_i} \xi u_i' \dot{u}_i \, d\xi\Big]_{t=0}^{t=T}$$
$$\overset{\text{by (5.10)}}{\geqslant} -4\sum_{i=1}^N \max\left\{1, \frac{(\ell_i^*)^2}{a(L_i)}, \frac{(\ell_i)^2}{\min\limits_{x\in[L_i,M_i]} a(x)}\right\} E_u(y_0, y_1, 0) \tag{5.18}$$

and

$$\frac{1}{2}\left|\int_0^{\ell_i} u_i \dot{u}_i \, d\xi\right| \leqslant \frac{1}{2}\int_0^{\ell_i} \left[\frac{1}{C_{i,a}} u_i^2 + C_{i,a}\dot{u}_i^2\right] d\xi$$
$$\overset{\text{by (2.7)}}{\leqslant} C_{i,a}\frac{1}{2}\int_0^{\ell_i} \left[\dot{u}_i^2 + a_i(u_i')^2\right] \, d\xi = C_{i,a} E_u(y_0, y_1, 0),$$

where $C_{i,a}$ is defined by (2.8), we see that

$$J_2 \geqslant -2\max\{\eta_{1,a}, \dots, \eta_{N,a}\} \sum_{i=1}^N C_{i,a}\, E_u(y_0, y_1, 0). \tag{5.19}$$

As a result, the announced estimate (5.15) immediately follows from (5.17)–(5.19). □

As a direct consequence of this theorem, we have the following criterion of boundary observability for the system (4.1)–(4.3).

Corollary 5.1. *Let $a : \Omega \to \mathbb{R}$ be a given weight function with properties (i)–(iii) and* (5.14)*. Then system* (4.1)–(4.3) *is boundary observable in time T provided that*

$$T > T_a := \frac{4\sum_{i=1}^N \max\left\{1, \frac{(\ell_i^*)^2}{a(L_i)}, \frac{(\ell_i)^2}{\min\limits_{x\in[L_i,M_i]} a(x)}\right\}}{(2 - \max\{\eta_{1,a}, \dots, \eta_{N,a}\})}$$
$$+ \frac{2\max\{\eta_{1,a}, \dots, \eta_{N,a}\}\sum_{i=1}^N C_{i,a}}{(2 - \max\{\eta_{1,a}, \dots, \eta_{N,a}\})}, \tag{5.20}$$

where $C_{i,a}$ are defined in (2.8)*. In this case*

$$C_T \geqslant \frac{C^*}{\max\{\ell_1 a(M_1), \dots, \ell_N a(M_N)\}} \quad \text{with } C^* \text{ given in (5.16).}$$

Example 5.1. For a given bounded domain Ω, let $G = \left\langle \{I_1, \dots, I_N\}, K \cup \{0\} \right\rangle$ be a planar network such that all segments I_i have the same lengths, i.e., $\ell_i = \ell$ for all $i = 1, \dots, N$. Let $a : \Omega \to \mathbb{R}$ be a given weight function with properties (i)–(iii) and (5.14). Assume that there exists an exponent $\theta \in (0, 2)$ such that

$$a_i(\xi) = a\big(\xi \frac{v_i}{|v_i|}\big) = \xi^\theta \quad \text{for each } i = 1, \dots, N.$$

Hence, $\ell_i^* = \ell_i$, $L_i = M_i$, and $a_i(\ell_i) = a(M_i) = \ell^\theta$ for all $i = 1, \dots, N$. These assumptions are obviously true provided Ω is an open ball in $\mathbb{R}^2$ and the weight function $a : \Omega \to \mathbb{R}$ is defined as $a(x) = |x|^\theta = \left(x_1^2 + x_2^2\right)^{\frac{\theta}{2}}$ in Ω.

Then direct calculations show that in this case, for each $i = 1, \dots, N$, we have

$$\eta_{i,a} = \theta, \quad \frac{\ell_i^2}{a(M_i)} = \frac{(\ell_i)^2}{\min\limits_{x \in [L_i, M_i]} a(x)} = \ell^{2-\theta}, \quad \forall\, i = 1, \dots, N.$$

Hence, as follows from estimate (5.20), system (4.1)–(4.3) is boundary observable in time T provided that $T > T_a$, where

$$T_a = \frac{4N \max\left\{1, \ell^{2-\theta}\right\}}{2-\theta} + \frac{2\theta N \sqrt{\min\left\{\frac{\ell^{2-\theta}}{2-\theta}, 4\max\left\{1, \ell^{2-\theta}\right\}\right\}}}{2-\theta}$$

It is worth to notice that $T_a \to 2N \max\left\{1, \ell^2\right\}$ as $\theta \to 0$. Hence, if $\ell = 1$ then the value T_a is equal to $2N$ and, therefore, it coincides with the classical observability time for the wave equations on the star-shaped planar network (see [7, Section 4.4]).

6. On Boundary Null Controllability

In this section the problem of boundary controllability of the degenerate linear hyperbolic system defined on star-shaped planar network is studied. The controls are assumed to act at the boundary points $x_i = M_i$, $i = 1, \dots, N$ through the Dirichlet conditions. So, we consider the following degenerate control system

$$u_{tt} - \operatorname{div}^\mu(a \nabla^\mu u) = 0 \quad \text{in } (0, \infty) \times \Omega, \tag{6.1}$$

$$u(t, M_i) = f_i(t) \quad \text{for a.a. } t \in (0, \infty) \text{ and } i = 1, \dots, M, \tag{6.2}$$

$$u(0, x) = y_0(x), \quad u_t(0, x) = y_1(x) \quad \text{for } \mu\text{-a.a. } x \in \Omega, \tag{6.3}$$

where $F = [f_1, f_2, \dots, f_N]^t \in L^2(0, T; \mathbb{R}^N)$ is a vector-valued control function.

By analogy with the previous section, for a given weight function $a : \Omega \to \mathbb{R}$ with properties (i)–(iii), we associate the Sobolev space $W^{1,2}_{a,0}(\Omega, d\mu)$. Let Φ : $W^{1,2}_{a,0}(\Omega, d\mu) \to X = \prod_{i=1}^N H^1_{a,0}(0, \ell_i)$ be the restriction operator onto the planar network G. For a given $i = 1, \dots, N$, let $H^{-1}_a(0, \ell_i)$ be the dual space to $H^1_{a,0}(0, \ell_i)$ with respect to the pivot space $L^2(0, \ell_i)$.

In order to make a precise definition of a solution to the boundary value problem (6.1)–(6.3), where $f_i \in L^2(0,T)$, $i = 1,\dots,N$, are the controls, and indicate its characteristic properties, we make use of the following notation. We say that a distribution y_1 belongs to the class $W_a^{-1,2}(\Omega, d\mu)$ if $\Phi(y_1) \in \prod_{i=1}^N H_a^{-1}(0,\ell_i)$.

Definition 6.1. System (6.1)–(6.3) is boundary null controllable in time $T > 0$ if, for every initial data $y_0 \in L^2(\Omega, d\mu)$, and $y_1 \in W_a^{-1,2}(\Omega, d\mu)$, the set of reachable states $(y(T), y_t(T))$, where y is a solution of (6.1)–(6.3) with $f_i \in L^2(0,T)$, $i = 1,\dots,N$, contains the element $(0,0)$.

Definition 6.2. System (6.1)–(6.3) is boundary exactly controllable in time $T > 0$ if, for every initial data $y_0 \in L^2(\Omega, d\mu)$, and $y_1 \in W_a^{-1,2}(\Omega, d\mu)$, the set of reachable states $(y(T), y_t(T))$, coincides with $L^2(\Omega, d\mu) \times W_a^{-1,2}(\Omega, d\mu)$.

Remark 6.1. Arguing as in Proposition 2.2.1 in [28], and utilizing the linearity and reversibility properties of system (6.1)–(6.3), it can be shown that this system is exactly controllable through the boundary Dirichlet conditions if and only if it is null controllable.

Following the standard approach and utilizing the Kirchhoff conditions (3.5)–(3.6), we define the solution of controlled system (6.1)–(6.3) by transposition.

Definition 6.3. Let $F = [f_1, f_2, \dots, f_N]^t \in L^2(0,T;\mathbb{R}^N)$, $y_0 \in L^2(\Omega, d\mu)$, and $y_1 \in W_a^{-1,2}(\Omega, d\mu)$ be given distributions. We say that $u \in C\left([0,\infty); L^2(\Omega, d\mu)\right)$ is a solution by transposition of the problem (6.1)–(6.3) if:

$u_i \in C^1\left([0,\infty); H_a^{-1}(0,\ell_i)\right)$ for each $i = 1,\dots,N$, where $u_i = u\left(t, \xi\frac{v_i}{|v_i|}\right)$ stands for the i-th slot of $\Phi(u) \in \mathbb{R}^N$;

the following equality

$$\begin{aligned}\left\langle \Phi\left(u_t(T)\right), \Phi\left(w_T^0\right)\right\rangle_{\prod\limits_{i=1}^N H_a^{-1}(0,\ell_i);\prod\limits_{i=1}^N H_{a,0}^1(0,\ell_i)} &- \int_\Omega u(T) w_T^1 \, d\mu \\ = \left\langle \Phi\left(y_1\right), \Phi\left(w(0)\right)\right\rangle&_{\prod\limits_{i=1}^N H_a^{-1}(0,\ell_i);\prod\limits_{i=1}^N H_{a,0}^1(0,\ell_i)} \\ - \int_\Omega & y_0 w_t(0)\, d\mu \\ - \sum_{i=1}^N & a(M_i) \int_0^T f_i(t) (w_i)_x(t,\ell_i)\, dt, \end{aligned} \tag{6.4}$$

holds true for all $T > 0$ and all $w_T^0 \in W_{a,0}^{1,2}(\Omega, d\mu)$ and $w_T^1 \in L^2(\Omega, d\mu)$, where $w_i = w\left(t, \xi\frac{v_i}{|v_i|}\right)$ and w is the solution of the backward homogeneous equation

$$w_{tt} - \operatorname{div}^\mu(a\nabla^\mu w) = 0 \quad \text{in} \quad (0,+\infty) \times \Omega \tag{6.5}$$

with the final conditions

$$w(T) = w_T^0, \quad w_t(T) = w_T^1 \quad \text{for } \mu\text{-a.a. } x \in \Omega \tag{6.6}$$

and the boundary conditions

$$w(t, M_i) = 0, \quad \text{for a.a. } t \in (0, \infty) \text{ and } i = 1, \dots, N. \tag{6.7}$$

Following the results of Section 4 and making the change of variable $u(t,x) = w(T-t,x)$, we see that the backward problem (6.5)–(6.7) admits a unique weak solution $w \in C^1([0,T]; L^2(\Omega, d\mu)) \cap C([0,T]; W_{a,0}^{1,2}(\Omega, d\mu))$, for each $T > 0$. Moreover, this solution w depends continuously on the data $(w_T^0, w_T^1) \in W_{a,0}^{1,2}(\Omega, d\mu) \times L^2(\Omega, d\mu)$. Besides, Theorem 3.1 and properties (i)–(iii) of the weight function $a : \Omega \to \mathbb{R}$ imply that the following transmission conditions at the origin hold true

$$\sum_{i=1}^N \lim_{\xi_i \searrow 0} \left[w\big(\xi_i \frac{v_i}{|v_i|}\big) a\Big(\xi_i \frac{v_i}{|v_i|}\Big)\right] = 0,$$
$$\sum_{i=1}^N \lim_{\xi_i \searrow 0} \left[a\Big(\xi_i \frac{v_i}{|v_i|}\Big) \frac{dw\big(\xi_i \frac{v_i}{|v_i|}\big)}{d\xi}\right] = 0.$$

Moreover, arguing as in Theorem 5.1, it can be shown that there exists a constant $C^* > 0$ such that

$$\sum_{i=1}^N \ell_i a(M_i) \int_0^T \left| \frac{\partial w(t, M_i)}{\partial v_i} \right|^2 dt \geqslant C^* E_w(T), \tag{6.8}$$

provided the weight function $a : \Omega \to \mathbb{R}$ satisfies properties (i)–(iii) and (5.14), where

$$\begin{aligned} E_w(t) &= \frac{1}{2} \int_\Omega \left[|w_t|^2 + a |\nabla^\mu w|^2 \right] d\mu \\ &= \frac{1}{2} \sum_{i=1}^N \int_0^{\ell_i} \left[\left| w_t\big(t, \xi \frac{v_i}{|v_i|}\big) \right|^2 + a\big(\xi \frac{v_i}{|v_i|}\big) \left| \frac{d}{d\xi} w\big(t, \xi \frac{v_i}{|v_i|}\big) \right|^2 \right] d\xi, \quad \forall\, t \in [0,T], \end{aligned}$$

is the energy of the weak solution w and it is conserved through time. Since $E_w(0) = E_w(T)$ and

$$E_w(T) = \frac{1}{2} \left[\|w_T^1\|^2_{L^2(\Omega, d\mu)} + \|\nabla^\mu w_T^0\|^2_{L^2(\Omega, a\, d\mu)^2} \right], \tag{6.9}$$

it follows from the direct inequality (5.4) that

$$\ell_i a(M_i) \int_0^T \left| \frac{\partial w(t, M_i)}{\partial v_i} \right|^2 dt \leqslant D\, E_w(0) = D\, E_w(T).$$

Thus, the right hand side of (6.4) defines a continuous linear form with respect to $(w_T^0, w_T^1) \in W_{a,0}^{1,2}(\Omega, d\mu) \times L^2(\Omega, d\mu)$ for each $T > 0$. Thus, a solution u by transposition of (6.1)–(6.3) is unique in $C\left([0,\infty); L^2(\Omega, d\mu)\right)$. The following theorem is a consequence of the classical results of existence and uniquencess of solutions of nonhomogeneous evolution equations. Full details can be found in [19] and [29].

Theorem 6.1. *For any* $F = [f_1, f_2, \ldots, f_N]^t \in L^2(0,T;\mathbb{R}^N)$, $y_0 \in L^2(\Omega, d\mu)$, *and* $y_1 \in W_a^{-1,2}(\Omega, d\mu)$ *transmission problem* (6.1)–(6.3) *has a unique solution defined by transposition*

$$u \in C\left([0,T]; L^2(\Omega, d\mu)\right), \; u_i \in C^1\left([0,\infty); H_a^{-1}(0,\ell_i)\right) \quad \text{for each } i = 1, \ldots, N.$$

Moreover, the map $(y_0, y_1, F) \mapsto \{u, u_t\}$ *is linear and there exists a constant* $C(T) > 0$ *such that*

$$\begin{aligned}\|u\|_{L^\infty(0,T;L^2(\Omega,d\mu))} &+ \sum_{i=1}^N \|(u_i)_t\|_{L^\infty(0,T;H_a^{-1}(0,\ell_i))} \\ &\leqslant C(T)\left[\|y_0\|_{L^2(\Omega,d\mu)} + \sum_{i=1}^N \|y_{1,i}\|_{H_a^{-1}(0,\ell_i)} + \sum_{i=1}^N \|f_i\|_{L^2(0,T)}\right].\end{aligned}$$

We are now in a position to prove the main result of this section.

Theorem 6.2. *Let* $a : \overline{\Omega} \to \mathbb{R}$ *be a weight function satisfying properties (i)–(iii) and* (5.14). *Let* T_a *be a value defined as in* (5.20). *Then, for any* $T > T_a$ *and* $(y_0, y_1) \in L^2(\Omega, d\mu) \times W_a^{-1,2}(\Omega, d\mu)$, *there exists a control function* $F = [f_1, f_2, \ldots, f_N]^t \in L^2(0,T;\mathbb{R}^N)$ *such that the corresponding solution of* (6.1)–(6.3) *(in the sense of transposition) satisfies condition* $(y(T), y_t(T)) \equiv (0,0)$, *i.e. the system* (6.1)–(6.3) *is boundary null controllable in time* $T > T_a$.

Proof. Let $\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} \in L^2(\Omega, d\mu) \times W_a^{-1,2}(\Omega, d\mu)$, $\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}$, $\begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in W_{a,0}^{1,2}(\Omega, d\mu) \times L^2(\Omega, d\mu)$ be arbitrary pairs. Let w and $\widehat{w}$ be the weak solutions of the backward problem (6.5)–(6.7) with final conditions $\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}$ and $\begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix}$, respectively. Let us define the bilinear form Λ on $W_{a,0}^{1,2}(\Omega, d\mu) \times L^2(\Omega, d\mu)$ as follows

$$\Lambda\left(\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}, \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix}\right) := \sum_{i=1}^N a(M_i) \int_0^T (w_i)_x(t,\ell_i)(\widehat{w}_i)_x(t,\ell_i)\,dt,$$
$$\forall \begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}, \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in W_{a,0}^{1,2}(\Omega, d\mu) \times L^2(\Omega, d\mu).$$

Then, in view of estimate (6.8), representation (6.9), and due to Theorem 5.1 and observability inequality (5.20), we deduce that the bilinear form

$$\Lambda : \left[W_{a,0}^{1,2}(\Omega, d\mu) \times L^2(\Omega, d\mu)\right]^2 \to \mathbb{R}$$

is continuous and coercive on $W^{1,2}_{a,0}(\Omega, d\mu) \times L^2(\Omega, d\mu)$ provided $T > T_a$. Thus, by the Lax-Milgram Lemma, variational problem

$$\Lambda\left(\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}, \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix}\right) = \left\langle \Phi\left(y_1\right), \Phi\left(\widehat{w}(0)\right)\right\rangle_{\prod_{i=1}^N H_a^{-1}(0,\ell_i); \prod_{i=1}^N H_{a,0}^1(0,\ell_i)} - \int_\Omega y_0 \widehat{w}_t(0)\, d\mu,$$
$$\forall \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in V^1_{a,0}(\Omega) \times L^2(\Omega)$$

has a unique solution $\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix} \in W^{1,2}_{a,0}(\Omega, d\mu) \times L^2(\Omega, d\mu)$. Then setting $f_i = (w_i)_x(t, \ell_i)$, for each $i = 1, \dots, N$, and $T > T_a$, where

$$w \in C^1([0,T]; L^2(\Omega, d\mu)) \cap C([0,T]; W^{1,2}_{a,0}(\Omega, d\mu))$$

is a weak solution of the backward problem (6.5)–(6.7) with $\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}$ as the final data, we see that

$$\begin{aligned}\sum_{i=1}^N a(M_i) \int_0^T f_i(t) (\widehat{w}_i)_x(t, \ell_i)\, dt &= \sum_{i=1}^N a(M_i) \int_0^T (w_i)_x(t, \ell_i) (\widehat{w}_i)_x(t, \ell_i)\, dt \\ &= \Lambda\left(\begin{bmatrix} w_T^0 \\ w_T^1 \end{bmatrix}, \begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix}\right) \\ &= \left\langle \Phi\left(y_1\right), \Phi\left(\widehat{w}(0)\right)\right\rangle_{\prod_{i=1}^N H_a^{-1}(0,\ell_i); \prod_{i=1}^N H_{a,0}^1(0,\ell_i)} - \int_\Omega y_0 \widehat{w}_t(0)\, d\mu, \quad (6.10)\end{aligned}$$

for all $\begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in W^{1,2}_{a,0}(\Omega, d\mu) \times L^2(\Omega, d\mu)$.

On the other hand, if y is the solution by transposition of the problem (6.1)–(6.3), then equality (6.4) implies that, for all $\begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in W^{1,2}_{a,0}(\Omega, d\mu) \times L^2(\Omega, d\mu)$, we have

$$\begin{aligned}&\sum_{i=1}^N a(M_i) \int_0^T f_i(t) (\widehat{w}_i)_x(t, \ell_i)\, dt \\ &\quad = -\left\langle \Phi\left(u_t(T)\right), \Phi\left(\widehat{w}_T^0\right)\right\rangle_{\prod_{i=1}^N H_a^{-1}(0,\ell_i); \prod_{i=1}^N H_{a,0}^1(0,\ell_i)} + \int_\Omega u(T) \widehat{w}_T^1\, d\mu \\ &\qquad + \left\langle \Phi\left(y_1\right), \Phi\left(\widehat{w}(0)\right)\right\rangle_{\prod_{i=1}^N H_a^{-1}(0,\ell_i); \prod_{i=1}^N H_{a,0}^1(0,\ell_i)} - \int_\Omega y_0 \widehat{w}_t(0)\, d\mu. \quad (6.11)\end{aligned}$$

Comparing the last relations (6.10)–(6.11), we obtain

$$-\left\langle \Phi\left(u_t(T)\right), \Phi\left(\widehat{w}_T^0\right)\right\rangle_{\prod_{i=1}^N H_a^{-1}(0,\ell_i); \prod_{i=1}^N H_{a,0}^1(0,\ell_i)} + \int_\Omega u(T) \widehat{w}_T^1\, d\mu = 0,$$

for all $\begin{bmatrix} \widehat{w}_T^0 \\ \widehat{w}_T^1 \end{bmatrix} \in W_{a,0}^{1,2}(\Omega, d\mu) \times L^2(\Omega, d\mu)$.

From this we finally deduce that $(u(T), u_t(T)) \equiv (0,0)$, i.e. the system (6.1)–(6.3) is boundary null controllable in time $T > T_a$. $\square$

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References

1. H.R. Beyer, *Beyond Partial Differential Equations: On Linear and Quasi-Linear Abstract Hyperbolic Evolution Equations*, Series: Lecture Notes in Mathematics, Vol. 1898, *Springer-Verlag*, Berlin, 2007.
2. P. Cannarsa, P. Martinez, J. Vancostenoble, *Carleman estimates and null controllability for boundary-degenerate parabolic operators*, *C. R. Math. Acad. Sci. Paris* **347** (2009), no. 3-4, 147–152.
3. F. Alabau-Boussouira, P. Cannarsa, G. Leugering, *Control and stabilization of degenerate wave equations*, *SIAM J. Control Optim.*, **55** (3) (2017), 2052–2087.
4. P. Cannarsa, P. Martinez, J. Vancostenoble, *Sharp estimate of the cost of controllability for a degenerate parabolic equation with interior degeneracy*, *Minimax Theory Appl.* **6** (2021), no. 2, 251–280.
5. S. Čanić, E.H. Kim, *A class of quasilinear degenerate elliptic problems*, *J. Differential Equations.*, **189**(2003), 71–98.
6. M. Colombo, A. Figalli, *Regularity results for very degenerate elliptic equations*, *J. Math. Pures Appl.*, **101**(2014), 94–117.
7. R. Dáger, E. Zuazua, *Wave Progagation, Observation and Control in* 1*-d Flexible Multi-Structures*, Series: Mathématiques and Applications, Vol. 50, *Springer-Verlag*, Berlin, 2006.
8. S. Ervedoza, E. Zuazua, *The Wave Equation: Control and Numerics*, In: Control of Partial Differential Equations. Lecture Notes in Mathematics, vol 2048. *Springer*, Berlin, Heidelberg, 2012.
9. Ch. Gaevski, K. Greger, K. Zacharias, *Nonlinear operator equations and operator differential equations*, *Mir*, Moskow, 1978 (in Russian).
10. M. Gueye, *Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations*, *SIAM J. Control Optim.*, **52**(2014), 2037–2054.
11. A. Greenleaf, Y. Kurylev, M. Lassas, G. Uhlmann, *Cloaking devices, electromagnetic wormholes, and transformation optics*, *SIAM Rev.*, **51**(2009), 3–33.
12. P.I. Kogut, G. Leugering, *Optimal and approximate boundary control of an elastic body with quasistatic evolution of damage*, *Mathematical Methods in the Applied Sciences*, **38**(2015), 2739–2760.
13. P.I. Kogut, G. Leugering, *Optimal Control Problems for Partial Differential Equations on Reticulated Domains. Approximation and Asymptotic Analysis*, Series: Systems and Control, *Birkhäuser Verlag*, Boston, 2011.
14. P. I. Kogut, O. P. Kupenko, G. Leugering, *Well-posedness and boundary observability of strongly degenerate hyperbolic systems on star-shaped planar network*, *Pure and Applied Functional Analysis PAFA*, 2022 **in press**.

15. P. I. Kogut, O. P. Kupenko, G. Leugering, *On boundary exact controllability of one-dimensional wave equations with weak and strong interior degeneration*, Mathematical Methods in the Applied Sciences., 2021. The article is published online. The link: http://doi.org/10.1002/mma.7811.
16. J.E. Lagnese, G. Leugering, and E.J.P.G. Schmidt, *Modelling, analysis and control of multi-link flexible structures*, Series: Systems and Control: Foundations and Applications, *Birkhäuser*, Basel, 1994.
17. I. Lasiecka, J. Sokolowski, *Sensitivity analysis of optimal control problems for wave equations, SIAM J. Control Optim.*, **29** (1991), 1128–1149.
18. G. Leugering, and E. Zuazua, *On exact controllability of generic trees*, in: Contrôle des Systémes Gouvernés par des Equations aux Dérivées Partielles (Nancy, 1999), ESAIM Proc. 8, Soc. Math. Appl. Indust., Paris, 2000, 95–105.
19. J.-L. Lions, E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, Vol.1, *Dunon*, Paris, 1968.
20. G. Lumer, R.S. Phillips, *Dissipative operators in a Banach space*, *Pacific Journal of Mathematics*, **11** (2) (1961), 679–698.
21. G. Marinoschi, R. M. Mininni, S. Romanelli, *An optimal control problem in coefficients for a strongly degenerate parabolic equation with interior degeneracy*, *J. Optim. Theory Appl*, **173**(2017), 56–77.
22. V.A. Rukavishnikov, E.I. Rukavishnikova, *Dirichlet problem with degeneration of the input data on the boundary of the domain*, *Differential Equations*, **52**(5)(2016), 681–685.
23. F. Santambrogio, V. Vespri, *Continuity in two dimensions for a very degenerate elliptic equation*, *Nonlinear Anal.*, **73**(2010), 3832–3841.
24. V.V. Zhikov, *Connection and averaging. Examples of fractal conductivity*, *Mat. Sb.*, **187** (8) (1996), 3–40.
25. V.V. Zhikov, *On an extension and an application of the method of two-scale convergence*, *Mat. Sb.*, **191** (7) (2000), 31–72.
26. V.V. Zhikov, *Averaging of problems in elasticity theory on singular structures*, *Izv. Ross. Akad. Nauk, Ser. Mat.*, **66** (2) (2002), 81–148.
27. V.V. Zhikov, *A note on Sobolev spaces*, *Journal of Mathematical Sciences*, **129** (1) (2005), 3593–3595.
28. E. Zuazua, *Controllability of Partial Differential Equations*, 3rd cycle. Castro Urdiales (Espagne), 2006.
29. E. Zuazua, *Controlabilidad exacta y estabilización la ecuación de ondas*, Textos de métodos matemáticos 23, Universidad Federal do Rio de Janeiro, 1990.

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