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Arithmetic

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Chapter 1

Arithmetic

Arithmetic introduces the elementary mathematical operators (addition, subtraction, multiplication, and division), along with more advanced notions such as exponents, decimal notation, and percentages. Operations such as the absolute value and factorial are also defined here.

- Whole numbers
  - The *whole numbers* are created using the *digits* 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
  - *Place value* is used to represent whole numbers exceeding nine. Reading from right to left, the meaning of the digits are ones, tens, hundreds, thousands, etc., for example,

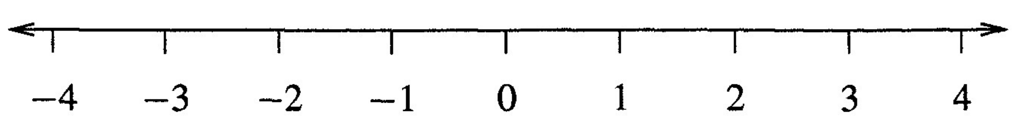
    $$9876 = 9 \cdot 1000 + 8 \cdot 100 + 7 \cdot 10 + 6 \cdot 1$$

    which is read as "nine thousand, eight hundred, seventy-six."
  - For readability, commas are used to place digits into groups of three proceeding from right to left for whole numbers containing four or more digits, for example, the speed of light is approximately

    $$299,792,458$$

    meters per second

- Number lines
  - A *number line* typically places *negative numbers* to the left of 0 and *positive numbers* to the right of 0



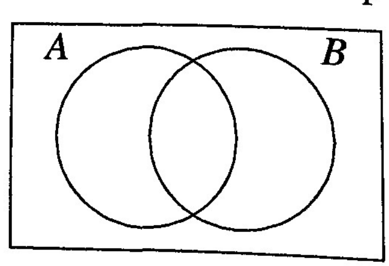
  - The point on a number line at 0 is known as the *origin*
  - The + sign is generally dropped for brevity, for example, +4 is usually written as simply 4

- Two different numbers that are equal distance from the origin (for example, -3 and 3) are known as *opposites*
- The $<$ symbol is read "is less than," for example, $1 < 4$ means 1 lies to the left of 4 on the number line
- The $>$ symbol is read "is greater than," for example, $-2 > -4$ means that -2 lies to the right of -4 on the number line

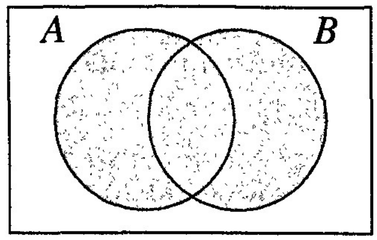
- Common mathematical symbols
  - The $=$ symbol is read "equals" or "is equal to." Two quantities on either side of an equals sign have the same value.
  - The $\neq$ symbol is read "is not equal to," for example, $3 \neq 7$.
  - The $\pm$ symbol is read "plus or minus," for example, $x = \pm 3$ means that x is equal to 3 or x is equal to -3
- Sets
  - A *set* is a collection of objects
  - The objects that comprise a set are often known as *elements* of the set
  - Uppercase letters are typically used to denote sets
  - Sets are typically defined by placing their elements in curly braces, for example,

    $$A = \{\text{Cubs, Sox, Mets}\} \qquad B = \{1, 3, 5, 7\} \qquad C = \{\pm 1, \pm 3, \pm 5, \ldots\}$$

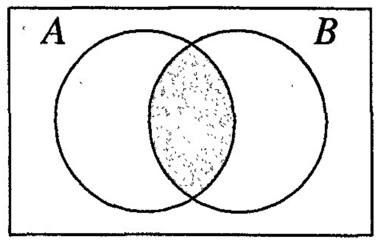
    where the ellipsis (...) indicates repetition (etc. or "and so on")
  - The number of elements in a set is known as its *cardinality*, for example, $N(A) = 3$, $N(B) = 4$, and $N(C)$ is "countably infinite"
  - If every element of a set B is also an element of a set C, as is true for the sets B and C defined above, then B is called a *subset* of C. This is denoted by: $B \subset C$.
  - The symbol $\in$ is read as "is an element of," for example, $3 \in C$
  - The ordering of elements in a set is not significant, for example, $\{2, 4, 6\} = \{6, 2, 4\}$
  - Two sets that have no elements in common are called *disjoint* or *mutually exclusive*
  - The *empty set* or *null set* $\emptyset$ is a set that contains no elements
  - *Venn diagrams* can be used to visualize the relationship between sets



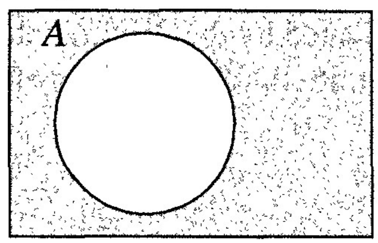
- Operations on sets
  * Union: $A \cup B$ is read "A union B" and is the set of all elements in either A or B



  * Intersection: $A \cap B$ is read "A intersect B" and is the set of all elements in both A and B



  * Complement: A' is read "A complement" and is the set of all elements *not* in A



- The integers and some special subsets of the integers

Integers:	$Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
Natural numbers (positive integers):	$N = Z^+ = \{1, 2, \ldots\}$
Whole numbers (nonnegative integers):	$\{0, 1, 2, \ldots\}$
Negative integers:	$Z^- = \{-1, -2, \ldots\}$
Even integers:	$\{0, \pm 2, \pm 4, \ldots\}$
Odd integers:	$\{\pm 1, \pm 3, \pm 5, \ldots\}$

- Basic arithmetic operations
  - Addition: "Two plus three equals five" is written as

$$2 + 3 = 5$$

where 2 and 3 are the *addends* and 5 is the *sum*. The number line interpretation of this addition problem is "beginning at 2 and moving 3 units to the right results in 5." Addition can be commuted, for example, $2 + 3 = 5$ and $3 + 2 = 5$.

  - Subtraction: "Five minus three equals two" is written as

$$5 - 3 = 2$$

where 5 is the *minuend*, 3 is the *subtrahend*, and 2 is the *difference*. The number line interpretation of this subtraction problem is "beginning at 5 and moving 3 units to the left results in 2."

- Multiplication: "Five times three equals fifteen" is written as

$$5 \cdot 3 = 15 \qquad \text{or} \qquad 5 \times 3 = 15 \qquad \text{or} \qquad (5)(3) = 15$$

  where 5 and 3 are the *factors* and 15 is the *product*. The interpretation of multiplication is repeated addition ($5+5+5=15$ or $3+3+3+3+3=15$). Multiplication can be commuted, for example, $5 \cdot 3 = 15$ and $3 \cdot 5 = 15$. The product of two negative numbers is a positive number, for example, $(-5)(-3) = 15$. The product of a positive number and a negative number is a negative number, for example, $(5)(-3) = (-5)(3) = -15$.

- Division: "Fifteen divided by three equals five" is written as

$$15 \div 3 = 5 \qquad \text{or} \qquad 15/3 = 5 \qquad \text{or} \qquad \frac{15}{3} = 5$$

  where 15 is the *numerator* and 3 is the *denominator* and 5 is the *quotient* (or *ratio*). The interpretation of division is dividing the number in the numerator into the number of equal parts indicated by the denominator, then noting the size of the parts. Division by zero is not permitted. The quotient of two negative numbers is a positive number, for example,

$$\frac{-15}{-3} = 5$$

  The quotient of a positive number and a negative number is a negative number, for example,

$$\frac{-15}{3} = \frac{15}{-3} = -5$$

  The *remainder* is the amount remaining after the division of two integers, for example, when 17 is divided by 3, the remainder is 2.

- Multiples
  - A *multiple* is the result of multiplying a number by an integer
  - The *multiples* of 7 are 7, 14, 21, 28, 35, ...
  - The *least common multiple* (LCM) of two or more natural numbers is the smallest common multiple, for example, $\text{LCM}(4, 6) = 12$

- Factors
  - A *factor* is a natural number that can be evenly divided (divided with a remainder of zero) into another natural number. The *factors* of 24, for example, are 1, 2, 3, 4, 6, 8, 12, and 24
  - The *greatest common factor* (GCF) is the largest factor common to two or more numbers, for example, $\text{GCF}(24, 36) = 12$

- Prime numbers
  - *Prime numbers* have exactly two *distinct* factors, for example, 17 has factors 1 and 17
  - The first ten primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29
  - There is no largest prime number
  - The *prime factorization* of an integer writes it as a product of primes, for example,

$$120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$$

  - *Composite numbers* are positive integers greater than 1 that are not prime
  - The first ten composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, 18

- Factorial: for a whole number n, the product of the consecutive integers from 1 to n is known as "n factorial" and is denoted by

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$$

  for example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. By definition, $0! = 1$.

- Modulo: the *modulo operator* returns the remainder when the second integer is divided into the first integer, for example,

$$57 \bmod 10 = 7$$
$$48 \bmod 8 = 0$$
$$225 \bmod 2 = 1$$

- Fractions
  - The fraction a/b is defined for any a and any $b \neq 0$
  - The fraction a/b is sometimes referred to as the *ratio* of a to b
  - If $a > b$ then the fraction a/b is an *improper fraction*, for example, $17/3$
  - A *mixed number* is the sum of an integer and a fraction, for example, $2\frac{1}{5} = 2 + \frac{1}{5}$
  - To convert an improper fraction to a mixed number, divide the numerator by the denominator, which gives the whole number; the remainder in this division is the numerator of the fractional portion of the mixed number, and the denominator is the same as in the original improper fraction, for example,

$$\frac{20}{3} = \frac{18}{3} + \frac{2}{3} = 6\frac{2}{3}$$

  - To convert a mixed number to an improper fraction, multiply the denominator by the whole number, then add the numerator, which gives the numerator of the improper fraction. The denominator stays the same, for example,

$$6\frac{2}{3} = \frac{3 \cdot 6 + 2}{3} = \frac{20}{3}$$

- Basic arithmetic operations
  * Addition (identical denominators)

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$

  * Addition (different denominators)

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

  * Subtraction (identical denominators)

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

  * Subtraction (different denominators)

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

  * Multiplication

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

  * Division ("invert and multiply")

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

    or, equivalently

$$\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$$

- A fraction can be expressed in *lowest terms* by writing the *prime factorization* of the numerator and denominator and canceling equal factors, for example,

$$\frac{24}{36}=\frac{\not{2}\cdot\not{2}\cdot 2\cdot\not{3}}{\not{2}\cdot\not{2}\cdot 3\cdot\not{3}}=\frac{2}{3}$$

- Fractions can also be reduced to *lowest terms* by dividing the numerator and the denominator by the greatest common factor of the numerator and denominator, for example,

$$\frac{24}{36}=\frac{24\div 12}{36\div 12}=\frac{2}{3}$$

- The *least common denominator* (LCD) is the smallest positive integer divisible by each of the denominators of a group of fractions, for example, the least common denominator of the fractions

$$\frac{1}{3},\frac{5}{6},\frac{2}{9}$$

is 18. This is useful for adding and subtracting fractions, for example,

$$\frac{1}{3}+\frac{5}{6}-\frac{2}{9}=\frac{6}{18}+\frac{15}{18}-\frac{4}{18}=\frac{6+15-4}{18}=\frac{17}{18}$$

The LCD of a group of fractions is the LCM of their denominators.

- Two numbers are *reciprocals* if their product is 1, for example, $5/7$ and $7/5$ are reciprocals, -3 and $-1/3$ are reciprocals

- Rational numbers, irrational numbers, and real numbers
  - A *rational number* is the ratio of two integers (the denominator can't be zero), for example, $7/4$, 17, $-2/3$, 0
  - The set of all rational numbers is denoted by Q, as in Quotient
  - An *irrational number* is a real number that is not rational, for example, $\sqrt{17}$
  - Two important irrational numbers are

$$\pi = 3.141592653589793\ldots \qquad \text{and} \qquad e = 2.718281828459045\ldots$$

  - The *real numbers* are comprised of the rational and irrational numbers
  - A *real number* can be expressed as an infinite decimal
  - The set of all real numbers is denoted by $\mathbb{R}$

- Decimals
  - A number expressed as a *decimal* often results from performing a division
  - Whole numbers are placed to the left of the *decimal point* (which is formatted as a period); fractional values are placed to the right of the decimal point
  - The digits to the left of the decimal point (reading right to left) are ones, tens, hundreds, thousands, ...
  - The digits to the right of the decimal point (reading left to right) are tenths, hundredths, thousandths, ...
  - Examples of decimals:

$$21\frac{3}{10}=21.3 \qquad 757\frac{38}{100}=757.38 \qquad \frac{7}{100}=0.07$$

Including the leading 0 on 0.07 is optional, that is, 0.07 and .07 are acceptable ways to write the decimal

  - A bar is used to denote repeating digits, for example,

$$1/22 = 0.045454545\ldots = 0.0\overline{45}$$

- A rational number a/b can be expressed as a *terminating decimal* if, in the long division process, one eventually gets a remainder of 0, for example,

$$1/5 = 0.2 \qquad 11/8 = 1.375$$

- A rational number a/b can be expressed as a *repeating decimal* if b divides a so that the decimal has a repeating pattern of integer digits, for example,

$$1/3 = 0.3333\ldots \qquad 2/7 = 0.\overline{285714}$$

- Rational numbers can be written as terminating decimals or repeating decimals
- Irrational numbers have a decimal form that is non-terminating and non-repeating, for example,

$$\pi = 3.141592653589793\ldots \qquad \sqrt{2} = 1.414213562373095\ldots$$

- A decimal can be *truncated* by eliminating certain right-hand digits, for example, 365.242199 truncated to four decimal places beyond the decimal point is 365.2421
- A decimal can be *rounded* by eliminating certain right-hand digits after inspecting the left-most digit eliminated, for example, 365.242199 rounded to four decimal places beyond the decimal point is 365.2422

• Rounding

- Rounding a numerical value is the process of finding an approximately equal value that has a simpler expression to a prescribed level of accuracy
  * 337 rounded to the nearest 10 is "rounded up" to 340
  * 337 rounded to the nearest 100 is "rounded down" to 300
  * $8/3$ rounded to the nearest integer is 3
  * $8/3$ rounded to the nearest hundredth is 2.67
  * \$12.3457 rounded to the nearest penny is \$12.35
  * \$12.35 rounded to the nearest dollar is \$12
  * π rounded to the nearest hundred thousandth is 3.14159
  * π rounded to the nearest ten thousandth is 3.1416
- When the numerical value being rounded falls exactly halfway between two rounded values, one common practice is to round up
  * 42.5 rounded to the nearest integer is 43
  * −42.5 rounded to the nearest integer is −42
- Rounding is useful for quickly determining an approximate solution to an arithmetic problem, for example, the sum of 4017 and 2976 is *approximately* $4000 + 3000 = 7000$

- Absolute value
  - The *absolute value* of a real number a is the distance between a and the origin on a number line. Equivalently, the absolute value is the numerical part of a number, ignoring its sign.
  - For any real number a, the absolute value of a is

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

  - The absolute value of $a-b$, denoted $|a-b|$, is the distance between a and b on a number line
  - Absolute value properties
    * $|a| \geq 0$
    * $|-a| = |a|$
    * $|a-b| = |b-a|$
    * $|ab| = |a| \cdot |b|$
    * $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

- Floor and ceiling functions
  - The floor and ceiling functions convert real numbers to nearby integers
  - The *floor function*, also known as the *greatest integer function*, returns the largest integer that is less than or equal to its argument, for example,

$$\lfloor 3.7 \rfloor = 3 \qquad \left\lfloor \frac{17}{2} \right\rfloor = 8 \qquad \lfloor 17 \rfloor = 17 \qquad \lfloor -5.2 \rfloor = -6$$

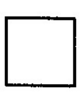
  - The *ceiling function*, also known as the *least integer function*, returns the smallest integer that is greater than or equal to its argument, for example,

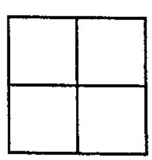
$$\lceil 3.7 \rceil = 4 \qquad \left\lceil \frac{17}{2} \right\rceil = 9 \qquad \lceil 17 \rceil = 17 \qquad \lceil -5.2 \rceil = -5$$

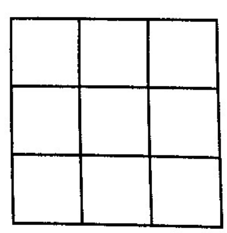
- Percentages
  - A percentage is a part of a whole expressed in hundredths, for example, the fraction $17/100$, the decimal 0.17, and the percentage 17% are all equal
  - To convert a fraction to a percentage, multiply by 100
  - To convert a percentage to a fraction, divide by 100
  - Use fractions or decimals when finding percentages, for example, to find 30% of 200, compute $(0.3)(200) = 60$

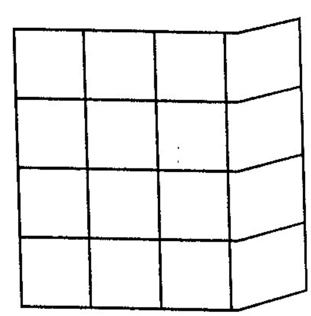
- The *percentage change* in a quantity is the difference between the final and original amounts, divided by the original amount. The change from \$100 to \$120, for example, is
$$\frac{\text{final} - \text{original}}{\text{original}} = \frac{120-100}{100} = \frac{20}{100} = 0.2$$
which corresponds to a 20% increase. The change from \$100 to \$80 is
$$\frac{\text{final} - \text{original}}{\text{original}} = \frac{80-100}{100} = -\frac{20}{100} = -0.2$$
which corresponds to a 20% decrease.

- Exponents
  - The expression x^n means multiply x by itself n times, provided that n is a natural number, for example, $5^3 = 5 \cdot 5 \cdot 5 = 125$
  - In the expression x^n, x is the *base*
  - In the expression x^n, n is the *exponent* or *power*
  - The n in the expression x^n is typeset as a *superscript*
  - Exponents of two and three are termed "squared" and "cubed"
    * The expression 5^2 is read "five squared"
    * The expression 7^3 is read "seven cubed"
    * The expression 10^4 is read "ten raised to the fourth power"
  - A leading negative sign is not included in the exponentiation unless parentheses are included, for example,
$$-3^2 = -9 \qquad (-3)^2 = 9$$
  - Exponents can be used to write a prime factorization more compactly, for example
$$120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 = 2^3 \cdot 3 \cdot 5$$
  - The *perfect squares* $1, 4, 9, 16, 25, \ldots$ are the squares of the positive integers. These correspond to the areas of squares with integer side lengths.

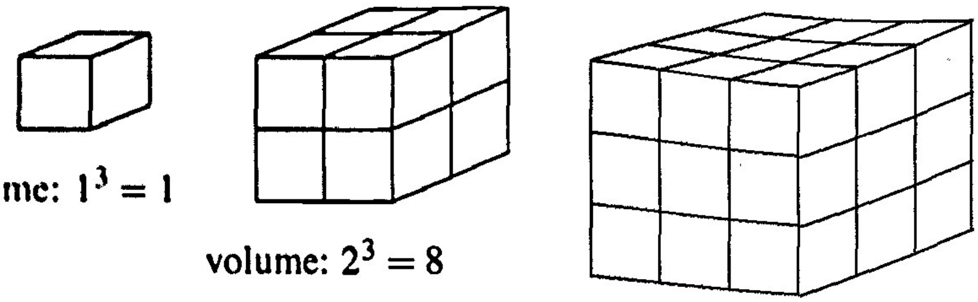

area: $1^2 = 1$


area: $2^2 = 4$


area: $3^2 = 9$


area: $4^2 = 16$

- The *perfect cubes* 1, 8, 27, 64, 125, ... are the cubes of the positive integers. These correspond to the volumes of cubes with integer side lengths.



volume: $1^3 = 1$

volume: $2^3 = 8$

volume: $3^3 = 27$

- Radicals
  - The square root of x, written as $\sqrt{x}$, denotes the number which, when multiplied by itself, equals x, for example, $\sqrt{16} = 4$
  - The cube root of x, written as $\sqrt[3]{x}$, denotes the number which, when multiplied by itself three times, equals x, for example, $\sqrt[3]{-8} = -2$
  - A radical implies a single *principal root*, for example, $\sqrt{9} = 3$, not ± 3

- *Measurement systems:* make sure units cancel properly when converting from one unit of measure to another, for example, the number of seconds in a non-leap year is

$$365 \text{ days} \times \frac{24 \text{ hours}}{\text{day}} \times \frac{60 \text{ minutes}}{\text{hour}} \times \frac{60 \text{ seconds}}{\text{minute}} = 31,536,000 \text{ seconds}$$

The factors that are fractions which are used to convert units are known as *unit multipliers*.

- Order of operations ("Please Excuse My Dear Aunt Sally" or PEMDAS): The priority order of operations when evaluating a complicated expression is
  - Parentheses (or brackets $[\cdot]$ or braces $\{\cdot\}$)
  - Exponents
  - Multiplication, Division
  - Addition, Subtraction

Work from left to right for operations with the same priority, for example,

$$\begin{aligned}
(2+3)\cdot 4 + 6^2/9 - 5 &= 5\cdot 4 + 6^2/9 - 5 && \text{parentheses} \\
&= 5\cdot 4 + 36/9 - 5 && \text{exponents} \\
&= 20 + 36/9 - 5 && \text{multiplication} \\
&= 20 + 4 - 5 && \text{division} \\
&= 24 - 5 && \text{addition} \\
&= 19 && \text{subtraction}
\end{aligned}$$

- Scientific notation
  - Very large and very small numbers are efficiently written using *scientific notation*, which is helpful for avoiding writing all of the leading or trailing zeros
  - Normalized form: $\pm a \times 10^n$, where $1 \leq a < 10$ and n is an integer (the 10 and exponent are typically omitted when $n = 0$); $n < 0$ for very small numbers (negative exponents are defined in the next chapter) and $n > 0$ for very large numbers, for example, 6.02×10^{23}

- Summation notation and formulas
  - The summation symbol $\sum$ includes an index (which is x in the example below); the lower limit of the index is given below the summation symbol; the upper limit is given above the summation symbol, for example,

$$\sum_{x=3}^{5} \sqrt{2x} = \sqrt{6} + \sqrt{8} + \sqrt{10}$$

  - One can show that the sum of the first n positive integers is

$$\sum_{x=1}^{n} x = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

  - One can show that the sum of the squares of the first n positive integers is

$$\sum_{x=1}^{n} x^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

- Arithmetic sequences and series
  - A *sequence* is a list of numbers (referred to as *terms*) in a specific order
  - The terms in a sequence are traditionally separated by commas, for example,

$$1, 4, 9, 16, 25, 36, \ldots$$

  - An *arithmetic sequence* is a list of numbers in a specific order with the difference between any two consecutive terms always the same, for example,

$$1, 3, 5, 7, 9, 11, \ldots \text{(common difference between consecutive terms is 2)}$$

    or

$$8, 5, 2, -1, -4, -7, \ldots \text{(common difference between consecutive terms is } -3\text{)}$$

  - The terms in an arithmetic sequence are denoted by $a_1, a_2, a_3, \ldots$

- The integers that index the a values are known as *subscripts*
- The *common difference* between consecutive terms in an arithmetic sequence is denoted by d
- A *recursive formula* for finding the nth term in an arithmetic sequence is

$$a_n = a_{n-1} + d$$

- A *general formula* for finding the nth term in an arithmetic sequence is

$$a_n = a_1 + (n-1)d$$

- An *arithmetic series* is a running sum of the terms in an arithmetic sequence, for example, the sum of the first four terms in the arithmetic sequence $1, 3, 5, 7, 9, 11, \ldots$ is

$$1 + 3 + 5 + 7 = 16$$

- The sum of the first n terms in an arithmetic sequence is

$$\begin{aligned}\sum_{i=1}^{n} a_i &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + \big(a_1 + (n-1)d\big) \\ &= \frac{n(a_1 + a_n)}{2}\end{aligned}$$

- Geometric sequences and series
  - A *geometric sequence* is a list of numbers in a specific order with the ratio between any two consecutive terms always the same, for example,

$$3, 6, 12, 24, 48, 96, \ldots \text{(common ratio between consecutive terms is 2)}$$

or

$$27, 9, 3, 1, \frac{1}{3}, \frac{1}{9}, \ldots \text{(common ratio between consecutive terms is } 1/3\text{)}$$

  - The terms in a geometric sequence are denoted by $a_1, a_2, a_3, \ldots$
  - The *common ratio* between consecutive terms in a geometric sequence is denoted by r
  - A *recursive formula* for finding the nth term in a geometric sequence is

$$a_n = a_{n-1} r$$

  - A *general formula* for finding the nth term in a geometric sequence is

$$a_n = a_1 r^{n-1}$$

- A *geometric series* is a running sum of the terms in a geometric sequence, for example, the sum of the first four terms in the geometric sequence 3, 6, 12, 24, 48, 96, ... is

$$3+6+12+24=45$$

- The sum of the first n terms in a geometric sequence with $r \neq 1$ is

$$\begin{aligned}\sum_{i=1}^{n} a_i &= a_1+a_2+a_3+\cdots+a_n \\ &= a_1+a_1r+a_1r^2+\cdots+a_1r^{n-1} \\ &= \frac{a_1(1-r^n)}{1-r}\end{aligned}$$

- When the common ratio r lies between -1 and 1, the *infinite geometric series* converges to

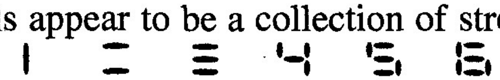
$$\sum_{i=1}^{\infty} a_i = \frac{a_1}{1-r} \qquad |r|<1$$

- Counting techniques
  - *Multiplication Rule (Fundamental Theorem of Counting):* If there are n_1 ways to make decision 1, n_2 ways to make decision 2, ..., n_k ways to make decision k, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways to make all of the decisions
  - *Permutations:* An arrangement of r objects taken from a set of n objects in a definite order is a *permutation.* The number of such permutations is

$$n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot(n-r+1)=\frac{n!}{(n-r)!}$$

  - *Combinations:* An arrangement of r objects taken from a set of n objects is a *combination.* The number of such combinations is

$$\binom{n}{r}=\frac{n!}{r!(n-r)!}$$

- Digits
  - The *Hindu–Arabic* numerals currently in use appear to have evolved from the *Brahmi* numerals
  - The first six numerals appear to be a collection of strokes consistent with the number that they represent: 
  - The symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are known as *digits*

- Number systems
  - The *decimal* number system, also known as the base 10 number system, uses digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
    - The choice of 10 as a base corresponds with the number of digits (fingers and thumbs) on both hands
    - *Place value* and the use of a sign (– for negative numbers) allows any number to be written in decimal using a sequence of digits, for example, denoting 365 in the decimal system as 365_{10},

$$365_{10} = 3 \cdot 10^2 + 6 \cdot 10^1 + 5 \cdot 10^0 = 3 \cdot 100 + 6 \cdot 10 + 5 \cdot 1$$

      where the subscript 10 on 365 denotes the base
  - The *octal* number system, also known as the base 8 number system, uses digits 0, 1, 2, 3, 4, 5, 6, 7
    - The choice of 8 as a base corresponds to ignoring opposing thumbs and just using the eight fingers
    - Octal is a useful number system for digital computers
    - *Place value* and the use of a sign (– for negative numbers) allows any number to be written in octal, for example 365_{10} can be written in octal as

$$555_8 = 5 \cdot 8^2 + 5 \cdot 8^1 + 5 \cdot 8^0 = 5 \cdot 64 + 5 \cdot 8 + 5 \cdot 1$$

  - The *binary* number system, also known as the base 2 number system, uses digits 0 and 1
    - The choice of 2 as a base corresponds to counting arms or legs, rather than fingers and thumbs
    - Binary is a useful number system for digital computers
    - *Place value* and the use of a sign (– for negative numbers) allows any number to be written in binary, for example $365_{10} = 555_8$ can be written in binary as

$$\begin{aligned} 101101101_2 &= 1 \cdot 2^8 + 0 \cdot 2^7 + 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\ &= 1 \cdot 256 + 0 \cdot 128 + 1 \cdot 64 + 1 \cdot 32 + 0 \cdot 16 + 1 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 1 \cdot 1 \end{aligned}$$

    - The digits in binary are known as *bits* when stored on a computer
    - It is easy to convert between octal and binary because one octal digit corresponds to three consecutive binary digits
  - Roman numerals comprise another number system that was supplanted by the Hindu–Arabic numerals. The meaning of the symbols is given in the table below.

Roman numeral symbol	I	V	X	L	C	D	M
Value	1	5	10	50	100	500	1000

* There is no zero in the Roman numeral system
* The convention is to place the Roman numerals in descending order, for example, MMXII is $1000+1000+10+1+1=2012$
* When a lesser-value Roman numeral precedes a greater-value Roman numeral, the lesser value is subtracted from the greater value. For example, MMCMXLIV is

  $$1000+1000+(1000-100)+(50-10)+(5-1)=2000+900+40+4=2944$$

* The first ten Roman numerals are I, II, III, IV, V, VI, VII, VIII, IX, X

Exercises

1.1 What is the prime factorization of 140?

1.2 Express $\frac{5}{6}+\frac{7}{9}-\frac{3}{2}$ as a single fraction.

1.3 Calculate $3(4-7)+|16/(6-8)|$.

1.4 Jill earns 10% in a stock mutual fund during a one-year period, then moves her funds to a second stock mutual fund where she earns 30% during the second year. What is Jill's overall rate of return for the two years? How does her overall rate of return compare to Logan's, if Logan earned 30% during the first year and 10% in the second year?

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1.1 What is the prime factorization of 140?

The prime factorization of 140 is

$$140 = 2 \cdot 2 \cdot 5 \cdot 7.$$

1.2 Express $\frac{5}{6} + \frac{7}{9} - \frac{3}{2}$ as a single fraction.

The first step to adding these fractions is to find the least common denominator.

- The multiples of 6 are 6, 12, 18, 24, ...
- The multiples of 9 are 9, 18, 27, 36, ...
- The multiples of 2 are 2, 4, 6, 8, ...

The least common multiple of 6, 9, and 2 is $\text{LCM}(6, 9, 2) = 18$, which is the least common denominator. Converting each of these fractions so that they each have denominator 18 results in

$$\frac{5}{6} + \frac{7}{9} - \frac{3}{2} = \frac{15}{18} + \frac{14}{18} - \frac{27}{18} = \frac{15 + 14 - 27}{18} = \frac{2}{18} = \frac{1}{9}.$$

1.3 Calculate $3(4-7) + |16/(6-8)|$.

First perform the operations within parentheses, then perform the multiplication and division, then apply the absolute value operator, and finally perform the addition. The steps are shown below.

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Since the answer does not depend on how much money Jill invests, it is acceptable to assume that she starts with \$100. Jill earns $(0.1)(\$100) = \$10$ during the first year, so she withdraws \$110 from the first stock mutual fund. She earns $(0.3)(\$110) = \$33$ during the second year, bringing her total to $\$110 + \$33 = \$143$. So the overall rate of return for the two years is 43%. The common error here is to simply add 10% and 30%, but this ignores the effect of *compounding*, which is making a return during the second year on the return that she made during the first year. Using the same logic on Logan's investment, he earns $(0.3)(\$100) = \$30$ at the end of the first year, and earns $(0.1)(\$130) = \$13$ during the second year, bringing his total to $\$130 + \$13 = \$143$. Their rates of return are identical.

1.5 What are the three numbers that should be placed in the blanks below so that the difference between consecutive numbers in the sequence is the same?

$$19, \_\_, \_\_, \_\_, 75$$

1.6 Express $3.\overline{87}$ as a ratio of integers.

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Since the difference between consecutive numbers in the sequence is the same, this is an arithmetic sequence. Using the arithmetic sequence notation, the first term is $a_1 = 19$. In order to find the common difference d, the fifth term of the arithmetic sequence can be expressed as

$$a_5 = a_1 + (5-1)d$$

or

$$75 = 19 + 4d$$

Solving this expression for d gives the common difference $d = 14$. The missing terms are 33, 47, and 61.

1.6 Express $3.\overline{87}$ as a ratio of integers.

Let x denote the fraction of interest, that is

$$x = 3.\overline{87}.$$

Since there are two repeating digits in $3.\overline{87}$, both sides of this equation should be multiplied by $10^2 = 100$:

$$100x = 387.\overline{87}.$$

When the first equation is subtracted from the second equation, the repeating portions of the numbers that lie to the right of the decimal point cancel, resulting in

$$100x - x = 387.\overline{87} - 3.\overline{87} = 384$$

or

$$99x = 384.$$

Finally, dividing by 99, the original repeating decimal can be expressed as

$$x = \frac{384}{99}.$$

1.7 What is the last digit in 3^{999}?

A calculator is not of use in solving this problem because of overflow. One approach is to write out the first few powers of 3:

$$3^0 = 1, \quad 3^1 = 3, \quad 3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 81, \quad 3^5 = 243, \quad \ldots.$$

Examining the last digits, one can conclude that they fall in a cycle with a period of 4 (that is, 1, 3, 9, 7, 1, 3, 9, 7, 1, ...). This means that every exponent that is evenly divisible by 4 (that is, 0, 4, 8, ...) is associated with a power of 3 that has last digit 1. Thus 3^{1000} has last digit 1. Moving back one in the sequence, the last digit in 3^{999} is 7.

1.8 A restaurant offers three appetizers, four entrees, and two desserts. How many ways are there to order one appetizer, one entree, and one dessert?

1.9 If 140% of a number is 280, then what is 60% of that number?

1.10 How many of the first 1000 positive integers are multiples of neither 6 nor 9?

1.8 A restaurant offers three appetizers, four entrees, and two desserts. How many ways are there to order one appetizer, one entree, and one dessert?

Using the multiplication rule, there are $3 \cdot 4 \cdot 2 = 24$ different ways to order one appetizer, one entree, and one dessert.

1.9 If 140% of a number is 280, then what is 60% of that number?

Since 140% of the number is 280, the number must be

$$\frac{280}{1.4} = 200$$

because 140% is equivalent to 1.4. Finally, 60% of 200 is

$$(0.6)(200) = 120$$

because 60% is equivalent to 0.6.

1.10 How many of the first 1000 positive integers are multiples of neither 6 nor 9?

There are 166 multiples of 6, which are

$$6, 12, 18, \ldots, 996$$

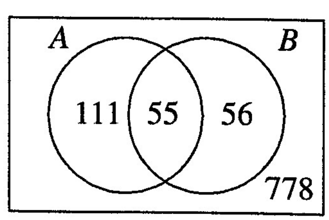
because $166 \cdot 6 = 996$. Likewise, there are 111 multiples of 9, which are

$$9, 18, 27, \ldots, 999$$

because $111 \cdot 9 = 999$. An integer is a multiple of both 6 and 9 if it is a multiple of the least common multiple of 6 and 9, which is $\text{LCM}(6, 9) = 18$. The 55 integers between 1 and 1000 that are multiples of both 6 and 9 are

$$18, 36, 54, \ldots, 990$$

because $55 \cdot 18 = 990$. Letting the set A denote the multiples of 6 and the set B denote the multiples of 9, the Venn diagram below shows the counts of the various numbers of integers in the four regions partitioned by the sets A and B.



To answer the original question, there are 778 integers between 1 and 1000 that are multiples of neither 6 nor 9.

1.11 How many ways are there to select a committee of three people from a group of ten people?

1.12 How many of the first 5000 positive integers are neither perfect squares nor perfect cubes?

1.11 How many ways are there to select a committee of three people from a group of ten people?

Since the order that the people are selected for the committee is not important, combinations are used to conclude that there are

$$\binom{10}{3} = \frac{10!}{3!7!} = \frac{10\cdot 9\cdot 8}{3\cdot 2\cdot 1} = \frac{720}{6} = 120$$

different committees that can be selected.

1.12 How many of the first 5000 positive integers are neither perfect squares nor perfect cubes?

Let S be the set of all perfect squares in the first 5000 positive integers and let C be the set of all perfect cubes in the first 5000 positive integers. The perfect squares are

$$1, 4, 9, 16, \ldots.$$

The number of perfect squares in the first 5000 positive integers is

$$N(S) = \lfloor\sqrt{5000}\rfloor = 70.$$

The perfect cubes are

$$1, 8, 27, 64, \ldots.$$

The number of perfect cubes in the first 5000 positive integers is

$$N(C) = \lfloor\sqrt[3]{5000}\rfloor = 17.$$

The positive integers that are both perfect squares and cubes are

$$1, 64, 729, 4096, \ldots,$$

that is, the positive integers raised to the sixth power. The number of perfect cubes in the first 5000 positive integers is

$$N(S\cap C) = \lfloor\sqrt[6]{5000}\rfloor = 4.$$

Using a well-known result from set theory, the number of elements of $S\cup C$ is

$$N(S\cup C) = N(S) + N(C) - N(S\cap C) = 70 + 17 - 4 = 83.$$

The number of integers between 1 and 5000 (inclusive) that are neither perfect squares nor perfect cubes is

$$N\left(S'\cap C'\right) = 5000 - N(S\cup C) = 5000 - 83 = 4917.$$

1.13 Find the number of distinct positive integer-valued solutions to the equation

$$a+b+c=9.$$

1.14 Find the number of trailing 0's at the end of 1000!.

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$$a+b+c=9.$$

One rather tedious solution to this problem is to enumerate all of the possible solutions, that is

$$a=1, b=1, c=7,$$
$$a=1, b=7, c=1,$$

etc. A more elegant and efficient solution to the problem is to frame the question in terms of nine balls (•, •, •, •, •, •, •, •, •) and two dividers (|, |). The solution to the equation corresponding to the arrangement

$$\bullet\bullet\bullet \mid \bullet\bullet\bullet\bullet\bullet \mid \bullet$$

for example, is

$$a=3, b=5, c=1,$$

because there are three balls to the left of the first divider, five balls between the two dividers and one ball to the right of the second divider. Since there are eight gaps between the balls in which to place the two dividers and the dividers are indistinguishable, there are

$$\binom{8}{2} = \frac{8 \cdot 7}{2 \cdot 1} = \frac{56}{2} = 28$$

different ways to arrange the bars between the balls. Therefore, there are 28 different positive integer-valued solutions to the equation $a+b+c=9$.

1.14 Find the number of trailing 0's at the end of 1000!.

A calculator is not of use here due to overflow. Consider the prime factorization of 1000!. Every trailing zero corresponds to a five and two pair that appears in the prime factorization of 1000!. Since there are more twos than fives in the prime factorization, the problem reduces to finding the number of fives in the prime factorization of 1000!. There is

- one five in the factors 5, 10, 15, 20, 25, 30, ..., 1000,
- a second five in the factors 25, 50, 75, 100, 125, 150, ..., 1000,
- a third five in the factors 125, 250, 375, 500, 625, 750, 875, 1000,
- a fourth five in the factor 625.

Thus, the number of trailing 0's at the end of 1000! is

$$\left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{5^2} \right\rfloor + \left\lfloor \frac{1000}{5^3} \right\rfloor + \left\lfloor \frac{1000}{5^4} \right\rfloor,$$

which is the number of occurrences of the factor 5 in the prime factorization. This simplifies to

$$200+40+8+1=249$$

trailing zeros.