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AN ALGORITHM FOR COMPUTING OPTIONS ON THE MAXIMUM OR MINIMUM OF SEVERAL ASSETS

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ABSTRACT

An approximate method is developed for computing options on the maximum or the minimum of several assets. The method is very fast and is accurate for parameter ranges that are often of the most interest. The approach casts the problem in terms of order statistics and can be used to handle situations where the initial asset prices, the asset variances and the covariances are all unequal. Numerical values are given to illustrate the accuracy of the method.

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1. Introduction

Options on the maximum and options on the minimum of several assets are of both theoretical and practical interest. Stulz [1982] developed closed form expressions for European options in the case of two underlying assets. Johnson [1987] extended these results to handle European options in the case of n assets. Boyle, Evnine and Gibbs [1988] have used a multinomial lattice method to value American options when there are several underlying assets. However, when the number of assets exceeds two the computations quickly become very burdensome.

Since a number of corporate securities contain option features of this nature there is some interest in obtaining methods to price these options. In addition the quality option which is present in a number of important futures contracts can be valued in terms of options on the minimum of the set of deliverable assets. Under the quality option the short position can deliver any one of a set of acceptable assets and the existence of this option reduces the futures price. Several recent papers have examined the impact of the quality option. Gay and Manaster [1984] analyzed its impact in the case of wheat futures contracts by assuming that there were just two deliverable types of wheat. Boyle [1989] extended this analysis to the case of n deliverable assets and obtained numerical results by imposing very strong symmetry conditions on the problem. He assumed that the assets had all the same initial price, the same variance and that all the covariances were equal.

The quality option is also of considerable importance in the case of the Treasury Bond Futures contract. Cheng [1985], Hemler [1988] and Chowdry [1986], among others, have analyzed this situation. For tractability these authors assumed a multivariate lognormal diffusion process. Carr [1988] in tackling this problem uses a more realistic process for the stochastic behavior of bond prices.

The aim of the present paper is to present an approximation method which computes the value of options on the maximum or the minimum of several assets. The method is applicable to the situation where the asset prices follow a multivariate lognormal distribution. It can handle cases where the asset prices are unequal and it does not require that the variance-covariance matrix of the asset returns has any particular structure. The basic idea is to analyze the problem of valuing options on the maximum or the minimum in terms of order statistics. The algorithm uses an approximation method due to Clark [1961] for computing the moments of the maximum of n jointly normal random variables. Lerman and Manski [1981] provide evidence of the accuracy of the Clark approach.

In the next section we describe the operation of the Clark approach. It is a recursive procedure which only involves the computation of the univariate normal cumulative distribution function at each step. This method can be used to approximate the first four moments of the extreme order statistics of a set of multivariate normal random variables. We show how to derive the corresponding results in the case of a multivariate lognormal distribution. As shown in Section 3, this can then be used to derive the value of the quality option since the futures price in this case is related to a call option on the minimum of n assets with a strike price of zero. This call option can be computed in terms of the expectation of the lowest order statistic. Some numerical values are given and compared with those computed by other methods.

In Section 4 we describe how the Clark approach can be modified to deal with censored distributions. The objective here is to obtain a method to value options with a non-zero strike price. The details are given in the Appendix. We present numerical results to illustrate the accuracy of the approximation. For plausible parameter values the algorithm gives values that are within 1 or 2% of the accurate value. We indicate the range of accuracy of the approximation. The final section contains some concluding comments.

2. The Clark Algorithm

In this section we describe the Clark algorithm and indicate how it can be used to obtain the first four moments of the maximum of a set of normal variates.

The Clark algorithm provides exact expressions for the first four moments of the maximum of a pair of jointly normal variates as well as the correlation coefficient between the maximum of the pair and a third normal variate. Suppose we have three variables with a multivariate normal distribution. Assume that we know the expected values and the variances of the first two variates and their correlation coefficient. Clark [1961] obtained explicit expressions for the first four moments of the distribution of the maximum of these two variables.

If the correlations between each member of this pair and the third normal variate are also known then Clark also provided an explicit expression for the correlation between the maximum of the first two variates and the third variate.

This result can be used to approximate the first four moments of the maximum of a set of n normal variates. The method proceeds recursively and the computations at each stage are very simple. Although the results are approximate previous research (Clark [1961] and Lerman and Manski [1981]) attests to their accuracy over a range of assumptions.

Assume we have n jointly normal variates: $X_1, X_2, \ldots, X_n$ with known means, variances and correlation coefficients. Let Y denote the maximum of these n variates. The following definitions are useful.

$$Y_1 = \mathrm{Max}[X_1, X_2]$$

$$Y_2 = \mathrm{Max}[X_1, X_2, X_3] = \mathrm{Max}[Y_1, X_3]$$

$$\cdot$$

$$\cdot$$

$$Y_i = \mathrm{Max}[X_1, X_2, \ldots, X_{i+1}] = \mathrm{Max}[Y_{i-1}, X_{i+1}].$$

Hence

$$Y = Y_{n-1} = \mathrm{Max}[Y_{n-2}, X_n].$$

By applying Clark's algorithm at each step we can set up a recursive procedure to compute the first four moments of Y. We begin by computing the mean and variance of Y_1. In addition we obtain the correlation coefficients of Y_1 with the remaining (n-2) variates. We now assume the joint distribution of Y_1 and the variates X_3, ..., X_n is multivariate normal. This assumption is obviously not correct but the virtue of Clark's method is that nonetheless it enables us to obtain quite accurate answers. We proceed in an iterative fashion until $Y_{n-1} = Y$. At this stage we apply Clark's algorithm to obtain the first four moments of Y.

3. The Quality Option

The algorithm just described can be extended to compute the value of the futures price in the presence of a quality option. It can be shown that in some circumstances this futures price can be expressed in terms of a European call option, with a zero strike price, on the minimum of the assets in the deliverable set. Boyle [1989] uses this approach and we follow his notations and assumptions. The European call, with the strike price equal to zero, on the minimum of the n assets can be expressed in terms of the expected value of the minimum of the n assets. Since the asset returns are assumed to follow a log-normal distribution we can modify the procedure described above for the normal distribution to obtain the result. Clark's procedure is used to obtain the first four moments of the multivariate normal distribution of the asset returns. These moments are used to derive a Taylor series expression for the expected value of the minimum of the n assets. This corresponds to the lowest extreme order statistic.

The European call option on the minimum of n assets, with zero strike price is denoted by

$$EC[t,(A_1(t),A_2(t),\ldots,A_n(t)),0,t+T]$$

where t denotes current time, (t+T) denotes the expiry date of the option and $A_i(t)$ denotes the current price of asset i. This European call can be written as the discounted expectation of the minimum of the n assets at the expiry date; i.e.,

$$e^{-RT}\hat{E}[Min(A_1(t+T),A_2(t+T),\ldots,A_n(t+T))]$$

where R is the (assumed constant) riskless rate and $\hat{E}$ denotes the expectation over the risk adjusted distribution of terminal asset prices.

To simplify the notation we let

$$A_i(t+T) = A_i \qquad 1 \leq i \leq n$$

$$B_i = \ell n(A_i) \qquad 1 \leq i \leq n.$$

The B_i variates have thus a multivariate normal distribution. The required expectation becomes

$$\hat{E}[Min(e^{B_1},e^{B_2},\ldots,e^{B_n})]$$

$$= \hat{E}[exp(Min(B_1,B_2,\ldots,B_n))]$$

$$= \hat{E}[exp(-Max(-B_1,-B_2,\ldots,-B_n))]$$

$$= \hat{E}[exp(-W)]$$

where $W = Max(-B_1,-B_2,\ldots,-B_n)$.

Since $B_1, B_2, \ldots, B_n$ are jointly normal, $-B_1, -B_2, \ldots, -B_n$ are also jointly normal. We can use Clark's procedure to compute the first four moments of W. We denote the mean of W by μ and higher order moments about the mean by μ_i, for i = 2, 3 and 4. The required expectation can be written as a Taylor series expansion in terms of these moments, i.e.,

$$\hat{E}(\exp[-W]) = \exp(-\mu)[1 + \frac{\mu_2}{2} - \frac{\mu_3}{6} + \frac{\mu_4}{24}]. \tag{1}$$

Table 1 provides some numerical comparisons between the values obtained using equation (1) and results obtained by Boyle [1989] using a different method. For these computations all assets have the same initial value of $40 and the same standard deviation of 25%. In addition the correlation between each pair of assets is assumed to be equal to 0.95. Table 1 compares the two methods as the number of assets in the deliverable set increases.

For small numbers of deliverable assets the agreement is exceptionally good and even for 50 assets the difference is only 0.06%. One advantage of the procedure developed in this paper is that it can handle non-equal variances, covariances and initial asset prices. The procedure proposed by Boyle [1989] imposes strong symmetry in that the variances and correlations are assumed to be equal.

4. Options on the Maximum and Minimum of Several Assets

The procedure developed in the previous Section to compute the value of a European call, with a zero strike price, on the minimum of several assets could also be used to compute the price of a European call, with a zero strike price, on the maximum of n assets. In addition the procedure can be extended to value European options on the maximum or the minimum of several assets when the strike price is non-zero. In this case we need to compute the moments of the extreme order statistics of a censored distribution. The technical development of the procedure is given in the Appendix. To illustrate the method we compare the results with the accurate values obtained by integrating the multivariate normal density. The present method is much simpler from a computational viewpoint.

The results obtained in the case of three assets for different parameter values are given in Tables 2 through 7. We note that the agreement between the approximate results and the accurate values is very good. There does not appear to be any discernible pattern in the bias. It should be pointed out, however, that the results are based on one particular ordering of the assets. With unequal current asset prices, volatilities or correlation coefficients, the algorithm is not invariant to the ordering of the assets. In general, if the average of the results for different orderings is taken as the approximation, the accuracy is further improved. Similar results were obtained when we used four assets and compared the approximate values with the accurate ones. We found that the method does not give good results for long-dated options, for example, options with 10 years to maturity. In this case the distribution of asset returns becomes strongly skewed and deviates considerably from the normal.

5. Concluding Remarks

The approximate method described in this note is very convenient for evaluating options on the minimum of several assets when the strike price is zero. Such options can be used to compute the price of certain futures contracts when there is a quality option. The method was extended to evaluate European options on the maximum or the minimum of several assets when the strike price was non-zero. We provided numerical examples to illustrate the accuracy of the procedure. It is hoped this approach may be a useful supplement to the more accurate methods available which involve extensive computation when there are several assets.

TABLE 1. Option values, with zero strike price, for different numbers of deliverable assets: Comparison of results obtained using equation (1) with those of Boyle [1989]

Number of Deliverable Assets	Option Value Equation (1)	Option Value Boyle [1989]
2	38.908	38.908
3	38.371	38.374
4	38.029	38.033
5	37.782	37.786
10	37.102	37.100
15	36.752	36.746
20	36.522	36.511
25	36.351	36.338
30	36.217	36.201
35	36.107	36.089
40	36.013	35.994
45	35.933	35.912
50	35.862	35.840

TABLE 2. Comparison of European call prices on the maximum and the minimum of three assets with accurate values. Interest rate 10% p.a. continuously compounded. Time to expiration nine months. Equal Asset Prices; Equal Volatilities; Equal Correlations.

	One	Two	Three
Current Asset Prices	40	40	40
Volatilities	30%	30%	30%
Correlation Matrix	1.0	0.9	0.9
	0.9	1.0	0.9
	0.9	0.9	1.0

Strike Price	Call on Maximum		Call on Minimum	
	Approx	Accurate	Approx	Accurate
30	16.351	16.351	10.396	10.405
35	12.383	12.384	7.086	7.094
40	8.984	8.986	4.581	4.588
45	6.267	6.270	2.835	2.840
50	4.226	4.229	1.694	1.698

TABLE 3. Comparison of European call prices on the maximum and the minimum of three assets with accurate values. Interest rate 10% p.a. continuously compounded. Time to expiration nine months. Equal Asset Prices; Unequal Volatilities; Equal Correlations.

	One	Two	Three
Current Asset Prices	40	40	40
Volatilities	25%	30%	35%
Correlation Matrix	1.0	0.9	0.9
	0.9	1.0	0.9
	0.9	0.9	1.0

Strike Price	Call on Maximum		Call on Minimum	
	Approx	Accurate	Approx	Accurate
30	16.703	16.687	10.172	10.178
35	12.682	12.661	6.914	6.917
40	9.235	9.223	4.441	4.427
45	6.490	6.496	2.715	2.681
50	4.438	4.462	1.593	1.545

TABLE 4. Comparison of European call prices on the maximum and the minimum of three assets with accurate values. Interest rate 10% p.a. continuously compounded. Time to expiration nine months. Equal Asset Prices; Equal Volatilities; Unequal Correlations.

	One	Two	Three
Current Asset Prices	40	40	40
Volatilities	30%	30%	30%
Correlation Matrix	1.0	0.6	0.4
	0.6	1.0	0.6
	0.4	0.6	1.0

Strike Price	Call on Maximum		Call on Minimum	
	Approx	Accurate	Approx	Accurate
30	20.046	20.018	7.184	7.214
35	15.758	15.730	4.323	4.345
40	11.855	11.832	2.408	2.419
45	8.536	8.520	1.259	1.262
50	5.901	5.895	0.626	0.626

TABLE 5. Comparison of European call prices on the maximum and the minimum of three assets with accurate values. Interest rate 10% p.a. continuously compounded. Time to expiration nine months. Equal Asset Prices; Unequal Volatilities; Unequal Correlations.

	One	Two	Three
Current Asset Prices	40	40	40
Volatilities	25%	30%	35%
Correlation Matrix	1.0	0.6	0.4
	0.6	1.0	0.6
	0.4	0.6	1.0

Strike Price	Call on Maximum		Call on Minimum	
	Approx	Accurate	Approx	Accurate
30	20.185	20.153	7.172	7.172
35	15.885	15.847	4.326	4.311
40	11.969	11.929	2.409	2.379
45	8.644	8.611	1.258	1.220
50	6.013	5.995	0.625	0.589

TABLE 6. Comparison of European call prices on the maximum and the minimum of three assets with accurate values. Interest rate 10% p.a. continuously compounded. Time to expiration nine months. Unequal Asset Prices; Equal Volatilities; Equal Correlations.

	One	Two	Three
Current Asset Prices	40	45	50
Volatilities	30%	30%	30%
Correlation Matrix	1.0	0.9	0.9
	0.9	1.0	0.9
	0.9	0.9	1.0

Strike Price	Call on Maximum		Call on Minimum	
	Approx	Accurate	Approx	Accurate
30	19.431	19.448	9.076	9.084
35	15.419	15.436	6.194	6.200
40	11.866	11.882	4.057	4.061
45	8.872	8.888	2.569	2.570
50	6.466	6.479	1.584	1.584

TABLE 7. Comparison of European call prices on the maximum and the minimum of three assets with accurate values. Interest rate 10% p.a. continuously compounded. Time to expiration nine months. Unequal Asset Prices; Equal Volatilities; Unequal Correlations.

	One	Two	Three
Current Asset Prices	40	45	50
Volatilities	30%	30%	30%
Correlation Matrix	1.0	0.6	0.4
	0.6	1.0	0.6
	0.4	0.6	1.0

Strike Price	Call on Maximum		Call on Minimum	
	Approx	Accurate	Approx	Accurate
30	22.511	22.510	6.538	6.600
35	18.248	18.245	4.031	4.078
40	14.325	14.321	2.342	2.373
45	10.891	10.889	1.296	1.314
50	8.037	8.038	0.690	0.699

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Appendix

Evaluation of European Call Option on $Max(A_1,...,A_n)$ and $Min(A_1,...,A_n)$

First consider option on $Max(A_1,...,A_n)$. Let K be the strike price and P be the payoff of the call option at maturity. Then

$$
\begin{aligned}
P &= Max[Max(A_1,...,A_n)-K,0] \\
&= -K + Max[Max(A_1,...,A_n),K] \\
&= -K + Max[e^V,K]
\end{aligned}
$$

where

$$V = Max(B_1,...,B_n). \tag{A.1}$$

Hence $\hat{E}(P) = -K + \hat{E}[Max(e^V,K)]$ and the value of the call option is $e^{-RT}\hat{E}(P)$. Thus, we need to evaluate $\hat{E}[Max(e^V,K)]$.

We use Clark's algorithm to evaluate the mean and variance of V, and denote these by μ_V and σ_V^2, respectively. With the standardization transformation $Z = (V-\mu_V)/\sigma_V$, we have

$$
\begin{aligned}
&\hat{E}[Max(e^V,K)] \\
&= \hat{E}[Max(e^{\mu_V+\sigma_V Z},K)] \\
&= \hat{E}(e^{\mu_V+\sigma_V Z^*}] \qquad (A.2)
\end{aligned}
$$

where Z^* is a censored random variable defined by

$$Z^* = \begin{cases} Z & \text{if } Z \geq K^* \\ K^* & \text{if } Z < K^* \end{cases}$$

with $K^* = (\ln(K)-\mu_V)/\sigma_V$. If we denote $\mu^* = \hat{E}(Z^*)$ and $\mu_i^* = \hat{E}(Z^*-\mu^*)^i$ for i = 2, 3 and 4, by a Taylor series expansion (A.2) can be written as

$$\hat{E}[e^{\mu_V+\sigma_V Z^*}] = e^{\mu_V}\{e^{\sigma_V\mu^*}[1 + \frac{\mu_2^*\sigma_V^2}{2} + \frac{\mu_3^*\sigma_V^3}{6} + \frac{\mu_4^*\sigma_V^4}{24}]\}. \qquad (A.3)$$

To compute the moments of Z^*, we approximate the density function of Z by a Gram-Charlier approximation (see Kendall and Stuart [1969]). We denote the density and distribution function of a standard normal variate by $\phi(\cdot)$ and $\Phi(\cdot)$, respectively. The third and fourth central moments of Z, denoted by $\bar{\mu}_3$ and $\bar{\mu}_4$, respectively, can be computed from the moments of V. The Gram-Charlier expansion approximates the density function of Z, $f(\cdot)$, by the equation

$$f(z) = \phi(z)[1 + \frac{\bar{\mu}_3}{6}(z^3-3z) + \frac{\bar{\mu}_4-3}{24}(z^4-6z^2+3)]. \qquad (A.4)$$

Then the moments μ^* and μ_i^* for i = 2, 3 and 4, can be evaluated from the integrals

$$\hat{E}(Z^{*i}) = K^{*i}\int_{-\infty}^{K^*} f(z)dz + \int_{K^*}^{\infty} z^i f(z)dz \qquad i = 1,\ldots,4. \qquad (A.5)$$

Straightforward integration shows that (A.5) can be calculated using the formulae

$$\hat{E}(Z^{*i}) = H_{i0} + \frac{\bar{\mu}_3}{6}(H_{i3}-3H_{i1}) + \frac{\bar{\mu}_4-3}{24}(H_{i4}-6H_{i2}+3H_{i0})$$

$$i = 1,\ldots,4 \qquad (A.6)$$

where $H_{ij} = K^{*i}J_j + I_{i+j}$ for $i = 1,\ldots,4$ and $j = 0,\ldots,4$, with I_i and J_i given by

$$I_0 = 1-\Phi(K^*)$$

$$I_1 = \phi(K^*)$$

$$I_{i+1} = iI_{i-1} + K^{*i}\phi(K^*) \qquad i = 1,\ldots,7$$

$$J_0 = \Phi(K^*)$$

$$J_1 = -I_1$$

$$J_2 = 1 - I_2$$

$$J_3 = -I_3$$

$$J_4 = 3 - I_4.$$

Finally, to evaluate the option on $\mathrm{Min}(A_1,\ldots,A_n)$ we only need to replace V by $-W = \mathrm{Max}(-B_1,\ldots,-B_n)$.

A listing of a FORTRAN program for the above algorithm can be obtained from the authors on request.