

A framework for the analysis of fully coupled normal and tangential contact problems with complex interfaces

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Abstract

An extension to the interface finite element with eMbedded Profile for Joint Roughness (MPJR interface finite element) is herein proposed for solving the frictional contact problem between a rigid indenter of any complex shape and an elastic body under generic oblique load histories. The actual shape of the indenter is accounted for as correction of the gap function. A regularized version of the Coulomb friction law is employed for modeling the tangential contact response, while a penalty approach is introduced in the normal contact direction. The development of the finite element (FE) formulation stemming from its variational formalism is thoroughly derived and the model is validated in relation to challenging scenarios for standard (alternative) finite element procedures and analytical methods, such as the contact with multi-scale rough profiles. The present framework enables the comprehensive investigation of the system response due to the occurrence of tangential tractions, which are at the origin of important phenomena such as wear and fretting fatigue, together with the analysis of the effects of coupling between normal and tangential contact tractions. This scenario is herein investigated in relation to challenging physical problems involving arbitrary loading histories.

Keywords: coupled problems, contact mechanics, roughness, waviness, friction, finite element method.

1. Introduction

The analysis of the tangential tractions arising during the frictional contacts between a rigid and an elastic body plays a central role in physics and engineering. Mathematical models based on different friction laws have been developed during the past years in parallel with the progresses in the field of contact mechanics [1, 2]. In general, due to the intrinsic characteristics of the frictional contact problem, non uniqueness issues might arise, depending on the coefficient of friction and the regularization of the Coulomb law adopted [3, 4, 5, 6].

From the mathematical point of view, the problem can be cast as a set of two integral equations, coupled by the first and second Dundurs' bimaterial constants α and β , and equipped with a proper friction law [7]. From the physical standpoint, the effect of coupling causes a vertical pressure to induce horizontal displacements, thus influencing the tangential tractions distribution which in its turn modifies vertical displacements, having so an appreciable effect on the vertical tractions and the contact area distribution. As far as analytical models are concerned, the inversion of the mentioned integral equations is possible in few selected cases only, considering the

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bodies as half planes and under simplifying hypotheses like simple contacting geometries and uncoupling or semi-coupling between normal and tangential tractions. The half plane approximation is particularly relevant since, under the invocation of this assumption, problems involving the contact of two elastic bodies can be addressed, via the introduction of *ad-hoc* elastic moduli.

Under these hypotheses, the pioneering investigations of Cattaneo [8] and Mindlin [9] are particularly relevant. These studies seminally concern with the solution of the problem of two contacting elastic spheres. They independently showed that the tangential tractions distribution caused by friction could be expressed as a superposition of two normal traction distributions. Later, still within the framework of uncoupled approaches, Jäger [10, 11] and Ciavarella [12, 13] extended Cattaneo-Mindlin results to the contact problem of half planes with generic non compact boundaries under oblique loading. In [14], Goodman introduced the hypothesis of semi-coupling, taking into account only the effect of normal tractions on horizontal displacements, but not vice-versa, allowing the independent treatment of the vertical pressures with the usual methods employed for frictionless problems, and then solving separately for tangential tractions. This framework has been exploited by Nowell *et al.* [15] who obtained the analytical solution for the special case of a quadratic indenter acting upon an elastic half plane. In the same reference, they also developed a numerical scheme based on the inversion of the governing integral equations for solving the related fully coupled problem with generic tangential loading histories. Under the hypothesis of full coupling, Spence [16] solved the problem of the indentation of an elastic half space by an axisymmetric punch for a monotonic normal load. In particular, he solved the problem for a flat punch and a Hertzian indenter, and proved, thanks to a property of the governing equations, that the solution for a generic power law profile could be directly derived from that corresponding to the flat punch. As a consequence, under the same intensity of the normal load, the extension of the slip area is found to be the same for every indenter with a power-law profile.

More recently, numerical methods have experienced a considerable development in order to overcome the limitations of the analytical approaches. Without the aim of providing an exhausting literature review on this topic, credit to the main contributions in the development of this subject can be given to Klarbring [17] and Klarbring and Björkman [18], Kalker [19, 20] and Ahn and Barber [21]. Numerical schemes based on the Boundary Element Method (BEM) have been proposed and widely used for the quantitative evaluation of tangential contact problems. This framework has proved to be very efficient from a computational point of view, since it only requires the discretization of the domain boundary, without the accounting for the bulk. Many implementations are available, see *e.g.* Zhao *et al.* [22], Pohrt and Li [23] and Willner [24]. On the other hand, BEM-based algorithms are limited by the assumptions of linear elasticity and homogeneity of the materials, and their extension to inhomogeneities [25], non-linear interface constitutive response [26, 27, 28] or finite size geometries [29, 30] are sometimes possible but yet exceptions.

In line with the previous discussion, The Finite Element Method (FEM) would represent a suitable modeling tool for overcoming the inherent shortcomings of BEM, since it can easily account, for example, for geometrical or material non-linearities and finite-size geometries [31, 32]. The main problem related to FEM when applied to frictional contact problems is represented by the treatment of complex geometries in contact, since a discretization of both the interface and the bulk is requested by the method. In case of roughness, a lack of convergence often arises in node-to-segment and segment-to-segment contact strategies, due to corner cases experienced by the contact search algorithms to detect the points in contact, for very irregular (rough) interfaces. Therefore, regularization

techniques can be employed [31, 33], with the drawback of artificially smoothing out real high-frequency profile features relevant for the physics of contact.

In this work, a relevant extension of the interface finite element with eMbedded Profile for Joint Roughness (MPJR) interface element presented in [34] is herein extended to model the fully coupled frictional contact problem between a deformable body and an indenter, with a significant gain in the discretization of the contacting interface obtained embedding any arbitrary shape of the contacting profile. The main idea striving the current developments concerns re-casting the original geometry of the contacting profiles via a macroscopically smooth interface, which enables the generation of a straightforward meshing with linear finite elements, albeit preserving the actual geometry of the rough profile. This information is stored in terms of its analytical expression through the correction to the initial gap function with the exploitation of the assumption of a rigid indenting profile. The mathematical formulation is detailed in Sec. 2. The advantage of the present approach over alternative methodologies relies on the fact that the shape of the profile, including waviness or even roughness, does not need to be explicitly modeled by the finite element discretization. In the proposed scheme, the exact interface geometry is embedded into a nominally flat interface finite element through an analytical and exact correction of the normal gap function. In such a way the finite element discretization and boundary geometry can be taken as nominally smooth, with a significant advantage in terms of regularity of the boundary, simplified finite element discretization, albeit preserving the exact profile shape in the computations. The full derivation of the proposed finite element is presented in Sec. 3, and has been implemented as a user finite element in the research finite element analysis program FEAP [35]. The forthcoming contents of the manuscript is arranged as follows. The validation of the proposed model is carried out in Sec. 4, in reference to a benchmark problem, namely an analytical solution of a Hertzian contact problem obtained by assuming semi-coupling between the normal and the tangential directions, under different load scenarios. In Sec. 5, the method is exploited to solve cases for which no analytical solutions are available, and that are particularly challenging according to standard finite element techniques. In particular, coupled normal and tangential frictional contact problems with indenters having Weierstrass profiles as boundaries and increasing number of length scales, representative of multi-scale waviness, are investigated.

2. Variational formulation for frictional contact problems with embedded roughness

In the present section, the variational formulation which governs the normal and tangential contact of two bodies across a rough interface is presented. First of all, the strong differential formulation is recalled, with the equations describing the mechanics of the two bodies together, along with the Hertz-Signorini-Moreau conditions for normal contact and the Coulomb law for tangential contact at the interface. Subsequently, the weak form is derived, and the contact conditions are treated. A penalty approach is employed for the normal contact, while a regularized friction law is employed for tangential interactions, in line with finite element procedures [36]. The weak form provides the starting point for the derivation of the interface finite element, which is then presented in the following subsection. The current formulation is valid for 2D domains, although the proposed framework might be extended to 3D in a straightforward way.

2.1. General framework

Let us assume that two deformable bodies define the domains $\mathcal{B}_i \in \mathbb{R}^2$, $i = [1, 2]$ in the undeformed configuration defined by the Cartesian reference system Oxy . The boundary $\partial\mathcal{B} = \bigcup_{i=1,2} \partial\mathcal{B}_i$ of the domain can be split into three distinct parts, Fig. 1:

- (i) a region where displacements are imposed, *i.e.* the Dirichlet boundary $\partial\mathcal{B}_i^D$;
- (ii) a region where tractions are imposed, *i.e.* the Neumann boundary $\partial\mathcal{B}_i^N$;
- (iii) an interface $\partial\mathcal{B}^C$, common to the two bodies, where contact might take place, for which specific boundary conditions must be specified in order to model the stress and deformation fields generated by normal and tangential contact.

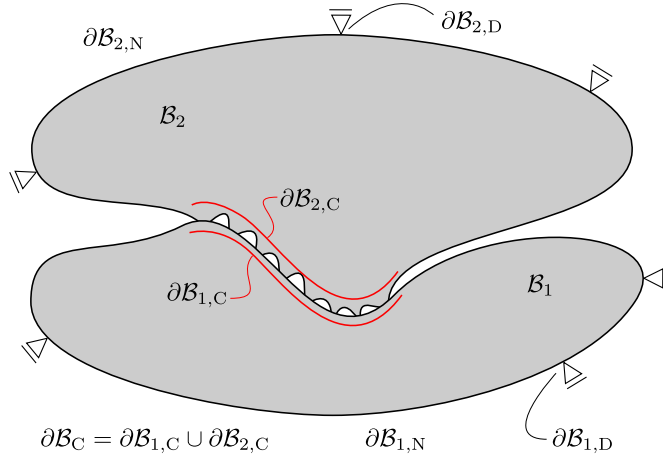


Figure 1: domains Ω_i ($i = 1, 2$), their Dirichlet ($\partial\Omega_i^D$) and Neumann ($\partial\Omega_i^N$) boundaries, and the contact interface $\partial\mathcal{B}^C$.

As customary, a displacement field $\mathbf{u}_i(\mathbf{x}, t) = [u_i(\mathbf{x}, t), v_i(\mathbf{x}, t)]^T$, which maps the displacements of the points of \mathcal{B}_i from the reference to the current configuration is assumed, identifying u_i and v_i the displacements along x and y directions, respectively, and recalling continuous and differentiable functions of the position vector \mathbf{x} and time t . Further, a small deformation strain tensor is defined as the symmetric part of the deformation gradient, $\boldsymbol{\varepsilon}(\mathbf{u})_i := \nabla^S \mathbf{u}_i$ (with ∇^S standing the symmetric part of the gradient operator), which, in standard Voigt notation, reads $\boldsymbol{\varepsilon}_i = [\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}]_i^T$.

At the interface $\partial\mathcal{B}^C$, the configuration of the system is described by the relative displacement field, common for the two bodies, called *gap field* across the interface, defined as $\mathbf{g} = \Delta\mathbf{u}$, which is the projection of the relative displacements of both bodies ($\mathbf{u}_2 - \mathbf{u}_1$) along the tangential and normal directions of the interface defined by the corresponding unit vectors \mathbf{t} and \mathbf{n} , respectively. In components, the gap field vector reads $\mathbf{g} = [g_t, g_n]^T = [\Delta u_t, \Delta u_n]^T$.

2.2. Definition of the equivalent contacting geometry

The innovation of the proposed approach lies in the definition of the normal gap g_n . In this framework, regardless of the actual topology, e.g. Fig. 2, the contact interface is assumed to be *nominally smooth*, with uniquely defined normal and tangential unit vectors, while its exact variation from planarity is analytically taken

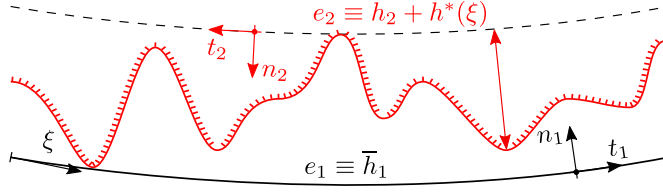


Figure 2: Composite topography of the interface embedding a rough profile

into account as a correction of the normal gap function. First of all, a local coordinate system is introduced for both the boundaries of the two bodies, being $\xi_i = \xi_i(\mathbf{x})$ a curvilinear coordinate defining a point-wise correspondence with the coordinates of the same point in the global reference system. Tangential and normal unit vectors $\mathbf{t}_1(\xi_1)$ and $\mathbf{n}_1(\xi_1)$ can also be uniquely defined, the latter pointing outwards \mathcal{B}_1 . If for every point of the contacting interface, the radius of curvature of the indenter is negligible with respect to the one of the deformable body, *i.e.* $R_1(\xi_1) \gg R_2(\xi_2)$, a *smoother line* $\bar{h}_2(\xi_2)$ can be set, parallel to $\partial\mathcal{B}_1^C$ and passing through the lowest point of elevation of $\partial\mathcal{B}_2^C$. A *roughness function* $h^*(\xi_2)$ is then used to account for the actual profile elevation from the reference datum set in correspondence of $\bar{h}_2(\xi_2)$. Therefore, the actual profile elevations are described, in the local curvilinear reference system, by the *elevation function* $e_2(\xi_2) = \bar{h}_2(\xi_2) + h^*(\xi_2)$. The smoothing line is parallel to $\partial\mathcal{B}_1^C$, and a set of unit vectors \mathbf{t}_2 and \mathbf{n}_2 can now be defined, equal and opposite to their counterparts characterized by index 1. With this transformation, a *zero thickness interface* $\partial\mathcal{B}_C^*$ is introduced, see Fig. 3. It is defined by

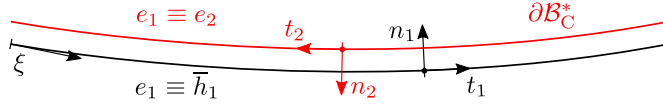


Figure 3: Zero-thickness interface model defining the equivalent interface $\partial\mathcal{B}_C^*$.

two distinct lines which at the initial time $t = 0$ are perfectly overlapping, and both defined by the function $e_1(\xi)$. The original geometry is stored in the function $h^*(\xi)$, which modifies the original gap field vector, representing the actual separation between the interfaces, such that now it reads:

$$\mathbf{g} = \begin{bmatrix} g_t \\ g_n \end{bmatrix} = \begin{bmatrix} \Delta u_t \\ \Delta u_n + \max_{\xi}[h^*(\xi)] - h^*(\xi) \end{bmatrix}, \quad (1)$$

because material 2 is rigid and its geometry is kept undeformed. The portion of the boundary in contact is now determined by the value of the modified function g_n , in accordance with the condition $\partial\mathcal{B}_C^* \iff g_n = 0$. At the present stage, a negative value for g_n is not admissible since it would imply material compenetration.

2.3. Composite topography

If the contacting bodies can be seen as half spaces, *e.g.* if the contact area is small compared to the bulks, the same formulation, *i.e.* rigid indenter over deformable body, can be interpreted as the solution of a contact problem involving two dissimilar elastic bodies, both potentially characterized by a geometrically complex interface. This problem can be re-cast in the already proposed framework by defining for the deformable body the composite elastic parameters E^* and ν , respectively given by:

$$\frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}, \quad \nu = \frac{1 - 2\beta}{2(1 - \beta)}, \quad (2)$$

in which E_i and ν_i are the original Young's moduli and Poisson's ratios, and β is the second Dundurs' constant. A composite topography can be defined [37, 38], combining the geometries of the two contacting boundaries. Again two sets of curvilinear coordinates $\xi_i = \xi_i(\mathbf{x})$ can be defined, together with normal and tangential unit vectors $\mathbf{t}_i(\xi_i)$ and $\mathbf{n}_i(\xi_i)$, the latter pointing outwards \mathcal{B}_i . For each of the two profiles, a *smoother line* $\bar{h}_i(\xi_i)$ is set, parallel to the average height of the original distribution and passing through its lowest point of elevation. Finally, the actual elevation of $\partial\mathcal{B}_1$ is flattened out, so that we have $e_1(\xi_1) \equiv \bar{h}_1(\xi_1)$, while the elevation of the indenter is defined as $e_2(\xi) = \bar{h}_2(\xi) + h^*(\xi)$, where now $h^*(\xi) = \max_{\xi} [h_1(\xi) + h_2(\xi)] - [h_1(\xi) + h_2(\xi)]$, and $h_i(\xi_i)$ are the elevations measured from the respective datum, as shown in Fig. 4. The common coordinate ξ , which is coincident with ξ_1 is here introduced, thus recovering the original case of Fig. 2.

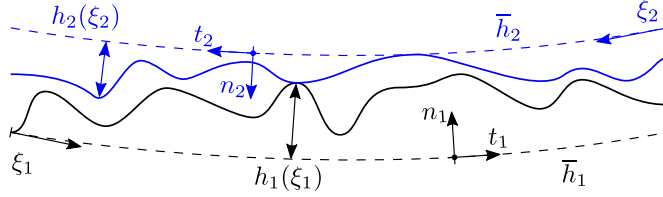


Figure 4: Parametrization of two rough profiles composing an interface Γ_C .

2.4. Governing equations and strong form

The linear momentum balance equation for both \mathcal{B}_1 and \mathcal{B}_2 , along with Dirichlet and Neumann boundary conditions on $\partial\mathcal{B}_i^D$ and $\partial\mathcal{B}_i^N$, can now be recalled for obtaining the strong form of equilibrium for the contacting bodies, enhanced by the conditions for normal contact on $\partial\mathcal{B}_C^*$, and the classic Coulomb friction law for tangential contact, which determines its further partition as $\partial\mathcal{B}_C^* = \partial\mathcal{B}_{C,st}^* \cup \partial\mathcal{B}_{C,sl}^*$, where the subscripts *st* and *sl* stand, respectively for the *stick* zone, where no tangential relative displacements between the two bodies occur, and the *slip* zone, where irreversible tangential displacements, *i.e.* *gross slip* or *sliding*, take place [36]:

$$\nabla \cdot \boldsymbol{\sigma}_i = \mathbf{0} \text{ in } \mathcal{B}_i, \quad (3a)$$

$$\mathbf{u}_i = \bar{\mathbf{u}} \text{ on } \partial\mathcal{B}_i^D, \quad (3b)$$

$$\boldsymbol{\sigma}_i \cdot \mathbf{n} = \mathbf{T} \text{ on } \partial\mathcal{B}_i^N, \quad (3c)$$

$$g_n = 0, p_n < 0 \text{ on } \partial\mathcal{B}_C^* \quad (3d)$$

$$g_t = 0, |p_t| \leq \mu p_n \text{ on } \partial\mathcal{B}_{C,st}^* \quad (3e)$$

$$p_t = -\mu |p_n| \frac{\dot{g}_t}{|\dot{g}_t|}, |p_t| > \mu p_n \text{ on } \partial\mathcal{B}_{C,sl}^* \quad (3f)$$

where $\bar{\mathbf{u}}$ denotes the imposed displacement and \mathbf{T} the traction vector.

2.5. Interface constitutive response

Since not only stick occurs at the interface level, a single penalty approach cannot be used for both normal and tangential responses, thus requiring a distinction for the two different directions. In the normal one, a standard penalty approach has been used, which is characterized by a penalty parameter ε_n chosen high enough to reduce compenetration at the minimum. In the tangential direction, a regularization of the Coulomb's law is introduced,

as in [36]. Given that, the two components of the interface traction vector $\mathbf{p} = [p_t, p_n]^T$ are:

$$p_n = \begin{cases} \varepsilon_n g_n, & \text{if } g_n < 0, \\ 0, & \text{if } g_n \geq 0, \end{cases} \quad (4a)$$

$$p_t = \mu |p_n| \tanh \frac{\dot{g}_t}{\varepsilon_t}. \quad (4b)$$

Eq. (4b) is one of the possible regularizations of Coulomb's friction law which enables a smooth transition from stick to slip condition. Correspondingly, if, on the one hand, it does not lead to a sharp transition from stick to slip domains, it has the great advantage of resolving the continuity issue of the Coulomb's law at the onset of slip. Finally, the presence of the regularization variable ε_t allows tuning the curve in order to reproduce as closely as possible the real trend of the classic law. It be remarked that even with a regularized law, the introduction of friction at the interface level is still a source of non linearity for the problem, since tractions do depend on the displacements and therefore the corresponding solution of the problem.

2.6. Weak form

The normal and tangential contact conditions on $\partial\mathcal{B}_C^*$, which modify the strong form expressed by Eq. (3) with respect to the classic elastostatic set of equations, also determine the modification of the variational equality representing the weak form of the problem, resulting into the following variational inequality:

$$\sum_{\gamma=1}^2 \left[\int_{\mathcal{B}_\gamma} \boldsymbol{\sigma}_\gamma(\mathbf{u}_\gamma) : \boldsymbol{\varepsilon}_\gamma(\mathbf{v}_\gamma) d\mathcal{V}_\gamma - \int_{\partial\mathcal{B}_\gamma^N} \bar{\mathbf{t}} \cdot \mathbf{v}_\gamma d\mathcal{A}_\gamma \right] \geq 0, \quad (5)$$

which is derived from the application of the principle of virtual work together with the contact constraints. In Eq. (5), \mathbf{v}_γ is the test function (virtual displacement field) for bodies 1 and 2, which fulfills the condition $\mathbf{v}_\gamma = \mathbf{0}$ on $\partial\mathcal{B}_\gamma^D$. Since body 2 is rigid, its contribution to the integral could be omitted, however it has been taken into account for considering also the more general case of two dissimilar elastic bodies, easily recovering the limiting condition of body 2 rigid simply setting, in the practical application, a suitable ratio of their respective Young's moduli. If the contact status is known, *e.g.* if the contact problem is conformal or an active set strategy has been implemented for identifying the contact domain, then Eq. (5) can be recast in the form of an equality by adding the terms related to the normal and tangential contact constraints:

$$\sum_{\gamma=1}^2 \left[\int_{\mathcal{B}_\gamma} \boldsymbol{\sigma}_\gamma(\mathbf{u}_\gamma) : \boldsymbol{\varepsilon}_\gamma(\mathbf{v}_\gamma) d\mathcal{V}_\gamma - \int_{\partial\mathcal{B}_\gamma^N} \bar{\mathbf{t}} \cdot \mathbf{v}_\gamma d\mathcal{A}_\gamma \right] - C_n - C_t = 0 \quad (6)$$

where, given the interface constitutive response chosen, the contributions related to normal contact, C_n , and to friction, C_t , read:

$$\begin{aligned} C_n &= \int_{\partial\mathcal{B}_C^*} p_n(\mathbf{u}) \delta g_n(\mathbf{v}) d\mathcal{A}, \\ C_t &= \int_{\partial\mathcal{B}_C^*} p_t(\mathbf{u}) \delta g_t(\mathbf{v}) d\mathcal{A}. \end{aligned} \quad (7)$$

The displacement field \mathbf{u}_i solution of the weak form Eq.(6) is such that it corresponds to the minimum of the energy for any choice of the test functions \mathbf{v}_i .

3. Interface finite element with eMbedded Profile for Joint Roughness (MPJR interface finite element) for frictional contact

The numerical solution of the variational problem described by Eq. (6) in the framework of the finite element method requires the geometrical approximation of the two bulks, \mathcal{B}_γ^h , and of the interface, $\partial\mathcal{B}_C^*$, and their discretization into finite elements:

$$\mathcal{B}_\gamma \approx \mathcal{B}_\gamma^h = \bigcup_{e=1}^{n_\Omega} \Omega_{\gamma,e}, \quad (8a)$$

$$\partial\mathcal{B}_C^* \approx \partial\mathcal{B}_{C,h}^* = \bigcup_{e=1}^{n_\Gamma} \Gamma_e^*. \quad (8b)$$

The bulk has been modeled using standard linear quadrilateral isoparametric finite elements [35], even though there is no restriction on the finite element topology, provided that it is consistent with that of the MPJR interface finite element used.

If the contact deformation is small, a conforming discretization of the bulk and the interface is enough to guarantee the presence of pairs of nodes which are expected to come into contact. This assumption holds also in case of friction and relative tangential displacements, since slip will be infinitesimal too. Given that, a four nodes interface finite element Γ_e^* can be introduced, as a special case of a collapsed 4 nodes quadrilateral element. Its kinematics is borrowed from the formulation common in non-linear fracture mechanics for cohesive crack growth, see [39, 40, 41, 42, 43] and then specialised for the present case of frictional contact in the presence of complex surfaces. The interface element is defined by 4 nodes, each pair belonging to the boundary of one of the two bodies:

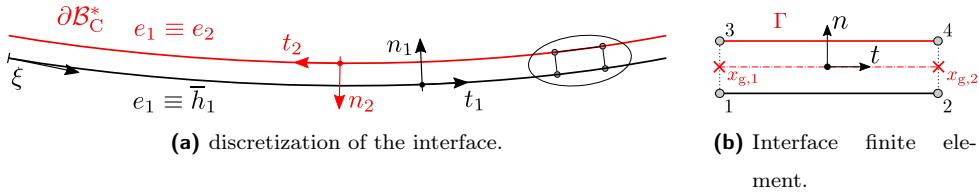


Figure 5: FEM approximation of the interface.

1 and 2 to $\partial\mathcal{B}_{1,C}^*$ and 3 and 4 to $\partial\mathcal{B}_{2,C}^*$, see Fig. 5. The contribution of the interface to the weak form is expressed by Eq. (7), whose geometrical approximation, together with the finite element discretization, reads:

$$\int_{\partial\mathcal{B}_C^*} p_n(\mathbf{u})\delta g_n(\mathbf{v}) d\mathcal{A} \approx \int_{\partial\mathcal{B}_{C,h}^*} p_n(\mathbf{u})\delta g_n(\mathbf{v}) d\mathcal{A}_h = \bigcup_{e=1}^{n_\Gamma} \int_{\Gamma_e^*} p_n(\hat{\mathbf{u}}_e)\delta g_n(\hat{\mathbf{v}}_e) d\Gamma_e^*, \quad (9a)$$

$$\int_{\partial\mathcal{B}_C^*} p_t(\mathbf{u})\delta g_t(\mathbf{v}) d\mathcal{A} \approx \int_{\partial\mathcal{B}_{C,h}^*} p_t(\mathbf{u})\delta g_t(\mathbf{v}) d\mathcal{A}_h = \bigcup_{e=1}^{n_\Gamma} \int_{\Gamma_e^*} p_t(\hat{\mathbf{u}}_e)\delta g_t(\hat{\mathbf{v}}_e) d\Gamma_e^*. \quad (9b)$$

In the equations above, the symbol $(\hat{\cdot})$ denotes, as customary, the difference between the exact displacement and its finite element approximation, while the subscripts e spans through the total number of elements n_Γ employed for the interface discretization.

The normal gap g_n and the tangential gap g_t are stored in the local gap vector $\mathbf{g}_l = [g_t, g_n]^T$. To evaluate them at every point of the interface element, the nodal displacement vector is introduced, defined as $\hat{\mathbf{u}}_e = [u_1, v_1, \dots, u_4, v_4]^T$, being u_i and v_i the respective horizontal and vertical displacements of the four nodes. A

linear matrix operator \mathbf{L} is deputed to evaluate the relative displacement $\Delta\hat{\mathbf{u}}_e = [\Delta u_{e,t}, \Delta u_{e,n}]^T$ between nodes 1 and 4, and 2 and 3 respectively. A linear interpolation is then performed by the matrix multiplication with $\mathbf{N}(\xi)$, which collects the shape functions $N_1(\xi)$ and $N_2(\xi)$. The final step is the multiplication by the rotation matrix \mathbf{Q} , for moving from the global to the local reference system, centred in, and aligned to, the interface element. In formulae, it results:

$$\Delta\hat{\mathbf{u}}_e = \mathbf{QNL}\hat{\mathbf{u}}_e, \quad (10)$$

and the matrix operators take the form:

$$\mathbf{Q} = \begin{bmatrix} t_x & t_y \\ n_x & n_y \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} N_1(\xi) & 0 & N_2(\xi) & 0 \\ 0 & N_1(\xi) & 0 & N_2(\xi) \end{bmatrix},$$

$$\mathbf{L} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & +1 & 0 & 0 \end{bmatrix},$$

$$\hat{\mathbf{u}}_e = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4],$$

being n_x, n_y, t_x, t_y the components of the unit vectors normal and perpendicular to the local reference system. The final tangential gap directly corresponds to $\Delta\hat{u}_{e,t}$, the final normal gap is given by $g_n = \Delta\hat{u}_{e,n} + h^*$, a correction which accounts for the actual complex surface of the rigid indenter. At this point it is possible to evaluate tractions p_n and p_t from by Eq.s (4a) and (4b), respectively.

The contribution of a single interface element to Eq. (6) can now be evaluated. Recalling the traction vector $\mathbf{p} = [p_t, p_n]^T$, it is possible to condense Eq.s (7), which for a single interface element read:

$$\delta\Pi_e = \int_{\Gamma_e^*} \delta\mathbf{g}(\hat{\mathbf{v}}_e)^T \mathbf{p}(\hat{\mathbf{u}}_e) d\Gamma_e^*, \quad (11)$$

where the variation of the local gap is given, in matrix notation, by:

$$\delta\mathbf{g}(\hat{\mathbf{v}}_e) = \frac{\partial\mathbf{g}}{\partial\hat{\mathbf{v}}_e} \delta\hat{\mathbf{v}}_e = \mathbf{QNL}\delta\hat{\mathbf{v}}_e \quad (12)$$

finally, the variation can be set to vanish, and the residual vector can be obtained:

$$\delta\Pi_e = \delta\hat{\mathbf{v}}_e^T \int_{\Gamma_e^*} \mathbf{L}^T \mathbf{N}^T \mathbf{Q}^T \mathbf{p}(\hat{\mathbf{u}}_e) d\Gamma_e^* = 0, \quad (13)$$

which gives:

$$\mathbf{R}(\hat{\mathbf{u}}_e) = \int_{\Gamma_e^*} \mathbf{L}^T \mathbf{N}^T \mathbf{Q}^T \mathbf{p}(\hat{\mathbf{u}}_e) d\Gamma_e^* = \mathbf{0}. \quad (14)$$

A Newton-Cotes integration formula is exploited for exactly evaluating the integral in \mathbf{R} , which requires the sampling of the integrand at points which coincide to the abscissae of nodes 1 and 2, obtaining:

$$\int_{\Gamma_e^*} \mathbf{L}^T \mathbf{N}^T \mathbf{Q}^T \mathbf{p}(\hat{\mathbf{u}}_e) d\Gamma_e^* = \sum_{k=1}^2 w_k \mathbf{L}^T \mathbf{N}_k^T \mathbf{Q}^T \mathbf{p}_k(\hat{\mathbf{u}}_e) j_e(\xi_k), \quad (15)$$

being w_k the weights and $j_e(\xi)$ the determinant of the transformation mapping the change of coordinate from the global to the natural reference system. To light up the notation, hereinafter the vector of nodal displacement $\hat{\mathbf{u}}_e$ will be replaced by \mathbf{u} .

Given the constitutive laws employed, Eq. (14) represents a set of non-linear transient equations. A full Newton Raphson iterative-incremental scheme is used to solve the system resulting from the discretization of the rate problem relative to the bulk and the interface. The problem to be solved is therefore the following:

$$\mathbf{R}(\mathbf{u}, \dot{\mathbf{u}}) = \mathbf{0}, \quad (16)$$

The residual vector is linearized introducing the tangent matrix, and the resulting set of linear equations reads:

$$\mathbf{S}^{(i)} d\mathbf{u}^{(i)} = -\mathbf{R}^{(i)} \quad (17)$$

where the superscript i denotes the iteration inside a Newton Raphson loop and \mathbf{S} is the tangent matrix defined as:

$$\mathbf{S} = -\frac{\partial \mathbf{R}}{\partial \mathbf{u}} - \frac{\partial \mathbf{R}}{\partial \dot{\mathbf{u}}} \frac{\partial \dot{\mathbf{u}}}{\partial \mathbf{u}} = c_1 \mathbf{K} + c_2 \mathbf{C}, \quad (18)$$

being \mathbf{K} the stiffness matrix, \mathbf{C} the damping matrix and c_1 and c_2 scalar coefficients that involve the time step Δt and the parameters of the selected time integration scheme. For every cycle, the solution is updated:

$$\mathbf{u}^{(i+1)} = \mathbf{u}^{(i)} + d\mathbf{u}^{(i)}, \quad (19)$$

until the convergence criterion $d\mathbf{u}^{(i)} \cdot \mathbf{R}^{(i)} < \varepsilon$ is met. The stiffness and damping matrix resulting from the linearization of the residual vector for a given iteration i read:

$$\mathbf{K}^{(i)} = \sum_{k=1}^2 w_k \mathbf{L}^T \mathbf{N}_k^T \mathbf{Q}^T \mathbb{K}_k \mathbf{Q} \mathbf{N}_k \mathbf{L} j_e(\xi_k), \quad (20a)$$

$$\mathbf{C}^{(i)} = \sum_{k=1}^2 w_k \mathbf{L}^T \mathbf{N}_k^T \mathbf{Q}^T \mathbb{C}_k \mathbf{Q} \mathbf{N}_k \mathbf{L} j_e(\xi_k) \quad (20b)$$

where \mathbb{K} and \mathbb{C} are the linearised interface constitutive matrices:

$$\mathbb{K} = \begin{bmatrix} \frac{\partial p_t}{\partial g_t} & \frac{\partial p_t}{\partial g_n} \\ \frac{\partial g_t}{\partial p_n} & \frac{\partial g_n}{\partial p_n} \end{bmatrix}, \quad \mathbb{C} = \begin{bmatrix} \frac{\partial p_t}{\partial \dot{g}_t} & \frac{\partial p_t}{\partial \dot{g}_n} \\ \frac{\partial \dot{g}_t}{\partial \dot{p}_n} & \frac{\partial \dot{g}_n}{\partial \dot{p}_n} \end{bmatrix}. \quad (21a)$$

\mathbb{K} and \mathbb{C} are derived by analytical differentiation if contact is detected, while on the other hand, every term of the matrices is set equal to zero if a positive g_n , *i.e.* an open gap, is detected. As a concluding remark for this section, is important to stress the fact that the normal gap is evaluated at every Gauss point, so two times for every element. When discretizing the contact zone, the number of interface element employed should be high enough in order to adequately sample and reproduce the elevation profile of the embedded contacting shapes.

4. Model validation

4.1. Frictional Hertzian contact problem between a cylinder and a half plane under a monotonic normal load

A semi-coupled¹ Hertz contact problem is used for validating the model, Fig. 6. This benchmark is found to be particularly suitable for testing the validity of the proposed implementation for two specific reasons. The

¹Hereinafter, the label *semi-coupled* will be referred to a system in which the effect of the tangential tractions over the vertical pressure is neglected, but not the opposite. This hypothesis has been introduced by Goodman [14].

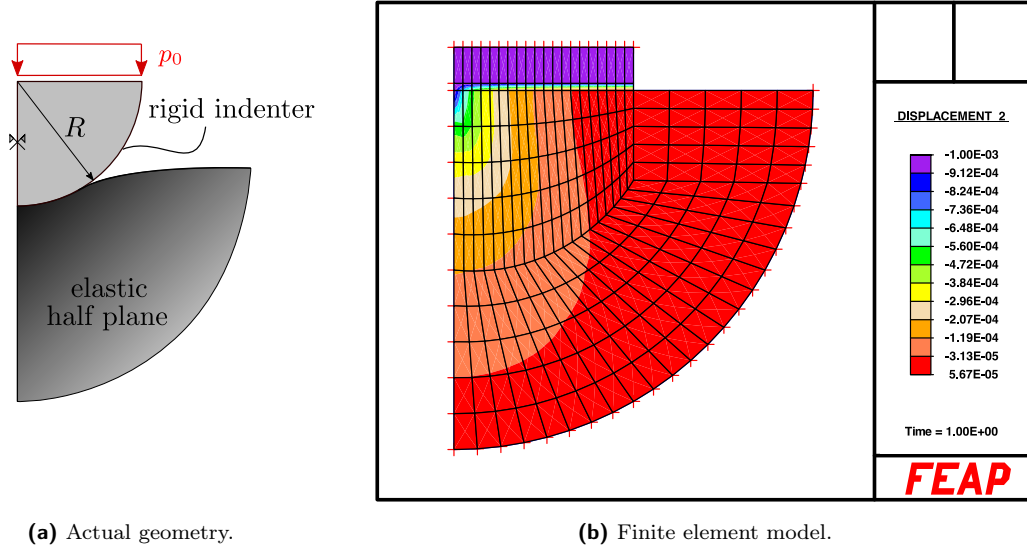


Figure 6: Actual geometry of the benchmark contact problem (a), and its finite element model set up with the present approach based on the MPJR interface finite element (b).

first underlying motivation, already thoroughly explained in [34], is that in spite of its simplicity, the solution of this problem via a standard FEM discretization of the curved geometry requires very refined meshes, specially at the edges of the contact strips. The second reason is that the present problem yields to a closed solution, thus providing an easy to implement and fast way of comparison and verification. Under the assumptions of purely normal monotonic load and neglecting the influence of tangential tractions $q_x(x)$ on the normal contact traction distribution $p_z(x)$, the solution can be written as [15]:

$$q_x(x) = \frac{\mu p_0}{K(c)} \left(\sqrt{1 - \frac{x^2}{a^2}} F(\theta, c) - \frac{x}{2b} \log \frac{1 - \sqrt{1 - x^2/b^2}}{1 + \sqrt{1 - x^2/b^2}} \right), \quad b \leq |x| \leq a \quad (22)$$

where p_0 is the maximum value of the vertical tractions, a and b are the extension of the contact radius and slip zone respectively, $c = b/a$, $\sin(\theta) = x/b$ and $K(c)$ and $F(\theta, c)$ are the first complete and incomplete elliptic integral of the first kind, respectively. The extent of the slip zone can be evaluated using Spence's relation [16], which is valid for every initial contact gap defined by a power law relation, and is given by:

$$\mu K(c) = \beta K(\sqrt{1 - c^2}), \quad (23)$$

where β is the second Dundurs' constant. This parameter governs the level of coupling of the system: for $\beta = 0$, the problem is uncoupled, while its maximum admissible value is 0.5. In the limiting case of a contact problem involving a rigid parabolic profile indenting an elastic half plane, we have $\beta = (1 - 2\nu)/[2(1 - \nu)]$, being ν the Poisson's ratio of the half plane. A value of 0.29 is herein chosen, which corresponds to $\nu = 0.3$. While this value is kept constant, four different values for the friction coefficient have been used to investigate the accuracy of the model predictions. The comparison has been carried out in terms of normal and tangential tractions exploiting Eq. (22), comparing our fully coupled numerical approach with the semi-coupled analytical solution.

4.2. FEM implementation and results

In the present framework, the actual shape of the contacting profile is embedded in the MPJR interface finite element. The profile elevation is given by its analytical form and evaluated at every Newton-Cotes integration point. Since one of the contacting profile, *i.e.* Γ_1^* , is flat, the composite topography exposed in Sec. 2 simply reduces to the one of Γ_2^* : $e_2(\xi) = \bar{h}_2(\xi) - \xi^2/2R$ and we have a normal gap in the initial undeformed condition which is given by $g_n = \Delta u_n + \xi^2/2R$, being R the radius of the cylinder. We analyze the problem under plane strain assumptions. The mesh is structured based on three different levels:

- the lower models \mathcal{B}_1 , which in the present setting is a half space, approximated by a circular sector clamped along the curved side and free to undergo vertical displacements in correspondence of the vertical side. An extension of its radius of $2R$ has been found to be enough for mimicking the elastic properties of a semi-infinite half plane, under plane strain assumptions. A Young's modulus $E_1 = 100$ and a Poisson's ratio $\nu_1 = 0.3$ has been assigned to the standard quadrilateral linear finite elements employed for the bulk discretization;
- the interface $\partial\mathcal{B}_C^*$ is modelled using a single layer of MPJR elements, discretized using $n_\Gamma = 100$ elements; The penalty stiffness has been set to $\varepsilon_n = 10^2 E_1/R$, that can be considered as sufficiently high in order to avoid material interpenetration.
- finally, the geometry of the indenting cylinder which represents \mathcal{B}_2 can be replaced by a regular array of quadrilateral linear finite elements, with an assigned elastic modulus $E_2 = 10^3 E_1$; Neumann boundary conditions are applied as a uniform distribution of vertical pressure p_0 resulting in a unitary vertical force P_z .

The results of the simulations are shown in Fig. 7a for $\beta = 0.29$, in terms of dimensionless vertical tractions $a_0 p_z(x)/P_z$ (red curves) and dimensionless tangential tractions $a_0 q_x(x)/\mu P_z$ (black curves), being a_0 the radius of contact related to the semi-coupled case and P_z the vertical force applied. For both the analytical semi-coupled and the FEM model, five different values of the coefficient of friction, $\mu = [0.1, 0.2, 0.3, 0.4, 0.5]$ are applied. In the semi-coupled case, the distribution of vertical tractions is coincident regardless the coefficient of friction employed, and is highlighted by circle markers, while when coupling is considered, slight differences in the normal tractions can be observed. For what concerns the tangential tractions, an excellent accordance can be observed for all the values of the coefficient of friction employed. The coupling effect observed in the curve related to the numerical approach is in line with the theory, which predicts a stiffening effect that results into an increase of the maximum value of the vertical tractions, compensated by a slight decrease of the contact radius. The slight deviation between the two sets of $q_x(x)$ that can be observed in Fig. 7a can be again ascribed to coupling. In fact, the FEM model predicts smaller values for the ratio b/a with respect to the semi-coupled approach and the effect can be quantitatively analyzed, with results shown in Fig. 7b. In this figure, the black solid curve represents the values of b/a for the semi-coupled case, as expressed by Eq. (23), corresponding to the dashed black curves of Fig. 7a; the red stars are the values for the coupled case, evaluated using an asymptotic solution provided in [15] and the blue circles the outcomes of the simulation. A very good accordance is found between the slip/contact strip width ratio calculated in the fully coupled case and the one obtained by the numerical simulation; the deviation between the tangential tractions of Fig. 7a can be justified in this way. The importance of the outcome of Fig. 7a

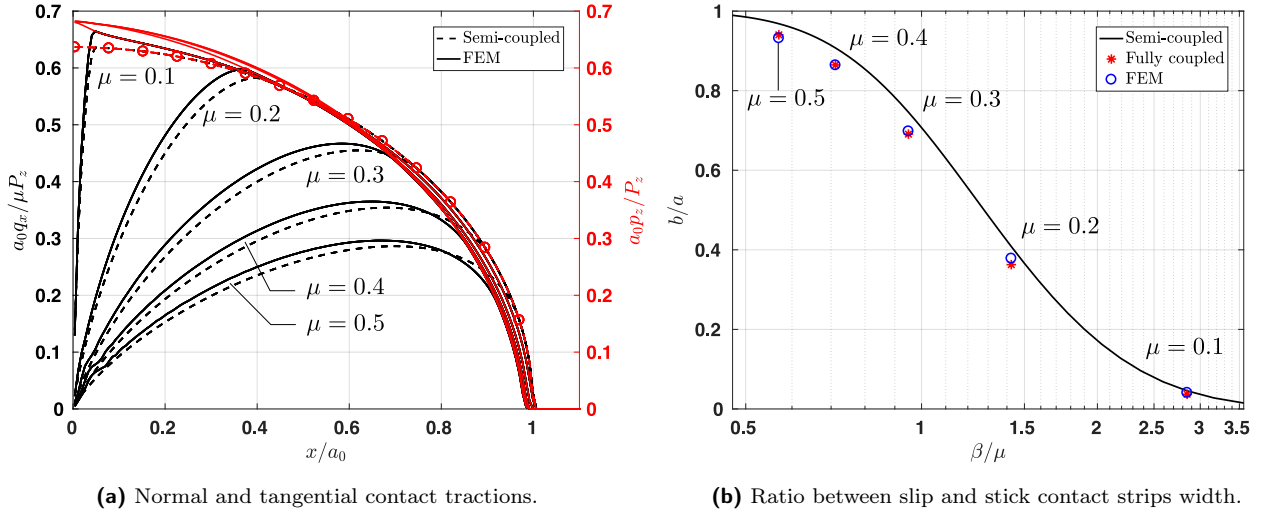


Figure 7: benchmark test with low values of coupling, $\beta = 0.29$.

lies in the fact that, for some specific instances, care should be taken in neglecting coupling effects, since this could lead to underestimating the magnitude of tractions. As a final remark, it can be noticed that even if a regularized friction law has been used, an adequate choice of the stiffness parameter ε_t guarantees results that are very close to the solution based on the Coulomb friction law, which has been exploited by the semi-analytical model.

4.3. Hertzian contact problem between a cylinder and a half plane under constant normal loading and cyclic tangential loading

As compared to the previous test problem, now a downward displacement is imposed on the cylinder, starting from zero and linearly increasing up to a maximum value of $\Delta_{z,0}$. At this point, the vertical displacement is held constant and a tangential load is applied, in terms of a horizontal far field displacement which harmonically oscillates with amplitude $\Delta_{x,0} = 0.8\mu\Delta_{z,0}$. A normalized load history plot is shown in Fig. 8a, together with the corresponding values of total normal load P_z and tangential load Q_x , evaluated as the resultant of the interface tractions. The imposed displacements are also plotted in the load space $\Delta_x - \Delta_z$, in which the black solid curve represents the variation of the tangential load with respect to the normal one, while the blue dashed curves represent the limit of gross sliding. In the present case the load path consists in a curve which is a straight line lying on the horizontal axis, from the origin to point (a) and then becomes a collapsed ellipse with the major axis passing through the points (b) and (c), Fig. 8b. The results in term of traction distributions are shown in Fig. 9. In Fig. 9a, the normal load is linearly increased from zero to its maximum value, and a self-similar symmetric central stick area encompassed by two regions of forward slip (positive tangential tractions) and backward slip (negative tangential tractions) develops. Then, Fig. 9b, the tangential load is applied (point (a) of Fig. 8a and 8b), which results in:

- (i) an increase in the tangential tractions at the leading edge;
- (ii) a reduction of the tangential tractions at the trailing edge;

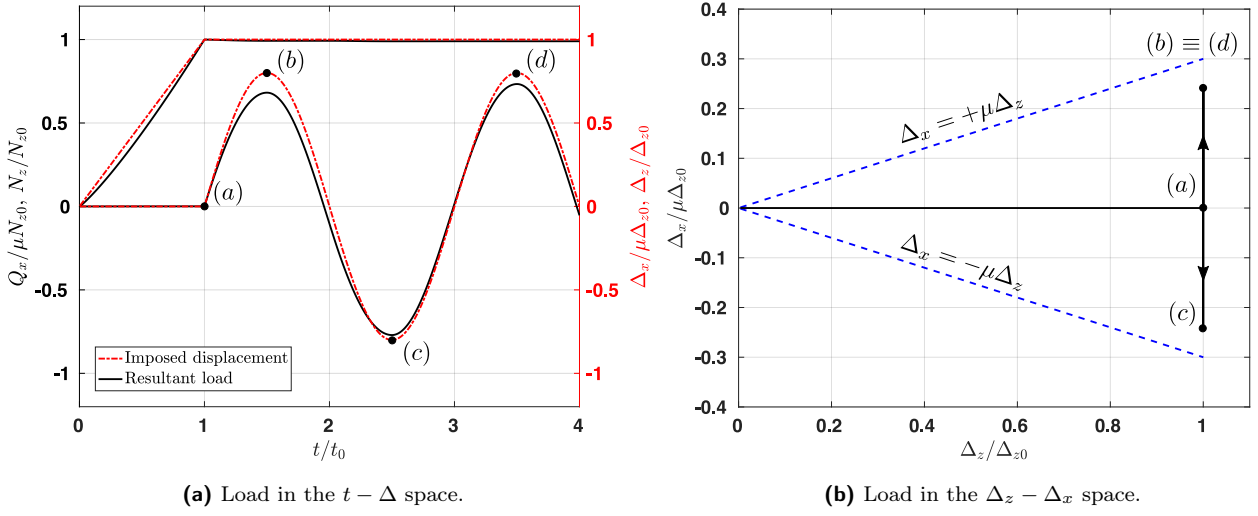
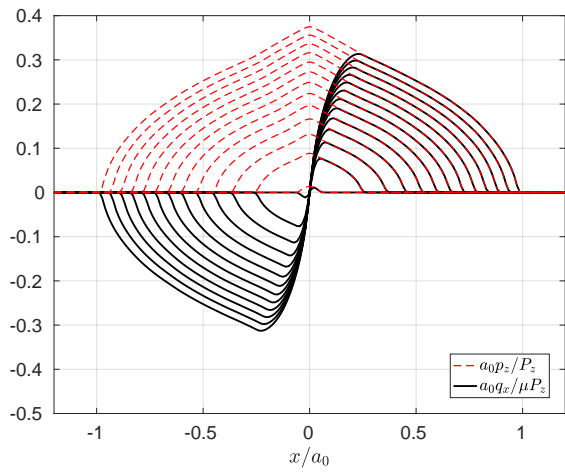


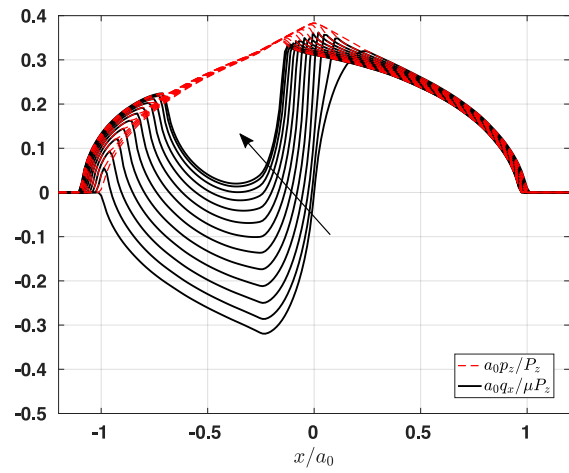
Figure 8: tangential cycling load history.

(iii) an instantaneous violation of the Coulomb law, which leads to a sudden change from backward slip to stick at the trailing edge, from which tractions start increasing again, but with opposite sign. At the same time, the stick zone shrinks, reaching its minimum at the point (b) of the loading history.

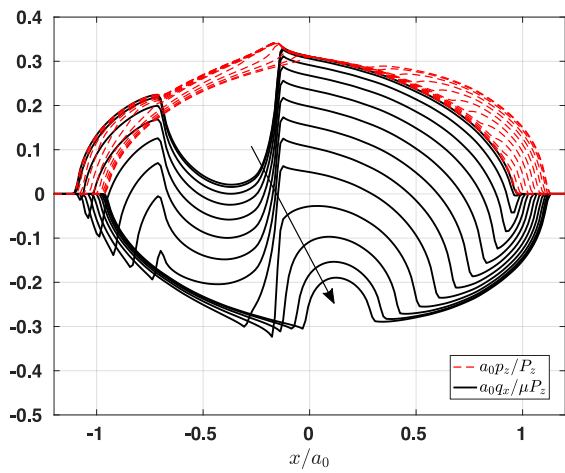
If now the load is reversed, an instantaneous stick zone is created again, which shrinks to a minimum value in correspondence to point (c), Fig. 9c. The system reaches a stationary state condition after point (d), but with a steady state value of the stick area which is, for a positive load, sensibly different from the one related to the first application of the load, Fig. 9d. In particular, it retains almost the same extension, but it is shifted towards the center of the contact zone. As a final remark, the system maintains a significant difference between the positive and negative stick areas, even though the steady state is reached after one cycle of loading. It can be noticed that this is directly related to the level of coupling. The current results are obtained with a high level of coupling, corresponding to $\beta = 0.50$. The same load history, applied to a system with $\beta = 0.29$ shows greater similarity between the positive and negative steady stick area, Fig. 10, showing again that the degree of coupling has an important effect on the contact response.



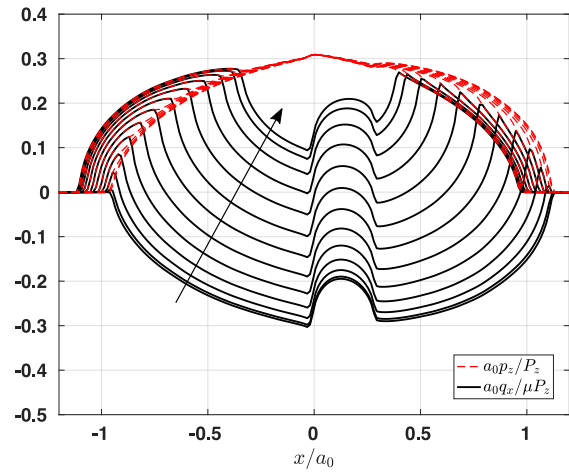
(a)



(b)

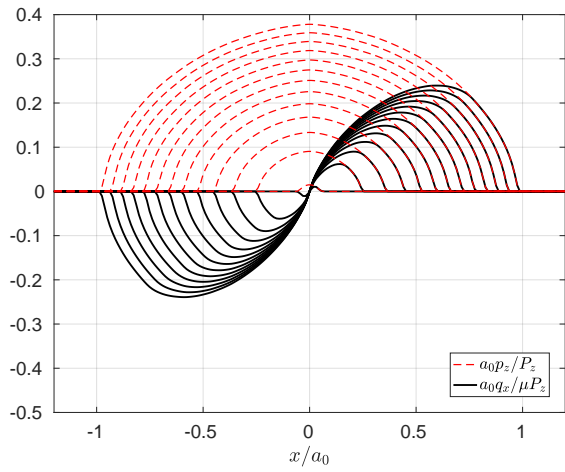


(c)

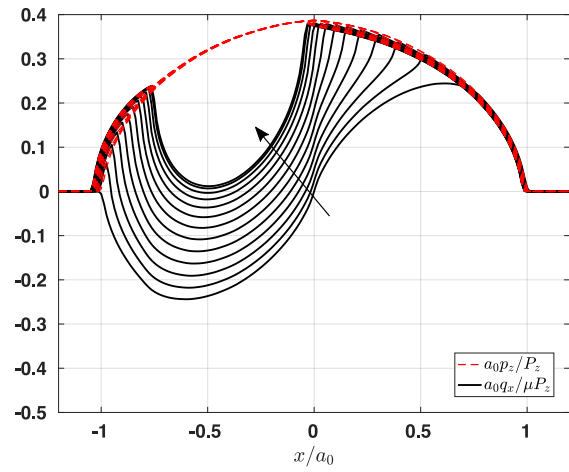


(d)

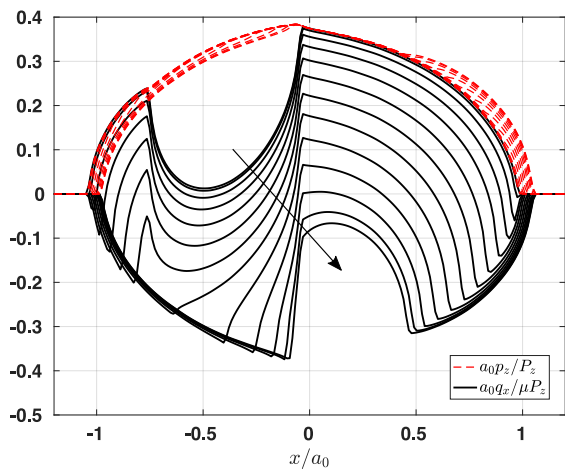
Figure 9: tangential cycling load history for high values of coupling, $\beta = 0.50$.



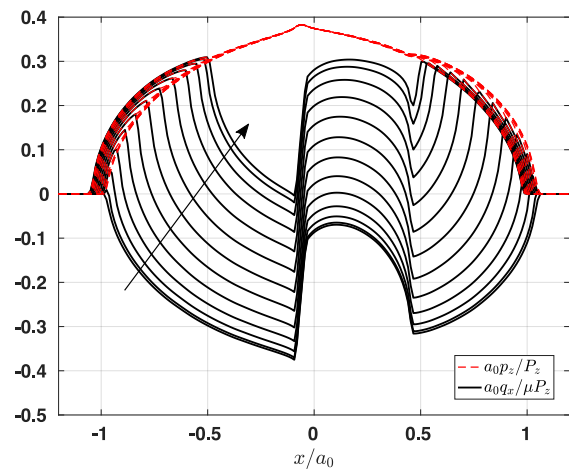
(a)



(b)



(c)



(d)

Figure 10: tangential cycling load history for low values of coupling, $\beta = 0.29$.

5. Contact between a rough profile and an elastic layer

As shown in [34], the strength of the formulation is particularly evident in the case of contact between a rough profile and an elastic layer. In spite of the fact that any kind of elevation field might be taken into consideration in the computational framework, without any restriction, even numerically generated, a Weierstrass profile is herein used as a possible example. The height field is generated by Eq. (24):

$$z(x) = g_0 \sum_{n=0}^{\infty} \gamma^{(D-2)n} \cos\left(2\pi \frac{\gamma^n x}{\lambda_0}\right), \quad (24)$$

where in practical applications the summation is carried on up to a certain n_w , thus obtaining a *pre-fractal* profile [44, 45] which consists of the superposition of n_w sinusoidal functions, each of them presenting a decreasing wavelength $\lambda_n = \lambda_0/\gamma^n$ and amplitude $g_n = g_0\gamma^{(D-2)n}$, where γ and D are parameters chosen such that $\gamma > 1$ and $1 \leq D \leq 2$.

In this section, three different indentation problems are solved, in which the contacting profiles exhibit such heights distribution. Each of them is tested against a rectangular elastic block with a height-to-width ratio $t/\lambda_0 = 0.5$. Such block presents the same elastic parameters employed for the model validation of Sec. 4, *i.e.* $E_1 = 100$ and $\nu_1 = 0.3$. Each indenter profile can be considered rigid, and is made of the superposition of one, two, or three terms respectively, according to Eq. (24), shown in Fig. 11. As in the previous section, the lower boundary, *i.e.* $\partial\mathcal{B}_{C,1}$, is flat, and the elevation field reduces to the one of $\partial\mathcal{B}_{C,2}$: $e_2(\xi) = \bar{h}_2(\xi) - z(\xi)$, with an initial normal gap in the undeformed condition given by $g_n = \Delta u_n + z(\xi)$. Finally, in order to simulate a contact problem which is indefinite in the x -direction, periodic boundary conditions have been applied to both the vertical edges of the mesh, at a distance of λ_0 . A classical ironing-type load history is applied for solving the contact problem. First, a purely normal far-field displacement Δ_z is imposed, starting from zero up to $5g_0$. Then, a horizontal tangential displacement Δ_x is applied, linearly varying from zero to 3 times the maximum value of Δ_x , which is the value that guarantees an incipient gross slip for the single harmonics profile. A full parametric study of the problem should involve a thorough evaluation of the sensitivity of the system with respect to the main governing physical parameters, such as β , μ , λ_0/t , g_0 , γ , Δ_x and Δ_z , but this is left for further investigation, since the main purpose is to show the feasibility of treating complex interface problems within the present finite element framework.

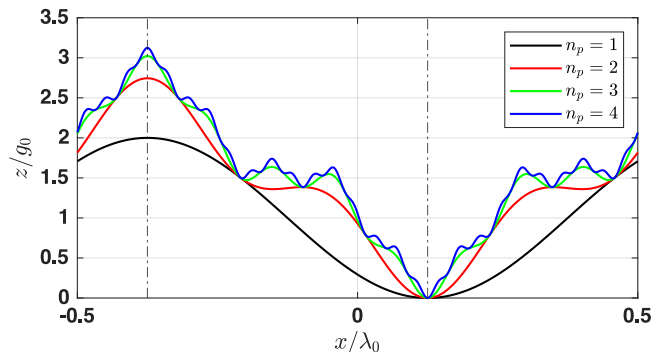


Figure 11: Weierstrass multi-scale profile.

5.1. Single harmonics profile in full contact

The case of $n_w = 0$ is shown in Fig. 12. The main difference with the results obtained for the previous test problems is that now we are dealing with an infinitely long profile which makes contact at an infinite set of spots. Under purely normal loading, the vertical tractions $p_z(x)$ present two axes of symmetry, highlighted by dash-dotted lines in the figure, which corresponds to axes of anti-symmetry for the shearing tractions $q_x(x)$. Since in this condition shearing tractions must be strictly null in each of these points, they result in having a lower intensity, and the absence of the characteristic backward and forward slip zones which are typical of the Hertzian problem.

Tangential tractions grow in intensity and extension until the full contact condition is reached, corresponding to a value of $p_z(x)$ which is highlighted by the blue dashed curve in Fig. 12a. After that point, since a full stick condition holds, $q_x(x)$ remains constant until the maximum value of the vertical far-field displacement is reached. After that point, the horizontal far field displacement is applied, and the horizontal tractions grow until a condition of partial slip is reached, again blue dashed line in Fig. 12b. As expected, the last point of the interface coming into contact is also the first one which undergoes partial slip. After this point, the state of the system is such that there is an alternation of shrinking stick islands bordered by increasing zones of full slip. When the transient regime ends, a perfect overlapping between $\mu p_z(x)$ and $q_x(x)$ is observed.

5.2. Multiple harmonics profile contact

The addition of a length-scale in the Weierstrass function has the immediate effect of increasing the peak values of the normal tractions, which are localized in correspondence of the local maxima of the profile, and of reducing the contact area for a given level of the external load: the full contact condition is still achieved, but after a higher number of time steps, see Fig. 13a. The same considerations on the distribution of the tangential tractions can be made also in this case, with the difference that now the contact domain is no longer compact.

In this case, an approximated study of $q_x(x)$ and of the extension of the stick and slip areas can still be possible, exploiting, *e.g.* the Ciavarella-Jäger theorem, but under the limiting assumptions of uncoupling and equality of the Green functions in x and z directions. An interesting feature that could be appreciated thanks to having taken

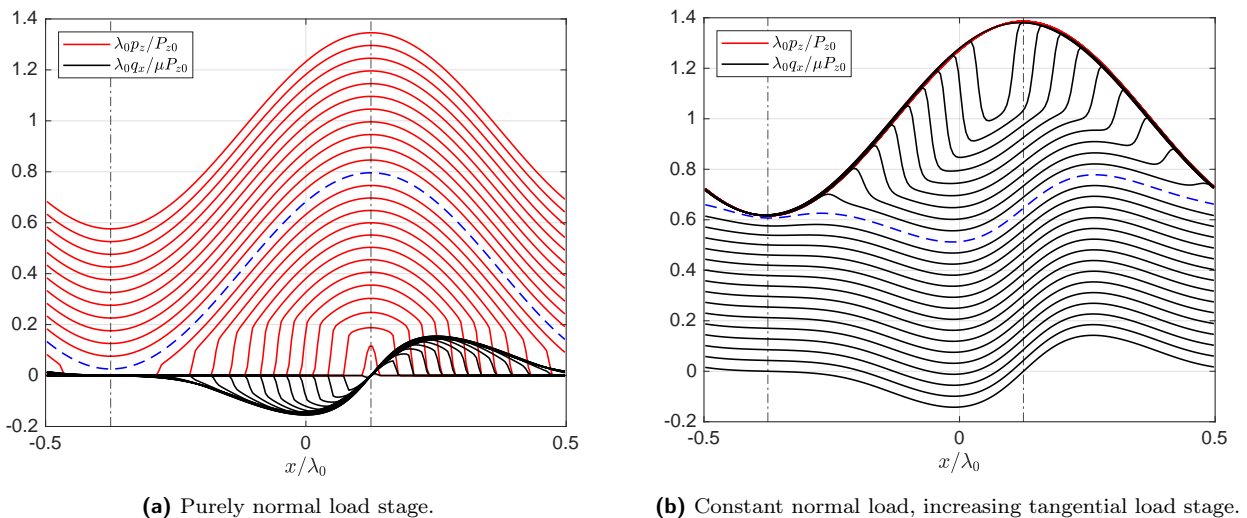


Figure 12: ironing test, single harmonic profile.

into account coupling effects is that, still under purely normal loading, if two asperities, each of them generating a separate contact island, are characterized by a severe gradient in terms of vertical tractions, the less pressed one might experience a horizontal traction distribution which is very far from the one typical of an isolated asperity, *i.e.* anti-symmetric. If the vertical tractions in the leading asperity are high enough, the horizontal displacements generated by them might be so high as compared to the ones generated by the secondary asperity that the latter are negligible, thus resulting in horizontal tractions which are all negative or positive valued from the beginning. With suitable boundary conditions or values of μ , there might also be a condition of gross slip from the beginning of the contact process. At the same time, when the second asperity comes into contact, it exerts a stiffening effect on the bulk, which reflects in a decrease of the magnitude of the increment rate of the horizontal displacements towards the high asperity, which in the final place determines a relaxing of the tangential tractions at the level of the leading asperity. This characteristics is depicted in Fig. 14a and 14b. The tangential tractions over the leading asperity, green dash-dotted curves, increase in extension and magnitude as long as the second asperity comes into contact, where they reach their maximum value, blue dashed line. After that moment, they continue growing in extension, since the contact area is still increasing, but they decrease in magnitude, due to the interaction between the different contact islands. Finally, when also the tangential far field displacement is applied, they start growing in magnitude again, gross slip starts, and the transition between full stick and full slip takes place, see Fig. 13b. The same trend can be observed also for the profile characterised by $n_w = 2$, Fig. 15, where the same comments apply as well for the increase in vertical tractions, the reduction of the contact area and the evolution of the stick and slip zones.

6. Discussion and conclusion

The proposed formulation provides a way to overcome the shortcomings and the difficulties which are encountered during the solution of contact problem between rough or generically complex profiles, when friction is considered. A standard approach of explicitly modeling the geometry of the interface using higher order interpo-

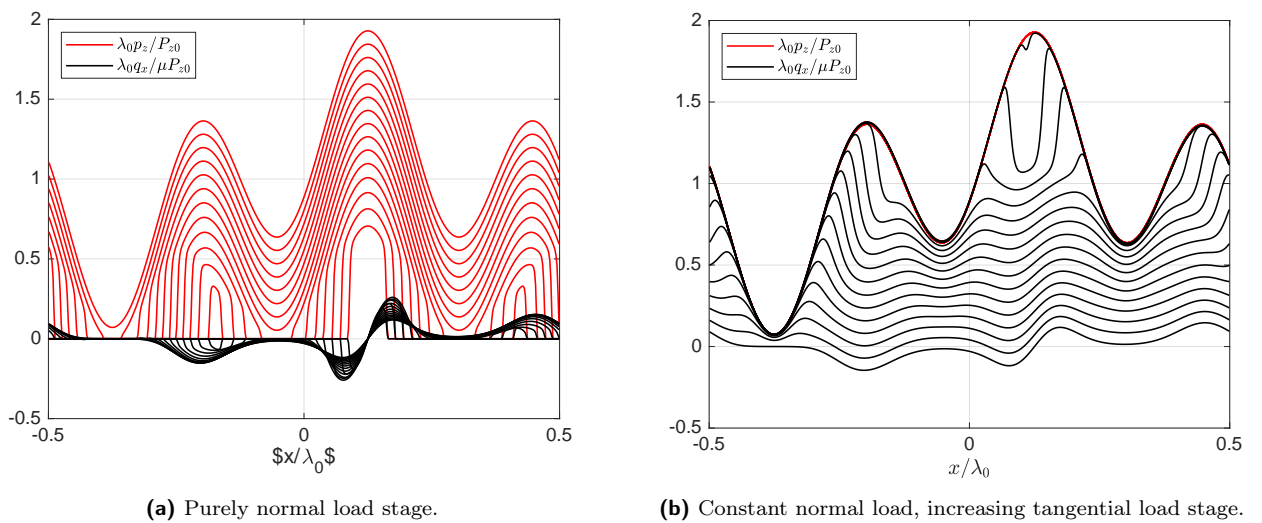


Figure 13: ironing test for double harmonics profile.

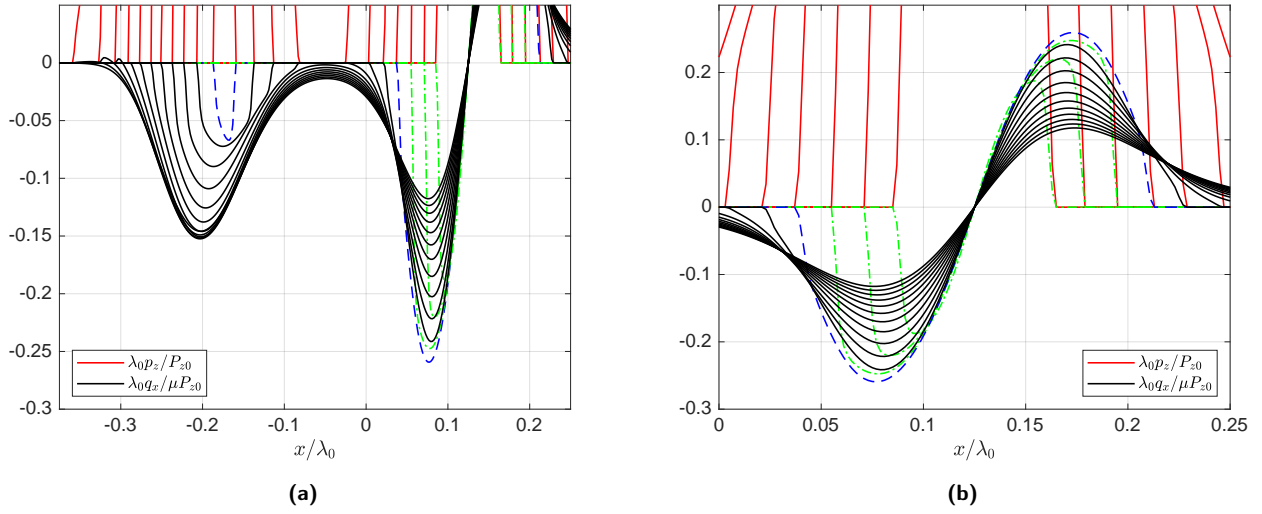


Figure 14: Evolution of contact tractions.

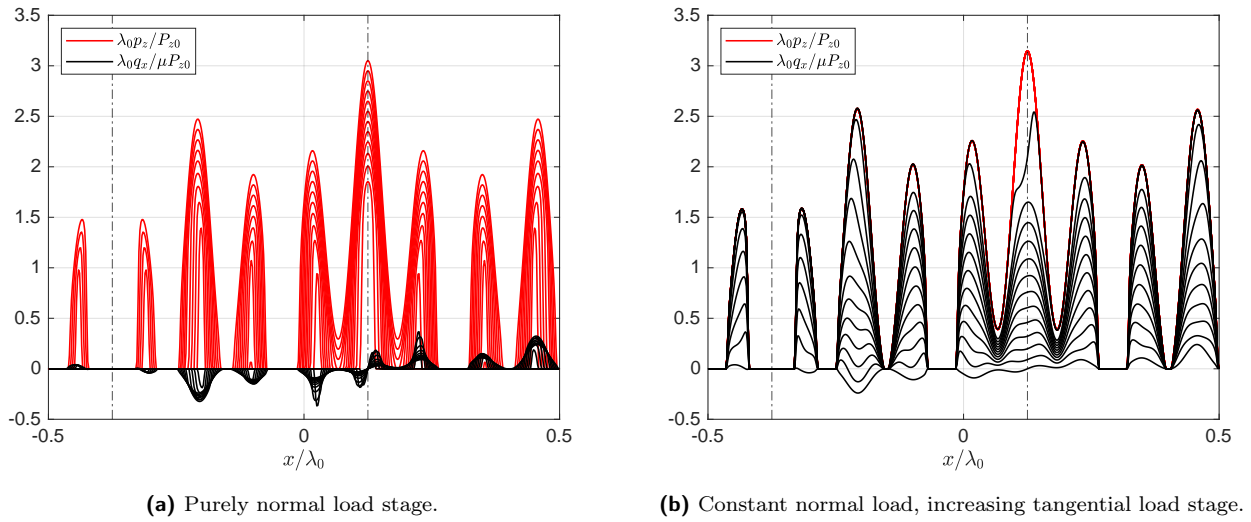


Figure 15: ironing test for a triple harmonics profile.

lation schemes such as Bezier curves, adaptive mesh refinement, or NURBS, is more suitable when the interface consists of regular and smooth profiles, while it can be difficult to exploit or it could be very expensive from a computational point of view when rough profiles have to be analyzed.

In this work, the comprehensive and challenging extension to frictional contact problem of the framework which has been set up in [34] (denominated as eMbedded Profile for Joint Roughness (MPJR) interface finite element)) has been proposed. The fundamental idea was to re-cast the original geometry of the contacting profiles, obtaining a macroscopically smooth interface which allows for a straightforward meshing with linear finite elements, while the actual geometry is stored in terms of its analytical expression and passed to the system as a correction to the initial gap function, thanks to the assumption of having a rigid indenting profile. The major advantages obtained by exploiting this approach are the use of a low order finite element interpolation scheme, with a significant reduction of nodal degrees of freedom and save of computational time.

The classical benchmark tests which are usually employed for testing the capability of higher order interpolation schemes have been used for validating the model, with excellent results obtained even for relatively coarse interface discretizations. Finally, the method has been successfully tested in relation to more complex scenarios of contact problems involving a Weierstrass profile with multiple harmonics, resulting in a useful tool for the investigation of the behavior of idealized 2D fractal rough surfaces under the non trivial assumption of full coupling between normal and tangential traction fields.

The natural development of the presented interface element, which by itself stems from [34], includes its extension to three dimensions and possibly the inclusion of other interface phenomena which could be much more difficult to be analyzed using standard FEM or BEM approaches, such as the interplay of friction and adhesion, or friction and plasticity. The presented implementation also allows for the possibility of more complex friction laws to be used, as, for example, the ones employed in [46].

Results have highlighted the important role of coupling between normal and tangential contact problems, with a special focus on rough surfaces, which is an open research topic also for precision engineering applications.

Finally, forthcoming studies would encompass the corresponding formulation to geometrically nonlinear effects and prospective coupling of contact-induced fracture events. Such developments are beyond the scope of the present investigation, deserving a careful attention.

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Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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