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On Cournot-Nash Equilibria in Generalized Qualitative Games with a Continuum of Players*

by

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Abstract. We present a result on the existence of Cournot-Nash equilibria in games with a measure space of players each with non-ordered preferences and with strategy sets in a separable Banach space. We prove our result through the use of a new selection theorem that has independent interest.

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1. Introduction

In recent years the Nash-Debreu theorem [28, 8] of 1951-52 has been the subject of two particularly far-reaching generalizations. The first is a neglected contribution of Ma [24] and is a result on the existence of Cournot-Nash equilibria in games with an arbitrary set of players each of whom has strategy sets in an arbitrary Hausdorff topological vector space. The second is a result of Toussaint [32] and it can be viewed as a generalization of Ma's theorem to a setting where each player's pay-off is generated by a non-ordered relation. Both of these results are elegant and simple enough that they can be fully stated.

The following result was proved in 1969.

Theorem (Ma) A: Let $\{X_t\}_{t\varepsilon T}$ be a family, finite or infinite, of nonempty compact convex sets each in a Hausdorff topological vector space. Let $\{f_t\}_{t\varepsilon T}$ be a family of real-valued continuous functions defined on $X = \prod_{t\varepsilon T} X_t$. If for each $t \varepsilon T$ and for any fixed[1] $\hat{x}_t \varepsilon \prod_{j\neq t} X_j = X_{-t}$, $f(x_t, \hat{x}_t)$ is a quasi-concave function of $x_t \varepsilon X_t$, then there exists a point $y \varepsilon X$ such that for any $t \varepsilon T$

$$f_t(y) = \operatorname*{Max}_{z\varepsilon X_t} f_t(z, y_{-t})$$

where y_{-t} is the projection of y in X_{-t}.

Ma's theorem can be viewed, in a sense, as the culmination of research initiated by Nash [28] and Debreu [8] and one in which the results of Browder [5] and Fan [12] played a leading role. It is worth underscoring the fact that the underlying space of strategies is not even assumed to be locally convex in Theorem A.

Browder's fixed point theorem [8, Theorem 1] also constituted an essential ingredient in the following theorem proved in 1984.

Theorem (Toussaint) B: Let $\{X_t\}_{t\epsilon T}$ be as in[2] Theorem A. Let $\{P_t\}_{t\epsilon T}$ and $\{A_t\}_{t\epsilon T}$ be a family of set-valued mappings from X into $\mathcal{P}(X_t)$ such that for any $t \in T$,

(i) (a) for all $x \in X$, $x_t \notin \text{co } P_t(x)$ and $P_t(x)$ is open in X_t,

(b) for all $y \in X_t$, $P_t^{-1}(y)$ is open in X,

(ii) (a) for all $x \in X$, int $A_t(x)$ is nonempty and convex,

(b) for all $y \in X_t$, int $A_t^{-1}(y)$ is open in X,

(c) A_t has closed graph in $X_t \times X$.

Then there exists a point $y \in X$ such that for any $t \in T$

$$y_t \in A_t(y) \quad \text{and} \quad P_t(y) \cap A_t(y) = \{\phi\}$$

where y_t is the projection of y in X_t and co and int denote convex hull and relative interior respectively.

Toussaint's theorem is heavily influenced by work on economies with non-ordered preferences initiated by Mas-Colell [26] and with subsequent contributions of Gale-Mas-Colell [14], Shafer-Sonnenschein [31], Borglin-Keiding [4] and Yannelis-Prabhakar [33]. The last bears special mention; see Theorem 6.1 in [33].

Powerful as these results are, their generality is also their principal weakness. Put simply, the complete lack of structure on the set of agents implies a corresponding lack of structure on the equilibrium point of the game. To state the matter differently, the generality of the set of players does not permit the formulation, in the context of infinite player games, of the notion of an <u>average response</u>, a concept that is of especial significance in the study of pure strategy equilibria (see, for example, Schmeidler [30]), as well as in other contexts (see, for example, Dubey, Mas-Colell and Shubik [9]). It thus seems desirable to ask whether by specializing the set of players to (say) an abstract measure space, one can show the existence of an equilibrium point that is measurable or even integrable over the set of players. We study this question here in the context of the generalized qualitative games[3] examined in Theorem B above by Toussaint.

The result we report in this paper, however, restricts this generality in two important ways. First, instead of an arbitrary Hausdorff topological vector space, we work in the setting of a separable Banach space. Second, we also require that our measure space be a locally compact subset of a metric space with a countably generated σ-field. It is of interest that Bewley [2] also made a similar restriction on the space of agents in his equivalence theorem. Even with such restrictions, however, the solution to our problem requires substantial mathematical machinery and poses technical problems whose resolution may have independent interest. In particular, in Section 4 below we present a new selection theorem of the Caratheodory type but with a stochastic domain. In order to motivate the need for such a theorem as well as to give the reader an appreciation of the difficulties engendered by a fairly straightforward economic problem, we present, in Section 3 below, an informal introduction to our proof. Section 2 presents the model and principal results and Section 5 the formal proofs.

We conclude this introduction with two further observations. Firstly, there has already been some recent work on the existence of Cournot-Nash equilibria in games with a measure space of players; see, for example, Schmeidler [30], Khan-Vohra [20] and Khan [17,18]. The principal result presented in this paper can be seen as an extension of this work to games with non-ordered preferences, as in the case of [30, 17, 18], or to games with infinite dimensional strategy sets, as in the case of [20, 30]. Indeed, the problem solved here was identified as an open problem in the survey of these results, see [19].

Secondly, it needs to be pointed out, though certainly not overstated, that the problem we solve in this paper has a bearing on the question of the existence of competitive equilibria in economies with an infinity of commodities and an infinity of agents--a problem on which substantial progress has remained elusive for some time. Of course, our setting is much simpler, as we do not have to concern ourselves with (i) price systems or alternatively, the existence of non-trivial continuous linear functionals,[4] (ii) an ordered structure to deal with concepts such as desirability and free disposal, and (iii) with showing compactness of the space of attainable allocations. In this sense, the problem of this paper is a double infinity problem with a minimum of complications.

2. The Model and Results

2.1 Preliminary Definitions and Notation

Let T be a locally compact metric space, Σ a countably generated complete σ-field and μ a real valued, non negative countably additive measure defined on Σ. (T,Σ,μ) is the measure space we shall be working with.

Let E denote a separable Banach space over the real numbers R and E* its topological dual. The norm in E will be denoted by $\|\cdot\|$. $\mathcal{P}_f(E)$ will denote the set of nonempty, closed subsets of E and $\mathcal{P}_k(E)$ the set of nonempty, compact subsets of E. A w in front of f or k will mean that the closedness or compactness is with respect to the weak topology on E. A c after f or k will denote that the set is in addition convex. We shall also use the conventional notation $\mathcal{P}(E)$ to denote the space of subsets of E and $\{\phi\}$ to denote the empty set.

For any A, B in $\mathcal{P}(E)$, we shall use the following notation.

$\bar{A}$: closure of A in the norm topology

co A: convex hull of A

$\overline{\text{co}}$ A: closed convex hull of A

A^c: compliment of A in E

A/B: set-theoretic subtraction

For the definitions of <u>upper semi-continuous</u> (u.s.c.) and <u>lower semi-continuous</u> (l.s.c.) multifunctions, the reader is referred to [3, Chapter VI]. We shall prefix the terms by a w to indicate that the relevant topology is the weak topology. We shall also have occassion to use Hausdorff continuity of a multifunction. We shall abbreviate it by h-continuity and shall not distinguish situations when we are working with the Hausdorff pseudo-metric rather than a metric--it shall be clear from the context. However, h_w-continuous will denote the fact that the Hausdorff metric has been derived from a metrizable weak topology. Finally, when we refer to joint continuity, we shall mean continuity in the product topology.

L_E^1 will denote the space of all (equivalence classes of) E-valued Bochner integrable functions defined on T with $\|f\| = \int_T \|f(t)\| d\mu t$. We shall abreviate L_R^1 by L^1. For any multifunction F: $T \to \mathcal{P}(E)/\{\phi\}$, we shall denote by S_F^1 the set $\{f \in L^1: f(t) \in F(t)$ a.e. in $T\}$.

A multifunction F: $T \to \mathcal{P}(E)$ is said to be measurable if the graph of F, $GrF = \{(t,x) \in TxE: x \in F(t)\}$, is an element of $\Sigma \otimes \mathcal{B}(E)$ where $\mathcal{B}(E)$ is the set of Borel subsets of E and $\otimes$ denotes the product σ-field. A measurable multifunction F: $T \to \mathcal{P}(E)$ is said to be integrably bounded if there exists $g \in L^1$ such that $\sup\{\|x\|: x \in F(t)\} \leq g(t)$ a.e. in T.

2.2 The Results

We now have all the terminology we need to present the principal results of our paper.

Theorem 1: If

1) X: $T \to \mathcal{P}_{wkc}(E)$ is integrably bounded,

2) P: $T \times S_X^1 \to \mathcal{P}(E)/\{\phi\}$ has norm open values and such that

(i) $\overline{co}$ P is jointly w-u.s.c.,

(ii) $(t_n,x_n) \to (t,x)$ implies $w\text{-}\overline{\lim}_{n\to\infty}$ co $P(t_n,x_n) \subseteq$ co $P(t,x)$,

(iii) $P(t,\cdot)$ is h-continuous for all t in T,

(iv) for all $x \in S_X^1$, $x(t) \notin$ co $P(t,x)$ a.e. in T,

3) A: $T \times S_X^1 \to \mathcal{P}_{wfc}(E)$ is such that for all $(t,x) \in T \times S_X^1$, $A(t,x) \subseteq X(t)$ and such that

(i) A is jointly w-u.s.c.,

(ii) $A(t,x)$ <u>has nonempty norm interior for all</u> $(t,x) \in T \times S_X^1$,

(iii) $A(t,\cdot)$ <u>is h-continuous for all</u> t in T, <u>then there exists</u> $f^* \in S_X^1$ <u>such that a.e. in</u> T,

$$f^*(t) \in A(t,f^*) \quad \text{and} \quad A(t,f^*) \cap P(t,f^*) = \{\phi\}.$$

The above result can be used to prove the existence of Cournot-Nash equilibria with only measure-theoretic hypotheses on the dependence of P and A on T. We shall refer to f* satisfying the conclusion of Theorem 1 as a Cournot-Nash equilibrium.

<u>Theorem 2</u>. <u>If</u>

1) $X: T \to \mathcal{P}_{wfc}(E)$ <u>is such that for all</u> t <u>in</u> T, $X(t) \subseteq Q$; $Q \in \mathcal{P}_{wkc}$,

2) $P: T \times S_X^1 \to \mathcal{P}(E)/\{\phi\}$ <u>is such that for all</u> $(t,x) \in T \times S_X^1$, $P(t,x) \subseteq X(t)$ <u>and such that</u>

(i) $P(\cdot,x)$ <u>is measurable for all</u> $x \in S_X^1$,

(ii) $P(t,\cdot)$ <u>is</u> h_w<u>-continuous for all</u> $t \in T$,

(iii) $P(t,x)$ <u>has norm open values for all</u> $(t,x) \in T \times S_X^1$,

(iv) $P^{-1}(t,y) = \{x \in S_X^1: \ y \in P(t,x)\}$ <u>is w-open in</u> S_X^1 <u>for all</u> $y \in X(t)$, <u>for all</u> $t \in T$,

(v) <u>for all</u> $x \in S_X^1$, $x(t) \notin \text{co } P(t,x)$ <u>a.e. in</u> T,

3) $A: T \times S_X^1 \to \mathcal{P}_{wfc}$ <u>is such that for all</u> $(t,x) \in T \times S_X^1$, $A(t,x) \subseteq X(t)$ <u>and such that</u>

(i) $A(\cdot,x)$ <u>is measurable for all</u> $x \in S_X^1$,

(ii) $A(t,\cdot)$ <u>is</u> h_w<u>-continuous for all</u> $t \in T$,

(iii) $A(t,x)$ <u>has nonempty norm interior for all</u> $(t,x) \in T \times S_X^1$,

(iv) $A^{-1}(t,y) = \{x \in S^1_X: \ y \in A(t,x)\}$ <u>is w-open in</u> S^1_X <u>for all</u>

$y \in X(t)$, <u>for all</u> $t \in T$,

<u>then there exists a Cournot-Nash equilibrium.</u>

2.3 Interpretation of the Results

T represents the set of players, Σ the set of all permissible coalitions and μ the size of a particular coalition. A coalition is numerically negligible if its μ measure is zero. This formalization of a continuum of players is now standard in the mathematical economics literature; see [15] for a comprehensive treatment.

The multifunction X represents the strategy sets with X(t) the strategy set of player t. The fact that the range of X is a separable Banach space allows us to consider infinite dimensional strategy sets.

Only a subset of the strategy set is available to a particular player and this subset depends on the choices of the rest of the players. This is formalized by the multifunction A. This feature is not present in the earlier literature on Cournot-Nash equilibria, as for example in Theorem A above, and is the reason for the terminology <u>qualitative game</u>. Following Shafer-Sonnenschein [31], such games are also referred to as <u>abstract economies</u>. We avoid this terminology here primarily because of our interest in the game as a worthwhile object of study in its own right rather than as a vehicle for showing the existence of competitive equilibria.

The multifunction P represents a formalization of the pay-off functions where P(t,x) represents all strategies preferred by player t to the strategy x(t) and that these preferred strategies depend on the choices of all the remaining players. Such a set is also referred to as the "better-than-set" in the economic literature. With this interpretation it is intuitively reasonable to suppose that $x(t) \notin P(t,x)$ and that $P(t,x)$ is an open set. Note, however, that even though P is a well-defined mathematical object, the interpretation we give it makes sense only for almost all players rather than for all players. This is simply a consequence of the fact that x is chosen from a space of equivalence classes and that it may be perturbed on sets of measure zero without changing its value. This difficulty, if it can be termed as such, is a consequence of a measure-theoretic formulation of a noncooperative solution concept of an infinite game and is discussed[5] at some length in [19].

We can now interpret our results. Consider first the Cournot-Nash equilibrium f* whose existence is asserted in Theorem 1. For all except a negligible set of players, f*(t) is in the t-th player's restricted strategy set A(t,f*) and there is no strategy in A(t,f*) which is preferred by him to f*(t). Both of these statements are contingent on the strategy choices of all other players being summarized by f*. Thus, in the language of Ma's theorem, almost everyone is choosing their best strategy given the choices of the others.

The disagreeable aspect of Theorem 1 is in the requirement that A and P are jointly w-u.s.c. on $T \times S_X^1$. Such an assumption makes essential use of the topology on T and allows statements such as "a sequence $\{t_n\}$ chosen from T tends to a limit t in T." This makes no sense from an economic point of view since the only basis for two players being "close" to each other lies in their characteristics being "close" and cannot be imposed from the outside in any other essentially ad-hoc manner.

Theorem 2 remedies this deficiency of Theorem 1. However, as will become clear from the section below, we still cannot prove Theorem 2 in the generality of an abstract measure space of players and need a topology on T. This is obviously a restriction though it is worth pointing out that there is a substantial literature in mathematical economics which considers the unit interval endowed with Lebesgue measure as the underlying space of economic agents.

Another requirement in Theorem 2 must be noted. This relates to hypothesis 1 and to the assumption that every player's strategy set must lie in the same weakly compact set Q. The technical reason for such an assumption will become clear from the discussion below; it suffices to state here that the assumption restricts variation in the distribution of the players' strategy sets.

Finally, in terms of the way we posed the problem in the introduction, it may be appropriate to draw the reader's attention to the fact that f* is Bochner integrable[6] and that $\int_T f^*(t)d\mu(t)$ denotes the equilibrium average response of the players.

3. An Introduction to the Proofs

There are now two basic approaches to deal with the problems arising out of non-ordered preferences. The first is due to Gale-Mas-Colell [14] and is essentially based on Michael's selection theorem [27, Theorem 3.1'''] while the second, due to Shafer-Sonnenschein [31], exploits the fact that one can construct a pseudo-utility function corresponding to each preference relation. Both approaches also use Kakutani's fixed point theorem. In the context of our problem, Yannelis-Prabhakar [33] and Khan-Vohra [20] respectively follow the two approaches rather closely whereas Toussaint [32], though clearly inspired by [14] by way of [4], dispenses with the selection theorem and develops a proof more directly based on a theorem of Browder [5, Theorem 1]. Our proof modifies and extends the approach of [33] but before we discuss our method of proof, it may be instructive to examine why methods based on other approaches fail.

As discussed in [19], once we have a pseudo-utility function, one can essentially follow the argument originally furnished by Debreu [8]. The only additional wrinkle lies in showing that the fixed point is indeed a Cournot-Nash equilibrium and for this one uses the fact that the preferences are irreflexive, i.e., $x(t) \notin \text{co } P(t,x)$. Of course, with non-ordered preferences, the set of maximizers of the pseudo-utility function is not convex but to overcome this one simply takes the convex hull of this set and, in a finite dimensional set-up, this operation disturbs neither the upper semicontinuity nor the measurability of the relevant mappings; for details the reader is referred to [19]. However, both of these become insurmountable difficulties in an infinite-dimensional context and since one cannot exclude that $x(t) \in \overline{\text{co}}\ P(t,x)$, the argument cannot be advanced by taking the closed convex hull.

The approach of Toussant [32] falls victim to measure-theoretic difficulties. Its elegance and power lies in reducing the entire argument to a single player and this cannot be done in a situation where negligible sets of players are being neglected to begin with. In terms of a little more detail, Toussaint extends each multifunction P_t: $X \to X_t$

to a multifunction $P_t': X \to X$ by defining $y \in X$ to be in $P_t'(x)$ iff $y_t \in X_t$. Then she applies Browder's theorem to a map $P: X \to X$ where for any x in X

$$P(x) = \begin{cases} \bigcap_{t \in T(x)} P_t'(x) & \text{if } T(x) = \{t \in T: P(x) \neq \phi\} \neq \phi \\ \{\phi\} & \text{otherwise.} \end{cases}$$

An examination of the proof of[7] Theorem 2.4 in [32] reveals that an essential step in showing that P satisfies the conditions for Browder's theorem one uses the fact that

$$P(z) \neq \{\phi\} \Longrightarrow \exists\, i_0 \text{ such that } P_{i_0}'(z) \neq \{\phi\}$$

in which case $P(z) = \bigcap_{t \in T(z)} P_t'(z) \subseteq P_{i_0}'(z)$ and the argument can be pursued in terms of player i_0. It is clear that this argument fails at the outset on account of the fact that P_t cannot be extended in the measure-theoretic context to P_t'.

Thus, the only argument which to us has the possibility of being successful is that due to Yannelis-Prabhakar. However, it has its own set of difficulties and before we go into them, a brief outline of their proof is warranted. We stay with the notation of Theorem B. YannelisPrabhakar construct a multifunction $F = \prod_{t \in T} F_t$ from X into X where $F_t: X \to X_t$ and such that

$$F_t(x) = \begin{cases} \{f_t(x)\} & \text{if } x \in U_t = \{x \in X: A_t(x) \cap \text{co } P_t(x) \neq \phi\} \\ A_t(x) & \text{otherwise} \end{cases}$$

Here f_t is a continuous function from U_t to X_t and is a selection guaranteed by the Michael selection theorem. The Ky-Fan fixed point theorem is applied to F and the fixed point is easily shown to be a Cournot-Nash equilibrium by using, in particular, the irreflexibility condition on preferences.

The extension of the multifunction F to our setting raises the obvious difficulty that we cannot simply take the Cartesian product of $F_t(\cdot)$ ($F(t,\cdot)$ in the notation of Section 2) over the set of players. We thus have to ensure that each F_t is measurable over t or, to go to the nub of the matter, to ensure that the continous selection $f_t(\cdot)$ is also measurable over t. To see it slightly differently, we have to get a selection of the correspondence $A_t(x) \cap \text{co } P_t(x)$ that is simultaneously continuous in x and measurable in t. Such selectors are called Caratheodory selectors in this branch of the mathematical literature and we are naturally led to them. However, still two difficulties remain. The first is the problem that we already met in the discussion of the Shafer-Sonnenschein approach and this is the fact that $A_t(x) \cap \text{co } P_t(x)$ is not necessarily closed. The second is that we are looking for a selector of a multifunction that is not defined on $T \times X$, i.e., on a Cartesian product of two spaces, but on the set U_t which varies with t. In summary then, we need a selection theorem of the Caratheodory type for a multifunction whose values cannot be assumed to be closed and which has a stochastic domain.

We present such a theorem in the next section and show how it relates to other such results in the mathematical literature. However, we could only prove it under a joint upper semicontinuity hypothesis on the underlying multifunction. This is required in order to appeal to Ma's [25] generalization of Dugundji's extension theorem. This is also the reason for such a hypothesis in our Theorem 1.

However, once we have Theorem 1, it is natural to ask whether we can use a Lusin type theorem to eliminate the continuity hypothesis on T. In particular, can one approximate a measurable correspondence by a continuous one, apply Theorem 1 to show the existence of an approximate Cournot-Nash equilibrium, and finally show that the limit of these equilibria is indeed a bona-fide Cournot-Nash equilibrium? The answer to all of these questions is yes but it is obvious that such a program requires a topology on the set of players and this is the reason for such a hypothesis even in Theorem 2.

It should be underscored that our formulation of Theorem 2 makes an essential use of the Hausdorff metric derived from the weak topology--indeed this seems to us to be the only essentially additional hypothesis in relation to previous work. However, as is well known, the weak topology is not globally metrizable and hence we have to include all our strategy sets in the weakly compact set Q. This also has the additional advantage of allowing us to go from Hausdorff continuity to upper semi-continuity.

In conclusion, it may be worth mentioning that the Lusin-type theorem that works for the problem at hand is a result due to Scorza-Dragoni [6] and that in showing that the limit of the approximate equilibria is an equilibrium, we use the Khan-Majumdar [21] extension of Artstein's characterization of weak sequential convergence in L^1_E.

4. A Selection Theorem of the Caratheodory Type

We begin with a definition. A function $f\colon\ T \times E \to E$ is said to be a *Caratheodory function* if $f(\cdot,x)$ is measurable in T for any x in E and if $f(t,\cdot)$ is continuous in E for any t in T. It is clear that the definition of a Caratheodory function carries over to any abstract measure space T and any topological space E. The following Lemma is a generalization of Lemma 1 in [22] and is a useful characterization of Caratheodory functions.

Lemma. Let $(\Omega, \mathcal{F}, \mu)$ *be a complete σ-finite measure space,* Z *a locally compact separable metric space and* Y *a metric space. Then* $f\colon\ \Omega \times Z \to Y$ *is Caratheodory iff* $\omega \to r(\omega)(\cdot) = f(\omega,\cdot)$ *from* Ω *into* C(Z,Y) *is measurable where* C(Z,Y) *is the space of continuous functions from* Z *into* Y *and endowed with the compact-open topology.*

The above Lemma allows us to prove

Theorem 3. If

1) K *is a w-compact subset of a separable Banach space,*

2) $A \subseteq T \times K$ *is closed* (K *with w-topology),*

3) $F\colon\ A \to \mathcal{P}_{wkc}(E)$ *and such that*

   (i) $F(t,x)$ *has nonempty norm interior for all* $(t,x) \in A$,

   (ii) $\overline{F}$ *is jointly w-u.s.c.*

   (iii) $F(t,\cdot)$ *is h-continuous on* $A_t = \{x \in K\colon\ (t,x) \in A\}$,

then there exists a Caratheodory function $f\colon\ A \to E$ *such that* $f(t,x) \in F(t,x)$ *for all* $(t,x) \in A$.

Remark: Theorem 3 is true if K is any locally compact metric space instead of being a w-compact subset of a separable Banach space.

The above theorem can be usefully compared to the Caratheodory type selection theorems of Kucia [22] and to those of Fryszkowski [13], the latter already generalizing earlier work of Castaing. Note that Fryszkowski's theorems are also restricted to a topological measure space but they deal with multifunctions with closed values and whose domain is a Cartesian product of two spaces, i.e., whose domain is not stochastic. As such they cannot be used for our problem. Kucia's work deals with abstract measure spaces but does not contain generalizations along the directions relevant to our problem.

5. Proofs of the Theorems

We begin with a proof of the Lemma.

Proof of Lemma.

Necessity. Let B be a basis element for the compact-open topology on $C(X,Y)$. Hence there exist $K \subseteq X$ compact, $V \subseteq Y$ open such that $B = \{q(\cdot) \varepsilon C(X): q(K) \subseteq V\}$. We need to show that $r^{-1}(B) \varepsilon \Sigma$. We have $r^{-1}(B) = \{\omega \varepsilon \Omega: r(\omega)(\cdot) \varepsilon B\} = \{\omega \varepsilon \Omega: r(\omega)(K) \subseteq V\} = \{\omega \varepsilon \Omega: f(\omega,K) \subseteq V\}$. Note that $f(\omega,K)$ is a compact subset of V. Let $\{z_n\}_{n\geq 1}$ be a dense set in K. Then exploiting the continuity of $f(\omega,\cdot)$ we can write

$$r^{-1}(B) = \bigcap_{n\geq 1} \{\omega \varepsilon \Omega: f(\omega,z_n) \varepsilon V\} \varepsilon \Sigma$$

since for all $x \varepsilon X$, $f(\cdot,z)$ is measurable.

Sufficiency. Let (r,id): $\Omega \times Z \to C(Z,Y) \times Z$ be defined by

$$(\omega,z) \to (r(\omega)(\cdot),z)$$

This is a measurable map. Let e be the evaluation map on $C(Z,Y) \times Z$ from [10, theorem 2.4, p. 260] we know that this is continuous. Now consider u: $\Omega \times Z \to Y$ defined by: $(\omega,z) \to u(\omega,z) = [e\cdot(r,id)](\omega,z)$. This is measurable. But $e\cdot(r,id)(\omega,z) = e(r(\omega)(\cdot),z) = r(\omega)(z) = f(\omega,z)$ implies $u \equiv f$ which shows that f is a Caratheodory function. ∎

We can now provide a

Proof of Theorem 3.

Consider the multifunction $\overline{F}$: $A \to \mathcal{P}_{wkc}(E)$. On applying [25, Theorem 2.1, p. 7] we can find

$$G: T \times K \to \mathcal{P}_{wkc}(E)$$

which is an u.s.c. extension of $F(\cdot,\cdot)$. From the proof of theorem 2.1 of Ma [25], we know that $G(\cdot,\cdot)$ has the following form

$$F(t,x) = \begin{cases} F(t,x) & \text{for } (t,x) \in A \\ \sum_{i \in I} a_i(t,x)F(\overline{t},\overline{x}) & \text{for } (t,x) \in A^c,\ (\overline{t},\overline{x}) \in A \end{cases}$$

where $\{a_i(\cdot,\cdot)\}_{i\in I}$ is an appropriate partition of unity for A^c (for details see Ma [25]). But then this expression suggests that $G(\cdot,\cdot)$ still has nonempty interior and that for every $t \in T$, $G(t,\cdot)$ is h-continuous. This also implies that int $G(t,\cdot)$ is h-continuous and hence int $G(t,\cdot)$ is lower semicontinuous.

Now we work with this globally defined multifunction $G(\cdot,\cdot)$. Consider the multifunction M: $T \to \mathcal{P}(C(K,X))$

$$M(t) = \{f \in C(K;X): \ f(x) \in \text{int } G(t,x) \text{ for all } x \in K\}$$

where $C(K;X)$ is the space of all continuous functions from K into X, endowed with the compact-open topology. Recalling that K being a weakly compact subset of separable Banach space, is metrizable for the weak topology (see Dunford-Schwartz [11, theorem 3, p. 434]) and so perfectly normal, we can apply theorem 3.1‴ (c) of Michael [27] and deduce that for all $t \in T$, $M(t) \neq \phi$. Now rewrite $M(\cdot)$ as follows

$$M(t) = \{f \in C(K;X): \ d_{\partial G(t,x)}(f(x)) > 0 \text{ for all } x \in K\} \cap \{C(\overline{G(t,\cdot)})\}$$

where $\partial G(t,x)$ denotes the boundary for the set $G(t,x)$ where $d_A(x)$ denotes the distance of x from the set A and $C(\overline{G(t,\cdot)})$ is the nonempty set of continuous selections from $\overline{G(t,\cdot)}$.

Next we claim that for fixed $t \in T$ the map $x \to d_{\partial G(t,x)}(f(x)$ is continous from K into R_+. To show that we proceed as follows. Since, by construction, $G(t,\cdot)$ is Hausdorff continuous, proposition 2.1 of DeBlasi-Pianigiani [7] tells us that $\partial G(t,\cdot)$ is Hausdorff too. So for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_w(x',x) < \delta$ ($d_w(\cdot,\cdot)$ is the metric that makes the weak topology on K metrizable) implies $h(\partial G(t,x'),\partial G(t,x)) < \varepsilon/2$. Also since $f \in C(K,X)$ we can take $\delta > 0$ so that $\|f(x') - f(x)\| < \varepsilon/2$.

Using the fact that the distance function is nonexpansive and the definition of the Hausdorff metric, we can write for $x,x' \in K$, $d_w(x',x) < \delta$:

$$\left| d_{\partial G(t,x')}(f(x')) - d_{\partial G(t,x)}(f(x)) \right| \leq \left| d_{\partial G(t,x')}(f(x')) - d_{\partial G(t,x')}(f(x)) \right| + \left| d_{\partial G(t,x')}(f(x)) - d_{\partial G(t,x)}(f(x)) \right|$$

$$\leq \| f(x') - f(x) \| + h(\partial G(t,x'),\partial G(t,x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

Hence we get that $x \to d_{\partial G(t,x)}(f(x))$ is continuous.

Also since $t \to G(t,x)$ is u.s.c. it is automatically measurable and so theorem 4.6 (iv) of Himmelberg [16] tells us that $t \to \partial G(t,x)$ is measurable and so $t \to d_{\partial G(t,x)}(f(x))$ is measurable.

Thus we have shown that the map

$$(t,x) \to d_{\partial G(t,x)}(f(x))$$

from $T \times K$ into R_+ is a Caratheodory map.

Now consider the map $L: T \times C(K;X) \to C(K;R_+)$ defined by

$$L(t,f(\cdot)) = d_{\partial G(t,\cdot)}(f(\cdot))$$

Since $(t,x) \to d_{\partial G(t,x)}(f(x))$ is Caratheodory, the lemma tells us that $t \to d_{\partial G(t,\cdot)}(f(\cdot))$ is measurable from T into $C(K;R)$. Also theorem 1 of Kuratowski [22, p. 93] tells us that $C(K;X)$ with the compact-open topology is metrizable.

So let $f^n(\cdot) \xrightarrow{C(K;X)} f(\cdot)$ as $n \to \infty$. Then for $x_n \to$ we have

$$\left|L(t,f^n(\cdot)(x_n) - L(t,f(\cdot))(x)\right| = \left|\frac{d(f^n(x_n) - d(f(x))}{\partial G(t,x_n^n) \quad \partial G(t,x)}\right|$$

$$\leq \left|\frac{d(f^n(x_n) - d(f(x))}{\partial G(f,x_n^n) \quad \partial G(t,x_n)}\right| + \left|\frac{d(f(x)) - d(f(x))}{\partial G(t,x_n) \quad \partial G(t,x)}\right|$$

$$\leq \|f^n(x_n) - f(x)\| + h(\partial G(t,x_n), \partial G(t,x))$$

Note that $h(\partial G(t,x_n), \partial G(t,x)) \to 0$ as $n \to \infty$. Also $f^n(\cdot) \overset{C(K;X)}{\to} f(\cdot)$. But since X is a metric space, the compact-open topology is equivalent to the topology of uniform convergence on compact a and the latter is in turn equivalent to continuous convergence (see theorem 7.5 of Dugundji [10, p. 268]). So $\|f^n(x_n) - f(x)\| \to 0$ as $n \to \infty$. Thus $L(t,\cdot)$ is continuous for the compact-open topology on $C(K;X)$.

Therefore we have just proved that $L(\cdot,\cdot)$ is Caratheodory. Since $C(K;X)$ is separable (see theorem 3 in Kuratowski [22, p. 94]), we deduce that $L(\cdot,\cdot)$ is in fact jointly measurable. Hence we have

$$\text{Gr } M = \{(t,f(\cdot)) \in T \times C(\overline{G(\cdot,\cdot)}):\ L(t,f(\cdot)) \in \text{int } C_+(K;R_+)\} \in \Sigma \otimes \mathcal{B}(C(K;X)$$

On applying Aumann's selection Theorem [15, Theorem 1, p. 54] we can find a measurable function $r: T \to C(K,X)$ such that for all $t \in T$ we have that $r(t) \in M(t)$. Let $r(t)(\cdot) = f(t,\cdot)$. Using the lemma we have that $f(\cdot,\cdot)$ is a Caratheodory function such that $f(t,x) \in F(t,x)$ for all $(t,x) \in A$. This is the desired Caratheodory selector of $F(\cdot,\cdot)$. ∎

Proof of Remark:

It can be easily checked that nothing is changed in the proof if instead of taking K to be a w-compact subset of a separable Banach space, we let it be any locally compact metric space. ∎

Proof of Theorem 1.

From [29, Theorem 4.2], we know that S_X^1 is a w-compact subset of L_E^1. Since Σ is countably generated and E is separable, then L_E^1 is separable and so S_X^1 endowed with the weak topology is metrizable.

Let $\psi(t,x) = A(t,x) \cap \text{co } P(t,x)$ and set $M(t) = \{x \in S_X^1 : \psi(t,x) \neq \phi\}$. We now claim that GrM is closed in $T \times S_X^1$, S_X^1 endowed always with the weak topology. To see this, let $\{(t_n,x_n)\}_{n\geq 1} \subseteq$ GrM such that $(t_n,x_n) \to (t,x)$. Then we have

$$\text{w-}\overline{\lim_{n\to\infty}}\, \psi(t_n,x_n) = \text{w-}\overline{\lim_{n\to\infty}}\, [A(t_n,x_n) \cap \text{co } P(t_n,x_n)]$$

$$\subseteq \text{w-}\overline{\lim_{n\to\infty}}\, A(t_n,x_n) \cap \text{w-}\overline{\lim_{n\to\infty}}\, \text{co } P(t_n,x_n).$$

But by hypothesis A is jointly w-u.s.c. and by the hypothesis on P, we obtain

$$\text{w-}\overline{\lim_{n\to\infty}}\, A(t_n,x_n) \subseteq A(t,x) \text{ and } \text{w-}\overline{\lim_{n\to\infty}}\, \text{co } P(t_n,x_n) \subseteq \text{co } P(t_n,x_n).$$

Thus we have

$$\text{w-}\overline{\lim}\, \psi(t_n,x_n) \subseteq A(t,x) \cap \text{co } P(t,x) = \psi(t,x).$$

Again, by hypothesis, $\psi(t_n,x_n) \subseteq \bigcup_{n\geq 1} B(t_n)$ which by [3, Theorem 3, p. 110] is a weakly compact set. Let $z_n \in \psi(t_n,x_n)$. Then the Eberlein-Smulian theorem [11, Theorem V.6.1, p. 430] tells us that we can find $(z_{n_k}) \xrightarrow{w} z$ which implies that $z \in \text{w-}\overline{\lim}\, \psi(t_n,x_n) \subseteq \psi(t,x)$. This implies that $\psi(t,x) \neq \phi$ which ensures that $(t,x) \in$ GrM, i.e., GrM is closed.

Now $\overline{\psi}(t,x) = \overline{A(t,x) \cap \text{co } P(t,x)} = A(t,x) \cap \overline{\text{co}}\ P(t,x)$ is w-u.s.c. from GrM into E, see Theorem 8 in Aubin-Ekeland [1, p. 110].

Finally, since $P(\cdot,\cdot)$ has open values and $A(\cdot,\cdot)$ has a nonempty norm interior, $A(t,x) \cap \overline{\text{co}}\ P(t,x)$ has a nonempty norm interior for all x in M(t). We can now appeal to [7, Proposition 2.3] to assert that $\overline{\psi}$ is h-continuous on M(t).

We are now in a position to apply Theorem 3 and find a Caratheodory function $f: T \times S_X^1 \to E$ such that

$$f(t,x) \in \psi(t,x) = A(t,x) \cap \text{co } P(t,x) \text{ for all } (t,x) \in GrM.$$

Next define

$$G(t,x) = \begin{cases} f(t,x) & \text{for } (t,x) \in GrM \\ A(t,x) & \text{for } (t,x) \notin GrM \end{cases}$$

Clearly $G: T \times S_X^1 \to \mathcal{P}_{wkc}(E)$.

Let $\alpha: S_X^1 \to \mathcal{P}(S_X^1)$ be defined by:

$$\alpha(x) = \{y(\cdot) \in S_X^1 : y(t) \in G(t,x) \text{ a.e. in } T\}$$

Clearly $\alpha(x) \neq \phi$ for all $x \in S_X^1$. It is also easy to see that $\alpha(\cdot)$ has closed and convex values. We will now show that it is w-u.s.c.

Since S_X^1 with the weak topology is a compact, metric space it suffices to show that $Gr\alpha$ is closed. So let $(x_n, y_n) \in Gr\alpha$, $(x_n, y_n) \xrightarrow{wxw} (x,y)$. Then we have

$$y_n(t) \in G(t,x_n) \text{ a.e.}$$

Invoking Mazur's theorem [11, V.3.14, p. 422] we can find $z_n(\cdot) \in \operatorname{conv} \bigcup_{k \geq n} y_k(\cdot)$ such that $z_n(\cdot) \xrightarrow{L_E^1} y(\cdot)$. By passing to a subsequence if necessary we may assume that $z_n(t) \xrightarrow{L_E^1} y(t)$ for all $t \in T/N$ with $\mu(N) = 0$. Fix $t \in T/N$. Observe that $G(t,\cdot)$ is w-u.s.c. from S_X^1 into E. So for every $\varepsilon > 0$ we can find $n \geq 1$ such that for all $k \geq n$ we have

$$G(t,x_k) \subseteq G(t,x) + \varepsilon B_1$$

where B_1 is the unit ball of E. Hence

$$\operatorname{con} \bigcup_{k \geq n} G(t,x_k) \subseteq G(t,x) + \varepsilon B_1 \Rightarrow z_n(t) \in G(t,x) + \varepsilon B_1$$

$$\Rightarrow y(t) \in G(t,x) + \varepsilon B_1$$

On letting ε go to zero, we conclude that $y(t) \in G(t,x)$. Since $t \in T/N$ was arbitrary, we also conclude that $y(t) \in G(t,x)$ a.e. in T. Thus $y \in \alpha(x)$, i.e., $\mathrm{Gr}\alpha$ is closed and hence $\alpha(\cdot)$ is w-u.s.c. Then Ky Fan's fixed point theorem [3, p. 251] tells us that $\hat{x}(\cdot) \in S_X^1$ such that $\hat{x} \in \alpha(\hat{x})$. It is easy to see that this is the desired Nash equilibrium. ∎

Finally, we can furnish a proof of our principal result.

<u>Proof of Theorem 2.</u>

View A and $\overline{\mathrm{co}}\, P$ as mappings from $T \times S_X^1$ into the space of nonempty, closed convex subsets of a weakly compact subset Q of E. Since E is separable, the weak topology on Q is metrizable and we can correspondingly view A and $\overline{\mathrm{co}}\, P$ as mappings which take values in a separable metric space. We can now apply the Scorza-Dragoni theorem [6, Theorem 3.1, p. 97] to find, for any $\varepsilon > 0$, a compact $T_\varepsilon \in \Sigma$ such that $\mu(T/T_\varepsilon) < \varepsilon$ and A and $\overline{co}$ P restricted to $T_\varepsilon \times S_X^1$ to be both jointly h_w-continuous. However, since both A and $\overline{co}$ P take compact values, we can appeal to [3, Theorem 1, p. 126] to assert that A and $\overline{co}$ P are w-u.s.c. In addition, since $\rho(U,V) = \rho(\overline{U},\overline{V})$ for any subsets U, V of E and ρ denotes the Hausdorff metric derived from the weak topology, we can assert that co P is also jointly h_w-continuous. But this clearly implies

$$w\text{-}\overline{\lim}\ \text{co}\ P(t_n,x_n) \subseteq \text{co}\ P(t,x) \text{ as } (t_n,x_n) \to (t,x)$$

Finally, observe that $P(t,\cdot)$ and $A(t,\cdot)$ are h-continuous on S_X^1 for all t in T on account of the fact that the weak topology is weaker than the norm topology.

Now extend $A|_{T_\varepsilon x S_X^1}$, $P|_{T_\varepsilon x S_X^1}$ to $T \times S_X^1$ by choosing the same, convex, subset Q^1 of Q for all t in T/T_ε and for all x in S_X^1 and such that Q^1 has nonempty norm interior. All the hypotheses of Theorem 1 are satisfied and for any n in N, we can appeal to Theorem 1 to assert the existence of a compact set $T_n \in \Sigma$, $\mu(T/T_n) < \varepsilon/2^n$, $f_n \in S_X^1$ such that for all t in (T/T_n)

$$f_n(t) \in A(t,f_n) \quad \text{and} \quad A(t,f_n) \cap P(t,f_n) = \{\phi\}.$$

We can thus manufacture a sequence $\{f_n\}_{n\geq 1}$ each of whose elements lie in S_X^1. Since S_X^1 is w-compact, there exists a subsequence, also denoted by $\{f_n\}$, such that w-lim $f_n = f^*$ and $f^* \in S^1_X$. We can now assert that a.e. in T,

(i) $f^*(t) \in A(t,f^*)$ and (ii) $A(t,f^*) \cap P(t,f^*) = \{\phi\}$.

Suppose not. Without loss of generality, let $S \in \Sigma$, $\mu(S) > 0$ be the set of players for which both (i) and (ii) are violated. Focus on the violation of (i). Find an n^1 such that for all $n \geq n^1$, $(\varepsilon/2^n) < (\mu(S)/4)$. Then $\mu(\bigcup_{n \geq n^1} T_n) < (\mu(S)/2)$.

Now by [21, Theorem 1], there exists $B \in \Sigma$, $\mu(B) = 0$ such that for all t in (T/B), $f^*(t) \in \text{co} \lim\sup \{f_n(t)\}$. Pick $\tau \in (S/B \bigcup_{n \geq n^1} T_n)$.

Since $A(\tau,\cdot)$ is h_w-continuous and takes weakly compact values, $A(\tau,\cdot)$ is w-u.s.c. Furthermore, since the domain and range of $A(\tau,\cdot)$ are compact sets, by [3, Corollary, p. 122], $A(\tau,\cdot)$ has a closed graph. Hence any cluster point $\bar{f}(\tau)$ of $\{f_n(\tau)\}$ lies in $A(\tau,f^*)$. Since this is true for any cluster point and since $A(t,f^*)$ is convex, we can assert that $f^*(\tau) \in \text{co} \{\bar{f}(\tau)\} \subseteq A(t,f^*)$, a contradiction to the fact that τ is in S.

Now focus on (ii). Suppose there exists y in $A(\tau,f^*) \cap P(\tau,f^*)$. Since both A and P have weakly open lower sections, there exists n_0 such that for all $n \geq n_0$, $y \in A(\tau,f^*_n) \cap P(\tau,f^*_n)$. Now choose n to be greater than Max (n_0,n^1), and we obtain a contradiction. ∎

Footnotes

[1]Strictly speaking, $(x_t, \hat{x}_t)$ in $f(x_t, \hat{x}_t)$ refers to a vector whose t-th coordinate is x_t. However no confusion should result.

[2]Toussaint emphasizes that her results do not even require the underlying topological space to be Hausdorff, see [32, Footnote 3, p. 100].

[3]These are also termed "abstract economies" by Shafer-Sonnenschein [31]; also see below.

[4]Hence the absence of the local convexity requirement on the topological vector space in Theorems A and B.

[5]As is pointed out in [19], this difficulty is also present in Schmeidler's work [30] and has nothing to do with non-ordered preferences as such.

[6]This is in view of the fact that X is integrably bounded.

[7]The reader can check that this is the essential result in the proof of her Theorem B quoted in the Introduction above.

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