12-2021

Tractor Connections for Killing Tensors and Their Generalizations

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TRACTOR CONNECTIONS FOR KILLING TENSORS AND THEIR GENERALIZATIONS

by

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A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Mathematics

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UTAH STATE UNIVERSITY
Logan, UT

2021

Abstract

Tractor Connections for Killing Tensors and their Generalizations

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We create new symbolic software tools for the analysis of Killing tensors. Central to our work is the construction of the tractor connection defined on the tractor bundle, which allows one to obtain information about the space of Killing tensors without solving the Killing equations–an approach termed the tractor approach. We give a new application of the tractor approach which allows one to more easily check explicitly for linear independence of a given set of Killing tensors. We develop software to implement such methods in the case of rank 2 Killing tensors; similarly, we develop software to implement analogous methods in the study of Killing-Yano tensors and conformal Killing vectors. Using our newly developed software, we find examples of rank 2 irreducible Killing tensors for exact solutions to Einstein's field equations. We also make an in-depth study of various other methods of constructing Killing tensors of rank 2 and find that these algorithms most often do not produce Killing tensors which are linearly independent of the reducible Killing tensors and the metric, with the Kerr metric being one of the only known sources of examples.

(220 pages)

Public Abstract

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We create new symbolic software tools for the analysis of Killing tensors. Central to our work is the construction of the tractor connection defined on the tractor bundle, which allows one to obtain information about the space of Killing tensors without solving the Killing equations–an approach termed the tractor approach. We give a new application of the tractor approach which allows one to more easily check explicitly for linear independence of a given set of Killing tensors. We develop software to implement such methods in the case of rank 2 Killing tensors; similarly, we develop software to implement analogous methods in the study of Killing-Yano tensors and conformal Killing vectors. Using our newly developed software, we find examples of rank 2 irreducible Killing tensors for exact solutions to Einstein's field equations. We also make an in-depth study of various other methods of constructing Killing tensors of rank 2 and find that these algorithms most often do not produce Killing tensors which are linearly independent of the reducible Killing tensors and the metric, with the Kerr metric being one of the only known sources of examples.

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1 Introduction

In this thesis, we create new symbolic software tools for the analysis of Killing tensors. Central to our work is the construction of the tractor connection defined on the tractor bundle, which allows one to obtain information about the space of Killing tensors without solving the Killing equations–an approach termed the tractor approach. We give a new application of the tractor approach which allows one to more easily check explicitly for linear independence of a given set of Killing tensors. We develop software to implement such methods in the case of rank 2 Killing tensors; similarly, we develop software to implement analogous methods in the study of Killing-Yano tensors and conformal Killing vectors. Using our newly developed software, we find examples of rank 2 irreducible Killing tensors for exact solutions to Einstein's field equations.

Killing vectors are quantities of remarkable interest in differential geometry and mathematical physics. Named after Wilhelm Killing, Killing vectors on Riemannian or pseudo-Riemannian manifolds are vector fields which Lie-differentiate the metric to zero. Additionally, the set of all Killing vector fields is isomorphic to the Lie algebra of the isometry group of the metric. Explicitly finding the Killing vectors for a given metric can prove to be elusive, as doing so requires one to solve a system of linear, first order, partial differential equations known as the Killing equations for Killing vectors.

Killing tensors are generalizations of Killing vectors. Killing tensors appear, among other places, as first integrals of the geodesic equation (Stephani et al., 2003). Additionally, they have been used in the separation of variables for the Hamilton-Jacobi equation (Kalnins and Miller, 1981) and in the separation of variables for the Dirac equation (Carignano et al., 2011). Notwithstanding their utility in physics, finding them explicitly involves solving a system of linear, first order, partial differential equations–as with Killing vectors–which we refer to as the Killing equations for Killing tensors. However, Killing tensors are generally more difficult to solve for than Killing vectors. In the case of Killing vectors, the Killing equations become a system of finite type differential equations after one differentiation: that is, all second derivatives of the components of the Killing vectors are written in terms of lower order derivatives. With Killing tensors of rank k, however, k derivatives must be taken before the Killing equations become a system of finite type: that is, k derivatives must be taken before all derivatives at order k are written in terms of lower-order derivatives (Houri et al., 2018). On a manifold of dimension 4, the Killing equations for Killing vectors become a finite system of 10 equations, whereas the Killing equations for Killing tensors of rank 2 become a finite system of 50 equations.

The two other generalizations of Killing vectors examined in this thesis are conformal Killing vectors and Killing-Yano tensors. Both have utility, among other uses, in explicitly constructing Killing tensors of rank 2 (Popa and Ovidiu, 2007; Edgar et al., 2004). Killing-Yano tensors have also been used in the separation of the Dirac equation (Carter and McLenaghan, 1979; Fels and Kamran, 1990). As with Killing vectors and Killing tensors, finding conformal Killing vectors and Killing-Yano tensors requires solving a system of linear, first order, partial differential equations–known also as Killing equations–and so they too can prove difficult to solve for explicitly.

Other generalizations of Killing vectors include higher rank conformal Killing forms, conformal Killing tensors, and Killing spinors (M. Walker and R. Penrose, 1970). These will not be treated in this thesis, though they too are of interest in differential geometry and mathematical physics.

Having identified objects of interest, and having pointed to the difficulty of explicitly finding those objects due to the requirement of solving the associated Killing equations, we will present a method for studying solutions of systems of differential equations which need not involve solving the Killing equations directly (Houri et al., 2018). This method is sometimes referred to, in the mathematical literature, as the tractor approach, as it will be referred to in this thesis.

The tractor approach entails the construction of a vector bundle known as the tractor bundle, as well as the construction of a linear connection defined on the bundle known as the tractor connection. The tractor approach allows us to write the original system of equations as the equations which define a smooth, parallel section on the tractor bundle with respect to the tractor connection. A section in the nullspace of the k^{th} and lower order derivatives of the curvature tensor is called quasi-parallel to order k, and the set of quasi-parallel sections of order k form what is known as the k^{th} order reduced tractor bundle. The dimension of the nullspace of the curvature tensor and its derivatives give us an upper bound on the number of parallel sections. The curvature tensor and its derivatives determine the holonomy algebra of the tractor connection.

In some cases, the tractor approach also allows one to get the independent solutions explicitly where a direct approach to solving the Killing equations fails. In cases where a sub-maximal number of independent solutions exist, the quasi-parallel sections as well as the condition of parallelism with respect to the tractor connection allow one to form a reduced system of equations which may be more practical to solve explicitly than the original system of equations.

The reduced system of equations is generated as follows. After computing a basis for the k^{th} order reduced tractor bundle, one forms an arbitrary linear combination of the basis elements using unknown functions as scalars. A system of equations for the unknown functions is then generated by applying the condition of parallelism. The resulting system of differential equations is thought to be easier to solve explicitly, since it may contain fewer unknown functions than the parallel equations, which equations are equivalent to the original Killing equations.

It is known that the set of Killing tensors of a particular metric forms an algebra with respect to the symmetric tensor product (Thompson, 1986). Killing tensors which cannot be algebraically generated by Killing tensors of lower rank are known as irreducible Killing tensors. Accordingly, one seeks to find all irreducible Killing tensors for a particular metric. To our knowledge, this has only been accomplished for metrics of constant sectional curvature (Thompson, 1986). As irreducible Killing tensors seem to be scarce (Kruglikov and Matveev, 2016), the discovery of an irreducible Killing tensor is of significance.

A novel application of the tractor approach is the ability to more easily check the linear independence of a set of Killing tensors. The issue of determining the number of independent, reducible Killing tensors of a given rank is, conceptually, an elementary question of basic linear algebra. And yet, it can be difficult in practice[1] to check linear independence when the components of the Killing tensor fields are not rational functions of the coordinates. However, our novel application is that a set of Killing tensors is linearly independent over $\mathbb{R}$ if and only if their lifts to the tractor bundle are linearly independent at a single point. Thus, with the tractor approach, checking linear independence for Killing tensor fields can be reduced to checking linear independence for vectors in $\mathbb{R}^n$.

This novel application is of paramount importance in the search for metrics which admit irreducible Killing tensors. The tractor approach is known, as we have explained, to produce an upper bound on the number of linearly independent Killing tensors of a given rank. On the other hand, the number of independent, reducible Killing tensors, together with the metric, which may or may not be reducible, gives us a lower bound on the number of linearly independent Killing tensors. For example, we consider the case of rank 2 Killing tensors. If a metric admits p Killing vectors, we can generate $\frac{p(p+1)}{2}$ reducible Killing tensors of rank 2 by means of the symmetric tensor products of the (covariant) Killing vectors. Coupled with the metric, we can

[1] i.e. in a computer algebra system.

easily generate $\frac{p(p+1)}{2}+1$ Killing tensors of rank 2, and, for the purposes of this illustration, we call this set S. Each element of S can be lifted up to the tractor bundle to form the set $\tilde{S}$. The set $\tilde{S}$ evaluated at a point x_0 can be checked for linear independence: in fact, the number of linearly independent elements of $\tilde{S}$ evaluated at x_0 can be computed in Maple using rudimentary principles of linear algebra. Suppose this number is q: our novel application then informs us that the dimension of the span of S is q, thus providing a lower bound on the number of Killing tensors admitted by a particular metric. If, by means of the tractor approach, we determine that an upper bound on the number of Killing tensors of rank 2 is k, then we know that there are at most $k-q$ Killing tensors of rank 2 which are not in the span of S. Thus, this novel application will allow us to determine which metrics may admit irreducible Killing tensors of rank 2, and it will provide an upper bound on the number of independent, irreducible Killing tensors of rank 2 which can be admitted.

One more application of the tractor approach stems from homogeneous spaces. If a manifold M is a homogeneous space G/H and the metric is G-invariant, then the tractor approach allows one to determine the number of Killing tensors without the need to introduce coordinates. Thus, the tractor approach can produce meaningful information about the space of Killing tensors where one cannot find the Killing tensors explicitly.

The tractor approach has been successfully applied to Killing vectors, and some examples are included in this thesis. While the equations that define the tractor connection for Killing vectors are well known, lesser-known are the equations which define the tractor connections for Killing tensors of rank 2, Killing-Yano tensors, and conformal Killing vectors. These equations are presented in this thesis, and software is created which constructs the tractor connections explicitly.

There are also existing formulas for constructing Killing tensors of rank 2 from objects such as Killing-Yano tensors (Popa and Ovidiu, 2007) and conformal Killing vectors (M. Walker and R. Penrose, 1970), as well as for type D vacuum solutions to the Einstein field equations (Stephani et al., 2003). We have tested these formulas extensively and found that in almost every case, each Killing tensor which was produced was a linear combination of reducible Killing tensors and the associated metric. Only one Killing tensor of rank 2 was produced which was not a linear combination of reducible Killing tensors and the metric, and the metric which admitted this Killing tensor was the Kerr metric.

This thesis is organized as follows. In chapter 2, we give our conventions and offer a description of some important properties of Killing tensors, Killing vectors, Killing-Yano tensors, and conformal Killing vectors. In chapter 3, we give a more detailed description of the tractor approach and illustrate the tractor approach with regard to Killing vectors and to a particular system of partial differential equations. The fourth chapter of this thesis details our application of the tractor approach to Killing vectors, and the fifth chapter details the tractor approach applied to conformal Killing vectors. In the sixth chapter, we recover the equations used to define the tractor connection (Thompson, 1986). We then identify many metrics which cannot admit rank 2 irreducible Killing tensors, and we find new examples of metrics which admit rank 2 irreducible Killing tensors. We identify other metrics which may admit rank 2 irreducible Killing tensors, though we do not find these Killing tensors explicitly.

Also included in chapter 6 is a treatment of Killing tensors of rank 2 for metrics in the plane. We apply the Darboux-Koenings theorem and offer necessary and sufficient conditions for the existence of precisely four Killing tensors of rank 2 for a metric in the plane with a single Killing vector. We also find that the maximum number of Killing tensors of rank 2 for a plane metric with no Killing vectors is three.

In chapter 7, we apply the tractor approach to Killing-Yano tensors of rank 2, first reproducing the known (Houri et al., 2018) equations which define the tractor connection. For many exact solutions, we obtain a count of the number of Killing-

Yano tensors, and, in some cases, we obtain them explicitly in cases for which a direct approach appears to be more problematic.

Finally, we allude to future projects of interest and then give a brief overview of how our software programs are to be used. After we offer a few software demonstrations, the thesis is concluded after the inclusion of the source code for the software programs that have been developed.

2 Conventions and Basic Properties

2.1 Conventions

The conventions used in this thesis are those used in the Differential Geometry software package (Anderson and Torre, 2016). Let ∇ be a connection on an n-dimensional manifold M. With respect to a system of local coordinates x^β, the connection coefficients $\Gamma^\gamma{}_{\beta\mu}$ are given as

$$\nabla_{\partial_{x^\mu}} \partial_{x^\beta} = \Gamma^\gamma{}_{\beta\mu} \partial_{x^\gamma}.$$

Let T be a type $\binom{r}{s}$ tensor defined on M. We denote the covariant derivative of T with respect to a connection using the "semicolon" notation, so that the first covariant derivative of T is written, in terms of the components, as

$$T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s ; \gamma} = \frac{\partial}{\partial x^\gamma} \left(T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s} \right) + \sum_{t=1}^{r} \Gamma^{\alpha_t}{}_{\nu\gamma} T^{\alpha_1 \ldots \alpha_{t-1} \nu \alpha_{t+1} \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s}$$
$$- \sum_{w=1}^{s} \Gamma^\nu{}_{\beta_w \gamma} T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_{w-1} \nu \beta_{w+1} \ldots \beta_s}.$$

When more derivatives are taken, more indices appear on the right of the semicolon. For example, the second covariant derivative of T is written as $T^{\alpha_1 \ldots \alpha_r}{}_{\beta_1 \ldots \beta_s ; \gamma\delta}$.

If M is taken to be a Riemannian or pseudo-Riemannian manifold endowed with a metric g, covariant differentiation is taken with respect to the Christoffel symbols $\left\{ {}_{\beta}{}^{\gamma}{}_{\mu} \right\}$, which are the components of the unique, torsion free connection for which the metric is covariantly constant. In local coordinates, these components can be expressed in terms of the metric by

$$\left\{ {\gamma \atop \beta\ \mu} \right\} = \frac{1}{2} g^{\alpha\gamma} \left(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} \right),$$

where the "comma" notation denotes partial differentiation. Let X be a contravariant vector field defined on M. The curvature tensor $R^{\alpha}{}_{\beta\gamma\delta}$ of the connection defined by the Christoffel symbols is defined by the commutator of the covariant derivatives. For a contravariant vector field, we have

$$X^{\alpha}{}_{;\beta\gamma} - X^{\alpha}{}_{;\gamma\beta} = R^{\alpha}{}_{\nu\gamma\beta} X^{\nu},$$

while the commutator of the covariant derivative of a covariant vector field Y is given as

$$Y_{\alpha;\beta\gamma} - Y_{\alpha;\gamma\beta} = R^{\nu}{}_{\alpha\beta\gamma} Y_{\nu}\,.$$

The two preceding equations give rise to the following Bianchi identities:

$$R^{\alpha}{}_{\beta\gamma\delta} + R^{\alpha}{}_{\delta\beta\gamma} + R^{\alpha}{}_{\gamma\delta\beta} = 0,$$

$$R^{\alpha}{}_{\beta\mu\nu;\lambda} + R^{\alpha}{}_{\beta\lambda\mu;\nu} + R^{\alpha}{}_{\beta\nu\lambda;\mu} = 0.$$

Applying the commutator formula to the metric, we find that

$$g_{\alpha\beta;\gamma\delta} - g_{\alpha\beta;\gamma\delta} = g_{\mu\beta} R^{\mu}{}_{\alpha\gamma\delta} + g_{\alpha\mu} R^{\mu}{}_{\beta\gamma\delta},$$

giving us, since $g_{\alpha\beta;\gamma} = 0$,

$$R_{\beta\alpha\gamma\delta} + R_{\alpha\beta\gamma\delta} = 0.$$

We use $\delta^{\alpha\beta}_{\mu\nu}$ to denote the generalized Kronecker-Delta symbol, which can be written in terms of the ordinary Kronecker-Delta symbols as follows:

$$\delta^{\alpha\beta}_{\mu\nu} = \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}.$$

Parenthesis will be used to denote the symmetrization of the indices of a tensor, which for a type $\binom{0}{2}$ tensor T is given as

$$T_{(\alpha\beta)} = \frac{1}{2}\left(T_{\alpha\beta} + T_{\beta\alpha}\right).$$

Square brackets will be used to denote skew-symmetrization, so that

$$T_{[\alpha\beta]} = \frac{1}{2}\left(T_{\alpha\beta} - T_{\beta\alpha}\right).$$

In the case of a type $\binom{0}{3}$ tensor T, the symmetrization formulas are given as

$$T_{(\alpha\beta\gamma)} = \frac{1}{3!}\left(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha} + T_{\beta\alpha\gamma} + T_{\gamma\beta\alpha} + T_{\alpha\gamma\beta}\right)$$

and

$$T_{[\alpha\beta\gamma]} = \frac{1}{3!}\left(T_{\alpha\beta\gamma} + T_{\gamma\alpha\beta} + T_{\beta\gamma\alpha} - T_{\beta\alpha\gamma} - T_{\gamma\beta\alpha} - T_{\alpha\gamma\beta}\right).$$

It is sometimes convenient to use an alternate means of indicating the symmetrization of tensors. In such cases, the operator Y_t is used, where t is a Young tableau from which the symmetrization is determined: one first applies (symmetric) symmetrization according to the rows of the tableau, then subsequently skew-symmetrization according to the columns of the tableau. For example, consider the following symmetrization operator applied to a tensor F of type $\binom{0}{4}$:

$$Y_{\begin{array}{|c|c|}\hline \beta & \alpha \\ \hline \gamma & \delta \\ \hline\end{array}} F_{\alpha\beta\gamma\delta}.$$

To compute this explicitly, one first computes the symmetrization over the pairs (β, α)

and (γ, δ). The result is then skew-symmetrized over the pairs (β, γ) and (α, δ):

$$\begin{aligned}
G_{\alpha\beta\gamma\delta} &= Y_{\begin{smallmatrix}\beta & \alpha \\ \gamma & \delta\end{smallmatrix}} F_{\alpha\beta\gamma\delta} \\
&= \frac{1}{4} Y_{\begin{smallmatrix}\beta \\ \gamma\end{smallmatrix}} Y_{\begin{smallmatrix}\alpha \\ \delta\end{smallmatrix}} \left(F_{\alpha\beta\gamma\delta} + F_{\beta\alpha\gamma\delta} + F_{\alpha\beta\delta\gamma} + F_{\beta\alpha\delta\gamma}\right) \\
&= \frac{1}{8} Y_{\begin{smallmatrix}\beta \\ \gamma\end{smallmatrix}} \left(F_{\alpha\beta\gamma\delta} + F_{\beta\alpha\gamma\delta} + F_{\alpha\beta\delta\gamma} + F_{\beta\alpha\delta\gamma} - F_{\delta\beta\gamma\alpha} - F_{\beta\delta\gamma\alpha} - F_{\delta\beta\alpha\gamma} - F_{\beta\delta\alpha\gamma}\right) \\
&= \frac{1}{16} \left(F_{\alpha\beta\gamma\delta} + F_{\beta\alpha\gamma\delta} + F_{\alpha\beta\delta\gamma} + F_{\beta\alpha\delta\gamma} - F_{\delta\beta\gamma\alpha} - F_{\beta\delta\gamma\alpha} - F_{\delta\beta\alpha\gamma} - F_{\beta\delta\alpha\gamma}\right) \\
&\quad - \frac{1}{16} \left(F_{\alpha\gamma\beta\delta} + F_{\gamma\alpha\beta\delta} + F_{\alpha\gamma\delta\beta} + F_{\gamma\alpha\delta\beta} - F_{\delta\gamma\beta\alpha} - F_{\gamma\delta\beta\alpha} - F_{\delta\gamma\alpha\beta} - F_{\gamma\delta\alpha\beta}\right).
\end{aligned}$$

The tensor G satisfies a cyclic identity on any three indices:

$$G_{\alpha\beta\gamma\delta} + G_{\alpha\delta\beta\gamma} + G_{\alpha\gamma\delta\beta} = 0.$$

We also note that

$$\begin{aligned}
Y_{\begin{smallmatrix}\beta \\ \alpha \\ \gamma \\ \delta\end{smallmatrix}} F_{\alpha\beta\gamma\delta} &= F_{[\alpha\beta\gamma\delta]} \\
&= \frac{1}{12} \left(F_{\alpha\beta\gamma\delta} + F_{\delta\alpha\beta\gamma} + F_{\gamma\delta\alpha\beta} + F_{\beta\gamma\delta\alpha}\right) \\
&\quad - \frac{1}{12} \left(F_{\beta\alpha\gamma\delta} + F_{\delta\beta\alpha\gamma} + F_{\gamma\delta\beta\alpha} + F_{\alpha\gamma\delta\beta}\right) \\
&\quad - \frac{1}{12} \left(F_{\gamma\beta\alpha\delta} + F_{\delta\gamma\beta\alpha} + F_{\alpha\delta\gamma\beta} + F_{\beta\alpha\delta\gamma}\right),
\end{aligned}$$

so that if $F_{\alpha\beta\gamma\delta} = F_{[\alpha\beta\gamma]\delta}$,

$$F_{[\alpha\beta\gamma\delta]} = \frac{1}{4} \left(F_{\alpha\beta\gamma\delta} + F_{\delta\alpha\beta\gamma} + F_{\gamma\delta\alpha\beta} + F_{\beta\gamma\delta\alpha}\right). \tag{2.1}$$

Equation (2.1) will be referenced in chapter 7. Our final demonstration using the

Young symmetrization operators is the following, applied to a type $\binom{0}{3}$ tensor $L_{\alpha\beta\gamma}$:

$$
\begin{aligned}
Y_{\begin{array}{|c|c|}\hline \alpha & \gamma \\ \hline \beta \\ \cline{1-1}\end{array}} L_{\alpha\beta\gamma} &= \frac{1}{2} Y_{\begin{array}{|c|}\hline \alpha \\ \hline \beta \\ \hline\end{array}} \left(L_{\alpha\beta\gamma} + L_{\gamma\beta\alpha} \right) \\
&= \frac{1}{4} \left(L_{\alpha\beta\gamma} - L_{\beta\alpha\gamma} + L_{\gamma\beta\alpha} - L_{\gamma\alpha\beta} \right) .
\end{aligned}
$$

These Young symmetrization operators map tensors into the various irreducible representations of the general linear group.

Let T be a tensor of type $\binom{0}{r}$. The tensor T is said to be a symmetric tensor of type $\binom{0}{r}$ if

$$T_{\alpha_1 \ldots \alpha_r} = T_{(\alpha_1 \ldots \alpha_r)} .$$

The tensor T is said to be a skew-symmetric tensor of type $\binom{0}{r}$ if

$$T_{\alpha_1 \ldots \alpha_r} = T_{[\alpha_1 \ldots \alpha_r]} .$$

The dimension of the space of rank r symmetric tensors on a manifold of dimension n is $\dfrac{(n+r-1)!}{r!(n-1)!}$, and the dimension of the space of rank r skew-symmetric tensors on a manifold of dimension n is $\dfrac{n!}{r!(n-r)!}$.

Now let S be a tensor of type $\binom{r}{0}$: S is said to be symmetric if

$$S^{\alpha_1 \ldots \alpha_r} = S^{(\alpha_1 \ldots \alpha_r)}.$$

Similarly, S is said to be skew-symmetric if

$$S^{\alpha_1 \ldots \alpha_r} = S^{[\alpha_1 \ldots \alpha_r]}.$$

2.2 The Theorem of Frobenius

In this section, we will present a theorem which is attributed to Frobenius, which theorem can be found on page 254 of (Spivak, 1979). This theorem will eventually lead to our novel application of the tractor approach.

Let $V \subset \mathbb{R}^n$ be open, and let U be an open neighborhood of $0 \in \mathbb{R}^m$. For $i = 1, \ldots, m$, let $f_i : U \times V \to \mathbb{R}^n$ be $\mathcal{C}^\infty$ functions. Let $\alpha : W \to V$, where W is a neighborhood of $0 \in \mathbb{R}^m$. We consider a system of equations, defined for all $t \in W$, of the form

$$\frac{\partial \alpha(t)}{\partial t^j} = f_j(t, \alpha(t)), \tag{2.2}$$

with initial conditions $\alpha(0) = x$. We will refer to such a system of equations as a Frobenius system of equations.

Theorem 2.1. *For every $x \in V$, there is at most one function $\alpha : W \to V$ satisfying equation (2.2): that is, any two functions α_1 and α_2 defined on neighborhoods W_1 and W_2, respectively, agree on the component of $W_1 \cap W_2$ containing 0. Moreover, such a function exists and is automatically $\mathcal{C}^\infty$ if and only if there is a neighborhood of $(0, x) \in U \times V$ on which the following equation is satisfied, for $i, j = 1, \ldots, m$:*

$$\frac{\partial f_j}{\partial t^i} - \frac{\partial f_i}{\partial t^j} + \sum_{k=1}^{n} \frac{\partial f_j}{\partial x^k} f_i^k - \sum_{k=1}^{n} \frac{\partial f_i}{\partial x^k} f_j^k = 0.$$

The proof of theorem 2.1 is documented in the literature (Spivak, 1979) and has been omitted in this thesis. The theorem does, however, allow us to make the following observation.

Corollary 2.1.1. *Let each $f_j(t, \alpha)$ as in equation (2.2) be linear in α, and suppose that $\alpha(0) = 0$. Then the unique solution satisfying (2.2) is $\alpha(t) = 0$.*

Proof. Because each f_j is linear in α, $\alpha(t) = 0$ is a solution of equation (2.2) with initial conditions $\alpha(0) = 0$. By theorem (2.1), the solution $\alpha(t) = 0$ is unique. $\square$

2.3 Properties of Killing tensors and their Generalizations

In this section, we will present important properties of Killing tensors, Killing-Yano tensors, and conformal Killing vectors.

2.3.1 Properties of Killing tensors

Let K be a symmetric tensor of type $\binom{0}{r}$. K is said to be a rank r Killing tensor of a connection ∇ if (Stephani et al., 2003)

$$K_{(\alpha_1 \ldots \alpha_r;\alpha_{r+1})} = 0. \tag{2.3}$$

This is known as the Killing equation. A special case of this is when $r = 1$, giving us

$$K_{\alpha;\beta} + K_{\beta;\alpha} = 0. \tag{2.4}$$

A rank 1 tensor satisfying equation (2.4) is a Killing tensor of rank 1, and if the index of K is raised, it is called a Killing vector. In general, if the indices α_i are raised in equation (2.3), K is called a contravariant Killing tensor of rank r. One reason Killing tensors are of interest is due to the fact that they can be associated with first integrals of the geodesic equation as follows (Stephani et al., 2003). Let $x^\alpha(s)$ be the coordinates of an affinely parameterized geodesic. Then

$$\frac{D}{Ds}\left(\frac{dx^\beta}{ds}\right) = \frac{d^2x^\beta}{ds^2} + \left\{ {}^{\;\beta}_{\alpha\;\gamma} \right\} \frac{dx^\alpha}{ds}\frac{dx^\gamma}{ds} = 0. \tag{2.5}$$

Equation (2.5) is the geodesic equation. We recall that the absolute differential of a type $\binom{0}{2}$ tensor field K is given as

$$DK_{\alpha\beta} = K_{\alpha\beta;\gamma}dx^\gamma,$$

so that if K is a Killing tensor of rank 2, we get

$$\frac{D}{Ds}\left(K_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\right) = K_{\alpha\beta;\gamma}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0.$$

Thus, $K_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$ is a quadratic first integral of the geodesic equation. Similarly, Killing tensors of higher rank define cubic and higher first integrals.

Let T be a covariant tensor of rank p, and let S be a covariant tensor of rank q. The symmetric tensor product of T and S, denoted $T \odot S$, is the completely symmetric part of the tensor product of T and S:

$$T \odot S = T_{(\alpha_1 \dots \alpha_p} S_{\beta_1 \dots \beta_q)}.$$

Similarly, the symmetric tensor product is defined for two contravariant tensors. Now suppose that T and S are symmetric tensors: it is evident that the symmetric tensor product of two symmetric tensors is a symmetric tensor. Thus, the set of symmetric tensors on M, together with the operations of scalar multiplication, addition of tensors, and the symmetric tensor product, forms an infinite dimensional, commutative algebra (Thompson, 1986).

The notion of the Lie bracket of vector fields generalizes to the Schouten bracket on the algebra of symmetric (contravariant) tensor fields (Thompson, 1986). Let T be a contravariant symmetric tensor of rank p, and let S be a contravariant symmetric tensor of rank q. The Schouten bracket is defined as

$$[T, S]^{\beta_1 \dots \beta_{p+q-1}} = pT^{\alpha(\beta_1 \dots \beta_{p-1}} S^{\beta_p \dots \beta_{p+q-1})}{}_{,\alpha} - qS^{\alpha(\beta_1 \dots \beta_{q-1}} T^{\beta_q \dots \beta_{p+q-1})}{}_{,\alpha}. \tag{2.6}$$

The tensorial nature of equation (2.6) is due to the fact that, on a pseudo-Riemannian manifold, one can replace the partial derivatives with covariant derivatives with respect to the Christoffel symbols, since the connection is torsion free. Additionally, if the tensors T and S are defined on a Riemannian or a pseudo-Riemannian manifold, the Schouten bracket can also be defined for covariant symmetric tensors by the lowering of indices.

The interaction of the structure of the symmetric tensor product and the structure of the Schouten bracket is captured in the following formula, for symmetric tensors T, S, and Y (Thompson, 1986):

$$[T \odot S, Y] = [T, Y] \odot S + T \odot [S, Y]. \tag{2.7}$$

Additional properties of the Schouten bracket include skew-symmetry,

$$[T, S] = -[S, T],$$

and the Jacobi identity (Woodhouse, 1975)

$$[T, [S, Y]] + [Y, [T, S]] + [S, [Y, T]] = 0.$$

Thus, it is clear that the Schouten bracket endows the space of symmetric tensors with the structure of a real, infinite dimensional Lie algebra (Thompson, 1986).

Killing tensors are symmetric tensors themselves, and so it is natural to consider the operations of the symmetric tensor product and of the Schouten bracket on the space of Killing tensors of a particular metric. Let T and S be Killing tensors of a metric g, and let $W = T \odot S$.

$$W_{(\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_p ; \gamma)} = \frac{1}{p+q} \left(q T_{\alpha_1 \ldots \alpha_p} S_{(\beta_1 \ldots \beta_q ; \gamma)} + p T_{(\alpha_1 \ldots \alpha_p ; \gamma)} S_{\beta_1 \ldots \beta_q} \right) = 0,$$

and so the symmetric tensor product of two Killing tensors is a Killing tensor. Thus, the symmetric tensor product defined on the space of symmetric tensors restricts to the space of Killing tensors of a particular metric, making the set of Killing tensors of a given metric an infinite dimensional, commutative algebra.

A Killing tensor of g may be defined as a tensor which commutes (with respect to the Schouten bracket) with g (Thompson, 1986; Woodhouse, 1975). Let T and S be two Killing tensors of g. The notion that the Schouten bracket endows the space of symmetric tensors with the structure of a Lie algebra, along with the notion that Killing tensors of g commute with g, give rise to the following:

$$[g, [T, S]] = -[S, [g, T]] - [T, [S, g]] = 0.$$

Thus, the Schouten bracket of two Killing tensors of g is also a Killing tensor of g, and so the space of Killing tensors of g forms an infinite dimensional Lie subalgebra of the Lie algebra of symmetric tensors (Woodhouse, 1975).

A Killing tensor is called a reducible Killing tensor if it can be written as a linear combination of the symmetric tensor products of Killing tensors of lower rank. An irreducible Killing tensor is a Killing tensor which is not reducible. For the special case of rank 2 Killing tensors, we say that *a Killing tensor is metric reducible if it can be written as a linear combination of reducible Killing tensors and the metric itself.* We introduce the term *metric reducible* due to the fact that the metric itself may be expressible as a linear combination of the reducible Killing tensors: a Killing tensor is not particularly interesting if it is metric reducible. A Killing tensor which is not metric reducible will be called *metric irreducible*.

By definition, the set of irreducible Killing tensors of a particular metric does not form an algebra with respect to the symmetric tensor product. Additionally, the Schouten bracket of two irreducible Killing tensors is, in general, not an irreducible Killing tensor. However, the set of Killing tensors is algebraically generated by the irreducible Killing tensors, which set is presumably finite.

2.3.2 Properties of Killing-Yano tensors

Equation (2.3) may be seen as one possible generalization of equation (2.4). There is another generalization, applicable to skew-symmetric tensors. Let F be a skew-symmetric tensor of rank p: F is said to be a rank p Killing-Yano tensor with respect to a connection ∇ if (Houri et al., 2018)

$$F_{\alpha_1 \ldots (\alpha_p;\beta)} = 0. \tag{2.8}$$

As with Killing tensors, the collection of Killing-Yano tensors of rank p has the structure of a real vector space. It is natural to consider what additional algebraic structure can be put on the set of Killing-Yano tensors, since the Schouten-Nijenhuis bracket for skew-symmetric tensors is given, for skew-symmetric, contravariant tensors A and B of ranks p and q, respectively, as (Kastor et al., 2007)

$$[A, B]^{\alpha_1 \ldots \alpha_{p+q-1}} = pA^{\beta[\alpha_1 \ldots \alpha_{p-1}} B^{\alpha_p \ldots \alpha_{p+q-1}]}{}_{,\beta} + q\,(-1)^{pq}\, B^{\beta[\alpha_1 \ldots \alpha_{q-1}} A^{\alpha_q \ldots \alpha_{p+q-1}]}{}_{,\beta}.$$

However, Killing-Yano tensors do not, in general, form a Lie algebra with respect to the Schouten-Nijenhuis bracket (Kastor et al., 2007). In any case, Killing-Yano tensors are of interest (Popa and Ovidiu, 2007) due to the fact that if $x^\alpha(s)$ are coordinates of an affinely parameterized geodesic, the $p-1$ form field $F_{\alpha_1 \ldots \alpha_{p-1}\beta} \dfrac{dx^\beta}{ds}$ is parallel transported along affine geodesics (Stephani et al., 2003):

$$\nabla_{\dot{x}^\gamma} \left(F_{\alpha_1 \ldots \alpha_{p-1}\beta} \frac{dx^\beta}{ds} \right) = F_{\alpha_1 \ldots \alpha_{p-1}\beta;\gamma} \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds} = 0.$$

Killing-Yano tensors have also been used in the separation of the Dirac equation (Carter and McLenaghan, 1979; Fels and Kamran, 1990). Additionally, when the connection is defined in terms of Christoffel symbols of a metric, the product of two

Killing-Yano tensors is of interest in the search for Killing tensors. This is due to the fact that the following product of Killing-Yano tensors F and $\tilde{F}$ defines a Killing tensor of rank 2 which is not necessarily reducible (Popa and Ovidiu, 2007):

$$A_{\gamma\beta} = \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{(\gamma} F_{\beta)\alpha_1\ldots\alpha_{p-1}} . \tag{2.9}$$

It can be checked that the symmetric tensor A is a Killing tensor of rank 2:

$$\begin{aligned}
A_{(\gamma\beta;\mu)} &= \frac{1}{3}\left(A_{\gamma\beta;\mu} + A_{\mu\gamma;\beta} + A_{\beta\mu;\gamma}\right) \\
&= \frac{1}{6}\left(\tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\gamma} F_{\beta\alpha_1\ldots\alpha_{p-1};\mu} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\mu} F_{\gamma\alpha_1\ldots\alpha_{p-1};\beta} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\beta} F_{\mu\alpha_1\ldots\alpha_{p-1};\gamma}\right) \\
&+\frac{1}{6}\left(\tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\beta} F_{\gamma\alpha_1\ldots\alpha_{p-1};\mu} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\gamma} F_{\mu\alpha_1\ldots\alpha_{p-1};\beta} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\mu} F_{\beta\alpha_1\ldots\alpha_{p-1};\gamma}\right) \\
&+\frac{1}{6}\left(\tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\gamma;\mu} F_{\beta\alpha_1\ldots\alpha_{p-1}} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\mu;\beta} F_{\gamma\alpha_1\ldots\alpha_{p-1}} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\beta;\gamma} F_{\mu\alpha_1\ldots\alpha_{p-1}}\right) \\
&+\frac{1}{6}\left(\tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\beta;\mu} F_{\gamma\alpha_1\ldots\alpha_{p-1}} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\gamma;\beta} F_{\mu\alpha_1\ldots\alpha_{p-1}} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\mu;\gamma} F_{\beta\alpha_1\ldots\alpha_{p-1}}\right) \\
&= \frac{(-1)^{p-1}}{3}\left(\tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\gamma} F_{\alpha_1\ldots\alpha_{p-1}(\beta;\mu)} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\mu} F_{\alpha_1\ldots\alpha_{p-1}(\gamma;\beta)}\right) \\
&+\frac{(-1)^{p-1}}{3}\tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{\beta} F_{\alpha_1\ldots\alpha_{p-1}(\mu;\gamma)} \\
&+\frac{1}{3}\left(\tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{(\gamma;\mu)} F_{\beta\alpha_1\ldots\alpha_{p-1}} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{(\mu;\beta)} F_{\gamma\alpha_1\ldots\alpha_{p-1}} + \tilde{F}^{\alpha_1\ldots\alpha_{p-1}}{}_{(\beta;\gamma)} F_{\mu\alpha_1\ldots\alpha_{p-1}}\right), \\
&= 0.
\end{aligned}$$

2.3.3 Properties of conformal Killing vectors

There is another generalization of Killing vectors, namely conformal Killing vectors, for which the defining equation (Ashtekar and Magnon-Ashtekar, 1978) is, for $\lambda = \lambda(x)$,

$$X_{(\alpha;\beta)} = \lambda g_{\alpha\beta}. \tag{2.10}$$

By taking the trace of equation (2.10), it can be shown that $\lambda = \frac{1}{n} X^{\gamma}{}_{;\gamma}$. While conformal Killing vectors are of interest in their own right (U. Semmelmann, 2002), it has been shown that they can be used to construct Killing tensors of rank 2 (Stephani et al., 2003; Edgar et al., 2004) as follows. If X is a non-null conformal Killing vector field (i.e. equation (2.10) is satisfied and $g_{\alpha\beta} X^{\alpha} X^{\beta} \neq 0$) which is not also a Killing vector field, and if

$$X_{\alpha;\beta} X^{\beta} = \zeta X_{\alpha} \tag{2.11}$$

for some smooth function $\zeta = \zeta(x)$–that is, if X is geodesic–then the following defines a Killing tensor of rank 2 (Edgar et al., 2004):

$$K_{\alpha\beta} = X_{\alpha} X_{\beta} - X^{\gamma} X_{\gamma} g_{\alpha\beta}. \tag{2.12}$$

To show this, we first note that by contracting equation (2.10) with $X^{\alpha} X^{\beta}$ and applying equation (2.11), we get

$$\zeta X_{\alpha} X^{\alpha} = \lambda X_{\alpha} X^{\alpha},$$

so that, since X is not null, $\zeta = \lambda$. Next, we note that

$$X_{\alpha;\beta} = X_{[\alpha;\beta]} + X_{(\alpha;\beta)},$$

so that, using equation (2.10),

$$X_{\alpha;\beta} X^{\beta} = X_{[\alpha;\beta]} X^{\beta} + \lambda g_{\alpha\beta} X^{\beta}.$$

Applying equation (2.11), and the fact that $\zeta = \lambda$, we get

$$\lambda X_{\alpha} = X_{[\alpha;\beta]} X^{\beta} + \lambda X_{\alpha},$$

which implies that

$$X_{[\alpha;\beta]}X^{\beta} = 0.$$

Thus,

$$X_{\alpha;\beta}X^{\beta} = X_{\beta;\alpha}X^{\beta}. \tag{2.13}$$

Now, we apply the Killing equation to (2.12):

$$\begin{aligned}
K_{(\alpha\beta;\mu)} &= \frac{1}{3}\left(K_{\alpha\beta;\mu} + K_{\mu\alpha;\beta} + K_{\beta\mu;\alpha}\right) \\
&= \frac{1}{3}\left(X_{\alpha;\mu}X_{\beta} + X_{\alpha}X_{\beta;\mu} - X^{\gamma}{}_{;\mu}X_{\gamma}g_{\alpha\beta} - X^{\gamma}X_{\gamma;\mu}g_{\alpha\beta}\right) \\
&+\frac{1}{3}\left(X_{\mu;\beta}X_{\alpha} + X_{\mu}X_{\alpha;\beta} - X^{\gamma}{}_{;\beta}X_{\gamma}g_{\mu\alpha} - X^{\gamma}X_{\gamma;\beta}g_{\mu\alpha}\right) \\
&+\frac{1}{3}\left(X_{\beta;\alpha}X_{\mu} + X_{\beta}X_{\mu;\alpha} - X^{\gamma}{}_{;\alpha}X_{\gamma}g_{\beta\mu} - X^{\gamma}X_{\gamma;\alpha}g_{\beta\mu}\right) \\
&= \frac{2}{3}\left(X_{\beta}X_{(\alpha;\mu)} + X_{\alpha}X_{(\beta;\mu)} + X_{\mu}X_{(\alpha;\beta)}\right) \\
&-\frac{2}{3}\left(X^{\gamma}X_{\gamma;\mu}g_{\alpha\beta} + X^{\gamma}X_{\gamma;\beta}g_{\mu\alpha} + X^{\gamma}X_{\gamma;\alpha}g_{\beta\mu}\right).
\end{aligned}$$

Applying equations (2.10), (2.11), and (2.13), we get

$$K_{(\alpha\beta;\mu)} = \frac{\lambda - \zeta}{3}\left(X_{\beta}g_{\alpha\mu} + X_{\alpha}g_{\beta\mu} + X_{\mu}g_{\alpha\beta}\right) = 0, \tag{2.14}$$

since $\lambda = \zeta$. In chapter 5 of this Thesis, we use equation (2.12) in an effort to construct Killing tensors from conformal Killing vectors.

Conformal Killing vectors also generalize to conformal Killing tensors (Edgar et al., 2004). A conformal Killing tensor Q of rank r is a symmetric tensor which satisfies, for some rank $r-1$ tensor λ,

$$Q_{(\alpha_1 \ldots \alpha_r;\beta)} = \lambda_{(\alpha_1 \ldots \alpha_{(r-1)}} g_{\alpha_r \beta)}.$$

Conformal Killing tensors, along with other generalizations of Killing vectors, such as conformal Killing forms (U. Semmelmann, 2002) and Killing spinors (M. Walker and R. Penrose, 1970), while of interest in their own right, are not treated in this thesis.

2.4 Literature Review

The purpose of this section is to summarize the list of references used. We will begin with our general references, the first of which is the Differential Geometry Software Package (Anderson and Torre, 2016), after which we pattern our own conventions. We find useful properties of Killing tensors in chapter 35 of the second edition of *Exact Solutions to Einstein's Field Equations* (Stephani et al., 2003), including the algorithm for constructing Killing tensors for Petrov type D vacuum solutions, which formula is proven in (M. Walker and R. Penrose, 1970). The theorem of Frobenius is found in (Spivak, 1979). We apply the tractor approach to several exact solutions contained in (Stephani et al., 2003) as well as to some exact solutions contained in chapter 5 of *The Large Scale Structure of Space-Time* (Hawkings and Ellis, 1973). Other references deal with additional useful properties of Killing tensors, including their algebraic structure and the notion that the Schouten bracket endows the space of contravariant Killing tensors with the structure of an infinite dimensional Lie algebra (Thompson, 1986; Woodhouse, 1975). As it is known that one can associate Killing tensors with first integrals of the geodesic equation (Stephani et al., 2003), another set of references deals with the use of Killing tensors in the separation of variables for the Hamilton-Jacobi equations (Kalnins and Miller, 1981; Woodhouse, 1975) as well as for the Dirac equation (Carignano et al., 2011; Fels and Kamran, 1990).

The next set of references deals with Killing tensors in spaces of constant curvature. The first of these contains the theorem which specifies the maximum number of Killing tensors of rank n on a manifold of dimension m (Thompson, 1986), which maximum is achieved on spaces of constant curvature. In the case of rank 2 Killing tensors, this has been independently verified by means of directly examining the structure equations (Hauser and Malhiot, 1975b).

The next set of references deal with what we are referring to as the tractor approach. For one reference, we find general information about parallel sections on vector bundles (Atkins, 2011). The equations which define the tractor connection for Killing vectors, as well as the equations which define the tractor connection for conformal Killing vectors, are given (Ashtekar and Magnon-Ashtekar, 1978) and are called the "Killing data:" we pattern our own construction according to these equations. This reference also gives the maximum number of conformal Killing vectors, which number coincides with the more general maximum number of conformal Killing p-forms (U. Semmelmann, 2002). Similar equations are given in the case of Killing tensors of ranks 1 and 2 (Hauser and Malhiot, 1975a), from which we pattern our own tractor approach. Another reference (Houri et al., 2018) gives an alternate way of constructing the tractor connection for Killing tensors using Young decomposition, though from this reference we use only the equations which define the tractor connection for Killing-Yano tensors. This reference (Houri et al., 2018) also gives a procedure for constructing the tractor connection for higher rank Killing tensors, though the equations are only explicitly given through rank 3. Other attempts have also been made to construct the tractor connection for Killing tensors of higher rank (Wolf, 1998).

The interest in Killing tensors of higher rank is evident not just in the attempts made to construct the tractor connections for them. For one particular metric in dimension 4, all Killing tensors have been found through rank 6 (Kruglikov and Matveev, 2012), though no irreducible Killing tensors were identified. In fact, it appears that irreducible Killing tensors are generally rare (Kruglikov and Matveev, 2016). Notwithstanding, irreducible Killing tensors have been explicitly identified for several *pp*-wave spacetimes (Keane and Tupper, 2010).

Our next set of references deal primarily with Killing-Yano tensors. Killing-Yano tensors are found to be parallel propagated along affine geodesics (Popa and Ovidiu, 2007). In contrast to Killing tensors, Killing-Yano tensors do not appear to form a Lie algebra (Kastor et al., 2007), though a particularly interesting reference (Popa and Ovidiu, 2007) uses Killing-Yano tensors of rank 3 to construct irreducible Killing tensors of rank 2: in this reference, we also learn that Killing-Yano tensors of any rank can be used to construct Killing tensors of rank 2.

A few other references give us information about conformal Killing vectors. From the reference that gives the equations which define the tractor connection for Killing vectors (Ashtekar and Magnon-Ashtekar, 1978), we also find the maximum number of conformal Killing vectors which a manifold of dimension $n \geq 2$ can admit: this number coincides with the more general maximum number of conformal Killing p-forms (U. Semmelmann, 2002). One reference outlines the method by which certain conformal Killing vectors can be used to construct Killing tensors of rank 2 (Edgar et al., 2004).

Other references deal primarily with 2 or 3 dimensions. From one reference (Kruglikov, 2008), we get the statement and proof of the Darboux-Koenig theorem, as well as criterion for whether a metric is Liouville. We also find that there are many examples of plane metrics which admit irreducible Killing tensors (Darboux, 1972). Systems of finite type with regard to Killing tensors have been treated in dimension 2 (G. Thompson, 1999), and elsewhere we find the criterion for the existence of Killing vectors in dimension 3 (Kruglikov and Tomoda, 2018).

2.5 The Barbance-Delong-Takeuchi-Thompson Theorem on the Maximum number of Killing Tensors.

Let g be a metric on a Riemannian or pseudo-Riemannian manifold M of dimension m. The maximum dimension of the space of Killing tensors of g is known (Thompson, 1986): the purpose of this section is to give the formula and a summary of its derivation. The following Theorem has been given by Thompson, but has also been attributed to Barbance, Delong, and Takeuchi (Houri et al., 2018).

Theorem 2.2. *The set of Killing tensors of rank n of the metric g is a vector space with dimension less than or equal to*

$$\frac{1}{n}\binom{m+n}{n+1}\binom{m+n-1}{n},$$

where we have equality in the case of constant curvature.

We have previously explained that the set of Killing tensors of rank n is a vector space, and so we will give an outline of the proof that the maximum dimension of this space is given as in Theorem (2.2). Equation (2.3) can be written as

$$K_{(i_1 \ldots i_n; i_{n+1})} = n K_{j(i_1 \ldots i_{n-1}} \Gamma^{j}{}_{i_n i_{n+1})}. \tag{2.15}$$

Next, we consider the set of equations obtained by differentiating equation (2.15) at most n times: the result is a homogeneous system of linear equations with unknowns

$$K_{i_1 \ldots i_n; j_1}, \quad K_{i_1 \ldots i_n; j_1 j_2}, \quad \ldots, \quad K_{i_1 \ldots i_n; j_1 \ldots j_{n+1}}.$$

The number of unknowns is given as

$$\sum_{r=0}^{n+1} \binom{m+r-1}{r}\binom{m+n-1}{n}, \tag{2.16}$$

while the number of independent linear equations is given as

$$\sum_{r=1}^{n+1} \binom{m+r-2}{r-1} \binom{m+n}{n+1}. \tag{2.17}$$

Subtracting equation (2.17) from equation (2.16) gives us

$$\sum_{r=0}^{n+1} \binom{m+r-1}{r} \binom{m+n-1}{n} - \sum_{r=1}^{n+1} \binom{m+r-2}{r-1} \binom{m+n}{n+1}$$
$$= \frac{(m+n-1)!(m+n)!}{(m-1)!m!n!(n+1)!},$$

from which the conclusion follows. Thus, the dimension of the space of Killing tensors of rank n is given as in the statement of Theorem (2.2).

In general, compatibility conditions constrain the second and higher order derivatives of the tensor $K_{i_1 \ldots i_n}$, due to the fact that the second covariant derivative of the Killing tensor may be written in terms of the curvature tensor and the Killing tensor itself: thus, in general, the dimension of the space of Killing tensors is less than the given formula. However, in a space in which the curvature tensor is zero, the compatibility conditions are satisfied identically, and so the dimension of the space of Killing tensors is given by the above formula. It is also shown that the compatibility conditions are satisfied identically in spaces of constant curvature (Thompson, 1986).

Theorem 2.3. *Let (M, g) be a Riemannian or pseudo-Riemannian manifold of constant curvature. Any Killing tensor on (M, g) consists of sums of symmetrized products of Killing vectors.*

That is to say, there are no irreducible Killing tensors in spaces of constant curvature (Thompson, 1986).

3 The Tractor Approach

The purpose of this section is to illustrate the method for constructing the tractor connection from a given Frobenius system of equations and to illustrate the utility of doing so.

Let M be a manifold, and let $\mathcal{X}(M)$ denote the vector space of smooth vector fields on M. Let $\pi : E \to M$ be a vector bundle: $E_x = \pi^{-1}(x)$ is the fiber of E at the point x. A map $\sigma : M \to E$ is said to be a section of the vector bundle if $\pi \circ \sigma$ is the identity map on M. Let $\mathcal{S}(E)$ denote the vector space of smooth sections of the vector bundle. A connection on the bundle is a linear mapping $\tilde{\nabla} : \mathcal{X}(M) \times \mathcal{S}(E) \to \mathcal{S}(E)$ such that, for any smooth function f on M, any vector field $X \in \mathcal{X}(M)$, and any smooth section $\sigma \in \mathcal{S}(E)$,

$$\tilde{\nabla}_{fX}(\sigma) = f\tilde{\nabla}_X(\sigma), \qquad \tilde{\nabla}_X(f\sigma) = X(f)\sigma + f\tilde{\nabla}_X(\sigma). \tag{3.1}$$

If $\{E_i\}$ is a local basis of sections of E, and if $\{x^\alpha\}$ are local coordinates for M, then the connection coefficients $\tilde{\Gamma}^j{}_{i\alpha}$ are defined by

$$\tilde{\nabla}_{\partial_{x^\alpha}} E_i = \tilde{\Gamma}^j{}_{i\alpha} E_j. \tag{3.2}$$

Let $\sigma = S^i E_i$. Equation (3.1) implies that

$$\tilde{\nabla}_X \sigma = \tilde{\nabla}_{X^\alpha \partial_{x^\alpha}} \left(S^i E_i\right) = X^\alpha \partial_{x^\alpha}(S^i) E_i + X^\alpha S^i \tilde{\Gamma}^j{}_{i\alpha} E_j. \tag{3.3}$$

In terms of the components of σ, equation (3.3) gives us

$$S^i{}_{;\alpha} = \frac{\partial S^i}{\partial x^\alpha} + \tilde{\Gamma}^i{}_{j\alpha} S^j. \tag{3.4}$$

A section σ is said to be parallel if, for all $X \in \mathcal{X}(M)$, $\tilde{\nabla}_X \sigma = 0$. By equations

(3.3) and (3.4), the condition of parallelism results in the following condition on the coefficients of σ:

$$\frac{\partial S^i}{\partial x^\alpha} + \tilde{\Gamma}^i{}_{j\alpha} S^j = 0. \tag{3.5}$$

Let $L(\mathcal{S})$ denote the set of linear mappings from $\mathcal{S}(E)$ to itself. Given the connection $\tilde{\nabla}$, we can introduce the curvature tensor $\tilde{K} : \mathcal{X}(M) \times \mathcal{X}(M) \to L(\mathcal{S})$ by virtue of

$$\tilde{K}(X,Y)(\sigma) = \tilde{\nabla}_X \tilde{\nabla}_Y(\sigma) - \tilde{\nabla}_Y \tilde{\nabla}_X(\sigma) - \tilde{\nabla}_{[X,Y]}(\sigma), \tag{3.6}$$

where $Y \in \mathcal{X}(M)$. We can also write $\tilde{K}$ as a matrix of 2-forms:

$$\tilde{K}^i{}_j = \tilde{K}^i{}_{j\alpha\beta} dx^\alpha \wedge dx^\beta = \tilde{\Gamma}^i{}_{j\alpha,\beta} dx^\alpha \wedge dx^\beta - \tilde{\Gamma}^i{}_{k\alpha} \tilde{\Gamma}^k{}_{j\beta} dx^\alpha \wedge dx^\beta. \tag{3.7}$$

If a section σ is parallel, then it is clear from equation (3.6) that

$$\tilde{K}(X,Y)(\sigma) = 0. \tag{3.8}$$

In coordinates, this condition can be written as

$$S^i K^j{}_{i\alpha\beta} = 0.$$

Equation (3.8) implies that, for any positive integer r, for $1 \leq i \leq r$, and for $Z_i \in \mathcal{X}(M)$,

$$\tilde{\nabla}_{Z_r} \tilde{\nabla}_{Z_{r-1}} \ldots \tilde{\nabla}_{Z_1} \left(\tilde{K}(X,Y)(\sigma) \right) = 0. \tag{3.9}$$

Due to the fact that $\tilde{\nabla}\sigma = 0$, equation (3.9) represents additional *algebraic* constraints on S^i.

Definition 3.1. *A section σ is said to be quasi-parallel to order k if, for each integer $0 \leq r \leq k$,*

$$\tilde{\nabla}_{Z_r} \tilde{\nabla}_{Z_{r-1}} \dots \tilde{\nabla}_{Z_1} \left(\tilde{K}(X,Y)(\sigma) \right) = 0.$$

Definition 3.2. *Let $\pi : E \to M$ be a vector bundle, let r be a non-negative integer, and let $E_x^r \subset E_x$ denote the subspace of elements which are quasi-parallel to order r. We assume that E_x^r has constant dimension, and we call $\pi : E^r \to M$ the r^{th} order reduced vector bundle.*

These definitions motivate the following inclusion statement.

Corollary 3.0.1. *Let $\mathcal{S}^{\tilde{\nabla}}(E) \subseteq \mathcal{S}(E)$ be the vector space for which a basis is the set of parallel sections with respect to the connection $\tilde{\nabla}$. Then for every non-negative integer r, we have*

$$\mathcal{S}^{\tilde{\nabla}}(E) \subseteq \mathcal{S}(E^r) \subseteq \mathcal{S}\left(E^{r-1}\right) \subseteq \dots \subseteq \mathcal{S}\left(E^1\right) \subseteq \mathcal{S}\left(E^0\right) \subseteq \mathcal{S}(E).$$

The utility of this observation is that it allows one to obtain an increasingly tighter upper bound on the number of independent parallel sections by computing bases for sets of quasi-parallel sections of increasingly higher order. The iterative process of doing so terminates, due to the following (Atkins, 2011):

Lemma 3.1. *There is a non-negative integer k such that*

$$\mathcal{S}^{\tilde{\nabla}}(E) = \mathcal{S}\left(E^k\right).$$

The question of what non-negative integer k is required so that the dimension of $\mathcal{S}(E^k)$ is that of the dimension of $\mathcal{S}^{\tilde{\nabla}}(E)$ is addressed as follows.

Suppose that a matrix A is smoothly parameterized by $t \in (a,b)$, and suppose that $A(t)$ has constant rank r on (a,b). It can be shown that the nullspace of A, denoted

$\{N_i(t)\}$ for $i = 1, \ldots N$ (where $r + N$ is the number of columns of A), consists of vectors which depend smoothly on t in the open interval (a, b).

Now suppose that the nullspace of $A(t)$ is contained within the nullspace of $\dot{A}(t)$. Then for every N_i in the nullspace of $A(t)$,

$$(AN_i)' = \dot{A}N_i + A\dot{N}_i = 0,$$

which implies that $A\dot{N}_i = 0$, since N_i is contained within the nullspace of $\dot{A}$: this in turn implies that $\dot{A}\dot{N}_i = 0$. On the other hand, we have

$$(\dot{A}N_i)' = \ddot{A}N_i + \dot{A}\dot{N}_i = 0,$$

which implies that $\ddot{A}N_i = 0$, since $\dot{A}\dot{N}_i = 0$. In the context of our reduced tractor bundles, we have the following.

Lemma 3.2. *Suppose that there is a non-negative integer r such that*

$$\mathcal{S}\left(E^r\right) = \mathcal{S}\left(E^{r+1}\right).$$

Then $\mathcal{S}\left(E^{r+2}\right) = \mathcal{S}\left(E^r\right)$.

Proof. Let $\sigma \in \mathcal{S}\left(E^r\right)$. By assumption,

$$\tilde{\nabla}_{Z_r} \ldots \tilde{\nabla}_{Z_1}\left(\tilde{K}(X,Y)(\sigma)\right) = 0, \tag{3.10}$$

and

$$\tilde{\nabla}_{Z_{r+1}}\tilde{\nabla}_{Z_r} \ldots \tilde{\nabla}_{Z_1}\left(\tilde{K}(X,Y)(\sigma)\right) = 0. \tag{3.11}$$

Equation (3.11) implies that

$$\tilde{\nabla}_{Z_{r+2}}\left(\tilde{\nabla}_{Z_{r+1}}\tilde{\nabla}_{Z_r} \ldots \tilde{\nabla}_{Z_1}\left(\tilde{K}(X,Y)(\sigma)\right)\right) = 0, \tag{3.12}$$

so that

$$\tilde{\nabla}_{Z_{r+2}} \dots \tilde{\nabla}_{Z_1} \left(\tilde{K}(X,Y)(\sigma)\right) + \tilde{\nabla}_{Z_{r+1}} \dots \tilde{\nabla}_{Z_1} \left(\tilde{K}(X,Y)\right) \tilde{\nabla}_{Z_{r+2}} (\sigma) = 0. \tag{3.13}$$

On the other hand, equation (3.10) implies that

$$\tilde{\nabla}_{Z_{r+2}} \left(\tilde{\nabla}_{Z_r} \dots \tilde{\nabla}_{Z_1} \left(\tilde{K}(X,Y)(\sigma)\right)\right) = 0, \tag{3.14}$$

so that

$$\tilde{\nabla}_{Z_{r+2}} \tilde{\nabla}_{Z_r} \dots \tilde{\nabla}_{Z_1} \left(\tilde{K}(X,Y)(\sigma)\right) + \tilde{\nabla}_{Z_r} \dots \tilde{\nabla}_{Z_1} \left(\tilde{K}(X,Y)\right) \tilde{\nabla}_{Z_{r+2}} (\sigma) = 0. \tag{3.15}$$

Since $\sigma \in \mathcal{S}\left(E^r\right) = \mathcal{S}\left(E^{r+1}\right)$, equation (3.15) gives us

$$\tilde{\nabla}_{Z_r} \dots \tilde{\nabla}_{Z_1} \left(\tilde{K}(X,Y)\right) \tilde{\nabla}_{Z_{r+2}} (\sigma) = 0. \tag{3.16}$$

Equation (3.16) implies that $\tilde{\nabla}_{Z_{r+2}}(\sigma) \in \mathcal{S}\left(E^r\right)$, so that, by assumption, $\tilde{\nabla}_{Z_{r+2}}(\sigma) \in \mathcal{S}\left(E^{r+1}\right)$. Coupled with equation (3.13), this implies that

$$\tilde{\nabla}_{Z_{r+2}} \dots \tilde{\nabla}_{Z_1} \left(\tilde{K}(X,Y)(\sigma)\right) = 0, \tag{3.17}$$

so that $\sigma \in \mathcal{S}\left(E^{r+2}\right)$. This implies that $\mathcal{S}\left(E^r\right) \subseteq \mathcal{S}\left(E^{r+2}\right)$. The conclusion then follows from corollary (3.0.1). □

Theorem 3.1. *If, for some non-negative r, $E^r = E^{r+1}$, then*

$$\mathcal{S}^{\tilde{\nabla}}(E) = \mathcal{S}(E^r).$$

Proof. Since there is a non-negative integer k such that $\mathcal{S}^{\tilde{\nabla}}(E) = \mathcal{S}\left(E^k\right)$ by lemma

(3.1), it follows from lemma (3.2) that if $k \geq r$, then $\mathcal{S}^{\tilde{\nabla}}(E) = \mathcal{S}\left(E^k\right) = \mathcal{S}\left(E^r\right)$. If $k < r$, the conclusion follows from corollary (3.0.1). $\square$

Thus, when for some non-negative integer r we have $E^r = E^{r+1}$, the number of independent parallel sections of $\tilde{\nabla}$ is the dimension of $\mathcal{S}(E^r)$.

The ultimate purpose of the tractor approach applied to Killing tensors is to establish a one-to-one correspondence between the solutions of the Killing equations and the parallel sections of a vector bundle E with respect to a connection $\tilde{\nabla}$ on E. This is done by creating a vector bundle where the components of the smooth sections of the vector bundle represent the unspecified functions in the Killing equation.

We will now illustrate the tractor approach to Killing tensors in the simpliest of cases, which is the case of rank 1 Killing tensors on a pseudo-Riemannian manifold of dimension 2. Let M be such a manifold with local coordinates (u, v), let g be the metric on M, let ∇ be the connection defined by the Christoffel symbols, and let R be the curvature tensor of ∇. The tensor X is a rank 1 Killing tensor of the metric g if and only if

$$X_{\alpha;\beta} + X_{\beta;\alpha} = 0. \tag{3.18}$$

However, equation (3.18) is not a Frobenius system of equations. This is due to the fact that there are only three independent equations and yet four first order derivatives. Thus, we will need to derive a Frobenius system from equation (3.18).

We define the tensor $\omega_{\alpha\beta}$ as follows:

$$\omega_{\alpha\beta} = X_{[\alpha;\beta]}. \tag{3.19}$$

Thus,

$$\omega_{\alpha\beta;\gamma} = X_{[\alpha;\beta]\gamma}. \tag{3.20}$$

We note that, using the commutator of covariant derivatives,

$$\left(X_{\alpha;\beta\gamma} - X_{\alpha;\gamma\beta}\right) + \left(X_{\beta;\alpha\gamma} - X_{\beta;\gamma\alpha}\right) + \left(X_{\gamma;\alpha\beta} - X_{\gamma;\beta\alpha}\right) \tag{3.21}$$

$$= \left({R^{\nu}}_{\alpha\beta\gamma} + {R^{\nu}}_{\beta\alpha\gamma} + {R^{\nu}}_{\gamma\alpha\beta}\right) X_{\nu}\,.$$

If X is a Killing vector, we can apply equation (3.18) for the left hand of equation (3.21), and, using the Bianchi identity on the right hand side, we get

$$2X_{[\gamma;\alpha]\beta} = 2X_{\gamma;\alpha\beta} = 2{R^{\nu}}_{\beta\alpha\gamma} X_{\nu}\,. \tag{3.22}$$

Thus, the Frobenius system for (covariant) Killing vectors is given as

$$\begin{cases} X_{\alpha;\beta} = \omega_{\alpha\beta} & \text{(3.23a)} \\ w_{\alpha\beta;\gamma} = {R^{\nu}}_{\gamma\beta\alpha} X_{\nu}\,. & \text{(3.23b)} \end{cases}$$

This system is a Frobenius system of partial differential equations due to the fact that the derivatives of X and ω are fully specified: compare with equation (2.2).

We note that in two dimensions, $\omega_{\alpha\beta}$ has only one independent component, namely ω_{21}. We also note that the curvature tensor has only one independent component, which we denote as ${R^{1}}_{212} = \kappa$. Using equation (3.23a), we find that

$$X_{2;1} = X_{2,1} - \left\{ {1 \atop 2\,1} \right\} X_1 - \left\{ {2 \atop 2\,1} \right\} X_2 = \omega_{21}\,. \tag{3.24}$$

Similarly, we find that

$$X_{1;2} = X_{1,2} - \left\{ {1 \atop 1\,2} \right\} X_1 - \left\{ {2 \atop 1\,2} \right\} X_2 = \omega_{12}\,. \tag{3.25}$$

Subtracting equation (3.25) from equation (3.24) gives us

$$\omega_{21} - \omega_{12} = X_{2,1} - X_{1,2} - \left\{ {1 \atop 2\,1} \right\} X_1 - \left\{ {2 \atop 2\,1} \right\} X_2 + \left\{ {1 \atop 1\,2} \right\} X_1 + \left\{ {2 \atop 1\,2} \right\} X_2, \tag{3.26}$$

which implies, due to the symmetry of the Christoffel symbols and the fact that $\omega_{12} = -\omega_{21}$, that

$$\omega_{21} = \frac{1}{2}\left(X_{2,1} - X_{1,2}\right). \tag{3.27}$$

The tractor bundle is $\pi : \mathbb{T} \to M$, where $\mathbb{T} = T^*(M) \oplus \bigwedge^2(M)$. The Killing tensor $X = p(u,v)du + q(u,v)dv$ on M is lifted to the local section $\tilde{X} = p(u,v)E_1 + q(u,v)E_2 + a(u,v)E_3$ on $\mathbb{T}$, where $a(u,v) = \dfrac{1}{2}\left(q_u - p_v\right)$.

The system of equations (3.23) becomes

$$\begin{cases} p_u - \left\{ {1 \atop 1\,1} \right\} p - \left\{ {2 \atop 1\,1} \right\} q = 0 & \text{(3.28a)} \\ p_v - \left\{ {1 \atop 1\,2} \right\} p - \left\{ {2 \atop 1\,2} \right\} q + a = 0 & \text{(3.28b)} \\ q_u - \left\{ {1 \atop 2\,1} \right\} p - \left\{ {2 \atop 2\,1} \right\} q - a = 0 & \text{(3.28c)} \\ q_v - \left\{ {1 \atop 2\,2} \right\} p - \left\{ {2 \atop 2\,2} \right\} q = 0 & \text{(3.28d)} \\ a_u - \left\{ {2 \atop 2\,1} \right\} a - \left\{ {1 \atop 1\,1} \right\} a + \kappa q = 0 & \text{(3.28e)} \\ a_v - \left\{ {2 \atop 2\,2} \right\} a - \left\{ {1 \atop 1\,2} \right\} a - \kappa p = 0. & \text{(3.28f)} \end{cases}$$

Matching system (3.28) with equation (3.5), we find that

$$\tilde{\Gamma}^j{}_{i1} = \begin{bmatrix} -\left\{ {1 \atop 1\,1} \right\} & -\left\{ {2 \atop 1\,1} \right\} & 0 \\ -\left\{ {1 \atop 2\,1} \right\} & -\left\{ {2 \atop 2\,1} \right\} & -1 \\ 0 & \kappa & -\left(\left\{ {2 \atop 2\,1} \right\} + \left\{ {1 \atop 1\,1} \right\} \right) \end{bmatrix},$$

$$\tilde{\Gamma}^j{}_{i2} = \begin{bmatrix} -\left\{ {1 \atop 1\,2} \right\} & -\left\{ {2 \atop 1\,2} \right\} & 1 \\ -\left\{ {1 \atop 2\,2} \right\} & -\left\{ {2 \atop 2\,2} \right\} & 0 \\ -\kappa & 0 & -\left(\left\{ {2 \atop 2\,2} \right\} + \left\{ {1 \atop 1\,2} \right\} \right) \end{bmatrix}.$$

Using equation (3.2), we find that

$$\tilde{\nabla}_{\partial_u} E_1 = -\left\{ {1 \atop 1\,1} \right\} E_1 - \left\{ {1 \atop 2\,1} \right\} E_2, \qquad \tilde{\nabla}_{\partial_v} E_1 = -\left\{ {1 \atop 1\,2} \right\} E_1 - \left\{ {1 \atop 2\,2} \right\} E_2 - \kappa E_3, \tag{3.29}$$

$$\tilde{\nabla}_{\partial_u} E_2 = -\left\{ {2 \atop 1\,1} \right\} E_1 - \left\{ {2 \atop 2\,1} \right\} E_2 + \kappa E_3, \qquad \tilde{\nabla}_{\partial_v} E_2 = -\left\{ {2 \atop 1\,2} \right\} E_1 - \left\{ {2 \atop 2\,2} \right\} E_2,$$

$$\tilde{\nabla}_{\partial_u} E_3 = -E_2 - \left(\left\{ {2 \atop 2\,1} \right\} + \left\{ {1 \atop 1\,1} \right\} \right) E_3, \qquad \tilde{\nabla}_{\partial_v} E_3 = E_1 - \left(\left\{ {2 \atop 2\,2} \right\} + \left\{ {1 \atop 1\,2} \right\} \right) E_3,$$

thus defining the tractor connection $\tilde{\nabla}$ on $\mathbb{T}$. *By construction, X is a rank 1 Killing tensor of g if and only if $\tilde{X}$ is parallel with respect to $\tilde{\nabla}$.*

Now, suppose that the metric g could be written as

$$g = \lambda du^2 + \lambda dv^2 \tag{3.30}$$

for some smooth function $\lambda = \lambda(u, v)$. The non-vanishing Christoffel symbols are

$$\left\{ {1 \atop 1\,1} \right\} = \left\{ {2 \atop 1\,2} \right\} = \left\{ {2 \atop 2\,1} \right\} = -\left\{ {1 \atop 2\,2} \right\} = \frac{\lambda_u}{2\lambda}, \quad \left\{ {1 \atop 1\,2} \right\} = \left\{ {1 \atop 2\,1} \right\} = -\left\{ {2 \atop 1\,1} \right\} = \left\{ {2 \atop 2\,2} \right\} = \frac{\lambda_v}{2\lambda}.$$

In this example, we find that

$$\tilde{\Gamma}^j{}_{i1} = \begin{bmatrix} -\frac{\lambda_u}{2\lambda} & \frac{\lambda_v}{2\lambda} & 0 \\ -\frac{\lambda_v}{2\lambda} & -\frac{\lambda_u}{2\lambda} & -1 \\ 0 & \kappa & -\frac{\lambda_u}{\lambda} \end{bmatrix}, \qquad \tilde{\Gamma}^j{}_{i2} = \begin{bmatrix} -\frac{\lambda_v}{2\lambda} & -\frac{\lambda_u}{2\lambda} & 1 \\ \frac{\lambda_u}{2\lambda} & -\frac{\lambda_v}{2\lambda} & 0 \\ -\kappa & 0 & -\frac{\lambda_v}{\lambda} \end{bmatrix}.$$

Thus, the curvature matrix is given as

$$\tilde{K}^i{}_j = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda k_u & -\lambda k_v & 0 \end{bmatrix} du \wedge dv, \tag{3.31}$$

where $k = k(u, v)$ is the sectional curvature of g, which is given, in this case, as

$$k = \frac{\kappa}{\lambda}.$$

If k is a constant, we see that the dimension of $\mathbb{T}^0$, the 0^{th} order reduced tractor bundle, is three, since the curvature matrix is identically zero. The derivatives of the curvature matrix would also be identically zero, making the dimension of $\mathbb{T}^1$ three as well. Thus, $\mathbb{T}^0 = \mathbb{T}^1$, which, by Theorem (3.1), implies that the dimension of $\mathcal{S}^{\tilde{\nabla}}(\mathbb{T})$ is three, which in turn implies that the metric g admits precisely 3 Killing tensors of rank 1.

On the other hand, if $k_u^2 + k_v^2 \neq 0$, a basis of the local sections of $\mathbb{T}^0$ is given as

$$\{k_v E_1 - k_u E_2, E_3\},$$

which implies that g admits no more than two rank 1 Killing tensors. Having obtained an upper bound at curvature order 0, we now endeavor to obtain an upper bound at curvature order 1. In order to calculate a basis for the local sections of $\mathbb{T}^1$, we will need to find all sections $\tilde{X} = b(u,v)\,(k_v E_1 - k_u E_2) + c(u,v) E_3$ for which the following conditions hold:

$$\tilde{K}^i{}_j \tilde{X}^j = 0, \qquad \tilde{\nabla}_{\partial_u} \tilde{K}^i{}_j \tilde{X}^j = 0, \qquad \tilde{\nabla}_{\partial_v} \tilde{K}^i{}_j \tilde{X}^j = 0. \tag{3.32}$$

The matrices $\tilde{K}^i{}_j$, $\tilde{\nabla}_{\partial_u} \tilde{K}^i{}_j$, and $\tilde{\nabla}_{\partial_v} \tilde{K}^i{}_j$ can also be stacked, so that equation (3.32) is equivalent to the following:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\lambda k_u & -\lambda k_v & 0 \\ 0 & 0 & 0 \\ \lambda k_u & \lambda k_v & 0 \\ -\lambda k_{uu} - \dfrac{\lambda_u k_u + \lambda_v k_v}{2} & -\lambda k_{uv} + \dfrac{\lambda_v k_u - \lambda_u k_v}{2} & -\lambda k_v \\ -\lambda k_u & -\lambda k_v & 0 \\ 0 & 0 & 0 \\ -\lambda k_{uv} + \dfrac{\lambda_u k_v - \lambda_v k_u}{2} & -\lambda k_{vv} - \dfrac{\lambda_u k_u + \lambda_v k_v}{2} & \lambda k_u \end{bmatrix} \begin{bmatrix} k_v b \\ -k_u b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{3.33}$$

Equation (3.33) is equivalent to $Ay = 0$, where

$$A = \begin{bmatrix} \left(-\lambda k_{uv} + \dfrac{\lambda_u k_v - \lambda_v k_u}{2}\right) k_v + \left(\lambda k_{vv} + \dfrac{\lambda_u k_u + \lambda_v k_v}{2}\right) k_u & \lambda k_u \\ -\left(\lambda k_{uu} + \dfrac{\lambda_u k_u + \lambda_v k_v}{2}\right) k_v + \left(\lambda k_{uv} + \dfrac{\lambda_u k_v - \lambda_v k_u}{2}\right) k_u & -\lambda k_v \end{bmatrix}, \tag{3.34}$$

and where $y = (b, c)$. Thus, the number of independent rank 1 Killing tensors of g is no greater than the dimension of the nullspace of A. If the rank of A is 2, then we must have $b = 0$ and $c = 0$, which means that there are no (non-zero) rank 1 Killing tensors of g. The rank of A cannot be zero, since this would imply that $k_u = 0$ and $k_v = 0$ and, subsequently, that $k_u^2 + k_v^2 = 0$, which we are assuming is not true. Therefore, the rank of A must be 1 or 2, and g cannot admit more than a single rank 1 Killing tensor.

If the rank of A is 1, $\det(A) = 0$, giving us, along with $k_u^2 + k_v^2 \neq 0$, the following additional condition:

$$\lambda^2 k_v^2 k_{uv} - \frac{\lambda k_v^3 \lambda_u}{2} - \lambda^2 k_v k_u k_{vv} - \frac{\lambda k_v \lambda_u k_u^2}{2} + \lambda^2 k_u k_v k_{uu} + \frac{\lambda k_u \lambda_v k_v^2}{2} - \lambda^2 k_u^2 k_{uv} + \frac{\lambda k_u^3 \lambda_v}{2} = 0. \tag{3.35}$$

Dividing equation (3.35) by λ^2, since $\lambda \neq 0$, our conditions can be written as

$$\frac{1}{2} W^{\alpha}{}_{;\alpha} - r_u - s_v = 0, \tag{3.36}$$

where $W^1 = r(u, v) = -k_v(k_u^2 + k_v^2)$, $W^2 = s(u, v) = k_u(k_u^2 + k_v^2)$, and $k_u^2 + k_v^2 \neq 0$. If these conditions are satisfied, then g admits a single rank 1 Killing tensor. If $k_u^2 + k_v^2 \neq 0$ but equation (3.36) is not satisfied, g admits no rank 1 Killing tensors. If $k_u^2 + k_v^2 = 0$, $k_u = 0$ and $k_v = 0$, and so g admits three independent rank 1 Killing tensors. This result confirms the findings of previous dealings with Killing vectors in two dimensions (Kruglikov, 2008).

The utility of the tractor approach extends beyond the study of Killing tensors, which we illustrate with a simple example. Suppose we had the following system of equations for the functions $f = f(u, v)$ and $h = h(u, v)$, where $F = F(u)$ is given:

$$\begin{cases} f_u - Ff = 0 & \text{(3.37a)} \\ f_v = 0 & \text{(3.37b)} \\ h_u = 0 & \text{(3.37c)} \\ h_v - f = 0. & \text{(3.37d)} \end{cases}$$

Formally, the tractor bundle is $\pi : \mathbb{T} \to \mathbb{R}^2$, with coordinates (u, v, E_1, E_2), and where $\mathbb{T} = \mathbb{R} \oplus \mathbb{R}$. A pair (f, h) is lifted to the section $fE_1 + hE_2$, and, using the methods described above, we see that

$$\tilde{\Gamma}^j{}_{i1} = \begin{bmatrix} -F & 0 \\ 0 & 0 \end{bmatrix}, \qquad \tilde{\Gamma}^j{}_{i2} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}. \tag{3.38}$$

Thus, the tractor connection is defined by

$$\tilde{\nabla}_{\partial_u} E_1 = -FE_1, \qquad \tilde{\nabla}_{\partial_v} E_1 = -E_2. \tag{3.39}$$

At this point, we see that the condition that the section $\tilde{X} = fE_1 + hE_2$ is parallel with respect to $\tilde{\nabla}$ is equivalent to the original system of equations, since the covariant derivative of $\tilde{X}$ is given as

$$(f_u - Ff)\, E_1 \otimes du + f_v E_1 \otimes dv + h_u E_2 \otimes du + (h_v - f)\, E_2 \otimes dv. \tag{3.40}$$

We find that the curvautre matrix of 2-forms is given as

$$\begin{bmatrix} 0 & 0 \\ -F & 0 \end{bmatrix} du \wedge dv. \tag{3.41}$$

The nullspace of this matrix is a basis for the set of sections which are quasi-parallel to order 0. Assuming $F \neq 0$, this basis is generated by a single element, namely E_2. This in turn implies that there is at most one independent parallel section, which implies that the space of solutions of (3.37) is at most 1-dimensional.

The tractor approach can also be used to reduce the original system of equations. We begin by constructing the section Y as follows:

$$Y = q(u,v)E_2;$$

that is, in general, a linear combination of the basis elements of $\mathcal{S}(E^0)$ using unknown function coefficients as scalars. The condition $\tilde{\nabla}Y = 0$ gives us the following system of equations:

$$\begin{cases} q_u = 0 & \text{(3.42a)} \\ q_v = 0. & \text{(3.42b)} \end{cases}$$

We see that $q = c_1$ for some constant c_1, so that Y is a parallel section if and only if $Y = c_1 E_2$. As the parallel sections are in one-to-one correspondence with the solutions of the original system of equations, we see that the general solution is $f = 0$, $h = c_1$.

4 The Tractor Connection for Killing vectors

4.1 Constructing the Tractor Connection

The tractor equations for Killing vectors are relatively well known, and are given in existing literature (Ashtekar and Magnon-Ashtekar, 1978; Hauser and Malhiot, 1975a). We derived these formulas in the previous chapter–see equation (3.23)–and they are, for a Killing vector X, given as

$$\begin{cases} X_{\alpha;\beta} = \omega_{\alpha\beta} & \text{(4.1a)} \\ w_{\alpha\beta;\gamma} = R^{\nu}{}_{\gamma\beta\alpha} X_{\nu}\,, & \text{(4.1b)} \end{cases}$$

where the tensor ω is defined as

$$\omega_{\alpha\beta} = X_{[\alpha;\beta]}\,.$$

Let M be a manifold with local coordinates x^{α}. For (covariant) Killing vectors defined on M, the tractor bundle is $\pi : \mathbb{T} \to M$, where $\mathbb{T} = T^{*}(M) \oplus \bigwedge^{2}(M)$. Coordinates for $\mathbb{T}$ are $(x^{\alpha}, a_{\alpha}, b_{\alpha\beta})$, where $b_{\alpha\beta} = -b_{\beta\alpha}$. Thus, it is clear that the fibers of $\mathbb{T}$ have dimension $n + \binom{n}{2} = n(n+1)/2$. If X is a Killing vector, then the lift of X to a section $\tilde{X}$ on $\mathbb{T}$ is given by $a_{\alpha}(x) = X_{\alpha}(x)$, $b_{\alpha\beta}(x) = \omega_{\alpha\beta}(x)$: by construction, this is a parallel section. Conversely, given a parallel section $(a_{\alpha}, b_{\alpha\beta})$, the (covariant) vector defined by $X_{\alpha}(x) = a_{\alpha}(x)$ is a Killing vector, since $a_{(\alpha;\beta)} = b_{(\alpha\beta)} = 0$. The tractor connection and the tractor curvature are defined by equations (3.2), (3.7), and (3.5).

Lemma 4.1. *Let X be a Killing vector, and suppose that $\tilde{X}$ is the lift of X up to the tractor bundle. $\tilde{X}$ vanishes at a point if and only if $X = 0$.*

This is due to the fact that equations ($4.1a$) and ($4.1b$) define a system of equations which is Frobenius in the sense of equation (2.2): the lemma follows as a direct result of Corollary (2.1.1).

Lemma 4.1 gives rise to the following useful application of the tractor approach.

Corollary 4.0.1. *The Killing vectors $X_1 \ldots X_k$ are linearly independent over $\mathbb{R}$ if and only if their lifts up to the tractor bundle are linearly independent at a single point.*

Proof. Let $X_1 \ldots X_k$ be Killing vectors, and let $\tilde{X}_1 \ldots \tilde{X}_k$ be their lifts up to the tractor bundle at a point. By the previous lemma, $a_1\tilde{X}_1 + a_2\tilde{X}_2 + \ldots a_k\tilde{X}_k = 0$ if and only if $a_1X_1 + a_2X_2 + \ldots a_kX_k = 0$. Thus, the linear independence of one set implies the linear independence of the other. □

Once Killing vector fields are explicitly identified, the task of determining their linear independence is, in principle, a rudimentary linear algebra problem when the coefficient functions are rational functions. If the coefficient functions are, for instance, trigonometric, exponential, or square-root functions, then a direct approach often fails, even when symbolic software is used. In this case, Corollary (4.0.1) can be used to complete the task.[2]

4.2 Killing vectors on spacetimes

In this section, we apply the tractor approach to finding Killing vectors for a few metrics taken from the literature (Stephani et al., 2003; Hawkings and Ellis, 1973). Our goal is to determine the number of Killing vectors which are admitted by these metrics, which goal is obtained as follows. Using the tractor approach, we will obtain an upper bound on the number of linearly independent Killing vectors which can exist: when possible, we will use Theorem (3.1) to determine the precise number.

[2]We will demonstrate this novel application for Killing tensors of rank 2 in Appendix F.

If we cannot apply Theorem (3.1), we will obtain a lower bound on the number of Killing vectors which exist by exhibiting solutions to the Killing equation. A summary is given in Table 1.

	Known Isometry	$\dim(\mathbb{T}^0)$	$\dim(\mathbb{T}^1)$	$\dim(\mathbb{T}^2)$	Direct pdsolve	Tractor pdsolve	T
5.29	2	4	x	x	x	✓	2
35.80(i)	1	5	1	-	x	✓	1
35.80(ii)	3	6	4	3	✓	✓	3
35.80(iii)	3	6	4	3	✓	✓	3
35.80(iv)	4	6	4	4	✓	✓	4
35.80(v)	4	6	4	4	✓	✓	4
35.80(vi)	1	6	2	1	x	✓	1
12.32	4	4	4	-	x	x	4

Table 1: The tractor approach for Killing vectors for a few metrics.

We indicate the number of Killing vectors known previous to the application of the tractor approach by means of the "Known Isometry" column. The "Direct pdsolve" column indicates whether the Killing vectors could be found by solving the Killing equations directly in Maple using the "pdsolve" command: a checkmark indicates that the Killing equations were solved directly, and an "x" indicates that the Maple computation was aborted. Maple computations were aborted either due to the seemingly abnormal amount of required time or due to memory constraints imposed by the computer used. The "Tractor pdsolve" column indicates whether the Killing vectors could be found by means of the reduced Killing equations. A checkmark in this column indicates that this computation was achieved by means of the Maple "pdsolve" command, while an "x" indicates that this Maple computation was aborted. The column T indicates the number of independent Killing vectors we obtain using the tractor approach. Dashes indicate that the associated computation is not needed.

The metric in the first entry is the Kerr metric, given explicitly in local coordinates (t, r, θ, ϕ) as (Hawkings and Ellis, 1973)

$$-\frac{A-2mr}{A}dt^2 - \frac{4amr\sin^2(\theta)}{A}dtd\phi + \frac{A}{a^2-2mr+r^2}dr^2 + Ad\theta^2$$
$$+\frac{\sin^2(\theta)\left(-2a^2mr\cos^2(\theta)+2a^2mr+A(a^2+r^2)\right)}{A}d\phi^2,$$

where a and m are parameters, and where $A = r^2 + a^2\cos^2(\theta)$. This metric will appear again in section 6.2. It can be easily determined that two Killing vectors of this metric are ∂_t and ∂_ϕ, since the components of the metric neither depend on t nor on ϕ. This observation does not answer the question, however, of whether the metric admits more independent Killing vectors. Our method for determining the precise number of independent Killing vectors for this metric is as follows.

First, we find a basis for the local sections of $\mathbb{T}^0$: that is, a basis for the local sections of the 0^{th} order reduced tractor bundle. There are four basis elements, which we denote as W_1, W_2, W_3, and W_4. Unfortunately, we were not able to compute a basis for the local sections of $\mathbb{T}^1$ for the Kerr metric: notwithstanding, we still attempt to find the Killing vectors. We construct a linear combination of the basis elements of the local sections of $\mathbb{T}^0$ (the coefficients being smooth functions):

$$S = q_1(t,r,\theta,\phi)W_1 + q_2(t,r,\theta,\phi)W_2 + q_3(t,r,\theta,\phi)W_3 + q_4(t,r,\theta,\phi)W_4.$$

The section S is a parallel section if and only if its covariant derivative with respect to the tractor connection $\tilde{\nabla}$ vanishes. The equations generated by $\tilde{\nabla}S = 0$ can be solved explicitly using Maple, and it is determined that the dimension of the space of solutions is 2. Thus, ∂_t and ∂_ϕ constitute a basis for the space of Killing vectors.

The metric 35.80(i) is simply 35.80 (Stephani et al., 2003), which is given in local coordinates (x, y, u, v) as

$$dx^2 + dy^2 - 2H(x, y, u)du^2 - 2dudv.$$

It is clear that ∂_v is a Killing vector of this metric, and our methods demonstrate that, absent additional conditions on the function H, no additional Killing vectors can exist. However, conditions on H can be imposed which grant the metric additional Killing vectors. 35.80(ii) is 35.80 with the condition that $H_y = 0$: in this case, our methods show that the Killing vectors are

$$\partial_v, \qquad \partial_y, \qquad u\partial_y + y\partial_v.$$

Similarly, if $H_x = 0$, which condition is imposed in 35.80(iii), we get the following Killing vectors:

$$\partial_v, \qquad \partial_x, \qquad u\partial_x + x\partial_v.$$

The metric 35.80(iv) has $H_y = 0$ and $H_u = 0$, giving us the Killing tensor ∂_u in addition to those specified for entry 35.80(ii). The metric 35.80(v) has $H_x = 0$ and $H_u = 0$, giving us a Killing vector of ∂_u in addition to those in entry 35.80(iii).

Our last examined case of 35.80 is 35.80(vi), where we impose the condition $H_{xx} + H_{yy} = 0$. We find that, at order 2, the metric admits a maximum of 1 Killing vector, so that ∂_v is the single Killing tensor. It should also be pointed out that there are other conditions which can be imposed which may yield a higher number of Killing vectors; for example, $H_x = 0$ and $H_y = 0$.

For the metric 12.32 (Stephani et al., 2003), we have $s = \sqrt{2}$ and $a = 1$, so that $\beta^2 = -3$, $k = \dfrac{\sqrt{3}}{2}$, $b = 2$, and $F = -1$.

5 The Tractor Connection for Conformal Killing Vectors.

5.1 Constructing the Tractor Connection

Let M be a pseudo-Riemannian manifold of dimension $n > 2$.[3] A conformal Killing vector is a vector X which satisfies

$$X_{(\alpha;\beta)} = \frac{1}{n} X^{\gamma}{}_{;\gamma} g_{\alpha\beta}. \tag{5.1}$$

The equations which define the tractor connection for conformal Killing vectors can be found in the literature (Ashtekar and Magnon-Ashtekar, 1978) and are also presented in this section. We define the skew-symmetric tensor Y as

$$Y_{\alpha\beta} = X_{[\alpha;\beta]}, \tag{5.2}$$

and the scalar function $F = \frac{1}{n} X^{\gamma}{}_{;\gamma}$, so that for a conformal Killing vector X,

$$X_{\alpha;\beta} = X_{[\alpha;\beta]} + X_{(\alpha;\beta)} = Y_{\alpha\beta} + F g_{\alpha\beta}. \tag{5.3}$$

With our definition of Y, we also have

$$Y_{\alpha\beta;\gamma} = \frac{1}{2}\left(X_{\alpha;\beta\gamma} - X_{\beta;\alpha\gamma}\right),$$

which becomes, after using the formula for the commutator of the covariant derivative for X,

$$Y_{\alpha\beta;\gamma} = \frac{1}{2}\left(X_{\alpha;\gamma\beta} - X_{\beta;\gamma\alpha} + R^{\delta}{}_{\alpha\beta\gamma} X_{\delta} - R^{\delta}{}_{\beta\alpha\gamma} X_{\delta}\right).$$

[3]An account of the $n = 2$ case will be given in section 5.1.1.

Using equation (5.1), we interchange the first two indices of the terms involving X to rewrite this as

$$Y_{\alpha\beta;\gamma} = 2g_{\gamma[\alpha}F_{;\beta]} + \frac{1}{2}\left(X_{\gamma;\beta\alpha} - X_{\gamma;\alpha\beta} + R^{\nu}{}_{\gamma\beta\alpha}X_{\nu}\right). \tag{5.4}$$

Lastly, we use the commutator of the covariant derivatives of X to obtain

$$Y_{\alpha\beta;\gamma} = 2g_{\gamma[\alpha}F_{;\beta]} + R^{\nu}{}_{\gamma\beta\alpha}X_{\nu},$$

so that

$$Y_{\alpha\beta;\gamma} = 2g_{\gamma[\alpha}Z_{\beta]} + R^{\nu}{}_{\gamma\beta\alpha}X_{\nu}, \tag{5.5}$$

where we have defined the tensor Z_{α} as

$$Z_{\alpha} = F_{;\alpha}. \tag{5.6}$$

This definition also implies, using the commutator of covariant derivatives of X, that

$$\begin{aligned} nZ_{\beta;\alpha} &= X^{\mu}{}_{;\mu\beta\alpha} \\ &= \left(X^{\mu}{}_{;\beta\mu} + R^{\mu}{}_{\gamma\beta\mu}X^{\gamma}\right)_{;\alpha} \\ &= X^{\mu}{}_{;\beta\alpha\mu} + R^{\mu}{}_{\gamma\alpha\mu}X^{\gamma}{}_{;\beta} + R^{\gamma}{}_{\beta\mu\alpha}X^{\mu}{}_{;\gamma} + R^{\mu}{}_{\gamma\beta\mu;\alpha}X^{\gamma} + R^{\mu}{}_{\gamma\beta\mu}X^{\gamma}{}_{;\alpha}. \end{aligned} \tag{5.7}$$

We find, using equations (5.3) and (5.2), that

$$\begin{aligned} X^{\mu}{}_{;\beta\alpha\mu} &= \left(Y^{\mu}{}_{\beta;\alpha} + g^{\mu}{}_{\beta}F_{;\alpha}\right)_{;\mu} \\ &= \left(R^{\nu}{}_{\alpha\beta}{}^{\mu}X_{\nu} + g_{\alpha}{}^{\mu}F_{;\beta} - g_{\alpha\beta}F_{;\delta}\,g^{\delta\mu} + g^{\mu}{}_{\beta}F_{;\alpha}\right)_{;\mu} \\ &= R^{\nu}{}_{\alpha\beta}{}^{\mu}{}_{;\mu}X_{\nu} + R^{\nu}{}_{\alpha\beta}{}^{\mu}X_{\nu;\mu} + g_{\alpha}{}^{\mu}Z_{\beta;\mu} - g_{\alpha\beta}Z_{\delta;\mu}g^{\delta\mu} + g^{\mu}{}_{\beta}Z_{\alpha;\mu}. \end{aligned}$$

Applying this to equation (5.7), we get

$$nZ_{\beta;\alpha} - g_{\alpha}{}^{\mu} Z_{\beta;\mu} + g_{\alpha\beta} Z_{\delta;\mu} g^{\delta\mu} - g^{\mu}{}_{\beta} Z_{\alpha;\mu} = R^{\nu}{}_{\alpha\beta}{}^{\mu}{}_{;\mu} X_{\nu} + R^{\nu}{}_{\alpha\beta}{}^{\mu} X_{\nu;\mu} \tag{5.8}$$

$$+R^{\mu}{}_{\gamma\alpha\mu} X^{\gamma}{}_{;\beta} + R^{\gamma}{}_{\beta\mu\alpha} X^{\mu}{}_{;\gamma} + R^{\mu}{}_{\gamma\beta\mu;\alpha} X^{\gamma} + R^{\mu}{}_{\gamma\beta\mu} X^{\gamma}{}_{;\alpha}.$$

The left side of equation (5.8) can be written as

$$Z_{\delta;\mu}\left(n\delta^{\delta}_{\beta}\delta^{\mu}_{\alpha} - g_{\alpha}{}^{\mu}\delta^{\delta}_{\beta} + g_{\alpha\beta}g^{\delta\mu} - g^{\mu}{}_{\beta}\delta^{\delta}_{\alpha}\right)$$

$$= Z_{\delta;\mu}\left(n\delta^{\delta}_{\beta}\delta^{\mu}_{\alpha} - \delta^{\mu}_{\alpha}\delta^{\delta}_{\beta} + g_{\alpha\beta}g^{\delta\mu} - \delta^{\mu}_{\beta}\delta^{\delta}_{\alpha}\right)$$

$$= Z_{\delta;\mu}\left((n-2)\delta^{\delta}_{\beta}\delta^{\mu}_{\alpha} + g_{\alpha\beta}g^{\delta\mu}\right),$$

so that

$$g^{\alpha\beta} Z_{\delta;\mu}\left((n-2)\delta^{\delta}_{\beta}\delta^{\mu}_{\alpha} + g_{\alpha\beta}g^{\delta\mu}\right) = (2n-2)Z^{\mu}{}_{;\mu}.$$

On the other hand, the right side of equation (5.8) can be written, using equation (5.3), as

$$-R_{\alpha\beta;\nu}X^{\nu} - 2R_{\alpha\beta}F - R^{\gamma}{}_{\alpha}Y_{\gamma\beta} - R^{\gamma}{}_{\beta}Y_{\gamma\alpha}.$$

Thus,

$$(2n-2)Z^{\mu}{}_{;\mu} = -R_{;\nu}X^{\nu} - 2FR,$$

and so, for $n \neq 2$,

$$Z_{\beta;\alpha} = \tag{5.9}$$

$$\frac{1}{n-2}\left(-X^{\nu}\left(R_{\alpha\beta}-\frac{1}{2n-2}Rg_{\alpha\beta}\right)_{;\nu}-2F\left(R_{\alpha\beta}-\frac{1}{2n-2}Rg_{\alpha\beta}\right)+2R^{\gamma}{}_{(\alpha}Y_{\beta)\gamma}\right).$$

Equations (5.3), (5.5), (5.6), and (5.9) define the Frobenius system from which the tractor connection is constructed. In the present case, and for a manifold M with coordinates x^{α}, the tractor bundle is $\pi : \mathbb{T} \to M$, with $\mathbb{T} = T^{*}(M) \oplus \bigwedge^{2}(M) \oplus \mathbb{R} \oplus T^{*}(M)$. The coordinates are $(x^{\alpha}, a_{\alpha}, b_{\alpha\beta}, c, d_{\alpha})$, where $b_{\alpha\beta} = -b_{\alpha\beta}$: thus, the dimension of the fibers is $n(n+3)/2+1$ for $n > 2$.

The lift up to the tractor bundle is given by $a_{\alpha}(x) = X_{\alpha}(x)$, $b_{\alpha\beta}(x) = Y_{\alpha\beta}(x)$, $c(x) = F(x)$, and $d_{\alpha}(x) = Z_{\alpha}(x)$. If X_{α} is a conformal Killing vector, the lift is a parallel section by construction. Conversely, given the parallel section $(a_{\alpha}, b_{\alpha\beta}, c, d_{\alpha})$, the covariant vector defined by $X_{\alpha} = a_{\alpha}$ is a covariant conformal Killing vector.

As with Killing vectors, we have an important application of the lift.

Lemma 5.1. *Let X be a conformal Killing vector, and suppose that $\tilde{X}$ is the lift of X up to the tractor bundle. $\tilde{X}$ vanishes at a point if and only if $X = 0$.*

As in the case of Killing vectors, the equations which define the tractor connection for conformal Killing vectors is Frobenius. Thus, lemma (5.1) follows from corollary (2.1.1).

Corollary 5.0.1. *The conformal Killing vectors $X_1 \ldots X_k$ are linearly independent over $\mathbb{R}$ if and only if their lifts up to the tractor bundle are linearly independent at a single point.*

Proof. Let $X_1 \ldots X_k$ be conformal Killing vectors, and let $\tilde{X}_1 \ldots \tilde{X}_k$ be their lifts up to the tractor bundle at a point. By the previous lemma, $a_1\tilde{X}_1 + a_2\tilde{X}_2 + \ldots a_k\tilde{X}_k = 0$ if and only if $a_1X_1 + a_2X_2 + \ldots a_kX_k = 0$. Thus, the linear independence of one set implies the linear independence of the other. $\square$

As with Killing vectors, this simplifies the issue of determining linear independence for conformal Killing vector fields.

5.1.1 Conformal Killing vectors in dimension 2

Our discussion of the tractor connection for conformal Killing vectors has been limited to the case where the dimension of the manifold is strictly greater than 2. We now briefly consider conformal Killing vectors on a manifold of dimension $n = 2$.

For a metric g in the plane, there exist coordinates (x, y) so that g can be written in the form

$$g = \lambda \left(dx^2 + dy^2\right) \tag{5.10}$$

for some smooth function $\lambda = \lambda(x, y)$. A metric g_2 are said to be conformally equivalent to a metric g_1 if there exists a positive function ϕ such that $g_1 = \phi g_2$ (Dairbekov and Sharafutdinov, 2011). Thus, every metric in the plane is conformally equivalent to the metric

$$\tilde{g} = dx^2 + dy^2.$$

In particular, any metric in the plane will have the same conformal Killing vectors as $\tilde{g}$. We define X as

$$X = pdx + qdy$$

for $p = p(x, y)$ and $q = q(x, y)$. In light of equation (5.1), X is a conformal Killing vector of $\tilde{g}$ if and only if the following system of equations is satisfied:

$$\begin{cases} p_x - q_y = 0 & (5.11a) \\ p_y + q_x = 0. & (5.11b) \end{cases}$$

(5.11) is equivalent to the Cauchy-Riemann equations for $z(x, y) = p(x, y) + iq(x, y)$. Thus, the solution space for (5.11) is infinite dimensional.

On the other hand, the number of parallel sections for any linear connection on a finite-dimensional vector bundle is always finite. Therefore, for $n = 2$, equation (5.1) cannot give us the interpretation of parallel sections of a finite dimensional vector bundle.

5.2 Conformal Killing vectors on non-conformally flat solutions

We will now employ the tractor approach to the study of conformal Killing vectors for exact solutions appearing in section 35.4.4 of *Exact Solutions to Einstein's Field Equations* (Stephani et al., 2003). Our goal is to determine the precise number of conformal Killing vectors which various metrics admit. As with Killing vectors, we will employ the tractor approach not only to determine an upper bound on the number of conformal Killing vectors, but also in hopes of applying theorem (3.1) to get the precise count. If theorem (3.1) cannot be applied, we will obtain a lower bound on the number of conformal Killing vectors by exhibiting solutions to the Killing equation for conformal Killing vectors. Our calculations in Maple are summarized in Table 2.

	Killing vectors	$\dim(\mathbb{T}^0)$	$\dim(\mathbb{T}^1)$	$\dim(\mathbb{T}^2)$	Direct pdsolve	Tractor pdsolve	C	T
35.74	4	6	6	-	✓	✓	2	0
35.75(i)	4	6	6	-	✓	✓	2	r
35.75(ii)	4	6	6	-	✓	✓	2	r
35.76(i)	4	6	5	5	✓	✓	1	0
35.76(ii)	4	6	5	5	✓	✓	1	0
35.77	2	4	4	-	✓	✓	2	0
35.78	1	3	3	-	✓	✓	2	0
35.79	0	4	1	0	-	-	0	0
35.80	1	8	x	-	x	x		

Table 2: Metrics from section 35.4.4.

We indicate, for each metric, the number of independent Killing vectors, as well as the dimensions of $\mathbb{T}^n$ for $n = 0, 1, 2$. We indicate whether the conformal Killing vectors were found by solving the Killing equation directly as well as whether they were found by solving the tractor-simplified Killing equations with the columns "Direct pdsolve" and "Tractor pdsolve," respectively. A checkmark indicates the affirmative result in the respective column, whereas an "x" indicates that the Maple calculation was aborted after a reasonable time with no results. Similarly, an "x" in other columns indicates that the Maple computation was aborted after some time. Column C denotes the number, as determined by the tractor approach, of conformal Killing vectors which are not also Killing vectors. Column T indicates the number of Killing tensors that were generated from the conformal Killing vectors using equation (2.12): an "r" indicates that only metric reducible Killing tensors were found.

Metric 35.80 appeared in section 4.2, and it was determined that the space of Killing vectors had dimension 1, at least where there are no additional constraints on the function $H(u, x, y)$. We find that the metric 35.80 admits at most 8 conformal Killing vectors, including the Killing vector itself, though we are unable to determine whether the dimension of the space of conformal Killing vectors is precisely 8.

For metric 35.74, we have chosen $n = 2$. 35.75(i) is metric 35.75 with $k = 1$, while 35.75(ii) has $k = -1$. 35.76(i) is metric 35.76 with $b = a$, $c = 4a(1 - a)(1 - 2a)$; 35.76(ii) has $b = (a - 1)/(2a - 1)$ and $c = 4a$. For 35.76(i) and 35.76(ii), $a = 2$. 35.77 has $a = 1 = b$, and 35.78 has $\alpha = 2$.

Metric 35.75(i) is given as

$$\frac{6}{\Lambda r^2 + 3} dr^2 + r^2 dx^2 + r^2 \sin^2(x) dy^2 - r^2 dt^2.$$

The covariant conformal Killing vector from which we constructed a Killing tensor is

$$\frac{r}{\sqrt{\Lambda r^2+3}}dr,$$

and the associated Killing tensor is

$$-\frac{r^4}{6}dx^2-\frac{r^4\sin^2(x)}{6}dy^2+\frac{r^4}{6}dt^2.$$

It can be shown that this Killing tensor is expressible as a linear combination of the reducible Killing tensors and the metric itself. Metric 35.75(ii) is similar, and is given explicitly as

$$\frac{6}{\Lambda r^2-3}dr^2+r^2dx^2+r^2\sinh^2(x)dy^2-r^2dt^2.$$

The covariant conformal Killing vector from which we constructed a Killing tensor is

$$\frac{r}{\sqrt{\Lambda r^2-3}}dr,$$

and the associated Killing tensor is

$$-\frac{r^4}{6}dx^2-\frac{r^4\sinh^2(x)}{6}dy^2+\frac{r^4}{6}dt^2.$$

As in the previous case, it can be shown that this Killing tensor is expressible as a linear combination of the reducible Killing tensors and the metric itself.

6 Killing tensors of rank two

6.1 Constructing the Tractor Connection

The equations that define the Tractor connection for Killing tensors of rank two are already known (Hauser and Malhiot, 1975a). In this section, we rederive these equations for arbitrary torsion free connections. The tractor connection equations are then further simplified in the case of metric connections.

Let K be a symmetric tensor of rank 2. By definition, K is a Killing tensor of rank 2 if

$$K_{(\alpha\beta;\gamma)} = 0. \tag{6.1}$$

We now define the tensor L as

$$L_{\alpha\beta\gamma} = -2Y_{\begin{smallmatrix}\alpha & \gamma \\ \beta & \end{smallmatrix}} K_{\gamma\alpha;\beta}, \tag{6.2}$$

and the tensor M as

$$M_{\alpha\beta\gamma\delta} = -2Y_{\begin{smallmatrix}\alpha & \gamma \\ \beta & \delta\end{smallmatrix}} K_{\gamma\alpha;\beta\delta}. \tag{6.3}$$

We note that $L_{[\alpha\beta]\gamma} = L_{\alpha\beta\gamma}$, and that $L_{[\alpha\beta\gamma]} = 0$. Similarly, we note that the tensor M has the symmetries of the Riemann curvature tensor:

$$M_{\alpha\beta\gamma\delta} = M_{[\alpha\beta][\gamma\delta]} = M_{\gamma\delta\alpha\beta}. \tag{6.4}$$

The derivation of the tractor equations has been organized into three parts. In the first part, we give the covariant derivative of K in terms of L. Next, we give the covariant derivative of L in terms of K and M, valid for any torsion free connection–

the equations for the metric connection are then derived. Lastly, we give the covariant derivative of M in terms of both K and L, first for any torsion free connection and then in the case of the metric connection.

6.1.1 The First Tractor Equation

We will now assume that K is a Killing tensor, and we will derive the formula for the covariant derivative of K. From the definition of the tensor L,

$$L_{\alpha\beta\gamma} = K_{\beta\gamma;\alpha} - K_{\alpha\gamma;\beta}. \tag{6.5}$$

From equation (6.5), we have

$$K_{\beta\gamma;\alpha} = L_{\alpha\beta\gamma} + K_{\alpha\gamma;\beta}. \tag{6.6}$$

Using the fact that $K_{\alpha\beta} = K_{\beta\alpha}$ as well as the fact that $K_{(\alpha\beta;\gamma)} = 0$, we get

$$K_{\beta\gamma;\alpha} = L_{\alpha\beta\gamma} - K_{\alpha\beta;\gamma} - K_{\beta\gamma;\alpha}. \tag{6.7}$$

By equation (6.5), $K_{\alpha\beta;\gamma} = -L_{\alpha\gamma\beta} + K_{\gamma\beta;\alpha}$, so that equation (6.7) becomes

$$K_{\beta\gamma;\alpha} = L_{\alpha\beta\gamma} + L_{\alpha\gamma\beta} - K_{\gamma\beta;\alpha} - K_{\beta\gamma;\alpha}, \tag{6.8}$$

which implies that

$$K_{\beta\gamma;\alpha} = \frac{2}{3} L_{\alpha(\beta\gamma)}. \tag{6.9}$$

Equation (6.9) is the first equation which will define the tractor connection for Killing tensors of rank 2. We note that this equation is valid when covariant differentiation is taken with respect to any symmetric connection.

6.1.2 The Second Tractor Equation

We now turn our attention to the covariant derivative of L. Due to equation (6.9), and by the definition of the tensor M in (6.3), we have

$$\begin{aligned} M_{\alpha\beta\gamma\delta} &= -2Y_{\begin{array}{|c|c|}\hline \alpha&\gamma\\\hline \beta&\delta\\\hline\end{array}} K_{\gamma\alpha;\beta\delta} = -\frac{2}{3}Y_{\begin{array}{|c|c|}\hline \alpha&\gamma\\\hline \beta&\delta\\\hline\end{array}} L_{\beta(\gamma\alpha);\delta} \\ &= -\frac{2}{3}Y_{\begin{array}{|c|}\hline \alpha\\\hline \beta\\\hline\end{array}} Y_{\begin{array}{|c|}\hline \gamma\\\hline \delta\\\hline\end{array}} \left(L_{\beta(\gamma\alpha);\delta} + L_{\delta(\gamma\alpha);\beta}\right) \\ &= \frac{1}{3}\left(L_{\alpha\beta[\gamma;\delta]} + L_{\gamma\delta[\alpha;\beta]}\right) + \frac{1}{3}Y_{\begin{array}{|c|}\hline \alpha\\\hline \beta\\\hline\end{array}} Y_{\begin{array}{|c|}\hline \gamma\\\hline \delta\\\hline\end{array}} \left(L_{\gamma\beta\alpha;\delta} + L_{\alpha\delta\gamma;\beta}\right). \end{aligned} \tag{6.10}$$

Using the fact that $L_{[\alpha\beta\gamma]} = 0$, we get

$$\begin{aligned} &\frac{1}{3}Y_{\begin{array}{|c|}\hline \alpha\\\hline \beta\\\hline\end{array}} Y_{\begin{array}{|c|}\hline \gamma\\\hline \delta\\\hline\end{array}} \left(L_{\gamma\beta\alpha;\delta} + L_{\alpha\delta\gamma;\beta}\right) \\ &= \frac{1}{12}\left(L_{\gamma\beta\alpha;\delta} - L_{\gamma\alpha\beta;\delta} - L_{\delta\beta\alpha;\gamma} + L_{\delta\alpha\beta;\gamma} + L_{\alpha\delta\gamma;\beta} - L_{\beta\delta\gamma;\alpha} - L_{\alpha\gamma\delta;\beta} + L_{\beta\gamma\delta;\alpha}\right) \\ &= \frac{1}{6}\left(L_{\alpha\beta[\gamma;\delta]} + L_{\gamma\delta[\alpha;\beta]}\right). \end{aligned} \tag{6.11}$$

Combining equations (6.10) and (6.11), we can express the tensor M in terms of the tensor L as follows:

$$M_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(L_{\alpha\beta[\gamma;\delta]} + L_{\gamma\delta[\alpha;\beta]}\right). \tag{6.12}$$

Next, we note that, using equation (6.5),

$$L_{\alpha\beta(\gamma;\delta)} = \frac{1}{2}\left(K_{\beta\gamma;\alpha\delta} - K_{\alpha\gamma;\beta\delta} + K_{\beta\delta;\alpha\gamma} - K_{\alpha\delta;\beta\gamma}\right), \tag{6.13}$$

and since $K_{(\alpha\gamma;\delta)} = 0 = -K_{(\beta\gamma;\delta)}$,

$$L_{\alpha\beta(\gamma;\delta)} = \frac{1}{2}\left(K_{\beta\gamma;\alpha\delta} - K_{\alpha\gamma;\beta\delta} + K_{\beta\delta;\alpha\gamma} - K_{\alpha\delta;\beta\gamma} + 3K_{(\alpha\gamma;\delta)\beta} - 3K_{(\beta\gamma;\delta)\alpha}\right), \quad (6.14)$$

which, after expanding the terms $K_{(\alpha\gamma;\delta)\beta)}$ and $K_{(\beta\gamma;\delta)\alpha)}$, can we written as

$$L_{\alpha\beta(\gamma;\delta)} = K_{\beta\gamma;[\alpha\delta]} + K_{\alpha\gamma;[\delta\beta]} + K_{\delta\alpha;[\gamma\beta]} + K_{\beta\delta;[\alpha\gamma]} + K_{\gamma\delta;[\alpha\beta]}. \quad (6.15)$$

The terms on the right are simply commuting covariant derivatives of $K_{\beta\gamma}$:

$$K_{\beta\gamma;[\alpha\delta]} = \frac{1}{2}\left(R^{\nu}{}_{\beta\alpha\delta}K_{\nu\gamma} + R^{\nu}{}_{\gamma\alpha\delta}K_{\beta\nu}\right). \quad (6.16)$$

Thus, equation (6.15) becomes

$$\begin{aligned} L_{\alpha\beta(\gamma;\delta)} = &\frac{1}{2}\left(R^{\nu}{}_{\beta\alpha\delta}K_{\nu\gamma} + R^{\nu}{}_{\gamma\alpha\delta}K_{\beta\nu}\right) + \frac{1}{2}\left(R^{\nu}{}_{\alpha\delta\beta}K_{\nu\gamma} + R^{\nu}{}_{\gamma\delta\beta}K_{\alpha\nu}\right) \\ &+\frac{1}{2}\left(R^{\nu}{}_{\delta\gamma\beta}K_{\nu\alpha} + R^{\nu}{}_{\alpha\gamma\beta}K_{\delta\nu}\right) + \frac{1}{2}\left(R^{\nu}{}_{\beta\alpha\gamma}K_{\nu\delta} + R^{\nu}{}_{\delta\alpha\gamma}K_{\beta\nu}\right) \\ &+\frac{1}{2}\left(R^{\nu}{}_{\gamma\alpha\beta}K_{\nu\delta} + R^{\nu}{}_{\delta\alpha\beta}K_{\gamma\nu}\right). \end{aligned}$$

Collecting the terms above on $K_{\alpha\beta}$, and using the symmetries of the Riemann curvature tensor, we find that

$$L_{\alpha\beta(\gamma;\delta)} = K_{\nu\gamma}R^{\nu}{}_{\delta\alpha\beta} + K_{\nu\delta}R^{\nu}{}_{\gamma\alpha\beta} + 2R^{\nu}{}_{(\gamma\delta)[\beta}K_{\alpha]\nu}. \quad (6.17)$$

On the other hand, equation (6.5) gives us

$$L_{\alpha\beta[\gamma;\delta]} - L_{\gamma\delta[\alpha;\beta]}$$

$$= \frac{1}{2}\left(K_{\beta\gamma;\alpha\delta} - K_{\alpha\gamma;\beta\delta} - K_{\beta\delta;\alpha\gamma} + K_{\alpha\delta;\beta\gamma}\right) - \frac{1}{2}\left(K_{\delta\alpha;\gamma\beta} - K_{\gamma\alpha;\delta\beta} - K_{\delta\beta;\gamma\alpha} + K_{\gamma\beta;\delta\alpha}\right),$$

which can be written as

$$L_{\alpha\beta[\gamma;\delta]} - L_{\gamma\delta[\alpha;\beta]} = K_{\beta\gamma;[\alpha\delta]} - K_{\alpha\gamma;[\beta\delta]} + K_{\alpha\delta;[\beta\gamma]} - K_{\beta\delta;[\alpha\gamma]}\,.$$

Using equation (6.16) and the symmetries of the Riemann curvature tensor, we find that

$$L_{\alpha\beta[\gamma;\delta]} - L_{\gamma\delta[\alpha;\beta]} = \frac{1}{2}\left(K_{\gamma\nu}R^{\nu}{}_{\delta\alpha\beta} + K_{\delta\nu}R^{\nu}{}_{\gamma\beta\alpha} + K_{\beta\nu}R^{\nu}{}_{\alpha\gamma\delta} + K_{\alpha\nu}R^{\nu}{}_{\beta\delta\gamma}\right). \quad (6.18)$$

By equation (6.12),

$$L_{\alpha\beta[\gamma;\delta]} + L_{\gamma\delta[\alpha;\beta]} = 2M_{\alpha\beta\gamma\delta}\,. \quad (6.19)$$

Adding equations (6.18) and (6.19) gives us

$$L_{\alpha\beta[\gamma;\delta]} = M_{\alpha\beta\gamma\delta} + \frac{1}{4}\left(K_{\gamma\nu}R^{\nu}{}_{\delta\alpha\beta} + K_{\delta\nu}R^{\nu}{}_{\gamma\beta\alpha} + K_{\beta\nu}R^{\nu}{}_{\alpha\gamma\delta} + K_{\alpha\nu}R^{\nu}{}_{\beta\delta\gamma}\right). \quad (6.20)$$

Adding equations (6.20) and (6.17) gives us

$$\begin{aligned} L_{\alpha\beta\gamma;\delta} &= K_{\nu\gamma}R^{\nu}{}_{\delta\alpha\beta} + K_{\nu\delta}R^{\nu}{}_{\gamma\alpha\beta} + 2R^{\nu}{}_{(\gamma\delta)[\beta}K_{\alpha]\nu} + M_{\alpha\beta\gamma\delta} \\ &+\frac{1}{4}\left(K_{\gamma\nu}R^{\nu}{}_{\delta\alpha\beta} + K_{\delta\nu}R^{\nu}{}_{\gamma\beta\alpha} + K_{\beta\nu}R^{\nu}{}_{\alpha\gamma\delta} + K_{\alpha\nu}R^{\nu}{}_{\beta\delta\gamma}\right), \end{aligned}$$

which can be rewritten as

$$L_{\alpha\beta\gamma;\delta} = M_{\alpha\beta\gamma\delta} + \frac{5}{4}K_{\nu\gamma}R^{\nu}{}_{\delta\alpha\beta} + \frac{3}{4}K_{\nu\delta}R^{\nu}{}_{\gamma\alpha\beta} + \frac{3}{2}R^{\nu}{}_{\gamma\delta[\beta}K_{\alpha]\nu} + \frac{1}{2}R^{\nu}{}_{\delta\gamma[\beta}K_{\alpha]\nu}\,. \quad (6.21)$$

Equation (6.21) is the second equation which will define the tractor connection, and is valid for any symmetric connection. Before addressing the third and final equation, we wish to show how equation (6.21) can be rewritten in the presence of a metric connection. Returning to equation (6.15), we write

$$L_{\alpha\beta(\gamma;\delta)} = \frac{1}{2}\left(K^{\mu}{}_{\beta}R_{\mu\gamma\alpha\delta} + K^{\mu}{}_{\gamma}R_{\mu\beta\alpha\delta}\right) + \frac{1}{2}\left(K^{\mu}{}_{\alpha}R_{\mu\gamma\delta\beta} + K^{\mu}{}_{\gamma}R_{\mu\alpha\delta\beta}\right) \tag{6.22}$$

$$+\frac{1}{2}\left(K^{\mu}{}_{\delta}R_{\mu\alpha\gamma\beta} + K^{\mu}{}_{\alpha}R_{\mu\delta\gamma\beta}\right) + \frac{1}{2}\left(K^{\mu}{}_{\beta}R_{\mu\delta\alpha\gamma} + K^{\mu}{}_{\delta}R_{\mu\beta\alpha\gamma}\right)$$

$$+\frac{1}{2}\left(K^{\mu}{}_{\gamma}R_{\mu\delta\alpha\beta} + K^{\mu}{}_{\delta}R_{\mu\gamma\alpha\beta}\right).$$

This can be re-written as

$$L_{\alpha\beta(\gamma;\delta)} = \frac{1}{2}K^{\mu}{}_{\delta}\left(R_{\mu\alpha\gamma\beta} + R_{\mu\beta\alpha\gamma} + R_{\mu\gamma\alpha\beta}\right) + \frac{1}{2}K^{\mu}{}_{\gamma}\left(-R_{\beta\mu\alpha\delta} - R_{\alpha\mu\delta\beta} - R_{\delta\mu\alpha\beta}\right) \tag{6.23}$$

$$+\frac{1}{2}\left(K^{\mu}{}_{\beta}R_{\mu\gamma\alpha\delta} + K^{\mu}{}_{\alpha}R_{\mu\gamma\delta\beta} + K^{\mu}{}_{\alpha}R_{\mu\delta\gamma\beta} + K^{\mu}{}_{\beta}R_{\mu\delta\alpha\gamma}\right),$$

which can be written, using the Bianchi identity, as

$$L_{\alpha\beta(\gamma;\delta)} = K^{\mu}{}_{\gamma}R_{\delta\mu\beta\alpha} + K^{\mu}{}_{\delta}R_{\gamma\mu\beta\alpha} + K^{\mu}{}_{\alpha}R_{\beta(\gamma\delta)\mu} - K^{\mu}{}_{\beta}R_{\alpha(\delta\gamma)\mu}, \tag{6.24}$$

or simply

$$L_{\alpha\beta(\gamma;\delta)} = 2K^{\mu}{}_{(\gamma}R_{\delta)\mu\beta\alpha} + 2K^{\mu}{}_{[\alpha}R_{\beta](\gamma\delta)\mu}. \tag{6.25}$$

On the other hand, equation (6.5) gives us

$$L_{\alpha\beta[\gamma;\delta]} - L_{\gamma\delta[\alpha;\beta]} \tag{6.26}$$

$$= \frac{1}{2}\left(K_{\beta\gamma;\alpha\delta} - K_{\alpha\gamma;\beta\delta} - K_{\beta\delta;\alpha\gamma} + K_{\alpha\delta;\beta\gamma}\right) - \frac{1}{2}\left(K_{\delta\alpha;\gamma\beta} - K_{\gamma\alpha;\delta\beta} - K_{\delta\beta;\gamma\alpha} + K_{\gamma\beta;\delta\alpha}\right),$$

which can be written as

$$L_{\alpha\beta[\gamma;\delta]} - L_{\gamma\delta[\alpha;\beta]} = K_{\beta\gamma;[\alpha\delta]} - K_{\alpha\gamma;[\beta\delta]} + K_{\alpha\delta;[\beta\gamma]} - K_{\beta\delta;[\alpha\gamma]}\,. \tag{6.27}$$

Applying equation (6.16) and the Bianchi identity to equation (6.27) gives us

$$L_{\alpha\beta[\gamma;\delta]} - L_{\gamma\delta[\alpha;\beta]} = \frac{1}{2}\left(K^{\mu}{}_{\delta}R_{\alpha\beta\gamma\mu} + K^{\mu}{}_{\gamma}R_{\beta\alpha\delta\mu} + K^{\mu}{}_{\beta}R_{\gamma\delta\mu\alpha} + K^{\mu}{}_{\alpha}R_{\delta\gamma\mu\beta}\right), \tag{6.28}$$

which can be written as

$$L_{\alpha\beta[\gamma;\delta]} - L_{\gamma\delta[\alpha;\beta]} = R_{\gamma\delta\mu[\alpha}K_{\beta]}{}^{\mu} - R_{\alpha\beta\mu[\gamma}K_{\delta]}{}^{\mu}. \tag{6.29}$$

By equation (6.12),

$$L_{\alpha\beta[\gamma;\delta]} + L_{\gamma\delta[\alpha;\beta]} = 2M_{\alpha\beta\gamma\delta}\,. \tag{6.30}$$

Adding equations (6.29) and (6.30) implies that

$$L_{\alpha\beta[\gamma;\delta]} = \frac{1}{2}R_{\gamma\delta\mu[\alpha}K_{\beta]}{}^{\mu} - \frac{1}{2}R_{\alpha\beta\mu[\gamma}K_{\delta]}{}^{\mu} + M_{\alpha\beta\gamma\delta}\,. \tag{6.31}$$

Adding equations (6.31) and (6.25) gives us

$$L_{\alpha\beta\gamma;\delta} = 2K^{\mu}{}_{(\gamma}R_{\delta)\mu\beta\alpha} + 2K^{\mu}{}_{[\alpha}R_{\beta](\gamma\delta)\mu} + \frac{1}{2}R_{\gamma\delta\mu[\alpha}K_{\beta]}{}^{\mu} - \frac{1}{2}R_{\alpha\beta\mu[\gamma}K_{\delta]}{}^{\mu} + M_{\alpha\beta\gamma\delta}\,. \tag{6.32}$$

We will now rewrite this expression, starting with expanding the right hand side:

$$L_{\alpha\beta\gamma;\delta} = M_{\alpha\beta\gamma\delta} + 2K^{\mu}{}_{(\gamma}R_{\delta)\mu\beta\alpha} + \frac{1}{2}K^{\mu}{}_{\alpha}R_{\beta\gamma\delta\mu} + \frac{1}{2}K^{\mu}{}_{\alpha}R_{\beta\delta\gamma\mu} - \frac{1}{2}K^{\mu}{}_{\beta}R_{\alpha\gamma\delta\mu} \quad (6.33)$$

$$-\frac{1}{2}K^{\mu}{}_{\beta}R_{\alpha\delta\gamma\mu} + \frac{1}{4}R_{\gamma\delta\mu\alpha}K_{\beta}{}^{\mu} - \frac{1}{4}R_{\gamma\delta\mu\beta}K_{\alpha}{}^{\mu} - \frac{1}{4}R_{\alpha\beta\mu\gamma}K_{\delta}{}^{\mu} + \frac{1}{4}R_{\alpha\beta\mu\delta}K_{\gamma}{}^{\mu}.$$

Using the symmetries of the Riemann curvature tensor, we write this as

$$\begin{aligned} L_{\alpha\beta\gamma;\delta} = M_{\alpha\beta\gamma\delta} &+ 2K^{\mu}{}_{(\gamma}R_{\delta)\mu\beta\alpha} - \frac{1}{2}K^{\mu}{}_{\delta}R_{\alpha\beta\gamma\mu} - \frac{1}{2}K^{\mu}{}_{\gamma}R_{\alpha\beta\delta\mu} \\ &+\frac{3}{4}\left(K^{\mu}{}_{\delta}R_{\alpha\beta\gamma\mu} + K^{\mu}{}_{\beta}R_{\delta\alpha\gamma\mu} + K^{\mu}{}_{\alpha}R_{\beta\delta\gamma\mu}\right) \\ &+\frac{1}{4}\left(K^{\mu}{}_{\gamma}R_{\alpha\beta\delta\mu} + K^{\mu}{}_{\beta}R_{\gamma\alpha\delta\mu} + K^{\mu}{}_{\alpha}R_{\beta\gamma\delta\mu}\right). \end{aligned} \quad (6.34)$$

Finally, we write equation (6.34) as

$$L_{\alpha\beta\gamma;\delta} = M_{\alpha\beta\gamma\delta} + 3R_{\alpha\beta\mu(\gamma}K_{\delta)}{}^{\mu} + \frac{9}{4}K^{\mu}{}_{[\delta}R_{\alpha\beta]\gamma\mu} + \frac{3}{4}K^{\mu}{}_{[\gamma}R_{\alpha\beta]\delta\mu}. \quad (6.35)$$

Equation (6.35) bears a stronger resemblance than equation (6.21) to that which is found in existing literature (Hauser and Malhiot, 1975a). Nevertheless, our software program will incorporate equation (6.21), which is valid for any torsion free connection.

6.1.3 The Third Tractor Equation

Finally, we will derive the formula for the covariant derivative of M. Differentiating equation (6.17) gives us

$$L_{\alpha\beta(\gamma;\mu)\delta} = K_{\nu\gamma;\delta}R^{\nu}{}_{\mu\alpha\beta} + K_{\nu\gamma}R^{\nu}{}_{\mu\alpha\beta;\delta} + K_{\nu\mu;\delta}R^{\nu}{}_{\gamma\alpha\beta} + K_{\nu\mu}R^{\nu}{}_{\gamma\alpha\beta;\delta} + K_{\alpha\nu}R^{\nu}{}_{(\gamma\mu)\beta;\delta}$$

$$+R^{\nu}{}_{(\gamma\mu)\beta}K_{\alpha\nu;\delta} - K_{\beta\nu}R^{\nu}{}_{(\gamma\mu)\alpha;\delta} - K_{\beta\nu;\delta}R^{\nu}{}_{(\gamma\mu)\alpha},$$

so that, using equation (6.9) and simplifying, we get

$$L_{\alpha\beta(\gamma;\mu)\delta} - L_{\alpha\beta(\delta;\mu)\gamma} = R^{\nu}{}_{\mu\alpha\beta}L_{\delta\gamma\nu} + 2R^{\nu}{}_{\mu\alpha\beta;[\delta}K_{\gamma]\nu} + R^{\nu}{}_{\gamma\alpha\beta}L_{\delta(\mu\nu)} - R^{\nu}{}_{\delta\alpha\beta}L_{\gamma(\mu\nu)} \quad (6.36)$$

$$+R^{\nu}{}_{(\gamma\mu)\beta}L_{\delta(\nu\alpha)} - R^{\nu}{}_{(\delta\mu)\beta}L_{\gamma(\nu\alpha)} - R^{\nu}{}_{(\gamma\mu)\alpha}L_{\delta(\nu\beta)} + R^{\nu}{}_{(\delta\mu)\alpha}L_{\gamma(\nu\beta)}$$

$$+K_{\nu\mu}\left(R^{\nu}{}_{\gamma\alpha\beta;\delta} - R^{\nu}{}_{\delta\alpha\beta;\gamma}\right) + K_{\alpha\nu}\left(R^{\nu}{}_{(\gamma\mu)\beta;\delta} - R^{\nu}{}_{(\delta\mu)\beta;\gamma}\right) - K_{\beta\nu}\left(R^{\nu}{}_{(\gamma\mu)\alpha;\delta} - R^{\nu}{}_{(\delta\mu)\alpha;\gamma}\right).$$

Meanwhile, by expanding each term and, subsequently, writing commuting covariant derivatives in terms of the Riemann curvature tensor, we find that

$$L_{\alpha\beta(\gamma;\mu)\delta} - L_{\alpha\beta(\delta;\mu)\gamma} = \frac{1}{2}L_{\alpha\beta\gamma;\mu\delta} + \frac{1}{2}L_{\alpha\beta\mu;\gamma\delta} - \frac{1}{2}L_{\alpha\beta\delta;\mu\gamma} - \frac{1}{2}L_{\alpha\beta\mu;\delta\gamma} \quad (6.37)$$

$$= L_{\alpha\beta\mu;[\gamma\delta]} + \frac{1}{2}L_{\alpha\beta\gamma;\mu\delta} - \frac{1}{2}L_{\alpha\beta\delta;\mu\gamma}$$

$$= L_{\alpha\beta\mu;[\gamma\delta]} + \frac{1}{2}L_{\alpha\beta\gamma;\mu\delta} - \frac{1}{2}L_{\alpha\beta\delta;\mu\gamma} - \frac{1}{2}L_{\alpha\beta\gamma;\delta\mu} + \frac{1}{2}L_{\alpha\beta\gamma;\delta\mu} + \frac{1}{2}L_{\alpha\beta\delta;\gamma\mu} - \frac{1}{2}L_{\alpha\beta\delta;\gamma\mu}$$

$$= L_{\alpha\beta\mu;[\gamma\delta]} + L_{\alpha\beta\gamma;[\mu\delta]} + L_{\alpha\beta\delta;[\gamma\mu]} + L_{\alpha\beta[\gamma;\delta]\mu}$$

$$= \frac{1}{2}\left(L_{\nu\beta\mu}R^{\nu}{}_{\alpha\gamma\delta} + L_{\alpha\nu\mu}R^{\nu}{}_{\beta\gamma\delta} + L_{\nu\beta\gamma}R^{\nu}{}_{\alpha\mu\delta} + L_{\alpha\nu\gamma}R^{\nu}{}_{\beta\mu\delta} + L_{\nu\beta\delta}R^{\nu}{}_{\alpha\gamma\mu} + L_{\alpha\nu\delta}R^{\nu}{}_{\beta\gamma\mu}\right)$$

$$+L_{\alpha\beta[\gamma;\delta]\mu} + L_{\alpha\beta\nu}R^{\nu}{}_{\mu\gamma\delta}.$$

Differentiating equation (6.20) and applying equation (6.9) gives us

$$L_{\alpha\beta[\gamma;\delta]\mu} = \frac{1}{4}\left(R^{\nu}{}_{\delta\alpha\beta}L_{\mu(\nu\gamma)} + K_{\gamma\nu}R^{\nu}{}_{\delta\alpha\beta;\mu} + R^{\nu}{}_{\gamma\beta\alpha}L_{\mu(\nu\delta)} + K_{\delta\nu}R^{\nu}{}_{\gamma\beta\alpha;\mu}\right) \quad (6.38)$$

$$+\frac{1}{4}\left(R^{\nu}{}_{\alpha\gamma\delta}L_{\mu(\nu\beta)}+K_{\beta\nu}R^{\nu}{}_{\alpha\gamma\delta;\mu}+R^{\nu}{}_{\beta\delta\gamma}L_{\mu(\nu\alpha)}+K_{\alpha\nu}R^{\nu}{}_{\beta\delta\gamma;\mu}\right)+M_{\alpha\beta\gamma\delta;\mu}.$$

Combining equations (6.36), (6.37), and (6.38), we get

$$M_{\alpha\beta\gamma\delta;\mu}=R^{\nu}{}_{\mu\alpha\beta}L_{\delta\gamma\nu}+2R^{\nu}{}_{\mu\alpha\beta;[\delta}K_{\gamma]\nu}+R^{\nu}{}_{\gamma\alpha\beta}L_{\delta(\mu\nu)}-R^{\nu}{}_{\delta\alpha\beta}L_{\gamma(\mu\nu)} \tag{6.39}$$

$$+R^{\nu}{}_{(\gamma\mu)\beta}L_{\delta(\nu\alpha)}-R^{\nu}{}_{(\delta\mu)\beta}L_{\gamma(\nu\alpha)}-R^{\nu}{}_{(\gamma\mu)\alpha}L_{\delta(\nu\beta)}+R^{\nu}{}_{(\delta\mu)\alpha}L_{\gamma(\nu\beta)}$$

$$+K_{\nu\mu}\left(R^{\nu}{}_{\gamma\alpha\beta;\delta}-R^{\nu}{}_{\delta\alpha\beta;\gamma}\right)+K_{\alpha\nu}\left(R^{\nu}{}_{(\gamma\mu)\beta;\delta}-R^{\nu}{}_{(\delta\mu)\beta;\gamma}\right)-K_{\beta\nu}\left(R^{\nu}{}_{(\gamma\mu)\alpha;\delta}-R^{\nu}{}_{(\delta\mu)\alpha;\gamma}\right).$$

$$-\frac{1}{2}\left(L_{\nu\beta\mu}R^{\nu}{}_{\alpha\gamma\delta}+L_{\alpha\nu\mu}R^{\nu}{}_{\beta\gamma\delta}+L_{\nu\beta\gamma}R^{\nu}{}_{\alpha\mu\delta}+L_{\alpha\nu\gamma}R^{\nu}{}_{\beta\mu\delta}+L_{\nu\beta\delta}R^{\nu}{}_{\alpha\gamma\mu}+L_{\alpha\nu\delta}R^{\nu}{}_{\beta\gamma\mu}\right)$$

$$-\frac{1}{4}\left(R^{\nu}{}_{\delta\alpha\beta}L_{\mu(\nu\gamma)}+K_{\gamma\nu}R^{\nu}{}_{\delta\alpha\beta;\mu}+R^{\nu}{}_{\gamma\beta\alpha}L_{\mu(\nu\delta)}+K_{\delta\nu}R^{\nu}{}_{\gamma\beta\alpha;\mu}\right)$$

$$-\frac{1}{4}\left(R^{\nu}{}_{\alpha\gamma\delta}L_{\mu(\nu\beta)}+K_{\beta\nu}R^{\nu}{}_{\alpha\gamma\delta;\mu}+R^{\nu}{}_{\beta\delta\gamma}L_{\mu(\nu\alpha)}+K_{\alpha\nu}R^{\nu}{}_{\beta\delta\gamma;\mu}\right)-L_{\alpha\beta\nu}R^{\nu}{}_{\mu\gamma\delta}.$$

Equation (6.39) is the third and final equation which defines the tractor connection for Killing tensors of rank 2, and is valid for any torsion free connection. As with the second equation, we wish to rewrite this equation under the assumption that the connection on the base manifold is the metric connection. Differentiating equation (6.25) gives us

$$L_{\alpha\beta(\gamma;\mu)\delta}=\left(2K^{\nu}{}_{(\gamma}R_{\mu)\nu\beta\alpha}+2K^{\nu}{}_{[\alpha}R_{\beta](\gamma\mu)\nu}\right)_{;\delta} \tag{6.40}$$

$$=\frac{1}{2}K^{\nu}{}_{\alpha;\delta}\left(R_{\beta\gamma\mu\nu}+R_{\beta\mu\gamma\nu}\right)-\frac{1}{2}K^{\nu}{}_{\beta;\delta}\left(R_{\alpha\gamma\mu\nu}+R_{\alpha\mu\gamma\nu}\right)$$

$$+\frac{1}{2}K^{\nu}{}_{\alpha}\left(R_{\beta\gamma\mu\nu;\delta}+R_{\beta\mu\gamma\nu;\delta}\right)-\frac{1}{2}K^{\nu}{}_{\beta}\left(R_{\alpha\gamma\mu\nu;\delta}+R_{\alpha\mu\gamma\nu;\delta}\right)$$

$$+K^{\nu}{}_{\gamma;\delta}R_{\mu\nu\beta\alpha}+K^{\nu}{}_{\gamma}R_{\mu\nu\beta\alpha;\delta}+K^{\nu}{}_{\mu;\delta}R_{\gamma\nu\beta\alpha}+K^{\nu}{}_{\mu}R_{\gamma\nu\beta\alpha;\delta}.$$

This implies, after rearranging terms, that

$$L_{\alpha\beta(\gamma;\mu)\delta} - L_{\alpha\beta(\delta;\mu)\gamma} = R_{\mu\nu\beta\alpha}\left(K^{\nu}{}_{\gamma;\delta} - K^{\nu}{}_{\delta;\gamma}\right) \tag{6.41}$$

$$+K^{\nu}{}_{\mu}\left(R_{\gamma\nu\beta\alpha;\delta} - R_{\delta\nu\beta\alpha;\gamma}\right) + \left(K^{\nu}{}_{\gamma}R_{\mu\nu\beta\alpha;\delta} - K^{\nu}{}_{\delta}R_{\mu\nu\beta\alpha;\gamma}\right)$$

$$+\frac{1}{2}K^{\nu}{}_{\alpha}\left(R_{\mu\nu\beta\gamma;\delta} + R_{\beta\mu\gamma\nu;\delta} - R_{\mu\nu\beta\delta;\gamma} - R_{\beta\mu\delta\nu;\gamma}\right)$$

$$+\frac{1}{2}K^{\nu}{}_{\beta}\left(R_{\mu\nu\alpha\gamma;\delta} + R_{\alpha\mu\gamma\nu;\delta} - R_{\mu\nu\alpha\delta;\gamma} - R_{\alpha\mu\delta\nu;\gamma}\right)$$

$$-R_{\alpha\beta\gamma}{}^{\nu}K_{\mu\nu;\delta} + R_{\alpha\beta\delta}{}^{\nu}K_{\mu\nu;\gamma} + R_{\beta(\mu\gamma)}{}^{\nu}K_{\nu\alpha;\delta}$$

$$-R_{\alpha(\mu\gamma)}{}^{\nu}K_{\nu\beta;\delta} - R_{\beta(\mu\delta)}{}^{\nu}K_{\nu\alpha;\gamma} + R_{\alpha(\mu\delta)}{}^{\nu}K_{\nu\beta;\gamma}.$$

Using equation (6.5), the fact that the tensor K is symmetric, and symmetries of the Riemann curvature tensor, we rewrite this as

$$L_{\alpha\beta(\gamma;\mu)\delta} - L_{\alpha\beta(\delta;\mu)\gamma} = R_{\alpha\beta\mu}{}^{\nu}L_{\gamma\delta\nu} - R_{\alpha\beta\gamma\delta;\nu}K^{\nu}{}_{\mu} + 2R_{\mu\nu\alpha\beta;[\gamma}K_{\delta]}{}^{\nu} \tag{6.42}$$

$$+R_{\mu\nu\gamma\delta;[\alpha}K_{\beta]}{}^{\nu} + K^{\nu}{}_{[\alpha}R_{\beta]\mu\gamma\delta;\nu}$$

$$-R_{\alpha\beta\gamma}{}^{\nu}K_{\mu\nu;\delta} + R_{\alpha\beta\delta}{}^{\nu}K_{\mu\nu;\gamma} + R_{\beta(\mu\gamma)}{}^{\nu}K_{\nu\alpha;\delta}$$

$$-R_{\alpha(\mu\gamma)}{}^{\nu}K_{\nu\beta;\delta} - R_{\beta(\mu\delta)}{}^{\nu}K_{\nu\alpha;\gamma} + R_{\alpha(\mu\delta)}{}^{\nu}K_{\nu\beta;\gamma}.$$

However, using equation (6.9), we see that

$$\frac{1}{3}\delta^{\varphi\chi}_{\alpha\beta}\delta^{\psi\omega}_{\gamma\delta}\left(R_{\varphi\chi\psi}{}^{\nu}L_{\omega(\mu\nu)} + 2R_{\varphi(\mu\psi)}{}^{\nu}L_{\omega(\nu\chi)}\right) \tag{6.43}$$

$$= -R_{\alpha\beta\gamma}{}^{\nu}K_{\mu\nu;\delta} + R_{\alpha\beta\delta}{}^{\nu}K_{\mu\nu;\gamma} + R_{\beta(\mu\gamma)}{}^{\nu}K_{\nu\alpha;\delta}$$

$$-R_{\alpha(\mu\gamma)}{}^{\nu}K_{\nu\beta;\delta} - R_{\beta(\mu\delta)}{}^{\nu}K_{\nu\alpha;\gamma} + R_{\alpha(\mu\delta)}{}^{\nu}K_{\nu\beta;\gamma},$$

so that equation (6.42) can be written as

$$L_{\alpha\beta(\gamma;\mu)\delta} - L_{\alpha\beta(\delta;\mu)\gamma} = R_{\alpha\beta\mu}{}^{\nu} L_{\gamma\delta\nu} - R_{\alpha\beta\gamma\delta;\nu} K^{\nu}{}_{\mu} + 2R_{\mu\nu\alpha\beta;[\gamma} K_{\delta]}{}^{\nu} + R_{\mu\nu\gamma\delta;[\alpha} K_{\beta]}{}^{\nu} \tag{6.44}$$

$$+K^{\nu}{}_{[\alpha} R_{\beta]\mu\gamma\delta;\nu} - \frac{1}{3}\delta^{\varphi\chi}_{\alpha\beta}\delta^{\psi\omega}_{\gamma\delta}\left(R_{\varphi\chi\psi}{}^{\nu} L_{\omega(\mu\nu)} + 2R_{\varphi(\mu\psi)}{}^{\nu} L_{\omega(\nu\chi)}\right).$$

Meanwhile, by expanding each term, we see that

$$L_{\alpha\beta(\gamma;\mu)\delta} - L_{\alpha\beta(\delta;\mu)\gamma} = \frac{1}{2}L_{\alpha\beta\gamma;\mu\delta} + \frac{1}{2}L_{\alpha\beta\mu;\gamma\delta} - \frac{1}{2}L_{\alpha\beta\delta;\mu\gamma} - \frac{1}{2}L_{\alpha\beta\mu;\delta\gamma} \tag{6.45}$$

$$= L_{\alpha\beta\mu;[\gamma\delta]} + \frac{1}{2}L_{\alpha\beta\gamma;\mu\delta} - \frac{1}{2}L_{\alpha\beta\delta;\mu\gamma}.$$

However,

$$L_{\alpha\beta\mu;[\gamma\delta]} + \frac{1}{2}L_{\alpha\beta\gamma;\mu\delta} - \frac{1}{2}L_{\alpha\beta\delta;\mu\gamma} = \tag{6.46}$$

$$L_{\alpha\beta\mu;[\gamma\delta]} + \frac{1}{2}L_{\alpha\beta\gamma;\mu\delta} - \frac{1}{2}L_{\alpha\beta\delta;\mu\gamma} - \frac{1}{2}L_{\alpha\beta\gamma;\delta\mu} + \frac{1}{2}L_{\alpha\beta\gamma;\delta\mu} + \frac{1}{2}L_{\alpha\beta\delta;\gamma\mu} - \frac{1}{2}L_{\alpha\beta\delta;\gamma\mu}$$

$$= L_{\alpha\beta\mu;[\gamma\delta]} + L_{\alpha\beta\gamma;[\mu\delta]} + L_{\alpha\beta\delta;[\gamma\mu]} + L_{\alpha\beta[\gamma;\delta]\mu}.$$

$$= L_{\alpha\beta[\gamma;\delta]\mu} + \frac{1}{2}L_{\alpha\beta\nu}\left(R^{\nu}{}_{\mu\gamma\delta} + R^{\nu}{}_{\gamma\mu\delta} + R^{\nu}{}_{\delta\gamma\mu}\right)$$

$$+\frac{1}{2}\left(L_{\nu\beta\mu}R^{\nu}{}_{\alpha\gamma\delta} + L_{\alpha\nu\mu}R^{\nu}{}_{\beta\gamma\delta} + L_{\nu\beta\gamma}R^{\nu}{}_{\alpha\mu\delta} + L_{\alpha\nu\gamma}R^{\nu}{}_{\beta\mu\delta} + L_{\nu\beta\delta}R^{\nu}{}_{\alpha\gamma\mu} + L_{\alpha\nu\delta}R^{\nu}{}_{\beta\gamma\mu}\right).$$

We see that we can rewrite the second term in the final expression on equation (6.46) as follows:

$$\frac{1}{2}L_{\alpha\beta\nu}\left(R^{\nu}{}_{\mu\gamma\delta} + R^{\nu}{}_{\gamma\mu\delta} + R^{\nu}{}_{\delta\gamma\mu}\right) = \frac{1}{2}L_{\alpha\beta}{}^{\nu}\left(R_{\mu\nu\delta\gamma} + R_{\gamma\nu\delta\mu} + R_{\delta\nu\mu\gamma}\right) \tag{6.47}$$

$$= \frac{1}{2}L_{\alpha\beta}{}^{\nu}\left(R_{\mu\nu\delta\gamma} - R_{\mu\nu\gamma\delta}\right) = -R_{\gamma\delta\mu}{}^{\nu} L_{\alpha\beta\nu}.$$

Additionally, we see that

$$\begin{aligned}&\frac{1}{2}\delta^{\varphi\chi}_{\alpha\beta}\delta^{\psi\omega}_{\gamma\delta}\left(R_{\mu\psi\varphi}{}^{\nu}L_{\chi\nu\omega}-\frac{1}{2}R_{\psi\omega\varphi}{}^{\nu}L_{\chi\nu\mu}\right)\\
&=\frac{1}{2}\left(R_{\mu\gamma\alpha}{}^{\nu}L_{\beta\nu\delta}-R_{\mu\delta\alpha}{}^{\nu}L_{\beta\nu\gamma}-R_{\mu\gamma\beta}{}^{\nu}L_{\alpha\nu\delta}+R_{\mu\delta\beta}{}^{\nu}L_{\alpha\nu\gamma}\right)\\
&-\frac{1}{4}\left(R_{\gamma\delta\alpha}{}^{\nu}L_{\beta\nu\mu}-R_{\delta\gamma\alpha}{}^{\nu}L_{\beta\nu\mu}-R_{\gamma\delta\beta}{}^{\nu}L_{\alpha\nu\mu}+R_{\delta\alpha\beta}{}^{\nu}L_{\alpha\nu\mu}\right)\\
&=-\frac{1}{2}\left(L_{\nu\beta\mu}R^{\nu}{}_{\alpha\gamma\delta}+L_{\alpha\nu\mu}R^{\nu}{}_{\beta\gamma\delta}+L_{\nu\beta\gamma}R^{\nu}{}_{\alpha\mu\delta}+L_{\alpha\nu\gamma}R^{\nu}{}_{\beta\mu\delta}+L_{\nu\beta\delta}R^{\nu}{}_{\alpha\gamma\mu}+L_{\alpha\nu\delta}R^{\nu}{}_{\beta\gamma\mu}\right).\end{aligned} \tag{6.48}$$

Applying equations (6.47) and (6.48) to equation (6.46), we are left with

$$L_{\alpha\beta(\gamma;\mu)\delta}-L_{\alpha\beta(\delta;\mu)\gamma}=-R_{\gamma\delta\mu}{}^{\nu}L_{\alpha\beta\nu}+L_{\alpha\beta[\gamma;\delta]\mu}-\frac{1}{2}\delta^{\varphi\chi}_{\alpha\beta}\delta^{\psi\omega}_{\gamma\delta}\left(R_{\mu\psi\varphi}{}^{\nu}L_{\chi\nu\omega}-\frac{1}{2}R_{\psi\omega\varphi}{}^{\nu}L_{\chi\nu\mu}\right). \tag{6.49}$$

Differentiating equation (6.31) gives us

$$\begin{aligned}L_{\alpha\beta[\gamma;\delta]\mu}&=M_{\alpha\beta\gamma\delta;\mu}+\frac{1}{4}R_{\gamma\delta\nu\alpha;\mu}K_{\beta}{}^{\nu}+\frac{1}{4}R_{\gamma\delta\nu\alpha}K^{\nu}{}_{\beta;\mu}-\frac{1}{4}R_{\gamma\delta\nu\beta;\mu}K_{\alpha}{}^{\nu}-\frac{1}{4}R_{\gamma\delta\nu\beta}K^{\nu}{}_{\alpha;\mu}\\
&-\frac{1}{4}R_{\alpha\beta\nu\gamma;\mu}K_{\delta}{}^{\nu}-\frac{1}{4}R_{\alpha\beta\nu\gamma}K^{\nu}{}_{\delta;\mu}+\frac{1}{4}R_{\alpha\beta\nu\delta;\mu}K_{\gamma}{}^{\nu}+\frac{1}{4}R_{\alpha\beta\nu\delta}K^{\nu}{}_{\gamma;\mu}.\end{aligned} \tag{6.50}$$

Now, we apply equation (6.9) to write the terms involving the covariant derivative of K as terms involving the tensor L. The result is

$$\begin{aligned}L_{\alpha\beta[\gamma;\delta]\mu}&=M_{\alpha\beta\gamma\delta;\mu}-\frac{1}{6}R_{\gamma\delta\alpha}{}^{\nu}L_{\mu(\nu\beta)}+\frac{1}{6}R_{\gamma\delta\beta}{}^{\nu}L_{\mu(\nu\alpha)}+\frac{1}{6}R_{\alpha\beta\gamma}{}^{\nu}L_{\mu(\nu\delta)}-\frac{1}{6}R_{\alpha\beta\delta\nu}L_{\mu(\nu\gamma)}\\
&-\frac{1}{4}\left(R_{\gamma\delta\nu\alpha;\mu}K_{\beta}{}^{\nu}+R_{\gamma\delta\beta\nu;\mu}K_{\alpha}{}^{\nu}+R_{\alpha\beta\gamma\nu;\mu}K_{\delta}{}^{\nu}+R_{\alpha\beta\nu\delta;\mu}K_{\gamma}{}^{\nu}\right).\end{aligned} \tag{6.51}$$

We now combine equations (6.44), (6.49), and (6.51) to obtain the following:

$$
\begin{aligned}
M_{\alpha\beta\gamma\delta;\mu} &= -R_{\alpha\beta\gamma\delta;\nu}K^{\nu}{}_{\mu} \qquad (6.52)\\
&+\delta^{\varphi\chi}_{\alpha\beta}\delta^{\psi\omega}_{\gamma\delta}\left(\frac{1}{2}R_{\mu\psi\varphi}{}^{\nu}L_{\chi\nu\omega}-\frac{1}{4}R_{\psi\omega\varphi}{}^{\nu}L_{\chi\nu\mu}-\frac{1}{3}R_{\varphi\chi\psi}{}^{\nu}L_{\omega(\mu\nu)}-\frac{2}{3}R_{\varphi(\mu\psi)}{}^{\nu}L_{\omega(\nu\chi)}\right)\\
&+R_{\gamma\delta\mu}{}^{\nu}L_{\alpha\beta\nu}+R_{\alpha\beta\mu}{}^{\nu}L_{\gamma\delta\nu}+\frac{1}{6}R_{\gamma\delta\alpha}{}^{\nu}L_{\mu(\nu\beta)}-\frac{1}{6}R_{\gamma\delta\beta}{}^{\nu}L_{\mu(\nu\alpha)}-\frac{1}{6}R_{\alpha\beta\gamma}{}^{\nu}L_{\mu(\nu\delta)}+\frac{1}{6}R_{\alpha\beta\delta}{}^{\nu}L_{\mu(\nu\gamma)}\\
&+2R_{\mu\nu\alpha\beta;[\gamma}K_{\delta]}{}^{\nu}+K^{\nu}{}_{[\alpha}R_{\beta]\mu\gamma\delta;\nu}+R_{\mu\nu\gamma\delta;[\alpha}K_{\beta]}{}^{\nu}+\frac{1}{2}\left(K^{\nu}{}_{[\beta}R_{\alpha]\nu\gamma\delta;\mu}+K^{\nu}{}_{[\gamma}R_{\delta]\nu\alpha\beta;\mu}\right).
\end{aligned}
$$

While equation (6.52) is valid if covariant differentiation is taken with respect to the metric connection, and more closely resembles the equations found in existing literature (Hauser and Malhiot, 1975a), it is equation (6.39) that is implemented in the software program. This is due to the fact that equation (6.39) is valid not only for the metric connection, but for any torsion free connection.

6.1.4 Summary

For Killing tensors of rank 2, and for a manifold $\mathcal{M}$ with coordinates x^α, the tractor bundle is $\pi : \mathbb{T} \to \mathcal{M}$, where $\mathbb{T} = S^2(\mathcal{M}) \oplus Y_3(\mathcal{M}) \oplus Y_4(\mathcal{M})$, Y_3 is the set of type $(0,3)$ tensors whose symmetry is that of $L_{\alpha\beta\gamma}$, and where Y_4 is the set of type $(0,4)$ tensors whose symmetry is that of $M_{\alpha\beta\gamma\delta}$. For a general torsion free connection, the equations which define the tractor connection are (6.9), (6.21), and (6.39). The coordinates are $(x^\alpha, a_{\alpha\beta}, b_{\alpha\beta\gamma}, c_{\alpha\beta\gamma\delta})$, where $a_{\alpha\beta} = a_{\beta\alpha}$, and the symmetries of $b_{\alpha\beta\gamma}$ and $c_{\alpha\beta\gamma\delta}$ are that of $L_{\alpha\beta\gamma}$ and $M_{\alpha\beta\gamma\delta}$, respectively. The dimension of the fibers of $\mathbb{T}$ is $n(n+1)^2(n+2)/12$, which for $n = 2$ is 6 and for $n = 4$ is 50.

The lift onto the tractor bundle is then given by $a_{\alpha\beta}(x) = K_{\alpha\beta}(x)$, $b_{\alpha\beta\gamma}(x) = L_{\alpha\beta\gamma}(x)$, and $c_{\alpha\beta\gamma\delta}(x) = M_{\alpha\beta\gamma\delta}(x)$. By construction, this lift is a parallel section if $K_{\alpha\beta}$ is a Killing tensor. Conversely, given a parallel section $(a_{\alpha\beta}, b_{\alpha\beta\gamma}, c_{\alpha\beta\gamma\delta})$, the tensor $K_{\alpha\beta}$ defined by $K_{\alpha\beta}(x) = a_{\alpha\beta}(x)$ is a Killing tensor of rank 2. This is due to the fact that, by equation (6.9),

$$K_{(\alpha\beta;\gamma)} = a_{(\alpha\beta;\gamma)} = \frac{2}{3} Y_{\boxed{\alpha}\boxed{\beta}\boxed{\gamma}}\, b_{\alpha(\beta\gamma)} = 0$$

by the symmetries of $b_{\alpha\beta\gamma}$.

As with Killing vectors and conformal Killing vectors, we have an important application of the lift of Killing tensors (of rank 2):

Lemma 6.1. *Let X be a Killing tensor of rank 2, and suppose that $\tilde{X}$ is the lift of X up to the tractor bundle. $\tilde{X}$ vanishes at a point if and only if $X = 0$.*

As in the case of Killing vectors and conformal Killing vectors, the equations which define the tractor connection for Killing tensors of rank 2–namely (6.9), (6.21), and (6.39)–form a Frobenius system of equations in the sense of equation (2.2). Thus, lemma (6.1) follows from corollary (2.1.1).

Corollary 6.0.1. *The rank 2 Killing tensors $X_1 \ldots X_k$ are linearly independent over $\mathbb{R}$ if and only if their lifts up to the tractor bundle are linearly independent at a single point.*

Proof. Let $X_1 \ldots X_k$ be rank 2 Killing tensors, and let $\tilde{X}_1 \ldots \tilde{X}_k$ be their lifts up to the tractor bundle at a point. By the previous lemma, $a_1\tilde{X}_1 + a_2\tilde{X}_2 + \ldots a_k\tilde{X}_k = 0$ if and only if $a_1X_1 + a_2X_2 + \ldots a_kX_k = 0$. Thus, the linear independence of one set implies the linear independence of the other. □

This result is of particular importance to us: though it closely resembles results for Killing vectors and conformal Killing vectors, it is of particular importance in the study of Killing tensors of rank 2. When we apply the tractor approach to Killing tensors, we will examine metrics for which the Killing vectors are already known. One can always use the (covariant) Killing vectors to generate a set of Killing tensors; however, it is not always clear whether the resulting set is linearly independent. As the tractor approach allows us to obtain upper bounds on the number of independent

Killing tensors, it is of paramount importance for us to know the dimension of the space of reducible Killing tensors, as it allows us to identify those metrics for which irreducible Killing tensors may exist. Though computing a basis for the space of reducible Killing tensors is often impractical using a direct approach, corollary (6.0.1) provides a computationally efficient way to find the number of basis elements for the set of reducible Killing tensors. When this number matches the upper bound obtained by the tractor approach, no additional Killing tensors can exist.

6.2 Rank 2 Killing tensors on type D vacuum solutions

Rank 2 Killing tensors are of particular interest in the study of General Relativity, and not only for the reason of being first integrals of the geodesic equation (Kalnins and Miller, 1981). We will examine Rank 2 Killing tensors in General Relativity in this section, the next, and again in section 7.2. In this section, we will calculate rank 2 Killing tensors for various Petrov type D vacuum solutions of the Einstein equations. In particular, we will determine whether the Killing tensors generated by equation (35.51) in the second edition of *Exact Solutions to Einstein's Field Equations* (Stephani et al., 2003) are metric irreducible, and we will attempt to determine whether there are Killing tensors beyond those found from this equation.

We begin by recalling the definition of a null tetrad. Let M be a Lorentzian manifold, and let g be the metric on M. Let l^α, n^α, m^α, and $\bar{m}^\alpha$ be null vectors, where $\bar{m}^\alpha$ is the complex conjugate of m^α. The set $\{l^\alpha, k^\alpha, m^\alpha, \bar{m}^\alpha\}$ is called a null tetrad if it is a basis for the space M and the following relations hold:

$$g_{\alpha\beta} l^\alpha m^\beta = g_{\alpha\beta} l^\alpha \bar{m}^\beta = g_{\alpha\beta} k^\alpha m^\beta = g_{\alpha\beta} k^\alpha \bar{m}^\beta = 0;$$

$$g_{\alpha\beta} l^\alpha k^\beta = 1, \qquad g_{\alpha\beta} m^\alpha \bar{m}^\beta = -1.$$

This gives rise to the following derivative operators: $D = \nabla_l$, $\Delta = \nabla_n$, $\delta = \nabla_m$, and $\bar{\delta} = \nabla_{\bar{m}}$. Let W be the Weyl tensor of g. The Weyl scalars are defined as:

$$\Psi_0 = -W_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma m^\delta, \qquad \Psi_1 = -W_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma m^\delta, \qquad \Psi_2 = -W_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta \bar{m}^\gamma n^\delta,$$

$$\Psi_3 = -W_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta, \qquad \Psi_4 = -W_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta.$$

A principle null direction is a null vector k which satisfies the following equation:

$$k_{[\mu} W_{\alpha]\beta\gamma[\delta} k_{\nu]} k^{\beta} k^{\gamma} = 0.$$

For a metric of Petrov type D, there are two distinct principal null directions, which are important in the study of Killing tensors due to the following, previously known (Stephani et al., 2003) method of constructing Killing tensors.

Theorem 6.1. *Let g be a Petrov type D vacuum solution which is not the charged Kerr metric, and let k^{α} and l^{α} be the two distinct principle null directions. The following defines a Killing tensor of g,*

$$K_{\alpha\beta} = \left(A^2 + B^2\right)\left(l_{\alpha} k_{\beta} + k_{\alpha} l_{\beta}\right) + B^2 g_{\alpha\beta},$$

with $A + iB = const\,(\Psi_2)^{-\frac{1}{3}}$ and $DA = \Delta A = \delta B = 0$.

The proof of this theorem makes use of the two-component spinor formalism (M. Walker and R. Penrose, 1970). We note that this theorem does not imply that all Killing tensors of a given type D vacuum solution take this form, nor does it imply that the Killing tensor so defined is irreducible.

Accordingly, we examine the utility of this result by examining several known vacuum type D solutions of the Einstein field equations, producing, for each metric, Killing tensors of rank 2 using theorem (6.1). We have examined metrics from *Exact Solutions to Einstein's Field Equations* (Stephani et al., 2003) and from *The Large Scale Structure of Space-Time* (Hawkings and Ellis, 1973). We will list the examined metrics by chapter and equation number before explaining the results.

We will begin with metrics from *Exact Solutions to Einstein's Field Equations.* From chapter 13, equation 49; from chapter 15, equations 19, 22, 23, 24, 26, 27, 29, and 30; from chapter 28, equations 21, 24, and 25. The metrics from *The Large Scale Structure of Space-Time* come from chapter 5, and are equations 21 and 29.

With only one exception, each of the Killing tensors produced using theorem (6.1) are metric reducible, the exception being the Kerr metric in equation 29 from chapter 5 of *The Large Scale Structure of Space-Time*. Explicitly, this metric is given in coordinates (t, r, θ, ϕ) as

$$g = -\frac{A - 2Mr}{A}dt^2 - \frac{4aMr\sin^2(\theta)}{A}dtd\phi + \frac{A}{a^2 - 2Mr + r^2}dr^2 + Ad\theta^2 \tag{6.53}$$

$$+\frac{\sin^2(\theta)\left(-2a^2Mr\cos^2(\theta) + 2a^2Mr + A(a^2 + r^2)\right)}{A}d\phi^2,$$

where a and M are parameters, and where $A = r^2 + a^2\cos^2(\theta)$. The null tetrad is given with k and l defined by

$$k = \frac{a^2 + r^2}{\sqrt{(2A(a^2 - 2Mr + r^2))}}\partial_t + \sqrt{\frac{a^2 - 2Mr + r^2}{2A}}\partial_r$$

$$+\frac{a}{\sqrt{2A(a^2 - 2Mr + r^2)}}\partial_\phi,$$

$$l = \frac{a^2 + r^2}{\sqrt{(2A(a^2 - 2Mr + r^2))}}\partial_t - \sqrt{\frac{a^2 - 2Mr + r^2}{2A}}\partial_r$$

$$+\frac{a}{\sqrt{2A(a^2 - 2Mr + r^2)}}\partial_\phi,$$

and with m defined as

$$m = \frac{ia\sin(\theta)}{\sqrt{2A}}\partial_t + \frac{1}{\sqrt{2A}}\partial_\theta + \frac{i}{\sin(\theta)\sqrt{2A}}\partial_\phi.$$

The principle null directions are k and l. Using theorem (6.1), we recover the following Killing tensor:

$$K_{\alpha\beta} = \tag{6.54}$$

$$\frac{a^2\left(A - 2mr\cos^2(\theta)\right)}{A}dt^2 - 2\left(\frac{a\sin^2(\theta)\left(-2a^2mr\cos^2(\theta) + a^2A + r^2A\right)}{A}\right)dtd\phi$$

$$-\frac{a^2\cos^2(\theta)A}{a^2-2mr+r^2}dr^2+r^2Ad\theta^2$$

$$-\frac{\sin^2(\theta)\left(2a^4mr\cos^2(\theta)\sin^2(\theta)-a^4A\sin^2(\theta)-2a^2r^2A+a^4r^2\cos^4(\theta)-r^6\right)}{A}d\phi^2.$$

Knowing from section 4.2 that the metric given in (6.53) admits precisely two independent Killing vectors, namely ∂_t and ∂_ϕ, we can construct the reducible Killing tensors and find, using Maple, that the Killing tensor defined in equation (6.54) is indeed metric irreducible (see Appendix E).

Our conclusion for this section is that while theorem (6.1) has been verified to produce Killing tensors for several known Petrov type D vacuum solutions, it may not be an efficient tool in the search for metric irreducible Killing tensors as all but one of the Killing tensors produced were found to be metric reducible.

6.3 Rank 2 Killing tensors on Homogeneous Exact solutions

In this section, we will present the results of examining many of the homogeneous spacetimes found in chapter 12 of *Exact Solutions to Einstein's Field Equations* (Stephani et al., 2003). For each metric, we have calculated the tractor connection as well as the dimensions of $\mathbb{T}^n$ for $n = 0, 1, 2$, where applicable. We have looked for additional Killing tensors in cases where $\dim(\mathbb{T}^n)$ is greater than the number of known, linearly independent, Killing tensors.

We found that, of the 39 metrics examined, 33 of them were found to admit no metric irreducible Killing tensors. Of the remaining 6, the dimension of the corresponding reduced tractor bundle $\mathbb{T}^2$ exceeds the number of metric reducible rank 2 Killing tensors, allowing for the possibility of metric irreducible Killing tensors. Of these 6 metrics, we have been able to establish the existence of metric irreducible Killing tensors for 3, namely metric 12.12 with $\epsilon = 0$ and $\gamma = 0$ (known as metric 12.12(ii)), metric 12.13, and metric 12.37 with $C(u)^2 = u^{\frac{3}{2}} + 2u + u^{\frac{1}{2}}$. Of these three metrics, we have explicitly identified the single metric irreducible tensor associated with metric 12.12(ii). No other metric irreducible Killing tensors have been identified explicitly.

We organize our calculations based on the dimensions of the isometry group. Table 3 contains the summary of the calculations for metrics with precisely 4 Killing tensors, Table 4 contains metrics with isometry dimension 5, Table 5 contains metrics with isometry dimension 6, and Table 6 contains the metrics with isometry dimension 7. The first column of each table denotes the dimension of the space of metric reducible Killing tensors of rank 2. The next three columns indicate the dimension of $\mathbb{T}^n$ for $n = 0, 1, 2$: an entry of "x" indicates that the Maple computation was aborted, either due to computational memory constraints or due to the computation seeming to take an abnormal amount of time. A dash in any column indicates that the associated computation was not attempted.

A checkmark or an x in the "Direct pdsolve" column indicates that the Killing equation was solved (resp. unable to be solved) directly in Maple; these symbols are also used to indicate, in the "Tractor pdsolve" column, whether the reduced Killing equations were solved. In the final column, denoted T, we indicate the number of metric irreducible Killing tensors which our methods demonstrate exist.

For a few of the metrics we have examined admitting 6 or 7 Killing vectors, we have reproduced the known results of (Keane and Tupper, 2010). Therein, rank 2 metric irreducible Killing tensors have been considered and, in many cases, explicitly calculated, specifically for *pp*-wave spacetimes of the form

$$g = -2H(u, y, z)du^2 - 2dudv + dy^2 + dz^2. \tag{6.55}$$

The results of (Keane and Tupper, 2010) also show that metric 12.12 with $\epsilon = 1$ and $\gamma = 0$ (known as 12.12(iii)) admits no metric irreducible Killing tensors, which result we obtain in Table 5.

6.3.1 Isometry dimension 4

We begin by examining a few metrics from chapter 12 of (Stephani et al., 2003) admitting precisely four, previously known Killing vectors. From the Killing vectors, we generate $\frac{4 \cdot 5}{2} = 10$ Killing tensors, which combine with the metric itself to produce 11 independent Killing tensors: in each case, the metric is irreducible.

A summary of our results is given in Table 3. We note that for metric 12.30, we have made choices for the constants appearing in the metric, which choices are $A = 1$, $B = 2$, and $F = 3$. For these metrics, we conclude that no Killing tensors of rank 2 beyond that of the reducible Killing tensors and the metric itself can exist.

	Reducibles and metric	$\dim(\mathbb{T}^0)$	$\dim(\mathbb{T}^1)$	$\dim(\mathbb{T}^2)$	Direct pdsolve	Tractor pdsolve	T
12.21	11	15	11	-	x	✓	0
12.14	11	16	11	-	x	✓	0
12.30	11	13	11	-	x	✓	0
12.35	11	18	11	-	✓	✓	0

Table 3: Metrics from chapter 12 with isometry dimension 4.

6.3.2 Isometry dimension 5

We now turn our attention to metrics from chapter 12 of (Stephani et al., 2003) which admit precisely five Killing vectors, which are previously known. These metrics are 12.26, 12.34, 12.36, and 12.38. Four variations of the metric 12.38 are examined. We denote 12.38 with $k = 1$ as 12.38(ii); 12.38 with $\epsilon = 1$ and $k = 2$ as 12.38(iii); 12.38 with $\epsilon = -1$ and $k = 2$ as 12.38(iv); 12.38 with $k = 3/2$ as 12.38(v).

We can build $\frac{5 \cdot 6}{2} = 15$ Killing tensors from products of rank 1 Killing tensors. For metric 12.26, the metric is reducible, and in all other cases, the metric is irreducible: therefore, we can build 15 or 16 independent, metric reducible Killing tensors, respectively.

We find that in each case, the Killing equations can be dealt with directly, without the need of the tractor construction. For these cases, we verify that the number of independent Killing tensors is no more than the dimension of metric reducible Killing tensors, which is also the number of Killing tensors we find by solving the Killing equations directly. A summary is made in Table 4.

6.3.3 Isometry dimension 6

We now turn our attention to metrics from chapter 12 (Stephani et al., 2003) which admit precisely 6 Killing vectors. These metrics are 12.6, 12.18, 12.19, 12.8, 12.16, 12.37 (generically), 12.9, 12.12, and 12.13.

We examine eight variations of the metric 12.8. 12.8(i) has $\Sigma(x, k) = \sin(x)$

	Reducibles and metric	dim($\mathbb{T}^0$)	dim($\mathbb{T}^1$)	dim($\mathbb{T}^2$)	Direct pdsolve	Tractor pdsolve	T
12.26	15	23	15	-	✓	✓	0
12.34	16	26	16	-	✓	✓	0
12.36	16	33	17	16	✓	✓	0
12.38(ii)	16	31	17	16	✓	✓	0
12.38(iii)	16	26	16	-	✓	✓	0
12.38(iv)	16	26	16	-	✓	✓	0
12.38(v)	16	26	16	-	✓	✓	0

Table 4: Metrics from chapter 12 with isometry dimension 5.

and $\Sigma(z,k) = \sin(z)$; 12.8(ii) has $\Sigma(x,k) = \sin(x)$ and $\Sigma(z,k) = z$; 12.8(iii) has $\Sigma(x,k) = \sin(x)$ and $\Sigma(z,k) = \sinh(z)$; 12.8(iv) has $\Sigma(x,k) = x$ and $\Sigma(z,k) = \sin(z)$; 12.8(v) has $\Sigma(x,k) = x$ and $\Sigma(z,k) = \sinh(z)$; 12.8(vi) has $\Sigma(x,k) = \sinh(x)$ and $\Sigma(z,k) = \sin(z)$; 12.8(vii) has $\Sigma(x,k) = \sinh(x)$ and $\Sigma(z,k) = z$; 12.8(viii) has $\Sigma(x,k) = \sinh(x)$ and $\Sigma(z,k) = \sinh(z)$.

We also examine three variations of the metric 12.9. 12.9(i) has $\Sigma(r,k) = \sin(r)$; 12.9(ii) has $\Sigma(r,k) = r$; 12.9(iii) has $\Sigma(r,k) = \sinh(r)$. For these metrics, we make no restrictions on the function $a(t)$ at this time, though it should be noted that it is possible to obtain different results for different choices of $a(t)$. 12.12(iii) is 12.12 with $\varepsilon = 1$ and $\gamma = 0$.

For metrics with isometry dimension 6, we can generate a maximum of $\frac{6 \cdot 7}{2} + 1 = 22$ independent Killing tensors. However, it appears that, with the exceptions of 12.6 and 12.12(iii), only 21 of these Killing tensors are independent.

Two of our candidates for admitting metric irreducible Killing tensors have isometry dimension 6, namely the metrics 12.37(i) and 12.13. 12.37(i) is 12.37 where $C(u)$ is treated as an arbitrary function: as such, our findings concerning this metric are rather inconclusive and results may vary greatly depending on the function $C(u)$[4]. Thus, it is recommended that a particular $C(u)$ be chosen for future study.[5]

[4]in fact, the number of Killing vectors themselves appears to depend on the choice of the function $C(u)$.

[5]Metric 12.37 is a special case of metric 12.7, though 12.7 is not examined in this thesis.

	Reducibles and metric	$\dim(\mathbb{T}^0)$	$\dim(\mathbb{T}^1)$	$\dim(\mathbb{T}^2)$	Direct pdsolve	Tractor pdsolve	T
12.6	22	26	22	-	✓	✓	0
12.18	21	34	22	21	✓	✓	0
12.19	21	34	22	21	✓	✓	0
12.8(i)	21	34	22	21	x	-	0
12.8(ii)	21	36	24	21	x	-	0
12.8(iii)	21	34	22	21	x	-	0
12.8(iv)	21	36	24	21	x	-	0
12.8(v)	21	36	24	21	x	-	0
12.8(vi)	21	34	22	21	x	-	0
12.8(vii)	21	36	24	21	x	-	0
12.8(viii)	21	34	22	21	x	-	0
12.16	21	34	22	21	x	-	0
12.37(i)	21	35	28	24	-	-	
12.9(i)	21	31	21	-	x	-	0
12.9(ii)	21	31	21	-	x	-	0
12.9(iii)	21	31	21	-	x	-	0
12.12(iii)	22	31	22	-	x	-	0
12.13	21	31	22	22	x	x	1

Table 5: Metrics from chapter 12 with isometry dimension 6.

At the time of writing, we are unable to explicitly identify the metric irreducible Killing tensors for metric 12.13, though our methods demonstrate that precisely one exists, given that $\dim(\mathbb{T})^1 = \dim(\mathbb{T})^2 = 22$, whereas the dimension of the space of metric reducible Killing tensors is 21. We find that the remaining metrics in Table 5 cannot admit metric irreducible Killing tensors.

6.3.4 Isometry dimension 7

Finally, we examine several metrics with isometry dimension 7. Metric 12.9 was examined previously, but if we now choose $a(t)$ to be a constant, we find that 12.9(i) and 12.9(iii) admit seven Killing vectors. 12.9(ii) admits 10 Killing vectors if $a(t)$ is constant, and is thus a space of constant curvature. 12.12(ii) is 12.12 with $\varepsilon = 0$ and $\gamma = 0$, and 12.12(iv) is equation 12.12 with $\varepsilon = 1$ and $a = 0$. We also examine a few variations of the metric 12.37. 12.37(iii) is 12.37 with $C(u) = \sinh(\sqrt{-2b}u)$; 12.37(iv) has $C(u) = \sin(\sqrt{2}bu)$; 12.37(v) has $C(u)^2 = u^{\frac{3}{2}} + 2u + u^{\frac{1}{2}}$; 12.37(vii) has $C(u) = \sqrt{u}$; 12.37(viii) has $C(u) = \sqrt{u}\ln(u)$; 12.37(ix) has $C(u) = \sqrt{u}\sin(c\ln(u))$.

The metric 12.12(ii) is given as

$$-2b^2\zeta\bar{\zeta}du^2 - 2dudv + 2d\zeta d\bar{\zeta},$$

and we have identified the following Killing tensor as being the single metric irreducible Killing tensor of this metric:

$$-\left(b^2\zeta\bar{\zeta}u - v\right)du^2 - ududv - \bar{\zeta}dud\zeta + ud\zeta d\bar{\zeta}.$$

We find that other metrics of isometry dimension 7 may admit metric irreducible Killing tensors, though at the time of writing we are unable to identify them explicitly. These metrics are 12.37(iii) and 12.37(iv). Metric 12.37(v) is shown to admit precisely 6 metric irreducible Killing tensors, since the dimension of $\mathbb{T}^1$ is that of $\mathbb{T}^2$, which is 6 greater than the dimension of the space of metric reducible Killing tensors.

	Reducibles and metric	$\dim(\mathbb{T}^0)$	$\dim(\mathbb{T}^1)$	$\dim(\mathbb{T}^2)$	Direct pdsolve	Tractor pdsolve	T
12.9(i)	27	37	28	27	x	-	0
12.9(iii)	27	37	28	27	✓	✓	0
12.12(ii)	27	37	29	28	✓	✓	1
12.12(iv)	28	35	29	28	✓	✓	0
12.37(iii)	27	37	29	28	x	x	
12.37(iv)	27	37	29	28	x	x	
12.37(v)	28	35	34	34	x	x	6
12.37(vii)	28	35	29	28	x	-	0
12.37(viii)	28	35	29	28	✓	✓	0
12.37(ix)	28	35	29	28	x	-	0

Table 6: Metrics from chapter 12 with isometry dimension 7.

We now point to results we have recovered from (Keane and Tupper, 2010). First, we have recovered the result that metric 12.12(iv) does not (generically) admit metric irreducible Killing tensors, which result is evident in Table 6. We have also confirmed the result that metric 12.12(iii) does not admit metric irreducible Killing tensors, which result is evident in Table 5.

We note that our discovery of the metric irreducible Killing tensor of 12.12(ii) confirms the result that a metric in the form of equation (6.55) with $2H = ay^2 + byz + cz^2$ admits an irreducible Killing tensor, where a, b, and c are constants: indeed metric 12.12(ii) takes this form under a convenient coordinate transformation.

Our conclusion for this section is that the tractor approach is useful for determining an upper bound for the number of independent Killing tensors for at least certain exact solutions. This upper bound is useful in the search for metrics which admit metric irreducible Killing tensors, since a lower bound can be obtained from known Killing vectors. There are even a number of metrics for which the tractor approach can be used to simplify the Killing equations themselves and allow one to obtain the Killing tensors explicitly, where solving the Killing equations without such simplification may be less practical.

6.4 Killing tensors in dimension 2

Killing tensors of rank 2, though useful in the context of general relativity, have also been considered exclusively in dimension 2 (G. Thompson, 1999). In fact, there are many examples of metrics in the plane which are known to admit irreducible Killing tensors (Darboux, 1972). In this section, we will apply the tractor connection to metrics in the plane in order to derive differential conditions from which the number of Killing tensors may be inferred. It is expected that the derivation of such conditions is more simple in dimension 2 than in dimension 4, since the dimension of the fibers of the tractor bundle is 6 instead of 50.

In section 3 of this thesis, we reestablished the result that a plane metric admits either 3, 1, or 0 Killing vectors (Kruglikov, 2008). By theorem (2.2), a plane metric has 3 Killing vectors when the space is one of constant curvature. Accordingly, we will examine the cases in which the metric admits either 1 or 0 Killing vectors.

When there is a single Killing vector, the result is given by the Darboux-Koening theorem (Kruglikov, 2008). In section 6.4.1, we give a partial proof of the Darboux-Koening theorem using the tractor approach, arriving at differential conditions which guarantee the existence of precisely 4 Killing tensors of rank 2. In section 6.4.2, we give a proof of the Darboux-Koening theorem using a more conventional approach–that is, dealing with the Killing equations directly.

The case of no Killing vectors does not seem to be resolved in existing literature. In section 6.4.3, we prove that a plane metric with no Killing vectors has a maximum of 3 Killing tensors of rank 2. Examples of metrics with 3, 2, and 1 Killing tensor(s) are explicitly given.

6.4.1 One Killing vector: the tractor approach

We begin with a normal form for metrics in the plane with a single Killing vector.

Lemma 6.2. *If g is a metric in the plane which admits a Killing vector X, then there exist coordinates (u, v) such that $X = \partial_v$ and*

$$g = \lambda du^2 + \lambda dv^2$$

for some nonzero function $\lambda = \lambda(u)$.

Proof (Sketch). Since X is a Killing vector, there exist coordinates (x, y) such that

$$X = m(x, y)\partial_x + n(x, y)\partial_y.$$

We now apply a change of coordinates $u = A(x, y)$, $v = B(u, v)$, where $mA_x + nB_y = 0$ and $mB_x + nA_y = 1$. This gives us

$$X(A) = mA_x + nB_y = 0 = X(u), \qquad X(B) = mB_x + nA_y = 1 = X(v),$$

and so $X = \partial_v$ in (u, v) coordinates. This in turn implies that the components of the metric do not depend on v. Subsequently, it can be shown that the metric can be written as in the statement of the lemma.

□

Accordingly, we will study metrics in this form. Also of interest is the following theorem, which is referred to as the Darboux-Koening theorem (Kruglikov, 2008):

Theorem 6.2. *Let g be a metric in the plane which admits a single Killing vector. The metric g admits precisely 4 Killing tensors of rank 2 if and only if it admits a Killing tensor of rank 2 which is not algebraically generated by the (covariant) Killing vector and the metric itself.*

In this section, we will use the tractor approach to provide a partial proof of this theorem. In the next section, we will provide a proof of this theorem using a more conventional approach.

For $n = 2$, we note that with respect to the general equations (6.9), (6.21), and (6.39), the tensor K has 3 independent components, the tensor L has two independent components (namely L_{211} and L_{212}), and the tensor M has a single independent component (namely M_{2112}), having the same symmetries as the Riemann curvature tensor. Thus, the equations which define the tractor connection are given as

$$\begin{cases} K_{11;1} = 0 & (6.56a) \\ K_{11;2} - \frac{2}{3}L_{211} = 0 & (6.56b) \\ K_{12;1} + \frac{1}{3}L_{211} = 0 & (6.56c) \\ K_{12;2} - \frac{1}{3}L_{212} = 0 & (6.56d) \\ K_{22;1} + \frac{2}{3}L_{212} = 0 & (6.56e) \\ K_{22;2} = 0 & (6.56f) \\ L_{211;1} - 3k\lambda K_{12} = 0 & (6.56g) \\ L_{211;2} + \frac{3k\lambda}{2}(K_{11} - K_{22}) + M_{2112} = 0 & (6.56h) \\ L_{212;1} + \frac{3k\lambda}{2}(K_{11} - K_{22}) - M_{2112} = 0 & (6.56i) \\ L_{212;2} + 3k\lambda K_{12} = 0 & (6.56j) \\ M_{2112;1} + \frac{3k'\lambda}{2}(K_{11} - K_{22}) + 3k\lambda L_{212} = 0 & (6.56k) \\ M_{2112;2} + 3k'\lambda K_{12} - 3k\lambda L_{211} = 0, & (6.56l) \end{cases}$$

where $k = \dfrac{\lambda''\lambda - (\lambda')^2}{2\lambda^3}$ is the sectional curvature of g expressible in terms of the single independent component of the curvature tensor. After writing out the covariant derivatives in terms of the partial derivatives and the Christoffel symbols, we can construct the matrices which define the tractor connection. The columns of the matrix are associated with the unknown functions in (6.56) as follows:

$$\begin{bmatrix} K_{11} & K_{12} & K_{22} & L_{211} & L_{212} & M_{2112} \end{bmatrix}.$$

The matrices which define the tractor connection are given explicitly as

$$\tilde{\Gamma}^i{}_{j1} = \begin{bmatrix} -\frac{\lambda'}{\lambda} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\lambda'}{\lambda} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{\lambda'}{\lambda} & 0 & \frac{2}{3} & 0 \\ 0 & -3k\lambda & 0 & -\frac{3\lambda'}{2\lambda} & 0 & 0 \\ \frac{3k\lambda}{2} & 0 & -\frac{3k\lambda}{2} & 0 & -\frac{3\lambda'}{2\lambda} & -1 \\ \frac{3k'\lambda}{2} & 0 & -\frac{3k'\lambda}{2} & 0 & 3k\lambda & -\frac{2\lambda'}{\lambda} \end{bmatrix},$$

$$\tilde{\Gamma}^i{}_{j2} = \begin{bmatrix} 0 & -\frac{\lambda'}{\lambda} & 0 & -\frac{2}{3} & 0 & 0 \\ \frac{\lambda'}{2\lambda} & 0 & -\frac{\lambda'}{\lambda} & 0 & -\frac{1}{3} & 0 \\ 0 & \frac{\lambda'}{\lambda} & 0 & 0 & 0 & 0 \\ \frac{3k\lambda}{2} & 0 & -\frac{3k\lambda}{2} & 0 & -\frac{\lambda'}{2\lambda} & 1 \\ 0 & 3k\lambda & 0 & \frac{\lambda'}{2\lambda} & 0 & 0 \\ 0 & 3k'\lambda & 0 & -3k\lambda & 0 & 0 \end{bmatrix}.$$

We find the curvature matrix to be given as

$$\tilde{K}^{i}{}_{j} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\lambda k'' - 3k'\lambda' & 0 & -5k'\lambda & 0 & 0 \end{bmatrix}.$$

Consequently, if k is constant, then $\tilde{K}^{i}{}_{j} = 0$ and the metric g admits precisely 6 Killing tensors of rank 2. If $k' \neq 0$, the rank of the curvature matrix is 1, which implies that g admits no more than 5 rank 2 Killing tensors. Having obtained an upper bound at curvature order 0, we now endeavor to obtain an upper bound at curvature order 1. In order to count the number of basis elements for the local sections of $\mathbb{T}^1$, we will need to find the rank of the stacked matrix

$$\begin{bmatrix} \tilde{K}^{i}{}_{j} \\ \tilde{\nabla}_{\partial_u} \tilde{K}^{i}{}_{j} \\ \tilde{\nabla}_{\partial_v} \tilde{K}^{i}{}_{j} \end{bmatrix}.$$

Removing rows of zeros and duplicate rows, the stacked matrix becomes

$$A = \begin{bmatrix} 0 & 3\lambda k'' - 3k'\lambda' & 0 & -5k'\lambda & 0 & 0 \\ 0 & P_1 & 0 & -6\lambda k'' - \dfrac{3\lambda' k'}{2} & 0 & 0 \\ P & 0 & -P & 0 & \lambda k'' - \dfrac{7\lambda' k'}{2} & 5k'\lambda \end{bmatrix},$$

where

$$P = \frac{3\left(5kk'\lambda^3 + k'(\lambda')^2 - k''\lambda\lambda'\right)}{2\lambda}$$

and

$$P_1 = -9kk'\lambda^2 + 3\lambda k''' - 3\lambda' k''.$$

Lemma 6.3. *Let Q_1 be defined as*

$$Q_1 = -45k(k')^2\lambda^3 + 15k'''k'\lambda^2 + \frac{9(k')^2(\lambda')^2 - 3k'k''\lambda\lambda'}{2} - 18(k'')^2\lambda^2.$$

The rank of A is 3 if and only if $Q_1 \neq 0$. The rank of A is 2 if and only if $Q_1 = 0$.

Proof. The third row of the matrix A gives us a pivot, since $A_{36} = 5k'\lambda$, which is nonzero by assumption. The number of pivots we get from the first and second rows of A is controlled by the number of pivots in the matrix

$$\tilde{A} = \begin{bmatrix} A_{12} & A_{14} \\ A_{22} & A_{24} \end{bmatrix} = \begin{bmatrix} 3\lambda k'' - 3k'\lambda' & -5k'\lambda \\ P_1 & -6\lambda k'' - \dfrac{3\lambda' k'}{2} \end{bmatrix}.$$

Thus, $\text{Rank}(A) = 1 + \text{Rank}(\tilde{A})$. The matrix $\tilde{A}$ is not identically zero, since $\tilde{A}_{12} = -5k'\lambda \neq 0$: thus, the rank of $\tilde{A}$ is at least 1. The rank of $\tilde{A}$ is 2 if and only if $\det(\tilde{A}) \neq 0$. The conclusion follows since $\det(\tilde{A}) = Q_1$. □

If the rank of A is 3, the metric admits no more than 3 Killing tensors of rank 2. If the rank of A is 2, the metric admits no more than 4 Killing tensors of rank 2.

We now wish to consider the following stacked matrix, the nullspace of which is representative of $\mathbb{T}^2$:

$$B = \begin{bmatrix} \tilde{K}^i{}_j \\ \tilde{\nabla}_{\partial_u}\tilde{K}^i{}_j \\ \tilde{\nabla}_{\partial_v}\tilde{K}^i{}_j \\ \tilde{\nabla}_{\partial_u}\tilde{\nabla}_{\partial_u}\tilde{K}^i{}_j \\ \tilde{\nabla}_{\partial_u}\tilde{\nabla}_{\partial_v}\tilde{K}^i{}_j \\ \tilde{\nabla}_{\partial_v}\tilde{\nabla}_{\partial_u}\tilde{K}^i{}_j \\ \tilde{\nabla}_{\partial_v}\tilde{\nabla}_{\partial_v}\tilde{K}^i{}_j \end{bmatrix}.$$

Removing rows of zeros and duplicate rows, we have the matrix

$$\tilde{B} = \begin{bmatrix} 0 & 3\lambda k'' - 3k'\lambda' & 0 & -5k'\lambda & 0 & 0 \\ 0 & P_1 & 0 & -6\lambda k'' - \dfrac{3\lambda' k'}{2} & 0 & 0 \\ P & 0 & -P & 0 & \lambda k'' - \dfrac{7\lambda' k'}{2} & 5k'\lambda \\ 0 & P_2 & 0 & 12\lambda k'' + \dfrac{11k'\lambda'}{2} & 0 & 0 \\ 0 & P_3 & 0 & P_4 & 0 & 0 \\ 0 & P_5 & 0 & -6\lambda k'' - 4k'\lambda' & 0 & 0 \\ P_6 & 0 & -P_6 & 0 & P_7 & 6\lambda k'' + \dfrac{3k'\lambda'}{2} \\ 0 & P_8 & 0 & P_9 & 0 & 0 \end{bmatrix},$$

where

$$P_2 = \frac{3\left(12\lambda^3 k'k - 4\lambda^2 k''' + 3k''\lambda\lambda' + k'(\lambda')^2\right)}{2\lambda},$$

$$P_3 = -30\lambda^2 kk'' - 9\lambda^2 (k')^2 - \frac{9\lambda kk'\lambda'}{2} + 3\lambda k'''' - 3\lambda' k''',$$

$$P_4 = \frac{84\lambda^3 k'k - 28\lambda^2 k''' - 14k''\lambda\lambda' - 3k'(\lambda')^2}{4\lambda},$$

$$P_5 = -\frac{3\left(6\lambda^3 k'k - 2\lambda^2 k''' + k''\lambda\lambda' + k'(\lambda')^2\right)}{2\lambda},$$

$$P_6 = \frac{3\left(12\lambda^3 kk'' + 9\lambda^2 kk'\lambda' - 2\lambda\lambda' k''' + k''(\lambda')^2\right)}{4\lambda},$$

$$P_7 = \frac{-12\lambda^3 k'k + 4\lambda^2 k''' - 16k''\lambda\lambda' - 3k'(\lambda')^2}{4\lambda},$$

$$P_8 = -3\frac{8k''\lambda^4 k + 10\lambda^4 (k')^2 - 23\lambda^3 k'\lambda' k + 2k''\lambda(\lambda')^2 - 2k'(\lambda')^3}{2\lambda^2},$$

and

$$P_9 = \frac{140\lambda^3 k'k - 6k''\lambda\lambda' + 11k'(\lambda')^2}{4\lambda}.$$

Lemma 6.4. *If the rank of A is 3, then the rank of $\tilde{B}$ is 4.*

Proof. The first three rows of $\tilde{B}$ are the three rows of A, and so the first three rows of $\tilde{B}$ yield three pivots. The only other row of $\tilde{B}$ which can yield a pivot is row 7. We consider the matrix which consists of the fifth and sixth columns of the third and seventh rows of $\tilde{B}$:

$$C = \begin{bmatrix} \lambda k'' - \dfrac{7\lambda' k'}{2} & 5k'\lambda \\ P_7 & 6\lambda k'' + \dfrac{3k'\lambda'}{2} \end{bmatrix}.$$

The determinant of C is $3Q_1$. Thus, the seventh row of $\tilde{B}$ yields another pivot if and only if $Q_1 \neq 0$, and so the conclusion follows from lemma 6.3. □

This allows us to make the following observation.

Corollary 6.2.1. *If $Q_1 \neq 0$, then the metric g admits precisely 2 Killing tensors of rank 2, namely the metric itself and the square of the single (covariant) Killing vector.*

Proof. $Q_1 \neq 0$ implies that $\mathrm{Rank}(\tilde{B}) = 4$ by lemma 6.3 and lemma 6.4, so that the metric g admits no more than 2 Killing tensors of rank 2. However, g admits at least two Killing tensors of rank 2, namely $\lambda^2 dv^2$ and the metric itself. □

Proposition 6.1. *Let*

$$Q_2 = 60kk''\lambda^4 k' + 90k\lambda^3\lambda'(k')^2 - 100\lambda^4(k')^3 - 6\lambda^2\lambda'(k'')^2 - 3\lambda(\lambda')^2 k'k'' + 9(\lambda')^3(k')^2.$$

If $Q_1 = 0$ and $Q_2 = 0$, the rank of $\tilde{B}$ is 2.

Proof. Since $Q_1 = 0$, there is precisely one pivot between the third and seventh rows of $\tilde{B}$ by the proof of lemma 6.4. Let B_2, B_4, B_5, B_6, and B_8 denote the 2×2 matrices where the first rows consist of the first two non-zero entries in the first row of $\tilde{B}$ and where the second rows are constructed from the first two non-zero entries of rows 2, 4, 5, 6, and 8 of $\tilde{B}$, respectively. Explicitly, these matrices are given as

$$B_2 = \begin{bmatrix} 3\lambda k'' - 3k'\lambda' & -5k'\lambda \\ P_1 & -6\lambda k'' - \dfrac{3\lambda' k'}{2} \end{bmatrix} = \tilde{A},$$

$$B_4 = \begin{bmatrix} 3\lambda k'' - 3k'\lambda' & -5k'\lambda \\ P_2 & 12\lambda k'' + \dfrac{11\lambda' k'}{2} \end{bmatrix} \qquad B_5 = \begin{bmatrix} 3\lambda k'' - 3k'\lambda' & -5k'\lambda \\ P_3 & P_4 \end{bmatrix},$$

$$B_6 = \begin{bmatrix} 3\lambda k'' - 3k'\lambda' & -5k'\lambda \\ P_5 & -6\lambda k'' - 4\lambda' k' \end{bmatrix}, \qquad B_8 = \begin{bmatrix} 3\lambda k'' - 3k'\lambda' & -5k'\lambda \\ P_8 & P_9 \end{bmatrix}.$$

It can be shown that if $Q_1 = 0$, the determinants of B_2, B_4, B_5, and B_6 are identically zero. It can also be shown that the determinant of B_8 is identically Q_2. Thus, if $Q_1 = 0$ and $Q_2 = 0$, there is precisely one pivot among rows 1, 2, 4, 5, 6, and 8 of $\tilde{B}$. Thus, there are two pivots of $\tilde{B}$, and the rank of $\tilde{B}$ is 2. □

Corollary 6.2.2. *If $Q_1 = 0$ and $Q_2 = 0$, then the metric g admits precisely 4 Killing tensors of rank 2.*

Proof. Since the ranks of A and $\tilde{B}$ are both 2 by assumption, $\dim(\mathbb{T}^1) = \dim(\mathbb{T}^2) = 4$, and so there are precisely 4 Killing tensors of rank 2 by theorem 3.1. □

Thus, we have established necessary and sufficient conditions for the existence of precisely 4 Killing tensors. However, the tractor approach has, so far, only led us to a partial proof of the Darboux-Koening theorem as it is not clear that the conditions $Q_1 = 0$ and $Q_2 = 0$ are degenerate. The remainder of this section will be devoted to showing that if $Q_2 = 0$, $Q_1 = 0$.

Let f_1, f_2, and h be defined as follows:

$$f_1 = \frac{3\left(-20\lambda^3 k' k + (\lambda')^2 k' + 4k''\lambda'\lambda\right)}{20\lambda^2 k'}, \qquad f_2 = \frac{3}{10\lambda k'}, \qquad h = 7k'\lambda' + 8\lambda k''.$$

It can be shown that

$$(Q_2)' - hf_2Q_2 - f_1Q_1 = 0. \tag{6.57}$$

Lemma 6.5. *If $k' \neq 0$, The conditions $Q_1 = 0$ and $h = 0$ are incompatible.*

Proof. Assume that $k' \neq 0$, $h = 0$ and $Q_1 = 0$. The assumption that $h = 0$ allows us to write

$$k'' = -\frac{7k'\lambda'}{8\lambda}. \tag{6.58}$$

The expression above can be substituted into the equation $Q_1 = 0$, which expression can be simplified to the following:

$$\frac{15(k')^2\left(-192k\lambda^3 + 71(\lambda')^2 - 56\lambda''\lambda\right)}{64} = 0,$$

which implies, since $k' \neq 0$, that

$$-192k\lambda^3 + 71(\lambda')^2 - 56\lambda''\lambda = 0. \tag{6.59}$$

First of all, equation (6.59) implies that

$$k = \frac{71(\lambda')^2 - 56\lambda''\lambda}{192\lambda^3}. \tag{6.60}$$

Secondly, the expression for k in terms of λ can be applied to equation (6.59), leaving us with

$$\frac{75\lambda''\lambda}{8} - \frac{375(\lambda')^2}{64} = 0,$$

which implies that

$$\lambda'' = \frac{5(\lambda')^2}{8\lambda}. \tag{6.61}$$

Meanwhile, the assumption that $Q_1 = 0$ and that $h = 0$ implies, in light of equation (6.57), that $(Q_2)' = 0$. Applying equations (6.60) and (6.61), we find that

$$Q_2 = \frac{297,675}{262,144}\left(\frac{\lambda'}{\lambda}\right)^9. \tag{6.62}$$

Thus,

$$(Q_2)' = 0 = \frac{2,679,075}{262,144}\frac{(\lambda')^8}{\lambda^{10}}\left((\lambda')^2 - \lambda''\lambda\right). \tag{6.63}$$

Equation (6.63) is satisfied if and only if $(\lambda')^2 - \lambda'' = 0$, which occurs if and only if $k = 0$: thus, assuming that $k' \neq 0$, we cannot have both $Q_1 = 0$ and $h = 0$. □

Lemma 6.6. *Suppose that $k' \neq 0$. If $Q_2 = 0$, then $Q_1 = 0$.*

Proof. If $Q_2 = 0$, equation (6.57) becomes

$$-f_1 Q_1 = 0. \tag{6.64}$$

Now suppose that $Q_1 \neq 0$. By equation (6.64), we must have $f_1 = 0$. Additionally, by lemma 6.5, $h = 0$. Thus, equation (6.58) can be applied to the equation $f_1 = 0$, resulting in the following:

$$\frac{3\left(8k\lambda^3 + (\lambda')^2\right)}{8\lambda^2} = 0. \tag{6.65}$$

This implies that

$$k = -\frac{(\lambda')^2}{8\lambda^3}, \tag{6.66}$$

and, after writing k in terms of λ,

$$\lambda'' = \frac{5(\lambda')^2}{4\lambda}. \tag{6.67}$$

However, applying equations (6.66) and (6.67) into the equation $Q_2 = 0$ results in the following:

$$-\frac{75(\lambda')^9}{4096\lambda^9} = 0, \tag{6.68}$$

thus requiring that $\lambda' = 0$ and thereby contradicting the assumption that $k' \neq 0$. Thus, if $k' \neq 0$ and $Q_2 = 0$, $Q_1 = 0$. □

Corollary 6.2.3. *If $Q_2 = 0$, the metric g admits precisely 4 Killing tensors of rank 2.*

Proof. The metric g has the property that $k' \neq 0$ since g admits a single Killing vector. Thus, if $Q_2 = 0$, $Q_1 = 0$ by lemma 6.6. The conclusion follows from corollary 6.2.1. □

6.4.2 One Killing vector: a conventional approach

We now wish to compare the results of the tractor approach in the case of a single Killing vector to that of a more conventional approach. In doing so, we will prove the Darboux-Koening theorem. Let the symmetric tensor T be defined by

$$T = pdu^2 + qdudv + rdv^2,$$

where $p = p(u, v)$, $q = q(u, v)$, and $r = r(u, v)$ are smooth functions. Applying $T_{(\alpha\beta;\gamma)} = 0$ results in the following system of equations:

$$\begin{cases} r_v + \dfrac{\lambda' q}{2\lambda} = 0 & (6.69a) \\ \dfrac{p_v}{3} - \dfrac{\lambda' q}{2\lambda} + \dfrac{q_u}{3} = 0 & (6.69b) \\ \dfrac{q_v\lambda + r_u\lambda - 2\lambda' r + \lambda' p}{3\lambda} = 0 & (6.69c) \\ p_u - \dfrac{\lambda' p}{\lambda} = 0. & (6.69d) \end{cases}$$

Proposition 6.2.

The assumption that $q \neq 0$ is equivalent to the assumption that T is a metric irreducible, rank 2 Killing tensor.

Proof. Let us assume that $q = 0$, so that

$$T = pdu^2 + rdv^2.$$

Applying the Killing equation to T, we get the following system of equations.

$$\begin{cases} r_u = -\dfrac{\lambda'(p - 2r)}{\lambda} & (6.70a) \\ r_v = 0 & (6.70b) \\ p_v = 0 & (6.70c) \\ p_u = \dfrac{\lambda' p}{\lambda}. & (6.70d) \end{cases}$$

This system can be solved directly for p and r, the solution of which is $p = c_1\lambda$, $r = c_1\lambda + c_2\lambda^2$. Thus, if $q = 0$, T is a linear combination of the metric and the reducible Killing tensor $\lambda^2 dv^2$. It is evident that if T is such a linear combination, then $q = 0$. □

Proposition 6.3.

If T is a metric irreducible Killing tensor, then there is a constant d such that λ satisfies the following equation:

$$4\lambda^2 d^2 - 10\lambda\lambda'' d + 15(\lambda')^2 d - 3\lambda'\lambda''' + 4(\lambda'')^2 = 0. \tag{6.71}$$

Proof. Equation (6.69d) can be solved to obtain $p = \delta\lambda$, where δ is a function of v. We will now rewrite (6.69) with this substitution.

$$\begin{cases} r_v = \dfrac{\lambda' q}{2\lambda} & \text{(6.72a)} \\ \dfrac{\delta'\lambda}{3} - \dfrac{\lambda' q}{2\lambda} + \dfrac{q_u}{3} = 0 & \text{(6.72b)} \\ \dfrac{\delta\lambda'\lambda + r_u\lambda - 2\lambda' r + q_v\lambda}{3\lambda} = 0. & \text{(6.72c)} \end{cases}$$

We can algebraically solve for δ in equation (6.72c), then differentiate with respect to v. The result, combined with equation (6.72a), is

$$\delta' = \frac{-2q_{vv}\lambda^2 + \lambda'' q\lambda - 3(\lambda')^2 q + \lambda' q_u \lambda}{2\lambda^2\lambda'}. \tag{6.73}$$

On the other hand, we know δ' directly from equation (6.72b):

$$\delta' = \frac{3\lambda' q - 2q_u\lambda}{2\lambda^2}. \tag{6.74}$$

Thus, subtracting (6.73) from (6.74) and multiplying by $2\lambda^2$, we get

$$2q_{vv}\lambda^2 - q(\lambda''\lambda - 6(\lambda')^2) - 3\lambda' q_u\lambda = 0. \tag{6.75}$$

However, we can also differentiate equation (6.74) with respect to u to obtain, after multiplying by $2\lambda^3$,

$$-2q_{uu}\lambda^2 + q(3\lambda''\lambda - 6(\lambda')^2) + 5\lambda' q_u\lambda = 0. \tag{6.76}$$

Now, we will take the second derivative of (6.75) with respect to u and add this to the second derivative of (6.76). Then, we will substitute the values of q_{vv} and q_{uu}

from (6.75) and (6.76). After multiplying the result by λ^2, we obtain the following:

$$q_u = -\frac{q(\lambda''''\lambda^3 - 12\lambda'''\lambda^2\lambda' - 4(\lambda'')^2\lambda^2 + 45\lambda''\lambda(\lambda')^2 - 30(\lambda')^4)}{5\lambda(3(\lambda')^3 - 4\lambda''\lambda'\lambda + \lambda'''\lambda^2)}. \tag{6.77}$$

We note that for our metric g, the Ricci Scalar is

$$K = -\frac{\lambda''\lambda - (\lambda')^2}{\lambda^3}. \tag{6.78}$$

With this, (6.77) can be written as

$$q_u = -\frac{q(K''\lambda - 6K'\lambda')}{5\lambda K'}, \tag{6.79}$$

which motivates us to define $A = 5\lambda K'$, $B = K''\lambda - 6K'\lambda'$, and $C = B/A$. With this, we can rewrite (6.79), (6.75), and (6.76) to obtain the following system of equations:

$$\begin{cases} q_{vv} = -\dfrac{q(K\lambda^3 + 3C\lambda\lambda' + 5(\lambda')^2)}{2\lambda^2} & \text{(6.80a)} \\ q_{uu} = -\dfrac{q(3K\lambda^3 + 5C\lambda\lambda' + 3(\lambda')^2)}{2\lambda^2} & \text{(6.80b)} \\ q_u = -qC & \text{(6.80c)} \\ q_{uv} = -q_v C, & \text{(6.80d)} \end{cases}$$

where (6.80d) has been obtained by differentiating (6.80c) with respect to v. Since we are assuming that T is a metric irreducible rank 2 Killing tensor, $q \neq 0$ by proposition 6.2, and so we can rewrite (6.80c) as

$$\frac{q_u}{q} = -C, \tag{6.81}$$

or

$$\frac{\partial}{\partial v}\left(\frac{q_u}{q}\right) = 0. \tag{6.82}$$

Solving (6.82) for q, we get $q = \alpha\beta$, where α is a function of u and β is a function of v. Now, we will substitute $q = \alpha\beta$ into (6.80a):

$$\alpha\beta'' = -\frac{\alpha\beta(K\lambda^3 + 3C\lambda\lambda' + 5(\lambda')^2)}{2\lambda^2}. \tag{6.83}$$

Since we are assuming that $q \neq 0$, $\alpha \neq 0$ and $\beta \neq 0$. Thus,

$$\frac{\beta''}{\beta} = \frac{-K\lambda^3 - 3C\lambda\lambda' - 5(\lambda')^2}{2\lambda^2}. \tag{6.84}$$

Since the left hand side of equation (6.84) is a function only of v, and since the right hand side is a function only of u,

$$\frac{\beta''}{\beta} = d, \tag{6.85}$$

and

$$\frac{-K\lambda^3 - 3C\lambda\lambda' - 5(\lambda')^2}{2\lambda^2} = d \tag{6.86}$$

for some constant d. Now if we substitute $q = \alpha\beta$ into (6.80b), we get

$$\frac{\alpha''}{\alpha} = \frac{-3K\lambda^3 - 5C\lambda\lambda' - 3(\lambda')^2}{2\lambda^2}. \tag{6.87}$$

Now, we will substitute the value of K from (6.86) into (6.87):

$$\frac{\alpha''}{\alpha} = \frac{2C\lambda\lambda' + 3d\lambda^2 + 6(\lambda')^2}{\lambda^2}. \tag{6.88}$$

However, we know that $C = q_u/q = \alpha'/\alpha$, and so (6.88) can be written as

$$\frac{\alpha''}{\alpha} = \frac{3d\lambda^2\alpha + 6(\lambda')^2\alpha - 2\lambda'\alpha'\lambda}{\alpha\lambda^2}. \tag{6.89}$$

With equation (6.89) in hand, we will now backtrack to obtain a solution for α, which solution will subsequently be substituted into equation (6.89). Substituting $q = \alpha\beta$ into system (6.72), we obtain

$$\begin{cases} r_u = \dfrac{-\alpha\beta'\lambda - \delta\lambda'\lambda + 2\lambda' r}{\lambda} & \text{(6.90a)} \\ r_v = -\dfrac{\lambda'\alpha\beta}{2\lambda} & \text{(6.90b)} \\ \delta' = \dfrac{\beta(3\lambda'\alpha - 2\alpha'\lambda)}{2\lambda^2}. & \text{(6.90c)} \end{cases}$$

Since we are assuming that $\beta \neq 0$, equation (6.90c) can be written as

$$\frac{\delta'}{\beta} = \frac{3\lambda'\alpha - 2\alpha'\lambda}{2\lambda^2}. \tag{6.91}$$

The left hand side of this equation is a function only of v, while the right hand side is a function only of u. Thus,

$$\frac{\delta'}{\beta} = c, \tag{6.92}$$

and

$$\frac{3\lambda'\alpha - 2\alpha'\lambda}{2\lambda^2} = c, \tag{6.93}$$

for some constant c. We now cross-differentiate equations (6.90a) and (6.90b): that is, we take the v-derivative of equation (6.90a) and subtract from it the u-derivative of equation (6.90b), giving us

$$\frac{-2\alpha\beta''\lambda^2 + \beta\alpha\lambda\lambda'' - 3\beta\alpha(\lambda')^2 + \beta\alpha'\lambda\lambda' - 2\delta'\lambda^2\lambda'}{2\lambda^2} = 0. \tag{6.94}$$

However, we know δ' from equation (6.90c), which we substitute into (6.94) to get

$$\frac{-2\alpha\beta''\lambda^2 + \beta(\lambda''\alpha\lambda + 3\lambda'\alpha'\lambda - 6(\lambda')^2\alpha)}{2\lambda^2} = 0. \tag{6.95}$$

Since we are assuming that α and β are nonzero, we can rewrite this as

$$\frac{\beta''}{\beta} = \frac{(\lambda''\alpha\lambda + 3\lambda'\alpha'\lambda - 6(\lambda')^2\alpha)}{2\alpha\lambda^2}. \tag{6.96}$$

Using our familiar trick, both sides of this equation are equal to a constant (d, in fact): the left hand side is only a function of v, while the right hand side is only a function of u. While the fact that the left hand side is a constant is nothing new for us, we now know that

$$\frac{\lambda''\alpha\lambda + 3\lambda'\alpha'\lambda - 6(\lambda')^2\alpha}{2\alpha\lambda^2} = d. \tag{6.97}$$

From this equation, we can obtain the following solution for α, since $\lambda' \neq 0$:

$$\alpha = c_1 e^{\int \frac{2d\lambda^2 - \lambda''\lambda + 6(\lambda')^2}{3\lambda'\lambda} du} \tag{6.98}$$

for some constant c_1, which solution we now substitute into equation (6.89). After we simplify, we are left with

$$4\lambda^2 d^2 - 10\lambda\lambda'' d + 15(\lambda')^2 d - 3\lambda'\lambda''' + 4(\lambda'')^2 = 0, \tag{6.99}$$

as in the statement of the proposition. □

Proposition 6.4.

(i) If the scalar curvature of g is not a constant, then g admits a metric irreducible, rank 2 Killing tensor if and only if there is a constant d such that equation (6.99) *is satisfied.*

(ii) If the scalar curvature of g is not a constant, then there is at most one constant

d such that equation (6.99) *is satisfied.*

(iii) If the scalar curvature of g is not a constant, then the space of rank 2 Killing tensors is at most four.

Proof. (i) Our previous work demonstrates that if T is an irreducible Killing tensor of rank 2, then there is a constant d such that equation (6.99) is satisfied. Now, we will prove that if there is a constant d such that equation (6.99) is satisfied, there is a rank two Killing tensor which is not a linear combination of the metric or the product of λdv with itself. Let $\beta(v)$ be a non-zero solution of equation (6.85). Now, recall equations (6.93) and (6.97), which can each be algebraically solved for α':

$$\alpha' = \frac{\alpha(2d\lambda^2 - \lambda''\lambda + 6(\lambda')^2)}{3\lambda'\lambda} = \frac{-2c\lambda^2 + 3\lambda'\alpha}{2\lambda}. \tag{6.100}$$

Solving algebraically for α, we find that

$$\alpha = -\frac{6c\lambda^2\lambda'}{4d\lambda^2 - 2\lambda''\lambda + 3(\lambda')^2}. \tag{6.101}$$

Of course, the reader may well be concerned that the denominator could be zero. However, it can be shown that if this is the case, the Ricci scalar of the metric is constant, and so we can assume that the denominator in equation (6.101) is nonzero. Let $\alpha(u)$ be defined as in equation (6.101), with $c = 1$, and define $q = \alpha\beta$. Since both α and β are nonzero by construction, q is nonzero, and so the tensor we are constructing will not be a linear combination of the metric and $\lambda^2 dv^2$ by proposition (6.2).

Let $p = \delta\lambda$, where δ is defined by equation (6.92), with $c = 1$. Finally, let r be defined to be

$$r = -\frac{\lambda'\alpha\delta}{2\lambda}.$$

With $T = pdu^2 + qdudv + rdv^2$, our previous work shows that T is an irreducible

Killing tensor. □

Proof. (ii) Let d_1 and d_2 be two constants such that equation (6.99) is satisfied (for the same metric). This occurs if and only if

$$4\lambda^2 d_1^2 - 10\lambda\lambda'' d_1 + 15(\lambda')^2 d_1 - 3\lambda'\lambda''' + 4(\lambda'')^2 \tag{6.102}$$

$$= 4\lambda^2 d_2^2 - 10\lambda\lambda'' d_2 + 15(\lambda')^2 d_2 - 3\lambda'\lambda''' + 4(\lambda'')^2.$$

However, equation (6.102) holds if and only if

$$(d_1 - d_2)(4d_1\lambda^2 + 4d_2\lambda^2 - 10\lambda''\lambda + 15(\lambda')^2) = 0, \tag{6.103}$$

which is true if and only if either $d_1 = d_2$ or

$$4d_1\lambda^2 + 4d_2\lambda^2 - 10\lambda''\lambda + 15(\lambda')^2 = 0. \tag{6.104}$$

However, it can be shown that if λ satisfies (6.104), then the Ricci scalar of the metric is constant. Therefore, if the metric has nonconstant scalar curvature, then there is only one constant d such that equation (6.99) is satisfied. □

Proof. (iii) By (ii), there is one constant d such that equation (6.99) is satisfied. As in the proof of (i), we can construct a Killing tensor; however, we can construct precisely two independent Killing tensors, since there are precisely two independent solutions for β in equation (6.85), and only one constant d. With the tensor $\lambda^2 dv^2$, along with the metric itself, we see that the space of rank two Killing tensors is at most 4. □

Corollary 6.2.4. *If g admits three rank 2 Killing tensors, then g admits four rank 2 Killing tensors.*

Proof. If g admits three rank 2 Killing tensors, then by our previous work, one of them is a metric irreducible Killing tensor. By proposition 6.4, there is a constant d

such that equation (6.99) is satisfied. However, there are two independent solutions for the equation $\beta'' = d\beta$, and so two (independent), metric irreducible, rank 2 Killing tensors. Thus, where the known irreducible Killing tensor will incorporate one solution to $\beta'' = d\beta$, we can construct another irreducible Killing tensor from the other solution, and so if g admits three rank 2 Killing tensors, g admits four rank 2 Killing tensors. □

The proof of the Darboux-Koening theorem is now evident. We have shown that the metric g admits precisely 4 Killing tensors of rank 2 if and only if there is a constant d such that equation (6.99) is satisfied, and that g admits precisely 2 Killing tensors of rank 2 otherwise.

We will now demonstrate the utility of equation (6.99) in two examples taken from existing literature (Darboux, 1972). First, let

$$g = u\left(du^2 + dv^2\right). \tag{6.105}$$

It can be shown that udv is the only rank 1 Killing tensor for this metric. It can also be shown that $\lambda = u$ satisfies equation (6.99) for $d = 0$: Thus, $\alpha = -2u^2$, by equation (6.101), and we can have either $\beta = v$ or $\beta = 1$. In the case where $\beta = v$, $\delta = v^2/2$, and so $p = \frac{uv^2}{2}$. We can find r from the original system of equations, since all other functions are known: $r = 2u^3 + \frac{uv^2}{2}$. In the case where $\beta = 1$, $\delta = v$, $p = uv$, and $r = uv$. Thus, the dimension of the space of rank 2 Killing tensors of the metric given in equation (6.105) is 4.

As our second example, consider the metric

$$g = \frac{a\cos(u/2) + b}{4\sin^2(u/2)}\left(du^2 + dv^2\right), \tag{6.106}$$

where a and b constants, with $a \neq 0$. It can be shown that equation (6.99) is satisfied for $d = \frac{1}{4}$. Using the formulation described above, it can be shown that the following symmetric tensors are the metric irreducible, rank 2 Killing tensors, where the minus signs are taken in the case of the second:

$$\pm\frac{e^{\pm v/2}(a\cos(u/2)+b)}{2\sin^2(u/2)}du^2+\frac{e^{\pm v/2}(a\cos(u/2)+b)}{a\sin^3(u/2)}dudv$$
$$\pm\frac{e^{\pm v/2}(a\cos^2(u/2)+2b\cos(u/2)+a)(a\cos(u/2)+b)}{2a\sin^4(u/2)}dv^2.$$

We will conclude this section with a short comparison of the tractor approach with the conventional approach. With the tractor approach, there was no need to explicitly solve any differential equations, whereas the conventional approach ultimately resorted to doing so. With the tractor approach, we offer two equations which, combined with the condition that the sectional curvature is not a constant, constitute necessary and sufficient conditions for the existence of precisely 4 Killing tensors of rank 2. These conditions are conditions of derivative orders 5 (Q_1) and 4 (Q_2) with respect to λ, since they involve the third and second derivatives of the scalar curvature, respectively. On the other hand, equation (6.99) is a third order condition: however, equation (6.99) may be more difficult to check due to the requirement of solving the equation for the constant d. Equation (6.99) offers a distinct advantage, however, in that if the constant d can be solved for, the irreducible Killing tensors can be constructed explicitly.

In summary, the tractor approach can be of use when a more direct approach may avail us nothing; however, there are appear to be certain metrics for which a more direct approach can be fruitful.

6.4.3 No Killing vectors

We now turn our attention to metrics of the form

$$g = \lambda du^2 + \lambda dv^2$$

with $\lambda = \lambda(u, v)$. We will also assume that g admits no Killing vectors. By equation (3.34), the assumption that g admits no Killing vectors is equivalent to the assumption that the following 2×2 matrix has full rank:

$$A = \begin{bmatrix} \left(-\lambda k_{uv} + \dfrac{\lambda_u k_v - \lambda_v k_u}{2}\right) k_v + \left(\lambda k_{vv} + \dfrac{\lambda_u k_u + \lambda_v k_v}{2}\right) k_u & \lambda k_u \\ -\left(\lambda k_{uu} + \dfrac{\lambda_u k_u + \lambda_v k_v}{2}\right) k_v + \left(\lambda k_{uv} + \dfrac{\lambda_u k_v - \lambda_v k_u}{2}\right) k_u & -\lambda k_v \end{bmatrix},$$

where $k = k(u, v)$ is the sectional curvature of g. Equivalently,

$$\lambda^2 k_v^2 k_{uv} - \frac{\lambda k_v^3 \lambda_u}{2} - \lambda^2 k_v k_u k_{vv} - \frac{\lambda k_v \lambda_u k_u^2}{2} + \lambda^2 k_u k_v k_{uu} + \frac{\lambda k_u \lambda_v k_v^2}{2} - \lambda^2 k_u^2 k_{uv} + \frac{\lambda k_u^3 \lambda_v}{2} \neq 0. \tag{6.107}$$

This condition is, of course, in addition to the condition that $k_u^2 + k_v^2 \neq 0$.

We begin, as in the case of a single Killing vector, by computing the tractor connection for Killing tensors of rank 2 for the metric above. We then find the curvature matrix to be given as

$$\tilde{K}^i{}_j = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ P_1 & P_2 & -P_1 & -5k_u\lambda & -5k_v\lambda & 0 \end{bmatrix}, \tag{6.108}$$

where

$$P_1 = \frac{3}{2}\left(k_v\lambda_u + k_u\lambda_v\right) - 3k_{uv}\lambda, \qquad P_2 = 3(k_v\lambda_v - k_{vv}\lambda + k_{uu}\lambda - k_u\lambda_u).$$

We now consider the following stacked matrix:

$$\begin{bmatrix} \tilde{K}^i{}_j \\ \tilde{\nabla}_{\partial_u}\tilde{K}^i{}_j \\ \tilde{\nabla}_{\partial_v}\tilde{K}^i{}_j \end{bmatrix}.$$

After removing rows of zeros as well as duplicate rows, we have

$$B = \begin{bmatrix} P_1 & P_2 & -P_1 & -5k_u\lambda & -5k_v\lambda & 0 \\ P_3 & P_4 & -P_3 & O_1 & O_2 & -5k_v\lambda \\ P_5 & P_6 & -P_5 & O_3 & O_4 & 5k_u\lambda \end{bmatrix}, \qquad (6.109)$$

where

$$P_3 = \frac{1}{2\lambda}\left(9\lambda^3 kk_v - 6\lambda^2 k_{uuv} + 6\lambda\lambda_v k_{uu} - 3\lambda\lambda_v k_{vv} + 3\lambda\lambda_{uv}k_u + 3\lambda\lambda_u k_{uv} - 3\lambda\lambda_{vv}k_v\right)$$

$$+\frac{1}{2\lambda}\left(6(\lambda_v)^2 k_v - 6\lambda_v\lambda_u k_u\right),$$

$$P_4 = \frac{1}{\lambda}\left(-9\lambda^3 kk_u + 3\lambda^2 k_{uuu} - 3\lambda^2 k_{uvv} + 9\lambda\lambda_v k_{uv} + 3\lambda\lambda_{uv}k_v - 3\lambda\lambda_u k_{uu} + 3\lambda\lambda_{vv}k_u\right)$$

$$-\frac{1}{\lambda}\left(6(\lambda_v)^2 k_u + 6\lambda_v\lambda_u k_v\right),$$

$$P_5 = \frac{3}{2\lambda}\left(5\lambda^3 kk_u - 2\lambda^2 k_{uvv} + \lambda\lambda_v k_{uv} + \lambda\lambda_{uv}k_v - \lambda\lambda_u k_{uu} + 2\lambda\lambda_u k_{vv} + \lambda\lambda_{vv}k_u\right)$$

$$+\frac{3}{2\lambda}\left(-(\lambda_v)^2 k_u + (\lambda_u)^2 k_u - 2\lambda_v\lambda_u k_v\right),$$

$$P_6 = \frac{1}{\lambda}\left(3\lambda^2 k_{uuv} - 3\lambda^2 k_{vvv} - 3\lambda\lambda_{uv}k_u + 3\lambda\lambda_{vv}k_v - 9\lambda\lambda_u k_{uv} + 3\lambda\lambda_v k_{vv} + 15\lambda^3 kk_v\right)$$

$$+\frac{1}{\lambda}\left(3(\lambda_u)^2 k_v - 3(\lambda_v)^2 k_v + 6\lambda_v\lambda_u k_u\right),$$

$$O_1 = -6\lambda k_{uu} + k_{vv}\lambda - \frac{7\lambda_v k_v + 3\lambda_u k_u}{2},$$

$$O_2 = -7\lambda k_{uv} + \frac{7\lambda_v k_u - 3\lambda_u k_v}{2},$$

$$O_3 = -7\lambda k_{uv} - \frac{3\lambda_v k_u - 7\lambda_u k_v}{2},$$

and

$$O_4 = \lambda k_{uu} - 6k_{vv}\lambda - \frac{3\lambda_v k_v + 7\lambda_u k_u}{2}.$$

Proposition 6.5. *The metric g admits no more than three Killing tensors of rank 2.*

Proof. The submatrix C of B given by

$$C = \begin{bmatrix} B_{14} & B_{15} & B_{16} \\ B_{24} & B_{25} & B_{26} \\ B_{34} & B_{35} & B_{36} \end{bmatrix} = \begin{bmatrix} -5k_u\lambda & -5k_v\lambda & 0 \\ O_1 & O_2 & -5k_v\lambda \\ O_3 & O_4 & 5k_u\lambda \end{bmatrix}$$

has full rank, since

$$\det(B) = 175\lambda \det(A).$$

Therefore, the rank of B is at least 3, and so the metric g admits no more than three Killing tensors of rank 2. □

It is natural to consider whether a tighter upper bound on the number of Killing tensors of the metric g exists. We will now provide an example from the literature (Kruglikov, 2008) of a metric in the plane which, despite admitting no Killing vectors, admits precisely three Killing tensors of rank 2. We will subsequently provide an example of a metric with no Killing vectors which admits precisely two Killing tensors, followed by an example which admits only one Killing tensor. For the first example, the metric is g with $\lambda(u, v) = u^2 + 4v^2$. The matrices which define the tractor connection are given as follows:

$$\tilde{\Gamma}^i{}_{j1} =$$

$$\begin{bmatrix} -\frac{2u}{\lambda} & \frac{8v}{\lambda} & 0 & 0 & 0 & 0 \\ -\frac{4v}{\lambda} & -\frac{2u}{\lambda} & \frac{4v}{\lambda} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{8v}{\lambda} & -\frac{2u}{\lambda} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{9u^2-36v^2}{\lambda^2} & 0 & -\frac{3u}{\lambda} & \frac{4v}{\lambda} & 0 \\ -\frac{9u^2-36v^2}{2\lambda^2} & 0 & \frac{9u^2-36v^2}{2\lambda^2} & -\frac{4v}{\lambda} & -\frac{3u}{\lambda} & -1 \\ \frac{18u(u^2-8v^2)}{\lambda^3} & \frac{288v(u^2-2v^2)}{\lambda^3} & -\frac{18u(u^2-8v^2)}{\lambda^3} & 0 & -\frac{9u^2-36v^2}{2\lambda^2} & -\frac{4u}{\lambda} \end{bmatrix},$$

$$\tilde{\Gamma}^i{}_{j2} =$$

$$\begin{bmatrix} -\frac{8v}{\lambda} & -\frac{2u}{\lambda} & 0 & -\frac{2}{3} & 0 & 0 \\ \frac{u}{\lambda} & -\frac{8v}{\lambda} & -\frac{u}{\lambda} & 0 & -\frac{1}{3} & 0 \\ 0 & \frac{2u}{\lambda} & -\frac{8v}{\lambda} & 0 & 0 & 0 \\ -\frac{9u^2-36v^2}{2\lambda^2} & 0 & \frac{9u^2-36v^2}{2\lambda^2} & -\frac{12v}{\lambda} & -\frac{u}{\lambda} & 1 \\ 0 & -\frac{9u^2-36v^2}{\lambda^2} & 0 & \frac{u}{\lambda} & -\frac{12v}{\lambda} & 0 \\ -\frac{144v(u^2-2v^2)}{\lambda^3} & \frac{36u(u^2-8v^2)}{\lambda^3} & \frac{144v(u^2-2v^2)}{\lambda^3} & \frac{9u^2-36v^2}{\lambda^2} & 0 & -\frac{16v}{\lambda} \end{bmatrix}.$$

The curvature matrix is given as equation (6.108) with

$$P_1 = \frac{2160u^3v - 8640uv^3}{\lambda^4}, \qquad P_2 = -\frac{540(u^4 - 28u^2v^2 + 32v^4)}{\lambda^4},$$

$$5k_u\lambda = \frac{60u(u^2-8v^2)}{\lambda^3}, \qquad 5k_v\lambda = \frac{480v(u^2-2v^2)}{\lambda^3}.$$

With the tractor connection and curvature matrix, we can compute the matrices $\tilde{\nabla}_{\partial_u}\tilde{K}^i{}_j$, $\tilde{\nabla}_{\partial_v}\tilde{K}^i{}_j$, $\tilde{\nabla}_{\partial_u}\tilde{\nabla}_{\partial_u}\tilde{K}^i{}_j$, $\tilde{\nabla}_{\partial_u}\tilde{\nabla}_{\partial_v}\tilde{K}^i{}_j$, $\tilde{\nabla}_{\partial_v}\tilde{\nabla}_{\partial_u}\tilde{K}^i{}_j$, and $\tilde{\nabla}_{\partial_v}\tilde{\nabla}_{\partial_v}\tilde{K}^i{}_j$. These matrices can be stacked, and a basis for the nullspace of the resulting stacked matrix–a basis for the local sections of $\mathbb{T}^2$–is given as

$$\{W_1, W_2, W_3\},$$

where

$$W_1 = E_1 + \frac{12v}{\lambda}E_4 + \frac{3u}{\lambda}E_5 - \frac{9}{2\lambda}E_6,$$
$$W_2 = E_2 - \frac{9u^2 + 12v^2}{u\lambda}E_4 + \frac{24v}{\lambda}E_5 + \frac{36v}{u\lambda}E_6,$$

and

$$W_3 = E_3 - \frac{12v}{\lambda}E_4 - \frac{3u}{\lambda}E_5 + \frac{9}{2\lambda}E_6.$$

We have already shown that $\dim(\mathbb{T}^1) \leq 3$, and so, since it is now apparent that $\dim(\mathbb{T}^2) = 3$, the metric g with $\lambda = u^2 + 4v^2$ has precisely three Killing tensors of rank 2. We will now identify them explicitly. This is done by imposing the condition of parallelism on an arbitrary linear combination of the basis elements of $\mathbb{T}^2$ and solving for the coefficient functions. Let

$$S = q_1(u,v)W_1 + q_2(u,v)W_2 + q_3(u,v)W_3.$$

The condition $\tilde{\nabla}S = 0$ results in a system of first order, linear, partial differential equations in the functions q_1, q_2, and q_3. The general solution is found, using Maple, to be

$$q_1 = \left(-4c_1vu^2 - 8c_1v^3 + \frac{c_2}{2}v^2 + c_3\right)\lambda, \qquad q_2 = c_1u\lambda^2,$$

$$q_3 = -\frac{\lambda}{8}\left(16c_1vu^2 + c_2u^2 - 8c_3\right),$$

for constants c_1, c_2, and c_3: thus, there are three independent solutions. A basis for $\mathcal{S}^{\tilde{\nabla}}(\mathbb{T})$ is given as $\{S_1, S_2, S_3\}$, where each S_i is generated by setting $c_i = 1$ and $c_j = 0$ for $i \neq j$:

$$S_1 = -4\lambda\left(vu^2 + 2v^3\right)E_1 + u\lambda^2 E_2 - 2\lambda vu^2 E_3 - 9\lambda^2 E_4 + 18\lambda uv E_5 + 45\lambda v E_6,$$

$$S_2 = \frac{\lambda v^2}{2}E_1 - \frac{\lambda u^2}{8}E_3 + \frac{3\lambda v}{2}E_4 + \frac{3\lambda u}{8}E_5 - \left(\frac{9u^2}{16} + \frac{9v^2}{4}\right)E_6,$$

$$S_3 = \lambda E_1 + \lambda E_3.$$

The Killing tensors associated with these parallel sections are, repectively,

$$\left(u^2 + 4v^2\right)\left(-4v\left(u^2 + 2v^2\right)du^2 + 2u\left(u^2 + 4v^2\right)dudv - 2vu^2dv^2\right),$$

$$\left(u^2 + 4v^2\right)\left(\frac{v^2}{2}du^2 - \frac{u^2}{8}dv^2\right),$$

$$\left(u^2 + 4v^2\right)\left(du^2 + dv^2\right).$$

Our next example is the metric g with $\lambda = uv$. The matrices which define the tractor connection are given as follows:

$$\tilde{\Gamma}^i{}_{j1} = \begin{bmatrix} -\frac{1}{u} & \frac{1}{v} & 0 & 0 & 0 & 0 \\ -\frac{1}{2v} & -\frac{1}{u} & \frac{1}{2v} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{v} & -\frac{1}{u} & 0 & \frac{2}{3} & 0 \\ 0 & -\frac{3u^2+3v^2}{2u^2v^2} & 0 & -\frac{3}{2u} & \frac{1}{2v} & 0 \\ \frac{3u^2+3v^2}{4u^2v^2} & 0 & -\frac{3u^2+3v^2}{4u^2v^2} & -\frac{1}{2v} & -\frac{3}{2u} & -1 \\ -\frac{3u^2+9v^2}{4u^3v^2} & -\frac{9u^2+3v^2}{2u^2v^3} & \frac{3u^2+9v^2}{4u^3v^2} & 0 & \frac{3u^2+3v^2}{2u^2v^2} & -\frac{4}{u} \end{bmatrix},$$

$$\tilde{\Gamma}^i{}_{j2} = \begin{bmatrix} -\frac{1}{v} & -\frac{1}{u} & 0 & -\frac{2}{3} & 0 & 0 \\ \frac{1}{2u} & -\frac{1}{v} & -\frac{1}{2u} & 0 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{u} & -\frac{1}{v} & 0 & 0 & 0 \\ \frac{3u^2+3v^2}{4u^2v^2} & 0 & -\frac{3u^2+3v^2}{4u^2v^2} & -\frac{3}{2v} & -\frac{1}{2u} & 1 \\ 0 & -\frac{3u^2+3v^2}{2u^2v^2} & 0 & \frac{1}{2u} & -\frac{3}{2v} & 0 \\ \frac{9u^2+3v^2}{4u^2v^3} & -\frac{3u^2+9v^2}{2u^3v^2} & -\frac{9u^2+3v^2}{4u^2v^3} & -\frac{3u^2+3v^2}{2u^2v^2} & 0 & -\frac{2}{v} \end{bmatrix}.$$

The curvature matrix is given in equation (6.108) with

$$P_1 = -\frac{15u^2+15v^2}{2u^3v^3}, \qquad P_2 = -\frac{45u^4-45v^4}{2u^4v^4},$$
$$5k_u\lambda = \frac{5u^2+15v^2}{2v^2u^3}, \qquad 5k_v\lambda = \frac{15u^2+5v^2}{2u^2v^3}.$$

As in the example before, a basis for the local sections of $\mathbb{T}^2$ can be computed.

We find this basis to be given as

$$\{W_1, W_2\},$$

where

$$W_1 = E_1 + E_3, \qquad W_2 = E_2 - \frac{3}{u}E_4 + \frac{3}{v}E_5 + \frac{3}{uv}E_6.$$

With $S = q_1(u,v)W_1 + q_2(u,v)W_2$, the condition $\tilde{\nabla}S = 0$ can be solved for q_1 and q_2. The general solution is

$$q_1 = -\frac{uv\left(c_1(u^2+v^2) - 2c_2\right)}{2}, \qquad q_2 = c_1 v^2 u^2,$$

for constants c_1 and c_2. Thus, a basis for the parallel sections of $\mathbb{T}$ is given as $\{S_1, S_2\}$, where

$$S_1 = -\frac{uv(u^2+v^2)}{2}E_1 + u^2v^2E_2 - \frac{uv(u^2+v^2)}{2}E_3 - 3v^2uE_4 + 3vu^2E_5 + 3uvE_6,$$

$$S_2 = uvE_1 + uvE_3.$$

The Killing tensors associated with these parallel sections are given, respectively, as

$$-\frac{uv(u^2+v^2)}{2}du^2 + 2u^2v^2dudv - \frac{uv(u^2+v^2)}{2}dv^2,$$

$$uv(du^2+dv^2).$$

For the metric g with $\lambda = uv$, it should also be noted that under the coordinate change $u = x+y$ and $v = x-y$, the metric transforms to become

$$\left(2x^2 - 2y^2\right)\left(dx^2 + dy^2\right).$$

Thus, the metric g with $\lambda(u,v) = uv$ is a Liouville metric, as any metric which can be transformed into the following form is considered to be a Liouville metric (Kruglikov, 2008):

$$(f+h)\left(dx^2+dy^2\right),$$

where $f = f(x)$ and $h = h(y)$. The tractor approach can be applied to metrics in this form, so that, absent additional conditions on the functions $f(x)$ and $g(y)$[6], the dimension of $\mathbb{T}^0$ is 5, the dimension of $\mathbb{T}^1$ is 3, and the dimension of $\mathbb{T}^2$ is 2. A local basis of $\mathbb{T}^2$ is given by

$$\{W_1, W_2\},$$

where

$$\begin{aligned} W_1 &= E_1 + \frac{3h'}{2(f+h)}E_4 + \frac{3f'}{2(f+h)}E_5 + \frac{3f''-3h''}{4(f+h)}E_6, \\ W_2 &= E_3 - \frac{3h'}{2(f+h)}E_4 - \frac{3f'}{2(f+h)}E_5 - \frac{3f''-3h''}{4(f+h)}E_6. \end{aligned}$$

From the local basis of $\mathbb{T}^2$, we find that the metric as well as the following symmetric tensor are Killing tensors of rank 2 for any Liouville metric:

$$(f+h)\left(hdx^2+fdy^2\right).$$

Our final example is the metric g with $\lambda = (uv)^{-\frac{2}{3}}$. The matrices which define the tractor connection are given as follows:

[6]Note that the metric g with $\lambda = u^2 + 4v^2$ is also Liouville.

$$\tilde{\Gamma}^i{}_{j1} = \begin{bmatrix} \frac{2}{3u} & -\frac{2}{3v} & 0 & 0 & 0 & 0 \\ \frac{1}{3v} & \frac{2}{3u} & -\frac{1}{3v} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3v} & \frac{2}{3u} & 0 & \frac{2}{3} & 0 \\ 0 & \frac{u^2+v^2}{u^2v^2} & 0 & \frac{1}{u} & -\frac{1}{3v} & 0 \\ -\frac{u^2+v^2}{2u^2v^2} & 0 & \frac{u^2+v^2}{2u^2v^2} & \frac{1}{3v} & \frac{1}{u} & -1 \\ \frac{-u^2+2v^2}{3u^3v^2} & \frac{4u^2-2v^2}{3u^2v^3} & \frac{u^2-2v^2}{3u^3v^2} & 0 & -\frac{u^2+v^2}{u^2v^2} & \frac{4}{3u} \end{bmatrix},$$

$$\tilde{\Gamma}^i{}_{j2} = \begin{bmatrix} \frac{2}{3v} & \frac{2}{3u} & 0 & -\frac{2}{3} & 0 & 0 \\ -\frac{1}{3u} & \frac{2}{3v} & \frac{1}{3u} & 0 & -\frac{1}{3} & 0 \\ 0 & -\frac{2}{3u} & \frac{2}{3v} & 0 & 0 & 0 \\ -\frac{u^2+v^2}{2u^2v^2} & 0 & \frac{u^2+v^2}{2u^2v^2} & \frac{1}{v} & \frac{1}{3u} & 1 \\ 0 & -\frac{u^2+v^2}{u^2v^2} & 0 & -\frac{1}{3u} & \frac{1}{v} & 0 \\ \frac{-2u^2+v^2}{3u^2v^3} & \frac{-2u^2+4v^2}{3u^3v^2} & \frac{2u^2-v^2}{3u^2v^3} & \frac{u^2+v^2}{u^2v^2} & 0 & \frac{4}{3v} \end{bmatrix}.$$

The curvature matrix is given with

$$P_1 = -\frac{10u^2+10v^2}{9u^3v^3}, \qquad P_2 = -\frac{20u^4-20v^4}{9u^4v^4},$$

$$5k_u\lambda = \frac{10u^2-20v^2}{9v^2u^3}, \qquad 5k_v\lambda = -\frac{20u^2-10v^2}{9u^2v^3}.$$

As in the previous examples, we will now compute a basis for the local sections of $\mathbb{T}^2$. We find this basis to be given as $\{W_1\}$, where $W_1 = E_1 + E_3$. Thus, $\dim(\mathbb{T}^2) = 1$, and the metric itself is the only independent Killing tensor for g with $\lambda = (uv)^{-\frac{2}{3}}$.

However, and rather interestingly, this metric admits the following Killing tensor of rank 3:

$$\frac{1}{uv^2}du^3 - \frac{1}{vu^2}dv^3.$$

We now summarize the topic of Killing tensors of rank 2 for metrics in the plane by means of the following proposition.

Proposition 6.6. *Let g be a metric in the plane. If g admits one Killing vector, g admits either two or four Killing tensors of rank 2. If g admits no Killing vectors, g admits either one, two, or three Killing tensors of rank 2.*

7 Killing-Yano tensors

7.1 Constructing the Rank 2 Tractor Connection

For Killing-Yano tensors of any rank, the equations that define the tractor connection are known (Houri et al., 2018). In this section, we will derive the equations which define the connection for Killing-Yano tensors of rank 2, though we note that our software program for constructing the tractor connection is operative for Killing-Yano tensors of any rank.

Let ∇ be a connection on a manifold M. A Killing-Yano tensor of rank 2–hereafter denoted simply as a Killing-Yano tensor–is a skew symmetric tensor $F_{\alpha\beta}$ such that

$$F_{\alpha(\beta;\gamma)} = 0,$$

where differentiation is taken with respect to the connection ∇: we say that $F_{\alpha\beta}$ is a Killing-Yano tensor of ∇. If the connection is a metric connection, we say that $F_{\alpha\beta}$ is a Killing-Yano tensor of the associated metric. In this section, we will assume that ∇ is a torsion free connection. In general,

$$\begin{aligned} F_{\alpha\beta;\gamma} &= F_{(\alpha\beta;\gamma)} + F_{[\alpha\beta;\gamma]} \\ &+\frac{1}{3}\left(F_{\alpha\beta;\gamma} + F_{\beta\alpha;\gamma} - F_{\gamma\alpha;\beta} - F_{\gamma\beta;\alpha}\right) + \frac{1}{3}\left(F_{\alpha\beta;\gamma} + F_{\gamma\beta;\alpha} - F_{\beta\alpha;\gamma} - F_{\beta\gamma;\alpha}\right), \end{aligned}$$

but with $F_{\alpha(\beta;\gamma)} = 0$ and $F_{\alpha\beta} = -F_{\beta\alpha}$, we find that

$$F_{\alpha\beta;\gamma} = F_{[\alpha\beta;\gamma]}\,. \tag{7.1}$$

We define the skew symmetric tensor $F_{\alpha\beta\gamma}$ as

$$F_{\alpha\beta\gamma} = F_{[\alpha\beta;\gamma]}\,,$$

giving us, by equation (7.1).

$$F_{\alpha\beta\gamma;\delta} = F_{\alpha\beta;\gamma\delta}\,.$$

We note that by equation (2.1),

$$\begin{aligned}
Y_{\begin{smallmatrix}\beta\\ \alpha\\ \gamma\\ \delta\end{smallmatrix}} F_{\alpha\beta;\gamma\delta} &= \frac{1}{4}\left(F_{\alpha\beta;\gamma\delta} + F_{\delta\alpha;\beta\gamma} + F_{\gamma\delta;\alpha\beta} + F_{\beta\gamma;\delta\alpha}\right) \\
&= \frac{1}{4}\left(F_{\alpha\gamma;\delta\beta} + F_{\beta\delta;\alpha\gamma} + F_{\beta\gamma;\alpha\delta} + F_{\delta\beta;\gamma\alpha}\right) \\
&= \frac{1}{4}\left(F_{\gamma\alpha;\beta\delta} - F_{\gamma\alpha;\delta\beta} + F_{\delta\beta;\gamma\alpha} - F_{\delta\beta;\alpha\gamma}\right),
\end{aligned}$$

so that

$$Y_{\begin{smallmatrix}\beta\\ \alpha\\ \gamma\\ \delta\end{smallmatrix}} F_{\alpha\beta;\gamma\delta} = \frac{1}{4}\left(F_{\mu\beta}\, R^{\mu}{}_{\delta\gamma\alpha} + F_{\delta\mu}\, R^{\mu}{}_{\beta\gamma\alpha} + F_{\mu\gamma}\, R^{\mu}{}_{\alpha\delta\beta} + F_{\alpha\mu}\, R^{\mu}{}_{\gamma\delta\beta}\right), \tag{7.2}$$

where the symmetrization operators Y are defined in the Conventions section of this thesis. We also note that

$$Y_{\begin{smallmatrix}\beta & \alpha\\ \gamma\\ \delta\end{smallmatrix}} F_{\alpha\beta;\gamma\delta} = \frac{1}{2} Y_{\begin{smallmatrix}\beta\\ \gamma\\ \delta\end{smallmatrix}} \left(F_{\alpha\beta;\gamma\delta} + F_{\beta\alpha;\gamma\delta}\right) = 0, \tag{7.3}$$

since $F_{\alpha\beta} = -F_{\beta\alpha}$. Additionally,

$$Y_{\begin{smallmatrix}\beta & \gamma\\ \alpha\\ \delta\end{smallmatrix}} F_{\alpha\beta;\gamma\delta} = \frac{1}{2} Y_{\begin{smallmatrix}\beta\\ \alpha\\ \delta\end{smallmatrix}} \left(F_{\alpha\beta;\gamma\delta} + F_{\alpha\gamma;\beta\delta}\right) = 0, \tag{7.4}$$

since $F_{\alpha\beta;\gamma} = -F_{\alpha\gamma;\beta}$. We also find that

$$Y_{\begin{smallmatrix}\beta&\delta\\\alpha\\\gamma\end{smallmatrix}} F_{\alpha\beta;\gamma\delta} = \frac{1}{2} Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\end{smallmatrix}} \left(F_{\alpha\beta;\gamma\delta} + F_{\alpha\delta;\gamma\beta}\right) \tag{7.5}$$

$$= \frac{1}{2} F_{\alpha\beta;\gamma\delta} + \frac{1}{12}\left(F_{\alpha\delta;\gamma\beta} - F_{\alpha\delta;\beta\gamma}\right) + \frac{1}{12}\left(F_{\beta\delta;\alpha\gamma} - F_{\beta\delta;\gamma\alpha}\right) + \frac{1}{12}\left(F_{\gamma\delta;\beta\alpha} - F_{\gamma\delta;\alpha\beta}\right)$$

$$= \frac{1}{12}\left(F_{\mu\delta} R^{\mu}{}_{\alpha\gamma\beta} + F_{\alpha\mu} R^{\mu}{}_{\delta\gamma\beta} + F_{\mu\delta} R^{\mu}{}_{\beta\alpha\gamma} + F_{\beta\mu} R^{\mu}{}_{\delta\alpha\gamma} + F_{\mu\delta} R^{\mu}{}_{\gamma\beta\alpha} + F_{\gamma\mu} R^{\mu}{}_{\delta\beta\alpha}\right)$$

$$+ \frac{1}{2} F_{\alpha\beta;\gamma\delta}$$

$$= \frac{1}{2} F_{\alpha\beta;\gamma\delta} + \frac{1}{12}\left(F_{\alpha\mu} R^{\mu}{}_{\delta\gamma\beta} + F_{\beta\mu} R^{\mu}{}_{\delta\alpha\gamma} + F_{\gamma\mu} R^{\mu}{}_{\delta\beta\alpha}\right).$$

Since

$$F_{\alpha\beta;\gamma\delta} = Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\end{smallmatrix}} F_{\alpha\beta;\gamma\delta},$$

we find, using equations (7.3) and (7.4), that

$$F_{\alpha\beta;\gamma\delta} = Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\end{smallmatrix}} \left(Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\\\delta\end{smallmatrix}} + Y_{\begin{smallmatrix}\beta&\alpha\\\gamma\\\delta\end{smallmatrix}} + Y_{\begin{smallmatrix}\beta&\gamma\\\alpha\\\delta\end{smallmatrix}} + Y_{\begin{smallmatrix}\beta&\delta\\\alpha\\\gamma\end{smallmatrix}} \right) F_{\alpha\beta;\gamma\delta}$$

$$= Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\end{smallmatrix}} Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\\\delta\end{smallmatrix}} F_{\alpha\beta;\gamma\delta} + Y_{\begin{smallmatrix}\beta&\delta\\\alpha\\\gamma\end{smallmatrix}} F_{\alpha\beta;\gamma\delta}.$$

Using equations (7.2) and (7.5), we get

$$F_{\alpha\beta;\gamma\delta} = \frac{1}{4} Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\end{smallmatrix}} \left(F_{\mu\beta} R^{\mu}{}_{\delta\gamma\alpha} + F_{\delta\mu} R^{\mu}{}_{\beta\gamma\alpha} + F_{\mu\gamma} R^{\mu}{}_{\alpha\delta\beta} + F_{\alpha\mu} R^{\mu}{}_{\gamma\delta\beta}\right)$$

$$+ \frac{1}{2} F_{\alpha\beta;\gamma\delta} + \frac{1}{12}\left(F_{\alpha\mu} R^{\mu}{}_{\delta\gamma\beta} + F_{\beta\mu} R^{\mu}{}_{\delta\alpha\gamma} + F_{\gamma\mu} R^{\mu}{}_{\delta\beta\alpha}\right)$$

$$= \frac{1}{4} Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\end{smallmatrix}} F_{\mu\beta} R^{\mu}{}_{\delta\gamma\alpha} + \frac{1}{2} Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\end{smallmatrix}} Y_{\begin{smallmatrix}\alpha\\\gamma\end{smallmatrix}} F_{\mu\gamma} R^{\mu}{}_{\alpha\delta\beta} + \frac{1}{2} F_{\alpha\beta;\gamma\delta} + \frac{1}{4} Y_{\begin{smallmatrix}\beta\\\alpha\\\gamma\end{smallmatrix}} F_{\beta\mu} R^{\mu}{}_{\delta\alpha\gamma},$$

which implies that, using the Bianchi identity,

$$
\begin{aligned}
F_{\alpha\beta;\gamma\delta} &= Y_{\begin{array}{|c|}\hline \beta\\\hline \alpha\\\hline \gamma\\\hline\end{array}} F_{\mu\beta}\, R^{\mu}{}_{\delta\gamma\alpha} + Y_{\begin{array}{|c|}\hline \beta\\\hline \alpha\\\hline \gamma\\\hline\end{array}} Y_{\begin{array}{|c|}\hline \alpha\\\hline \gamma\\\hline\end{array}} F_{\mu\gamma}\, R^{\mu}{}_{\alpha\delta\beta} \\
&= Y_{\begin{array}{|c|}\hline \beta\\\hline \alpha\\\hline \gamma\\\hline\end{array}} \left(-F_{\mu\beta}\, R^{\mu}{}_{\gamma\delta\alpha} - F_{\mu\beta}\, R^{\mu}{}_{\alpha\delta\gamma}\right) + Y_{\begin{array}{|c|}\hline \beta\\\hline \alpha\\\hline \gamma\\\hline\end{array}} Y_{\begin{array}{|c|}\hline \alpha\\\hline \gamma\\\hline\end{array}} F_{\mu\gamma}\, R^{\mu}{}_{\alpha\delta\beta} \\
&= 2Y_{\begin{array}{|c|}\hline \beta\\\hline \alpha\\\hline \gamma\\\hline\end{array}} Y_{\begin{array}{|c|}\hline \alpha\\\hline \gamma\\\hline\end{array}} F_{\mu\gamma}\, R^{\mu}{}_{\alpha\delta\beta}.
\end{aligned}
$$

Therefore, the equations which define the tractor connection are the following:

$$F_{\alpha\beta;\gamma} = F_{\alpha\beta\gamma}\,, \tag{7.6}$$

$$F_{\alpha\beta\gamma;\delta} = 2Y_{\begin{array}{|c|}\hline \beta\\\hline \alpha\\\hline \gamma\\\hline\end{array}} Y_{\begin{array}{|c|}\hline \alpha\\\hline \gamma\\\hline\end{array}} F_{\mu\gamma}\, R^{\mu}{}_{\alpha\delta\beta}. \tag{7.7}$$

For Killing-Yano tensors of higher rank, it has been shown that similar formulas hold (Houri et al., 2018).

For Killing-Yano tensors of rank 2 on an n-dimensional manifold M with coordinates x^α and $n > 2$, the tractor bundle is $\pi : \mathbb{T} \to M$, where $\mathbb{T} = \bigwedge^2(M) \oplus \bigwedge^3(M)$. The coordinates are $(x^\alpha, a_{\alpha\beta}, b_{\alpha\beta\gamma})$, where $a_{\alpha\beta} = a_{[\alpha\beta]}$ and $b_{\alpha\beta\gamma} = b_{[\alpha\beta\gamma]}$. The lift is given as $a_{\alpha\beta}(x) = F_{\alpha\beta}(x)$ and $b_{\alpha\beta\gamma}(x) = F_{\alpha\beta\gamma}(x)$, and we see that the dimension of the fibers of $\mathbb{T}$ is given as $\binom{n}{2} + \binom{n}{3} = \dfrac{1}{6} n\,(n^2-1)$. If $F_{\alpha\beta}$ is a Killing-Yano tensor of rank 2, then $(a_{\alpha\beta}, b_{\alpha\beta\gamma})$ is a parallel section by construction. Conversely, given a parallel section $(a_{\alpha\beta}, b_{\alpha\beta\gamma})$, the tensor $F_{\alpha\beta} = a_{\alpha\beta}$ is a Killing tensor of rank 2, since $F_{\alpha(\beta;\gamma)} = a_{\alpha(\beta;\gamma)} = b_{\alpha(\beta\gamma)} = 0$.

Lemma 7.1. *Let X be a Killing-Yano tensor of rank 2, and suppose that $\tilde{X}$ is the lift of X up to the tractor bundle. $\tilde{X}$ vanishes at a point if and only if $X = 0$.*

As in the case of Killing vectors, conformal Killing vectors, and Killing tensors of rank 2, the equations which define the tractor connection for Killing-Yano tensors of rank 2 is Frobenius in the sense of equation (2.2). Thus, lemma (7.1) follows from corollary (2.1.1).

Corollary 7.0.1. *The rank 2 Killing-Yano tensors $X_1 \dots X_k$ are linearly independent over $\mathbb{R}$ if and only if their lifts up to the tractor bundle are linearly independent at a single point.*

Proof. Let $X_1 \dots X_k$ be rank 2 Killing-Yano tensors, and let $\tilde{X}_1 \dots \tilde{X}_k$ be their lifts up to the tractor bundle at a point. By the previous lemma, $a_1\tilde{X}_1 + a_2\tilde{X}_2 + \dots a_k\tilde{X}_k = 0$ if and only if $a_1X_1 + a_2X_2 + \dots a_kX_k = 0$. Thus, the linear independence of one set implies the linear independence of the other. □

Thus, as with Killing vectors, Killing tensors, and conformal Killing vectors, the tractor approach is fruitful for the purpose of determining linear independence.

7.2 Rank 2 Killing-Yano tensors in General Relativity

Our primary interest in Killing-Yano tensors is in the construction of Killing tensors of rank 2. While Killing-Yano tensors are of interest in the study of General Relativity for this reason, it should be noted that Killing-Yano tensors have also been used in the separation of the Dirac Equation (Carter and McLenaghan, 1979; Fels and Kamran, 1990), and so the utility of Killing-Yano tensors extends beyond that of constructing Killing tensors.

We have examined many metrics from chapter 14 of *Exact Solutions to Einstein's Field Equations* (Stephani et al., 2003) using the tractor approach for Killing-Yano tensors of rank 2. Using our methods, we have found two metrics admitting precisely one Killing-Yano tensor each: these Killing-Yano tensors have been identified explicitly, where solving the Killing equation directly using Maple appears to have failed in that the Maple computation was not able to be performed in a seemingly reasonable time frame. The metrics are 14.22 and 14.24. Another metric, 14.10, has been found to admit no Killing-Yano tensors, where arriving at this conclusion by attempting to solve the Killing equation using Maple appears to have failed. Tables 7, 8, and 9 summarize our calculations for metrics admitting 3, 4, and 6 Killing vectors (none of the metrics examined had precisely 5 Killing vectors).

For each table, the column "Known Killing-Yano tensors" denotes the number of Killing-Yano tensors obtained from solving the Killing equations directly: an entry of 0 indicates that either no attempt was made or that no solutions were found. The next two columns indicate the dimensions of $\mathbb{T}^n$ for $n = 0, 1$, where applicable: a dash indicates that this computation was not attempted. The "Direct pdsolve" column indicates whether the Killing equations were solved directly. A checkmark indicates that, using Maple, the Killing equations were solved directly, and an "x" indicates that the Maple computation was aborted either due to memory constraints or due to the computation appearing to take an unusually long amount of time. A dash indicates that no attempt was made. These same indicators–the checkmark, the "x", and the dash–are used in the "Tractor pdsolve" column, which column indicates whether the reduced system of equations was solved. Column Y indicates the number of rank 2 Killing-Yano tensors which our methods demonstrate exist. Column T indicates whether the Killing tensors generated from the Killing-Yano tensors, according to equation (2.9), are metric reducible: an entry of "r" indicates that all Killing tensors are metric reducible, while a numeric entry indicates the number of Killing tensors which are metric irreducible.

7.2.1 Isometry dimension 3

Beginning with metrics in chapter 14 (Stephani et al., 2003) which admit precisely 3 Killing vectors, we apply the tractor connection. We examine metrics 14.26, 14.27, 14.29, 14.30, 14.31, 14.32, 14.33, 14.34, 14.35, 14.41, 14.42, 14.44, 14.45, and 14.46. Table 7 summarizes our results.

	Known Killing-Yano tensors	$\dim(\mathbb{T}^0)$	$\dim(\mathbb{T}^1)$	Direct pdsolve	Tractor pdsolve	Y	T
14.26	0	0	-	-	-	0	-
14.27	0	0	-	-	-	0	-
14.29	0	0	-	-	-	0	-
14.30	0	0	-	-	-	0	-
14.31	0	0	-	-	-	0	-
14.32	0	0	-	-	-	0	-
14.33	0	0	-	-	-	0	-
14.34	0	0	-	-	-	0	-
14.35	0	0	-	-	-	0	-
14.41	0	0	-	-	-	0	-
14.42	0	0	-	-	-	0	-
14.44	0	0	-	-	-	0	-
14.45	0	x	-	x	x		
14.46	0	2	0	-	-	0	-

Table 7: Metrics from chapter 14 with isometry dimension 3.

All but one of the examined metrics admitting precisely three Killing vectors have been found to admit no Killing-Yano tensors. In most cases, this can be verified at curvature order 0. At the time of writing, we were not able to apply the tractor approach for metric 14.45 due to the inability to compute the tractor connection in a timely fashion.

7.2.2 Isometry dimension 4

We now present the summary of our calculations for metrics from chapter 14 (Stephani et al., 2003) which admit precisely 4 Killing vectors. We examine 14.14, 14.15, 14.16, 14.17, 14.18, 14.19, 14.20, 14.21, 14.22, 14.23, 14.24, and 14.25.

14.14(i) is metric 14.14, and 14.14(ii) interchanges $\sinh(u)$ and $\cosh(u)$. 14.16(i) has $k=-1$, while 14.16(ii) has $k=1$. 14.21(i), 14.21(ii), and 14.21(iii) have $k=1,0,$ and -1, respectively.

All of the examined metrics with precisely four Killing vectors admit either 1 or 0 Killing-Yano tensors. In all but two metrics admitting a Killing-Yano tensor, the Killing-Yano tensor has been found by solving the Killing equation directly. The two exceptional metrics are found to be 14.22 and 14.24. Table 8 summarizes our calculations.

The metric 14.22 is

$$a^2\left(g-b\dot{g}\right)^2dx^2+g^2e^{-2x}dy^2+g^2e^{2x}dz^2-dt^2,$$

where $g=g(bx+t)$. The Killing-Yano tensor is given as

$$g^3e^{-3x}dy\wedge dz,$$

from which the following Killing tensor of rank 2 is generated:

	Known Killing-Yano tensors	dim($\mathbb{T}^0$)	dim($\mathbb{T}^1$)	Direct pdsolve	Tractor pdsolve	Y	T
14.14(i)	0	0	-	-	-	0	-
14.14(ii)	0	0	-	-	-	0	-
14.15(a)	1	1	1	✓	✓	1	r
14.15(b)	1	1	1	✓	✓	1	r
14.16(i)	1	1	1	✓	✓	1	r
14.16(ii)	1	1	1	✓	✓	1	r
14.17	1	1	1	✓	✓	1	r
14.18(a)	0	0	-	-	-	0	-
14.18(b)	0	0	-	-	-	0	-
14.19	1	1	1	✓	✓	1	r
14.20	0	0	-	-	-	0	-
14.21(i)	0	0	-	-	-	0	-
14.21(ii)	0	0	-	-	-	0	-
14.21(iii)	0	0	-	-	-	0	-
14.22	0	1	1	x	✓	1	r
14.23	0	0	-	-	-	0	-
14.24	0	1	1	x	✓	1	r
14.25	0	0	-	-	-	0	-

Table 8: Metrics from chapter 14 with isometry dimension 4.

$$-e^{-4x}g^4\left(dy^2+dz^2\right).$$

However, this Killing tensor is reducible, since it is generated by the rank 1 Killing tensors $e^{-2x}g^2dy$ and $e^{-2x}g^2dz$.

The metric 14.24 is

$$-\frac{c^2\left(U'\right)^2}{a^2U^2}dt^2+\frac{c^2}{U^2}dx^2+\frac{c^2}{e^{2x}U^2}dy^2+\frac{c^2}{e^{2x}U^2}dz^2,$$

where $U=U(t+x)$. The Killing-Yano tensor is given as

$$\frac{1}{e^{3x}U^3}dy\wedge dz,$$

from which the following rank 2 Killing tensor is generated:

$$-\frac{1}{c^2 e^{4x} U^4}\left(dy^2 + dz^2\right).$$

However, this Killing tensor is reducible, since it can be generated by the following rank 1 Killing tensors:

$$\frac{1}{e^{2x}U^2}dy, \qquad \frac{1}{e^{2x}U^2}dz.$$

7.2.3 Isometry dimension 6

We now examine metrics from chapter 14 (Stephani et al., 2003) which admit precisely 6 Killing vectors. We examine metrics 14.7, 14.10, and 14.12. A summary is given in Table 9.

	Known Killing-Yano tensors	$\dim(\mathbb{T}^0)$	$\dim(\mathbb{T}^1)$	Direct pdsolve	Tractor pdsolve	Y	T
14.7	4	4	4	✓	✓	4	r
14.10	0	4	1	x	✓	0	-
14.12(a)	4	4	4	✓	✓	4	r
14.12(b)	4	4	4	✓	✓	4	r
14.12(c)	4	4	4	✓	✓	4	r

Table 9: Metrics from chapter 14 with isometry dimension 6.

With the exception of 14.10, all of the metrics admitting 6 Killing vectors were found to admit 4 Killing-Yano tensors by solving the Killing equation directly. For metric 14.10, we were unable to compute $\dim(\mathbb{T}^2)$; however, we solved the reduced system at at $\mathbb{T}^1$ directly to show that no non-zero solutions can exist.

It is apparent that although the Killing equations for Killing-Yano tensors of rank 2 can be solved directly more often than the Killing equations for Killing tensors of rank 2, the tractor approach may still grant useful insight with regard to Killing-Yano tensors. In the infrequent cases in which the Killing-Yano tensors cannot be solved for directly, the tractor approach can be used to obtain a count of the number of Killing-Yano tensors, and, in certain cases, the tractor approach can be used to obtain the Killing-Yano tensors explicitly.

It is also apparent that the ability to generate meaningful Killing tensors of rank 2 from Killing-Yano tensors of rank 2 is limited, as all of our examined cases yield only metric reducible Killing tensors. It is possible that meaningful Killing tensors (i.e. metric irreducible Killing tensors) are more easily generated by Killing-Yano tensors of rank 3, as examples exist in the literature of metric irreducible Killing tensors generated from Killing-Yano tensors of rank 3 (Popa and Ovidiu, 2007).

We will conclude this section by providing an example of a metric irreducible Killing tensor generated by a Killing-Yano tensor of rank 2. The Kerr metric given in equation (6.53) has been shown to admit the metric irreducible Killing tensor given in equation (6.54). This Killing tensor is also generated by the following Killing-Yano tensor, which Killing-Yano tensor we obtained by solving the Killing equation (for rank 2 Killing-Yano tensors) directly using Maple:

$$-a\cos(\theta)dt\wedge dr + ar\sin(\theta)dt\wedge d\theta - a^2\cos(\theta)\sin^2(\theta)dr\wedge d\phi + r(a^2+r^2)\sin(\theta)d\theta\wedge d\phi.$$

8 Future Work

We have applied the tractor approach to Killing vectors, Killing tensors of rank 2, Killing-Yano tensors of rank 2, and conformal Killing vectors. In each case, we have built supportive programs in Maple. We have also built software which is supportive of applying the tractor approach for Killing-Yano tensors of any rank up to the dimension of the base manifold. Formulas for tractor connections for Killing tensors of rank greater than 2 have been proposed (Houri et al., 2018), making the construction of computer programs capable of implementing them explicitly a project of interest. However, designing such a program is problematic, since the existing formulas which define the tractor connection are not explicitly given. Other formulas have been proposed (Wolf, 1998), and yet the explicit manifestation of the tractor connection formulas is likewise absent.

Notwithstanding, it may be possible to construct the required Frobenius system for higher rank Killing tensors if attention is restricted to a low-dimensional case, such as dimension 2, 3, or 4. Thus, the development of software for low-dimensional, higher-rank Killing tensors is of interest.

Constructing tractor connections for conformal Killing forms and tensors is also of interest, though we have yet to find any literature concerning the maximum number or the prolongation of conformal Killing tensors.

It may also be of interest to apply the tractor approach more completely to Killing tensors of rank 2. Finding irreducible Killing tensors for many other exact solutions of the Einstein equations may be of interest, and it may also be of interest to search for irreducible Killing tensors outside of the context of general relativity. In this case, the software tools developed in this thesis can serve as an exceptional aide.

Our apparent lack of success in constructing interesting Killing tensors of rank 2 from Killing-Yano tensors and conformal Killing vectors is troubling, and may be cause to suspect that such Killing tensors are a particularly rare phenomenon. An interesting project would be the examination of many exact solutions to Einstein's equations for the purpose of constructing Killing tensors from Killing-Yano tensors and conformal Killing vectors–Killing tensors which are not generated from Killing vectors or the metric itself. Special attention should be given to metrics which have been shown to admit metric irreducible Killing tensors, with the question in mind being whether the Killing tensors produced using Killing-Yano tensors and conformal Killing vectors are metric reducible or not.

It may also be of interest to write a program which constructs the tractor connection for Killing spinors (M. Walker and R. Penrose, 1970). Ideally, Killing spinors could then be identified without the need to solve the associated equations directly.

Another useful project is the optimization of existing software. In this thesis, we have built software which allows us to apply the tractor approach in many cases, but the nominal time required of Maple to construct the tractor connection for Killing tensors of rank 2 in dimension 4 is 55 seconds. While this amount of time may not seem to be impractical, the amount of time required to compute tractor connections for Killing tensors increases dramatically as the dimension of the base space increases. It would be useful to look for programming inefficiencies in the code, in hopes that the required time to complete the computations needed for the tractor approach can be significantly reduced.

As Killing vectors and conformal Killing vectors have been treated, it would also be of interest to examine homothetic Killing vectors. These are conformal Killing vectors for which, in light of equation (5.1), $X^{\gamma}{}_{;\gamma}$ is a constant. It is thought that homothetic Killing vectors can be treated as conformal Killing vectors, though with the lift onto the tractor bundle having $Z_{\alpha} = 0$, making the treatment of homothetic Killing vectors attainable, ideally, using existing methods.

In a few tables, it is evident that while we were often able to compute a basis for the 0^{th}-order reduced tractor bundle, the higher order reduced tractor bundles sometimes proved elusive on account of the need to check the nullspaces of additional matrices. In particular, it is desirable to compute the 1^{st}-order reduced tractor bundle for Killing vectors for the Kerr metric. It may be possible to use our basis for the 0^{th}-order reduced bundle to get the desired result for the Kerr metric, and, if this proves to be successful, it is reasonable to assume that whatever successful technique was used in this case can be applied more generally.

As we have stated earlier, it is only for metrics of constant curvature that the algebra of Killing tensors is completely known, and this due to the fact that no irreducible Killing tensors of rank 2 or greater can exist (Thompson, 1986). The problem of finding a generating set for the algebra of Killing tensors in general, however, is open. Thus, it may be of interest to attempt to, for a particular metric or for a class of metrics, find all irreducible Killing tensors.

9 Maple Programs

We will now describe the utility of the programs which were developed for this Thesis. For examples of their use or the source code itself, see the Appendices.

9.1 Programs for Killing vectors

- **rnk1TracConn(**Γ, N**)**. This program will construct the tractor connection for rank 1 Killing tensors. It requres a connection Γ defined on the base space as well as the bundle N itself.

- **liftrnk1(**X, Γ, N**)**. This program is intended to lift a rank 1 Killing tensor X to a section on N which is parallel with respect to the **rnk1TracConn** connection. This procedure also requires a connection Γ defined on the base space.

- **getRnk1(**X, N**)**. This program is intended to take a section X on the bundle N which is parallel with respect to the **rnk1TracConn** connection and output the associated rank 1 Killing tensor defined on the base space.

9.2 Programs for conformal Killing vectors

- **ConfTracConn(**g, N**)**. This program will construct the tractor connection for conformal Killing 1-forms. This program requires a metric g defined on the base space of the vector bundle N rather than a connection, since the equations which define the tractor connection for conformal Killing vectors are given in terms of the metric itself.

- **liftConfKV(**X, g, N**)**. This program is intended to lift a conformal Killing 1-form X to a section on N which is parallel with respect to the **ConfTracConn** connection. This procedure also requires a metric g as input.

- **getConfKV(X, N)**. This program is intended to take a section X on the bundle N which is parallel with respect to the **ConfTracConn** connection and output the associated conformal Killing 1-form.

- **CKVtoKT(X, g)**. This program takes as input a (covariant) conformal Killing vector X and metric g and constructs a tensor using equation (2.12). Certain conditions on X must be satisfied in order for the resulting tensor to be a Killing tensor.

9.3 Programs for Killing tensors of rank 2

- **HauserTractorConnection(Γ,N)**. This program takes as input a connection defined on the base space and the base space itself, and outputs the tractor connection on the bundle using the Hauser tractor equations(Hauser and Malhiot, 1975a).

- **HauserTractorLift2(K,Γ,N)**. This program is intended to perform the lift of a Killing tensor K of rank 2 to a parallel (with respect to the Hauser connection) section on the bundle N. A connection Γ on the base space is also required.

- **getHauserKT2(X, N)**. This program is intended to do the opposite of the **HauserTractorLift2** command: it is intended to take a parallel section X on the Tractor bundle N and push it down to a rank 2 Killing tensor on the base space.

9.4 Programs for Killing-Yano tensors

- **KYTracCon(Γ, k, N)**. This program will construct the tractor connection for Killing-Yano tensors(Houri et al., 2018) of rank k. This program requires a Connection Γ on the base space, an integer k, and an initialized vector bundle

N. It is not recommended that this program be used for Killing-Yano tensors of rank 1.

- **liftKY(**F, Γ, N**)**. This program is intended to lift a Killing-Yano tensor F to a section on N which is parallel with respect to the **KYTracCon** connection. This procedure also requires a connection Γ defined on the base space.

- **getKY(**X, k, N**)**. This program is intended to take a section X on the bundle N which is parallel with respect to the **KYTracCon** connection and output the associated Killing-Yano tensor of rank k.

- **KYtoKT(**$g, h, KY1, KY2$**)**. This command computes the rank 2 Killing tensor formed by Killing-Yano tensors $KY1$ and $KY2$ using equation (2.9). A metric g and the inverse metric h are also required.

9.5 Utility programs

- **MaxKT(**m, n**)**. This program takes as input two integers, namely the dimension of the base manifold (m) and the desired Killing tensor rank (n), and outputs the maximum number of Killing tensors of the desired rank in the desired dimension(Hauser and Malhiot, 1975a). Before initializing the vector bundle in Maple, this command is used to compute the required size of the fibers.

- **MaxKY(**m, n**)**. This program is entirely analogous to the program **MaxKT**, but for Killing-Yano tensors.

- **MaxCF(**m, n**)**. Another program analogous to **MaxKT**, but for conformal Killing forms of rank n.

- **MaxSym(**m, n**)**. This program calculates the number of independent components of a completely symmetric, rank n tensor in m dimensions.

- **MaxSkew(m, n).** This program calculates the number of independent components of a rank n, completely skew-symmetric tensor in m dimensions.

- **KillingTensorLibrary(n,name,output:=[]).** This command loads in known Killing tensors from a Killing tensor database. This procedure requires at least two arguments: an integer n and a name of the user's choosing. Specifying only these two arguments will return the metric of the database entry given by the integer on the procedure-initialized Manifold whose name is given as the second input value.

  A third possible input takes the form output = [], where the values in the list are given by the user. If the only input in the list is an integer, then the procedure will return the known irreducible Killing tensors of the associated database entry metric, the rank of which Killing tensors are equal to the integer specified. If the user inputs output = ["KillingTensors",k] for some integer k, then the program returns all known Killing tensors of rank k for the associated database entry. The user can also specify output = ["IrreducibleRank"] to generate a list of integers which will inform the user of what rank(s) of irreducible Killing tensors associated with the database entry metric are known. ["KillingYanoTensors",k] will return the known Killing-Yano tensors of rank k, and ["ConformalKillingForms",k] will return the known conformal Killing forms of rank k.

  The user may also specify the output as one of a few keywords: "Notes", "Reference", and "Coordinates" to display relevant information about the database entry. Some entries come from the MetricSearch library in the Differential Geometry software package[7]: for these database entries, the reference is given so that they may be retrieved from the MetricSearch library.

[7] For more information, see https://digitalcommons.usu.edu/dg/.

the output option may also be used to retrieve known tractor connections and their associated curvature tensors. [“TractorConnection”,k,Q] will return the tractor connection for Killing tensors of rank k on the procedure-initialized vector bundle Q, and [“TractorCurvature”,k,Q] retrieves the associated curvature tensor. Similar commands are available for Killing-Yano tensors and for conformal Killing forms by specifying the output as [“YanoTractorConnection”,k,Q], [“YanoTractorCurvature”,k,Q], [“ConformalFormTractorConnection”,k,Q], and [“ConformalFormTractorCurvature”,k,Q].

- **BundleLift(**T, N**)**. This command is intended to redefine the object T as an object in the environment N. The most common application for us is to take a metric g defined on a manifold M and redefine it on the base space of a vector bundle N by way of the following: BundleLift(g, N).

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APPENDICES

11 Appendix A: Software demonstration for Killing vectors

We will now provide a small demonstration of how our software may be implemented in the study of Killing vectors. Our example will be given on a 2-dimensional Riemannian manifold with metric $u^p(du^2 + dv^2)$. We will construct the tractor bundle and tractor connection, calculate the curvature tensor, and then find bases for the reduced tractor bundles to explicitly identify the Killing vectors themselves. We will verify the one-to-one correspondence of parallel sections on the tractor bundle and Killing vectors on the base space.

Appendix A: Software demonstration for Killing vectors

In this demonstration, we will illustrate the utility of the programs with the following names: MaxKT, BundleLift, rnk1TracConn, getRnk1, and liftrnk1.

First, we will read in the file which contains the programs we have written and load in other necessary packages.

```
> read "TractorPrograms.txt";
  with(DifferentialGeometry):
  with(Tensor):
```

Now, we will initialize a 2-dimensional manifold.

```
> DGEnvironment[Coordinate]([u,v], M);
```

Manifold: M **(1)**

Next, we will define a simple metric.

```
M > g := evalDG( u^p * (du &t du + dv &t dv) );
```

$$g := u^p\, du \otimes du + u^p\, dv \otimes dv \quad (2)$$

We need to know the required size of the fibers of the tractor bundle. We find this to be 3:

```
M > MaxKT(2, 1);
```

3 **(3)**

Now we initialize the required environments.

```
M > DGEnvironment[VectorSpace](3, V);
```

Vector Space: V **(4)**

```
V > DGEnvironment[VectorBundle](M, V, N);
```

Vector Bundle: N **(5)**

Now that the vector bundle has been initialized, it is convenient to redefine the metric on this bundle:

```
N > G := BundleLift(g, N);
```

$$G := u^p\, du \otimes du + u^p\, dv \otimes dv \quad (6)$$

We will also need the Christoffel symbols:

```
N > Gamma := Christoffel(G);
```

$$\Gamma := \nabla_{\partial_u} \partial_u = \frac{p}{2\,u}\,\partial_u, \nabla_{\partial_u} \partial_v = \frac{p}{2\,u}\,\partial_v, \nabla_{\partial_v} \partial_u = \frac{p}{2\,u}\,\partial_v, \nabla_{\partial_v} \partial_v = -\frac{p}{2\,u}\,\partial_u \qquad (7)$$

Now we can compute the tractor connection:

```
N > C := rnk1TracConn(Gamma, N);
```

$$C := \nabla_{\partial_u} E1 = -\frac{p}{2\,u}\,E1, \nabla_{\partial_u} E2 = -\frac{p}{2\,u}\,E2 + \frac{p}{2\,u^2}\,E3, \nabla_{\partial_u} E3 = -E2 - \frac{p}{u}\,E3$$
$$, \nabla_{\partial_v} E1 = \frac{p}{2\,u}\,E2 - \frac{p}{2\,u^2}\,E3, \nabla_{\partial_v} E2 = -\frac{p}{2\,u}\,E1, \nabla_{\partial_v} E3 = E1 \qquad (8)$$

We can also represent this connection as a matrix of 1-forms:

```
N > convert(C, DGMatrix);
```

$$\begin{bmatrix} -\frac{p}{2\,u}\,du & -\frac{p}{2\,u}\,dv & dv \\ \frac{p}{2\,u}\,dv & -\frac{p}{2\,u}\,du & -du \\ -\frac{p}{2\,u^2}\,dv & \frac{p}{2\,u^2}\,du & -\frac{p}{u}\,du \end{bmatrix} \qquad (9)$$

We now compute the Curvature tensor for the tractor connection.

```
N > K := CurvatureTensor(C);
```

$$K := \frac{p\,(2+p)}{2\,u^3}\,E3 \otimes \Theta 1 \otimes du \otimes dv - \frac{p\,(2+p)}{2\,u^3}\,E3 \otimes \Theta 1 \otimes dv \otimes du \qquad (10)$$

It is interesting to note that p=0 and p=-2 are the only values of p for which the curvature tensor vanishes identically.

We can represent this tensor as a collection of (1,1) tensors by contracting the curvature tensor with the coordinate vectors of the base manifold. In our 2-dimensional case, there is only one such (1,1) tensor, but there may be others in higher dimensions.

```
N > k1 := ContractIndices(K, evalDG(D_u &t D_v), [[3,1],[4,2]] );
```

$$k1 := \frac{p\,(2+p)}{2\,u^3}\,E3 \otimes \Theta 1 \qquad (11)$$

We can think of this as a matrix

```
N > K1 := convert(k1, DGMatrix);
```

$$K1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1}{2}\,\frac{p\,(2+p)}{u^3} & 0 & 0 \end{bmatrix} \tag{12}$$

and subsequently find the basis for the 0th order reduced tractor bundle:

```
N > IT0 := LieAlgebras:-InvariantTensors([K1], [seq(E||i,i=1..3)]
    ) ;
```

$$IT0 := [E2, E3] \tag{13}$$

This calculation is assuming that p is not 0 or 2: caution is advised when examining metrics with unknown constants as exponents. Assuming that p is not 0 or 2, there are a maximum of 2 Killing vectors of the metric g, since there are 2 invariants.

We will now try to get a tighter upper bound using the 1st order reduced tractor bundle. We begin by differentiating the curvature tensor.

```
N > dK1 := CovariantDerivative(k1, C);
```

$$dK1 := \frac{p\,(2+p)}{2\,u^3}\,E1 \otimes \Theta1 \otimes dv - \frac{p\,(2+p)}{2\,u^3}\,E2 \otimes \Theta1 \otimes du - \frac{p\,(2+p)\,(6+p)}{4\,u^4}\,E3 \otimes \Theta1 \otimes du + \frac{p^2\,(2+p)}{4\,u^4}\,E3 \otimes \Theta2 \otimes dv - \frac{p\,(2+p)}{2\,u^3}\,E3 \otimes \Theta3 \otimes dv \tag{14}$$

We now generate a set of (1,1) tensors by contraction:

```
N > D1K1 := ContractIndices(dK1, D_u, [[3,1]] );
```

$$D1K1 := -\frac{p\,(2+p)}{2\,u^3}\,E2 \otimes \Theta1 - \frac{p\,(2+p)\,(6+p)}{4\,u^4}\,E3 \otimes \Theta1 \tag{15}$$

```
N > D2K1 := ContractIndices(dK1, D_v, [[3,1]] );
```

$$D2K1 := \frac{p\,(2+p)}{2\,u^3}\,E1 \otimes \Theta1 + \frac{p^2\,(2+p)}{4\,u^4}\,E3 \otimes \Theta2 - \frac{p\,(2+p)}{2\,u^3}\,E3 \otimes \Theta3 \tag{16}$$

Now we will convert them to matrices.

```
N > Md1k1 := convert(D1K1, DGMatrix);
```

$$Md1k1 := \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2}\frac{p(2+p)}{u^3} & 0 & 0 \\ -\frac{1}{4}\frac{p(2+p)(6+p)}{u^4} & 0 & 0 \end{bmatrix} \quad (17)$$

```
N > Md2k1 := convert(D2K1, DGMatrix);
```

$$Md2k1 := \begin{bmatrix} \frac{1}{2}\frac{p(2+p)}{u^3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4}\frac{p^2(2+p)}{u^4} & -\frac{1}{2}\frac{p(2+p)}{u^3} \end{bmatrix} \quad (18)$$

And now we can compute a basis for the 1st order reduced tractor connection.

```
N > IT1 := LieAlgebras:-InvariantTensors([K1,Md1k1,Md2k1], [seq
    (E||i,i=1..3)] );
```

$$IT1 := \left[E2 + \frac{p}{2\,u}\,E3\right] \quad (19)$$

Thus, we get a maximum of 1 Killing vector. Let's be sure that our calculation doesn't depend on p not being 6, since p=6 appears, from looking at the matrices above, to be an exceptional value.

```
N > LieAlgebras:-InvariantTensors( eval([K1,Md1k1,Md2k1], p=6),
    [seq(E||i,i=1..3)]);
```

$$\left[E2 + \frac{3}{u}\,E3\right] \quad (20)$$

Now let's see if we can get the Killing vector explicitly. First, we form a linear combination of the 1st order reduced tractor basis elements with function coefficients.

```
N > s1 := DGzip([seq(q||i(u,v), i=1..nops(IT1))] ,IT1, "plus");
```

$$s1 := q1(u,v)\,E2 + \frac{q1(u,v)\,p}{2\,u}\,E3 \quad (21)$$

Next, we take the covariant derivative.

```
N > s2 := CovariantDerivative(s1, C);
```

(22)

$$s2 := -\left(-\left(\frac{\partial}{\partial u}\, q1(u,v)\right) + \frac{q1(u,v)\, p}{u}\right) E2 \otimes du + \frac{\partial}{\partial v}\, q1(u,v)\, E2 \otimes dv + \frac{p\left(\left(\frac{\partial}{\partial u}\, q1(u,v)\right) u - q1(u,v)\, p\right)}{2\, u^2}\, E3 \otimes du + \frac{\left(\frac{\partial}{\partial v}\, q1(u,v)\right) p}{2\, u}\, E3 \otimes dv \quad \textbf{(22)}$$

Let's look at the equations we need to solve.

```
N > s3 := DGinformation(s2, "CoefficientSet");
```

$$s3 := \left\{\frac{1}{2}\, \frac{p\left(\left(\frac{\partial}{\partial u}\, q1(u,v)\right) u - q1(u,v)\, p\right)}{u^2},\ \frac{1}{2}\, \frac{\left(\frac{\partial}{\partial v}\, q1(u,v)\right) p}{u},\ \frac{\partial}{\partial u}\, q1(u,v) - \frac{q1(u,v)\, p}{u},\ \frac{\partial}{\partial v}\, q1(u,v)\right\} \quad \textbf{(23)}$$

How many equations are there?

```
N > nops(s3);
```

$$4 \quad \textbf{(24)}$$

This is an easier system to solve than the Killing equations themselves, which has 3 equations and 2 unknown functions.

Let's get the solution:

```
N > s4 := pdsolve(s3, q||1(u,v));
```

$$s4 := \{q1(u,v) = \_C1\, u^p\} \quad \textbf{(25)}$$

Here is then what the parallel sections should look like:

```
N > s5 := DETools:-dsubs(s4, s1);
```

$$s5 := \_C1\, u^p\, E2 + \frac{\_C1\, u^p\, p}{2\, u}\, E3 \quad \textbf{(26)}$$

We may as well evaluate this at _C1=1.

```
N > s6 := eval(s5,_C1=1);
```

$$s6 := u^p\, E2 + \frac{u^p\, p}{2\, u}\, E3 \quad \textbf{(27)}$$

We check that it's a parallel section:

```
N > CovariantDerivative(s6, C);
```

$$0\, E1 \otimes du \tag{28}$$

Now we drop this parallel section down:

```
N > T := getRnk1(s6, N);
```

$$T := u^p\, dv \tag{29}$$

Let's check that this is a Killing tensor of rank 1.

```
N > CheckKillingTensor(G, T);
```

$$0\, du \otimes du \tag{30}$$

If the contravariant vector is desired, we can simply raise indices:

```
N > X := RaiseLowerIndices( InverseMetric(G), T, [1] );
```

$$X := \partial_v \tag{31}$$

Now let's check that we can lift the Killing tensors of the metric to parallel sections. We begin by calculating the Killing tensors conventionally.

```
N > kt1 := KillingTensors(G, 1);
```

$$kt1 := [u^p\, dv] \tag{32}$$

As there is only one, we will lift this individually rather than lift the list. Note that the Christoffel symbols of G are needed.

```
N > liftkt1 := liftrnk1(kt1[1], Gamma, N);
```

$$liftkt1 := u^p\, E2 + \frac{u^{p-1}\, p}{2}\, E3 \tag{33}$$

Let's check that this is a parallel section.

```
N > CovariantDerivative(liftkt1, C);
```

$$0\, E1 \otimes du \tag{34}$$

But are there more parallel sections?

```
N > CovariantlyConstantTensors(C, [seq(E||i,i=1..3)] );
```

$$\left[\frac{2\, u^p}{p}\, E2 + u^{p-1}\, E3\right] \tag{35}$$

Since there is only 1, which corresponds with our known parallel section, the parallel sections and the Killing vectors are in one-to-one correspondence.

12 Appendix B: Software demo for conformal Killing vectors

We will now feature a 3-dimensional example to demonstrate our software for the tractor approach for conformal Killing vectors. We will construct the tractor bundle and calculate the tractor connection and tractor curvature. Then we will calculate bases for the reduced tractor bundles of orders 0 and 1–we will then obtain the conformal Killing vectors explicitly. We will compare these computed conformal Killing vectors to the conformal Killing vectors calculated conventionally.

Appendix B: Software demo for conformal Killing vectors.

In this demonstration, we will illustrate the utility of the programs with the following names: MaxCF, BundleLift, ConfTracConn, getConfKV, and liftConfKV.

First, we will read in the file which contains the programs we have written and load in other required packages.

```
> read "TractorPrograms.txt";
  with(DifferentialGeometry):
  with(Tensor):
```

Now, we will initialize a 3-dimensional manifold.

```
> DGEnvironment[Coordinate]([x,y,z], M);
```

Manifold: M **(1)**

Next, we will define a simple metric.

```
M > g := evalDG(dx &t dx + dy &t dy + x * dz &t dz);
```

$$g := dx \otimes dx + dy \otimes dy + x\, dz \otimes dz \quad (2)$$

We need to know the required size of the fibers of the tractor bundle. We find this to be 10:

```
M > MaxCF(3, 1);
```

10 **(3)**

Now we initialize the required environments.

```
M > DGEnvironment[VectorSpace](10, V);
```

Vector Space: V **(4)**

```
V > DGEnvironment[VectorBundle](M, V, N);
```

Vector Bundle: N **(5)**

Now that the vector bundle has been initialized, it is convenient to redefine the metric on this bundle:

```
N > G := BundleLift(g, N);
```

$$G := dx \otimes dx + dy \otimes dy + x\, dz \otimes dz \quad (6)$$

Now we can compute the tractor connection:

```
N > C := ConfTracConn(G, N);
```

$$C := \nabla_{\partial_x} E1 = -\frac{1}{4x^3}E8, \nabla_{\partial_x} E3 = -\frac{1}{2x}E3 + \frac{1}{4x^2}E5, \nabla_{\partial_x} E4 = -E2 - \frac{1}{4x^2}E9, \nabla_{\partial_x} E5 = -E3 - \frac{1}{2x}E5, \nabla_{\partial_x} E6 = -\frac{1}{2x}E6, \nabla_{\partial_x} E7 = E1 - \frac{1}{4x^2}E8, \nabla_{\partial_x} E8 = E7, \nabla_{\partial_x} E9 = E4, \nabla_{\partial_x} E10 = E5 - \frac{1}{2x}E10, \nabla_{\partial_y} E1 = \frac{1}{4x^3}E9, \nabla_{\partial_y} E4 = E1 - \frac{1}{4x^2}E8, \nabla_{\partial_y} E6 = -E3 + \frac{1}{4x^2}E10, \nabla_{\partial_y} E7 = E2 + \frac{1}{4x^2}E9, \nabla_{\partial_y} E8 = -E4, \nabla_{\partial_y} E9 = E7, \nabla_{\partial_y} E10 = E6, \nabla_{\partial_z} E1 = \left(\frac{1}{2}\right)E3 - \frac{1}{4x}E5 - \frac{1}{4x^2}E10, \nabla_{\partial_z} E3 = -\frac{1}{2x}E1, \nabla_{\partial_z} E4 = -\left(\frac{1}{2}\right)E6, \nabla_{\partial_z} E5 = E1, \nabla_{\partial_z} E6 = E2 + \frac{1}{2x}E4 + \frac{1}{4x^2}E9, \nabla_{\partial_z} E7 = xE3 - \frac{1}{4x}E10, \nabla_{\partial_z} E8 = -xE5 + \left(\frac{1}{2}\right)E10, \nabla_{\partial_z} E9 = -xE6, \nabla_{\partial_z} E10 = E7 - \frac{1}{2x}E8 \tag{7}$$

We now compute the Curvature tensor for the tractor connection.

```
N > K := CurvatureTensor(C);
```

$$K := \frac{1}{8x^4}E8 \otimes \Theta3 \otimes dx \otimes dz - \frac{1}{8x^4}E8 \otimes \Theta3 \otimes dz \otimes dx + \frac{1}{4x^3}E8 \otimes \Theta4 \otimes dx \otimes dy - \frac{1}{4x^3}E8 \otimes \Theta4 \otimes dy \otimes dx - \frac{1}{4x^3}E8 \otimes \Theta5 \otimes dx \otimes dz + \frac{1}{4x^3}E8 \otimes \Theta5 \otimes dz \otimes dx - \frac{3}{4x^4}E9 \otimes \Theta1 \otimes dx \otimes dy + \frac{3}{4x^4}E9 \otimes \Theta1 \otimes dy \otimes dx - \frac{1}{8x^4}E9 \otimes \Theta3 \otimes dy \otimes dz + \frac{1}{8x^4}E9 \otimes \Theta3 \otimes dz \otimes dy + \frac{1}{4x^3}E9 \otimes \Theta5 \otimes dy \otimes dz - \frac{1}{4x^3}E9 \otimes \Theta5 \otimes dz \otimes dy - \frac{1}{2x^3}E9 \otimes \Theta6 \otimes dx \otimes dz + \frac{1}{2x^3}E9 \otimes \Theta6 \otimes dz \otimes dx - \frac{3}{4x^3}E9 \otimes \Theta7 \otimes dx \otimes dy + \frac{3}{4x^3}E9 \otimes \Theta7 \otimes dy \otimes dx + \frac{3}{4x^3}E10 \otimes \Theta1 \otimes dx \otimes dz - \frac{3}{4x^3}E10 \otimes \Theta1 \otimes dz \otimes dx + \frac{1}{4x^2}E10 \otimes \Theta4 \otimes dy \otimes dz - \frac{1}{4x^2}E10 \otimes \Theta4 \otimes dz \otimes dy - \frac{1}{2x^3}E10 \otimes \Theta6 \otimes dx \otimes dy + \frac{1}{2x^3}E10 \otimes \Theta6 \otimes dy \otimes dx + \frac{3}{4x^2}E10 \otimes \Theta7 \otimes dx \otimes dz - \frac{3}{4x^2}E10 \otimes \Theta7 \otimes dz \otimes dx \tag{8}$$

We can represent this tensor as a collection of (1,1) tensors by contracting the curvature tensor with the coordinate vectors of the base manifold.

```
N > k1 := ContractIndices(K, evalDG(D_x &t D_y), [[3,1],[4,2]] );
```

$$k1 := \frac{1}{4\,x^3}\,E8 \otimes \Theta 4 - \frac{3}{4\,x^4}\,E9 \otimes \Theta 1 - \frac{3}{4\,x^3}\,E9 \otimes \Theta 7 - \frac{1}{2\,x^3}\,E10 \otimes \Theta 6 \tag{9}$$

```
N > k2 := ContractIndices(K, evalDG(D_x &t D_z), [[3,1],[4,2]] );
```

$$k2 := \frac{1}{8\,x^4}\,E8 \otimes \Theta 3 - \frac{1}{4\,x^3}\,E8 \otimes \Theta 5 - \frac{1}{2\,x^3}\,E9 \otimes \Theta 6 + \frac{3}{4\,x^3}\,E10 \otimes \Theta 1 + \frac{3}{4\,x^2}\,E10 \otimes \Theta 7 \tag{10}$$

```
N > k3 := ContractIndices(K, evalDG(D_y &t D_z), [[3,1],[4,2]] );
```

$$k3 := -\frac{1}{8\,x^4}\,E9 \otimes \Theta 3 + \frac{1}{4\,x^3}\,E9 \otimes \Theta 5 + \frac{1}{4\,x^2}\,E10 \otimes \Theta 4 \tag{11}$$

We can think of these as matrices

```
N > K1 := convert(k1, DGMatrix);
    K2 := convert(k2, DGMatrix);
    K3 := convert(k3, DGMatrix);
```

$$K1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4\,x^3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{3}{4\,x^4} & 0 & 0 & 0 & 0 & 0 & -\frac{3}{4\,x^3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\,x^3} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K2 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{8\,x^4} & 0 & -\frac{1}{4\,x^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2\,x^3} & 0 & 0 & 0 & 0 \\ \frac{3}{4\,x^3} & 0 & 0 & 0 & 0 & 0 & \frac{3}{4\,x^2} & 0 & 0 & 0 \end{bmatrix}$$

$$K3 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{8\,x^4} & 0 & \frac{1}{4\,x^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4\,x^2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(12)}$$

and subsequently find a basis for the 0th order reduced tractor bundle.

```
N > IT0 := LieAlgebras:-InvariantTensors([K1,K2,K3], [seq(E||i,i=
    1..10)]);
```

$$IT0 := \left[E1 - \frac{1}{x}\,E7,\ E2,\ E3 + \frac{1}{2\,x}\,E5,\ E8,\ E9,\ E10\right] \quad \textbf{(13)}$$

How many are there?

```
N > nops(IT0);
```

$$6 \tag{14}$$

Thus, there are a maximum of 6 conformal Killing vectors of the metric g.

Now let's see if we can get the Killing vectors explicitly. We begin by forming a function-coefficient linear combination of these basis elements.

```
N > s1 := DGzip([seq(q||i(x,y,z), i=1..nops(IT0))], IT0, "plus");
```

$$s1 := q1(x, y, z)\, E1 + q2(x, y, z)\, E2 + q3(x, y, z)\, E3 + \frac{q3(x, y, z)}{2\, x}\, E5 - \frac{q1(x, y, z)}{x}\, E7 + q4(x, y, z)\, E8 + q5(x, y, z)\, E9 + q6(x, y, z)\, E10 \tag{15}$$

Next, we take the covariant derivative.

```
N > s2 := CovariantDerivative(s1, C);
```

$$s2 := \left(\frac{\partial}{\partial x}\, q1(x, y, z) - \frac{q1(x, y, z)}{x}\right) E1 \otimes dx + \frac{\partial}{\partial y}\, q1(x, y, z)\, E1 \otimes dy + \frac{\partial}{\partial z}\, q1(x, y, z)\, E1 \otimes dz + \frac{\partial}{\partial x}\, q2(x, y, z)\, E2 \otimes dx + \left(\frac{\partial}{\partial y}\, q2(x, y, z) - \frac{q1(x, y, z)}{x}\right) E2 \otimes dy + \frac{\partial}{\partial z}\, q2(x, y, z)\, E2 \otimes dz + \left(\frac{\partial}{\partial x}\, q3(x, y, z) - \frac{q3(x, y, z)}{x}\right) E3 \otimes dx + \frac{\partial}{\partial y}\, q3(x, y, z)\, E3 \otimes dy + \left(\frac{\partial}{\partial z}\, q3(x, y, z) - \frac{q1(x, y, z)}{2}\right) E3 \otimes dz + q5(x, y, z)\, E4 \otimes dx - q4(x, y, z)\, E4 \otimes dy + \left(\frac{\frac{\left(\frac{\partial}{\partial x}\, q3(x, y, z)\right) x}{2} - \frac{q3(x, y, z)}{2}}{x^2} + q6(x, y, z)\right) E5 \otimes dx + \frac{\frac{\partial}{\partial y}\, q3(x, y, z)}{2\, x}\, E5 \otimes dy \tag{16}$$

$$- \frac{4\, q4(x,y,z)\, x^2 + q1(x,y,z) - 2 \left(\frac{\partial}{\partial z} q3(x,y,z) \right)}{4\, x}\, E5 \otimes dz + q6(x,y,z)\, E6$$

$$\otimes dy - q5(x,y,z)\, x\, E6 \otimes dz + \left(\frac{-\left(\frac{\partial}{\partial x} q1(x,y,z) \right) x + q1(x,y,z)}{x^2} + q4(x,y,z) \right.$$

$$\left. \right) E7 \otimes dx + \left(-\frac{\frac{\partial}{\partial y} q1(x,y,z)}{x} + q5(x,y,z) \right) E7 \otimes dy + \left(-\frac{\frac{\partial}{\partial z} q1(x,y,z)}{x} \right.$$

$$\left. + q6(x,y,z) \right) E7 \otimes dz + \frac{\partial}{\partial x} q4(x,y,z)\, E8 \otimes dx + \frac{\partial}{\partial y} q4(x,y,z)\, E8 \otimes dy + \left(\frac{\partial}{\partial z} \right.$$

$$\left. q4(x,y,z) - \frac{q6(x,y,z)}{2\, x} \right) E8 \otimes dz + \frac{\partial}{\partial x} q5(x,y,z)\, E9 \otimes dx + \frac{\partial}{\partial y} q5(x,y,z)\, E9$$

$$\otimes dy + \frac{\partial}{\partial z} q5(x,y,z)\, E9 \otimes dz + \left(\frac{\partial}{\partial x} q6(x,y,z) - \frac{q6(x,y,z)}{2\, x} \right) E10 \otimes dx + \frac{\partial}{\partial y}$$

$$q6(x,y,z)\, E10 \otimes dy + \left(\frac{\partial}{\partial z} q6(x,y,z) + \frac{q4(x,y,z)}{2} \right) E10 \otimes dz$$

Let's look at the equations we need to solve.

```
N > s3 := DGinformation(s2, "CoefficientSet");
```

$$s3 := \left\{ \frac{1}{2} \frac{\frac{\partial}{\partial y} q3(x,y,z)}{x}, -\frac{1}{4} \frac{4\, q4(x,y,z)\, x^2 + q1(x,y,z) - 2 \left(\frac{\partial}{\partial z} q3(x,y,z) \right)}{x}, \right. \quad \textbf{(17)}$$

$$-q5(x,y,z)\, x,\ -q4(x,y,z),\ -\frac{\frac{\partial}{\partial y} q1(x,y,z)}{x} + q5(x,y,z),\ -\frac{\frac{\partial}{\partial z} q1(x,y,z)}{x}$$

$$+ q6(x,y,z),\ \frac{-\left(\frac{\partial}{\partial x} q1(x,y,z) \right) x + q1(x,y,z)}{x^2} + q4(x,y,z),$$

$$\frac{\frac{1}{2} \left(\frac{\partial}{\partial x} q3(x,y,z) \right) x - \frac{1}{2} q3(x,y,z)}{x^2} + q6(x,y,z),\ \frac{\partial}{\partial x} q1(x,y,z)$$

$$- \frac{q1(x,y,z)}{x},\ \frac{\partial}{\partial y} q2(x,y,z) - \frac{q1(x,y,z)}{x},\ \frac{\partial}{\partial x} q3(x,y,z) - \frac{q3(x,y,z)}{x},$$

$$\frac{\partial}{\partial z} q3(x,y,z) - \frac{1}{2} q1(x,y,z),\ \frac{\partial}{\partial z} q4(x,y,z) - \frac{1}{2} \frac{q6(x,y,z)}{x},\ \frac{\partial}{\partial x} q6(x,y,z)$$

$$-\frac{1}{2}\frac{q6(x,y,z)}{x}, \frac{\partial}{\partial z} q6(x,y,z)+\frac{1}{2} q4(x,y,z), \frac{\partial}{\partial y} q1(x,y,z), \frac{\partial}{\partial z} q1(x,y,z), \frac{\partial}{\partial x} q2(x,y,z), \frac{\partial}{\partial z} q2(x,y,z), \frac{\partial}{\partial y} q3(x,y,z), \frac{\partial}{\partial x} q4(x,y,z), \frac{\partial}{\partial y} q4(x,y,z), \frac{\partial}{\partial x} q5(x,y,z), \frac{\partial}{\partial y} q5(x,y,z), \frac{\partial}{\partial z} q5(x,y,z), \frac{\partial}{\partial y} q6(x,y,z), q5(x,y,z), q6(x,y,z)\}$$

How many equations are there?

```
N > nops(s3);
```

$$28 \quad \textbf{(18)}$$

This system may be easier to solve than the Killing equation itself. Let's get the solution.

```
N > s4 := pdsolve(s3, {seq(q||i(x,y,z), i=1..nops(IT0))} );
```

$$s4 := \{q1(x,y,z) = 2\,\_C1\,x,\ q2(x,y,z) = 2\,\_C1\,y + \_C3,\ q3(x,y,z) = (\_C1\,z + \_C2)\,x,\ q4(x,y,z) = 0,\ q5(x,y,z) = 0,\ q6(x,y,z) = 0\} \quad \textbf{(19)}$$

How many independent solutions are there?

```
N > has(s4,_C3);
```

$$true \quad \textbf{(20)}$$

```
N > has(s4,_C4);
```

$$false \quad \textbf{(21)}$$

Thus, there are 3 independent solutions. Here is then what the parallel sections should look like:

```
N > s5 := DETools:-dsubs(s4, s1);
```

$$s5 := 2\,\_C1\,x\,E1 + \left(2\,\_C1\,y + \_C3\right) E2 + (\_C1\,z + \_C2)\,x\,E3 + \left(\frac{\_C1\,z}{2} + \frac{\_C2}{2}\right) E5 - 2\,\_C1\,E7 + 0\,E8 + 0\,E9 + 0\,E10 \quad \textbf{(22)}$$

Now we will generate a list of parallel sections according to the independent solutions.

```
N > t1 := eval(s5, [_C1=1,_C2=0,_C3=0]);
    t2 := eval(s5, [_C1=0,_C2=1,_C3=0]);
    t3 := eval(s5, [_C1=0,_C2=0,_C3=1]);
```

$$t1 := 2\,x\,E1 + 2\,y\,E2 + z\,x\,E3 + \frac{z}{2}\,E5 - 2\,E7 + 0\,E8 + 0\,E9 + 0\,E10$$

$$t2 := 0\,E1 + 0\,E2 + x\,E3 + \left(\frac{1}{2}\right)\,E5 + 0\,E7 + 0\,E8 + 0\,E9 + 0\,E10$$

$$t3 := 0\,E1 + E2 + 0\,E3 + 0\,E5 + 0\,E7 + 0\,E8 + 0\,E9 + 0\,E10 \quad \textbf{(23)}$$

We check that they're parallel sections:

```
N > map(CovariantDerivative, [t1,t2,t3], C);
```

$$[0\,E1 \otimes dx,\, 0\,E1 \otimes dx,\, 0\,E1 \otimes dx] \quad \textbf{(24)}$$

Now we drop these parallel sections down to the base space:

```
N > T := map(getConfKV, [t1,t2,t3], N);
```

$$T := [2\,x\,dx + 2\,y\,dy + z\,x\,dz,\, x\,dz,\, dy] \quad \textbf{(25)}$$

These are the (covariant) conformal Killing vectors of the metric. Now let's check that we can lift the conformal Killing vectors of the metric to parallel sections. We begin by calculating them conventionally.

```
N > ckv := ConformalKillingVectors(G);
```

$$ckv := \left[2\,x\,\partial_x + 2\,y\,\partial_y + z\,\partial_z\right],\, \left[\partial_z,\, \partial_y\right] \quad \textbf{(26)}$$

Let's get the covariant versions:

```
N > ckt := ListTools:-FlattenOnce([map2(RaiseLowerIndices, G, ckv
    [1], [1]), map2(RaiseLowerIndices, G, ckv[2], [1])] );
```

$$ckt := [2\,x\,dx + 2\,y\,dy + z\,x\,dz,\, x\,dz,\, dy] \quad \textbf{(27)}$$

Now let's lift them to sections.

```
N > liftckt := map(liftConfKV, ckt, G, N);
```

$$liftckt := \left[2\,x\,E1 + 2\,y\,E2 + z\,x\,E3 + \frac{z}{2}\,E5 - 2\,E7,\, x\,E3 + \left(\frac{1}{2}\right)\,E5,\, E2\right] \quad \textbf{(28)}$$

Let's check that these are parallel sections.

```
N > map(CovariantDerivative, liftckt, C);
```

$$[0\,E1 \otimes dx,\, 0\,E1 \otimes dx,\, 0\,E1 \otimes dx] \quad \textbf{(29)}$$

But are there more parallel sections?

```
N > CovariantlyConstantTensors(C, [seq(E||i,i=1..10)]);
```

$$\left[-x\,E1 - y\,E2 - \frac{z\,x}{2}\,E3 - \frac{z}{4}\,E5 + E7,\ 2\,x\,E3 + E5,\ E2 \right] \tag{30}$$

```
N > nops(%);
```

$$3 \tag{31}$$

Thus, the parallel sections and the Killing tensors are in one-to-one correspondence.

13 Appendix C: Software demonstration for Killing tensors of rank 2

We will return to the metric $u^p(du^2 + dv^2)$ to demonstrate our software for implementing the tractor approach for Killing tensors. We will, as done previously, construct the bundle, compute the tractor connection, compute the tractor curvature, and compute bases for the reduced tractor bundles in order to explicitly identify the Killing tensors of rank 2. However, we will also illustrate the point that certain values of p may yield different results and we will identify an exceptional value of p.

Appendix C: Software demonstration for Killing tensors of rank 2.

In this demonstration, we will illustrate the utility of the programs with the following names: MaxKT, BundleLift, HauserTractorConnection, getHauserKT2, and HauserTractorLift2.

First, we will read in the file which contains the programs we have written and also load in the required packages.

```
> read "TractorPrograms.txt";
  with(DifferentialGeometry):
  with(Tensor):
```

Now, we will initialize a 2-dimensional manifold.

```
> DGEnvironment[Coordinate]([u,v], M);
```

Manifold: M **(1)**

Next, we will define a simple metric.

```
M > g := evalDG(u^p * (du &t du + dv &t dv) );
```

$$g := u^p\, du \otimes du + u^p\, dv \otimes dv \qquad \textbf{(2)}$$

We need to know the required size of the fibers of the tractor bundle. We find this to be 6:

```
M > MaxKT(2, 2);
```

6 **(3)**

Now we initialize the required environments.

```
M > DGEnvironment[VectorSpace](6, V);
```

Vector Space: V **(4)**

```
V > DGEnvironment[VectorBundle](M, V, N);
```

Vector Bundle: N **(5)**

Now that the vector bundle has been initialized, it is convenient to redefine the metric on this bundle:

```
N > G := BundleLift(g, N);
```

$$G := u^p\, du \otimes du + u^p\, dv \otimes dv \qquad \textbf{(6)}$$

We will also need the Christoffel symbols:

```
N > Gamma := Christoffel(G);
```

$$\Gamma := \nabla_{\partial_u} \partial_u = \frac{p}{2\,u}\,\partial_u, \nabla_{\partial_u} \partial_v = \frac{p}{2\,u}\,\partial_v, \nabla_{\partial_v} \partial_u = \frac{p}{2\,u}\,\partial_v, \nabla_{\partial_v} \partial_v = -\frac{p}{2\,u}\,\partial_u \tag{7}$$

Now we can compute the tractor connection:

```
N > C := HauserTractorConnection(Gamma, N);
```

$$C := \nabla_{\partial_u} E1 = -\frac{p}{u}\,E1 + \frac{3\,p}{4\,u^2}\,E5 - \frac{3\,p\,(2+p)}{4\,u^3}\,E6, \nabla_{\partial_u} E2 = -\frac{p}{u}\,E2 - \frac{3\,p}{2\,u^2}\,E4, \nabla_{\partial_u} E3 = -\frac{p}{u}\,E3 - \frac{3\,p}{4\,u^2}\,E5 + \frac{3\,p\,(2+p)}{4\,u^3}\,E6, \nabla_{\partial_u} E4 = \left(\frac{1}{3}\right) E2 - \frac{3\,p}{2\,u}\,E4, \nabla_{\partial_u} E5 = \left(\frac{2}{3}\right) E3 - \frac{3\,p}{2\,u}\,E5 + \frac{3\,p}{2\,u^2}\,E6, \nabla_{\partial_u} E6 = -E5 - \frac{2\,p}{u}\,E6, \nabla_{\partial_v} E1 = \frac{p}{2\,u}\,E2 + \frac{3\,p}{4\,u^2}\,E4, \nabla_{\partial_v} E2 = -\frac{p}{u}\,E1 + \frac{p}{u}\,E3 + \frac{3\,p}{2\,u^2}\,E5 - \frac{3\,p\,(2+p)}{2\,u^3}\,E6, \nabla_{\partial_v} E3 = -\frac{p}{2\,u}\,E2 - \frac{3\,p}{4\,u^2}\,E4, \nabla_{\partial_v} E4 = -\left(\frac{2}{3}\right) E1 + \frac{p}{2\,u}\,E5 - \frac{3\,p}{2\,u^2}\,E6, \nabla_{\partial_v} E5 = -\left(\frac{1}{3}\right) E2 - \frac{p}{2\,u}\,E4, \nabla_{\partial_v} E6 = E4 \tag{8}$$

We can also represent this connection as a matrix of 1-forms:

```
N > convert(C, DGMatrix);
```

$$\left[\left[-\frac{p}{u}\,du, -\frac{p}{u}\,dv, 0\,du, -\left(\frac{2}{3}\right) dv, 0\,du, 0\,du\right], \left[\frac{p}{2\,u}\,dv, -\frac{p}{u}\,du, -\frac{p}{2\,u}\,dv, \left(\frac{1}{3}\right) du, -\left(\frac{1}{3}\right) dv, 0\,du\right], \left[0\,du, \frac{p}{u}\,dv, -\frac{p}{u}\,du, 0\,du, \left(\frac{2}{3}\right) du, 0\,du\right], \left[\frac{3\,p}{4\,u^2}\,dv, -\frac{3\,p}{2\,u^2}\,du, -\frac{3\,p}{4\,u^2}\,dv, -\frac{3\,p}{2\,u}\,du, -\frac{p}{2\,u}\,dv, dv\right], \left[\frac{3\,p}{4\,u^2}\,du, \frac{3\,p}{2\,u^2}\,dv, -\frac{3\,p}{4\,u^2}\,du, \frac{p}{2\,u}\,dv, -\frac{3\,p}{2\,u}\,du, -du\right], \left[-\frac{3\,p\,(2+p)}{4\,u^3}\,du, -\frac{3\,p\,(2+p)}{2\,u^3}\,dv, \frac{3\,p\,(2+p)}{4\,u^3}\,du, -\frac{3\,p}{2\,u^2}\,dv, \frac{3\,p}{2\,u^2}\,du, -\frac{2\,p}{u}\,du\right]\right] \tag{9}$$

We now compute the Curvature tensor for the tractor connection.

```
N > K := CurvatureTensor(C);
```

$$K := \frac{3\,p\,(2+p)\,(3+2\,p)}{2\,u^4}\,E6 \otimes \Theta 2 \otimes du \otimes dv - \frac{3\,p\,(2+p)\,(3+2\,p)}{2\,u^4}\,E6 \otimes \Theta 2 \otimes dv \otimes du + \frac{5\,p\,(2+p)}{2\,u^3}\,E6 \otimes \Theta 4 \otimes du \otimes dv - \frac{5\,p\,(2+p)}{2\,u^3}\,E6 \otimes \Theta 4 \otimes dv \otimes du \quad (10)$$

It is interesting to note that p=0 and p=-2 are the only values of p for which the curvature tensor vanishes identically.

We can represent this tensor as a collection of (1,1) tensors by contracting the curvature tensor with the coordinate vectors of the base manifold. In our 2-dimensional case, there is only one such (1,1) tensor, but there may be others in higher dimensions.

```
N > k1 := ContractIndices(K, evalDG(D_u &t D_v), [[3,1],[4,2]] );
```

$$k1 := \frac{3\,p\,(2+p)\,(3+2\,p)}{2\,u^4}\,E6 \otimes \Theta 2 + \frac{5\,p\,(2+p)}{2\,u^3}\,E6 \otimes \Theta 4 \quad (11)$$

We can think of this as a matrix

```
N > K1 := convert(k1, DGMatrix);
```

$$K1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{3}{2}\,\frac{p\,(2+p)\,(3+2\,p)}{u^4} & 0 & \frac{5}{2}\,\frac{p\,(2+p)}{u^3} & 0 & 0 \end{bmatrix} \quad (12)$$

and subsequently find the basis for the 0th order reduced tractor bundle:

```
N > IT0 := LieAlgebras:-InvariantTensors([K1], [seq(E||i,i=1..6)]
    ) ;
```

$$IT0 := \left[E1,\ E2 - \frac{3\,(3+2\,p)}{5\,u}\,E4,\ E3,\ E5,\ E6\right] \quad (13)$$

How many are there?

```
N > nops(IT0);
```

$$5 \tag{14}$$

This calculation is assuming that p is not 0 or 2: caution is advised when examining metrics with unknown constants as exponents. Assuming that p is not 0 or 2, there are a maximum of 5 Killing tensors of the metric g..

We will now try to get a tighter upper bound. We begin by differentiating the curvature tensor.

```
N > dK1 := CovariantDerivative(k1, C);
```

$$dK1 := \frac{3\,p\,(2+p)\,(3+2\,p)}{2\,u^4}\,E4 \otimes \Theta 2 \otimes dv + \frac{5\,p\,(2+p)}{2\,u^3}\,E4 \otimes \Theta 4 \otimes dv - \frac{3\,p\,(2+p)\,(3+2\,p)}{2\,u^4}\,E5 \otimes \Theta 2 \otimes du - \frac{5\,p\,(2+p)}{2\,u^3}\,E5 \otimes \Theta 4 \otimes du - \frac{3\,p^2\,(2+p)\,(11+4\,p)}{8\,u^5}\,E6 \otimes \Theta 1 \otimes dv - \frac{3\,p\,(2+p)\,(4\,p^2+17\,p+24)}{4\,u^5}\,E6 \otimes \Theta 2 \otimes du + \frac{3\,p^2\,(2+p)\,(11+4\,p)}{8\,u^5}\,E6 \otimes \Theta 3 \otimes dv - \frac{9\,p\,(2+p)\,(4+p)}{4\,u^4}\,E6 \otimes \Theta 4 \otimes du + \frac{3\,p\,(2+p)\,(2+3\,p)}{4\,u^4}\,E6 \otimes \Theta 5 \otimes dv - \frac{5\,p\,(2+p)}{2\,u^3}\,E6 \otimes \Theta 6 \otimes dv \tag{15}$$

We now generate a set of (1,1) tensors by contraction:

```
N > D1K1 := ContractIndices(dK1, D_u, [[3,1]]);
```

$$D1K1 := -\frac{3\,p\,(2+p)\,(3+2\,p)}{2\,u^4}\,E5 \otimes \Theta 2 - \frac{5\,p\,(2+p)}{2\,u^3}\,E5 \otimes \Theta 4 - \frac{3\,p\,(2+p)\,(4\,p^2+17\,p+24)}{4\,u^5}\,E6 \otimes \Theta 2 - \frac{9\,p\,(2+p)\,(4+p)}{4\,u^4}\,E6 \otimes \Theta 4 \tag{16}$$

```
N > D2K1 := ContractIndices(dK1, D_v, [[3,1]]);
```

$$D2K1 := \frac{3\,p\,(2+p)\,(3+2\,p)}{2\,u^4}\,E4 \otimes \Theta 2 + \frac{5\,p\,(2+p)}{2\,u^3}\,E4 \otimes \Theta 4 - \frac{3\,p^2\,(2+p)\,(11+4\,p)}{8\,u^5}\,E6 \otimes \Theta 1 + \frac{3\,p^2\,(2+p)\,(11+4\,p)}{8\,u^5}\,E6 \otimes \Theta 3 + \frac{3\,p\,(2+p)\,(2+3\,p)}{4\,u^4}\,E6 \otimes \Theta 5 - \frac{5\,p\,(2+p)}{2\,u^3}\,E6 \otimes \Theta 6 \tag{17}$$

Now we will convert them to matrices.

```
N > Md1k1 := convert(D1K1, DGMatrix);
```

$$Md1k1 := \left[\left[0, 0, 0, 0, 0, 0\right],\right.$$

$$\left[0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0\right],$$

$$\left[0, -\frac{3}{2}\ \frac{p\,(2+p)\,(3+2\,p)}{u^4}, 0, -\frac{5}{2}\ \frac{p\,(2+p)}{u^3}, 0, 0\right],$$

$$\left.\left[0, -\frac{3}{4}\ \frac{p\,(2+p)\,\left(4\,p^2+17\,p+24\right)}{u^5}, 0, -\frac{9}{4}\ \frac{p\,(2+p)\,(4+p)}{u^4}, 0, 0\right]\right] \quad \textbf{(18)}$$

```
N > Md2k1 := convert(D2K1, DGMatrix);
```

$$Md2k1 := \left[\left[0, 0, 0, 0, 0, 0\right],\right. \quad \textbf{(19)}$$

$$\left[0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0\right],$$

$$\left[0, \frac{3}{2}\ \frac{p\,(2+p)\,(3+2\,p)}{u^4}, 0, \frac{5}{2}\ \frac{p\,(2+p)}{u^3}, 0, 0\right],$$

$$\left[0, 0, 0, 0, 0, 0\right],$$

$$\left[-\frac{3}{8}\ \frac{p^2\,(2+p)\,(11+4\,p)}{u^5}, 0, \frac{3}{8}\ \frac{p^2\,(2+p)\,(11+4\,p)}{u^5}, 0,\right.$$

$$\left.\left.\frac{3}{4}\ \frac{p\,(2+p)\,(2+3\,p)}{u^4}, -\frac{5}{2}\ \frac{p\,(2+p)}{u^3}\right]\right]$$

And now we can find a basis for the 1st order reduced tractor bundle:

```
N > IT1 := LieAlgebras:-InvariantTensors([K1,Md1k1,Md2k1], [seq
    (E||i,i=1..6)]);
```

$$IT1 := \left[E1 - \frac{3\,p\,(11+4\,p)}{20\,u^2}\ E6, E3 + \frac{3\,p\,(11+4\,p)}{20\,u^2}\ E6, E5 + \frac{3\,(2+3\,p)}{10\,u}\ E6\right] \quad \textbf{(20)}$$

```
N > nops(IT1);
```

$$3 \quad \textbf{(21)}$$

We are led to believe that there is a maximum of 3 Killing tensors. However, if p=1, we get 4 basis elements:

```
N > rIT1 := LieAlgebras:-InvariantTensors(eval([K1,Md1k1,Md2k1],
    p=1), [seq(E||i,i=1..6)]);
```

$$rIT1 := \left[E1 - \frac{9}{4\,u^2}\,E6,\, E2 - \frac{3}{u}\,E4,\, E3 + \frac{9}{4\,u^2}\,E6,\, E5 + \frac{3}{2\,u}\,E6\right] \quad \textbf{(22)}$$

```
N > nops(rIT1);
```

$$4 \quad \textbf{(23)}$$

Thus, it is not advisable to work with metrics which have unknown exponents, unless the exceptional values for those exponents are being sought.

Now let's see if we can get the Killing tensors explicitly for p=1. We begin by forming a function-coefficient linear combination of the basis elements of the 1st order reduced tractor bundle.

```
N > s1 := DGzip([seq(q||i(u,v), i=1..nops(rIT1))], rIT1, "plus");
```

$$s1 := q1(u,v)\,E1 + q2(u,v)\,E2 + q3(u,v)\,E3 - \frac{3\,q2(u,v)}{u}\,E4 + q4(u,v)\,E5 + \left(-\frac{9\,q1(u,v)}{4\,u^2} + \frac{9\,q3(u,v)}{4\,u^2} + \frac{3\,q4(u,v)}{2\,u}\right)E6 \quad \textbf{(24)}$$

Next, we take the covariant derivative.

```
N > s2 := CovariantDerivative(s1, eval(C, p=1) );
```

$$s2 := \left(\frac{\partial}{\partial u}\,q1(u,v) - \frac{q1(u,v)}{u}\right)E1 \otimes du + \left(\frac{\partial}{\partial v}\,q1(u,v) + \frac{q2(u,v)}{u}\right)E1 \otimes dv + \left(\frac{\partial}{\partial u}\,q2(u,v) - \frac{2\,q2(u,v)}{u}\right)E2 \otimes du + \frac{-2\,q4(u,v)\,u + 6\left(\frac{\partial}{\partial v}\,q2(u,v)\right)u + 3\,q1(u,v) - 3\,q3(u,v)}{6\,u}\,E2 \otimes dv + \left(\frac{\partial}{\partial u}\,q3(u,v) - \frac{q3(u,v)}{u} + \frac{2\,q4(u,v)}{3}\right)E3 \otimes du + \left(\frac{\partial}{\partial v}\,q3(u,v) + \frac{q2(u,v)}{u}\right)E3 \otimes dv + \frac{3\left(-\left(\frac{\partial}{\partial u}\,q2(u,v)\right)u + 2\,q2(u,v)\right)}{u^2}\,E4 \otimes du \quad \textbf{(25)}$$

$$- \frac{-2\, q4(u,v)\, u + 6 \left(\frac{\partial}{\partial v} q2(u,v)\right) u + 3\, q1(u,v) - 3\, q3(u,v)}{2\, u^2} E4 \otimes dv$$
$$+ \frac{\left(\frac{\partial}{\partial u} q4(u,v)\right) u^2 - 3\, q4(u,v)\, u + 3\, q1(u,v) - 3\, q3(u,v)}{u^2} E5 \otimes du + \frac{\partial}{\partial v} q4(u,v)\, E5 \otimes dv + \frac{1}{4\, u^3}\left(3\left(2\left(\frac{\partial}{\partial u} q4(u,v)\right) u^2 - 4\, q4(u,v)\, u - 3\left(\frac{\partial}{\partial u} q1(u,v)\right) u + 3\left(\frac{\partial}{\partial u} q3(u,v)\right) u + 9\, q1(u,v) - 9\, q3(u,v)\right)\right) E6 \otimes du$$
$$- \frac{3\left(-2\left(\frac{\partial}{\partial v} q4(u,v)\right) u + 3\left(\frac{\partial}{\partial v} q1(u,v)\right) - 3\left(\frac{\partial}{\partial v} q3(u,v)\right)\right)}{4\, u^2} E6 \otimes dv$$

Let's look at the equations we need to solve.

```
N > s3 := DGinformation(s2, "CoefficientSet");
```

$$s3 := \left\{ \frac{\left(\frac{\partial}{\partial u} q4(u,v)\right) u^2 - 3\, q4(u,v)\, u + 3\, q1(u,v) - 3\, q3(u,v)}{u^2}, \frac{3\left(-\left(\frac{\partial}{\partial u} q2(u,v)\right) u + 2\, q2(u,v)\right)}{u^2}, -\frac{3}{4}\, \frac{-2\left(\frac{\partial}{\partial v} q4(u,v)\right) u + 3\left(\frac{\partial}{\partial v} q1(u,v)\right) - 3\left(\frac{\partial}{\partial v} q3(u,v)\right)}{u^2}, -\frac{1}{2}\, \frac{-2\, q4(u,v)\, u + 6\left(\frac{\partial}{\partial v} q2(u,v)\right) u + 3\, q1(u,v) - 3\, q3(u,v)}{u^2}, \frac{1}{6}\, \frac{-2\, q4(u,v)\, u + 6\left(\frac{\partial}{\partial v} q2(u,v)\right) u + 3\, q1(u,v) - 3\, q3(u,v)}{u}, \frac{3}{4}\, \frac{1}{u^3}\left(2\left(\frac{\partial}{\partial u} q4(u,v)\right) u^2 - 4\, q4(u,v)\, u - 3\left(\frac{\partial}{\partial u} q1(u,v)\right) u + 3\left(\frac{\partial}{\partial u} q3(u,v)\right) u + 9\, q1(u,v) - 9\, q3(u,v)\right), \frac{\partial}{\partial u} q1(u,v) - \frac{q1(u,v)}{u}, \frac{\partial}{\partial v} q1(u,v) + \frac{q2(u,v)}{u}, \frac{\partial}{\partial u} q2(u,v) - \frac{2\, q2(u,v)}{u}, \frac{\partial}{\partial v} q3(u,v) + \frac{q2(u,v)}{u}, \frac{\partial}{\partial u} q3(u,v) - \frac{q3(u,v)}{u} + \frac{2}{3}\, q4(u,v), \frac{\partial}{\partial v} q4(u,v) \right\} \quad \textbf{(26)}$$

How many equations are there?

```
N > nops(s3);
```

$$12 \quad (27)$$

This system may be easier to solve than the Killing equation itself. Let's get the solution.

```
N > s4 := pdsolve(s3, {seq(q||i(u,v),i=1..nops(rIT1))} );
```

$$s4 := \left\{ q1(u, v) = \left(-\frac{1}{2}\_C1\, v^2 - \_C2\, v + \_C3\right) u,\ q2(u, v) = (\_C1\, v + \_C2)\, u^2,\right. \quad (28)$$
$$q3(u, v) = -\frac{1}{6}\, u\, (12\,\_C1\, u^2 + 3\,\_C1\, v^2 + 6\,\_C2\, v + 4\,\_C4\, u - 6\,\_C3),\ q4(u, v)$$
$$\left. = 6\,\_C1\, u^2 + \_C4\, u \right\}$$

How many independent solutions are there?

```
N > has(s4, _C4);
```

$$true \quad (29)$$

```
N > has(s4, _C5);
```

$$false \quad (30)$$

Thus, there are 4 independent solutions. Here is then what the parallel sections should look like:

```
N > s5 := DETools:-dsubs(s4, s1);
```

$$s5 := -\frac{(\_C1\, v^2 + 2\,\_C2\, v - 2\,\_C3)\, u}{2}\, E1 + (\_C1\, v + \_C2)\, u^2\, E2 \quad (31)$$
$$- \frac{u\, (12\,\_C1\, u^2 + 3\,\_C1\, v^2 + 6\,\_C2\, v + 4\,\_C4\, u - 6\,\_C3)}{6}\, E3 - 3\, u\, (\_C1\, v$$
$$+ \_C2)\, E4 + \left(6\,\_C1\, u^2 + \_C4\, u\right) E5 + \frac{9\, u\,\_C1}{2}\, E6$$

Now we will generate a list of parallel sections according to the independent solutions.

```
N > t1 := eval(s5, [_C1=1,_C2=0,_C3=0,_C4=0]);
    t2 := eval(s5, [_C1=0,_C2=1,_C3=0,_C4=0]);
    t3 := eval(s5, [_C1=0,_C2=0,_C3=1,_C4=0]);
    t4 := eval(s5, [_C1=0,_C2=0,_C3=0,_C4=1]);
```

$$t1 := -\frac{v^2\, u}{2}\, E1 + v\, u^2\, E2 - \frac{u\, (12\, u^2 + 3\, v^2)}{6}\, E3 - 3\, u\, v\, E4 + 6\, u^2\, E5 + \frac{9\, u}{2}\, E6$$

$$t2 := -u\,v\,E1 + u^2\,E2 - u\,v\,E3 - 3\,u\,E4 + 0\,E5 + 0\,E6$$
$$t3 := u\,E1 + 0\,E2 + u\,E3 + 0\,E4 + 0\,E5 + 0\,E6$$
$$t4 := 0\,E1 + 0\,E2 - \frac{2\,u^2}{3}\,E3 + 0\,E4 + u\,E5 + 0\,E6 \qquad \textbf{(32)}$$

We check that they're parallel sections:

```
N > map(CovariantDerivative, [t1,t2,t3,t4], eval(C, p=1) );
```

$$[0\,E1 \otimes du,\ 0\,E1 \otimes du,\ 0\,E1 \otimes du,\ 0\,E1 \otimes du] \qquad \textbf{(33)}$$

Now we drop these parallel sections down:

```
N > T := map(getHauserKT2, [t1,t2,t3,t4], N);
```

$$T := \left[-\frac{v^2\,u}{2}\,du \otimes du + v\,u^2\,du \otimes dv + v\,u^2\,dv \otimes du - \frac{u\,(4\,u^2+v^2)}{2}\,dv \otimes dv, -u\,v\,du \otimes du + u^2\,du \otimes dv + u^2\,dv \otimes du - u\,v\,dv \otimes dv,\ u\,du \otimes du + u\,dv \otimes dv,\ -\frac{2\,u^2}{3}\,dv \otimes dv\right] \qquad \textbf{(34)}$$

Let's check that these are Killing tensors of rank 2.

```
N > map2(CheckKillingTensor, eval(G, p=1), T);
```

$$[0\,du \otimes du \otimes du,\ 0\,du \otimes du \otimes du,\ 0\,du \otimes du \otimes du,\ 0\,du \otimes du \otimes du] \qquad \textbf{(35)}$$

Now let's check that we can lift the Killing tensors of the metric to parallel sections.
We begin by calculating the Killing tensors conventionally.

```
N > kt2 := KillingTensors(eval(G, p=1), 2);
```

$$kt2 := \left[\frac{v^2\,u}{2}\,du \otimes du - v\,u^2\,du \otimes dv - v\,u^2\,dv \otimes du + \left(\frac{1}{2}\,v^2\,u + 2\,u^3\right)\,dv \otimes dv,\ u\,v\,du \otimes du - u^2\,du \otimes dv - u^2\,dv \otimes du + u\,v\,dv \otimes dv,\ u\,du \otimes du + u\,dv \otimes dv,\ u^2\,dv \otimes dv\right] \qquad \textbf{(36)}$$

Note that the Christoffel symbols of G are needed for the lift.

```
N > liftkt2 := map(HauserTractorLift2, kt2, eval(Gamma, p=1), N);
```

$$liftkt2 := \left[\frac{v^2\,u}{2}\,E1 - v\,u^2\,E2 + \left(\frac{1}{2}\,v^2\,u + 2\,u^3\right)\,E3 + 3\,u\,v\,E4 - 6\,u^2\,E5 - \frac{9\,u}{2}\,E6,\ u\,v\,E1 - u^2\,E2 + u\,v\,E3 + 3\,u\,E4,\ u\,E1 + u\,E3,\ u^2\,E3 - \frac{3\,u}{2}\,E5\right] \qquad \textbf{(37)}$$

Let's check that these are parallel sections.

```
N > map(CovariantDerivative, liftkt2, eval(C, p=1) );
```

$$[0\,E1 \otimes du,\ 0\,E1 \otimes du,\ 0\,E1 \otimes du,\ 0\,E1 \otimes du] \tag{38}$$

But are there more parallel sections?

```
N > CovariantlyConstantTensors(eval(C, p=1), [seq(E||i,i=1..6)]);
```

$$\left[-\frac{v^2\,u}{9}\,E1 + \frac{2\,v\,u^2}{9}\,E2 - \frac{u\,(4\,u^2+v^2)}{9}\,E3 - \frac{2\,u\,v}{3}\,E4 + \frac{4\,u^2}{3}\,E5 + u\,E6,\ -\frac{2\,u^2}{3}\,E3 + u\,E5,\ \frac{u\,v}{3}\,E1 - \frac{u^2}{3}\,E2 + \frac{u\,v}{3}\,E3 + u\,E4,\ u\,E1 + u\,E3 \right] \tag{39}$$

```
N > nops(%);
```

$$4 \tag{40}$$

Thus, the parallel sections and the Killing tensors are in one-to-one correspondence.

14 Appendix D: Software demonstration for Killing-Yano tensors

Our last software demonstration illustrates the utility of our software in the study of Killing-Yano tensors of rank 2. We will demonstrate our software on a manifold of dimension 3, and we will use the same metric as in Appendix B. We will use the tractor approach to explicitly identify the single Killing-Yano tensor of rank 2.

Appendix D: Software demonstration for Killing-Yano tensors.

In this demonstration, we will illustrate the utility of the programs with the following names: KillingTensorLibrary, MaxKY, BundleLift, KYTracCon, getKY, liftKY, and KYtoKT.

First, we will load in the required packages and read in the file which contains the programs we have written.

```
> read "TractorPrograms.txt";
  with(DifferentialGeometry):
  with(Tensor):
```

Now, we will read in an example metric in 3 dimensions. In using the KillingTensorLibrary command, we will initialize a coordinate environment M.

```
> g1 := KillingTensorLibrary(3, M);
```

$$g1 := dx \otimes dx + dy \otimes dy + x^p\, dz \otimes dz \quad (1)$$

The metric we will use will have p=1.

```
M > g := eval(g1, p=1);
```

$$g := dx \otimes dx + dy \otimes dy + x\, dz \otimes dz \quad (2)$$

We need to know the required size of the fibers of the tractor bundle. We find this to be 10:

```
M > MaxKY(3, 2);
```

$$4 \quad (3)$$

Now we initialize the required environments.

```
M > DGEnvironment[VectorSpace](4, V);
```

Vector Space: V **(4)**

```
V > DGEnvironment[VectorBundle](M, V, N);
```

Vector Bundle: N **(5)**

Now that the vector bundle has been initialized, it is convenient to redefine the metric on this bundle:

```
N > G := BundleLift(g, N);
```

(6)

$$G := dx \otimes dx + dy \otimes dy + x\, dz \otimes dz \quad (6)$$

We also need the Christoffel symbols.

```
N > Gamma := Christoffel(G);
```

$$\Gamma := \nabla_{\partial_x} \partial_z = \frac{1}{2\,x}\,\partial_z, \nabla_{\partial_z} \partial_x = \frac{1}{2\,x}\,\partial_z, \nabla_{\partial_z} \partial_z = -\left(\frac{1}{2}\right)\partial_x \quad (7)$$

Now we can compute the tractor connection. Note that the rank of the Killing-Yano tensor must be specified.

```
N > C := KYTracCon(Gamma, 2, N);
```

$$C := \nabla_{\partial_x} E2 = -\frac{1}{2\,x}\,E2, \nabla_{\partial_x} E3 = -\frac{1}{2\,x}\,E3 - \frac{1}{8\,x^2}\,E4, \nabla_{\partial_x} E4 = E3 - \frac{1}{2\,x}\,E4 \quad (8)$$
$$, \nabla_{\partial_y} E4 = -E2, \nabla_{\partial_z} E1 = -\left(\frac{1}{2}\right) E3 - \frac{1}{8\,x}\,E4, \nabla_{\partial_z} E3 = \frac{1}{2\,x}\,E1, \nabla_{\partial_z} E4 = E1$$

We now compute the Curvature tensor for the tractor connection.

```
N > K := CurvatureTensor(C);
```

$$K := -\frac{1}{8\,x^2}\,E1 \otimes \Theta3 \otimes dx \otimes dz + \frac{1}{8\,x^2}\,E1 \otimes \Theta3 \otimes dz \otimes dx + \frac{1}{8\,x}\,E2 \otimes \Theta1 \otimes dy \otimes dz \quad (9)$$
$$- \frac{1}{8\,x}\,E2 \otimes \Theta1 \otimes dz \otimes dy - \frac{1}{8\,x^2}\,E2 \otimes \Theta3 \otimes dx \otimes dy + \frac{1}{8\,x^2}\,E2 \otimes \Theta3 \otimes dy \otimes dx$$
$$+ \frac{1}{8\,x}\,E3 \otimes \Theta1 \otimes dx \otimes dz - \frac{1}{8\,x}\,E3 \otimes \Theta1 \otimes dz \otimes dx + \frac{1}{4\,x^2}\,E4 \otimes \Theta1 \otimes dx \otimes dz$$
$$- \frac{1}{4\,x^2}\,E4 \otimes \Theta1 \otimes dz \otimes dx$$

We can represent this tensor as a collection of (1,1) tensors by contracting the curvature tensor with the coordinate vectors of the base manifold.

```
N > k1 := ContractIndices(K, evalDG(D_x &t D_y), [[3,1],[4,2]]);
```

$$k1 := -\frac{1}{8\,x^2}\,E2 \otimes \Theta3 \quad (10)$$

```
N > k2 := ContractIndices(K, evalDG(D_x &t D_z), [[3,1],[4,2]]);
```

$$k2 := -\frac{1}{8\,x^2}\,E1 \otimes \Theta3 + \frac{1}{8\,x}\,E3 \otimes \Theta1 + \frac{1}{4\,x^2}\,E4 \otimes \Theta1 \quad (11)$$

```
N > k3 := ContractIndices(K, evalDG(D_y &t D_z), [[3,1],[4,2]]);
```

$$k3 := \frac{1}{8\,x}\,E2 \otimes \Theta1 \quad (12)$$

We can think of these as matrices

```
> K1 := convert(k1, DGMatrix);
  K2 := convert(k2, DGMatrix);
  K3 := convert(k3, DGMatrix);
```

$$K1 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{8\,x^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K2 := \begin{bmatrix} 0 & 0 & -\frac{1}{8\,x^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{8\,x} & 0 & 0 & 0 \\ \frac{1}{4\,x^2} & 0 & 0 & 0 \end{bmatrix}$$

$$K3 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{8\,x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

and subsequently find the basis of the 0th order reduced tractor bundle:

```
> IT0 := LieAlgebras:-InvariantTensors([K1,K2,K3], [seq(E||i,i=
  1..4)]);
```

$$IT0 := [E2, E4] \tag{14}$$

How many are there?

```
> nops(IT0);
```

$$2 \tag{15}$$

Thus, there are a maximum of 2 rank 2 Killing Yano tensors.

Now let's see if we can get the Killing-Yano tensors explicitly. We begin by forming a function-coefficient linear combination of the basis elements.

```
N > s1 := DGzip([seq(q||i(x,y,z), i=1..nops(IT0))], IT0, "plus");
```

$$s1 := q1(x, y, z)\, E2 + q2(x, y, z)\, E4 \quad \textbf{(16)}$$

Next, we take the covariant derivative.

```
N > s2 := CovariantDerivative(s1, C);
```

$$s2 := q2(x, y, z)\, E1 \otimes dz + \left(\frac{\partial}{\partial x}\, q1(x, y, z) - \frac{q1(x, y, z)}{2\, x}\right) E2 \otimes dx + \left(\frac{\partial}{\partial y}\, q1(x, y, z) - q2(x, y, z)\right) E2 \otimes dy + \frac{\partial}{\partial z}\, q1(x, y, z)\, E2 \otimes dz + q2(x, y, z)\, E3 \otimes dx + \left(\frac{\partial}{\partial x}\, q2(x, y, z) - \frac{q2(x, y, z)}{2\, x}\right) E4 \otimes dx + \frac{\partial}{\partial y}\, q2(x, y, z)\, E4 \otimes dy + \frac{\partial}{\partial z}\, q2(x, y, z)\, E4 \otimes dz \quad \textbf{(17)}$$

Let's look at the equations we need to solve.

```
N > s3 := DGinformation(s2, "CoefficientSet");
```

$$s3 := \left\{ \frac{\partial}{\partial x}\, q1(x, y, z) - \frac{1}{2}\, \frac{q1(x, y, z)}{x}, \frac{\partial}{\partial y}\, q1(x, y, z) - q2(x, y, z), \frac{\partial}{\partial x}\, q2(x, y, z) - \frac{1}{2}\, \frac{q2(x, y, z)}{x}, \frac{\partial}{\partial z}\, q1(x, y, z), \frac{\partial}{\partial y}\, q2(x, y, z), \frac{\partial}{\partial z}\, q2(x, y, z), q2(x, y, z) \right\} \quad \textbf{(18)}$$

How many equations are there?

```
N > nops(s3);
```

$$7 \quad \textbf{(19)}$$

This system may be easier to solve than the Killing equation itself. Let's get the solution.

```
N > s4 := pdsolve(s3, {seq(q||i(x,y,z),i=1..nops(IT0))} );
```

$$s4 := \left\{ q1(x, y, z) = \_C1 \sqrt{x}, q2(x, y, z) = 0 \right\} \quad \textbf{(20)}$$

How many independent solutions are there?

```
N > has(s4, _C1);
```

true **(21)**

```
N > has(s4, _C2);
```

false **(22)**

Thus, there is 1 independent solution and, consequently, a single Killing-Yano tensor. Here is then what the parallel section should look like:

```
N > s5 := DETools:-dsubs(s4, s1);
```

$$s5 := \_C1\sqrt{x}\ E2 + 0\ E4 \quad (23)$$

We can evaluate this at _C1=1.

```
N > t1 := eval(s5, [_C1=1]);
```

$$t1 := \sqrt{x}\ E2 + 0\ E4 \quad (24)$$

We check that it's a parallel section:

```
N > CovariantDerivative(t1, C);
```

$$0\ E1 \otimes dx \quad (25)$$

Now we drop this parallel section down. Note that the rank of the Killing-Yano tensor must again be specified.

```
N > T := getKY(t1, 2, N);
```

$$T := \sqrt{x}\ dx \wedge dz \quad (26)$$

Now let's check that we can lift the Killing-Yano tensors of the metric to parallel sections. We begin by calculating them conventionally.

```
N > ky := KillingYanoTensors(G, 2);
```

$$ky := \left[\sqrt{x}\ dx \wedge dz\right] \quad (27)$$

Now let's lift it to a section. The rank is not required here.

```
N > liftky := liftKY(ky[1], Gamma, N);
```

$$liftky := \sqrt{x}\ E2 \quad (28)$$

Let's check that this is a parallel section.

```
N > CovariantDerivative(liftky, C);
```

$$0\ E1 \otimes dx \quad (29)$$

But are there more parallel sections?

```
N > CovariantlyConstantTensors(C, [seq(E||i,i=1..4)]);
```

$$\left[\sqrt{x}\ E2\right] \quad (30)$$

```
N > nops(%);
```

$$1 \quad (31)$$

Thus, the parallel sections and the Killing tensors are in one-to-one correspondence.

We will now construct a Killing tensor of rank 2 from the known Killing-Yano tensor.

```
N > KT := KYtoKT(G, InverseMetric(G), ky[1], ky[1]);
```

$$KT := -dx \otimes dx - x\, dz \otimes dz \quad \textbf{(32)}$$

Let's check that this is a Killing tensor of G.

```
N > CheckKillingTensor(G,KT);
```

$$0\, dx \otimes dx \otimes dx \quad \textbf{(33)}$$

The KillingTensorLibrary command can also be used to call other known quantities of a metric, such as the Killing tensors of rank 1:

```
N > kt1 := eval(map(BundleLift, KillingTensorLibrary(3, M,
    output=["KillingTensors", 1]), N), p=1);
```

$$kt1 := [x\, dz, dy] \quad \textbf{(34)}$$

Now, let's get a basis for the space of known Killing tensors, including the metric. In principle, this can be done using the tractor approach, but a more conventional command exists and alliviates the need to construct the tractor bundle for Killing tensors of rank 2.

```
N > reds := SymmetricProductsOfKillingTensors([kt1, [G]], 2);
```

$$reds := \left[x^2\, dz \otimes dz, \frac{x}{2}\, dy \otimes dz + \frac{x}{2}\, dz \otimes dy, dy \otimes dy, dx \otimes dx + dy \otimes dy + x\, dz \otimes dz\right] \quad \textbf{(35)}$$

Now we will determine, conventionally, whether the Killing tensor we've newly constructed is a linear combination of the known Killing tensors.

```
N > GetComponents(KT, reds);
```

$$[0, 0, 1, -1] \quad \textbf{(36)}$$

Thus, the Killing tensor so constructed is a linear combination of the Killing tensors which are already known. It is, however, not a linear combination of only the reducible Killing tensors, and is therefore irreducible--the metric itself is irreducible, in this case.

For the sake of curiosity, and since irreducible Killing tensors are of such interest, we will calculate all Killing tensors of rank 2 for the metric G. In principle, this can be done by means of the tractor approach; however, this particular metric presents no obstacles in computing the Killing tensors directly:

```
N > kt2 := KillingTensors(G, 2);
```

$$kt2 := \left[y\, dx \otimes dx - \frac{x}{2}\, dx \otimes dy - \frac{x}{2}\, dy \otimes dx - \frac{z\, x}{4}\, dy \otimes dz - \frac{z\, x}{4}\, dz \otimes dy + x\, y\, dz \otimes dz,\ dx \otimes dx + x\, dz \otimes dz,\ x^2\, dz \otimes dz,\ \frac{x}{2}\, dy \otimes dz + \frac{x}{2}\, dz \otimes dy,\ dy \otimes dy\right] \quad (37)$$

We now find which Killing tensors, if any, are not linear combinations of the metric and of the reducible Killing tensors.

```
N > irreds := IndependentKillingTensors(kt2, reds);
```

$$irreds := \left[y\, dx \otimes dx - \frac{x}{2}\, dx \otimes dy - \frac{x}{2}\, dy \otimes dx - \frac{z\, x}{4}\, dy \otimes dz - \frac{z\, x}{4}\, dz \otimes dy + x\, y\, dz \otimes dz\right] \quad (38)$$

Thus, the Killing tensor above is not a linear combination of the metric and the reducible Killing tensors. In particular, it is irreducible.

15 Appendix E: an irreducible Killing tensor for the Kerr metric

In section 6.2, we searched many vacuum type D solutions to the einstein field equations for Killing tensors which were not linear combinations of the reducible Killing tensors and the metric itself. We found such a Killing tensor in the case of the Kerr metric. Below is a demonstration of the technique that was used to conduct the search among the several metrics which were searched.

Appendix E: an irreducible Killing tensor for the Kerr metric.

We will now demonstrate the method by which vacuum type D solutions were searched for irreducible Killing tensors, as explained in section 6.2.

Here is the Kerr metric, which is found in the MetricSearch library in the Differential Geometry software package.

```
M > g := Library:-Retrieve("HawkingEllis", 1, [5,29,1], manifoldname=M, output=["Metric"])[1]
```

$$g := -\frac{\_a^2\cos(\theta)^2-2\,r\,\_m+r^2}{r^2+\_a^2\cos(\theta)^2}\,dt\otimes dt-\frac{2\,\_m\,r\,\_a\sin(\theta)^2}{r^2+\_a^2\cos(\theta)^2}\,dt\otimes d\phi + \frac{r^2+\_a^2\cos(\theta)^2}{\_a^2-2\,r\,\_m+r^2}\,dr\otimes dr+\left(r^2+\_a^2\cos(\theta)^2\right)d\theta\otimes d\theta - \frac{2\,\_m\,r\,\_a\sin(\theta)^2}{r^2+\_a^2\cos(\theta)^2}\,d\phi\otimes dt+\frac{1}{r^2+\_a^2\cos(\theta)^2}\left(\sin(\theta)^2\left(\cos(\theta)^2\,\_a^4 - 2\cos(\theta)^2\,\_a^2\,\_m\,r+\cos(\theta)^2\,\_a^2\,r^2+2\,r\,\_m\,\_a^2+\_a^2\,r^2+r^4\right)\right)d\phi\otimes d\phi \tag{1}$$

obtaining the coordinates will be useful later.

```
M > coords := Library:-Retrieve("HawkingEllis", 1, [5,29,1], manifoldname=M, output=["Coordinates"])[1]
```

$$coords := [t, r, \theta, \phi] \tag{2}$$

```
M > Coords := seq(coords[i], i=1..nops(coords));
```

$$Coords := t, r, \theta, \phi \tag{3}$$

Below are the Killing vectors for this metric:

```
M > kv := Library:-Retrieve("HawkingEllis", 1, [5,29,1], manifoldname=M, output=["KillingVectors"])[1]
```

$$kv := \left[\partial_t, \partial_\phi\right] \tag{4}$$

We now lower the indices to get the rank 1 Killing tensors.

```
M > kt := map2(RaiseLowerIndices, g, kv, [1]);
```

$$kt := \left[-\frac{\_a^2\cos(\theta)^2-2\,r\,\_m+r^2}{r^2+\_a^2\cos(\theta)^2}\,dt-\frac{2\,\_m\,r\,\_a\sin(\theta)^2}{r^2+\_a^2\cos(\theta)^2}\,d\phi,\right. \tag{5}$$

$$- \frac{2\_m\, r\, \_a \sin(\theta)^2}{r^2 + \_a^2 \cos(\theta)^2}\, dt + \frac{1}{r^2 + \_a^2 \cos(\theta)^2} \big(\sin(\theta)^2 \big(\cos(\theta)^2\, \_a^4$$

$$- 2 \cos(\theta)^2\, \_a^2\, \_m\, r + \cos(\theta)^2\, \_a^2\, r^2 + 2\, r\, \_m\, \_a^2 + \_a^2\, r^2 + r^4\big)\big)\, d\phi \Big]$$

The Null Tetrad for this metric has been computed previously, and is available in the MetricSearch library.

```
M > NT := Library:-Retrieve("HawkingEllis", 1, [5,29,1],
    manifoldname=M, output=["NullTetrad"])[1]
```

$$NT := \left[\frac{\sqrt{2}\, (\_a^2 + r^2)}{2 \sqrt{\_a^2 - 2\, r\, \_m + r^2}\, \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_t + \frac{\sqrt{2}\, \sqrt{\_a^2 - 2\, r\, \_m + r^2}}{2 \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_r \right. \tag{6}$$

$$+ \frac{\sqrt{2}\, \_a}{2 \sqrt{\_a^2 - 2\, r\, \_m + r^2}\, \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_\phi,$$

$$\frac{\sqrt{2}\, (\_a^2 + r^2)}{2 \sqrt{\_a^2 - 2\, r\, \_m + r^2}\, \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_t - \frac{\sqrt{2}\, \sqrt{\_a^2 - 2\, r\, \_m + r^2}}{2 \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_r$$

$$+ \frac{\sqrt{2}\, \_a}{2 \sqrt{\_a^2 - 2\, r\, \_m + r^2}\, \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_\phi, \frac{\frac{I}{2} \sqrt{2}\, \sin(\theta)\, \_a}{\sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_t$$

$$+ \frac{\sqrt{2}}{2 \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_\theta + \frac{\frac{I}{2} \sqrt{2}}{\sqrt{r^2 + \_a^2 \cos(\theta)^2}\, \sin(\theta)}\, \partial_\phi,$$

$$- \frac{\frac{I}{2} \sqrt{2}\, \sin(\theta)\, \_a}{\sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_t + \frac{\sqrt{2}}{2 \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_\theta$$

$$\left. - \frac{\frac{I}{2} \sqrt{2}}{\sqrt{r^2 + \_a^2 \cos(\theta)^2}\, \sin(\theta)}\, \partial_\phi \right]$$

We now compute the Principal Null directions.

```
M > PND := PrincipalNullDirections(NT, "D");
```

$$PND := \left[\frac{\sqrt{2}\, (\_a^2 + r^2)}{2 \sqrt{\_a^2 - 2\, r\, \_m + r^2}\, \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_t + \frac{\sqrt{2}\, \sqrt{\_a^2 - 2\, r\, \_m + r^2}}{2 \sqrt{r^2 + \_a^2 \cos(\theta)^2}}\, \partial_r \right. \tag{7}$$

$$+\frac{\sqrt{2}\ \_a}{2\sqrt{\_a^2-2\,r\,\_m+r^2}\ \sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_\phi,$$

$$\frac{\sqrt{2}\ (\_a^2+r^2)}{2\sqrt{\_a^2-2\,r\,\_m+r^2}\ \sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_t-\frac{\sqrt{2}\ \sqrt{\_a^2-2\,r\,\_m+r^2}}{2\sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_r+\frac{\sqrt{2}\ \_a}{2\sqrt{\_a^2-2\,r\,\_m+r^2}\ \sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_\phi\Big]$$

We will call the first one K,

```
M > K := PND[1];
```

$$K:=\frac{\sqrt{2}\ (\_a^2+r^2)}{2\sqrt{\_a^2-2\,r\,\_m+r^2}\ \sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_t+\frac{\sqrt{2}\ \sqrt{\_a^2-2\,r\,\_m+r^2}}{2\sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_r+\frac{\sqrt{2}\ \_a}{2\sqrt{\_a^2-2\,r\,\_m+r^2}\ \sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_\phi \quad \textbf{(8)}$$

and the second L:

```
M > L := PND[2];
```

$$L:=\frac{\sqrt{2}\ (\_a^2+r^2)}{2\sqrt{\_a^2-2\,r\,\_m+r^2}\ \sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_t-\frac{\sqrt{2}\ \sqrt{\_a^2-2\,r\,\_m+r^2}}{2\sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_r+\frac{\sqrt{2}\ \_a}{2\sqrt{\_a^2-2\,r\,\_m+r^2}\ \sqrt{r^2+\_a^2\cos(\theta)^2}}\,\partial_\phi \quad \textbf{(9)}$$

Now we will apply theorem 6.1 to construct Killing tensors. We will construct them as in the theorem, only keeping A and B arbitrary for the moment.

```
M > T := evalDG( A(Coords)*RaiseLowerIndices(g, K &s L, [1,2]) +
    B(Coords)*g);
```

$$T:=\frac{1}{2\,(r^2+\_a^2\cos(\theta)^2)}\big(-2\,B(t,r,\theta,\phi)\cos(\theta)^2\,\_a^2+A(t,r,\theta,\phi)\,\_a^2-2\,A(t,r,\theta,\phi)\,\_m\,r+A(t,r,\theta,\phi)\,r^2+4\,B(t,r,\theta,\phi)\,\_m\,r-2\,B(t,r,\theta,\phi)\,r^2\big)\,dt\otimes dt+\frac{1}{2\,(r^2+\_a^2\cos(\theta)^2)}\big((\cos(\theta)^2-1)\,\_a\,(A(t,r,\theta,\phi)\,\_a^2-2\,A(t,r,\theta,\phi)\,\_m\,r+A(t,r,\theta,\phi)\,r^2+4\,B(t,r,\theta,\phi)\,\_m\,r)\big)\,dt\otimes d\phi \quad \textbf{(10)}$$

$$- \frac{(r^2 + \_a^2 \cos(\theta)^2)\,(A(t, r, \theta, \phi) - 2\,B(t, r, \theta, \phi))}{2\,\_a^2 - 4\,r\,\_m + 2\,r^2}\,dr \otimes dr + B(t, r, \theta, \phi)\,(r^2 + \_a^2 \cos(\theta)^2)\,d\theta \otimes d\theta + \frac{1}{2\,(r^2 + \_a^2 \cos(\theta)^2)}\big((\cos(\theta)^2 - 1)\,\_a\,(A(t, r, \theta, \phi)\,\_a^2 - 2\,A(t, r, \theta, \phi)\,\_m\,r + A(t, r, \theta, \phi)\,r^2 + 4\,B(t, r, \theta, \phi)\,\_m\,r)\big)\,d\phi \otimes dt - \frac{1}{2\,(r^2 + \_a^2 \cos(\theta)^2)}\big(\sin(\theta)^2\,(A(t, r, \theta, \phi)\cos(\theta)^2\,\_a^4 - 2\,A(t, r, \theta, \phi)\cos(\theta)^2\,\_a^2\,\_m\,r + A(t, r, \theta, \phi)\cos(\theta)^2\,\_a^2\,r^2 - 2\,B(t, r, \theta, \phi)\cos(\theta)^2\,\_a^4 + 4\,B(t, r, \theta, \phi)\cos(\theta)^2\,\_a^2\,\_m\,r - 2\,B(t, r, \theta, \phi)\cos(\theta)^2\,\_a^2\,r^2 - A(t, r, \theta, \phi)\,\_a^4 + 2\,A(t, r, \theta, \phi)\,\_a^2\,\_m\,r - A(t, r, \theta, \phi)\,\_a^2\,r^2 - 4\,B(t, r, \theta, \phi)\,\_a^2\,\_m\,r - 2\,B(t, r, \theta, \phi)\,\_a^2\,r^2 - 2\,B(t, r, \theta, \phi)\,r^4)\big)\,d\phi \otimes d\phi$$

Now we will compute Killing tensors for the metric which have the form given in theorem 6.1 for arbitrary functions A and B. We find that there are 2 (a common trait with other metrics).

```
M > KT := KillingTensors(g, ansatz = T, unknowns = [A, B](Coords)
     ) ;
```

$$KT := \Big[-\frac{\_a^2 \cos(\theta)^2 - 2\,r\,\_m + r^2}{r^2 + \_a^2 \cos(\theta)^2}\,dt \otimes dt - \frac{2\,\_m\,r\,\_a \sin(\theta)^2}{r^2 + \_a^2 \cos(\theta)^2}\,dt \otimes d\phi + \frac{r^2 + \_a^2 \cos(\theta)^2}{\_a^2 - 2\,r\,\_m + r^2}\,dr \otimes dr + \left(r^2 + \_a^2 \cos(\theta)^2\right) d\theta \otimes d\theta - \frac{2\,\_m\,r\,\_a \sin(\theta)^2}{r^2 + \_a^2 \cos(\theta)^2}\,d\phi \otimes dt + \frac{1}{r^2 + \_a^2 \cos(\theta)^2}\big(\sin(\theta)^2\,(\cos(\theta)^2\,\_a^4 - 2\cos(\theta)^2\,\_a^2\,\_m\,r + \cos(\theta)^2\,\_a^2\,r^2 + 2\,r\,\_m\,\_a^2 + \_a^2\,r^2 + r^4)\big)\,d\phi \otimes d\phi, \tag{11}$$

$$\frac{\_a^2\,(\_a^2 \cos(\theta)^2 - 2\cos(\theta)^2\,\_m\,r + r^2)}{r^2 + \_a^2 \cos(\theta)^2}\,dt \otimes dt - \frac{1}{r^2 + \_a^2 \cos(\theta)^2}\big(\_a \sin(\theta)^2\,(\cos(\theta)^2\,\_a^4 - 2\cos(\theta)^2\,\_a^2\,\_m\,r + \cos(\theta)^2\,\_a^2\,r^2 + \_a^2\,r^2 + r^4)\big)\,dt \otimes d\phi - \frac{(r^2 + \_a^2 \cos(\theta)^2)\,\_a^2 \cos(\theta)^2}{\_a^2 - 2\,r\,\_m + r^2}\,dr \otimes dr + r^2\,(r^2 + \_a^2 \cos(\theta)^2)\,d\theta \otimes d\theta$$

$$- \frac{1}{r^2 + \_a^2 \cos(\theta)^2} \left(\_a \sin(\theta)^2 \left(\cos(\theta)^2 \_a^4 - 2 \cos(\theta)^2 \_a^2 \_m\, r + \cos(\theta)^2 \_a^2 r^2 + \_a^2 r^2 + r^4\right)\right) d\phi \otimes dt$$

$$- \frac{1}{r^2 + \_a^2 \cos(\theta)^2} \left(\sin(\theta)^2 \left(\cos(\theta)^4 \_a^6 - 2 \cos(\theta)^4 \_a^4 \_m\, r + \_a^4 r^2 \cos(\theta)^4 - \cos(\theta)^2 \_a^6 + 2 \cos(\theta)^2 \_a^4 \_m\, r - \cos(\theta)^2 \_a^4 r^2 - \_a^4 r^2 - 2\, \_a^2 r^4 - r^6\right)\right) d\phi \otimes d\phi \Big]$$

Having obtained all Killing tensors of the metric g which are of the form of theorem 6.1, we construct a basis of known Killing tensors from the Killing tensors of rank 1 and the metric itself.

```
M > S2 := SymmetricProductsOfKillingTensors([kt, [g]], 2):
```

We are expecting a maximum of 4, and in fact there are:

```
M > nops(S2);
```

$$4 \qquad \textbf{(12)}$$

We now ask: which of the Killing tensors constructed by theorem 6.1 can be written as a linear combination of the known Killing tensors?

```
M > GetComponents(KT[1], S2, method="real");
```

$$[0, 0, 0, 1] \qquad \textbf{(13)}$$

```
M > GetComponents(KT[2], S2, method="real");
```

$$[\ ] \qquad \textbf{(14)}$$

The first is apparently the metric, which although irreducible, is uninteresting. It is the second which is of interest, and which is included in section 6.2:

```
M > irredKT := KT[2];
```

$$irredKT := \frac{\_a^2 \left(\_a^2 \cos(\theta)^2 - 2 \cos(\theta)^2 \_m\, r + r^2\right)}{r^2 + \_a^2 \cos(\theta)^2} dt \otimes dt - \frac{1}{r^2 + \_a^2 \cos(\theta)^2} \left(\_a \sin(\theta)^2 \left(\cos(\theta)^2 \_a^4 - 2 \cos(\theta)^2 \_a^2 \_m\, r + \cos(\theta)^2 \_a^2 r^2 + \_a^2 r^2 + r^4\right)\right) dt \otimes d\phi - \frac{\left(r^2 + \_a^2 \cos(\theta)^2\right) \_a^2 \cos(\theta)^2}{\_a^2 - 2\, r\, \_m + r^2} dr \otimes dr + r^2 \left(r^2 + \_a^2 \cos(\theta)^2\right) d\theta \otimes d\theta \qquad \textbf{(15)}$$

$$
\begin{aligned}
&- \frac{1}{r^2 + \_a^2 \cos(\theta)^2} \left(\_a \sin(\theta)^2 \left(\cos(\theta)^2 \_a^4 - 2 \cos(\theta)^2 \_a^2 \_m\, r\right.\right. \\
&\left.\left. + \cos(\theta)^2 \_a^2 r^2 + \_a^2 r^2 + r^4\right)\right) d\phi \otimes dt \\
&- \frac{1}{r^2 + \_a^2 \cos(\theta)^2} \left(\sin(\theta)^2 \left(\cos(\theta)^4 \_a^6 - 2 \cos(\theta)^4 \_a^4 \_m\, r\right.\right. \\
&+ \_a^4 r^2 \cos(\theta)^4 - \cos(\theta)^2 \_a^6 + 2 \cos(\theta)^2 \_a^4 \_m\, r - \cos(\theta)^2 \_a^4 r^2 - \_a^4 r^2 \\
&\left.\left. - 2 \_a^2 r^4 - r^6\right)\right) d\phi \otimes d\phi
\end{aligned}
$$

16 Appendix F: Killing tensors on Frames

In this appendix, we will construct a frame on which the tractor approach will be applied. Then, we will demonstrate how the novel application of the tractor approach is used.

Appendix F: Killing tensors on Frames

In this worksheet, we will demonstrate the application of the tractor approach to an Anholonomic frame.

First, we read in the required programs.

```
> read "stage4_programs.txt";
```

Next, we load in a suitable candidate.

```
M > g26 := KillingTensorLibrary(26, M);
```

$$g26 := -du \otimes dv - dv \otimes du + \left(u^{1+-c} + 2\,u + u^{1--c}\right) dx \otimes dx + \left(u^{1+-c} + 2\,u + u^{1--c}\right) dy \otimes dy \quad (1)$$

We will actually evaluate this at c=1/2:

```
M > g36 := eval(g26, _c=1/2);
```

$$g36 := -du \otimes dv - dv \otimes du + \left(u^{3/2} + 2\,u + \sqrt{u}\right) dx \otimes dx + \left(u^{3/2} + 2\,u + \sqrt{u}\right) dy \otimes dy \quad (2)$$

This metric is also in the MetricSearch database:

```
M > KillingTensorLibrary(26, M, output=["Reference"]);
```

$$["Stephani", 1, [12, 37, 5]] \quad (3)$$

Thus, we can get the Killing vectors.

```
M > kvg36 := Library:-Retrieve("Stephani", 1, [12,37,5], manifoldname=M, output=["KillingVectors"])[1];
```

$$kvg36 := \left[\partial_v, \partial_x, \partial_y, -y\partial_x + x\partial_y, -\_c\,y\,\partial_v + \frac{1}{u^{-c}+1}\partial_y, -\_c\,x\,\partial_v + \frac{1}{u^{-c}+1}\partial_x, -u\,\partial_u + \left(v + \frac{1}{2}\_c^2 x^2 + \frac{1}{2}\_c^2 y^2\right)\partial_v + \frac{x\left(u^{-c} + u^{-c}\_c + 1 - \_c\right)}{2\left(u^{-c}+1\right)}\partial_x + \frac{y\left(u^{-c} + u^{-c}\_c + 1 - \_c\right)}{2\left(u^{-c}+1\right)}\partial_y\right] \quad (4)$$

We can also get the orhonormal tetrad.

```
M > L := eval(Library:-Retrieve("Stephani", 1, [12,37,5], manifoldname=M, output=["OrthonormalTetrad"])[1], _c=1/2);
```

$$L := \left[\frac{\sqrt{2}}{2}\,\partial_u + \frac{\sqrt{2}}{2}\,\partial_v,\ \frac{1}{\sqrt{u}\,\sqrt{\frac{1}{\sqrt{u}}+\sqrt{u}+2}}\,\partial_x,\ \frac{1}{\sqrt{u}\,\sqrt{\frac{1}{\sqrt{u}}+\sqrt{u}+2}}\,\partial_y,\ \frac{\sqrt{2}}{2}\,\partial_u - \frac{\sqrt{2}}{2}\,\partial_v\right] \tag{5}$$

Next, we will need to compute the lie brackets of our orthonormal tetrad.

M > DF36 := FrameData(L, P36);

$$DF36 := \left[E1, E2\right] = -\frac{\left(3\,u+4\sqrt{u}+1\right)\sqrt{2}}{8\,u^{3/2}\sqrt{\frac{1}{\sqrt{u}}+\sqrt{u}+2}\,\sqrt{\frac{\left(\sqrt{u}+1\right)^2}{\sqrt{u}}}}\,E2,\ \left[E1, E3\right] = -\frac{\left(3\,u+4\sqrt{u}+1\right)\sqrt{2}}{8\,u^{3/2}\sqrt{\frac{1}{\sqrt{u}}+\sqrt{u}+2}\,\sqrt{\frac{\left(\sqrt{u}+1\right)^2}{\sqrt{u}}}}\,E3,\ [E1, E4] = 0,\ [E2, E3] = 0,\ \left[E2, E4\right] = \frac{\sqrt{2}\left(3\,u+4\sqrt{u}+1\right)}{8\,u\left(\sqrt{u}+1\right)^2}\,E2,\ \left[E3, E4\right] = \frac{\sqrt{2}\left(3\,u+4\sqrt{u}+1\right)}{8\,u\left(\sqrt{u}+1\right)^2}\,E3 \tag{6}$$

Now, we can initialize our frame.

M > DGEnvironment[AnholonomicFrame](DF36);

Anholonomic Frame: P36 **(7)**

we define the identity transformation:

P36 > ID := Transformation(P36, M, [x=x, u=u, y=y, v=v]);

$$ID := u = u, v = v, x = x, y = y \tag{8}$$

and now we bring the metric onto this frame.

P36 > G36:=simplify(Pullback(ID, g26), useassumptions);

$$G36 := -\Theta1 \otimes \Theta1 + \frac{u^{3/2}+2\,u+\sqrt{u}}{\sqrt{u}\left(\sqrt{u}+1\right)^2}\,\Theta2 \otimes \Theta2 + \frac{u^{3/2}+2\,u+\sqrt{u}}{\sqrt{u}\left(\sqrt{u}+1\right)^2}\,\Theta3 \otimes \Theta3 + \Theta4 \otimes \Theta4 \tag{9}$$

Maple seems to refuse to simplify this expression: (the numerator and denominator are apparently equal)

```
P36 > simplify(expand(denom(DGinformation(G36,
      "CoefficientList", [[2,2]])[1])))
```

$$u^{3/2} + 2\,u + \sqrt{u} \tag{10}$$

```
P36 > numer(DGinformation(G36, "CoefficientList", [[2,2]])[1])
```

$$u^{3/2} + 2\,u + \sqrt{u} \tag{11}$$

Thus, we will help it along directly:

```
P36 > realG36 := _DG([["tensor", P36, [["cov_bas", "cov_bas"],
      []]], [[[1, 1], -1], [[2, 2], 1], [[3, 3], 1], [[4, 4],
      1]]]);
```

$$realG36 := -\Theta1 \otimes \Theta1 + \Theta2 \otimes \Theta2 + \Theta3 \otimes \Theta3 + \Theta4 \otimes \Theta4 \tag{12}$$

Now, we seek to build the tractor bundle. The required size of the fibers is 50, but we label the forms and vectors as W and Phi, respectively, to avoid a conflict of notation with the frame labels.

```
P36 > DGEnvironment[VectorSpace](50, V36, vectorlabels=[W],
      formlabels=[Phi]);
```

Vector Space: V36 **(13)**

```
P36 > DGEnvironment[VectorBundle](P36, V36, N36);
```

Vector Bundle: N36 **(14)**

Now we realize the metric on the base space.

```
N36 > G36n:=BundleLift(realG36, N36);
```

$$G36n := -\Theta1 \otimes \Theta1 + \Theta2 \otimes \Theta2 + \Theta3 \otimes \Theta3 + \Theta4 \otimes \Theta4 \tag{15}$$

Next, we get the Christoffel symbols.

```
N36 > Gamma:=eval(Christoffel(G36n), _c=1/2);
```

$$\Gamma := \nabla_{E2} E1 = \frac{\sqrt{2}\left(3\,u + 4\sqrt{u} + 1\right)}{8\,u\left(\sqrt{u} + 1\right)^2}\,E2, \nabla_{E2} E2 = \frac{\sqrt{2}\left(3\,u + 4\sqrt{u} + 1\right)}{8\,u\left(\sqrt{u} + 1\right)^2}\,E1 - \frac{\sqrt{2}\left(3\,u + 4\sqrt{u} + 1\right)}{8\,u\left(\sqrt{u} + 1\right)^2}\,E4, \nabla_{E2} E4 = \frac{\sqrt{2}\left(3\,u + 4\sqrt{u} + 1\right)}{8\,u\left(\sqrt{u} + 1\right)^2}\,E2, \nabla_{E3} E1 = \frac{\sqrt{2}\left(3\,u + 4\sqrt{u} + 1\right)}{8\,u\left(\sqrt{u} + 1\right)^2}\,E3, \nabla_{E3} E3 = \frac{\sqrt{2}\left(3\,u + 4\sqrt{u} + 1\right)}{8\,u\left(\sqrt{u} + 1\right)^2}\,E1 \tag{16}$$

$$-\frac{\sqrt{2}\left(3u+4\sqrt{u}+1\right)}{8u\left(\sqrt{u}+1\right)^2}E4, \nabla_{E3}E4 = \frac{\sqrt{2}\left(3u+4\sqrt{u}+1\right)}{8u\left(\sqrt{u}+1\right)^2}E3$$

We can now compute the tractor connection and the curvature tensor. We will suppress the output of each.

```
N36 > C36 := eval(HauserTractorConnection(Gamma, N36), _c=1/2):
N36 > K36 := CurvatureTensor(C36):
```

Let's be sure we have c=1/2 for the curvature tensor.

```
N36 > k36 := eval(K36, _c=1/2):
```

Applying the tractor approach

Below are the (1,1) tensors associated with the curvature tensor.

```
N36 > K1 := ContractIndices(k36, evalDG(E1 &t E2), [[3,1],[4,
        2]]):
N36 > K2 := ContractIndices(k36, evalDG(E1 &t E3), [[3,1],[4,
        2]]):
N36 > K3 := ContractIndices(k36, evalDG(E1 &t E4), [[3,1],[4,
        2]]):
N36 > K4 := ContractIndices(k36, evalDG(E2 &t E3), [[3,1],[4,
        2]]):
N36 > K5 := ContractIndices(k36, evalDG(E2 &t E4), [[3,1],[4,
        2]]):
N36 > K6 := ContractIndices(k36, evalDG(E3 &t E4), [[3,1],[4,
        2]]):
```

Next, we convert them to matrices.

```
N36 > IH0 := map(convert, [K1,K2,K3,K4,K5,K6], DGMatrix);
```

$$IH0 := \left[\left[\begin{array}{c} \textit{50 x 50 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: sparse} \\ \textit{Order: Fortran\_order} \end{array}\right], \left[\begin{array}{c} \textit{50 x 50 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: sparse} \\ \textit{Order: Fortran\_order} \end{array}\right], \right. \qquad \textbf{(1.1)}$$

$$\left[\begin{array}{c}50 \text{ x } 50 \text{ Matrix}\\ \text{Data Type: anything}\\ \text{Storage: sparse}\\ \text{Order: Fortran\_order}\end{array}\right], \left[\begin{array}{c}50 \text{ x } 50 \text{ Matrix}\\ \text{Data Type: anything}\\ \text{Storage: sparse}\\ \text{Order: Fortran\_order}\end{array}\right],$$

$$\left[\begin{array}{c}50 \text{ x } 50 \text{ Matrix}\\ \text{Data Type: anything}\\ \text{Storage: sparse}\\ \text{Order: Fortran\_order}\end{array}\right], \left[\begin{array}{c}50 \text{ x } 50 \text{ Matrix}\\ \text{Data Type: anything}\\ \text{Storage: sparse}\\ \text{Order: Fortran\_order}\end{array}\right]\Bigg]$$

Let's compute a basis for the shared nullspace.

```
N36 > LA0 := LieAlgebras:-InvariantTensors(eval(IH0, _c=1/2),
      [seq(W||i,i=1..50)]):
```

How many are there?

```
N36 > nops(LA0);
```

$$35 \tag{1.2}$$

Thus, the metric admits at most 35 Killing tensors of rank 2.

This is the method by which the tractor approach can be applied to frames.

We have applied the tractor approach to determine an upper bound on the dimension of the space of rank 2 Killing tensors. Now, we will use our novel application of the tractor approach to determine a lower bound.

We begin by recalling the Killing vectors, now making sure to evaluate at c=1/2.

```
M > Kvg36 := eval(kvg36, _c=1/2);
```

$$Kvg36 := \left[\partial_v, \partial_x, \partial_y, -y\,\partial_x + x\,\partial_y, -\frac{y}{2}\,\partial_v + \frac{1}{\sqrt{u}+1}\,\partial_y, -\frac{x}{2}\,\partial_v + \frac{1}{\sqrt{u}+1}\,\partial_x, -u\,\partial_u + \left(v + \frac{x^2}{8} + \frac{y^2}{8}\right)\partial_v + \frac{x\left(\frac{3\sqrt{u}}{2} + \frac{1}{2}\right)}{2\left(\sqrt{u}+1\right)}\,\partial_x + \frac{y\left(\frac{3\sqrt{u}}{2} + \frac{1}{2}\right)}{2\left(\sqrt{u}+1\right)}\,\partial_y\right] \tag{17}$$

Let's be sure each of these is a Killing vector.

```
M > LieDerivative(Kvg36, g36);
```

$$[0\,du \otimes du,\ 0\,du \otimes du,\ 0\,du \otimes du,\ 0\,du \otimes du,\ 0\,du \otimes du,\ 0\,du \otimes du,\ 0\,du \otimes du] \quad (18)$$

We will need to lower the indices of these.

```
M > kt1g36 := map2(RaiseLowerIndices, g36, Kvg36, [1]);
```

$$kt1g36 := \Big[-du,\ \left(u^{3/2}+2\,u+\sqrt{u}\right) dx,\ \left(u^{3/2}+2\,u+\sqrt{u}\right) dy,\ -\left(u^{3/2}+2\,u + \sqrt{u}\right) y\,dx + \left(u^{3/2}+2\,u+\sqrt{u}\right) x\,dy,\ \frac{y}{2}\,du + \frac{u^{3/2}+2\,u+\sqrt{u}}{\sqrt{u}+1}\,dy,\ \frac{x}{2}\,du + \frac{u^{3/2}+2\,u+\sqrt{u}}{\sqrt{u}+1}\,dx,\ -\left(v+\frac{x^2}{8}+\frac{y^2}{8}\right) du + u\,dv + \frac{\left(u^{3/2}+2\,u+\sqrt{u}\right) x \left(3\sqrt{u}+1\right)}{4\left(\sqrt{u}+1\right)}\,dx + \frac{\left(u^{3/2}+2\,u+\sqrt{u}\right) y \left(3\sqrt{u}+1\right)}{4\left(\sqrt{u}+1\right)}\,dy \Big] \quad (19)$$

Now we will form a set consisting of all symmetric tensor products of the (covariant) Killing vectors and the metric.

```
M > S := [seq( seq( evalDG( kt1g36[i] &s kt1g36[j] ), i=1..j) ,
    j=1..nops(kt1g36) ), g36]:
```

There should be 29 Killing tensors in this set.

```
M > nops(S);
```

$$29 \quad (20)$$

Now we will initialize the tractor bundle. Recall that the size of the fibers is 50.

```
M > DGEnvironment[VectorSpace](50,V);
```

Vector Space: V **(21)**

```
V > DGEnvironment[VectorBundle](M,V,N2);
```

Vector Bundle: N2 **(22)**

Now we will redefine the metric on the base space of the bundle.

```
N2 > g36n := BundleLift(g36, N2);
```

(23)

$$g36n := -du \otimes dv - dv \otimes du + \left(u^{3/2} + 2u + \sqrt{u}\right) dx \otimes dx + \left(u^{3/2} + 2u + \sqrt{u}\right) dy \otimes dy \quad \textbf{(23)}$$

We also need the Christoffel symbols.

```
N2 > Gamma2 := Christoffel(g36n);
```

$$\Gamma 2 := \nabla_{\partial_u} \partial_x = \frac{3u + 4\sqrt{u} + 1}{4\left(u^{3/2} + 2u + \sqrt{u}\right)\sqrt{u}} \partial_x, \nabla_{\partial_u} \partial_y = \frac{3u + 4\sqrt{u} + 1}{4\left(u^{3/2} + 2u + \sqrt{u}\right)\sqrt{u}} \partial_y, \nabla_{\partial_x} \partial_u = \frac{3u + 4\sqrt{u} + 1}{4\left(u^{3/2} + 2u + \sqrt{u}\right)\sqrt{u}} \partial_x, \nabla_{\partial_x} \partial_x = \frac{3u + 4\sqrt{u} + 1}{4\sqrt{u}} \partial_v, \nabla_{\partial_y} \partial_u = \frac{3u + 4\sqrt{u} + 1}{4\left(u^{3/2} + 2u + \sqrt{u}\right)\sqrt{u}} \partial_y, \nabla_{\partial_y} \partial_y = \frac{3u + 4\sqrt{u} + 1}{4\sqrt{u}} \partial_v \quad \textbf{(24)}$$

We will also need to redifine the set S on the base space of the bundle.

```
N2 > S2 := map(BundleLift, S, N2):
```

Now we will lift the set S2 to the set S3--the set of sections consisting of the lifted Killing tensors of S.

```
N2 > S3 := map(HauserTractorLift2, S2, Gamma2, N2):
```

Let us ensure that we didn't miss any: there should be 29 of these sections.

```
N2 > nops(S3);
```

29 **(25)**

Now, we evaluate S3 at a convenient point.

```
N2 > S4 := eval(S3, [u=1, v=0, x=0, y=0]):
```

And now we ask: how many are in a basis?

```
N2 > b1 := DGbasis(S4):
N2 > nops(b1);
```

28 **(26)**

Thus, the true minium number of Killing tensors is 28 instead of 29.

In this particular example, this novel application is not of paramount importance, since we can check for linear independence directly:

N2 > b2 := DGbasis(S2, method="real"):

N2 > nops(b2);

$$28 \tag{27}$$

However, this novel application is quite useful when the linear independence of the Killing tensor fields is difficult to compute directly.

We can also double-check that each of these sections is parallel. We must first calculate the tractor connection.

N2 > C2 := HauserTractorConnection(Gamma2, N2):

Now we can check that all of these sections (in S3) are parallel.

N2 > map(CovariantDerivative, S3, C2);

$$[0\, E1 \otimes du,\ 0\, E1 \otimes du] \tag{28}$$

17 Appendix G: Maple source code

We will now give the raw source code for the programs we have built to apply the tractor approach to Killing vectors, conformal Killing vectors, Killing tensors of rank 2, and Killing-Yano tensors. Additional refinements may be made before the programs are distributed.

```
rnk1TracConn:=proc(Gamma,T) local Bv,Bf,CT,dimbase,Fv,X1,Y1,
    numX,X,X2,Y2,numW,W,getsome,XA,WA,Comps,Comps2,Comps3,
    Comps4,realX,realW,XR,Xc2,Xc1,Wc2,Wc1,Xc1b,Xc1c,Wc1b,Wc1c,
    m,BigMat;

Bv:=DGinformation(T,"FrameBaseVectors");
Bf:=DGinformation(T,"FrameBaseForms");

CT:=CurvatureTensor(Gamma);

dimbase:=nops(Bv);
Fv:=DGinformation("FrameFiberVectors");

#First, create an arbitrary vector:

X1:=GenerateDGobjects[DGtensors]([["cov_bas"],[]]);
Y1:=DGbasis([seq(Tensor:-YoungSymmetrizer(a, Matrix([[1]])),
    a = X1)]);
numX:=nops(Y1);
X:=DGzip([seq(z||i, i=1..numX)],Y1,"plus");

#Now, create an arbitrary rank 2 tensor with the required
    symmetry.

X2:=GenerateDGobjects[DGtensors]([["cov_bas","cov_bas"],[]]);
Y2:=DGbasis([seq(Tensor:-YoungSymmetrizer(a, Matrix
    ([[1],[2]])), a = X2)]);
numW:=nops(Y2);
W:=DGzip([seq(z||i, i=1+numX..numX+numW)],Y2,"plus");

#The following procedure is a bit overkill for the Killing
    vector case. It is intended to give us the index list from
     which we pick off the independent components of each
    tensor defined above.
```

```
getsome:=proc(T,Y) local A,B,g,A2,inds,B2,thing,bracket,term;
A:=Array([]);
for thing in op(2,op(T)) do
if nops(op(1,op(1,thing[2]))) = 2 then ArrayTools:-Append(A,
   op(1,thing));
fi;
od;
B:=[seq(A[i],i=op(2,A))];
#we now have the bads.
g:=seq(op(2,op(Y[i])),i=1..nops(Y));
A2:=Array([]);
inds:=seq([seq(g[j][i][1],i=1..nops(g[j]))],j=1..nops([g]));
for bracket in inds do
for term in bracket do
if has(B,[term])=false then if has(A2,bracket)=false then
   ArrayTools:-Append(A2,[term]);
fi;
fi;
od;
od;
B2:=ListTools:-FlattenOnce([seq(A2[i],i=op(2,A2))]);
end;

XA:=getsome(X,Y1);
#if dimbase = 2 then XA:= [1,2] else XA:=getsome(X,Y1) fi;
if dimbase = 2 then WA:=[[1,2]] else WA:=getsome(W,Y2) fi;
#WA:=getsome(W,Y2);

#Now, we will get rid of the scalars: terms like z12/2 will
   be turned into y12.

Comps:=ListTools:-FlattenOnce([[seq(DGinformation(X,"
   CoefficientList",[a])[1],a=XA)],[seq(DGinformation(W,"
   CoefficientList",[a])[1],a=WA)]]);
Comps2:=[seq(y||i,i=1..nops(Comps))];
Comps3:=[seq(Comps[i] = Comps2[i], i=1..nops(Comps))];
Comps4:=solve(Comps3,{seq(z||i,i=1..nops(Comps))});

realX:=evalDG(simplify(subs(Comps4,X)));
realW:=evalDG(simplify(subs(Comps4,W)));

XR:=ContractIndices(CT,realX,[[1,1]]);

#Now, we build each piece to the matrix of 1-forms.
```

```
for m in seq(i,i=1..dimbase) do
Xc1b:=DirectionalCovariantDerivative(Bv[m],realX,Gamma);
Xc1c:=[seq( DGinformation(Xc1b,"CoefficientList",[[a[1]]])
    [1],a=XA)];
Xc2:=[seq(DGinformation(realW,"CoefficientList",[[a[1],m]])
    [1], a=XA)];
Xc1:=[seq(Xc2[i]+Xc1c[i],i=1..nops(Xc1c))];

Wc1b:=DirectionalCovariantDerivative(Bv[m],realW,Gamma);
Wc1c:=[seq( DGinformation(Wc1b,"CoefficientList",[[a[1],a
    [2]]])[1],a=WA)];
Wc2:=[seq(DGinformation(XR,"CoefficientList",[[m,a[2],a[1]]])
    [1], a=WA)];
Wc1:=[seq(Wc2[i] +Wc1c[i],i=1..nops(Wc1c))];

Eqns||m:=ListTools:-Flatten([Xc1,Wc1]);
Mat||m:=evalDG(LinearAlgebra:-GenerateMatrix(Eqns||m,Comps2)*
    Bf[m]);
od;

#Lastly, we will piece together the matrix and build the
    connection from it.

BigMat:=add(Mat||i,i=1..dimbase);
Connection(BigMat);
end:

###################################################################

liftrnk1:=proc(X,Gamma,Q) local Bf,dimbase,Bft,Fv,forms2,
    Xcomps,omega,Omega,Omegacomps,COMPS,liftedKV;
Bf:=DGinformation(Q,"FrameBaseForms");
dimbase:=nops(Bf);
Bft:=map(convert,Bf,DGtensor);
Fv:=DGinformation(Q,"FrameFiberVectors");
forms2:=GenerateForms(Bf,2);

Xcomps:=GetComponents(X,Bft);
#dummyX:=DGzip([seq(z||i,i=1..dimbase)],Bf,"plus");

omega:=evalDG(-CovariantDerivative(X,Gamma));
Omega:=convert(omega,DGform);
Omegacomps:=GetComponents(Omega,forms2);
```

```
COMPS:=ListTools:-FlattenOnce([Xcomps,Omegacomps]);
liftedKV:=DGzip(COMPS,Fv,"plus");
end:

###############################################################

getRnk1:=proc(X,Q) local Bf,dimbase,Fv,Xcomps,Xcomps2,realX;
Bf:=DGinformation(Q,"FrameBaseForms");
dimbase:=nops(Bf);
Fv:=DGinformation(Q,"FrameFiberVectors");
Xcomps:=GetComponents(X,Fv);
Xcomps2:=[seq(Xcomps[i],i=1..dimbase)];
realX:=DGzip(Xcomps2,Bf,"plus");
end:

################################################################

ConfTracConn:=proc(g,T) local Bv,Bf,Gamma,CT,RT,RS,dimbase,Fv
    ,X1,Y1,numX,X,X2,Y2,numW,W,getsome,XA,WA,Comps,Comps2,
    Comps3,Comps4,realX,realW,XR,Xc2,Xc1,Wc2,Wc1,Xc1b,Xc1c,
    Wc1b,Wc1c,m,BigMat,gin,Y,numY,F,Z,YA,realY,realF,realZ,YR,
    term,term1,term2,Xc3,Yc1b,Yc1c,Yc2,Yc3,Yc4,Yc1,Fc2,Fc1,
    Zc1b,Zc1c,Zc1,Zc2,Zc3,Zc4,Zc5;

Bv:=DGinformation(T,"FrameBaseVectors");
Bf:=DGinformation(T,"FrameBaseForms");

Gamma:=Christoffel(g);
CT:=CurvatureTensor(Gamma);
RT:=RicciTensor(CT);
RS:=RicciScalar(g);
gin:=InverseMetric(g);

dimbase:=nops(Bv);
Fv:=DGinformation("FrameFiberVectors");

#First, create an arbitrary vector:

X1:=GenerateDGobjects[DGtensors]([["cov_bas"],[]]);
#Y1:=DGbasis([seq(Tensor:-YoungSymmetrizer(a, Matrix([[1]])),
     a = X1)]);
#numX:=nops(Y1);
numX:=nops(X1);
#X:=DGzip([seq(z||i,i=1..numX)],Y1,"plus");
X:=DGzip([seq(z||i,i=1..numX)],X1,"plus");
```

```
#Now, create an arbitrary rank 2 tensor with the required
   symmetry.

X2:=GenerateDGobjects[DGtensors]([["cov_bas","cov_bas"],[]]);
Y2:=DGbasis([seq(Tensor:-YoungSymmetrizer(a, Matrix
   ([[1],[2]])), a = X2)]);
numY:=nops(Y2);
Y:=DGzip([seq(z||i, i=1+numX..numX+numY)],Y2,"plus");

F:=z||(1+numX+numY);

Z:=DGzip([seq(z||i, i=2+numX+numY..1+numX+numY+dimbase)],X1,"
   plus");

#The following procedure is a bit overkill for the Killing
   vector case. It is intended to give us the index list from
    which we pick off the independent components of each
   tensor defined above.

getsome:=proc(T,Y) local A,B,g,A2,inds,B2,thing,bracket,term;
A:=Array([]);
for thing in op(2,op(T)) do
if nops(op(1,op(1,thing[2]))) = 2 then ArrayTools:-Append(A,
   op(1,thing));
fi;
od;
B:=[seq(A[i],i=op(2,A))];
#we now have the bads.
g:=seq(op(2,op(Y[i])),i=1..nops(Y));
A2:=Array([]);
inds:=seq([seq(g[j][i][1],i=1..nops(g[j]))],j=1..nops([g]));
for bracket in inds do
for term in bracket do
if has(B,[term])=false then if has(A2,bracket)=false then
   ArrayTools:-Append(A2,[term]);
fi;
fi;
od;
od;
B2:=ListTools:-FlattenOnce([seq(A2[i],i=op(2,A2))]);
end;

XA:=getsome(X,X1);
#if dimbase = 2 then XA:= [1,2] else XA:=getsome(X,Y1) fi;
```

```
if dimbase = 2 then YA:=[[1,2]] else YA:=getsome(Y,Y2) fi;
#WA:=getsome(W,Y2);

#Now, we will get rid of the scalars: terms like z12/2 will
   be turned into y12.

Comps:=ListTools:-FlattenOnce([[seq(DGinformation(X,"
   CoefficientList",[a])[1],a=XA)],[seq(DGinformation(Y,"
   CoefficientList",[a])[1],a=YA)], F,[seq(DGinformation(Z,"
   CoefficientList",[a])[1],a=XA)]]);
Comps2:=[seq(y||i,i=1..nops(Comps))];
Comps3:=[seq(Comps[i] = Comps2[i], i=1..nops(Comps))];
Comps4:=solve(Comps3,{seq(z||i,i=1..nops(Comps))});

realX:=evalDG(simplify(subs(Comps4,X)));
realY:=evalDG(simplify(subs(Comps4,Y)));
realF:=evalDG(simplify(subs(Comps4,F)));
realZ:=evalDG(simplify(subs(Comps4,Z)));

XR:=ContractIndices(CT,realX,[[1,1]]);
YR:=ContractIndices(RaiseLowerIndices(gin,RT,[1]),realY
    ,[[1,2]]);
term:=evalDG(RT-evalDG((RS/(2*(dimbase-1)))*g));
CovariantDerivative(term,Gamma);
evalDG(CovariantDerivative(term,Gamma) &t realX);
gin;
ContractIndices(evalDG(CovariantDerivative(term,Gamma) &t
   realX),gin,[[1,1],[2,2]]);
term1:=evalDG(-(-1)*ContractIndices(evalDG(
   CovariantDerivative(term,Gamma) &t realX),gin
   ,[[3,1],[4,2]]));
term2:=evalDG(-2*realF*term);

#Now, we build each piece to the matrix of 1-forms.

for m in seq(i,i=1..dimbase) do
Xc1b:=DirectionalCovariantDerivative(Bv[m],realX,Gamma);
Xc1c:=[seq( DGinformation(Xc1b,"CoefficientList",[[a[1]]])
    [1],a=XA)];
Xc2:=[seq(DGinformation(realY,"CoefficientList",[[a[1],m]])
    [1], a=XA)];
Xc3:=[seq(DGinformation(evalDG(realF*g),"CoefficientList",[[m
    ,a[1]]])[1],a=XA)];
Xc1:=[seq(Xc2[i]+Xc3[i]+Xc1c[i],i=1..nops(Xc1c))];
```

```
Yc1b:=DirectionalCovariantDerivative(Bv[m], realY, Gamma);
Yc1c:=[seq( DGinformation(Yc1b,"CoefficientList",[[a[1],a
    [2]]])[1], a=YA)];
Yc2:=[seq(DGinformation(XR,"CoefficientList",[[m,a[2],a[1]]])
    [1],  a=YA)];
Yc3:=[seq(DGinformation(evalDG(g &t  realZ),"CoefficientList
    ",[[m,a[1],a[2]]])[1],a=YA)];
Yc4:=[seq(DGinformation(evalDG(-g &t  realZ),"CoefficientList
    ",[[m,a[2],a[1]]])[1],a=YA)];
Yc1:=[seq(Yc2[i] + Yc3[i] + Yc4[i] + Yc1c[i],i=1..nops(Yc1c))
    ];

#Fc1b:=DirectionalCovariantDerivative(Bv[m], realF, Gamma);
#Fc1c:=[seq(DGinformation(Fc1b,"CoefficientList",[[m]])[1],a=
   XA)];

Fc2:=[DGinformation(realZ,"CoefficientList",[[m]])[1]];
Fc1:=[seq(Fc2[i],i=1..nops(Fc2))];

Zc1b:=DirectionalCovariantDerivative(Bv[m], realZ, Gamma);
Zc1c:=[seq( DGinformation(Zc1b,"CoefficientList",[[a[1]]])
    [1],a=XA)];
Zc2:=[seq(DGinformation(evalDG((1)/(dimbase-2)*term1),"
    CoefficientList",[[b[1],m]])[1],  b=XA)];
Zc3:=[seq(DGinformation(evalDG((1)/(dimbase-2)*term2),"
    CoefficientList",[[b[1],m]])[1],  b=XA)];
Zc4:=[seq(DGinformation(evalDG((1)/(dimbase-2)*YR),"
    CoefficientList",[[b[1],m]])[1],  b=XA)];
Zc5:=[seq(DGinformation(evalDG((1)/(dimbase-2)*YR),"
    CoefficientList",[[m,b[1]]])[1],  b=XA)];
Zc1:=[seq(Zc2[i] + Zc3[i] + Zc4[i] + Zc5[i] + Zc1c[i],i=1..
    nops(Zc1c))];

Eqns||m:=ListTools:-Flatten([Xc1,Yc1,Fc1,Zc1]);
Mat||m:=evalDG(LinearAlgebra:-GenerateMatrix(Eqns||m,Comps2)*
    Bf[m]);
od;

#Lastly, we will piece together the matrix and build the
    connection from it.

BigMat:=add(Mat||i, i=1..dimbase);
Connection(BigMat);
end:
```

```
##############################################################

liftConfKV:=proc(X,g,Q) local Bf,dimbase,Bft,Fv,forms2,Gamma,
    gin,Xcomps,omega,Omega,Omegacomps,F,Fcomps,Z,Zcomps,COMPS,
    liftedKV;
Bf:=DGinformation(Q,"FrameBaseForms");
dimbase:=nops(Bf);
Bft:=map(convert,Bf,DGtensor);
Fv:=DGinformation(Q,"FrameFiberVectors");
forms2:=GenerateForms(Bf,2);

Gamma:=Christoffel(g);
gin:=InverseMetric(g);

Xcomps:=GetComponents(X,Bf);
#dummyX:=DGzip([seq(z||i,i=1..dimbase)],Bf,"plus");

omega:=SymmetrizeIndices(evalDG(-CovariantDerivative(X,Gamma)
    ),[1,2],"SkewSymmetric");
Omega:=convert(omega,DGform);
Omegacomps:=GetComponents(Omega,forms2);

F:=evalDG(-(1/dimbase)*ContractIndices(CovariantDerivative(
    RaiseLowerIndices(gin,X,[1]),Gamma),[[1,2]]));
Fcomps:=[F];

Z:=evalDG(-CovariantDerivative(F,Gamma));
Zcomps:=GetComponents(Z,Bft);

COMPS:=ListTools:-FlattenOnce([Xcomps,Omegacomps,Fcomps,
    Zcomps]);
liftedKV:=DGzip(COMPS,Fv,"plus");
end:

##############################################################

getConfKV:=proc(X,Q) local Bf,dimbase,Fv,Xcomps,Xcomps2,realX
    ;
Bf:=DGinformation(Q,"FrameBaseForms");
dimbase:=nops(Bf);
Fv:=DGinformation(Q,"FrameFiberVectors");
Xcomps:=GetComponents(X,Fv);
Xcomps2:=[seq(Xcomps[i],i=1..dimbase)];
realX:=DGzip(Xcomps2,Bf,"plus");
end:
```

```
#########################################################

HauserTractorConnection:=proc(Gamma,T) local Bv,Bf,CT,dimbase
    ,Fv,ktb,KA,ltb,Ltb,LA,mtb,Mtb,MA,numK,numL,numM,K1,K2,L1,
    L2,M1,M2,Comps,Comps2,Comps3,Comps4,realK,realL,realM,KR,
    Lc1,dR,KdR,RL2,RL3,Mc1,Eqns,Kc1a,Kc2,Kc3,Lc1a,Lc2,Lc3,Lc4,
    Lc5,Lc6,Lc7,Lc8,Lc9,Lc10,Mc1a,Mc2,Mc3,Mc4,Mc5,Mc6,Mc7,Mc8,
    Mc9,Mc10,Mc11,Mc12,Mc13,Mc14,Mc15,Mc16,Mc17,Mc18,Mc19,Mc20
    ,Mc21,Mc22,Mc23,Mc24,Mc25,Mc26,Mc27,Mc28,Mc29,Mc30,Mc31,
    Mc32,Mc33,Mc34,Mc35,Mc36,Mc37,Mc38,Mc39,Mc40,Mc41,Mc42,
    Mc43,Mc44,Mc45,Mc46,Mc47,Mc48,Mc49,Mc50,Mc51,Mc52,Mc53,
    Mc54,Mc55,Kc1,Kc1b,Kc1c,Lc1b,Lc1c,Mc1b,Mc1c,m,BigMat;

Bv:=DGinformation(T,"FrameBaseVectors");
Bf:=DGinformation(T,"FrameBaseForms");

CT:=CurvatureTensor(Gamma);

dimbase:=nops(Bv);
Fv:=DGinformation(T,"FrameFiberVectors");

#Here we get the independent components list for each tensor.

ktb:=YoungTableauBasis([2],dimbase,output="Matrix");
KA:=[seq([ktb[i][1][1],ktb[i][1][2]],i=1..nops(ktb))];
ltb:=YoungTableauBasis([2,1],dimbase,output="Matrix");
Ltb:=map(LinearAlgebra:-Transpose,ltb);
LA:=[seq([Ltb[i][1][1],Ltb[i][1][2],Ltb[i][2][1]],i=1..nops(
    Ltb))];
mtb:=YoungTableauBasis([2,2],dimbase,output="Matrix");
Mtb:=map(LinearAlgebra:-Transpose,mtb);
MA:=[seq([Mtb[i][1][1],Mtb[i][1][2],Mtb[i][2][1],Mtb[i
    ][2][2]],i=1..nops(Mtb))];

numK:=nops(KA);
numL:=nops(LA);
numM:=nops(MA);

K1:=_DG([["tensor", T, [["cov_bas", "cov_bas"], []]], [seq([
    KA[i],z||i],i=1..numK)]]);
K2:=YoungSymmetrizer(K1,Matrix([[1,2]]));

L1:=_DG([["tensor", T, [["cov_bas", "cov_bas","cov_bas"],
    []]], [seq([LA[i-numK],z||i],i=1+numK..numK+numL)]]);
```

```
L2:=YoungSymmetrizer(L1,Matrix([[1,3],[2]]));

M1:=_DG([["tensor", T, [["cov_bas", "cov_bas","cov_bas","
    cov_bas"], []]], [seq([MA[i-numL-numK],z||i],i=1+numK+numL
    ..numK+numL+numM)]]);
M2:=YoungSymmetrizer(M1,Matrix([[1,3],[2,4]]));

Comps:=ListTools:-FlattenOnce([[seq(DGinformation(K2,"
    CoefficientList",[a])[1],a=KA)],[seq(DGinformation(L2,"
    CoefficientList",[a])[1],a=LA)],[seq(DGinformation(M2,"
    CoefficientList",[a])[1],a=MA)]]);

Comps2:=[seq(y||i,i=1..nops(Comps))];
Comps3:=[seq(Comps[i] = Comps2[i], i=1..nops(Comps))];
Comps4:=solve(Comps3,{seq(z||i,i=1..nops(Comps))});

realK:=evalDG(simplify(subs(Comps4,K2)));
realL:=evalDG(simplify(subs(Comps4,L2)));
realM:=evalDG(simplify(subs(Comps4,M2)));

#Having constructed K, L, and M, we give the first structure
    equation.

KR:=ContractIndices( realK,CT,[[2,1]]):
dR:=CovariantDerivative(CT,Gamma);
KdR:=ContractIndices(realK,dR,[[2,1]]);
RL2:=ContractIndices(CT,realL,[[1,2]]);
RL3:=ContractIndices(CT,realL,[[1,3]]);

for m in seq(i,i=1..dimbase) do
Kc1b:=DirectionalCovariantDerivative(Bv[m],realK,Gamma);
Kc1c:=[seq( DGinformation(Kc1b,"CoefficientList",[[a[1],a
    [2]]])[1],a=KA)];
Kc2:=[seq(DGinformation(evalDG(1/3*realL),"CoefficientList
    ",[[m,a[1],a[2]]])[1], a=KA)];
Kc3:=[seq(DGinformation(evalDG(1/3*realL),"CoefficientList
    ",[[m,a[2],a[1]]])[1], a=KA)];
Kc1:=[seq(Kc2[i]+Kc3[i] + Kc1c[i],i=1..nops(Kc1c))];
#was the last sign a minus??

#End of first structure equation.

#The next Structure equation is as follows. First, we give
    all of the terms.
```

```
Lc1b:=DirectionalCovariantDerivative(Bv[m],realL,Gamma);
Lc1c:=[seq( DGinformation(Lc1b,"CoefficientList",[[a[1],a[2],
   a[3]]])[1],a=LA)];
Lc2:=[seq(DGinformation(evalDG(5/4*KR),"CoefficientList",[[a
   [3],m,a[1],a[2]]])[1], a=LA)];
Lc3:=[seq(DGinformation(evalDG(3/4*KR),"CoefficientList",[[m,
   a[3],a[1],a[2]]])[1], a=LA)];
Lc4:=[seq(DGinformation(evalDG(1/2*KR),"CoefficientList",[[a
   [1],m,a[3],a[2]]])[1], a=LA)];
Lc5:=[seq(DGinformation(evalDG(-1/2*KR),"CoefficientList",[[a
   [2],m,a[3],a[1]]])[1], a=LA)];
Lc6:=[seq(DGinformation(evalDG(1/2*KR),"CoefficientList",[[a
   [1],a[3],m,a[2]]])[1], a=LA)];
Lc7:=[seq(DGinformation(evalDG(-1/2*KR),"CoefficientList",[[a
   [2],a[3],m,a[1]]])[1], a=LA)];
Lc8:=[seq(DGinformation(evalDG(1/4*KR),"CoefficientList",[[a
   [2],a[1],a[3],m]])[1], a=LA)];
Lc9:=[seq(DGinformation(evalDG(-1/4*KR),"CoefficientList",[[a
   [1],a[2],a[3],m]])[1], a=LA)];
Lc10:=[seq(DGinformation(realM,"CoefficientList",[[a[1],a[2],
   a[3],m]])[1], a=LA)];

#Now we "zip" the terms together.

Lc1:=[seq(Lc2[i] + Lc3[i] + Lc4[i] + Lc5[i] + Lc6[i] + Lc7[i]
    + Lc8[i] + Lc9[i] + Lc10[i] + Lc1c[i],i=1..nops(Lc1c))];

#Having constructed the second structure equation, we
   construct the last.

#Now we give each term separately, as before.

#"easy" part:
Mc1b:=DirectionalCovariantDerivative(Bv[m],realM,Gamma);
Mc1c:=[seq( DGinformation(Mc1b,"CoefficientList",[[a[1],a[2],
   a[3],a[4]]])[1],a=MA)];

#KdR terms:
Mc2:=[seq(DGinformation(evalDG((-1)*KdR),"CoefficientList",[[
    m,a[3],a[1],a[2],a[4]  ]])[1], a=MA)];
Mc3:=[seq(DGinformation(evalDG((-1)*KdR),"CoefficientList",[[
    m,a[4],a[2],a[1],a[3]  ]])[1], a=MA)];
Mc4:=[seq(DGinformation(evalDG((-1)*KdR),"CoefficientList",[[
    a[4],m,a[2],a[1],a[3]  ]])[1], a=MA)];
Mc5:=[seq(DGinformation(evalDG(-(-1)*KdR),"CoefficientList
```

```
    ",[[ a[3],m,a[2],a[1],a[4] ]])[1], a=MA)];
#End of line 1
Mc6:=[seq(DGinformation(evalDG(1/2*(-1)*KdR),"CoefficientList
    ",[[ a[1],a[3],m,a[2],a[4] ]])[1], a=MA)];
Mc7:=[seq(DGinformation(evalDG(1/2*(-1)*KdR),"CoefficientList
    ",[[ a[1],a[4],a[2],m,a[3] ]])[1], a=MA)];
Mc8:=[seq(DGinformation(evalDG(1/2*(-1)*KdR),"CoefficientList
    ",[[ a[2],a[3],a[1],m,a[4] ]])[1], a=MA)];
Mc9:=[seq(DGinformation(evalDG(1/2*(-1)*KdR),"CoefficientList
    ",[[ a[2],a[4],m,a[1],a[3] ]])[1], a=MA)];
Mc10:=[seq(DGinformation(evalDG(1/2*(-1)*KdR),"
    CoefficientList",[[ a[2],m,a[4],a[3],a[1] ]])[1], a=MA)];
#End of line 2
Mc11:=[seq(DGinformation(evalDG(-1/2*(-1)*KdR),"
    CoefficientList",[[ a[1],m,a[4],a[3],a[2] ]])[1], a=MA)];
Mc12:=[seq(DGinformation(evalDG(1/4*(-1)*KdR),"
    CoefficientList",[[ a[2],a[1],a[4],a[3],m ]])[1], a=MA)];
Mc13:=[seq(DGinformation(evalDG(-1/4*(-1)*KdR),"
    CoefficientList",[[ a[1],a[2],a[4],a[3],m ]])[1], a=MA)];
Mc14:=[seq(DGinformation(evalDG(1/4*(-1)*KdR),"
    CoefficientList",[[ a[3],a[4],a[2],a[1],m ]])[1], a=MA)];
Mc15:=[seq(DGinformation(evalDG(-1/4*(-1)*KdR),"
    CoefficientList",[[ a[4],a[3],a[2],a[1],m ]])[1], a=MA)];
#End of line 3: end of KdR terms.

#Beginning of RL2 terms.
Mc16:=[seq(DGinformation(evalDG(1/2*RL2),"CoefficientList",[[
     a[1],a[3],m,a[2],a[4] ]])[1], a=MA)];
Mc17:=[seq(DGinformation(evalDG(-1/2*RL2),"CoefficientList
    ",[[ a[1],a[4],m,a[2],a[3] ]])[1], a=MA)];
Mc18:=[seq(DGinformation(evalDG(-1/2*RL2),"CoefficientList
    ",[[ a[2],a[3],m,a[1],a[4] ]])[1], a=MA)];
Mc19:=[seq(DGinformation(evalDG(1/2*RL2),"CoefficientList",[[
     a[2],a[4],m,a[1],a[3] ]])[1], a=MA)];
Mc20:=[seq(DGinformation(evalDG(-1/2*RL2),"CoefficientList
    ",[[ a[1],a[4],a[3],a[2],m ]])[1], a=MA)];
Mc21:=[seq(DGinformation(evalDG(1/2*RL2),"CoefficientList",[[
     a[2],a[4],a[3],a[1],m ]])[1], a=MA)];
#End of line 4

Mc22:=[seq(DGinformation(evalDG(-1/3*RL2),"CoefficientList
    ",[[ a[3],a[2],a[1],a[4],m ]])[1], a=MA)];
Mc23:=[seq(DGinformation(evalDG(1/3*RL2),"CoefficientList",[[
     a[4],a[2],a[1],a[3],m ]])[1], a=MA)];
```

```
#Mc22:=[seq(DGinformation(evalDG(−1/6∗RL2),"CoefficientList
    ",[[ a[1],a[2],a[3],a[4],m ]])[1], a=MA)];
#Mc23:=[seq(DGinformation(evalDG(1/6∗RL2),"CoefficientList
    ",[[ a[1],a[2],a[4],a[3],m ]])[1], a=MA)];
#Mc24:=[seq(DGinformation(evalDG(1/6∗RL2),"CoefficientList
    ",[[ a[2],a[1],a[3],a[4],m ]])[1], a=MA)];
#Mc25:=[seq(DGinformation(evalDG(−1/6∗RL2),"CoefficientList
    ",[[ a[2],a[1],a[4],a[3],m ]])[1], a=MA)];

#End of line 5
Mc26:=[seq(DGinformation(evalDG(−1/6∗RL2),"CoefficientList
    ",[[ a[3],m,a[1],a[4],a[2] ]])[1], a=MA)];
Mc27:=[seq(DGinformation(evalDG(−1/6∗RL2),"CoefficientList
    ",[[ m,a[3],a[1],a[4],a[2] ]])[1], a=MA)];
Mc28:=[seq(DGinformation(evalDG(1/6∗RL2),"CoefficientList",[[
     a[4],m,a[1],a[3],a[2] ]])[1], a=MA)];
Mc29:=[seq(DGinformation(evalDG(1/6∗RL2),"CoefficientList",[[
     m,a[4],a[1],a[3],a[2] ]])[1], a=MA)];
#End of line 6
Mc30:=[seq(DGinformation(evalDG(1/6∗RL2),"CoefficientList",[[
     a[3],m,a[2],a[4],a[1] ]])[1], a=MA)];
Mc31:=[seq(DGinformation(evalDG(1/6∗RL2),"CoefficientList",[[
     m,a[3],a[2],a[4],a[1] ]])[1], a=MA)];
Mc32:=[seq(DGinformation(evalDG(−1/6∗RL2),"CoefficientList
    ",[[ a[4],m,a[2],a[3],a[1] ]])[1], a=MA)];
Mc33:=[seq(DGinformation(evalDG(−1/6∗RL2),"CoefficientList
    ",[[ m,a[4],a[2],a[3],a[1] ]])[1], a=MA)];
#End of line 7
Mc34:=[seq(DGinformation(evalDG(1/12∗RL2),"CoefficientList
    ",[[ a[1],a[4],a[3],m,a[2] ]])[1], a=MA)];
Mc35:=[seq(DGinformation(evalDG(−1/12∗RL2),"CoefficientList
    ",[[ a[2],a[4],a[3],m,a[1] ]])[1], a=MA)];
Mc36:=[seq(DGinformation(evalDG(−1/12∗RL2),"CoefficientList
    ",[[ a[3],a[2],a[1],m,a[4] ]])[1], a=MA)];
Mc37:=[seq(DGinformation(evalDG(1/12∗RL2),"CoefficientList
    ",[[ a[4],a[2],a[1],m,a[3] ]])[1], a=MA)];
#End of line 8 and end of RL2 terms.

#Beginning of RL3 terms.

Mc38:=[seq(DGinformation(evalDG(−1/3∗RL3),"CoefficientList
    ",[[ a[3],a[2],a[1],a[4],m ]])[1], a=MA)];
Mc39:=[seq(DGinformation(evalDG(1/3∗RL3),"CoefficientList",[[
     a[4],a[2],a[1],a[3],m ]])[1], a=MA)];
```

```
#Mc38:=[seq(DGinformation(evalDG(−1/6*RL3),"CoefficientList
     ",[[  a[1],a[2],a[3],a[4],m  ]])[1],  a=MA)];
#Mc39:=[seq(DGinformation(evalDG(1/6*RL3),"CoefficientList
     ",[[  a[1],a[2],a[4],a[3],m  ]])[1],  a=MA)];
#Mc40:=[seq(DGinformation(evalDG(1/6*RL3),"CoefficientList
     ",[[  a[2],a[1],a[3],a[4],m  ]])[1],  a=MA)];
#Mc41:=[seq(DGinformation(evalDG(−1/6*RL3),"CoefficientList
     ",[[  a[2],a[1],a[4],a[3],m  ]])[1],  a=MA)];

#End of line 9
Mc42:=[seq(DGinformation(evalDG(−1/6*RL3),"CoefficientList
     ",[[  a[3],m,a[1],a[4],a[2]  ]])[1],  a=MA)];
Mc43:=[seq(DGinformation(evalDG(−1/6*RL3),"CoefficientList
     ",[[  m,a[3],a[1],a[4],a[2]  ]])[1],  a=MA)];
Mc44:=[seq(DGinformation(evalDG(1/6*RL3),"CoefficientList",[[
      a[4],m,a[1],a[3],a[2]  ]])[1],  a=MA)];
Mc45:=[seq(DGinformation(evalDG(1/6*RL3),"CoefficientList",[[
      m,a[4],a[1],a[3],a[2]  ]])[1],  a=MA)];
#End of line 10
Mc46:=[seq(DGinformation(evalDG(1/6*RL3),"CoefficientList",[[
      a[3],m,a[2],a[4],a[1]  ]])[1],  a=MA)];
Mc47:=[seq(DGinformation(evalDG(1/6*RL3),"CoefficientList",[[
      m,a[3],a[2],a[4],a[1]  ]])[1],  a=MA)];
Mc48:=[seq(DGinformation(evalDG(−1/6*RL3),"CoefficientList
     ",[[  a[4],m,a[2],a[3],a[1]  ]])[1],  a=MA)];
Mc49:=[seq(DGinformation(evalDG(−1/6*RL3),"CoefficientList
     ",[[  m,a[4],a[2],a[3],a[1]  ]])[1],  a=MA)];
#End of line 11
Mc50:=[seq(DGinformation(evalDG(1/12*RL3),"CoefficientList
     ",[[  a[1],a[4],a[3],m,a[2]  ]])[1],  a=MA)];
Mc51:=[seq(DGinformation(evalDG(−1/12*RL3),"CoefficientList
     ",[[  a[2],a[4],a[3],m,a[1]  ]])[1],  a=MA)];
Mc52:=[seq(DGinformation(evalDG(−1/12*RL3),"CoefficientList
     ",[[  a[3],a[2],a[1],m,a[4]  ]])[1],  a=MA)];
Mc53:=[seq(DGinformation(evalDG(1/12*RL3),"CoefficientList
     ",[[  a[4],a[2],a[1],m,a[3]  ]])[1],  a=MA)];
Mc54:=[seq(DGinformation(RL3,"CoefficientList",[[  m,a[4],a
    [3],a[1],a[2]  ]])[1],  a=MA)];
Mc55:=[seq(DGinformation(RL3,"CoefficientList",[[  m,a[2],a
    [1],a[3],a[4]  ]])[1],  a=MA)];
#End of line 12
#End of terms.

#Mc1:=[seq(Mc2[i] + Mc3[i] + Mc4[i] + Mc5[i] + Mc6[i] + Mc7[i
    ] + Mc8[i] + Mc9[i] + Mc10[i] + Mc11[i] + Mc12[i] + Mc13[i
```

```
] + Mc14[i] + Mc15[i] + Mc16[i] + Mc17[i] + Mc18[i] + Mc19
    [i] + Mc20[i] + Mc21[i] + Mc22[i] + Mc23[i] + Mc24[i] +
    Mc25[i] + Mc26[i] + Mc27[i] + Mc28[i] + Mc29[i] + Mc30[i]
    + Mc31[i] + Mc32[i] + Mc33[i] + Mc34[i] + Mc35[i] + Mc36[i
    ] + Mc37[i] + Mc38[i] + Mc39[i] + Mc40[i] + Mc41[i] + Mc42
    [i] + Mc43[i] + Mc44[i] + Mc45[i] + Mc46[i] + Mc47[i] +
    Mc48[i] + Mc49[i] + Mc50[i] + Mc51[i] + Mc52[i] + Mc53[i]
    + Mc54[i] + Mc55[i] + Mc1c[i],i=1..nops(Mc1c))];

Mc1:=[seq(Mc2[i] + Mc3[i] + Mc4[i] + Mc5[i] + Mc6[i] + Mc7[i]
     + Mc8[i] + Mc9[i] + Mc10[i] + Mc11[i] + Mc12[i] + Mc13[i]
     + Mc14[i] + Mc15[i] + Mc16[i] + Mc17[i] + Mc18[i] + Mc19[
    i] + Mc20[i] + Mc21[i] + Mc22[i] + Mc23[i] + Mc26[i] +
    Mc27[i] + Mc28[i] + Mc29[i] + Mc30[i] + Mc31[i] + Mc32[i]
    + Mc33[i] + Mc34[i] + Mc35[i] + Mc36[i] + Mc37[i] + Mc38[i
    ] + Mc39[i] + Mc42[i] + Mc43[i] + Mc44[i] + Mc45[i] + Mc46
    [i] + Mc47[i] + Mc48[i] + Mc49[i] + Mc50[i] + Mc51[i] +
    Mc52[i] + Mc53[i] + Mc54[i] + Mc55[i] + Mc1c[i],i=1..nops(
    Mc1c))];

Eqns||m:=ListTools:-Flatten([Kc1,Lc1,Mc1]);
Mat||m:=evalDG(LinearAlgebra:-GenerateMatrix(Eqns||m,Comps2)*
    Bf[m]);

od;

BigMat:=add(Mat||i, i=1..dimbase);
Connection(BigMat);

end:

#########################################################

HauserTractorLift2:=proc(K,Gamma,Q) local Bv,Bf,dimbase,Fv,
    ktb,ytb,Ytb,mtb,Mtb,KA,LA,MA,dK,L,numK,numL,numM,dL,dLs,M,
    Kcomps,Lcomps,Mcomps,COMPS,liftedKT;
Bv:=DGinformation(Q,"FrameBaseVectors");
Bf:=DGinformation(Q,"FrameBaseForms");

dimbase:=nops(Bv);
Fv:=DGinformation(Q,"FrameFiberVectors");

#Here we get the independent components list for each tensor.

ktb:=YoungTableauBasis([2],dimbase,output="Matrix");
```

```
KA:=[seq([ktb[i][1][1],ktb[i][1][2]],i=1..nops(ktb))];
ytb:=YoungTableauBasis([2,1],dimbase,output="Matrix");
Ytb:=map(LinearAlgebra:-Transpose,ytb);
LA:=[seq([Ytb[i][1][1],Ytb[i][1][2],Ytb[i][2][1]],i=1..nops(
    Ytb))];
mtb:=YoungTableauBasis([2,2],dimbase,output="Matrix");
Mtb:=map(LinearAlgebra:-Transpose,mtb);
MA:=[seq([Mtb[i][1][1],Mtb[i][1][2],Mtb[i][2][1],Mtb[i
    ][2][2]],i=1..nops(Mtb))];

dK:=CovariantDerivative(K,Gamma);

L:=evalDG(2*(-1)*RearrangeIndices(SymmetrizeIndices(dK
    ,[2,3],"SkewSymmetric"),[[3,1],[2]]));

dL:=CovariantDerivative(L,Gamma);
dLs:=SymmetrizeIndices(dL,[3,4],"SkewSymmetric");

M:=evalDG(1/2*(-1)*( dLs + RearrangeIndices(dLs
    ,[[3,1],[4,2]]) ));

Kcomps:=DGinformation(K,"CoefficientList",KA);
Lcomps:=DGinformation(L,"CoefficientList",LA);
Mcomps:=DGinformation(M,"CoefficientList",MA);
COMPS:=ListTools:-FlattenOnce([Kcomps,Lcomps,Mcomps]);

liftedKT:=DGzip(COMPS,Fv,"plus");
end:

#####################################################################

getHauserKT2:=proc(KT,Q) local Bv,dimbase,Fv,ktb,K1,K2,numK,
    Kcomps,KA,Comps,Comps2,Comps3,Comps4,Comps5,realK,RealK,
    ZERO;
Bv:=DGinformation(Q,"FrameBaseVectors");
dimbase:=nops(Bv);
Fv:=DGinformation(Q,"FrameFiberVectors");

ktb:=YoungTableauBasis([2],dimbase,output="Matrix");
KA:=[seq([ktb[i][1][1],ktb[i][1][2]],i=1..nops(ktb))];
numK:=nops(KA);
K1:=_DG([["tensor", Q, [["cov_bas", "cov_bas"], []]], [seq([
    KA[i],z||i],i=1..numK)]]);
K2:=YoungSymmetrizer(K1,Matrix([[1,2]]));
```

```
Kcomps:=[seq(GetComponents(KT,Fv)[i],i=1..numK)];

Comps:=[seq(DGinformation(K2,"CoefficientList",[a])[1],a=KA)
    ];
Comps2:=[seq(y||i,i=1..nops(Comps))];
Comps3:=[seq(Comps[i] = Comps2[i], i=1..nops(Comps))];
Comps4:=solve(Comps3,{seq(z||i,i=1..nops(Comps))});

realK:=evalDG(simplify(subs(Comps4,K2)));

Comps5:=[seq(Comps2[i] = Kcomps[i], i=1..nops(Kcomps))];

ZERO:=_DG([["tensor", Q, [["cov_bas", "cov_bas"], []]], [[[1,
     1], 0]]]);

RealK:=evalDG(evalDG(simplify(subs(Comps5,realK)))+ZERO);
end:

##################################################################

KYTracCon:=proc(Gamma,k,Q) local Bv,dimbase,Bf,Fl,numF,F1l,
   numF1,Ff,Ft,FA,F1f,F1t,F1A,CT,CTF,m,Fc1b,Fc1c,Fc2,Fc1,
   F1c1b,F1c1c,F1c1,F1c2,BigMat,f1c2a,f1c2b;

Bv:=DGinformation(Q,"FrameBaseVectors");
dimbase:=nops(Bv);
Bf:=DGinformation(Q,"FrameBaseForms");
Fl:=GenerateDGobjects[DGforms](Bf,k);
numF:=nops(Fl);

if dimbase=k then

#This first part deals with the case in which the dimension
   is equal to the rank.

Ff:=DGzip([seq(z||i,i=1..numF)],Fl,"plus");
Ft:=convert(Ff,DGtensor);
FA:=[seq(op(1,op(2,op(Fl[i]))[1]),i=1..numF)];

F1f:=DGzip([0],Fl,"plus");
F1t:=convert(F1f,DGtensor);

for m in seq(i,i=1..dimbase) do
Fc1b:=DirectionalCovariantDerivative(Bv[m],Ft,Gamma);
Fc1c:=[seq( DGinformation(Fc1b,"CoefficientList",[a])[1],a=FA
```

```
    )];
Fc2:=DGinformation(F1t,"CoefficientList",[seq(ListTools:-
    FlattenOnce([[m],a]),a=FA)]);
Fc1:=[seq(Fc2[i]+Fc1c[i],i=1..nops(Fc1c))];

Eqns||m:=ListTools:-Flatten([Fc1]);
Mat||m:=evalDG(LinearAlgebra:-GenerateMatrix(Eqns||m,[seq(z||
    i,i=1..numF)])*Bf[m]);
od;

#Lastly, we will piece together the matrix and build the
    connection from it.

BigMat:=add(Mat||i,i=1..dimbase);
Connection(BigMat);

#Now we handle all other cases.

else

F1l:=GenerateDGobjects[DGforms](Bf,k+1);
numF1:=nops(F1l);

Ff:=DGzip([seq(z||i,i=1..numF)],Fl,"plus");
Ft:=convert(Ff,DGtensor);
FA:=[seq(op(1,op(2,op(Fl[i]))[1]),i=1..numF)];

F1f:=DGzip([seq(z||i,i=1+numF..numF+numF1)],F1l,"plus");
F1t:=convert(F1f,DGtensor);
F1A:=[seq(op(1,op(2,op(F1l[i]))[1]),i=1..numF1)];

CT:=CurvatureTensor(Gamma);
CTF:=ContractIndices(CT,Ft,[[1,1]]);

for m in seq(i,i=1..dimbase) do
Fc1b:=DirectionalCovariantDerivative(Bv[m],Ft,Gamma);
Fc1c:=[seq( DGinformation(Fc1b,"CoefficientList",[a])[1],a=FA
    )];
Fc2:=DGinformation(F1t,"CoefficientList",[seq(ListTools:-
    FlattenOnce([[m],a]),a=FA)]);
Fc1:=[seq(Fc2[i]+Fc1c[i],i=1..nops(Fc1c))];

F1c1b:=DirectionalCovariantDerivative(Bv[m],F1t,Gamma);
F1c1c:=[seq( DGinformation(F1c1b,"CoefficientList",[a])[1],a=
    F1A)];
```

```
f1c2a:=evalDG((1)*SymmetrizeIndices(CTF,[seq(k+2-i,i=0..k-2)
    ,1],"SkewSymmetric"));
f1c2b:=evalDG(((-1)^k*(k+1)/k)*SymmetrizeIndices(f1c2a,[seq(k
    +2-i,i=0..k-2),2,1],"SkewSymmetric"));

F1c2:=[seq(DGinformation(evalDG(k*f1c2b),"CoefficientList",[[
    a[2],a[1],m,seq(a[i],i=3..k+1)]])[1], a=F1A)];

F1c1:=[seq(F1c2[i] +F1c1c[i],i=1..nops(F1c1c))];

Eqns||m:=ListTools:-Flatten([Fc1,F1c1]);
Mat||m:=evalDG(LinearAlgebra:-GenerateMatrix(Eqns||m,[seq(z||
    i,i=1..numF+numF1)])*Bf[m]);
od;

#Lastly, we will piece together the matrix and build the
    connection from it.

BigMat:=add(Mat||i,i=1..dimbase);
Connection(BigMat);
fi;
end:

########################################################################

liftKY:=proc(F,Gamma,Q) local Bv,dimbase,Bf,k,Fl,numF,F1l,
    numF1,Fv,FA,F1A,dF,dFs,F1,Fcomps,F1comps,COMPS,liftedKY;

Bv:=DGinformation(Q,"FrameBaseVectors");
dimbase:=nops(Bv);
Bf:=DGinformation(Q,"FrameBaseForms");
Fv:=DGinformation(Q,"FrameFiberVectors");
k:=op(1,op(F))[3];

if k=dimbase then

evalDG(DGinformation(F,"CoefficientSet")[1]*Fv[1]);

else

Fl:=GenerateDGobjects[DGforms](Bf,k);
numF:=nops(Fl);
F1l:=GenerateDGobjects[DGforms](Bf,k+1);
numF1:=nops(F1l);
```

```
Fv:=DGinformation(Q,"FrameFiberVectors");

FA:=[seq(op(1,op(2,op(Fl[i]))[1]),i=1..numF)];

F1A:=[seq(op(1,op(2,op(F1l[i]))[1]),i=1..numF1)];

dF:=CovariantDerivative(F,Gamma);
dFs:=evalDG(SymmetrizeIndices(dF,[seq(i,i=1..k+1)],"
    SkewSymmetric"));
F1:=RearrangeIndices(dFs,[seq(k+2-i,i=1..k+1)]);

Fcomps:=DGinformation(F,"CoefficientList",FA);
F1comps:=DGinformation(F1,"CoefficientList",F1A);
COMPS:=ListTools:-FlattenOnce([Fcomps,F1comps]);

liftedKY:=DGzip(COMPS,Fv,"plus");
fi;
end:

####################################################################

getKY:=proc(KY,k,Q)  local  Fv,  Bf,dimbase,numF,Fl,Fcomps,Ff,
    ZERO,RealF;
Fv:=DGinformation(Q,"FrameFiberVectors");
Bf:=DGinformation(Q,"FrameBaseForms");
dimbase:=nops(Bf);

Fl:=GenerateDGobjects[DGforms](Bf,k);
numF:=nops(Fl);
Fcomps:=[seq(GetComponents(KY,Fv)[i],i=1..numF)];

Ff:=DGzip([seq(z||i,i=1..numF)],Fl,"plus");

ZERO:=_DG([["form",Q,k],[[[1,1],0]]]);
RealF:=evalDG(subs([seq(z||i=Fcomps[i],i=1..numF)],Ff)+ZERO);

end:

####################################################################

MaxKT:=proc(m,n)
((m+n-1)!*(m+n)!)/((m-1)!*m!*n!*(n+1)!);
end:

####################################################################
```

```
MaxKY:=proc(n,k) local numF,numF1;
numF:=binomial(n,k);
numF1:=binomial(n,k+1);
numF+numF1;
end:

#####################################################################

MaxSym:=proc(m,n)
#Calculates the number of independent components of a
   completely symmetric tensor.
(m+n-1)!/(n!*(m-1)!)
end:

#####################################################################

MaxSkew:=proc(m,n)
#Calculates the number of independent components of a
   completely skew-symmetric tensor (should be the binomial
   formula).
m!/(n!*(m-n)!);
end:

#####################################################################

MaxCF:=proc(n,p);
#Calculates the maximum number of conformal Killing forms in
   n dimensions of rank p.
binomial(n+2,p+1);
end:

#####################################################################

KillingTensorLibrary:=module() export ModuleApply;

#The following read command would read in the entire database
   , which is now quite large. Thus, this program has been
   amended to read in only the file required.

#read "Database_table.txt";

ModuleApply := proc(n,name,{output:=[]}) local filename,
   filenamestr,C,V,out1,out2,name2,list,list2,list3; # name
   of manifold.
```

```
filename:= kt||n;
filenamestr:=cat("kt_entries/",cat(convert(filename,string)
    ,".txt"));
read filenamestr;

V := Ben[n]["Coordinates"];
DGEnvironment[Coordinate](V,name);
C := Ben[n]["Metric"];
if output <> [] then out1:=op(1,output) fi;
if nops(output) >= 2 then out2:=op(2,output) fi;
if nops(output) >= 3 then name2:=op(3,output) fi;

if output = [] then return
_DG([["tensor",name,[["cov_bas","cov_bas"],[]]],C]); #change
   the lprint to name;
fi;

if type(out1,integer) then #return #then return the
   irreducible killing tensors of order k.
list:=Ben[n]["IrreducibleKillingTensors",out1];
return [seq(_DG([["tensor",name,[[seq("cov_bas",i=1..out1)
   ],[]]],list[j]]),j=1..nops(list))];
fi;

if output = ["KillingTensors",out2] then
list:=Ben[n][out1,out2];
return [seq(_DG([["tensor",name,[[seq("cov_bas",i=1..out2)
   ],[]]],list[j]]),j=1..nops(list))];
fi;

if output = ["KillingYanoTensors",out2] then
list:=Ben[n][out1,out2];
return [seq(_DG([["form",name,out2],list[j]]),j=1..nops(list)
   )];
fi;

if output = ["ConformalKillingForms",out2] then
list:=Ben[n][out1,out2];
return [seq(_DG([["form",name,out2],list[j]]),j=1..nops(list)
   )];
fi;

if output = ["IrreducibleRank"] then
list:=[seq(lhs(op(op(Ben)[n])[i]),i=1..nops(op(op(Ben)[n])))
```

```
    ];
list2:=[ListTools:-SearchAll("IrreducibleKillingTensors",list
    )];
list3:=[seq(list[list2[i]+1],i=1..nops(list2))];
return list3;
fi;

if output = ["Notes"] then return Ben[n]["Notes"] fi;
if output = ["Reference"] then return Ben[n]["Reference"] fi;
if output = ["Coordinates"] then return Ben[n]["Coordinates"]
     fi;

if output = ["TractorConnection",out2,name2] then
DGEnvironment[VectorSpace](MaxKT(nops(Ben[n]["Coordinates"]),
    out2),name2_vs);
DGEnvironment[VectorBundle](name,name2_vs,name2);
return
_DG([["connection", name2, [["con_vrt", "cov_vrt", "cov_bas
    "], []]], Ben[n]["TractorConnection",out2]]);
fi;

if output = ["TractorCurvature",out2,name2] then
DGEnvironment[VectorSpace](MaxKT(nops(Ben[n]["Coordinates"]),
    out2),name2_vs);
DGEnvironment[VectorBundle](name,name2_vs,name2);
return
_DG([["tensor", name2, [["con_vrt", "cov_vrt", "cov_bas", "
    cov_bas"], []]], Ben[n]["TractorCurvature",out2]]);
fi;

if output = ["YanoTractorConnection",out2,name2] then
DGEnvironment[VectorSpace](MaxKY(nops(Ben[n]["Coordinates"]),
    out2),name2_vs);
DGEnvironment[VectorBundle](name,name2_vs,name2);
return
_DG([["connection", name2, [["con_vrt", "cov_vrt", "cov_bas
    "], []]], Ben[n]["YanoTractorConnection",out2]]);
fi;

if output = ["YanoTractorCurvature",out2,name2] then
DGEnvironment[VectorSpace](MaxKY(nops(Ben[n]["Coordinates"]),
    out2),name2_vs);
DGEnvironment[VectorBundle](name,name2_vs,name2);
return
_DG([["tensor", name2, [["con_vrt", "cov_vrt", "cov_bas", "
```

```
   cov_bas"], []]], Ben[n]["YanoTractorCurvature",out2]]);
fi;

if output = ["ConformalFormTractorConnection",out2,name2]
   then
DGEnvironment[VectorSpace](MaxCF(nops(Ben[n]["Coordinates"]),
   out2),name2_vs);
DGEnvironment[VectorBundle](name,name2_vs,name2);
return
_DG([["connection", name2, [["con_vrt", "cov_vrt", "cov_bas
   "], []]], Ben[n]["ConformalFormTractorConnection",out2]]);
fi;

if output = ["ConformalFormTractorCurvature",out2,name2] then
DGEnvironment[VectorSpace](MaxCF(nops(Ben[n]["Coordinates"]),
   out2),name2_vs);
DGEnvironment[VectorBundle](name,name2_vs,name2);
return
_DG([["tensor", name2, [["con_vrt", "cov_vrt", "cov_bas", "
   cov_bas"], []]], Ben[n]["ConformalFormTractorCurvature",
   out2]]);
fi;
end proc;

end module:

#####################################################

BundleLift:=proc(T,Q) local liftedT;
liftedT:=_DG([[op(T)[1][1],Q,op(T)[1][3]],op(T)[2]]);
end:

############################################################


KYtoKT:=proc(g,gin,KY1,KY2) local r,s1,s2;
r:=op(KY1)[1][3];
s1:=evalDG((-1)^(r-1)*TensorInnerProduct(g,KY1,KY2,
   inversemetric=gin,tensorindices=[seq(i,i=1..r-1)]));
s2:=SymmetrizeIndices(s1,[1,2],"Symmetric");
end:

########################################################

CKVtoKT:=proc(X,g) local s1,gin,Xu,F,s2,s3;
```

```
s1:=evalDG(X &t X);
gin:=InverseMetric(g);
Xu:=RaiseLowerIndices(gin,X,[1]);
F:=ContractIndices(Xu,X,[[1,1]]);
s2:=evalDG(F*g);
s3:=evalDG(s1-s2);
end:
```