BEBR
FACULTY WORKING PAPER NO. 1441

Computing Cournot-Nash Equilibria

Charles D. Kolstad
Lars Mathiesen

College of Commerce and Business Administration
Bureau of Economic and Business Research
University of Illinois, Urbana-Champaign

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March 1988

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Charles D. Kolstad, Assistant Professor
Department of Economics

Lars Mathiesen
Norwegian School of Economics and Business Administration, Bergen

This research was conducted in part while the first author was visiting the Center for Applied Research at the Norwegian School of Economics and Business Administration, and the Massachusetts Institute of Technology, whose support and hospitality are appreciated.

ABSTRACT

The paper presents an algorithm for the computation of Cournot-Nash economic equilibria. The method is based on formulating the equilibrium problem as that of finding a solution to a nonlinear complementarity problem, solved by sequential linearization and Lemke's algorithm. Conditions for local and global convergence are developed and the technique is applied to homogeneous, segmented and differentiated product markets.

I. INTRODUCTION

The computation of partial and general competitive equilibria has been a field of enormous empirical importance as well as a source of diverse and fundamental theoretical research questions. However, aside from the case of a pure monopolist, the question of the computation of equilibria under conditions of imperfect competition has been largely neglected.[1]

The purpose of this paper is to present an algorithm for the computation of Cournot-Nash (or, more simply, Cournot) equilibria. The method is based on formulating the equilibrium problem as that of finding a solution to a nonlinear complementarity problem (CP). We solve this CP by Newton's method whereby the CP is sequentially linearized and the resulting linear complementarity problems (LCP) solved using Lemke's algorithm.

This sequential LCP-algorithm (SLCP) has been used before in contexts other than the computation of imperfectly competitive equilibria. Specifically, Josephy (1979) and Friesz et al (1983) have solved competitive partial equilibrium models and Mathiesen (1985) has solved general equilibrium models using this method.

We shall demonstrate global convergence to a unique equilibrium. Assumptions on first, second and third derivatives of the profit functions are required. Roughly speaking, the condition that profits peak, yields existence; strict concavity of profits provides uniqueness as well as local convergence; and finally, concavity of marginal profits guarantees global convergence.

In the next section of the paper we review conditions for the existence of a unique equilibrium to the Cournot model. Then we proceed to prove convergence of the SLCP algorithm. In the subsequent sections we apply SLCP to a numerical example and then extend the method to other markets with Cournot-type producer behavior.

II. AN ALGORITHM FOR COMPUTING COURNOT EQUILIBRIA

A. Cournot Equilibrium: Existence and Uniqueness

Let there be N firms, each providing the same good. Denote the output of the ith firm by q_i, produced at cost $C_i(q_i)$. The firms face a market inverse demand function P(Q). Profits for the ith firm are then given by

$$\pi_i(q_i) = P(Q)q_i - C_i(q_i) \tag{1}$$

where $Q = \sum_{i=1}^{N} q_i$.

The Cournot equilibrium is defined in the conventional manner:

Defn: A Cournot equilibrium is a vector of outputs q^*, such that

$$\pi_i(q^*) = \max_{q_i} \pi_i(q_1^*,\ldots,q_{i-1}^*,\ q_i,\ q_{i+1}^*,\ldots,q_N^*),\ i = 1,\ldots,n$$

In other words, holding output of other firms constant, each firm can do no better.

We will assume inverse demand and costs are twice continuously differentiable and that profits are concave (pseudoconcavity[2] would suffice) with respect to own output. Thus, first-order stationarity conditions for a profit maximum are necessary and sufficient for a global (not necessarily unique) maximum of profit. Consequently, the set of Cournot equilibria is the same as the set of solutions to the following first order conditions for a profit maximum:

$$f_i \equiv -\frac{\partial \pi_i}{\partial q_i} = C_i'(q_i) - P\left(\sum_{j=1}^{N} q_j\right) - q_i P'\left(\sum_{j=1}^{N} q_j\right) \geq 0, \tag{2a}$$

$$\frac{\partial \pi_i}{\partial q_i} q_i = 0, \text{ and} \tag{2b}$$

$$q_i \geq 0, \text{ for } i = 1,\ldots,N. \tag{2c}$$

Eqn (2) is a complementarity problem which can be more compactly written as

$$\text{CP}(f): \quad \text{Find } q \in R^N \text{ such that}$$
$$f(q) \geq 0, \; q \geq 0 \;\text{ and }\; q^T f(q) = 0, \tag{3}$$

where $f\colon R^N_+ \to R^N$ and ^{T} denotes transpose. We can now formalize the relationship between Cournot equilibria and solutions to complementarity problems:

Lemma 1: If profit functions are twice continuously differentiable (C^2) and pseudoconcave with respect to own output, then q^* is a Cournot equilibrium if and only if q^* is a solution of the complementarity problem given by (3).

Proof: Straightforward application of necessary and sufficient conditions for maximization of a pseudoconcave function.

In the remainder of this paper we will take advantage of this correspondence and focus on solutions to the complementarity problem (3). The following two results regarding the existence and uniqueness of solutions to CP are known:

<u>Lemma 2</u>: (Karamardian, 1972): Let f: $R_+^N \to R^N$ be continuous. Assume there is a nonempty compact set $K \subset R^N$ such that for any $x \in R_+^N \setminus K$, there exists a $y \in K$ with $(x-y)f(x) > 0$. Then there exists a solution to CP(f).

The assumption on f is that eventually $f(x) > 0$. Observe that f may be positive for all $x \geq 0$. If so, $K = \{0\}$ and $x = 0$ is the solution.

<u>Lemma 3</u>: (Moré, 1974): If f is continuously differentiable and $\nabla f(x)$ is a P-matrix[3] for all $x \in R_+^N$, then there exists at most one solution to CP(f).

In order to assure that a Cournot equilibrium exists, we introduce the notion of a bound on industry output.

<u>Defn</u>: Industry output is said to be bounded by $Q > 0$ if output in excess of Q from any producer implies that the marginal profits of all producers are negative.

The definition merely states that for some industry output level, no firm is at a profit maximum and that further expanding the industry will not improve profits. That is, marginal profits become negative and the functions f(q) become positive.

Based on Lemmas 1 - 3, we can state conditions under which a unique Cournot equilibrium exists:

<u>Theorem 1</u>: Assume that

i) inverse demand and costs are C^2,

ii) industry output is bounded, and

iii) profits are strictly concave for all $q \geq 0$.

Then there exists a unique Cournot equilibrium.

Proof: Let f be the vector of negative marginal profits. Since profits are strictly concave, the Jacobian of f is positive definite and hence a P-matrix. Thus, by Lemma 3 there is at most one solution to (3). Because industry output is bounded, Lemma 2 implies that a solution exists (with y in the Lemma being any point each of whose coordinates exceeds the industry bound). Because profits are strictly concave with respect to own output, the Cournot equilibrium is unique by Lemma 1.

A special case is where the Jacobian matrix of f has a positive dominant diagonal and thus is a P-matrix. In this case the interpretation is that if own-effects on marginal profits dominate cross-effects, an equilibrium will be unique. It should be noted that theorem 1 is a sufficient condition for uniqueness and can be considerably relaxed (see Kolstad and Mathiesen, 1987). However, negative definiteness of the Jacobian will be used to prove local convergence of the SLCP algorithm applied to the Cournot model, so it is retained here.

B. The Algorithm

The SLCP algorithm applied to the function f involves linearizing f at some initial point x^0 and solving the resulting linear complementarity problem. f is then re-linearized at this solution and the process is continued until convergence is achieved. Eaves (1978) and Josephy (1979) were the first to suggest this method, and prove local convergence based on a norm-contraction approach.[4]

We will also establish local convergence based on the norm-contraction approach, and make some observations as to why this result and the norm-contraction approach in general, do not seem to be helpful in proving global convergence when applying SLCP to the Cournot model. Then we will state our global convergence theorem based on the monotone approach.

Let us define the linearization of f at y as the first order Taylor expansion:

$$Lf(x|y) = f(y) + \nabla f(y)(x - y). \tag{4}$$

The linear complementarity problem (LCP(f|y)) is

Find $x \in R^N_+$ such that

$$Lf(x|y) = q + Mx \geq 0, \; x \geq 0, \tag{5a}$$

$$x^T(q + Mx) = 0. \tag{5b}$$

where $q = [f(y) - \nabla f(y)y]$ and $M = \nabla f(y)$. A vector x is said to be feasible to LCP (f|y) when it satisfies (5a).

The SLCP algorithm

Step 0. Stipulate $x^0 \in R^N_+$.

Step k (k=1,2,...) Compute x^k, the solution to $LCP(f|x^{k-1})$.

Thus the solution to the linear complementarity problem for f at y is the solution to $CP(Lf(x|y))$. It should be pointed out that even if there is a unique solution to CP(f), there may be some y for which there is no solution or multiple solutions to $CP(Lf(x|y))$.[5] If the Jacobian matrix of f is everywhere a P-matrix however, then the Jacobian of each linearization of f is also a P-matrix. Thus, by Lemma 3 there will never be more than one solution to $CP(Lf(x|y))$. Furthermore, since in our case the Jacobian matrix of f is positive definite, Dorn's theorem (Karamardian, 1969) implies existence (as well as uniqueness) of solutions to $CP(Lf(x|y)))$. It is also known that Lemke's algorithm computes the solution to the LCP in this case; see Lemke (1965) or Cottle and Dantzig (1968).

Lemma 4: (Pang and Chan, 1982): Let K be a nonempty closed and convex subset of R^N. Let f: $R^N_+ \to R^N$ be continuously differentiable. Suppose that x* solves (3) and that $\nabla f(x^*)$ is positive definite. Then a) there exists a neighborhood of x* such that if the initial iterate x^0 is chosen there, the sequence $\{x^k\}$ of solutions to $CP(Lf(x|x^{k-1}))$ is well defined and converges to x*. Moreover, b) if ∇f is Lipschitz continuous[6] at x* then $\{x^k\}$ converges quadratically to x*.

Based on Lemma 4 we can state our local convergence result for SLCP applied to the Cournot model.

Theorem 2: Let q^* solve (2). Assume that each firm's profit, $\pi_i(q)$ is C^2 and locally concave at q^*. Then there exists a neighborhood of q^* such that when the initial iterate q^0 is chosen there, SLCP computes a sequence $\{q^k\}$ converging to q^* (quadratically if $\nabla^2\pi$ is Lipschitz continuous).

Proof. Concavity of π in a neighborhood of q^* implies $\nabla^2\pi(q^*)$ is negative definite. $\nabla f(q^*) = -\nabla^2\pi(q^*)$ is positive definite; thus Lemma 4 applies.

Pang and Chan also provide a global convergence result for the SLCP process based on norm-contraction, their corollary 2.10. For convergence over a set K, this requires

1) that $\nabla f(x)$ be positive definite for all $x \in K$,
2) the existence of a positive definite matrix G, such that $\nabla f(x) - G$ is positive semi-definite for all $x \in K$, and
3) for all $x, y \in K$

$$\|\nabla f(x) - \nabla f(y)\| \leq \| G \| \tag{6}$$

The problem in applying their result to the Cournot model is that these conditions are overly strong. A few examples will illustrate our point. Let

$$f(x) = x^2 - 1, \qquad 0 < x < 2, \tag{7}$$

whose solution is $x^* = 1$. SLCP converges to this solution for any $x^0 \in (0,2)$. There is, however, no G for which both conditions 2 and 3 are simultaneously fulfilled.

Consider now the modified problem

$$f(x) = \begin{cases} f_1(x) = x^2 - 1, & 0 < x \le 1, \\ f_2(x) = (x-1)(3-x), & 1 \le x < 2. \end{cases} \tag{8}$$

The solution to this is also $x^* = 1$. We note that $f_1(1) = f_2(1) = 0$ and $f_1'(1) = f_2'(1) = 2$; hence f is continuously differentiable. It is easily verified that SLCP converges for $x^0 \in (1/3, 5/3)$, cycles between the solutions 1/3 and 5/3 when x^0 is at either end point, and diverges for $x^0 \notin [1/3, 5/3]$. Certainly, requirements 2 and 3 are not both met for, say, $K = [0.3, 1.7]$ and $y = 0.3$ and $x = 1$. Note that over K, f' takes all values in [0.6,2]. Thus any G that satisfies condition 2 must be in [0,0.6]. But (6) requires

$$\|2(1) - 2(0.3)\| = \|1.4\| \le \|G\|$$

which cannot hold for $G \in [0,0.6]$.

Essentially, the requirements of 1) - 3) do not address all critical features of the function f. Both (7) and (8) have $f' > 0$ for $x \in K$; however, what seems to be the issue is the behavior of f''. This is exactly what is considered by the monotone approach which requires that f be convex and hence that $\nabla^2 f$ be positive definite. This condition rules out (8) where f is non-convex and predicts global convergence for (7) where f is convex.

Observe that the convexity of f is also too strong a requirement. To see this consider a third example

$$f(x) = \begin{cases} f_1(x) = (x-1)(3-x), & 0 < x \le 1 \\ f_2(x) = x^2 - 1, & 1 \le x < 2. \end{cases} \tag{9}$$

The solution is $x^* = 1$. f is continuously differentiable and again it is easily verified that SLCP converges for any $x^0 \varepsilon (0,2)$ (or in $(0,\infty)$). The graphs of functions (7) - (9) are shown in Figure 1a - 1c respectively.

Our global convergence result is based on Pang and Chan (1982, theorem 4.2). To state their result we recall some matrix definitions. Let M be a square real matrix. Then M is said to be a Z-matrix if its off-diagonal elements are nonpositive, and, as defined earlier, a P-matrix if its principal minors are all positive definite. If M is both Z and P it is said to be a K-matrix. M is said to have a positive dominant diagonal[7] if $q_{ii} > 0$ for all i and

$$a_{ii} > \sum_{j \neq i} |a_{ij}| \text{ for all } i. \tag{10}$$

If M has a positive dominant diagonal then it is positive definite. M has a negative dominant diagonal if -M has a positive dominant diagonal.

Lemma 5: (Pang and Chan, 1982): Let $f: R_+^N \to R^N$ be convex and differentiable. If there exists a K-matrix X such $\nabla f(x)$ X is Z for all x feasible to (3), then there exists a well defined sequence $\{x^k\}$ where each x^k solves $CP(Lf(x|x^{k-1}))$, such that $\{x^k\}$ converges to some solution of CP(f) for any feasible x^0.

In order to apply this lemma to our Cournot model, we shall assume that f, defined in (2), is a convex function; and that there exists a K-matrix X such that $\nabla f(q)X$ is Z for all feasible q.

Our assumption that $f_i \equiv C_i' - P - P'q_i$ is convex obviously holds if both C_i' and $-(P + P'q_i)$ are convex. But our assumption also allows either marginal cost or marginal revenue to be non-convex as long as the other is sufficiently more convex.

Note that $A = \nabla f(q)$ has diagonal elements $a_{ii} = C_i'' - 2P' - P''q_i$ and off-diagonal elements $a_{ij} = -P' - P''q_i$ for all $j \neq i$. Construct X to have diagonal elements $x_{ii} = N - 1 + \varepsilon$ and off-diagonal elements $x_{ij} = -1$ for all $j \neq i$. X is obviously Z, and for $\varepsilon > 0$ we have $x_{ii} = N - 1 + \varepsilon > \sum_{j \neq i} |x_{ij}| = N - 1$, which implies that X is diagonally dominant. X is then positive definite; hence it is a P-matrix. Thus X is a K-matrix.

Now consider the matrix $B = \nabla f(q)X$ and its off-diagonal elements b_{ij}, $j \neq i$:

$$\begin{aligned} b_{ij} &= -(C_i''-2P'-P''q_i) + (N-2)(P'+P''q_i) - (N-1+\varepsilon)(P'+P''q_i) \\ &= -C_i'' + P' - \varepsilon(P'+P''q_i). \end{aligned} \tag{11}$$

We can reasonably assume demand is downward sloping: $P' < 0$. Over a compact set for quantities, one can choose a positive ε small enough that the third term becomes unimportant; thus $b_{ij} \approx -C_i'' + P'$ which is negative if $C_i'' < P'$. That is, marginal costs may be downward-sloping, but not too much. Certainly, with $C_i'' > 0$ and $P' < 0$, the typical case, then $b_{ij} < 0$.

<u>Theorem 3</u>. Assume that

i) profits (π_i) are C^2,

ii) industry output is bounded,

iii) marginal profits (defined in eqn 2) are concave, and

iv) the Jacobian of marginal profits has a negative dominant diagonal.

Then there exists a well defined sequence $\{q^k\}$ where each q^k solves $CP(Lf(q|q^{k-1}))$, such that $\{q^k\}$ converges to the unique solution of CP for any q^0 feasible (at least one of which exists).

Proof: As shown above, construct the matrix X with diagonal elements $x_{ii} = N - 1 + \varepsilon$, and off-diagonal elements $x_{ij} = -1$. Then the matrix X is K. To apply the Lemma, we must show that off-diagonal elements of $\nabla f(x)X$ are nonpositive for all $q \varepsilon R^N_+$, not just over a compact set; i.e., that eqn (11) is nonpositive for all q. From positive diagonal dominance of $A = \nabla f(q)$, we have for all i, q, and $N \geq 2$:

$$C_i'' - 2P' - P''q_i > (N-1)|P'+P''q_i| > -(1-\varepsilon)(P'+P''q_i) \qquad (12a)$$

for $0 < \varepsilon < 1$. Rearranging terms yields

$$C_i'' - P' - \varepsilon(P'+P''q_i) > 0. \qquad (12b)$$

Thus from convexity of f, lemma 5 tells us that the process converges to a solution; from positive diagonal dominance of the Jacobian we know that profits are strictly concave and thus that this solution is the unique equilibrium of the Cournot model (from Theorem 1 and Lemma 1).

This theorem gives a restricted form of global convergence: starting at any <u>feasible</u> point, global convergence is assured. The assumption that industry output is bounded gives a natural feasible starting point: a vector of outputs for which all marginal profits are nonpositive. Two other significant restrictions of the theorem are that marginal profits be concave and the Jacobian of marginal profits has a negative dominant diagonal. Diagonal dominance implies that own-effects on marginal profits dominate cross-effects. So if everyone raises output equally, the change in marginal profits from increasing own-output dominates the change in marginal profits from the actions of other producers.

The concavity of marginal profits is probably the most severe restriction. Recall that marginal profits are the sum of marginal revenue and negative marginal costs. Negative marginal costs will typically be concave, even with increasing returns. Marginal revenue may or may not be concave. Concave demand yields concave marginal revenue. A constant elasticity of demand yields convex marginal revenue and thus possibly non-concave marginal profits.

III. NUMERICAL EXAMPLE AND ECONOMIC INTERPRETATION

Consider an oligopoly with five firms, each with marginal cost functions

$$C_i' = c_i + (q_i/L_i)^{\alpha_i}, \quad i = 1,\ldots,5.$$

Table 1 lists the parameters of these functions. The inverse demand curve is given by

$$P(Q) = (5000/Q)^{\beta}$$

where $\beta = 1/1.1$.

Murphy et al (1982) presented this example and Harker (1984) also used it to illustrate his algorithm. We observe that for i = 1, 2, 3, f_i are concave. Hence theorem 3 does not apply so we are not assured of convergence. For the sake of comparison, however, the SLCP-algorithm was initiated at $q^0 = (10, 10, 10, 10, 10)$, as reported by Murphy et al and Harker. As seen from the last column of Table 2 (L^∞ norm), good convergence was achieved in five iterations. Observe that because of the concavity of functions f_i, SLCP computed a monotonically increasing sequence $\{q^k\}$.

Obviously a full comparison of the computational effort of each of these algorithms involves more than an iteration count and one example. It is interesting to observe the well known fact that SLCP converges dramatically better than Harker's diagonalization (relaxation) algorithm near the equilibrium, but not necessarily so far away from the equilibrium. A combination of these two iterative schemes would probably outperform either.

IV. APPLICATIONS

The homogeneous product Cournot model where quantities are decision variables is a standard textbook model that can be extended in several directions. We will consider two such extensions: a spatial or segmented homogeneous product market and a differentiated product market. These models can be solved by the algorithm of the previous section under essentially similar conditions. In this section we describe these models and develop an economic interpretation of their assumptions.

A. The Homogeneous Product Segmented Market Cournot Model

In this application the N firms sell their product in M different segments of the market. The segmentation may be on the basis of a number of criteria, such as geographic location or consumer group. It is assumed that these segments are completely separated from each other and that there are no possibilities of arbitrage among segments. Hence the Cournot equilibrium will typically imply discriminatory pricing.

Major features of this model date back to Enke (1951) and Samuelson (1952) and are perhaps best known through Takayama and Judge (1971), and a long series of applications to a wide range of industries. All these models of competitive equilibrium are cast as optimization models where generally supply and demand are either fixed or step (LP-models) or described by linear functions (QP-models). MacKinnon (1976) allows for nonlinear supply and demand functions and uses a fixed point algorithm to compute the equilibrium.

Recently the behavioral assumption has been modified to that of a Cournot-Nash strategy. In these applications (Harker, 1984; Kolstad and Abbey, 1984; Kolstad and Burris, 1986; Lont and Mathiesen, 1984; and Okuguchi, 1983), the complementarity approach is generally followed, but using different iterative processes. Salant (1982) and Murphy <u>et al</u> (1982) find an equilibrium through a sequence of optimization problems.

Assume there are N firms and M segments. For firm i and segment j let

q_{ij} denote the quantity sold,

$C_i(\sum_{j=1}^{M} q_{ij})$ denote the cost function for firm i, and

$P_j(Q_j)$ denote the inverse demand function for segment j, where

$$Q_j = \sum_{i=1}^{N} q_{ij}.$$

Finally, let q denote the vector of all quantities; i.e., $q = (q_{11}, q_{21}, \ldots, q_{N1}, q_{12}, \ldots, q_{NM})$. Profits for the ith firm are then

$$\pi_i(q) = \sum_{j=1}^{M} P_j(Q_j) q_{ij} - C_i\left(\sum_{j=1}^{M} q_{ij}\right), \tag{13}$$

and the first-order conditions for a profit maximum are

$$-\frac{\partial \pi_i}{\partial q_{ij}} = C_i'\left(\sum_{j=1}^{M} q_{ij}\right) - P_j\left(\sum_{i=1}^{N} q_{ij}\right) - q_{ij} P_j'\left(\sum_{i=1}^{N} q_{ij}\right) \geq 0, \tag{14a}$$

$$-\frac{\partial \pi_i}{\partial q_{ij}} \cdot q_{ij} = 0. \tag{14b}$$

$$q_{ij} \geq 0. \tag{14c}$$

We make several assumptions about the market.

<u>Assumption S1</u>. Inverse demand functions $P_j(\cdot)$, j=1,...,M and cost functions $C_i(\cdot)$, i=1,...,N are C^2.

<u>Assumption S2</u>. Industry output is bounded.

<u>Assumption S3</u>. Marginal profits are concave.

Finally, consider the Jacobian of the negative marginal profit functions:

$$J = \nabla f = -\nabla^2 \pi = \begin{pmatrix} A_1 & D & \cdots & D \\ D & A_2 & \cdots & D \\ \cdot & \cdot\cdot & \cdots & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ D & D & & A_M \end{pmatrix} \tag{15}$$

where the matrix A_j has diagonal elements $C_i'' - 2P_j' - q_{ij}P_j''$ and off-diagonal elements $-(P_j' + q_{ij}P_j'')$ and where

$$D = \begin{pmatrix} C_1'' & 0 & \cdots & 0 \\ 0 & C_2'' & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & C_N'' \end{pmatrix}.$$

Observe that each A_j, ignoring the index j, is equivalent to the Jacobian of the negative marginal profit functions of the nonsegmented market model.

Assumption S4. The $(N\times M) \times (N\times M)$ matrix J given by (15) has a positive dominant diagonal:

$$C_i'' - 2P_j' - q_{ij}P_j'' > (N-1)|(P_j'+q_{ij}P_j'')| + (M-1)|C_i''| \text{ for all } i.$$

Compared to the nonsegmented market model, the last element $(M-1)|C_i''|$ is added. This is a stricter, though natural extension of positive-definiteness. The feedback effects on a firm's marginal profits from selling in segment j are now both from its (N-1) competitors in segment j and from its own selling in the (M-1) other segments.

Theorem 4. Under assumptions S1 - S4, the sequence $\{q^k\}$ generated by SLCP converges to a unique equilibrium q^* for any feasible q^0.

Proof: Parallels that of theorem 3.

B. The Differentiated Product Model

In this application, we consider N firms, each selling a similar, though not identical product; that is, each firm has its own product (or "brand" of a product). As the products are not perfect substitutes for one another, it makes sense to talk about demand for a given firm's product, and each firm may charge a different price. Let $p = (p_1, \ldots, p_N)$ be the vector of prices obtained by each firm and $q = (q_1, \ldots, q_N)$ their quantities.

The demand function of the ith firm is $d_i(p)$. It is assumed to be C^2 and satisfy gross substitutability; i.e., its first derivatives satisfy $d_{ii} < 0$, $d_{ij} > 0$, and $|d_{ii}| > \sum_{j \neq i} |d_{ij}|$. The interpretation is that demand for product i decreases when its price is increased; it increases when some other price is increased; and when all firms increase their prices by an equal amount, demand decreases.

Lemma 6. Let the demand function be $d(p) = (d^1(p), \ldots, d^N(p))$ with Jacobian $D(p)$. Assume $D(p)$ has a dominant negative diagonal and that $-D(p)$ is a K-matrix. Then the inverse demand function $p = d^{-1}(q) = P(q)$ exists and its first derivatives are all non-positive.

Proof. Since $D(p)$ has a negative dominant diagonal, it is negative definite, and thus nonsingular so the inverse exists. The signs on the derivatives follow from the fact that the inverse of a K-matrix is nonnegative.

As before, let $C_i(q_i)$ denote the cost function of the ith firm. The profit function of this firm is

$$\pi_i = P_i(q)q_i - C_i(q_i), \tag{16}$$

and the first-order conditions for a profit maximum are:

$$-\frac{\partial \pi_i}{\partial q_i} = C_i'(q_i) - (P_i^i(q)q_i + P_i) \geq 0, \tag{17a}$$

$$-\frac{\partial \pi_i}{\partial q_i} q_i = 0, \tag{17b}$$

$$q_i \geq 0, \tag{17c}$$

where superscripts on P_i indicate differentiation.

The Jacobian of the negative marginal profit functions $\nabla f(q)$ has in this case diagonal elements $C_i'' - 2P_i^i - P_i^{ii}q_i$ and off-diagonal (i,j) elements $-P_i^j - P_i^{ij}q_i$.

Assumption D1. Inverse demand functions $P_i(\cdot)$, $i=1,\ldots,N$ and cost functions $C_i(\cdot)$, $i=1,\ldots,N$ are C^2.

Assumption D2. Industry output is bounded.

Assumption D3. Marginal profits are concave.

Assumption D4. The $(N\text{x}N)$ matrix $\nabla f(q)$ of negative marginal profits has a positive dominant diagonal. That is, for $i=1,\ldots,N$

1. $C_i'' - 2P_i^i - P_i^{ii}q_i > 0$, and

2. $|C_i'' - 2P_i^i - P_i^{ii}q_i| > \sum_{j\neq i} |P_i^j + P_i^{ij}q_i|$.

Theorem 5. Under assumptions D1 - D4, the SLCP algorithm converges to a unique equilibrium q^* for any sequence $\{q^k\}$ generated by the feasible q^0.

Proof: Parallels that of Theorem 3.

V. CONCLUSIONS

The basic purpose of this paper has been to present an algorithm for finding Cournot-Nash equilibria and demonstrate conditions for global convergence of the algorithm. Within this context, we examined three generic classes of markets: homogeneous products; homogeneous product, segmented markets; and differentiated products. Clearly, there are other types of markets to which the algorithm can be applied but which are not considered here. Furthermore, it may be possible to weaken the conditions of Theorems 3-5 so that stronger convergence results may be obtained.

FOOTNOTES

[1]Recent exceptions include the work of Murphy *et al* (1982) and Harker (1984) for computing Cournot equilibria, and Salant (1982) for computing Cournot equilibria in the international oil market.

[2]Recall that f is pseudoconcave on $X \subset R^n$ if $(x_1-x_2)\nabla f(x_2) \geq 0$ implies $f(x_1) \leq f(x_2)$ for any x_1, $x_2 \in X$.

[3]A is a P-matrix if all of the principal minors of A are positive definite.

[4]See also Pang and Chan (1982) who establish both local and global results for this and other algorithms using norm-contraction as well as other approaches.

[5]See Mathiesen (1987).

[6]The function g is Lipschitz continuous if there is a β such that for all x and y, $\|f(x) - f(y)\| < \beta\|x - y\|$.

[7]Frequently the definition of diagonal dominance allows scaling of the matrix before imposing (10).

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TABLE 1 PARAMETERS OF MARGINAL COST FUNCTIONS

Firm i	c_i	L_i	α_i
1	10	5	1/1.2
2	8	5	1/1.1
3	6	5	1
4	6	5	1/0.9
5	2	5	1/0.8

TABLE 2 ITERATES OF SLCP

Iteration	q_1	q_2	q_3	q_4	q_5	Δ*
0	10	10	10	10	10	49.45
1	16.648	17.937	19.111	20.139	20.943	21.41
2	25.191	29.207	32.241	33.943	33.920	6.67
3	33.967	39.191	41.837	41.639	38.780	1.11
4	36.815	41.736	43.664	42.643	39.174	0.038
5	36.9324	41.8181	43.7065	42.6593	39.1790	3.8×10^{-5}
6	36.9325	41.8182	43.7066	42.6593	39.1790	9.5×10^{-7}

$*\Delta^K = \max_i \{|f_i(q^K)|\}$

(a): Eqn (7)

(b): Eqn (8)

(c): Eqn (9)

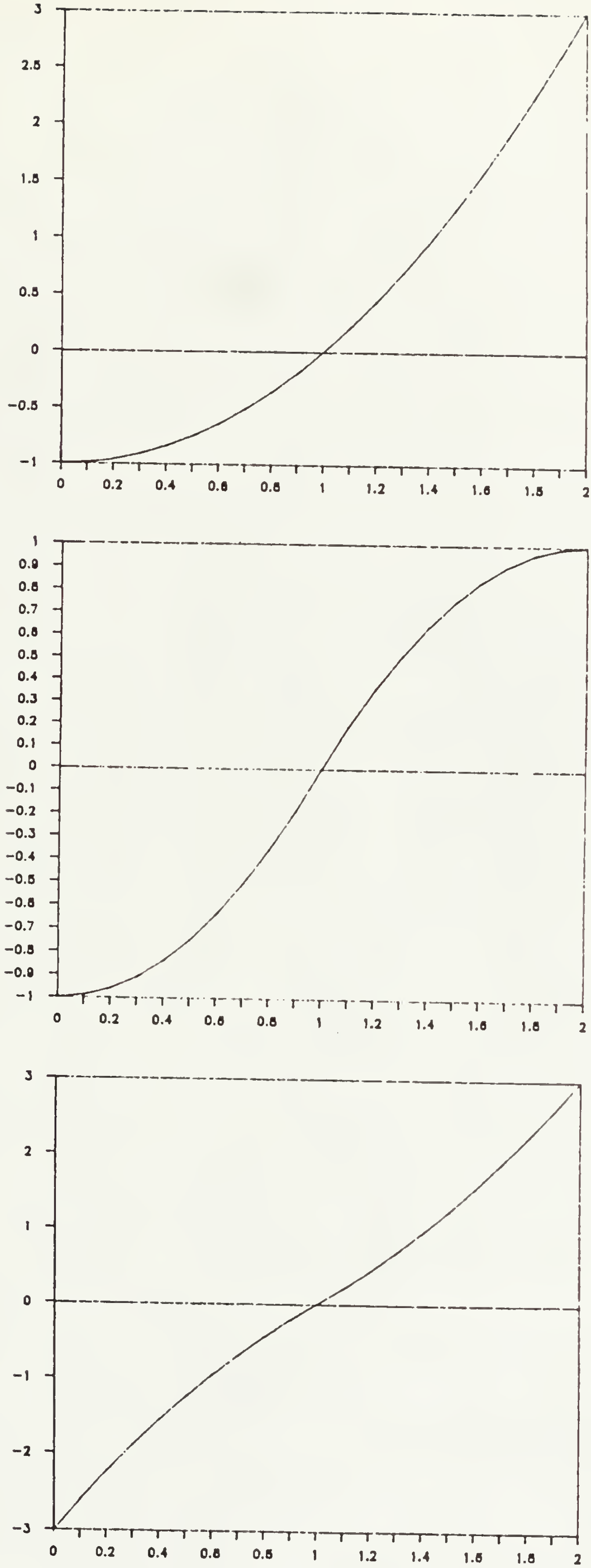

Fig. 1: Plots of Eqn. 7-9