

Exact equilibrium distributions in statistical quantum field theory with rotation and acceleration: scalar field

F. Becattini, M. Buzzegoli and A. Palermo

*Dipartimento di Fisica e Astronomia, Università di Firenze and INFN sezione di Firenze,
Via G. Sansone 1, Sesto Fiorentino I-50019, Firenze, Italy*

E-mail: becattini@fi.infn.it, matteo.buzzegoli@unifi.it,
andrea.palermo@unifi.it

ABSTRACT: We derive a general exact form of the phase space distribution function and the thermal expectation values of local operators for the free quantum scalar field at equilibrium with rotation and acceleration in flat space-time without solving field equations in curvilinear coordinates. After factorizing the density operator with group theoretical methods, we obtain the exact form of the phase space distribution function as a formal series in thermal vorticity through an iterative method and we calculate thermal expectation values by means of analytic continuation techniques. We separately discuss the cases of pure rotation and pure acceleration and derive analytic results for the stress-energy tensor of the massless field. The expressions found agree with the exact analytic solutions obtained by solving the field equation in suitable curvilinear coordinates for the two cases at stake and already — or implicitly — known in literature. In order to extract finite values for the pure acceleration case we introduce the concept of analytic distillation of a complex function. For the massless field, the obtained expressions of the currents are polynomials in the acceleration/temperature ratios which vanish at 2π , in full accordance with the Unruh effect.

KEYWORDS: Space-Time Symmetries, Thermal Field Theory, Boundary Quantum Field Theory

ARXIV EPRINT: [2007.08249](https://arxiv.org/abs/2007.08249)

Contents

1	Introduction	1
2	Thermal expectation values of annihilation and creation operators	4
2.1	Factorization of the density operator	4
2.2	Iterative solution with imaginary ϖ	6
3	The covariant Wigner function and the phase space distribution function	9
4	Properties of the distribution function	13
5	Mathematical interlude: analytic distillation and series resummation	15
6	Study of the series and comparison with known results: acceleration	19
6.1	Distillation and comparison with known results	21
7	Study of the series and comparison with known results: rotation	26
7.1	Comparison with known results	29
8	Thermodynamic equilibrium with rotation and acceleration: a new result	34
9	Summary	36
A	Recurrence formula	37
B	Solutions of the homogeneous equation	38
C	Analytic distillation for pure acceleration and the Unruh effect	39
D	Stress-energy tensor for a massless scalar field in Rindler coordinates	43
E	Analytic distillation of the stress-energy tensor for pure rotation	46

1 Introduction

The goal of thermal quantum field theory is to calculate mean values of physical quantities at thermodynamic equilibrium, henceforth denoted as thermal expectation values. This is a well known subject for the familiar thermodynamic equilibrium, described by the grand-canonical ensemble density operator:

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\hat{H}/T_0 + \mu_0 \hat{Q}/T_0 \right], \quad (1.1)$$

T_0 being the temperature and μ_0 the chemical potential coupled to a conserved charge \widehat{Q} , and Z the partition function.

However, this is not the only possible form of global¹ thermodynamic equilibrium in special relativity. It has become a topic of interest the calculation of thermal expectation values for the general form of global thermodynamic equilibrium in flat space-time, which is described by the density operator:

$$\widehat{\rho} = \frac{1}{Z} \exp \left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} + \zeta \widehat{Q} \right], \quad (1.2)$$

where the \widehat{P}^μ are the total four-momentum operators and $\widehat{J}^{\mu\nu}$ the total angular-momentum boost operators; b is a timelike four-vector and ϖ an antisymmetric constant tensor which is known as *thermal vorticity*. Its form shows that the most general global equilibrium density operator in Minkowski spacetime involves all 10 generators of its maximal symmetry group, the proper orthochronous Poincaré group, with 10 constant coefficients. The operator (1.2) and its derivation from the generally covariant form as well as the relations between thermodynamic and hydrodynamic fields and the parameters b , ϖ have been discussed in detail elsewhere [1–5]. Here we just recollect that (1.2) implies that the four-temperature field β is given by:

$$\beta_\mu = b_\mu + \varpi_{\mu\nu} x^\nu, \quad (1.3)$$

whose amplitude $\sqrt{\beta^2}$ is the inverse of the local comoving temperature T and its direction defines a four velocity u . The thermal vorticity can be decomposed into two space-like fields as:

$$\varpi_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} w^\rho u^\sigma + (\alpha_\mu u_\nu - \alpha_\nu u_\mu) \quad (1.4)$$

where, in the case of global equilibrium only:

$$\alpha^\mu = \frac{A^\mu}{T}, \quad w^\mu = \frac{\omega^\mu}{T}, \quad (1.5)$$

A^μ being the acceleration field and ω^μ the kinematic vorticity field.

The calculation of thermal expectation values of quantities such as the stress-energy tensor, conserved charge currents or spin density matrices with the operator (1.2) has a theoretical interest in the framework of statistical quantum field theory in general space-times [6–8] but it is also phenomenologically relevant for relativistic fluids undergoing very strong accelerations and rotations, such as the QCD plasma in high energy nuclear collisions [9]. Indeed, with b and ϖ depending on the space-time point x , the operator (1.2) is the lowest order approximation of the local thermodynamic equilibrium density operator [2, 5, 10–13] which is needed to calculate those mean values in relativistic fluids.

For free boson fields, it can be expected that any thermal expectation values arises from the Bose-Einstein distribution function in its covariant form (Jüttner distribution):

$$f(x, p) = \frac{1}{e^{\beta \cdot p - \zeta q} - 1} \quad (1.6)$$

¹We use global to distinguish it from local. The word global implies an actual equilibrium, with a density operator depending only on conserved quantities and with vanishing entropy production.

with β in (1.3) including all the effects of acceleration and rotation. In fact, it is known that a non-vanishing thermal vorticity implies the appearance of quantum corrections to the quantities obtained by integrating the (1.6). They have been calculated perturbatively [3, 4, 14–16] for small values of the adimensional thermal vorticity ϖ (or, equivalently, small values of the ratios (1.5)) with the operator expansion technique or the functional approach [17, 18]. However, not much is known about the full global equilibrium expression except for the two special cases of pure acceleration [19, 20] and rotation [21] where the exact solution is obtained by solving field equations in suitable curvilinear coordinates. Besides the theoretical value, the knowledge of exact solutions can be of relevance for the physics of the QCD plasma in that the ratios (1.5) can locally be $\mathcal{O}(1)$, according to the hydrodynamic simulations, and imply significant corrections to the leading order expression obtained in the perturbative expansion. The knowledge of the exact solution of the scalar field is also the first crucial step to derive an exact solution for the Dirac field. This can be of great importance to derive a complete expression of the fermion polarization in a relativistic fluid, a problem which has lately become phenomenologically very relevant.

In this work, we present a method to calculate the exact expression of the distribution function in statistical quantum field theory in global thermodynamic equilibrium with rotation and acceleration without solving the field equations in curvilinear coordinates, that is the exact form of (1.6) with all quantum corrections. The method is based on the factorization of (1.2) and an iterative procedure to determine the thermal expectation values of creation and annihilation operators for imaginary thermal vorticity. The physical solutions are thereafter obtained extracting the analytic part of the formal series found, a mathematical procedure introduced in this work and named *analytic distillation*, followed by an analytic continuation to real thermal vorticity. We then compare the obtained results with those calculated by solving field equations in two special cases — pure acceleration and pure rotation — and find complete agreement. Finally, we demonstrate the power of this method by calculating the thermal expectation value of the field squared in a newly explored global equilibrium case where both the acceleration and the vorticity vectors are non-vanishing.

The paper is organized as follows: in section 2 we introduce the factorization of the density operator and, after continuing to imaginary thermal vorticity, we iteratively compute the expectation value of creation and annihilation operators. In sections 3 and 4 we determine the exact form of the distribution function and study its properties. In section 5, we introduce the operation of *analytic distillation* which is necessary to obtain the exact results for physical acceleration and rotation presented in sections 6 and 7 respectively for the massless scalar free field. Finally, in section 8, a new result is derived for the free massless scalar field at equilibrium with both acceleration and rotation.

Notations. In this paper we adopt the natural units, with $\hbar = c = K = 1$. The Minkowskian metric g is $\text{diag}(1, -1, -1, -1)$; for the Levi-Civita symbol we use the convention $\epsilon^{0123} = 1$. We will use the relativistic notation with repeated indices assumed to be saturated. However, contractions of a single index, e.g. $\beta_\mu p^\mu$ will be sometimes denoted with a dot, i.e. $\beta \cdot p$; similarly, the contraction of two indices, e.g. $\varpi_{\mu\nu} J^{\mu\nu}$ will be sometimes

denoted by a colon, i.e. $\varpi : J$. Operators in Hilbert space will be denoted by an upper hat, e.g. \hat{O} .

2 Thermal expectation values of annihilation and creation operators

For free fields, the building block to calculate any statistical quantity (mean values, correlations) is the mean value of the combination of one creation and one annihilation operator of four-momentum eigenstates:

$$\text{Tr} \left(\hat{\rho} \hat{a}^\dagger(p) \hat{a}(p') \right). \tag{2.1}$$

Its value depends, of course, on the density operator; if $\hat{\rho} = (1/Z) \exp[-\hat{H}/T]$ the result is a well known one — the familiar Bose-Einstein and Fermi-Dirac distributions — and the derivation is based on the use of the known commutation relations between $\hat{\rho}$ and \hat{a}, \hat{a}^\dagger . However, if the density operator is (1.2), finding an exact expression is not trivial. Before setting out to do that, it should be first pointed out that any other combination of creation or annihilation operators, such as $\langle \hat{a}^\dagger(p) \hat{a}^\dagger(p') \rangle, \langle \hat{a}(p) \hat{a}(p') \rangle$ or combination of creation/annihilation operators of particles and antiparticles in case of a charged scalar field, will vanish. This happens because the density operator (1.2), involving just the generators of Lorentz transformation, translations and charge, does not change the number of particles and antiparticles. Hence:

$$\langle \hat{a}^\dagger(p) \hat{a}^\dagger(p') \rangle = \langle \hat{a}(p) \hat{a}(p') \rangle = \langle \hat{b}^\dagger(p) \hat{b}^\dagger(p') \rangle = \langle \hat{b}(p) \hat{b}(p') \rangle = \langle \hat{a}(p) \hat{b}(p') \rangle = \langle \hat{a}(p) \hat{b}^\dagger(p') \rangle = 0. \tag{2.2}$$

In this section we will present a general method to calculate (2.1) with the density operator (1.2). We will first set ϖ imaginary and factorize the density operator (1.2) by using a group theoretical method; then we will calculate (2.1) for imaginary ϖ by means of an iterative method and give the general solution in terms of a uniformly convergent series. Finally, we will discuss the analytic continuation to real ϖ .

2.1 Factorization of the density operator

As has been mentioned, we take advantage of Poincaré group relations which make it possible to factorize the density operator (1.2) into the product of two independent operators.

Let us start with the following very simple observation concerning the composition of translations and Lorentz transformations in Minkowski space-time. Let x be a four-vector and apply the combination

$$\mathbb{T}(a) \Lambda \mathbb{T}(a)^{-1},$$

$\mathbb{T}(a)$ being a translation of some four-vector a and Λ a Lorentz transformation. The effect of the above combination on x reads:

$$x \mapsto x - a \mapsto \Lambda(x - a) \mapsto \Lambda(x - a) + a = \Lambda(x) + (I - \Lambda)(a) = \mathbb{T}((I - \Lambda)(a))(\Lambda(x)).$$

Since x was arbitrary, we have:

$$\mathbb{T}(a) \Lambda \mathbb{T}(a)^{-1} = \mathbb{T}((I - \Lambda)(a)) \Lambda.$$

This relation has a representation of unitary operators in Hilbert space, which can be written in terms of the generators of the Poincaré group:

$$\exp[ia \cdot \hat{P}] \exp[-i\phi : \hat{J}/2] \exp[-ia \cdot \hat{P}] = \exp[i((1 - \Lambda)(a)) \cdot \hat{P}] \exp[-i\phi : \hat{J}/2], \quad (2.3)$$

where ϕ are the parameters of the Lorentz transformation. By taking ϕ infinitesimal, we can obtain a known relation about the effect of translations on angular momentum operators:

$$\exp[ia \cdot \hat{P}] \hat{J}_{\mu\nu} \exp[-ia \cdot \hat{P}] = \hat{T}(a) \hat{J}_{\mu\nu} \hat{T}(a)^{-1} = \hat{J}_{\mu\nu} - a_\mu \hat{P}_\nu + a_\nu \hat{P}_\mu.$$

The left hand side of (2.3) can now be worked out by using the above relation:

$$\begin{aligned} \exp[ia \cdot \hat{P}] \exp[-i\phi : \hat{J}/2] \exp[-ia \cdot \hat{P}] &= \hat{T}(a) \exp[-i\phi : \hat{J}/2] \hat{T}(a)^{-1} \\ &= \exp[-i\phi : \hat{T}(a) \hat{J} \hat{T}(a)^{-1} / 2] = \exp[-i\phi : (\hat{J} - a \wedge \hat{P}) / 2] = \exp[i\phi_{\mu\nu} a^\mu \hat{P}^\nu - i\phi_{\mu\nu} \hat{J}^{\mu\nu} / 2]. \end{aligned} \quad (2.4)$$

Hence, combining (2.4) with (2.3), we have obtained the factorization:

$$\exp[i\phi_{\mu\nu} a^\mu \hat{P}^\nu - i\phi_{\mu\nu} \hat{J}^{\mu\nu} / 2] = \exp[i((1 - \Lambda)(a)) \cdot \hat{P}] \exp[-i\phi : \hat{J}/2]. \quad (2.5)$$

Now, since it holds

$$i(1 - \Lambda)(a) = ia - i \sum_{k=0}^{\infty} \frac{(-i)^k}{2^k k!} (\phi : \mathbf{J})^k(a) = -i \sum_{k=1}^{\infty} \frac{(-i)^k}{2^k k!} (\phi : \mathbf{J})^k(a), \quad (2.6)$$

by setting:

$$V_\mu \equiv i\phi_{\mu\nu} a^\nu$$

and taking into account that:

$$(\mathbf{J}_{\mu\nu})_\beta^\alpha = i \left(\delta_\mu^\alpha g_{\nu\beta} - \delta_\nu^\alpha g_{\mu\beta} \right),$$

we have:

$$(\phi : \mathbf{J})(a)_\alpha = 2i\phi_{\alpha\beta} a^\beta = 2V_\alpha.$$

Therefore, the right hand side of the eq. (2.6) becomes:

$$-i \sum_{k=1}^{\infty} \frac{(-i)^k}{2^k k!} (\phi : \mathbf{J})^k(a) = -i \sum_{k=1}^{\infty} \frac{(-i)^k}{2^{k-1} k!} (\phi : \mathbf{J})^{k-1}(V) = - \sum_{k=0}^{\infty} \frac{(-i)^k}{2^k (k+1)!} (\phi : \mathbf{J})^k(V).$$

Finally, the eq. (2.5) becomes:

$$\exp[-V \cdot \hat{P} - i\phi : \hat{J}/2] = \exp[-\tilde{V}(\phi) \cdot \hat{P}] \exp[-i\phi : \hat{J}/2], \quad (2.7)$$

where we denoted

$$\tilde{V}(\phi)_\mu \equiv \sum_{k=0}^{\infty} \frac{(-i)^k}{2^k (k+1)!} (\phi : \mathbf{J})^k(V)_\mu = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{(\phi_{\mu\nu_1} \phi^{\nu_1\nu_2} \dots \phi_{\nu_{k-1}\nu_k})}_{k \text{ times}} V^{\nu_k}.$$

The eq. (2.7) can be read as the factorization of the exponential of a linear combinations of generators of the Poincaré group. For this reason, it must be derivable also by using the known formulae of the factorization of the exponential of the sum of matrices $\exp[A + B]$ in terms of exponentials of commutators of A and B . Indeed, it can be shown, by using the commutation relations of \hat{P} and \hat{J} , that one precisely gets the eq. (2.7) for *any* vector V and tensor ϕ , either real or complex. Hence, the formula (2.7) can be applied to factorize the density operator (1.2), neglecting \hat{Q} for the moment, by setting $\phi = i\varpi$:

$$\hat{\rho} = \frac{1}{Z} \exp[-b \cdot \hat{P} + \varpi : \hat{J}/2] = \frac{1}{Z} \exp[-\tilde{b}(\varpi) \cdot \hat{P}] \exp[\varpi : \hat{J}/2] \quad (2.8)$$

with:

$$\tilde{b}(\varpi)_\mu \equiv \sum_{k=0}^{\infty} \frac{1}{2^k (k+1)!} (\varpi : J)^k (b)_\mu = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \underbrace{(\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{k-1}\nu_k})}_{k \text{ times}} b^{\nu_k}. \quad (2.9)$$

It is important to point out that $\tilde{b}(\varpi)$ is in general a complex vector, however, according to eq. (2.9), it is *real* when ϖ is imaginary, that is when we deal with actual Lorentz transformations.

2.2 Iterative solution with imaginary ϖ

We can now take advantage of this factorized form to calculate the mean value (2.1). The idea is to seek the solution for pure imaginary ϖ first and then to analytically continue it to real ϖ , the physical case. The reason is that with imaginary $\varpi = -i\phi$ one deals with an actual Lorentz transformation, for which commutation relations with creation and annihilation operators are known. Specifically, a Lorentz transformation operator $\hat{\Lambda} = \exp[-i\phi : \hat{J}/2]$ acts as:²

$$\hat{\Lambda} \hat{a}^\dagger(p) \hat{\Lambda}^{-1} = \hat{a}^\dagger(\Lambda p). \quad (2.10)$$

For the translation-like operators, the commutation relations can be calculated regardless of whether the vector \tilde{b} is real or complex:

$$\exp[-\tilde{b} \cdot \hat{P}] \hat{a}^\dagger(p) \exp[\tilde{b} \cdot \hat{P}] = e^{-\tilde{b} \cdot p} \hat{a}^\dagger(p). \quad (2.11)$$

Let us then write the analytic continuation of the relation (2.8) to $\varpi = -i\phi$, so:

$$\hat{\rho} = \frac{1}{Z} \exp[-b \cdot \hat{P} - i\phi : \hat{J}/2] = \frac{1}{Z} \exp[-\tilde{b}(-i\phi) \cdot \hat{P}] \exp[-i\phi : \hat{J}/2] \equiv \frac{1}{Z} \exp[-\tilde{b}(-i\phi) \cdot \hat{P}] \hat{\Lambda}.$$

By using the relations (2.10) and (2.11), the analytic continuation of (2.1) can be worked out as follows:

$$\begin{aligned} \text{Tr}(\hat{\rho} \hat{a}^\dagger(p) \hat{a}(p')) &= \frac{1}{Z} \text{Tr}(\exp[-\tilde{b} \cdot \hat{P}] \hat{\Lambda} \hat{a}^\dagger(p) \hat{a}(p')) = \frac{1}{Z} \text{Tr}(\exp[-\tilde{b} \cdot \hat{P}] \hat{a}^\dagger(\Lambda p) \hat{\Lambda} \hat{a}(p')) \\ &= \frac{1}{Z} e^{-\tilde{b} \cdot \Lambda(p)} \text{Tr}(\hat{a}^\dagger(\Lambda p) \exp[-\tilde{b} \cdot \hat{P}] \hat{\Lambda} \hat{a}(p')) = e^{-\tilde{b} \cdot \Lambda(p)} \text{Tr}(\hat{a}(p') \hat{a}^\dagger(\Lambda p) \hat{\rho}) = e^{-\tilde{b} \cdot \Lambda(p)} \text{Tr}(\hat{\rho} \hat{a}(p') \hat{a}^\dagger(\Lambda p)), \end{aligned}$$

²Throughout this work, Λp is a shorthand for $\Lambda(p)$.

where, in the last equalities, we have used the cyclicity of trace. The above derivation can be written by using the shorthand of $\langle \rangle$ replacing the trace with the density operator:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = e^{-\tilde{b} \cdot \Lambda(p)} \langle \hat{a}(p') \hat{a}^\dagger(\Lambda p) \rangle.$$

We can now use the known commutation relation between creation and annihilation operators

$$[\hat{a}(p), \hat{a}^\dagger(p')] = \delta^3(\mathbf{p} - \mathbf{p}') 2\varepsilon',$$

where $\varepsilon = \sqrt{\mathbf{p}^2 + m^2}$, to obtain:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = e^{-\tilde{b} \cdot \Lambda p} \langle \hat{a}^\dagger(\Lambda p) \hat{a}(p') \rangle + \delta^3(\Lambda \mathbf{p} - \mathbf{p}') 2\varepsilon' e^{-\tilde{b} \cdot p'}, \quad (2.12)$$

where $\Lambda \mathbf{p}$ stand for the space part of the four-vector Λp . This is the basic equation we have to solve.

First, we observe that if $\Lambda = 1$, we get back to the familiar thermal field theory and the (2.12) can be solved algebraically leading to the well known Bose-Einstein distribution function, multiplied by 2ε in view of our adopted covariant commutation relations. In the more general case, we will obtain a consistent solution of (2.12) by an iterative method. The idea is to approximate the left hand side with the second term of the right hand side with the Dirac delta, and correct the expression obtained by inserting the first term calculated with the previous approximation. Let us see how this is accomplished in formulae. We start with:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle \simeq \delta^3(\Lambda \mathbf{p} - \mathbf{p}') 2\varepsilon' e^{-\tilde{b} \cdot p'}.$$

This approximated solution implies:

$$\langle \hat{a}^\dagger(\Lambda p) \hat{a}(p') \rangle \simeq \delta^3(\Lambda^2 \mathbf{p} - \mathbf{p}') 2\varepsilon' e^{-\tilde{b} \cdot p'}.$$

Now we can plug the above expression again in the (2.12) and get, after multiplying both sides by $e^{\tilde{b} \cdot p'}$:

$$e^{\tilde{b} \cdot p'} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle \simeq \delta^3(\Lambda^2 \mathbf{p} - \mathbf{p}') e^{-\tilde{b} \cdot \Lambda(p)} 2\varepsilon' + \delta^3(\Lambda \mathbf{p} - \mathbf{p}') 2\varepsilon'. \quad (2.13)$$

From eq. (2.13) we can now calculate an updated approximated expression of $\langle \hat{a}^\dagger(\Lambda(p)) \hat{a}(p') \rangle$:

$$e^{\tilde{b} \cdot p'} \langle \hat{a}^\dagger(\Lambda(p)) \hat{a}(p') \rangle \simeq \delta^3(\Lambda^3 \mathbf{p} - \mathbf{p}') e^{-\tilde{b} \cdot \Lambda^2(p)} 2\varepsilon' + \delta^3(\Lambda^2 \mathbf{p} - \mathbf{p}') 2\varepsilon'$$

and plugging it again in eq. (2.12); we obtain:

$$e^{\tilde{b} \cdot p'} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle \simeq \left[\delta^3(\Lambda^3 \mathbf{p} - \mathbf{p}') e^{-\tilde{b} \cdot \Lambda^2(p) - \tilde{b} \cdot \Lambda p} + \delta^3(\Lambda^2 \mathbf{p} - \mathbf{p}') e^{-\tilde{b} \cdot \Lambda p} + \delta^3(\Lambda \mathbf{p} - \mathbf{p}') \right] 2\varepsilon'.$$

Iterating, we eventually obtain the solution of (2.12) in the form of a series:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = 2\varepsilon' \sum_{n=1}^{\infty} \delta^3(\Lambda^n \mathbf{p} - \mathbf{p}') \exp \left[- \sum_{k=1}^n \tilde{b} \cdot \Lambda^k p \right]. \quad (2.14)$$

The expression (2.14) can be further transformed by using the orthogonality of Λ :

$$\Lambda x \cdot \Lambda y = x \cdot y.$$

Note that, the orthogonality relation holds for complex ϖ and complex vectors x, y , so this equality would not be affected by choosing ϖ real or imaginary. We can then rewrite eq. (2.14) as:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = 2\varepsilon' \sum_{n=1}^{\infty} \delta^3(\Lambda^n \mathbf{p} - \mathbf{p}') \exp \left[- \sum_{k=1}^n \Lambda^{-k} \tilde{b} \cdot p \right]. \quad (2.15)$$

It is possible to find alternative forms of the (2.15) without the appearance of Lorentz transformations in the exponent. Indeed, it can be shown that (see appendix A), for generally complex ϖ such that $\Lambda = \exp[\varpi : \mathbf{J}/2]$:

$$\Lambda^{-k} \tilde{b}(\varpi) = (1-k)\tilde{b}((1-k)\varpi) + k\tilde{b}(-k\varpi), \quad (2.16)$$

implying:

$$\sum_{k=1}^n \Lambda^{-k} \tilde{b}(\varpi) = n\tilde{b}(-n\varpi). \quad (2.17)$$

By substituting the (2.17) into the (2.15), we obtain its alternative form:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = 2\varepsilon' \sum_{n=1}^{\infty} \delta^3(\Lambda^n \mathbf{p} - \mathbf{p}') \exp[-n \tilde{b}(-n\varpi) \cdot p]. \quad (2.18)$$

Some comments are now in order. First of all, it can be checked that indeed the found series in either form (2.15) or (2.18) are actual solution of the equation (2.12) by direct substitution. Besides, it can be readily realized that they are an extension of the Bose distribution function by taking the limit $\varpi \rightarrow 0$, which implies $\Lambda \rightarrow \mathbb{1}$. Since, from (2.9) $\tilde{b}(0) = b$ one has, from (2.18):

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = 2\varepsilon' \delta^3(\mathbf{p} - \mathbf{p}') \sum_{n=1}^{\infty} \exp[-n b \cdot p] = 2\varepsilon' \frac{1}{e^{b \cdot p} - 1} \delta^3(\mathbf{p} - \mathbf{p}').$$

An important point is that, because of the Dirac deltas with argument $\Lambda \mathbf{p}$, the forms (2.15), (2.18) for finite ϖ make sense only if ϖ is imaginary, that is only if Λ is an actual Lorentz transformation. Therefore, it is possible to make analytic continuations only *after* the integration in the momentum p' is done. Once the integration is carried out, the convergence of the analytically continued series is not trivial (see later discussion). We just remark that if $\tilde{b}(\varpi)$ is time-like and future oriented, each exponent in the (2.12) is negative, what is more apparent in the form (2.14).

To conclude, we observe that, should the density operator include a chemical potential term coupled to a conserved charge like in eq. (1.2), the iterative derivation can be repeated and one has one more exponential factor:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = 2\varepsilon' \sum_{n=1}^{\infty} \delta^3(\Lambda^n \mathbf{p} - \mathbf{p}') \exp[-n \tilde{b}(-n\varpi) \cdot p + n \zeta]. \quad (2.19)$$

An important question is whether the (2.15), or one of its equivalent forms (2.18) (2.19), is the unique solution of equation (2.12) or if there are possibly more solutions. In general, it can be readily shown that any solution $S(p, p')$ of the (2.12) can be written as:

$$S(p, p') = H(p, p') + S_0(p, p')$$

where $S_0(p, p')$ is a particular solution of the (2.12), such as (2.19), and $H(p, p')$ is the general solution of the associated homogeneous equation:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = e^{-\tilde{b} \cdot \Lambda p} \langle \hat{a}^\dagger(\Lambda p) \hat{a}(p') \rangle. \quad (2.20)$$

We will see that the existence of non-trivial solutions of the homogeneous equation (2.20) depends on the Lorentz transformation Λ ; if Λ is a rotation, we will show in section 7 that there are no non-trivial solutions; conversely, if Λ is a pure boost, there are non-trivial solutions, as shown in section 6.

Nevertheless, it is a general feature that solutions of the homogeneous equation (2.20) are non-analytic in thermal vorticity around zero, that is for $\phi = i\varpi = 0$, as shown in section 5. Therefore, the particular solution found by iteration (2.12) may, in principle, contain non-analytic contributions in ϖ around zero which would obviously hinder a process of analytic continuation from imaginary to real ϖ . If we could single out an analytic part around $\Lambda = 1$ of the particular solution (2.12), the function would be unique and its analytic continuation possible by construction. In principle, there might be cases where no analytic solutions exist because $\varpi = 0$ is itself a singular point in the thermodynamics. However, to our knowledge, no instances of this phenomenon are known, hence the physical solution can be determined by extracting the analytic part. This method of *analytic distillation* is discussed in section 5.

3 The covariant Wigner function and the phase space distribution function

The expectation value of the quadratic combination (2.1) is all we need to calculate every field-related quantity, like the stress-energy tensor or any other conserved current. However, it is convenient to make use of a very useful concept, the covariant Wigner function [22]:

$$W(x, k) = \frac{2}{(2\pi)^4} \int d^4 y \langle : \hat{\psi}^\dagger(x + y/2) \hat{\psi}(x - y/2) : \rangle e^{-iy \cdot k}, \quad (3.1)$$

where “:” stands for the normal ordering of creation and annihilation operators and a density operator is understood in the mean value. Note that the covariant Wigner function is real, but it is not positive definite. The covariant Wigner function (3.1) allows to express the mean particle current as a four-dimensional integral:

$$j^\mu(x) = i \langle : \hat{\psi}^\dagger(x) \overleftrightarrow{\partial}^\mu \hat{\psi}(x) : \rangle = \int d^4 k k^\mu W(x, k). \quad (3.2)$$

Plugging the free scalar field expansion:

$$\hat{\psi}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3 p}{2\varepsilon} \left[e^{-ip \cdot x} \hat{a}(p) + e^{ip \cdot x} \hat{b}^\dagger(p) \right] \quad (3.3)$$

in the eq. (3.1) we get:

$$\begin{aligned}
W(x, k) &= \frac{2}{(2\pi)^7} \int \frac{d^3\mathbf{p}}{2\varepsilon} \frac{d^3\mathbf{p}'}{2\varepsilon'} \int d^4y \left[e^{-iy \cdot (k - (p+p')/2)} e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle + e^{-iy \cdot (k + (p+p')/2)} e^{-i(p-p') \cdot x} \langle \hat{b}^\dagger(p') \hat{b}(p) \rangle \right. \\
&\quad \left. + e^{-iy \cdot (k - (p-p')/2)} e^{i(p+p') \cdot x} \langle \hat{a}^\dagger(p) \hat{b}^\dagger(p') \rangle + e^{-iy \cdot (k + (p-p')/2)} e^{-i(p+p') \cdot x} \langle \hat{b}(p) \hat{a}(p') \rangle \right] \\
&= \frac{2}{(2\pi)^7} \int \frac{d^3\mathbf{p}}{2\varepsilon} \frac{d^3\mathbf{p}'}{2\varepsilon'} \int d^4y \left[e^{-iy \cdot (k - (p+p')/2)} e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle + e^{-iy \cdot (k + (p+p')/2)} e^{i(p-p') \cdot x} \langle \hat{b}^\dagger(p) \hat{b}(p') \rangle \right. \\
&\quad \left. + e^{-iy \cdot (k - (p-p')/2)} e^{i(p+p') \cdot x} \langle \hat{a}^\dagger(p) \hat{b}^\dagger(p') \rangle + e^{-iy \cdot (k + (p-p')/2)} e^{-i(p+p') \cdot x} \langle \hat{b}(p') \hat{a}(p) \rangle \right] \\
&= \frac{2}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{2\varepsilon} \frac{d^3\mathbf{p}'}{2\varepsilon'} \left[e^{i(p-p') \cdot x} \left(\delta^4(k - (p+p')/2) \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle + \delta^4(k + (p+p')/2) \langle \hat{b}^\dagger(p) \hat{b}(p') \rangle \right) \right. \\
&\quad \left. + \delta^4(k - (p-p')/2) \left(e^{i(p+p') \cdot x} \langle \hat{a}^\dagger(p) \hat{b}^\dagger(p') \rangle + e^{-i(p+p') \cdot x} \langle \hat{b}(p') \hat{a}(p) \rangle \right) \right].
\end{aligned} \tag{3.4}$$

In the above equalities, we have taken advantage of the symmetric integration in the variables p, p' . The expression (3.4) makes it apparent that the variable k of the Wigner function $W(x, k)$ is *not* on-shell, i.e. $k^2 \neq m^2$ even in the free case. This makes the definition of a particle distribution function $f(x, p)$ à la Boltzmann not straightforward in a quantum relativistic framework. From equation (3.4), we can infer that W is made up of three terms which can be distinguished for the characteristic of k . For future time-like $k = (p + p')/2$, only the first term involving particles is retained; for past time-like $k = -(p + p')/2$, only the second term involving antiparticles; finally, for space-like $k = (p - p')/2$ the last term with the mean values of two creation/annihilation operators is retained. In symbols:

$$W(x, k) = W(x, k) \theta(k^2) \theta(k^0) + W(x, k) \theta(k^2) \theta(-k^0) + W(x, k) \theta(-k^2) \equiv W_+(x, k) + W_-(x, k) + W_S(x, k). \tag{3.5}$$

Multiplying the previous expression by $\theta(k^0) \theta(k^2)$ selects the particle contribution to the covariant Wigner function and by $\theta(-k^0) \theta(k^2)$ the anti-particle one. It is thus possible to define the particle covariant Wigner function $W_+(x, k)$ as well as the antiparticle counterpart $W_-(x, k)$. The particle contribution to the current can be obtained from eq. (3.2) by using the (3.4):

$$\begin{aligned}
j_+^\mu(x) &= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{\varepsilon} \frac{d^3\mathbf{p}'}{2\varepsilon'} \frac{(p+p')^\mu}{2} e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle \\
&= \int \frac{d^3\mathbf{p}}{\varepsilon} p^\mu \operatorname{Re} \left(\frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}'}{2\varepsilon'} e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle \right).
\end{aligned} \tag{3.6}$$

To obtain the last expression, we have taken advantage of the hermiticity of the density operator, implying:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = \langle \hat{a}^\dagger(p') \hat{a}(p) \rangle^*$$

which makes it possible to swap the integration variables p and p' . The formula (3.6) identifies the particle distribution function or phase space density $f(x, p)$ as the real part of a *complex distribution function* $f_c(x, p)$:

$$f(x, p) = \operatorname{Re} f_c(x, p), \tag{3.7}$$

where

$$f_c(x, p) = \frac{1}{(2\pi)^3} \int \frac{d^3 p'}{2\varepsilon'} e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle. \quad (3.8)$$

For antiparticles the distribution function $\bar{f}(x, p)$ is obtained from (3.7) replacing $\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle$ with $\langle \hat{b}^\dagger(p) \hat{b}(p') \rangle$. Also note that with the density operator (1.2) all terms involving pairs of creation and annihilation operators in (3.4) vanish according to the (2.2), so that the current does not involve any space-like contribution from $W_S(x, k)$ and one is left with:

$$j^\mu(x) = \int d^4 k \, k^\mu W(x, k) = \text{Re} \int \frac{d^3 p}{2\varepsilon} p^\mu [f_c(x, p) - \bar{f}_c(x, p)],$$

which also shows that the current can be written as an integral of on-shell four-momenta when using the phase space densities and off-shell, equation (3.2), when using the Wigner function.

We can now turn to the calculation of the mean value of the stress-energy tensor. From the Lagrangian:

$$\mathcal{L} = \partial_\mu \hat{\psi}^\dagger \partial^\mu \hat{\psi} - m^2 \hat{\psi}^\dagger \hat{\psi}$$

one can obtain the so-called *canonical* stress-energy tensor operator:

$$\hat{T}_C^{\mu\nu} = \partial^\mu \hat{\psi}^\dagger \partial^\nu \hat{\psi} + \partial^\nu \hat{\psi}^\dagger \partial^\mu \hat{\psi} - g^{\mu\nu} \mathcal{L} \quad (3.9)$$

and so determine its normal ordered mean value $\langle : \hat{T}_C^{\mu\nu} : \rangle$. For this purpose, we again make use of the covariant Wigner function and the known expression [22]:

$$-\frac{1}{2} \langle : \hat{\psi}^\dagger \overset{\leftrightarrow}{\partial}^\mu \overset{\leftrightarrow}{\partial}^\nu \hat{\psi} : \rangle = \int d^4 k \, k^\mu k^\nu W(x, k). \quad (3.10)$$

Now, it can be shown by using the equations of motion of the free scalar field, that:

$$\hat{T}_C^{\mu\nu} = -\frac{1}{2} \hat{\psi}^\dagger \overset{\leftrightarrow}{\partial}^\mu \overset{\leftrightarrow}{\partial}^\nu \hat{\psi} + \frac{1}{2} \partial^\mu \partial^\nu (\hat{\psi}^\dagger \hat{\psi}) - \frac{1}{2} g^{\mu\nu} \square (\hat{\psi}^\dagger \hat{\psi}).$$

Since, from the covariant Wigner function definition (3.1):

$$\langle : \hat{\psi}^\dagger \hat{\psi} : \rangle = \frac{1}{2} \int d^4 k \, W(x, k), \quad (3.11)$$

we finally obtain, by using (3.10) and (3.11):

$$\langle : \hat{T}_C^{\mu\nu} : \rangle = \int d^4 k \, k^\mu k^\nu W(x, k) + \frac{1}{4} (\partial^\mu \partial^\nu - g^{\mu\nu} \square) \int d^4 k \, W(x, k). \quad (3.12)$$

We note in passing that, since $\hat{T}^{\mu\nu}$ is a quadratic operator of creation and annihilation operators of momentum eigenstates, just like the Wigner operator, its normally ordered mean value corresponds to the subtraction of the vacuum expectation value from the plain mean value, that is:

$$\langle : \hat{T}_C^{\mu\nu} : \rangle = \langle \hat{T}_C^{\mu\nu} \rangle - \langle 0 | \hat{T}_C^{\mu\nu} | 0 \rangle.$$

In order to derive the relation between the stress-energy tensor and the phase space distribution function, we first write down the derivative of f_c in the eq. (3.8):

$$\partial_\nu f_c(x, p) = \frac{1}{(2\pi)^3} \int \frac{d^3 p'}{2\varepsilon'} i(p - p')_\nu e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle,$$

whence:

$$\frac{1}{(2\pi)^3} \int \frac{d^3 p'}{2\varepsilon'} p'_\nu e^{i(p-p') \cdot x} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = i\partial_\nu f_c + p_\nu f_c. \quad (3.13)$$

Now we can plug the (3.4) in (3.12) and by the same method used for the current, we can express the stress-energy tensor as a function of the phase space distribution function:

$$\langle : \hat{T}_C^{\mu\nu} : \rangle = \text{Re} \left[\int \frac{d^3 p}{\varepsilon} \left(p^\mu p^\nu + \frac{1}{4} (ip^\mu \partial^\nu + ip^\nu \partial^\mu) + \frac{1}{4} (\partial^\mu \partial^\nu - g^{\mu\nu} \square) \right) (f_c(x, p) + \bar{f}_c(x, p)) \right]. \quad (3.14)$$

The above expression makes it apparent that the gradient terms are quantum correction to the “classical” expression of the stress-energy tensor in kinetic theory in that they require — after restoring the natural constants — \hbar, \hbar^2 factors to have the same dimension. Also, the phase space distribution functions may themselves have quantum corrections depending on \hbar . It should also be emphasized that the relation between stress-energy tensor and phase space distribution, unlike sometimes believed, depends on the particular stress-energy tensor operator. This relation indeed is non-invariant under pseudo-gauge transformations except at homogeneous thermodynamic equilibrium [23].

The equation (2.15) or its equivalent forms (2.18), (2.19) can now be used to calculate the complex phase space distribution function (3.8) for the general global equilibrium. It is important to keep in mind that the expectation values were obtained for imaginary ϖ and that an analytic continuation to real ϖ must be eventually carried out. By using the eq. (2.18), it turns out to be:

$$\begin{aligned} f_c(x, p) &= \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \exp[i(l - \Lambda^n)p \cdot x] \exp[-n\tilde{b}(-n\varpi) \cdot p + n\zeta] \\ &= \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \exp[(l - \Lambda^{-n})ix \cdot p] \exp[-n\tilde{b}(-n\varpi) \cdot p + n\zeta], \end{aligned} \quad (3.15)$$

where we have used the orthogonality of Λ and its linearity. Since:

$$l - \Lambda^{-n} = \sum_{k=0}^{n-1} \Lambda^{-k} (l - \Lambda^{-1}) \quad (3.16)$$

and:

$$\left(\frac{\varpi : J}{2} \right) x = i\varpi \cdot x, \quad (3.17)$$

we have:

$$(l - \Lambda^{-1})ix = - \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{-\varpi : J}{2} \right)^k ix = - \sum_{k=1}^{\infty} \frac{1}{k!} \left(\frac{-\varpi : J}{2} \right)^{k-1} \varpi \cdot x = - \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{-\varpi : J}{2} \right)^k (\varpi \cdot x) = -\tilde{c}(-\varpi), \quad (3.18)$$

where we have defined $c \equiv \varpi \cdot x$ and used the eq. (2.9) as a definition of tilde-transformed vector. Therefore, by using (3.16), (3.18) and (2.17), which hold for any tilde-transformed vector, we obtain:

$$(1 - \Lambda^{-n})ix \cdot p = - \sum_{k=0}^{n-1} \Lambda^{-k} \tilde{c}(-\varpi) \cdot p = -n\tilde{c}(-n\varpi) \cdot p.$$

We can substitute this result into the (3.15):

$$f_c(x, p) = \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \exp[-n(\tilde{b}(-n\varpi) + \tilde{c}(-n\varpi)) \cdot p + n\zeta].$$

Now note that, by definition of \tilde{b} and \tilde{c} :

$$\tilde{b}(\varpi) + \tilde{c}(\varpi) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{\varpi : J}{2}\right)^k (b + \varpi \cdot x) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{\varpi : J}{2}\right)^k \beta = \tilde{\beta}(\varpi), \quad (3.19)$$

where β is the four-temperature vector of global thermodynamic equilibrium in eq. (1.3). Like for the \tilde{b} in eq. (2.9), the vector $\tilde{\beta}$ can be written as:

$$\tilde{\beta}_\mu = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{\varpi : J}{2}\right)^k (\beta)_\mu = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \underbrace{(\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{k-1}\nu_k})}_{k \text{ times}} \beta^{\nu_k}. \quad (3.20)$$

Another very useful expression is:

$$\tilde{\beta}(\varpi) = \tilde{b}(\varpi) + i(1 - e^{\varpi:J/2})x \quad (3.21)$$

which is a consequence of (3.17) and (3.19).

We can finally write the complex phase space distribution function as:

$$\boxed{f_c(x, p) = \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \exp[-n\tilde{\beta}(-n\varpi) \cdot p + n\zeta].} \quad (3.22)$$

For antiparticles, the $\bar{f}_c(x, p)$ is obtained by replacing $\zeta \rightarrow -\zeta$. We are now going to study the above series and highlight some of its main features.

4 Properties of the distribution function

From the function (3.22) we can calculate most quantities of interest, once analytic continuation to real ϖ is made.

We first observe that in the limit $\varpi \rightarrow 0$ the series is real and it boils down to the Bose-Einstein distribution function:

$$f_c(x, p) = \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \exp(-nb \cdot p + n\zeta) = \frac{1}{(2\pi)^3} \frac{1}{e^{b \cdot p - \zeta} - 1},$$

where the last equality applies if ζ is lower than the mass so that the series converges. This suggests that the series expresses the familiar quantum statistics expansion and its first term is the Boltzmann limit.

Otherwise, if $\varpi \neq 0$, the (3.22) is a series of analytic functions of the various $\varpi^{\mu\nu}$ taken as complex variables and if the series uniformly converges in some region, it defines an analytic function therein. In fact, in the physical case with ϖ real, the series in eq. (3.22) is not always convergent, as we will see, whether $\tilde{b}(\varpi)$ is time-like or not. While this does not prevent an analytic continuation of the phase space distribution function, it requires a careful analysis to obtain it. Particularly, it turns out that the properties of the series are different according to which components appear in the thermal vorticity tensor ϖ and a separate treatment is needed.

In the physical case, it is worth pointing out that $\text{Re } \tilde{\beta}$ is itself a Killing vector, just like β . Looking at the equation in (3.20) we obtain:

$$\partial_\nu \tilde{\beta}_\mu = \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \underbrace{(\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{k-1}\nu_k})}_{k \text{ times}} \partial_\nu \beta^{\nu_k} = - \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} \underbrace{(\varpi_{\mu\nu_1} \varpi^{\nu_1\nu_2} \dots \varpi_{\nu_{k-1}\nu_k} \varpi^{\nu_k}_\nu)}_{k+1 \text{ times}}.$$

Each term of the last series is antisymmetric in $\mu \leftrightarrow \nu$ if k is even and symmetric if k is odd. Thus, because of i^k , the real part of the right hand side series selects antisymmetric terms and $\partial_\nu \text{Re } \tilde{\beta}_\mu$ turns out to be an antisymmetric tensor, which proves our statement. Conversely, the imaginary part of $\partial_\nu \tilde{\beta}_\mu$ is a symmetric tensor. This observation leads to the conclusion that $f(x, p)$ is not a solution of the Boltzmann equation. From the eq. (3.22), with $\tilde{\beta}_n \equiv \tilde{\beta}(-n\varpi)$:

$$\begin{aligned} \partial_\mu f &= -\frac{1}{(2\pi)^3} \text{Re} \sum_{n=1}^{\infty} n p^\nu \partial_\mu \tilde{\beta}_{n\nu} \exp[-n\tilde{\beta}_n \cdot p + n\zeta] \\ &= -\frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} n p^\nu \partial_\mu (\text{Re } \tilde{\beta}_{n\nu}) \text{Re} e^{-n\tilde{\beta}_n \cdot p + n\zeta} - n p^\nu \partial_\mu (\text{Im } \tilde{\beta}_{n\nu}) \text{Im} e^{-n\tilde{\beta}_n \cdot p + n\zeta}. \end{aligned}$$

If we now multiply by p^μ we get:

$$p^\mu \partial_\mu f = -\frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} n p^\nu p^\mu \partial_\mu (\text{Re } \tilde{\beta}_{n\nu}) \text{Re} e^{-n\tilde{\beta}_n \cdot p + n\zeta} - n p^\nu p^\mu \partial_\mu (\text{Im } \tilde{\beta}_{n\nu}) \text{Im} e^{-n\tilde{\beta}_n \cdot p + n\zeta}.$$

The first term on the right hand side vanishes as $\text{Re } \tilde{\beta}_n$ is a Killing vector, while the second term in general is non vanishing.

It is also important to point out that the form (3.22) is indeed a resummation of all possible quantum corrections at all orders in \hbar of the Bose-Einstein distribution function. Looking at (3.21) it can be realized that, restoring natural constants, β is a classical vector taking into account the eqs. (1.3) and (1.4), whereas the k -th term in the series is proportional to \hbar^k to make $\varpi : \mathbf{J}$ adimensional. Hence, an expansion in ϖ of the (3.22) with fixed β , such as (setting $\zeta = 0$ for simplicity):

$$\begin{aligned} f_c(x, p) &= \frac{1}{(2\pi\hbar)^3} \sum_{n=1}^{\infty} \exp[-n\tilde{\beta}(-n\varpi) \cdot p] = \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \exp \left[-n \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{-n\hbar\varpi : \mathbf{J}}{2} \right)^k \beta \cdot p \right] \\ &= \frac{1}{(2\pi\hbar)^3} \sum_{n=1}^{\infty} \prod_{k=0}^{\infty} \exp \left[(-n)^{k+1} \frac{1}{(k+1)!} \left(\frac{-\hbar\varpi : \mathbf{J}}{2} \right)^k \beta \cdot p \right] \\ &= \frac{1}{(2\pi\hbar)^3} \sum_{n=1}^{\infty} \exp[-n\beta \cdot p] \prod_{k=1}^{\infty} \sum_{l_k=0}^{\infty} \frac{1}{l_k!} \left[(-n)^{k+1} \frac{1}{(k+1)!} \left(\frac{-\hbar\varpi : \mathbf{J}}{2} \right)^k \beta \cdot p \right]^{l_k}, \end{aligned}$$

is in fact the full semi-classical expansion in \hbar , which has been purposely restored for the occasion; each power of ϖ involves a corresponding factor \hbar to the same power. The series above can be rearranged as an asymptotic power series in \hbar or ϖ , which, as we will see in sections 6 and 7, is mostly divergent. Notwithstanding, it could be used to work out the quantum corrections of, say, the stress-energy tensor, with the eq. (3.14), at some fixed order in \hbar or ϖ . It is worth remarking the similarity of this situation with that of the general, non-equilibrium, gradient expansions in relativistic hydrodynamics [24], whose the above expression is, in a sense, a special case (with vanishing shear tensor) because:

$$\varpi_{\mu\nu} = \frac{1}{2}(\partial_\nu\beta_\mu - \partial_\mu\beta_\nu)$$

from eq. (1.3).

5 Mathematical interlude: analytic distillation and series resummation

The solution of the equation (2.12) found by iteration, the eq. (2.19), may not be unique if the associated homogeneous equation (2.20) has non-trivial solutions, as we discussed in section 2. However, it is possible to show that if (2.20) has non-trivial solutions they are non-analytic about the identity, that is for $\phi = i\varpi = 0$, being $\Lambda = \exp[-i\phi : \mathbf{J}]$.

To prove it, suppose a non-trivial analytic solution around $\phi = 0$ exists, $F(p, p', \phi)$ (without loss of generality, we can assume that ϕ is a single-valued variable). We can then write:

$$F(p, p', \phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n F(p, p', \phi)}{\partial \phi^n} \right|_{\phi=0} \phi^n = \sum_{n=0}^{\infty} F_n(p, p') \phi^n$$

and equate both sides of (2.20):

$$\sum_{n=0}^{\infty} F_n(p, p') \phi^n = e^{-\tilde{b}(\phi) \cdot \Lambda(\phi)p} F(\Lambda(\phi)p, p', \phi).$$

We can now expand the right hand side around $\phi = 0$ and equate each power of ϕ separately. For the constant term at $\phi = 0$ we get:

$$F_0(p, p') = e^{-b \cdot p} F(p, p', 0) = e^{-b \cdot p} F_0(p, p'),$$

whose solution is $F_0(p, p') = 0$. At first order we have, taking into account that $F_0(p, p') = 0$:

$$\begin{aligned} F_1(p, p') &= e^{-b \cdot p} \frac{\partial}{\partial p^\mu} F(p, p', 0) (-iJ_\nu^\mu p^\nu) + e^{-b \cdot p} F_1(p, p') = e^{-b \cdot p} \frac{\partial}{\partial p^\mu} F_0(p, p') (-iJ_\nu^\mu p^\nu) + e^{-b \cdot p} F_1(p, p') \\ &= e^{-b \cdot p} F_1(p, p'), \end{aligned}$$

whence $F_1(p, p') = 0$. Iterating, by taking derivatives of higher order, it can be shown that all functions $F_n(p, p')$ vanish, hence the only analytic solution of the equation (2.20) around $\phi = 0$ is 0. In appendix B a particular non-analytic solution of the (2.20) is found.

It should be now quite clear that non-analytic terms in $\varpi = 0$ could be hidden in the particular solution (2.19) and all quantities derived therefrom. This makes analytic continuations quite problematic unless an unambiguous procedure to single out an analytic part is pinpointed. We call this procedure *analytic distillation* and we define it as follows:

Definition. Let $f(z)$ be a function on a domain D of the complex plane and $z_0 \in \bar{D}$ a point where the function may not be analytic. Suppose that asymptotic³ power series of $f(z)$ in $z - z_0$ exist in subsets $D_i \subset D$ such that $\cup_i D_i = D$:

$$f(z) \sim \sum_n a_n^{(i)} (z - z_0)^n$$

where n can take integer negative values. If the series formed with the common coefficients in the various subsets restricted to $n \geq 0$ has a positive radius of convergence, the analytic function defined by this power series is called analytic distillate of $f(z)$ in z_0 and it is denoted by $\text{dist}_{z_0} f(z)$.

It is worth dwelling upon this definition. First, note that asymptotic power series in a point of a function can be constructed [25] with iterative limits:

$$a_n = \lim_{z \rightarrow z_0} \frac{f(z) - \sum_{i=N_{\min}}^{n-1} a_i (z - z_0)^i}{(z - z_0)^n}.$$

However, these limits may depend upon the argument of the complex number z (e.g. $\exp(-1/z)$ for $z = 0$). Indeed, it is a well known fact that asymptotic expansions are different in different angular sectors centered in z_0 — the well known Stokes phenomenon [25]. The above definition prescribes to retain the part of the power series asymptotic expansion which is common to the various sectors or subsets D_i covering the domain where the function is originally defined:

$$\bar{a}_n \equiv \begin{cases} a_n^{(i)} & \text{if } a_n^{(i)} = a_n^{(j)}, \forall i, j \\ 0 & \text{otherwise} \end{cases} \quad \text{dist}_{z_0} f(z) \equiv \sum_n \bar{a}_n (z - z_0)^n;$$

if there is no common part, the distillate is simply zero. The strong requirement in the definition is the convergence of the asymptotic series, which is not usually the case, but it will apply in our cases of interest, as we will see. Also, from the definition it turns out that, if the function is analytic in z_0 , then $\text{dist}_{z_0} f(z) = f(z)$. If, on the other hand, the function has an isolated pole in z_0 , the distillation extracts the positive powers of the Laurent series.

A simple example of analytic distillation is that of a function which is the sum of an analytic function and a non-analytic function in $z = 0$, for instance:

$$f(z) = g(z) + e^{-1/z}.$$

The analytic distillation of f in zero is

$$\text{dist}_0 f(z) = g(z)$$

because it can be readily shown that $\text{dist}_0 e^{-1/z} = 0$, as its power series about zero in the domain $\text{Re}(z) > 0$ is vanishing.

The calculation of an analytic distillate involves that of an asymptotic power series, which is not always a simple task. When the function presents itself as a series of simple functions, however, there is an important result by D.B. Zagier [26, 27] which can be cast as follows:

³We denote asymptotic equality with the symbol \sim .

Theorem 1. Let $f : (0, +\infty) \rightarrow \mathbb{C}$ be a C^∞ complex valued function on the positive real axis with the asymptotic power series in $x = 0$

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n$$

and $\mathcal{O}(1/x^{1+\epsilon})$ for $x \rightarrow +\infty$ with $\epsilon > 0$. Then, the function defined by the series:

$$g(x) = \sum_{n=1}^{\infty} f(nx)$$

has the asymptotic expansion for $x \rightarrow 0^+$:

$$g(x) \sim \frac{I_f}{x} + \sum_{n=0}^{\infty} a_n \zeta(-n) x^n,$$

where

$$I_f = \int_0^{\infty} dx f(x)$$

and ζ is the Riemann Zeta function.

This theorem can be extended to a complex function. Let $F(z)$ be a function of the complex variable z with the asymptotic power series about $z = 0$ in a domain of the complex plane including the real positive axis:

$$F(z) \sim \sum_{n=0}^{\infty} A_n z^n.$$

Defining:

$$x = |z| \implies z = x e^{i\varphi},$$

we can apply the previous theorem to the function $f(x)$ defined as:

$$f(x) \equiv F(x e^{i\varphi}).$$

We have, from the above definition, for the function $f(x)$:

$$a_n = A_n e^{in\varphi}$$

and

$$I_f = \int_0^{\infty} dx f(x) = \int_0^{\infty} dx F(x e^{i\varphi}) = e^{-i\varphi} \int_{\Gamma_\varphi} dz F(z) \equiv I_{\Gamma_\varphi} \frac{x}{z}$$

where Γ_φ is the $\arg z = \varphi$ line on the complex plane. Therefore, for fixed φ we have, by using the theorem 1 and the above results:

$$G(z) \equiv \sum_{n=1}^{\infty} F(nz) = \sum_{n=1}^{\infty} F(nx e^{i\varphi}) = \sum_{n=1}^{\infty} f(nx) \sim \frac{I_f}{x} + \sum_{n=0}^{\infty} a_n \zeta(-n) x^n = \frac{I_{\Gamma_\varphi}}{z} + \sum_{n=0}^{\infty} A_n \zeta(-n) z^n$$

which concludes the proof. Now, according to the definition of distillate, if the asymptotic expansion of $G(z)$ is convergent, we can conclude that:

$$\text{dist}_0 \sum_{n=0}^{\infty} F(nz) = \text{dist}_0 G(z) = \sum_{n=0}^{\infty} A_n \zeta(-n) z^n.$$

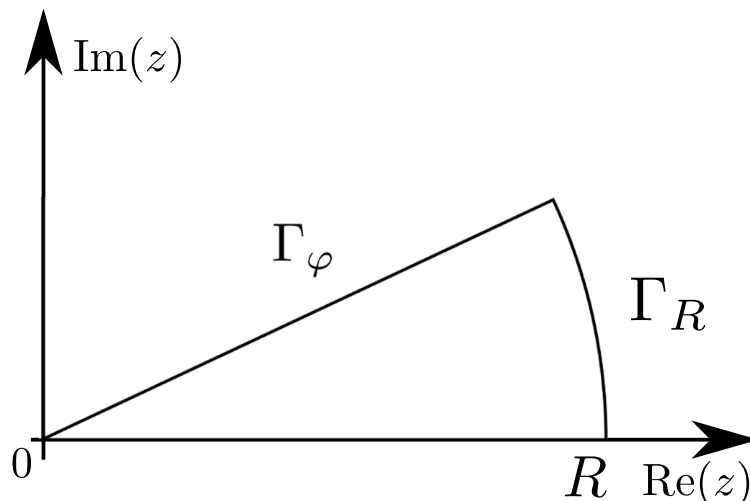


Figure 1. The path Γ_φ .

The integral I_{Γ_φ} , in general, depends on the phase of the complex variable z by construction. According to the definition of distillate then, under the same hypothesis of convergence:

$$\text{dist}_0 z^M G(z) = \text{dist}_0 \left(I_{\Gamma_\varphi} z^{M-1} + \sum_{n=0}^{\infty} A_n \zeta(-n) z^{n+M} \right) = \sum_{n=0}^{\infty} A_n \zeta(-n) z^{n+M}.$$

However, if the function $F(z)$ is analytic in the complex plane, including $z = 0$, then the integral I_{Γ_φ} does not depend on the integration path because the integral on the arc Γ_R vanishes for $R \rightarrow \infty$ owing to the hypothesis of fast decay of f in the theorem (see figure 1). In this case, the asymptotic expansions is independent of the domain or angular sector and we have:

$$\text{dist}_0 z^M G(z) = I_{\mathbb{R}^+} z^{M-1} + \sum_{n=0}^{\infty} A_n \zeta(-n) z^{n+M},$$

where

$$I_{\mathbb{R}^+} = I_{\Gamma_\varphi} = \int_0^\infty dz F(z).$$

The theorem 1 has an important extension to asymptotic expansions with negative powers [26]:

Theorem 2. *Let $f : (0, +\infty) \rightarrow \mathbb{C}$ be a C^∞ complex valued function on the positive real axis with the asymptotic power series in $x = 0$*

$$f(x) \sim \sum_{n=-M}^{\infty} a_n x^n$$

and $f(x) - a_{-1}/x = \mathcal{O}(1/x^{1+\epsilon})$ for $x \rightarrow +\infty$ with $\epsilon > 0$. Then, the function defined by the series:

$$g(x) = \sum_{n=1}^{\infty} f(nx)$$

has the asymptotic expansion for $x \rightarrow 0^+$:

$$g(x) \sim \frac{1}{x} (-a_{-1} \log x + I_f) + \sum_{\substack{n=-M \\ n \neq -1}}^{\infty} a_n \zeta(-n) x^n,$$

where

$$I_f = \int_0^{\infty} dx \left(f(x) - \sum_{n=-M}^{-2} a_n x^n - a_{-1} e^{-x}/x \right).$$

It can be seen that, if $a_{-1} \neq 0$, the asymptotic expansion for $x \rightarrow 0^+$ has a logarithmic term and it is thus not an asymptotic power series. Nevertheless, if $a_{-1} = 0$, this theorem provides, again, an asymptotic power series which is fit for analytic distillation according to the definition above. Furthermore, if $a_{-1} = 0$, the theorem 2 can be extended to complex variables with the same argument following theorem 1

$$G(z) \sim \frac{I_{\Gamma_\varphi}}{z} + \sum_{\substack{n=-M \\ n \neq -1}}^{\infty} A_n \zeta(-n) z^n \tag{5.1}$$

with:

$$I_{\Gamma_\varphi} = \int_{\Gamma_\varphi} dz \left(F(z) - \sum_{n=-M}^{-2} A_n z^n \right).$$

Again, if the function $F(z) - \sum_{n=-M}^{-2} A_n z^n$ is analytic, the integral can be done on any path and no dependence on the argument of z arises in the asymptotic expansion in the complex plane.

6 Study of the series and comparison with known results: acceleration

We are now going to study a special case of global equilibrium which is obtained by setting the constant vector b and the antisymmetric tensor ϖ in eq. (1.2) to

$$b_\mu = (1/T_0, 0, 0, 0), \quad \varpi_{\mu\nu} = (a/T_0)(g_{0\nu}g_{3\mu} - g_{3\nu}g_{0\mu}), \tag{6.1}$$

with T_0 and a real positive constants. The resulting density operator can be written:

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\hat{H}/T_0 + a\hat{K}_z/T_0 \right] \tag{6.2}$$

with $\hat{K}_z \equiv \hat{J}_{30}$ being the generator of a Lorentz boost along the z axis. The combination $\hat{H} - a\hat{K}_z$ can be seen as the generator of translations along its flow lines [28]. This form, and its relation with the Unruh effect, has been recently studied in some detail in refs. [14–16, 19, 20]; we only mention here that the constant a is the constant acceleration of the comoving observer with velocity $u = \beta/\sqrt{\beta^2}$ with a hyperbolic world-line through the origin and T_0 the temperature measured by a comoving thermometer with the same world-line.

From the equation (6.1) we get:

$$b = \frac{1}{T_0}(1, \mathbf{0}), \quad \varpi \cdot x = \frac{a}{T_0}(z, 0, 0, t), \quad \beta = \frac{1}{T_0}(1 + az, 0, 0, at). \tag{6.3}$$

In this case, there is a point $x_0 = (0, 0, 0, -1/a)$ such that:

$$b = -\varpi \cdot x_0.$$

Hence, from eqs. (3.17) and (2.9):

$$\tilde{b} = -i(1 - e^{a/T_0 K_z})x_0 \quad (6.4)$$

and, because of the (3.21):

$$\tilde{\beta} = i(1 - e^{a/T_0 K_z})(x - x_0). \quad (6.5)$$

Note that:

$$1 - e^{a/T_0 K_z} = \begin{pmatrix} 1 - \cos(a/T_0) & 0 & 0 & -i \sin(a/T_0) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i \sin(a/T_0) & 0 & 0 & 1 - \cos(a/T_0) \end{pmatrix}$$

so that if $a/T_0 = 2\pi$, at the Unruh temperature, the above matrix vanishes and so does $\tilde{\beta}$, making the distribution function $f_c(x, p)$ in (3.22) independent of x and p , though divergent.

An important result for global equilibrium (1.2) is that the mean values of local operators depend on x only through the four-temperature vector, that is [20]:

$$\langle \widehat{O}(x) \rangle_{b, \varpi} = \langle \widehat{O}(0) \rangle_{\beta(x), \varpi} \quad (6.6)$$

where the mean value is calculated with the density operator (1.2) with the parameters in the subscript and $\beta(x)$ given by the eq. (1.3). For instance, for a scalar operator, this implies that it may depend only on the scalars that can be formed with β and its only non-vanishing derivative, that is ϖ . Looking at the equation (1.4), (1.5), they are:

$$\beta^2, \quad \alpha^2, \quad w^2, \quad \alpha \cdot w. \quad (6.7)$$

However, in the pure acceleration case, we have $w = 0$ and we are just left with two independent scalars, β^2 and α^2 ; moreover, $\alpha^2 = a^2/T^2$ turns out to be uniform throughout [19]. Thus, once a scalar function is calculated in $x = 0$, it can be inferred everywhere by replacing $\beta^2(0)$ with $\beta^2(x)$. Therefore, the calculation of $\langle \widehat{O}(0) \rangle_{\beta(0)=b, \varpi}$ is sufficient to determine the whole function. For vector or general tensor fields, this conclusion holds, because it is possible to reduce the calculation of a vector or a tensor field to a set of scalar fields once decomposed onto the vectors β , the tensor ϖ and the metric tensor g . Altogether, in the pure acceleration case, we can determine every local mean value by setting $x = 0$ in the distribution function and study the series:

$$f_c(0, p) = \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \exp[-n\tilde{b}(-n\varpi) \cdot p],$$

where, from the (6.4):

$$\tilde{b} = \frac{1}{a} (\sin(a/T_0), 0, 0, i(1 - \cos(a/T_0))). \quad (6.8)$$

The resulting series is:

$$f_c(0, p) = \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \exp \left[-\frac{1}{a} \sin(na/T_0) \varepsilon \right] \exp \left[\frac{i}{a} (1 - \cos(na/T_0)) p_z \right], \quad (6.9)$$

where ε is the energy and p_z the contravariant component of the momentum along the z direction. The above series is not convergent for $a \neq 0$ because the limit of the sequence is undefined. Note that the above series, as it stands, is not an asymptotic series because the terms are not an asymptotic sequence for $a \rightarrow 0$ according to the definitions [25].

Since the simple replacement of the real acceleration in the phase space distribution function does not give rise to a convergent series, one may wonder whether an analytic continuation can be obtained through different methods, like e.g. Borel resummation. However, this would not cure the problem of (6.9). Moreover, we should consider that the non-convergence of the series in the physical case could arise from non-analytic terms in $\varpi = 0$, specifically $a/T_0 = 0$, which are hidden in the form (6.9), as discussed in some detail in section 5. Indeed, since in the acceleration case a non-vanishing vector $v = -ix_0 = (0, 0, 0, i/a)$ exists such that:

$$\tilde{b} = (1 - \Lambda(ia/T_0))v$$

according to the formula (6.4), there are non-trivial solutions of the homogeneous equation (2.20) and an instance can be easily found which is actually non-analytic for $a = 0$ (see appendix B). Therefore, the idea is to single out the physical solution through the method of analytic distillation in $a = 0$. We will not apply this procedure to the distribution function (6.9) itself, but to some integrals thereof, because a comparison can be done with calculations of thermal expectation values carried out with a completely independent method.

6.1 Distillation and comparison with known results

Quantum field theory at thermodynamic equilibrium with the density operator (6.2) can be approached by solving field equations in Rindler coordinates and calculating the expectation values of relevant creation and annihilation operators. Since Wigner function is defined with a normal ordering, all expressions obtained in section 3 and, consequently, the distribution function (3.22) involve a normal ordering in Cartesian coordinates. Therefore, we should obtain the same thermal expectation values in Rindler-based calculations with the subtraction of the Minkowski vacuum contribution; particularly, all obtained results should vanish when $a/T_0 = 2\pi$, that is at the Unruh temperature [19].

To start with, we will compare the mean value of $\hat{\psi}^\dagger \hat{\psi}$ for a single real uncharged scalar field in the massless neutral case (hence $\zeta = 0$) which has been calculated analytically in Rindler coordinates [20]. According to the (3.1), (3.4) and the definition of $f_c(x, p)$ in (3.8), we have:

$$\langle : \hat{\psi}^2(x) : \rangle = \frac{1}{2} \int d^4k W(x, k) = \int \frac{d^3p}{\varepsilon} f_c(x, p)$$

and from now on we again set $\tilde{\beta}_n \equiv \tilde{\beta}(-n\varpi)$. Note that the momentum integral of $f_c(x, p)$ is real, which can be proved from the definition (3.8) taking advantage of the hermiticity of

the density operator. As has been mentioned, it suffices to calculate the above integral at $x = 0$, and yet the series (6.9) expressing f_c is divergent in the physical case. However, we can write the integrand as a series for imaginary acceleration, that is setting $a/T_0 = -i\phi$ and obtain the analytically continued mean value of the field squared $\langle : \widehat{\psi}^2(x) : \rangle_I$:

$$\langle : \widehat{\psi}^2(x) : \rangle_I = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{p}}{\varepsilon} \sum_{n=1}^{\infty} \exp[-n\tilde{\beta}_n \cdot p] \tag{6.10}$$

where, by using (6.8):

$$\tilde{\beta}_n(0) = \tilde{b}_n = \left(\frac{\sinh(n\phi)}{nT_0\phi}, 0, 0, -2\frac{\sinh^2(n\phi/2)}{nT_0\phi} \right), \quad \tilde{\beta}_n(0)^2 = \tilde{\beta}_n \cdot \tilde{\beta}_n = \frac{4\sinh^2(n\phi/2)}{n^2T_0^2\phi^2}, \tag{6.11}$$

where the scalar product is the usual Minkowskian between complex vectors. Note that \tilde{b}_n is indeed real and future time-like for any real ϕ . With such a vector, the series in (6.10) is uniformly convergent for real values of ϕ and can be integrated term by term. The result is, for a massless field:

$$\int \frac{d^3\mathbf{p}}{\varepsilon} e^{-n\tilde{\beta}_n \cdot p} = \frac{4\pi}{n^2(\tilde{\beta}_n \cdot \tilde{\beta}_n)}; \tag{6.12}$$

therefore:

$$\langle : \widehat{\psi}^2(x) : \rangle_I = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2\tilde{\beta}_n \cdot \tilde{\beta}_n} = \frac{T_0^2}{8\pi^2} \sum_{n=1}^{\infty} \frac{\phi^2}{\sinh^2(n\phi/2)}$$

which has the correct known limit of $T_0^2/12$ for $\phi = 0$.

We now have to go back to the physical case, i.e. to continue the above expressions to imaginary ϕ or real accelerations of the function defined by the series:

$$S_2(\phi) = \sum_{n=1}^{\infty} \frac{\phi^2}{\sinh^2(n\phi/2)} = \phi^2 G_2(\phi). \tag{6.13}$$

However, as we have pointed out right before this subsection, the above function may include non-analytic contributions which should be subtracted away through the procedure of analytic distillation in $\phi = 0$ described in section 5. Indeed, the function $S_2(\phi)$ is peculiar on the complex plane; it is analytic if $\text{Re } \phi \neq 0$ because it can be readily shown that the series is uniformly convergent, whereas for imaginary ϕ one has:

$$\sum_{n=1}^{\infty} \frac{(\text{Im } \phi)^2}{\sin^2(n \text{Im } \phi/2)},$$

which is certainly divergent as $1/\sin^2(n \text{Im } \phi/2) \geq 1$; thus, the function $S_2(\phi)$ is non-analytic everywhere on the imaginary axis, although it is convergent for $\phi = 0$. Moreover, the series is not an asymptotic series for $\phi \rightarrow 0$ [25] and cannot even be resummed with the Borel method because zeroes of the denominators are overall dense. Nevertheless, for real ϕ , the series $G_2(\phi)$ fulfills the requirements of the theorem 2 of section 5 with $a_{-1} = 0$, hence an asymptotic power series around $\phi = 0$ can be found which can be extended in

the complex plane according to the discussion following theorem 2 and the equation (5.1). Applying the theorem 2 one finds, being $G_2(\phi)$ an even function:

$$G_2(\phi) \sim \begin{cases} \frac{2\pi^2}{3\phi^2} + \frac{1}{6} - \frac{2}{\phi} & \text{Re } \phi > 0 \\ \frac{2\pi^2}{3\phi^2} + \frac{1}{6} + \frac{2}{\phi} & \text{Re } \phi < 0 \end{cases}.$$

Note that the asymptotic power series of $G_2(\phi)$ is finite, because of the vanishing of the $\zeta(-n)$ for even n in the (5.1). This makes it possible to make an analytic distillation in $\phi = 0$ of the function S_2 :

$$\text{dist}_0 S_2(\phi) = \text{dist}_0 \phi^2 G_2(\phi) = \frac{2\pi^2}{3} + \frac{\phi^2}{6}.$$

The physical solution is then obtained by setting $\phi = ia/T_0$:

$$\langle : \widehat{\psi}^2(0) : \rangle = \frac{T_0^2}{8\pi^2} \left(\frac{2\pi^2}{3} - \frac{a^2}{6T_0^2} \right)$$

and, taking into account that $\beta^2(0) = 1/T_0^2$ and $a^2/T_0^2 = -\alpha^2$, we have, in a generic point x :

$$\langle : \widehat{\psi}^2(x) : \rangle = \frac{1}{\beta(x)^2} \left(\frac{1}{12} + \frac{\alpha^2(x)}{48\pi^2} \right). \quad (6.14)$$

This result coincides with the exact solution in Rindler coordinates found in refs. [19, 20] with subtraction of Minkowski vacuum contribution. Indeed, as we have pointed out in section 3, the normally ordered mean value of an operator which is a quadratic combination of creation and annihilation operators of momentum eigenstates coincides with the mean value with subtraction of the vacuum expectation value. Therefore we expect, according to the general features of the Unruh effect and the conclusions of ref. [19], the normally ordered thermal expectation value (6.14), as well as any other quadratic operator in the fields to vanish precisely at the Unruh temperature, that is $a/T_0 = \sqrt{-\alpha^2} = 2\pi$ and this actually occurs. Furthermore, since (6.10) is an integral of the complex distribution function $f_c(x, p)$, we argue that this characteristic must extend to *any* integral of the distribution function after analytic distillation and continuation; this will be demonstrated for a large class of integrals in the appendix C. As pointed out below eq. (6.5), for $a/T_0 = 2\pi$ the equation (6.5) yields $\tilde{\beta} = 0$, hence, for $\zeta = 0$ a complex distribution function (3.22) which is a divergent constant independent of x and p .

In order to confirm the agreement found, we have compared the expectation values of the canonical stress-energy tensor with an exact calculation in Rindler coordinates, reported in appendix D. If the distribution function were known, the stress-energy tensor could be calculated using the eq. (3.14), which select only the real part of a momentum integral. But we only know the distribution function through eq. (3.22) that is only valid for imaginary value of thermal vorticity. Therefore to obtain the physical expectation values we have to compute the momentum integrals by plugging the distribution function representation (3.22) into (3.14) for imaginary thermal vorticity, then continue it to physical

values and at last take the real part:

$$\langle : \widehat{T}_C^{\mu\nu} : \rangle = \text{Re} \left[\langle : \widehat{T}_C^{\mu\nu} : \rangle_I \right]_{\phi=ia/T_0}, \quad (6.15)$$

where the imaginary expectation value is the series:

$$\langle : \widehat{T}_C^{\mu\nu} : \rangle_I = \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \left[\int \frac{d^3p}{\varepsilon} \left(p^\mu p^\nu + \frac{1}{4} (ip^\mu \partial^\nu + ip^\nu \partial^\mu) + \frac{1}{4} (\partial^\mu \partial^\nu - g^{\mu\nu} \square) \right) e^{-n\tilde{\beta}_n \cdot p} \right]. \quad (6.16)$$

The momenta in the integrand of eq. (6.16) can be expressed as derivatives with respect to $\tilde{\beta}_n$:

$$\begin{aligned} \left(p^\mu p^\nu + \frac{1}{4} (ip^\mu \partial^\nu + ip^\nu \partial^\mu) + \frac{1}{4} (\partial^\mu \partial^\nu - g^{\mu\nu} \square) \right) e^{-n\tilde{\beta}_n \cdot p} = \\ \left(\frac{1}{n^2} \frac{\partial}{\partial \tilde{\beta}_\mu} \frac{\partial}{\partial \tilde{\beta}_\nu} - \frac{1}{4n} \left(i\partial^\nu \frac{\partial}{\partial \tilde{\beta}_\mu} + i\partial^\mu \frac{\partial}{\partial \tilde{\beta}_\nu} \right) + \frac{1}{4} (\partial^\mu \partial^\nu - g^{\mu\nu} \square) \right) e^{-n\tilde{\beta}_n \cdot p}. \end{aligned}$$

Now the momentum integral is the same as eq. (6.12) and we obtain:

$$\begin{aligned} \langle : \widehat{T}_C^{\mu\nu} : \rangle_I = \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \frac{\partial}{\partial \tilde{\beta}_{n\mu}} \frac{\partial}{\partial \tilde{\beta}_{n\nu}} - \frac{1}{4n} \left(i\partial^\nu \frac{\partial}{\partial \tilde{\beta}_{n\mu}} + i\partial^\mu \frac{\partial}{\partial \tilde{\beta}_{n\nu}} \right) + \frac{1}{4} (\partial^\mu \partial^\nu - g^{\mu\nu} \square) \right) \times \\ \times \frac{4\pi}{n^2(\tilde{\beta}_n \cdot \tilde{\beta}_n)}. \end{aligned} \quad (6.17)$$

The derivatives can be worked out by using:

$$\begin{aligned} \frac{\partial}{\partial \tilde{\beta}_\mu} \frac{1}{\tilde{\beta} \cdot \tilde{\beta}} = -2 \frac{\tilde{\beta}^\mu}{\tilde{\beta} \cdot \tilde{\beta}} \frac{1}{\tilde{\beta} \cdot \tilde{\beta}}; \quad \frac{\partial}{\partial \tilde{\beta}_\mu} \frac{\partial}{\partial \tilde{\beta}_\nu} \frac{1}{\tilde{\beta} \cdot \tilde{\beta}} = \left(8 \frac{\tilde{\beta}^\mu \tilde{\beta}^\nu}{(\tilde{\beta} \cdot \tilde{\beta})^2} - 2 \frac{g^{\mu\nu}}{\tilde{\beta} \cdot \tilde{\beta}} \right) \frac{1}{\tilde{\beta} \cdot \tilde{\beta}}; \\ \partial^\nu \frac{\partial}{\partial \tilde{\beta}_\mu} \frac{1}{\tilde{\beta} \cdot \tilde{\beta}} = \left(8 \frac{\tilde{\beta}^\mu \tilde{\beta}_\rho \partial^\nu \tilde{\beta}^\rho}{(\tilde{\beta} \cdot \tilde{\beta})^2} - 2 \frac{\partial^\nu \tilde{\beta}^\mu}{\tilde{\beta} \cdot \tilde{\beta}} \right) \frac{1}{\tilde{\beta} \cdot \tilde{\beta}}; \end{aligned}$$

and one eventually obtains:

$$\langle : \widehat{T}_C^{\mu\nu}(x) : \rangle_I = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} [t^{\mu\nu}(n) + t^{\nu\mu}(n) - g^{\mu\nu} t(n)], \quad (6.18)$$

where:

$$\begin{aligned} t^{\mu\nu}(n) = \left(\frac{4}{n^2} \frac{\tilde{\beta}_n^\mu \tilde{\beta}_n^\nu}{(\tilde{\beta}_n \cdot \tilde{\beta}_n)^2} - \frac{2i}{n} \frac{\tilde{\beta}_n^\mu \tilde{\beta}_{n\rho} \partial^\nu \tilde{\beta}_n^\rho}{(\tilde{\beta}_n \cdot \tilde{\beta}_n)^2} + \frac{i}{2n} \frac{\partial^\nu \tilde{\beta}_n^\mu}{\tilde{\beta}_n \cdot \tilde{\beta}_n} + \frac{1}{8} \partial^\mu \partial^\nu \right) \frac{1}{n^2(\tilde{\beta}_n \cdot \tilde{\beta}_n)} \\ t(n) = \left(\frac{2}{n^2} \frac{1}{\tilde{\beta}_n \cdot \tilde{\beta}_n} + \frac{1}{4} \square \right) \frac{1}{n^2(\tilde{\beta}_n \cdot \tilde{\beta}_n)}. \end{aligned} \quad (6.19)$$

As has been pointed out earlier in this section (see eq. (6.6)), it is sufficient to calculate the stress-energy tensor in $x = 0$. From the (6.5) one can write the vector field $\tilde{\beta}_n(x)$ as a function of the Minkowskian coordinates t and z :

$$\tilde{\beta}_n^\mu(x) = \frac{1}{n} \left(it(\cosh(n\phi)-1) + \frac{\sinh(n\phi)}{\phi} \left(\frac{1}{T_0} + iz\phi \right), 0, 0, it \sinh(n\phi) + \frac{(\cosh(n\phi)-1)(1+iT_0z\phi)}{T_0\phi} \right), \quad (6.20)$$

whence:

$$\tilde{\beta}_n(x) \cdot \tilde{\beta}_n(x) = -\frac{4 \sinh^2\left(\frac{n\phi}{2}\right) (T_0^2 \phi^2 (z^2 - t^2) - 2iT_0 z \phi - 1)}{n^2 T_0^2 \phi^2},$$

and:

$$\partial^\mu \tilde{\beta}_n^\nu(x) = \begin{pmatrix} \frac{i(\cosh(n\phi)-1)}{n} & 0 & 0 & \frac{i \sinh(n\phi)}{n} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{i \sinh(n\phi)}{n} & 0 & 0 & -\frac{i(\cosh(n\phi)-1)}{n} \end{pmatrix}.$$

The derivatives of the $\tilde{\beta}_n$ needed to work out the (6.19) in $x = 0$ are:

$$\left(\partial^\mu \partial^\nu \frac{1}{\tilde{\beta}_n \cdot \tilde{\beta}_n} \right) \Big|_{x=0} = \text{diag} \left(\frac{n^2 T_0^4 \phi^4}{1 - \cosh(n\phi)}, 0, 0, \frac{3n^2 T_0^4 \phi^4}{1 - \cosh(n\phi)} \right).$$

From the equations (6.18), (6.19) and using the above derivatives we get:

$$\langle : T^{\mu\nu}(0) : \rangle_I = \frac{1}{16\pi^2} \sum_{n=1}^{\infty} \begin{pmatrix} 3T_0^4 \phi^4 \sinh^{-4}(n\phi/2) & 0 & 0 & 0 \\ 0 & T_0^4 \phi^4 \frac{\cosh(n\phi)}{\sinh^4(n\phi/2)} & 0 & 0 \\ 0 & 0 & T_0^4 \phi^4 \frac{\cosh(n\phi)}{\sinh^4(n\phi/2)} & 0 \\ 0 & 0 & 0 & T_0^4 \phi^4 \sinh^{-4}(n\phi/2) \end{pmatrix}. \quad (6.21)$$

Because of the symmetries of the density operator (6.2) and the fact that the stress-energy tensor can only depend on β and ϖ and not explicitly on x , according to the discussion about equation (6.6), it may only have this form [20]:

$$\langle : T^{\mu\nu}(x) : \rangle = \rho u^\mu u^\nu - p (g^{\mu\nu} - u^\mu u^\nu) + \mathcal{A} \alpha^\mu \alpha^\nu, \quad (6.22)$$

where $u^\mu = \beta^\mu / \sqrt{\beta^2}$ and the thermal functions ρ , p and \mathcal{A} can only depend on the Lorentz scalars β^2 and α^2 . From the (6.21) and the (6.15) we then get the physical thermal functions:

$$\begin{aligned} \rho &= \frac{1}{16\pi^2} \text{Re} \left[\sum_{n=1}^{\infty} 3T_0^4 \phi^4 \sinh^{-4}(n\phi/2) \right]_{\phi=ia/T_0}, \\ p &= \frac{1}{16\pi^2} \text{Re} \left[\sum_{n=1}^{\infty} T_0^4 \phi^4 \frac{\cosh(n\phi)}{\sinh^4(n\phi/2)} \right]_{\phi=ia/T_0}, \\ \mathcal{A} &= \frac{1}{16\pi^2} \text{Re} \left[\sum_{n=1}^{\infty} 2T_0^4 \phi^4 \sinh^{-2}(n\phi/2) \right]_{\phi=ia/T_0}. \end{aligned}$$

The series are similar to the one in eq. (6.13) and they can be tackled in the same manner. The analytic distillation and the continuation to physical acceleration with $a = -iT_0\phi$ of

these series are discussed in appendix C. The result is:

$$\begin{aligned}\rho &= T_0^4 \left(\frac{\pi^2}{30} + \frac{a^2}{12T_0^2} - \frac{11a^4}{480\pi^2 T_0^4} \right), \\ p &= T_0^4 \left(\frac{\pi^2}{90} - \frac{a^2}{18T_0^2} + \frac{19a^4}{1440\pi^2 T_0^4} \right), \\ \mathcal{A} &= T_0^4 \left(\frac{1}{12} - \frac{a^2}{48\pi^2 T_0^2} \right).\end{aligned}$$

The form of the thermodynamic coefficient in any point x can be inferred from the previous result simply by taking into account that $\beta^2(0) = 1/T_0^2$ and $a^2/T_0^2 = -\alpha^2$, so we have:

$$\begin{aligned}\rho &= \beta(x)^{-4} \left(\frac{\pi^2}{30} - \frac{\alpha^2(x)}{12} - \frac{11\alpha^4(x)}{480\pi^2} \right), \\ p &= \beta(x)^{-4} \left(\frac{\pi^2}{90} + \frac{\alpha^2(x)}{18} + \frac{19\alpha^4(x)}{1440\pi^2} \right), \\ \mathcal{A} &= \beta(x)^{-4} \left(\frac{1}{12} + \frac{\alpha^2(x)}{48\pi^2} \right).\end{aligned}\tag{6.23}$$

These coefficients nicely coincide with those calculated by solving field equations in Rindler coordinates in the right Rindler wedge (reported in appendix D), where the four-temperature is a time-like vector. Moreover, they all vanish at the Unruh temperature $T_0 = a/2\pi$ that is for $|\alpha| = 2\pi$, as expected for a quadratic operator according to the discussion following equation (6.14).

These results precisely coincide with a perturbative expansion in acceleration to quadratic order in refs. [29] and to fourth order presented in refs. [14, 15] for the same statistical operator in (6.2). This result bears out the observation made in refs. [14] that the vanishing of the stress-energy tensor at the Unruh temperature is achieved, in the perturbative expansion in a/T at the fourth-order which led the same authors to conclude that the exact expression must be polynomial [15], namely that the (6.23) were indeed the complete solutions. These authors also pointed out that these same results can be derived within a non-perturbative approach in a quantum field theory over a space with a conical singularity [16], see for instance [30] and other references in [16].

7 Study of the series and comparison with known results: rotation

We are now going to study a special case of global equilibrium which corresponds to system rigidly rotating along the z axis and which is obtained from eq. (1.2) by setting:

$$b_\mu = (1/T_0, 0, 0, 0), \quad \varpi_{\mu\nu} = (\omega/T_0)(g_{1\mu}g_{2\nu} - g_{1\nu}g_{2\mu}), \tag{7.1}$$

where ω and T_0 are positive constants. More in general, without specifying the axis of rotation, we can choose

$$b = \frac{1}{T_0}(1, \mathbf{0}), \quad \varpi \cdot x = \frac{1}{T_0}(0, \boldsymbol{\omega} \times \mathbf{x}),$$

with $\boldsymbol{\omega}$ constant vector. It can be shown that $\boldsymbol{\omega}$ has the physical meaning of a constant angular velocity [31] and T_0 is the temperature measured on the axis of rotation. This state is also known as *rotational thermodynamic equilibrium*. The density operator becomes:

$$\hat{\rho} = \frac{1}{Z} \exp \left[-\frac{\hat{H}}{T_0} + \frac{1}{T_0} \boldsymbol{\omega} \cdot \hat{\mathbf{J}} \right]. \quad (7.2)$$

Because of equation (2.9), it turns out that $\tilde{b}(\varpi) = b$. In the physical case with real ϖ , that is real $\boldsymbol{\omega}$, the eq. (3.21) can be rewritten as:

$$\tilde{\beta}(\varpi) = b - i(1 - R(i\boldsymbol{\omega}/T_0))x,$$

where $R(i\boldsymbol{\omega}/T_0) = \exp[\boldsymbol{\omega} \cdot \mathbf{J}/T_0]$ is the rotation around the axis defined by $\boldsymbol{\omega}$ by an imaginary angle $i\boldsymbol{\omega}/T_0$. Thus:

$$\begin{aligned} \tilde{\beta}(\varpi) &= b + i[\sin(i\boldsymbol{\omega}/T_0)\hat{\boldsymbol{\omega}} \times \mathbf{x} + (1 - \cos(i\boldsymbol{\omega}/T_0))\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{x})] \\ &= b - \sinh(\boldsymbol{\omega}/T_0)\hat{\boldsymbol{\omega}} \times \mathbf{x} + i(1 - \cosh(\boldsymbol{\omega}/T_0))\hat{\boldsymbol{\omega}} \times (\hat{\boldsymbol{\omega}} \times \mathbf{x}) \end{aligned} \quad (7.3)$$

and, consequently:

$$\tilde{\beta}(\varpi) \cdot p = \frac{\varepsilon}{T_0} + \sinh\left(\frac{\boldsymbol{\omega}}{T_0}\right) \hat{\boldsymbol{\omega}} \cdot (\mathbf{x} \times \mathbf{p}) - i\left(1 - \cosh\left(\frac{\boldsymbol{\omega}}{T_0}\right)\right) (\hat{\boldsymbol{\omega}} \cdot \mathbf{x} \hat{\boldsymbol{\omega}} - \mathbf{x}) \cdot \mathbf{p}.$$

Therefore, the analytically continued distribution function is, from eq. (3.22):

$$f(x, p) = \text{Re} \frac{1}{(2\pi)^3} \sum_{n=1}^{\infty} \exp \left[-n \frac{\varepsilon}{T_0} + n \sinh\left(\frac{n\boldsymbol{\omega}}{T_0}\right) \hat{\boldsymbol{\omega}} \cdot (\mathbf{x} \times \mathbf{p}) + ni \left(1 - \cosh\left(\frac{n\boldsymbol{\omega}}{T_0}\right)\right) (\hat{\boldsymbol{\omega}} \cdot \mathbf{x} \hat{\boldsymbol{\omega}} - \mathbf{x}) \cdot \mathbf{p} + \zeta n \right] \quad (7.4)$$

which converges only if $(\mathbf{x} \times \mathbf{p}) \cdot \hat{\boldsymbol{\omega}} < 0$, that is only if the orbital angular momentum along the $\boldsymbol{\omega}$ direction is negative. Otherwise, the series is divergent, which seems to imply that there is an infinite number of particles with $(\mathbf{x} \times \mathbf{p}) \cdot \hat{\boldsymbol{\omega}} > 0$.

To understand the source of this problem, and to prove that the distribution function found is the actual exact solution, it is very convenient to diagonalize the density operator (7.2), following the method of ref. [32]. Setting $\hat{\boldsymbol{\omega}} = \hat{\mathbf{k}}$ as the direction of the z axis, this can be done by using single particle states with eigenvalues p_T, p_z (transverse and longitudinal momentum) and the eigenvalue of the angular momentum M which takes on integer values. The relation between momentum eigenstates and eigenstates $|p_T, p_z, M\rangle$ is well known and leads to the relations between creation operators:

$$\hat{a}^\dagger(p) = \sqrt{2\varepsilon} \sum_{M=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-iM\varphi} \hat{a}^\dagger(p_T, p_z, M) \quad (7.5)$$

along with its inverse:

$$\hat{a}^\dagger(p_T, p_z, M) = \frac{1}{\sqrt{2\pi}\sqrt{2\varepsilon}} \int_0^{2\pi} d\varphi e^{iM\varphi} \hat{a}^\dagger(p), \quad (7.6)$$

where φ is the azimuthal coordinate of \mathbf{p} . One can then plug the (7.5) in the field expansion in cartesian coordinates (3.3) and obtain the corresponding solution in cylindrical coordinates. The operator on the exponent of (7.2) can then be written as:

$$\hat{H} - \boldsymbol{\omega} \hat{J}_z = \sum_{M=-\infty}^{\infty} \int dp_T p_T dp_z (\varepsilon - M\boldsymbol{\omega}) \hat{a}^\dagger(p_T, p_z, M) \hat{a}(p_T, p_z, M) + H_0, \quad (7.7)$$

where H_0 is a divergent constant. Unlike the Hamiltonian, the spectrum of the operator (7.7) is not bounded from below because the single particle energy can be as low as the mass m , whereas the angular momentum eigenvalue M can take on any value, so $\varepsilon - M\omega$ has no lower bound. This causes most statistical mean to be hopeless divergent because the probability of large occupation numbers is exponentially large when $\varepsilon - M\omega < 0$ which is clearly the case when M is positive and the energy is low enough; this is the origin of the divergence of the distribution function $f(x, p)$. This problem is due to the absence of boundary conditions for the field; if the scalar field has Dirichlet boundary condition $\hat{\psi} = 0$ at a radius $R < 1/\omega$ the spectrum is bounded from below because the transverse momentum eigenvalues are discrete and depending on M , and the following inequality holds [32]:

$$\varepsilon = \sqrt{p_z^2 + m^2 + p_T(M)^2} > M\omega.$$

Curing the divergence inherent in the rotational equilibrium without field boundary conditions goes beyond the scope of this work. However, we can check that the solution found (2.18) is the same as the known one for the basis (p_T, p_z, M) . By using the same method as for $\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle$ one can calculate $\langle \hat{a}^\dagger(p_T, p_z, M) \hat{a}(p'_T, p'_z, M') \rangle$ with the density operator (7.2); it is a much easier task for \hat{J}_z is diagonal on the eigenstates created by $\hat{a}^\dagger(p_T, p_z, M)$ [32]. The result is diagonal in the mean values:

$$\langle \hat{a}^\dagger(p_T, p_z, M) \hat{a}(p'_T, p'_z, M') \rangle = \frac{1}{p_T} \delta(p_T - p'_T) \delta(p_z - p'_z) \delta_{MM'} \sum_{n=1}^{\infty} e^{-n(\varepsilon/T_0 - M\omega/T_0)}. \quad (7.8)$$

Note that the series is convergent only if $\varepsilon > M\omega$, see above discussion. For an imaginary $\omega/T_0 = -i\phi$, that is for an actual Lorentz transformation, the sum (7.8) is always convergent and the result is:

$$\langle \hat{a}^\dagger(p_T, p_z, M) \hat{a}(p'_T, p'_z, M') \rangle = \frac{1}{e^{\varepsilon/T_0 + iM\phi} - 1} \frac{1}{p_T} \delta(p_T - p'_T) \delta(p_z - p'_z) \delta_{MM'}. \quad (7.9)$$

The relation (7.8) can be used to calculate $\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle$ by means of the (7.5), for $\omega/T_0 = -i\phi$:

$$\begin{aligned} \langle \hat{a}^\dagger(p) \hat{a}(p') \rangle &= \sum_{M, M'} \frac{1}{2\pi} e^{-iM\varphi + iM'\varphi'} \langle \hat{a}^\dagger(p_T, p_z, M) \hat{a}(p'_T, p'_z, M') \rangle \\ &= \sum_M \frac{1}{2\pi} e^{-iM(\varphi - \varphi')} \sum_{n=1}^{\infty} e^{-n(\varepsilon/T_0 - iM\phi)} \frac{1}{p_T} \delta(p_T - p'_T) \delta(p_z - p'_z) \\ &= \sum_{n=1}^{\infty} e^{-n\varepsilon/T_0} \sum_M \frac{1}{2\pi} e^{-iM(\varphi - \varphi' + n\phi)} \frac{1}{p_T} \delta(p_T - p'_T) \delta(p_z - p'_z) \\ &= \sum_{n=1}^{\infty} e^{-n\varepsilon/T_0} \delta(\varphi - \varphi' + n\phi) \frac{1}{p_T} \delta(p_T - p'_T) \delta(p_z - p'_z). \end{aligned} \quad (7.10)$$

Let us now use take (2.19) with $\Lambda = \exp[-i\phi J_z]$. Since the rotation around the z axis leaves p_T and p_z unchanged, the vector $\Lambda^n \mathbf{p}$ has cylindrical coordinates p_T , $\varphi + n\phi$ and p_z . Hence:

$$\delta^3(\Lambda^n \mathbf{p} - \mathbf{p}') = \delta(\varphi + n\phi - \varphi') \frac{1}{p_T} \delta(p_T - p'_T) \delta(p_z - p'_z).$$

Therefore, being $\tilde{b} = b = (1/T_0, \mathbf{0})$ for rotations, as we have seen, the eq. (2.18) becomes:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle = \sum_{n=1}^{\infty} e^{-n\varepsilon/T_0} \delta(\varphi - \varphi' + n\phi) \frac{1}{p_T} \delta(p_T - p'_T) \delta(p_z - p'_z),$$

which is the same as (7.10). The proof is concluded.

A remarkable feature of the calculation with imaginary angular velocity is that the series involved in (7.9) is always convergent. Thereby, the problem of the divergence due to the lack of lower bound of the operator (7.7) is evaded by going to imaginary ω and it is possible that a suitable analytic continuation to real ω yields finite results. It should be stressed, however, that the physical meaning of the expressions found is limited because the divergence should be cured in the physical case with different methods.

This proof has another noteworthy consequence, that it is that there is no non-trivial solution of the homogeneous equation (2.20) for the density operator (7.2); in other words the (2.19) is the only solution of the equation (2.12). The reason is simply that $\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle$ is the only possible expression corresponding to the solution (7.9), which is in turn the only possible expression of $\langle \hat{a}^\dagger(p_T, p_z, M) \hat{a}(p'_T, p'_z, M') \rangle$ which can be obtained algebraically [32] in the basis diagonalizing the density operator. This is consistent with what we described as a sufficient condition to obtain a non-trivial solution of the homogeneous equation (2.20) as, in the case of rotation, there is no vector v such that $\tilde{b} = b = (I - \Lambda)(v)$ with b as in eq. (7.1) (see appendix B).

7.1 Comparison with known results

We now move to the calculation of thermal expectation values similar to those of the previous section. First we define a suitable tetrad for the rotational equilibrium with $\hat{\omega} = \hat{\mathbf{k}}$. Keeping in mind the decomposition (1.4), we define the following four-vectors (see also [3, 4]):

$$w^\mu = \frac{\beta^\mu}{\sqrt{\beta^2}}, \quad \alpha^\mu = \varpi^{\mu\nu} u_\nu, \quad w^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \varpi_{\nu\rho} u_\sigma, \quad l^\mu = \epsilon^{\mu\nu\rho\sigma} w_\nu \alpha_\rho u_\sigma.$$

The tetrad $\{u, \alpha, w, l\}$ is an orthogonal, non normalized basis of Minkowski spacetime built upon the four-temperature β and thermal vorticity ϖ and can be used to decompose all vectors and tensor fields. Moreover, denoting with $r^2 = x^2 + y^2$ the distance from the rotation axis we have that the time component of the four-velocity u reads:

$$u^0 \equiv \gamma = \frac{1}{\sqrt{1 - r^2 \omega^2}},$$

and:

$$\beta^2(x) = \frac{1}{T_0^2 \gamma^2}, \quad \alpha^2(x) = -(\gamma^2 - 1) \frac{\omega^2}{T_0^2}, \quad w^2(x) = -\gamma^2 \frac{\omega^2}{T_0^2}. \quad (7.11)$$

Formulae (7.11) allow to transform the widely used r^2, ω, T_0 variables into the Lorentz invariants discussed in the previous section. Note that, unlike in the acceleration case, in the rotational case the calculation of the thermal expectation value of a scalar operator at

$x = 0$, that is $r = 0$, is not sufficient to know its value everywhere because at $r = 0$ we have $\alpha^2 = 0$ and the dependence on one of the three independent Lorentz scalars is lost.

We now turn to the mean value of the field squared in the massless neutral case, see eq. (6.10). With $\phi = i\omega/T_0$, we have

$$\tilde{\beta}_n \cdot \tilde{\beta}_n = \frac{1}{T_0^2} + [\sin^2(n\phi) + (1 - \cos(n\phi))^2] \frac{r^2}{n^2} = \frac{1}{T_0^2} + 4 \sin^2\left(\frac{n\phi}{2}\right) \frac{r^2}{n^2} \quad (7.12)$$

and the series (6.10) becomes:

$$\langle : \hat{\psi}^2 : \rangle_I = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{T_0^2}{n^2 + 4 \sin^2\left(\frac{n\phi}{2}\right) T_0^2 r^2}. \quad (7.13)$$

For real ϕ the series is absolutely and uniformly convergent, because it is bounded by the series of $1/n^2$; this is an expected result according to the discussion on the convergence of the series (7.9). However, for complex ϕ the series is densely divergent. To show this, let $\text{Re } \phi$ be a rational multiple of 2π , so that $\text{Re } \phi = (N/L)2\pi$ with N and L irreducible integers. The n -th denominator of the series in (7.13) vanishes when $n = KL$ with K integer and if:

$$\text{Im } \phi = \pm \frac{2}{n} \text{asinh}\left(\frac{n}{2rT_0}\right) = \pm \frac{\text{Re } \phi}{KN\pi} \text{asinh}\left(\frac{KN}{\text{Re } \phi} \frac{\pi}{rT_0}\right), \quad (7.14)$$

making the series in (7.13) a divergent one. Since K and N are arbitrary integer numbers, the curves (7.14) are dense in the complex plane. Altogether, the function (7.13) cannot then be defined except on the real axis.

The divergence of the series (7.13) for non-real ϕ is evidently a consequence of the divergence of the distribution function (7.4) for $(\mathbf{x} \times \mathbf{p}) \cdot \boldsymbol{\omega} > 0$ and its physical reason has already been discussed. Even though there is no proper domain of existence of the complex function (7.13), thus no asymptotic expansion for complex ϕ , we can, notwithstanding, extract a finite analytic distillate of the function defined by the series (7.13) for real ϕ by using the theorems of the section 5, confining ourselves to the positive and negative real axis and keeping in mind that the physical meaning of the solution will be limited (this point will be resumed later on).

We start rewriting the right hand side of (7.13) in a form which is suitable for the application of the theorem 2. By using the auxiliary real parameter t we have:

$$\langle : \hat{\psi}^2 : \rangle_I = \frac{T_0^2}{2\pi^2} \sum_{n=1}^{\infty} \lim_{t \rightarrow \phi} \frac{\phi^2}{\phi^2 n^2 + 4 \sin^2\left(\frac{n\phi}{2}\right) t^2 T_0^2 r^2} = \frac{T_0^2}{2\pi^2} \lim_{t \rightarrow \phi} \sum_{n=1}^{\infty} \frac{\phi^2}{\phi^2 n^2 + 4 \sin^2\left(\frac{n\phi}{2}\right) t^2 T_0^2 r^2} \equiv \lim_{t \rightarrow \phi} R(t, \phi). \quad (7.15)$$

Exchanging the limit and the series, for real values of ϕ and t , is possible because as a function of t the (7.15) is a uniformly convergent series of continuous functions. We can now apply the theorem 2 to the function $g(\phi) = R(t, \phi)/\phi^2$ which leads to the following result:

$$R(t, \phi) \sim \frac{1}{12} \frac{T_0^2}{1 + r^2 T_0^2 t^2} - \frac{r^2 T_0^4 t^2}{48\pi^2 (1 + r^2 T_0^2 t^2)^2} \phi^2 \pm I_f(t^2) \frac{\phi T_0^2}{2\pi^2}, \quad (7.16)$$

where the upper sign applies to $\phi > 0$ and the lower to $\phi < 0$ and:

$$I_f(x^2) = \int_0^\infty \left[\frac{1}{y^2 + 4x^2 \sin^2(\frac{y}{2})} - \frac{1}{(1+x^2)y^2} \right] dy.$$

Note that, again, the term $a_{-1} = 0$, which excludes logarithmic terms in the asymptotic series. The asymptotic power series of $R(t, \phi)$ near $\phi = 0$ (7.16) has, again, a finite number of terms due to the fact that there are no odd powers in the expansion and $\zeta(-n) = 0$ for even $n \geq 2$. Now we can take the limit $t \rightarrow \phi$ on both sides of (7.16) and obtain the asymptotic power expansion of $\langle : \hat{\psi}^2 : \rangle_I$ about $\phi = 0$, which is unique as it is known. This is possible because the coefficients of the right hand side of (7.16) are analytic functions of t and $t = \phi$ is an analytic function of ϕ in $\phi = 0$. Therefore we have:

$$\langle : \hat{\psi}^2 : \rangle_I = \lim_{t \rightarrow \phi} R(t, \phi) \sim \frac{1}{12} \frac{T_0^2}{1 + r^2 T_0^2 \phi^2} - \frac{r^2 T_0^4 \phi^4}{48\pi^2 (1 + r^2 T_0^2 \phi^2)^2} \pm I_f(\phi^2) \frac{\phi T_0^2}{2\pi^2}. \quad (7.17)$$

It can be seen that the right hand side of (7.17) can be expanded as a power series in ϕ about $\phi = 0$ for $|\phi| < 1/rT_0$ by using the geometric series and thereby obtain the full asymptotic power series of $\langle : \hat{\psi}^2 : \rangle_I$. On the other hand, the (7.17) is a resummed form of this asymptotic power series which is now suitable for analytic distillation. The distillation just removes the last term in (7.17) and, going back to physical angular velocity with $\phi = i\omega/T_0$:

$$\langle : \hat{\psi}^2 : \rangle = \frac{1}{12} \frac{T_0^2}{1 - r^2 \omega^2} - \frac{r^2 \omega^4}{48\pi^2 (1 - r^2 \omega^2)^2}. \quad (7.18)$$

Finally, taking advantage of relations (7.11), the (7.15) and (7.17) yield:

$$\langle : \hat{\psi}^2(x) : \rangle = \frac{1}{12\beta(x)^2} + \frac{\alpha(x)^2}{48\pi^2 \beta(x)^2}, \quad (7.19)$$

which coincides with the expression found in the pure acceleration case (6.14).

This is indeed a remarkable and unexpected result. Starting from two considerably different statistical operators (6.2) and (7.2) we have obtained, after a long mathematical derivation, the very same expression of the mean value of the field squared in terms of the Lorentz invariant thermodynamic variables. On one hand, this confirms the covariant structure of the theory and how deep is the connection between angular velocity and acceleration in relativity. On the other hand, it is quite surprising that in the rotating case the (7.19) does not feature any dependence on w^2 . This could have happened, since a scalar function like $\langle : \hat{\psi}^2(x) : \rangle$ in principle depends on all the scalars in eq. (6.7) (see discussion thereabout) and only $\alpha \cdot w$ vanishes in the pure rotating case. The dependence of $\langle : \hat{\psi}^2(x) : \rangle$ on the latter will be worked out in section 8.

Like in the pure acceleration case, we are going to calculate the thermal expectation values of the stress-energy tensor. To compare with expressions found in literature [21, 33] we considered the general form of the stress-energy tensor operator of the scalar field, which is trace-less for $m = 0$ for $\xi = 1/6$:

$$\hat{T}_\xi^{\mu\nu} = \hat{T}_C^{\mu\nu} - \xi(\partial^\mu \partial^\nu - g^{\mu\nu} \square) \hat{\psi}^\dagger \hat{\psi}, \quad (7.20)$$

where the canonical stress-energy tensor is given by the (3.9). The mean values can be calculated with the same method of the previous section for pure acceleration: we study the series associated to a complex mean value $\langle : \widehat{T}_\xi^{\mu\nu} : \rangle_I$ for $\omega = -i\phi/T_0$, we take the analytic distillate and continue the result to physical angular velocity and lastly we take the real part. We just need to modify the relation (6.18) to account for the term proportional to ξ in the (7.20); this is simply done by adding the following terms to the quantities in (6.19):

$$\Delta t_\xi^{\mu\nu} = -\frac{\xi}{2} \partial^\mu \partial^\nu \frac{1}{n^2(\tilde{\beta}_n \cdot \tilde{\beta}_n)}, \quad \Delta t_\xi = -\xi \square \frac{1}{n^2(\tilde{\beta}_n \cdot \tilde{\beta}_n)}. \quad (7.21)$$

From the (7.3), with $\phi = i\omega/T_0$, we obtain:

$$\begin{aligned} \tilde{\beta}_n^\mu(x) &= \left(\frac{1}{T_0}, i \frac{x(\cos(n\phi) - 1) + y \sin(n\phi)}{n}, i \frac{y(\cos(n\phi) - 1) - x \sin(n\phi)}{n}, 0 \right), \\ \tilde{\beta}_n(x) \cdot \tilde{\beta}_n(x) &= \frac{1}{T_0^2} + 4 \sin^2 \left(\frac{n\phi}{2} \right) \frac{r^2}{n^2}, \\ \partial^\mu \tilde{\beta}_n^\nu(x) &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{i(\cos(n\phi)-1)}{n} & \frac{i \sin(n\phi)}{n} & 0 \\ 0 & -\frac{i \sin(n\phi)}{n} & -\frac{i(\cos(n\phi)-1)}{n} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ \left(\partial^\mu \partial^\nu \frac{1}{\tilde{\beta}_n \cdot \tilde{\beta}_n} \right) &= \text{diag} \left(0, \frac{4n^4 T_0^4 (\cos(n\phi) - 1) + 32n^2 T_0^6 (3x^2 - y^2) \sin^4 \left(\frac{n\phi}{2} \right)}{(n^2 + 2r^2 T_0^2 - 2r^2 T_0^2 \cos(n\phi))^3}, \right. \\ &\quad \left. -\frac{4n^4 T_0^4 (\cos(n\phi) - 1) + 32n^2 T_0^6 (3y^2 - x^2) \sin^4 \left(\frac{n\phi}{2} \right)}{(n^2 + 2r^2 T_0^2 - 2r^2 T_0^2 \cos(n\phi))^3}, 0 \right) \\ &\quad + \frac{128n^2 T_0^6 xy \sin^4 \left(\frac{n\phi}{2} \right)}{(n^2 + 2r^2 T_0^2 - 2r^2 T_0^2 \cos(n\phi))^3} \times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Plugging these expressions into the (6.18), (6.19) and (7.21) we can work out every component of $\langle : \widehat{T}_\xi^{\mu\nu} : \rangle_I$ and express it as a series; the explicit forms are given in appendix E. Each series has the same mathematical features as the one in (7.13) and can be resummed likewise, that is by using theorem 2 followed by the analytic distillation and continuation to the physical values. The details of the procedure and the results of each components in terms of the physical rotation are given in appendix E. Here we report the results in terms of the thermal coefficients associated to the stress-energy tensor. Indeed, the stress-energy tensor can be decomposed on the tetrad $\{u, \alpha, w, l\}$, which can only depend on the Lorentz scalars (6.7):

$$\begin{aligned} \langle : T^{\mu\nu}(x) : \rangle &= \rho u^\mu u^\nu - p \Delta^{\mu\nu} + W w^\mu w^\nu + \mathcal{A} \alpha^\mu \alpha^\nu + G^l l^\mu l^\nu + G (l^\mu u^\nu + l^\nu u^\mu) + \mathbb{A} (\alpha^\mu u^\nu + \alpha^\nu u^\mu) \\ &+ G^\alpha (l^\mu \alpha^\nu + l^\nu \alpha^\mu) + \mathbb{W} (w^\mu u^\nu + w^\nu u^\mu) + A^w (\alpha^\mu w^\nu + \alpha^\nu w^\mu) + G^w (l^\mu w^\nu + l^\nu w^\mu), \end{aligned} \quad (7.22)$$

with $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$. While a symmetric stress-energy tensor has ten degrees of freedom, the thus defined scalar coefficients are eleven. The redundancy of (7.22) is owing to the fact that the tetrad $\{u, \alpha, w, l\}$ is no longer a basis for $r = 0$ because there $\alpha = l = 0$. For this reason we introduced a term $p\Delta^{\mu\nu}$ which would be redundant at finite r . By projecting the stress-energy tensor onto the suitable pair of vectors (see appendix E), we obtain the expressions of each scalar function. The isotropic pressure is identified in the projection of $T^{\mu\nu}$ along the direction of l , which has the following continuation to physical values of ω :

$$\langle : T^{\mu\nu} : \rangle \hat{l}_\mu \hat{l}_\nu = G^l l^2 - p = \frac{1-70\xi}{240\pi^2\beta^4} (-\alpha^2 w^2) - \left[\frac{\pi^2}{90\beta^4} - \frac{\xi}{6\beta^4} w^2 + \frac{1-6\xi}{18\beta^4} \alpha^2 - \frac{\xi}{24\pi^2\beta^4} w^4 + \frac{19-120\xi}{1440\pi^2\beta^4} \alpha^4 \right].$$

Since $l^2 = -\alpha^2 w^2$, the identification of p and G^l from the above equation is unambiguous. Once p has been determined, all the other coefficients are obtained by subtraction and one finally finds:

$$\begin{aligned} \rho &= \frac{\pi^2}{30\beta^4} + \frac{4\xi-1}{12\beta^4} w^2 + \frac{6\xi-1}{12\beta^4} \alpha^2 + \frac{4\xi-1}{48\pi^2\beta^4} w^4 + \frac{60\xi-11}{480\pi^2\beta^4} \alpha^4 + \frac{270\xi-61}{720\pi^2\beta^4} \alpha^2 w^2, \\ p &= \frac{\pi^2}{90\beta^4} - \frac{\xi}{6\beta^4} w^2 + \frac{1-6\xi}{18\beta^4} \alpha^2 - \frac{\xi}{24\pi^2\beta^4} w^4 + \frac{19-120\xi}{1440\pi^2\beta^4} \alpha^4, \\ W &= \frac{2\xi-1}{12\beta^4} + \frac{2\xi-1}{48\pi^2\beta^4} w^2 + \frac{120\xi-29}{360\pi^2\beta^4} \alpha^2, \\ \mathcal{A} &= \frac{1-6\xi}{12\beta^4} + \frac{1}{360\pi^2\beta^4} w^2 + \frac{1-6\xi}{48\pi^2\beta^4} \alpha^2, \\ G^l &= \frac{1-70\xi}{240\pi^2\beta^4}, \\ G &= \frac{6\xi+1}{36\beta^4} + \frac{10\xi-1}{240\pi^2\beta^4} w^2 + \frac{30\xi-7}{720\pi^2\beta^4} \alpha^2, \\ \mathbb{A} &= 0, \\ G^\alpha &= 0, \\ G^w &= A^w = \mathbb{W} = 0. \end{aligned} \tag{7.23}$$

Taking into account the relations (7.11), these results can be compared with those calculated in refs. [21, 33] by solving the Klein-Gordon field equation in rotating coordinates without enforcing boundary conditions at finite radius r and, in spite of the different methods used, precise agreement is found.

Remarkably, by setting $w = 0$ and $l = 0$, as well as $\xi = 0$, the resulting canonical stress-energy tensor is the same function of β, α found in the pure acceleration case and particularly the functions in (6.23) are the same functions of β^2 and α^2 . This confirms the previous finding for $\langle : \hat{\psi}^2 : \rangle$ and the deep relation between the two examined cases. It appears, because of the covariant structure of (1.2), that from the study of the pure rotation, one can also deduce the exact solution of thermal expectation values in the pure acceleration case, which is certainly a remarkable and unexpected fact. It should be pointed out, however, that the physical meaning of the (7.23) is limited in the case of pure rotation;

these results have been obtained by removing the inherent divergences pertaining to the density operator (7.2) with the method of analytic distillation, and not by setting appropriate boundary conditions or other physical meaningful constraints which would make the operator $\widehat{H} - \omega \widehat{J}_z$ bounded from below. Indeed, the terms in α^2/β^2 , α^4/β^4 , w^4/β^4 and $\alpha^2 w^2/\beta^4$ are actually independent of T_0 and do not vanish in the limit $T_0 \rightarrow 0$ what would be expected in the pure rotation case for a well-behaved, bounded from below operator $\widehat{H} - \omega \widehat{J}_z$ in the formula (7.2).

8 Thermodynamic equilibrium with rotation and acceleration: a new result

So far we have shown that this method reproduces the exact results known or virtually known in literature, which were obtained by solving field equations in curvilinear coordinates. However, the main advantage of the proposed method is that it allows to obtain the exact expressions for thermal expectation values even when the solutions in curvilinear coordinates are not known or are difficult to handle, for instance when both acceleration and rotation in the equations (1.4) and (1.5) are non-vanishing. To prove it, as a case study, we compute the thermal expectation value $\langle : \widehat{\psi}^2 : \rangle$ in a thermodynamic equilibrium state with both rotation and acceleration along the z -axis, that is with thermal vorticity:

$$\varpi^{\mu\nu} = \frac{\omega}{T_0} (g^{\mu 1} g^{\nu 2} - g^{\mu 2} g^{\nu 1}) - \frac{a}{T_0} (g^{\mu 0} g^{\nu 3} - g^{\mu 3} g^{\nu 0}) \quad \implies \quad \beta^\mu = \frac{1}{T_0} (1 + az, -\omega y, \omega x, ta).$$

Contrary to the previous cases, $\langle : \widehat{\psi}^2 : \rangle$ may now depend on the Lorentz scalar $(\alpha \cdot w)$, which is non-vanishing:

$$\alpha \cdot w = -\frac{a\omega}{T_0^2},$$

whereas the other Lorentz scalars it may depend on (see equation (6.7) and foregoing discussion) turn out to be:

$$\begin{aligned} \beta^2 &= \frac{1}{T_0^2} \left((1 + az)^2 - a^2 t^2 - r^2 \omega^2 \right), & \alpha^2 &= -\frac{a^2 \left((1 + az)^2 - a^2 t^2 \right) + r^2 \omega^4}{\beta^2 T_0^4}, & (8.1) \\ w^2 &= -\frac{\omega^2 \left((1 + az)^2 - a^2 t^2 + a^2 r^2 (az + 1)^2 \right)}{T_0^4 \beta^2}, \end{aligned}$$

where $r^2 = x^2 + y^2$.

We start by computing the $\tilde{\beta}$ vector according to the equation (3.20). For this purpose, one can take advantage of the following relation:

$$\varpi_\alpha^\mu \varpi_\nu^\alpha = \frac{1}{T_0^2} \begin{pmatrix} a^2 & 0 & 0 & 0 \\ 0 & -\omega^2 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & a^2 \end{pmatrix}$$

which returns a diagonal matrix. The sum (3.20) can then be readily split into odd and even powers, leading to an analytic expression involving both hyperbolic and trigonometric

functions:

$$\tilde{\beta}(\varpi) = \left(\frac{(az + 1) \sin\left(\frac{a}{T_0}\right) - iat \left(\cos\left(\frac{a}{T_0}\right) - 1\right)}{a}, -y \sinh\left(\frac{\omega}{T_0}\right) - ix \left(\cosh\left(\frac{\omega}{T_0}\right) - 1\right), \right. \\ \left. x \sinh\left(\frac{\omega}{T_0}\right) - iy \left(\cosh\left(\frac{\omega}{T_0}\right) - 1\right), \frac{at \sin\left(\frac{a}{T_0}\right) - i(az + 1) \left(\cos\left(\frac{a}{T_0}\right) - 1\right)}{a} \right).$$

After some lengthy but simple calculations, the expectation value can be obtained (see eq. (6.10) and following) for imaginary acceleration and vorticity $a/T_0 \rightarrow -i\Phi$ and $\omega/T_0 \rightarrow -i\phi$:

$$\langle : \hat{\psi}^2(x) : \rangle_I = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2 \tilde{\beta}(-n\varpi) \cdot \tilde{\beta}(-n\varpi)} = \frac{1}{8\pi^2} \sum_{n=1}^{\infty} \frac{\Phi^2}{r^2 \Phi^2 \sin^2\left(\frac{n\phi}{2}\right) + \sinh^2\left(\frac{n\Phi}{2}\right) \left(t^2 \Phi^2 + \left(\frac{1}{T_0} - i\Phi z\right)^2\right)}.$$

The series as it stands, with ϕ and Φ real, is uniformly convergent. On the other hand, for general complex values of ϕ and/or Φ the series diverges and an analytic distillation procedure can be carried out by limiting ϕ and Φ to be real, much the same way as for equilibrium with rotation. It is thus convenient to write Φ and ϕ in polar coordinates $\Phi = \xi \cos \theta$ and $\phi = \xi \sin \theta$ and focus on the analytic distillation of the variable ξ whose limit $\xi \rightarrow 0$ corresponds to both ϕ and Φ vanishing. We first write the series in a form which is suitable for the application of the Zagier's theorem 2:

$$\langle : \hat{\psi}^2(x) : \rangle_I = \sum_{n=1}^{\infty} \lim_{B \rightarrow B(\xi, \theta)} \lim_{C \rightarrow C(\xi, \theta)} \frac{1}{8\pi^2} \frac{\xi^2 \cos^2 \theta}{r^2 C^2 \sin^2\left(\frac{n\xi \sin \theta}{2}\right) + B \sinh^2\left(\frac{n\xi \cos \theta}{2}\right)}, \quad (8.2)$$

where we introduced the auxiliary parameters B and C :

$$B(\xi, \theta) = t^2 \xi^2 \cos^2 \theta^2 + \left(\frac{1}{T_0} - i\xi \cos \theta z\right)^2, \quad C(\xi, \theta) = \xi \cos \theta. \quad (8.3)$$

Except for $B = 0$,⁴ the exchange of series with limits is legitimate because the (8.2) is a uniformly convergent series of continuous functions. Now we can apply the theorem 2 to the function

$$g(\xi, \theta) = \frac{1}{8\pi^2} \sum_{n=1}^{\infty} \frac{1}{r^2 C^2 \sin^2\left(\frac{n\xi \sin \theta}{2}\right) + B \sinh^2\left(\frac{n\xi \cos \theta}{2}\right)}$$

and obtain the asymptotic power series of g for $\xi \rightarrow 0$:

$$g(\xi, \theta) \sim \frac{1}{12(B \cos^2 \theta + C^2 r^2 \sin^2 \theta)} \frac{1}{\xi^2} - \frac{C^2 r^2 \sin^4 \theta - B \cos^4 \theta}{48\pi^2 (B \cos^2 \theta + C^2 r^2 \sin^2 \theta)^2} \pm \frac{I(\theta, B, C)}{\xi} \quad (8.4)$$

being:

$$I(\theta, B, C) = \int_0^\infty dy \left[\frac{1}{8\pi^2 \left(B \sinh^2\left(\frac{1}{2}y \cos \theta\right) + C^2 r^2 \sin^2\left(\frac{1}{2}y \sin \theta\right)\right)} - \frac{1}{2\pi^2 y^2 (B \cos^2 \theta + C^2 r^2 \sin^2 \theta)} \right].$$

⁴Since $T_0 > 0$, it is guaranteed that $B \neq 0$.

Since $B(\xi, \theta)$ and $C(\xi, \theta)$ are analytic functions of ξ , and the coefficients of the powers of ξ in the (8.4) are analytic functions of ξ in $\xi = 0$, we can obtain the full resummed asymptotic expression of $g(\xi, \theta)$ about $\xi = 0$ by taking the limits of B and C to $B(\xi, \theta)$ and $C(\xi, \theta)$ respectively:

$$g(\xi, \theta) \sim \frac{1}{12 \left(\left(t^2 \xi^2 \cos^2 \theta + \left(\frac{1}{T_0} - i \xi \cos \theta z \right)^2 \right) \cos^2 \theta + \xi^2 r^2 \cos^2 \theta \sin^2 \theta \right)} \frac{1}{\xi^2} + \frac{\xi^2 r^2 \cos^2 \theta \sin^4 \theta - \left(t^2 \xi^2 \cos^2 \theta + \left(\frac{1}{T_0} - i \xi \cos \theta z \right)^2 \right) \cos^4 \theta}{48 \pi^2 \left(\left(t^2 \xi^2 \cos^2 \theta + \left(\frac{1}{T_0} - i \xi \cos \theta z \right)^2 \right) \cos^2 \theta + \xi^2 r^2 \cos^2 \theta \sin^2 \theta \right)^2} \pm \frac{I(\xi, \theta)}{\xi}$$

much like in the rotation case in section 7. We are now in a position to apply the distillation process to the above function; we obtain:

$$\text{dist}_0 \xi^2 \cos^2 \theta g(\xi, \theta) = \frac{\cos^2 \theta}{12 \left(\left(t^2 \xi^2 \cos^2 \theta + \left(\frac{1}{T_0} - i \xi \cos \theta z \right)^2 \right) \cos^2 \theta + \xi^2 r^2 \cos^2 \theta \sin^2 \theta \right)} + \frac{\xi^2 r^2 \cos^2 \theta \sin^4 \theta - \left(t^2 \xi^2 \cos^2 \theta + \left(\frac{1}{T_0} - i \xi \cos \theta z \right)^2 \right) \cos^4 \theta}{48 \pi^2 \left(\left(t^2 \xi^2 \cos^2 \theta + \left(\frac{1}{T_0} - i \xi \cos \theta z \right)^2 \right) \cos^2 \theta + \xi^2 r^2 \cos^2 \theta \sin^2 \theta \right)^2} \xi^2 \cos^2 \theta.$$

Going back to the variables Φ and ϕ and analytically continuing to real acceleration and angular velocity we get:

$$\langle : \widehat{\psi}^2(x) : \rangle = \frac{T_0^2}{12 \left((1 + az)^2 - a^2 t^2 - r^2 \omega^2 \right)} - \frac{a^2 \left((1 + az)^2 - a^2 t^2 \right) + r^2 \omega^4}{48 \pi^2 \left((1 + az)^2 - a^2 t^2 - r^2 \omega^2 \right)^2}, \quad (8.5)$$

which, by using the equations (8.1), can be rewritten as:

$$\langle : \widehat{\psi}^2(x) : \rangle = \frac{1}{12 \beta(x)^2} + \frac{\alpha(x)^2}{48 \pi^2}. \quad (8.6)$$

The result is the same as in the case of pure acceleration (6.14) and pure rotation (7.19); there is no dependence on the possible argument $\alpha \cdot w$. Therefore, the equation (6.14) applies to any kind of global equilibrium for the massless scalar field, that is for any tensor ϖ . Achieving this conclusion by solving the field equations would have been extremely hard.

9 Summary

In summary, we have derived a general exact form of the phase space distribution function and the thermal expectation values of local operators of the free quantum scalar field in the most general case of thermodynamic equilibrium in Minkowski space-time, with the four-temperature being a Killing field, that is including rotation and acceleration. The presented derivation does not make use of the solutions of the Klein-Gordon equations in curvilinear coordinates but it is just based on the plane wave expansion of the field in Minkowski

space-time and it is therefore suitable for the general case including both rotation and linear acceleration. The crucial steps of the derivation are a factorization of the general density operator (1.2) using Poincaré group algebra and an iterative method to obtain the thermal expectation values of quadratic combinations of creation and annihilation operators. The general form of the phase space distribution function has been written as a formal series including all quantum corrections to the classical term. We have studied the series in two major cases of non-trivial equilibrium, the pure acceleration and the pure rotation and compared with exact known results obtained solving Klein-Gordon equation in Rindler and rotating coordinates respectively. Furthermore, we have obtained a completely new result, that is the thermal expectation value of the massless scalar field squared in the case of global equilibrium with both acceleration and rotation. The result shows that this quantity does not depend on the scalar product of acceleration and rotation vectors $A \cdot \omega$. It should be emphasized that it would have been extremely hard to obtain the same result by means of the solutions of Klein-Gordon equation in curvilinear coordinates. In the case of pure acceleration, the iterative method introduces undesired non-analytic terms to be subtracted, while in the rotation case the absence of boundary conditions at finite radius $< 1/\omega$ implies unavoidable physical divergences. The resummation of the series in these two cases required the introduction of a new operation on complex functions, defined as analytic distillation, in order to extract the analytic part in the pure acceleration case and analytically continue the finite solution for imaginary angular velocity in the pure rotation case. In the former case, the method of analytic distillation leads to expressions in the massless case which are the same as obtained in Rindler coordinates and, remarkably, automatically vanish at the Unruh temperature; mathematically, a new class of complex polynomials is generated by this physics problem, which all vanish for $z = 2\pi i$. In the case of pure rotation, on the other hand, it should be pointed out that the analytic distillation and continuation provides a finite result which is in agreement with analytic calculations in literature, but with limited physical meaning.

Acknowledgments

We are greatly indebted to D. Dorigoni for letting us know about Zagier's theorem and his work on Lambert series. We warmly thank D. Basile and D. Dorigoni for illuminating discussions about asymptotic series and the extraction of an analytic part. We thank F. Colomo and C. Dappiaggi for equally useful discussions.

A Recurrence formula

We want to work out $\Lambda^{-s} \tilde{b}(\varpi)$ for a non-negative integer s and $\Lambda = \exp[\varpi : J/2]$ being ϖ complex in general. By using the eqs. (2.9) and the expansion of Λ , we have:

$$\Lambda^{-s} \tilde{b}(\varpi) = \sum_{h=0}^{\infty} \left(\frac{-s\varpi : J}{2} \right)^h \frac{1}{h!} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{\varpi : J}{2} \right)^k b.$$

This expression can be worked out as follows:

$$\begin{aligned} \sum_{h=0}^{\infty} \left(\frac{-s\varpi : \mathbb{J}}{2}\right)^h \frac{1}{h!} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{\varpi : \mathbb{J}}{2}\right)^k b &= \sum_{h,k=0}^{\infty} (-s)^h \frac{1}{h!(k+1)!} \left(\frac{\varpi : \mathbb{J}}{2}\right)^{h+k} b \\ &= \sum_{n=0}^{\infty} \left(\frac{\varpi : \mathbb{J}}{2}\right)^n b \sum_{h=0}^n (-s)^h \frac{1}{h!(n+1-h)!} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{\varpi : \mathbb{J}}{2}\right)^n b \sum_{h=0}^n (-s)^h \frac{(n+1)!}{h!(n+1-h)!}, \end{aligned}$$

where we set $n = h+k$. We can write the last obtained expression with binomial coefficients:

$$\Lambda^{-s} \tilde{b}(\varpi) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\frac{\varpi : \mathbb{J}}{2}\right)^n b \sum_{h=0}^n (-s)^h \binom{n+1}{h}. \quad (\text{A.1})$$

Now, since:

$$\sum_{h=0}^n (-s)^h \binom{n+1}{h} = \sum_{h=0}^{n+1} (-s)^h \binom{n+1}{h} - (-s)^{n+1} = (1-s)^{n+1} + s(-s)^n$$

the eq. (A.1) can be written as:

$$\begin{aligned} \Lambda^{-s} \tilde{b}(\varpi) &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (1-s) \left(\frac{(1-s)\varpi : \mathbb{J}}{2}\right)^n b + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} s \left(\frac{(-s)\varpi : \mathbb{J}}{2}\right)^n b \\ &= (1-s) \tilde{b}((1-s)\varpi) + s \tilde{b}(-s\varpi). \end{aligned}$$

B Solutions of the homogeneous equation

We provide an instance of a non-analytic solution of equation (2.20) for $\phi = 0$. In general, if $\Lambda = \exp[-i\phi : \mathbb{J}]$ and there is a non-vanishing vector $v(\phi)$ such that:

$$\tilde{b} = (1 - \Lambda)v(\phi) \quad (\text{B.1})$$

where \tilde{b} is given by the (2.9) with $\phi = i\varpi$, then:

$$H(p, p') = G(p', \phi) \exp[v(\phi) \cdot p]$$

solves the equation (2.20) for any function $G(p', \phi)$. Note that, since \tilde{b} is real for real Λ , $v(\phi)$ must be real as well. Since:

$$\langle \hat{a}^\dagger(p) \hat{a}(p') \rangle_\phi = \frac{1}{Z} \text{Tr} \left(\exp[-b \cdot \hat{P} - i\phi : \hat{J}] \hat{a}^\dagger(p) \hat{a}(p') \right) = \frac{1}{Z} \text{Tr} \left(\hat{a}^\dagger(p') \hat{a}(p) \exp[-b \cdot \hat{P} + i\phi : \hat{J}] \right)^* = \langle \hat{a}^\dagger(p') \hat{a}(p) \rangle_{-\phi}^*$$

taking into account that v is real, we have:

$$G(p', \phi) \exp[v(\phi) \cdot p] = G(p, -\phi)^* \exp[v(-\phi) \cdot p'].$$

This functional equation, applying to any p, p' is solved by the combination:

$$H(p, p') = F(\phi) \exp[v(\phi) \cdot p + v(-\phi) \cdot p'] \quad (\text{B.2})$$

with $F(\phi)^* = F(-\phi)$.

If Λ is a rotation, then $\tilde{b} = b$ (see section 7) and since b is proportional to the time unit vector \hat{t} , there is no real vector $v(\phi)$ solving the (B.1). If, on the other hand, Λ is a pure boost, say along the z axis, then we have that (B.1) has non-trivial solutions. By using the equation (6.4) and the definition of $x_0 = (0, 0, 0, -1/a)$ (see section 6) it can be checked that:

$$v(\phi) = \frac{1}{T_0} \left(0, 0, 0, -\frac{1}{\phi} \right)$$

is a solution of the (B.1) with $\phi = ia/T_0$. In this case, it turns out that, according to the above general proof, $H(p, p')$ is non-analytic in $\phi = 0$.

C Analytic distillation for pure acceleration and the Unruh effect

Just like the thermal expectation value of the canonical stress-energy tensor (3.9), any local operator which is quadratic in the field and its derivatives has a thermal expectation value given by a four-momenta integral of the Wigner function (3.12) or equivalently of the phase space distribution function and its derivatives (3.14). Note that the appearance of the real part in eq. (3.14) is a consequence of the hermiticity of the stress-energy tensor and, more in general, of any physical observable. As already pointed out in the main text, our knowledge of the distribution function is limited to (3.22) which holds for imaginary vorticity. From (3.22), the thermal expectation value of local operators is obtained by considering integrals of this sort:

$$\int \frac{d^3\mathbf{p}}{\varepsilon} p^{\mu_1} \dots p^{\mu_N} \partial^{\nu_1} \dots \partial^{\nu_M} e^{-n\tilde{\beta}_n \cdot p} = \partial^{\nu_1} \dots \partial^{\nu_M} \frac{\partial}{\partial \tilde{\beta}_n^{\mu_1}} \dots \frac{\partial}{\partial \tilde{\beta}_n^{\mu_N}} \frac{(-1)^N}{n^N} \int \frac{d^3\mathbf{p}}{\varepsilon} e^{-n\tilde{\beta}_n \cdot p}. \quad (\text{C.1})$$

These integrals must be computed for imaginary vorticity, and the real part, implied by hermiticity, must be taken only after the analytic distillation and the continuation back to real vorticity.

We want to study such integrals for the pure acceleration case, with imaginary acceleration $a/T_0 = -i\phi$ and demonstrate explicitly that — in the massless neutral case — after analytic distillation and continuation they give rise to expressions vanishing at the Unruh temperature $T_0 = a/2\pi$. This might be a consequence of the distribution function (3.22) being independent of p and x when evaluated at the Unruh temperature, as shown in section 6. Since that is a general feature of the distribution function, we can expect that the expressions for the massive and charged case will also be vanishing, as dictated by the Unruh effect. Moreover, it is sufficient to calculate the integrals (C.1) in $x = 0$ because the whole dependence on x of any tensor field is fully constrained (see discussion after eq. (6.6)).

By using eq. (6.12) and eq. (6.20), it can be realized that the equation (C.1) in $x = 0$ gives rise to linear combinations of the following series:

$$S_{2m+2}(\phi) = \sum_{n=1}^{\infty} \frac{\phi^{2m+2}}{\sinh^{2m+2}(n\phi/2)}, \quad S_{2m+2}^{(1)}(\phi) = \sum_{n=1}^{\infty} \frac{\phi^{2m+2} \sinh(n\phi)}{\sinh^{2m+2}(n\phi/2)}. \quad (\text{C.2})$$

As shown in the text specifically for $S_2(\phi)$, these are series of analytic functions if $\text{Re } \phi \neq 0$ and they are uniformly convergent in the same domain, so they define analytic functions

when $\text{Re } \phi \neq 0$. Conversely, these series are divergent whenever ϕ is purely imaginary. Nevertheless, we can obtain asymptotic power series about $\phi = 0$.

As far as the series $S_{2m+2}^{(1)}(\phi)$ in (C.2) is concerned, it can be shown that its asymptotic expansion about $\phi = 0$ is given by a series of odd powers of ϕ , hence its contribution to the physical expectation values, after distillation, continuation, and extraction of the real part, vanishes.

We are then left with $S_{2m+2}(\phi)$ only. With $\phi > 0$ real and positive, we can use the theorem 2 to obtain an asymptotic power series of the function $G_{2m+2}(\phi)$:

$$G_{2m+2}(\phi) \equiv \phi^{-2m-2} S_{2m+2}(\phi) = \sum_{n=1}^{\infty} \frac{1}{\sinh^{2m+2}(n\phi/2)} \equiv \sum_{n=1}^{\infty} f(n\phi).$$

By using the generalized version of the Bernoulli polynomials $B_n^{(m)}(t)$ defined by [34]

$$\left(\frac{x}{e^x - 1}\right)^m e^{tx} = \sum_{n=0}^{\infty} \frac{B_n^{(m)}(t)}{n!} x^n,$$

the function f can be written as a power series about $\phi = 0$:

$$\begin{aligned} f(\phi) &= \frac{1}{\sinh^{2m+2}\left(\frac{\phi}{2}\right)} = 2^{2m+2} \phi^{-2m-2} \left(\frac{\phi}{e^{\phi} - 1}\right)^{2m+2} e^{(m+1)\phi} \\ &= 2^{2m+2} \sum_{n=0}^{\infty} \frac{B_n^{(2m+2)}(m+1)}{n!} \phi^{n-2m-2} = 2^{2m+2} \sum_{n=-2m-2}^{\infty} \frac{B_{2m+2+n}^{(2m+2)}(m+1)}{(2m+2+n)!} \phi^n, \end{aligned} \quad (\text{C.3})$$

which converges for any complex ϕ with $|\phi| < 2\pi$ [34]. Owing to the parity of the function f , the coefficient $B_n^{(2m+2)}(m+1)$ appearing in (C.3) must be vanishing for odd n . The conditions of the theorem 2 for its application to the function $G_{2m+2}(\phi)$ are fulfilled and we have, in its notation:

$$a_{-1} = 2^{2m+2} \frac{B_{2m+2-1}^{(2m+2)}(m+1)}{(2m+2-1)!} = 0$$

as well as

$$\begin{aligned} I_f &= \int_0^{\infty} \left[\frac{1}{\sinh^{2m+2}\left(\frac{\phi}{2}\right)} - 2^{2m+2} \sum_{n=-2m-2}^{-2} \frac{B_{2m+2+n}^{(2m+2)}(m+1)}{(2m+2+n)!} \phi^n \right] d\phi \\ &= \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^{\infty} \left[\frac{1}{\sinh^{2m+2}\left(\frac{\phi}{2}\right)} - 2^{2m+2} \sum_{n=-2m-2}^{-2} \frac{B_{2m+2+n}^{(2m+2)}(m+1)}{(2m+2+n)!} \phi^n \right] d\phi = (-1)^{m+1} \frac{4}{m+2}, \end{aligned}$$

where in the last step we iteratively used [35, Formula 1, section 1.4.5, p. 146]. Therefore, the asymptotic expansion of $G_{2m+2}(\phi)$ reads:

$$G_{2m+2}(\phi) \sim (-1)^{m+1} \frac{4}{m+2} \frac{1}{\phi} + 2^{2m+2} \sum_{n=-2m-2}^{\infty} \frac{B_{2m+2+n}^{(2m+2)}(m+1)}{(2m+2+n)!} \zeta(-n) \phi^n.$$

Since $a_{-1} = 0$ and the function f in eq. (C.3) is analytic in the region $\text{Re } \phi > 0$, we can extend the above expansion to complex ϕ in the same region according to the discussion following theorem 2 (see also figure 1 in section 5). Notice that the above sum effectively stops at $n = 0$ because, for positive n , the Zeta function is non-vanishing only when n is odd, but in this case the generalized Bernoulli polynomial coefficients vanish. Then, by taking advantage of the relation between the Riemann Zeta function and the Bernoulli numbers

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!},$$

the asymptotic expansion of the hyperbolic series can be written as:

$$S_{2m+2}(\phi) \sim -2^{2m+1} \sum_{k=0}^{m+1} (2\pi i)^{2k} \frac{B_{2k}}{(2k)!} \frac{B_{2m+2-2k}^{(2m+2)}(m+1)}{(2m+2-2k)!} \phi^{2m+2-2k} + (-1)^{m+1} \frac{4}{m+2} \phi^{2m+1}. \tag{C.4}$$

For instance, in the cases $m = 0, 1$ the asymptotic expansions are:

$$S_2(\phi) \sim \frac{2\pi^2}{3} + \frac{\phi^2}{6} - 2\phi,$$

$$S_4(\phi) \sim \frac{8\pi^4}{45} - \frac{4\pi^2}{9} \phi^2 - \frac{11}{90} \phi^4 + \frac{4}{3} \phi^3.$$

Repeating this procedure in the region where $\text{Re } \phi < 0$, we find that the asymptotic expansion of $S_{2m+2}(\phi)$ is the same as in eq. (C.4) with the opposite sign of the term proportional to ϕ^{2m+1} , which is hence removed by the distillate.

Now we can extract the analytic distillate of the function $S_{2m+2}(\phi)$ and continue it to imaginary values of ϕ , that is to a real acceleration. Like in section 6 the result of distillation are the polynomials:

$$\text{dist}_0 S_{2m+2}(\phi) = \mathcal{B}_{2m+2}(\phi) \equiv -2^{2m+1} \sum_{k=0}^{m+1} (2\pi i)^{2k} \frac{B_{2k}}{(2k)!} \frac{B_{2m+2-2k}^{(2m+2)}(m+1)}{(2m+2-2k)!} \phi^{2m+2-2k}, \tag{C.5}$$

which are to be continued to imaginary ϕ .

We now prove that the polynomials (C.5) have two zeroes in $\phi = \pm 2\pi i$ for any $m \geq 0$, hence their analytic continuation for real accelerations $a/T_0 = -i\phi$ vanish precisely at the Unruh temperature. If $\phi = 2\pi i$ is a zero of \mathcal{B}_{2m+2} then the following identity must be true:

$$\mathcal{B}_{2m+2}(2\pi i) = \sum_{k=0}^{m+1} \frac{B_{2k} B_{2m+2-2k}^{(2m+2)}(m+1)}{(2k)!(2m+2-2k)!} = 0. \tag{C.6}$$

To prove the identity (C.6), we start showing that for every integer $m > 0$ the generalized Bernoulli polynomials have a zero in m :

$$B_{2m}^{(2m+1)}(m) = 0. \tag{C.7}$$

With $z \in \mathbb{C}$ and $\alpha, \beta \geq 0$ integers, the following ratio of Euler Gamma functions has an exact power series representation [36]:

$$\frac{\Gamma(z + \alpha)}{\Gamma(z - \beta)} = \sum_{k=0}^{\alpha+\beta} \frac{(\alpha + \beta)!}{k!} \frac{B_{\alpha+\beta-k}^{(1+\alpha+\beta)}(\alpha)}{(\alpha + \beta - k)!} z^k. \tag{C.8}$$

Then, for $m > 0$ and setting $\alpha = \beta = m$, the above equation becomes:

$$\frac{\Gamma(z + m)}{\Gamma(z - m)} = \sum_{k=0}^m \frac{(2m)!}{k!(2m - 2k)!} B_{2m-k}^{(2m+1)}(m) z^k.$$

If we evaluate the previous expression in $z = 0$ and we use the reflection formula for the Gamma function, we have:

$$\frac{\Gamma(m)}{\Gamma(-m)} = \Gamma(m)\Gamma(m + 1) \frac{\sin(m\pi)}{\pi} = 0.$$

For $z = 0$ the sum (C.8) contains just one term, hence we have:

$$\frac{\Gamma(m)}{\Gamma(-m)} = B_{2m}^{(2m+1)}(m) = 0,$$

which proves the equation (C.7).

To proceed towards the proof of (C.6), let us now consider the function

$$f(x) = \left(\frac{x}{e^x - 1} + \frac{x}{2} \right) \left(\frac{x e^{x/2}}{e^x - 1} \right)^{2m+2}, \tag{C.9}$$

which involves the generators of the Bernoulli polynomials and has a power series expansion:

$$f(x) = \sum_{k=0}^{\infty} d_k x^k, \tag{C.10}$$

which is convergent for $|x| \leq 2\pi$. We are going to show that, for any integer $m \geq 0$:

$$d_{2m+2} = \sum_{l=0}^{m+1} \frac{B_{2l}}{(2l)!} \frac{B_{2m+2-2l}^{(2m+2)}(m+1)}{(2m+2-2l)!} = 0 \tag{C.11}$$

which proves the identity (C.6). Consider the even terms of (C.10), i.e. d_{2k} . Splitting the product in (C.9) as

$$f(x) = \frac{x}{e^x - 1} \left(\frac{x e^{x/2}}{e^x - 1} \right)^{2m+2} + \frac{x}{2} \left(\frac{x e^{x/2}}{e^x - 1} \right)^{2m+2},$$

it can be realized that the second term is odd and does not contribute to the d_{2k} while the first term gives rise to the generalized Bernoulli polynomials:

$$\frac{x}{e^x - 1} \left(\frac{x e^{x/2}}{e^x - 1} \right)^{2m+2} = \left(\frac{x}{e^x - 1} \right)^{2m+3} e^{(m+1)x} = \sum_{k=0}^{\infty} \frac{1}{k!} B_k^{(2m+3)}(m+1) x^k.$$

From this expansion and eq. (C.7) it follows that for every integer $m \geq 0$ the coefficient d_{2m+2} vanishes:

$$d_{2m+2} = \frac{1}{(2m+2)!} B_{2m+2}^{(2m+3)}(m+1) = 0.$$

To prove that d_{2m+2} equates the right hand side of (C.11), we need to write the full power series expansion of f in (C.10), which can be obtained with the product of the series of the two functions enclosed in the brackets of (C.9). The first factor in the right hand side of (C.9) can be expanded as:

$$\frac{x}{e^x - 1} + \frac{x}{2} = \sum_{i=0}^{\infty} \frac{B_{2i}}{(2i)!} x^{2i}; \tag{C.12}$$

while the second factor is the generating function of the generalized Bernoulli polynomials of order $2m+2$:

$$\left(\frac{x e^{x/2}}{e^x - 1} \right)^{2m+2} = \sum_{j=0}^{\infty} \frac{B_{2j}^{(2m+2)}(m+1)}{(2j)!} x^{2j}. \tag{C.13}$$

Hence, by making the Cauchy product of (C.12) and (C.13) in the (C.9) we obtain:

$$f(x) = \sum_{k=0}^{\infty} \sum_{l=0}^k \frac{B_{2l}}{(2l)!} \frac{B_{2k-2l}^{(2m+2)}(m+1)}{(2k-2l)!} x^{2k}.$$

Equating this expression with the (C.10), we obtain the expression (C.11) for the vanishing coefficient d_{2m+2} , which finally proves the identity (C.6).

Taking advantage of the (C.6), the polynomial (C.5) can also be written as:

$$\mathcal{B}_{2m+2}(\phi) = \sum_{i=0}^m \frac{2^{2m+1} B_{2i+2}}{(2i+2)!} \frac{B_{2m-2i}^{(2m+2)}(m+1)}{(2m-2i)!} \phi^{2(m-i)} \left(\phi^{2i+2} - (2\pi i)^{2i+2} \right),$$

which manifestly shows that \mathcal{B}_{2m+2} vanishes at $\phi = \pm 2\pi i$.

D Stress-energy tensor for a massless scalar field in Rindler coordinates

We calculate the thermal expectation value of the stress-energy tensor of the massless scalar field with the density operator (6.2) by using the solutions of the Klein-Gordon equation in Rindler coordinates, in order to compare with the results (6.23). The task will be accomplished by taking advantage of calculations presented in refs. [19, 20].

The symmetries of (6.2) constrain the mean value to be of the form (6.22). On the other hand, from quantum field theory and using the equation of motion for a neutral massless field, the canonical stress-energy tensor may be written as:

$$\hat{T}^{\mu\nu} = \nabla^\mu \hat{\psi} \nabla^\nu \hat{\psi} - \frac{1}{2} g^{\mu\nu} \nabla_\rho \hat{\psi} \nabla^\rho \hat{\psi} = \nabla^\mu \hat{\psi} \nabla^\nu \hat{\psi} - \frac{1}{4} g^{\mu\nu} \square \hat{\psi}^2.$$

We kept the covariant derivatives for future convenience. Combining this expression with (6.22) the following expression are obtained:

$$\rho_R = \langle : u_\mu u_\nu \widehat{T}^{\mu\nu} : \rangle_R = \langle : (u \cdot \nabla \widehat{\psi})^2 : \rangle_R - \frac{1}{4} \square \langle : \widehat{\psi}^2 : \rangle_R, \quad (\text{D.1a})$$

$$p_R - \frac{\mathcal{A}_R}{3} \alpha^2 = -\frac{1}{3} \Delta_{\mu\nu} \langle : \widehat{T}^{\mu\nu} : \rangle_R = \frac{1}{3} \left(\rho_R + \frac{1}{2} \square \langle : \widehat{\psi}^2 : \rangle_R \right), \quad (\text{D.1b})$$

$$-p_R \alpha^2 + \mathcal{A}_R (\alpha^2)^2 = \alpha_\mu \alpha_\nu \langle : \nabla^\mu \widehat{\psi} \nabla^\nu \widehat{\psi} : \rangle_R - \frac{\alpha^2}{4} \square \langle : \widehat{\psi}^2 : \rangle_R, \quad (\text{D.1c})$$

where u^μ is the four-velocity $\beta^\mu / \sqrt{\beta^2}$ and $\alpha^\mu = \sqrt{\beta^2} A^\mu$ where A^μ is the acceleration field, see section 1. Note that in this section the subscript R specifies that normal ordering applies to Rindler's creation and annihilation operators and not to Minkowski's, like it is understood in the rest of the paper. Since they have non-trivial Bogoliubov transformations, the two normal-ordering are not equivalent.

From the four-temperature in (6.3) it ensues (Cartesian components):

$$u^\mu = \frac{1}{k} (z', 0, 0, t), \quad A^\mu = \frac{1}{k^2} (t, 0, 0, z'),$$

where $z' = 1 + az$ and $k = \sqrt{(z'^2 - t^2)}$. The thermal expectation values of the fields can be calculated from the solution of the Klein-Gordon equation in Rindler coordinates. In fact, this calculation has already been partially carried out in [20], where it was found that:

$$\langle : (u \cdot \nabla \widehat{\psi})^2 : \rangle_R = \frac{\pi^2}{30\beta^4}, \quad \square \langle : \widehat{\psi}^2 : \rangle_R = \frac{\alpha^2}{3\beta^4},$$

with $\alpha^2 = \alpha_\mu \alpha^\mu$. With these results, the (D.1) becomes:

$$\rho_R = \frac{\pi^2}{30\beta^4} - \frac{\alpha^2}{12\beta^4}, \quad (\text{D.2a})$$

$$p_R - \frac{\mathcal{A}_R}{3} \alpha^2 = \frac{\pi^4}{90\beta^4} + \frac{\alpha^2}{36\beta^4}, \quad (\text{D.2b})$$

$$-p_R \alpha^2 + \mathcal{A}_R (\alpha^2)^2 = \alpha_\mu \alpha_\nu \langle : \nabla^\mu \widehat{\psi} \nabla^\nu \widehat{\psi} : \rangle - \frac{(\alpha^2)^2}{12\beta^4}, \quad (\text{D.2c})$$

and we just need to calculate one more thermal expectation value. Before doing that, it is useful to remind some useful relations in Rindler coordinates. The relation between Cartesian and Rindler coordinates (called here τ and ξ) is given by:

$$t = \frac{e^{a\xi}}{a} \sinh(a\tau), \quad z' = \frac{e^{a\xi}}{a} \cosh(a\tau).$$

Using these expressions, we can show that:

$$\frac{dx^\mu}{d\xi} = \frac{e^{2a\xi}}{a} A^\mu.$$

Also, in Rindler coordinates, $k^2 = a^{-2}e^{2a\xi}$ and $\beta^2 = e^{-2a\xi}T_0^{-2}$. Now we can work out the thermal expectation value in eq. (D.2c):

$$\langle : (\alpha_\mu \nabla^\mu \hat{\psi})^2 : \rangle_R = \beta^2 \langle : (A_\mu \nabla^\mu \hat{\psi})^2 : \rangle_R = \beta^2 a^2 e^{-4\xi a} \left\langle : \left(\frac{d\hat{\psi}}{d\xi} \right)^2 : \right\rangle_R = -\alpha^2 e^{-2\xi a} \left\langle : \left(\frac{d\hat{\psi}}{d\xi} \right)^2 : \right\rangle_R. \quad (\text{D.3})$$

In the right Rindler wedge, the scalar massless field is given by [37]:

$$\hat{\psi}(\tau, \xi, \mathbf{x}_T)^{(R)} = \int_0^\infty d\omega \int d^2 \mathbf{k}_T u_{\omega, \mathbf{k}_T} a_{\omega, \mathbf{k}_T}^{(R)} + u_{\omega, \mathbf{k}_T}^* a_{\omega, \mathbf{k}_T}^{\dagger (R)}$$

where the eigenfunctions u_{ω, \mathbf{k}_T} read:

$$u_{\omega, \mathbf{k}_T}^{(R)} = \sqrt{\frac{\sinh\left(\frac{\pi\omega}{a}\right)}{4\pi^4 a}} K_{i\frac{\omega}{a}} \left(\frac{k_T e^{a\xi}}{a} \right) e^{i\mathbf{k}_T \cdot \mathbf{x}_T} e^{-i\omega\tau},$$

$K_n(z)$ being the modified Bessel function of the second kind and \mathbf{k}_T and \mathbf{x}_T are the transverse momenta and coordinates. Using the mean value of the number operator in the right Rindler wedge [19]:

$$\langle a_{\omega, \mathbf{k}_T}^{\dagger (R)} a_{\omega', \mathbf{k}'_T}^{(R)} \rangle = \frac{\delta(\omega - \omega') \delta^2(\mathbf{k}_T - \mathbf{k}'_T)}{e^{\frac{\omega}{T_0}} - 1}$$

and the formula:

$$\frac{\partial K_\nu(x)}{\partial x} = -\frac{1}{2} (K_{\nu+1}(x) + K_{\nu-1}(x)),$$

we obtain the integrals:

$$\langle : \left(\frac{d\hat{\psi}}{d\xi} \right)^2 : \rangle_R = \frac{e^{2a\xi}}{4\pi^3 a} \int_0^\infty d\omega \frac{\sinh\left(\frac{\pi\omega}{a}\right)}{e^{\frac{\omega}{T_0}} - 1} \sum_{\pm} \int_0^\infty dk_T k_T^3 K_{\pm 1 + i\frac{\omega}{a}} \left(\frac{k_T e^{a\xi}}{a} \right) K_{\pm 1 - i\frac{\omega}{a}} \left(\frac{k_T e^{a\xi}}{a} \right),$$

where the sum runs over all possible combinations of signs \pm in the order of the Bessel functions. The integration over \mathbf{k}_T can be done by means of the known integral [38]:

$$\begin{aligned} \int_0^\infty dx x^{-\lambda} K_\mu(ax) K_\nu(ax) &= \frac{2^{-2-\lambda} a^{\lambda-1}}{\Gamma(1-\lambda)} \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \\ &\quad \times \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right), \end{aligned}$$

which applies to $\text{Re } a > 0$ and $\text{Re } \lambda < 1 - |\text{Re } \mu| - |\text{Re } \nu|$. After integrating over ω we get:

$$\langle : (\alpha_\mu \nabla^\mu \hat{\psi})^2 : \rangle_R = \frac{\pi^2 \alpha^2}{90\beta^4} - \frac{(\alpha^2)^2}{9\beta^4},$$

and, putting it in the (D.2), we have:

$$\begin{aligned} \rho_R &= \frac{\pi^2}{30\beta^4} - \frac{\alpha^2}{12\beta^4}, \\ p_R - \frac{\mathcal{A}_R}{3} \alpha^2 &= \frac{\pi^4}{90\beta^4} + \frac{\alpha^2}{36\beta^4}, \\ -p_R \alpha^2 + \mathcal{A}_R (\alpha^2)^2 &= -\alpha^2 \left(\frac{\pi^2}{90\beta^4} - \frac{\alpha^2}{9\beta^4} - \frac{\alpha^2}{12\beta^4} \right); \end{aligned}$$

so that, finally:

$$\rho_R = \frac{\pi^2}{30\beta^4} - \frac{\alpha^2}{12\beta^4}, \quad p_R = \frac{\pi^2}{90\beta^4} + \frac{\alpha^2}{18\beta^4}, \quad \mathcal{A}_R = \frac{1}{12\beta^4}.$$

The last step is to calculate the stress-energy tensor with subtraction of the Minkowski vacuum, which was implied in the normal-ordering used throughout the paper. As has been discussed in detail in ref. [19], the subtraction of Minkowski vacuum corresponds to the subtraction of the normally-ordered Rindler expression at the Unruh temperature which is such that $T_U^2 = -A_\mu A^\mu / 4\pi^2$. So, for the Minkowski normally-ordered quantities (the subscript is dropped) we have:

$$\begin{aligned} \rho &= \frac{\pi^2}{30\beta^4} - \frac{\alpha^2}{12\beta^4} - \frac{11\alpha^4}{480\pi^2\beta^4}, \\ p &= \frac{\pi^2}{90\beta^4} + \frac{\alpha^2}{18\beta^4} + \frac{19\alpha^4}{1440\pi^2\beta^4}, \\ \mathcal{A} &= \frac{1}{12\beta^4} + \frac{\alpha^2}{48\pi^2\beta^4}. \end{aligned}$$

which agree with the equations (6.23).

E Analytic distillation of the stress-energy tensor for pure rotation

Here we provide some details about how to derive the stress-energy tensor of the neutral scalar field in the pure rotation case, as it appears in (7.23). For an imaginary angular velocity to temperature ratio $\phi = i\omega/T_0$, by using the equations (6.18), (6.19) and (7.21) one can write each component of the stress-energy tensor as a series. Defining the components Θ with:

$$\langle : \hat{T}_\xi^{\mu\nu}(x) : \rangle_I = \sum_{n=1}^{\infty} \frac{T_0^4 \phi^4}{\pi^2 \left(n^2 \phi^2 + 4 \sin^2 \left(\frac{n\phi}{2} \right) r^2 T_0^2 \phi^2 \right)^3} \Theta_n^{\mu\nu}(x),$$

one obtains, for the non-vanishing ones:

$$\Theta_n^{00}(x) = (n\phi)^2(4\xi+2) + \cos(n\phi) \left[(n\phi)^2(1-4\xi) + 2(8\xi-1)r^2 T_0^2 \phi^2 \right] - \cos(2n\phi)(4\xi-1)r^2 T_0^2 \phi^2 - (12\xi-1)r^2 T_0^2 \phi^2,$$

$$\begin{aligned} \Theta_n^{xx}(x) &= (n\phi)^2(1-2\xi) + 2 \cos(n\phi) \left[(n\phi)^2 \xi + (4\xi-1)T_0^2 \phi^2 (x^2-3y^2) \right] - 2(3\xi-1)T_0^2 \phi^2 (x^2-3y^2) + \\ &+ 2T_0^2 \phi^2 \left[xy(\sin(2n\phi)-2 \sin(n\phi)) - \xi(x^2-3y^2) \cos(2n\phi) \right], \end{aligned}$$

$$\begin{aligned} \Theta_n^{yy}(x) &= (n\phi)^2(1-2\xi) + \cos(n\phi) \left[2(n\phi)^2 \xi - 2(4\xi-1)T_0^2 \phi^2 (3x^2-y^2) \right] + 2(3\xi-1)T_0^2 \phi^2 (3x^2-y^2) + \\ &+ 2T_0^2 \phi^2 \left[xy(2 \sin(n\phi) - \sin(2n\phi)) + \xi(3x^2-y^2) \cos(2n\phi) \right], \end{aligned}$$

$$\Theta_n^{zz}(x) = (n\phi)^2(2-4\xi) - \cos(n\phi) \left[(n\phi)^2(1-4\xi) - 2(4\xi-1)r^2 T_0^2 \phi^2 \cos(n\phi) + 2(8\xi-1)r^2 T_0^2 \phi^2 \right] + 8\xi r^2 T_0^2 \phi^2,$$

$$\Theta_n^{0x}(x) = \Theta_n^{x0}(x) = -2i(n\phi)(T_0\phi) \left[x(\cos(n\phi)-1) - 2y \sin(n\phi) \right],$$

$$\Theta_n^{0y}(x) = \Theta_n^{y0}(x) = 2i(n\phi)(T_0\phi) \left[-2x \sin(n\phi) - y(\cos(n\phi)-1) \right],$$

$$\Theta_n^{xy}(x) = \Theta_n^{yx}(x) = 4T_0^2 \phi^2 \sin^2 \left(\frac{n\phi}{2} \right) \left[4xy(1-2\xi+2\xi \cos(n\phi)) + (x-y)(x+y) \sin(n\phi) \right].$$

As in the case of the series (7.13), the poles of these series cover the whole complex plane. However, we can still obtain a proper asymptotic power series if we restrict ϕ to take on only real values. Notice that the variable ϕ never appears alone but always multiplied by either n or T_0 . So, we can replace ϕ with an auxiliary variable t every time it appears with a coefficient T_0 . By means of this replacement, new functions $R^{\mu\nu}(n\phi, t, x)$ are defined and the thermal expectation values of the stress-energy tensor can be obtained by taking the limit:

$$\langle : \widehat{T}_\xi^{\mu\nu}(x) : \rangle_I = \sum_{n=1}^{\infty} \lim_{t \rightarrow \phi} R^{\mu\nu}(n\phi, t, x),$$

as it was done in the main text for $\langle : \widehat{\psi}^2 : \rangle_I$. Indeed, these series are of the same kind of that in eq. (7.15). Also, it can be realized that for real values of ϕ , t and x they are all uniformly convergent series of continuous functions and, consequently, we can exchange the limit and the sum:

$$\langle : \widehat{T}_\xi^{\mu\nu}(x) : \rangle_I = \lim_{t \rightarrow \phi} \sum_{n=1}^{\infty} R^{\mu\nu}(n\phi, t, x). \quad (\text{E.1})$$

The form (E.1) fulfills the requirements of the theorem 2, which can then be used to obtain the asymptotic expansion of the thermal expectation values around $\phi = 0$. Thereafter, like for the eq. (7.15), the asymptotic expansion is used to obtain the analytic distillate of (E.1) in $\phi = 0$. Finally, the stress-energy tensor components are obtained restoring the physical angular velocity and taking the real part:

$$\begin{aligned} \langle : T^{00}(x) : \rangle &= \frac{\pi^2}{30} \frac{4\gamma^2 - 1}{3} \gamma^4 T_0^4 + \frac{1 - 4\xi}{12} \frac{6\gamma^4 - \gamma^2(24\xi + 1) + 12\xi - 2}{3 - 12\xi} \gamma^4 T_0^2 \omega^2 + \\ &\quad + \frac{4\xi - 1}{48\pi^2} \frac{120\gamma^6 - 3\gamma^4(240\xi + 17) + \gamma^2(720\xi - 58) - 120\xi + 19}{30(1 - 4\xi)} \gamma^4 \omega^4, \\ \langle : T^{xx}(x) : \rangle &= \frac{\pi^2}{90} \gamma^6 T_0^4 + \frac{\xi}{6} \frac{15(\gamma^2 + 4\xi - 1) - 4\pi^2 T_0^2 (x^2 - 3y^2)}{60\xi} \gamma^6 T_0^2 \omega^2 + \\ &\quad - \frac{\xi}{24\pi^2} \frac{3(5\gamma^4 + \gamma^2(30\xi - 9) - 20\xi + 4) + 10\pi^2 T_0^2 (3\gamma^2 + 12\xi - 2)(x^2 - 3y^2)}{30\xi} \gamma^6 \omega^4 + \\ &\quad + \frac{30\gamma^4 + 3\gamma^2(60\xi - 13) - 60\xi + 11}{1440\pi^2} (x^2 - 3y^2) \gamma^6 \omega^6, \\ \langle : T^{yy}(x) : \rangle &= \frac{\pi^2}{90} \gamma^6 T_0^4 + \frac{\xi}{6} \frac{15(\gamma^2 + 4\xi - 1) - 4\pi^2 T_0^2 (y^2 - 3x^2)}{60\xi} \gamma^6 T_0^2 \omega^2 + \\ &\quad - \frac{\xi}{24\pi^2} \frac{3(5\gamma^4 + \gamma^2(30\xi - 9) - 20\xi + 4) + 10\pi^2 T_0^2 (3\gamma^2 + 12\xi - 2)(y^2 - 3x^2)}{30\xi} \gamma^6 \omega^4 + \\ &\quad - \frac{30\gamma^4 + 3\gamma^2(60\xi - 13) - 60\xi + 11}{1440\pi^2} (3x^2 - y^2) \gamma^6 \omega^6, \\ \langle : T^{zz}(x) : \rangle &= \frac{\pi^2}{90} \gamma^4 T_0^4 + \frac{4\xi - 1}{12} \frac{\gamma^2(24\xi - 5) - 12\xi + 2}{3(4\xi - 1)} \gamma^4 T_0^2 \omega^2 + \\ &\quad + \frac{1 - 4\xi}{48\pi^2} \frac{15\gamma^4(48\xi - 11) + \gamma^2(154 - 720\xi) + 120\xi - 19}{30(4\xi - 1)} \gamma^4 \omega^4, \end{aligned}$$

$$\begin{aligned} \langle : T^{0x}(x) : \rangle &= \langle : T^{x0}(x) : \rangle = -\frac{2\pi^2\gamma^6}{45}yT_0^4\omega - \frac{(3\gamma^2-1)\gamma^6}{18}yT_0^2\omega^3 + \frac{(5\gamma^2-4)\gamma^8}{60\pi^2}y\omega^5, \\ \langle : T^{0y}(x) : \rangle &= \langle : T^{y0}(x) : \rangle = \frac{2\pi^2\gamma^6}{45}xT_0^4\omega + \frac{(3\gamma^2-1)\gamma^6}{18}xT_0^2\omega^3 - \frac{(5\gamma^2-4)\gamma^8}{60\pi^2}x\omega^5, \\ \langle : T^{xy}(x) : \rangle &= \langle : T^{yx}(x) : \rangle = -\frac{2\pi^2\gamma^6}{45}T_0^4xy\omega^2 - \frac{(3\gamma^2+12\xi-2)\gamma^6}{18}xyT_0^2\omega^4 + \\ &\quad + \frac{(30\gamma^4+3\gamma^2(60\xi-13)-60\xi+11)\gamma^6}{360\pi^2}xy\omega^6, \end{aligned}$$

where $\gamma = u^0 = 1/\sqrt{1-r^2\omega^2}$. The thermal coefficients defined by the decomposition (7.22), i.e.

$$\begin{aligned} \langle : T^{\mu\nu}(x) : \rangle &= \rho u^\mu u^\nu - p \Delta^{\mu\nu} + W w^\mu w^\nu + \mathcal{A} \alpha^\mu \alpha^\nu + G^l l^\mu l^\nu + G (l^\mu u^\nu + l^\nu u^\mu) + \mathbb{A} (\alpha^\mu u^\nu + \alpha^\nu u^\mu) \\ &\quad + G^\alpha (l^\mu \alpha^\nu + l^\nu \alpha^\mu) + \mathbb{W} (w^\mu u^\nu + w^\nu u^\mu) + A^w (\alpha^\mu w^\nu + \alpha^\nu w^\mu) + G^w (l^\mu w^\nu + l^\nu w^\mu), \end{aligned}$$

can be obtained by projecting the stress-energy tensor along the vectors of the tetrad:

$$\begin{aligned} u^\mu &= \gamma(1, -y\omega, x\omega, 0), & \alpha^\mu &= \left(0, -\frac{\gamma x\omega^2}{T_0}, -\frac{\gamma y\omega^2}{T_0}, 0\right), \\ w^\mu &= \left(0, 0, 0, \frac{\gamma\omega}{T_0}\right), & l^\mu &= \left(\frac{\gamma(\gamma^2-1)\omega^2}{T_0^2}, -\frac{\gamma^3 y\omega^3}{T_0^2}, \frac{\gamma^3 x\omega^3}{T_0^2}, 0\right). \end{aligned}$$

Finally, the scalar functions in the projections are written in terms of the scalars of (7.11), namely:

$$\beta^2(x) = \frac{1}{T_0^2\gamma^2}, \quad \alpha^2(x) = -(\gamma^2-1)\frac{\omega^2}{T_0^2}, \quad w^2(x) = -\gamma^2\frac{\omega^2}{T_0^2}.$$

The final expressions are reported in the equations (7.23).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] F. Becattini, *Covariant statistical mechanics and the stress-energy tensor*, *Phys. Rev. Lett.* **108** (2012) 244502 [[arXiv:1201.5278](https://arxiv.org/abs/1201.5278)] [[INSPIRE](https://inspirehep.net/literature/102000)].
- [2] F. Becattini, L. Bucciattini, E. Grossi and L. Tinti, *Local thermodynamical equilibrium and the beta frame for a quantum relativistic fluid*, *Eur. Phys. J. C* **75** (2015) 191 [[arXiv:1403.6265](https://arxiv.org/abs/1403.6265)] [[INSPIRE](https://inspirehep.net/literature/121000)].
- [3] F. Becattini and E. Grossi, *Quantum corrections to the stress-energy tensor in thermodynamic equilibrium with acceleration*, *Phys. Rev. D* **92** (2015) 045037 [[arXiv:1505.07760](https://arxiv.org/abs/1505.07760)] [[INSPIRE](https://inspirehep.net/literature/134000)].
- [4] M. Buzzegoli and F. Becattini, *General thermodynamic equilibrium with axial chemical potential for the free Dirac field*, *JHEP* **12** (2018) 002 [[arXiv:1807.02071](https://arxiv.org/abs/1807.02071)] [[INSPIRE](https://inspirehep.net/literature/164000)].
- [5] F. Becattini, M. Buzzegoli and E. Grossi, *Reworking the Zubarev's approach to non-equilibrium quantum statistical mechanics*, *Particles* **2** (2019) 197 [[arXiv:1902.01089](https://arxiv.org/abs/1902.01089)] [[INSPIRE](https://inspirehep.net/literature/164000)].

- [6] M. Gransee, N. Pinamonti and R. Verch, *KMS-like Properties of Local Equilibrium States in Quantum Field Theory*, *J. Geom. Phys.* **117** (2017) 15 [[arXiv:1508.05585](#)] [[INSPIRE](#)].
- [7] R. Panerai, *Global equilibrium and local thermodynamics in stationary spacetimes*, *Phys. Rev. D* **93** (2016) 104021 [[arXiv:1511.05963](#)] [[INSPIRE](#)].
- [8] K. Fredenhagen, T.-P. Hack and N. Pinamonti, *Thermodynamics of Quantum Fields in Nonstationary Spacetimes*, in *Particles, Strings and the Early Universe: The Structure of Matter and Space-Time*, J. Haller and M. Greife eds., DESY, Hamburg Germany (2018), pp. 363–378 [[arXiv:1809.08557](#)] [[INSPIRE](#)].
- [9] STAR collaboration, *Global Λ hyperon polarization in nuclear collisions: evidence for the most vortical fluid*, *Nature* **548** (2017) 62 [[arXiv:1701.06657](#)] [[INSPIRE](#)].
- [10] D.N. Zubarev, A.V. Prozorkevich and S.A. Smolyanskii, *Derivation of nonlinear generalized equations of quantum relativistic hydrodynamics*, *Theor. Math. Phys.* **40** (1979) 821.
- [11] C.G. van Weert, *Maximum entropy principle and relativistic hydrodynamics*, *Ann. Phys.* **140** (1982) 133.
- [12] M. Hongo, *Path-integral formula for local thermal equilibrium*, *Annals Phys.* **383** (2017) 1 [[arXiv:1611.07074](#)] [[INSPIRE](#)].
- [13] A. Harutyunyan, A. Sedrakian and D.H. Rischke, *Relativistic Dissipative Fluid Dynamics from the Non-Equilibrium Statistical Operator*, *Particles* **1** (2018) 155 [[arXiv:1804.08267](#)] [[INSPIRE](#)].
- [14] G.Y. Prokhorov, O.V. Teryaev and V.I. Zakharov, *Unruh effect for fermions from the Zubarev density operator*, *Phys. Rev. D* **99** (2019) 071901 [[arXiv:1903.09697](#)] [[INSPIRE](#)].
- [15] G.Y. Prokhorov, O.V. Teryaev and V.I. Zakharov, *Thermodynamics of accelerated fermion gases and their instability at the Unruh temperature*, *Phys. Rev. D* **100** (2019) 125009 [[arXiv:1906.03529](#)] [[INSPIRE](#)].
- [16] G.Y. Prokhorov, O.V. Teryaev and V.I. Zakharov, *Unruh effect universality: emergent conical geometry from density operator*, *JHEP* **03** (2020) 137 [[arXiv:1911.04545](#)] [[INSPIRE](#)].
- [17] P. Kovtun, *Thermodynamics of polarized relativistic matter*, *JHEP* **07** (2016) 028 [[arXiv:1606.01226](#)] [[INSPIRE](#)].
- [18] P. Kovtun and A. Shukla, *Kubo formulas for thermodynamic transport coefficients*, *JHEP* **10** (2018) 007 [[arXiv:1806.05774](#)] [[INSPIRE](#)].
- [19] F. Becattini, *Thermodynamic equilibrium with acceleration and the Unruh effect*, *Phys. Rev. D* **99** (2018) 085013 [[arXiv:1712.08031](#)] [[INSPIRE](#)].
- [20] F. Becattini and D. Rindori, *Extensivity, entropy current, area law and Unruh effect*, *Phys. Rev. D* **99** (2019) 125011 [[arXiv:1903.05422](#)] [[INSPIRE](#)].
- [21] V.E. Ambrus, *Quantum non-equilibrium effects in rigidly-rotating thermal states*, *Phys. Lett. B* **771** (2017) 151 [[arXiv:1704.02933](#)] [[INSPIRE](#)].
- [22] S.D. Groot, W. van Leeuwen and C. van Weert, *Relativistic Kinetic Theory*, North Holland, Amsterdam The Netherlands (1980).
- [23] F. Becattini and L. Tinti, *Thermodynamical inequivalence of quantum stress-energy and spin tensors*, *Phys. Rev. D* **84** (2011) 025013 [[arXiv:1101.5251](#)] [[INSPIRE](#)].
- [24] M.P. Heller, A. Serantes, M. Spaliński, V. Svensson and B. Withers, *The hydrodynamic gradient expansion in linear response theory*, [arXiv:2007.05524](#) [[INSPIRE](#)].

- [25] A. Erdelyi, *Asymptotic expansions*, Dover Publications (2010).
- [26] D. Zagier, *The Mellin transform and related analytic techniques*, in *Quantum Field Theory I: Basics in Mathematics and Physics. A Bridge Between Mathematicians and Physicists*, Springer (2006), pp. 305–323 and online pdf version at <http://people.mpim-bonn.mpg.de/zagier/files/tex/MellinTransform/fulltext.pdf>.
- [27] D. Dorigoni and A. Kleinschmidt, *Resurgent expansion of Lambert series and iterated Eisenstein integrals*, *Commun. Num. Theor. Phys.* **15** (2021) 1 [[arXiv:2001.11035](#)] [[INSPIRE](#)].
- [28] J.I. Korsbakken and J.M. Leinaas, *The Fulling-Unruh effect in general stationary accelerated frames*, *Phys. Rev. D* **70** (2004) 084016 [[hep-th/0406080](#)] [[INSPIRE](#)].
- [29] M. Buzzegoli, E. Grossi and F. Becattini, *General equilibrium second-order hydrodynamic coefficients for free quantum fields*, *JHEP* **10** (2017) 091 [*Erratum* *JHEP* **07** (2018) 119] [[arXiv:1704.02808](#)] [[INSPIRE](#)].
- [30] J.S. Dowker, *Quantum Field Theory on a Cone*, *J. Phys. A* **10** (1977) 115 [[INSPIRE](#)].
- [31] L.D. Landau and E.M. Lifshitz, *Statistical physics. Part 1*, in *Course of Theoretical Physics. Volume 5*, third edition, Pergamon (1969) [[INSPIRE](#)].
- [32] A. Vilenkin, *Quantum field theory at finite temperature in a rotating system*, *Phys. Rev. D* **21** (1980) 2260 [[INSPIRE](#)].
- [33] V.E. Ambrus, *Dirac fermions on rotating space-times*, Ph.D. Thesis, University of Sheffield, School of Mathematics and Statistics, Sheffield U.K. (2014).
- [34] N.E. Nörlund, *Vorlesungen über Differenzenrechnung*, in *Die Grundlehren der Mathematischen Wissenschaften* **13**, Springer (1924).
- [35] A.P. Prudnikov, Y.A. Brychkov, O.I. Marichev and N.M. Queen, *Integrals and Series. Volume 1. Elementary functions*, first edition, CRC (1998).
- [36] Y.L. Luke, *The special functions and their approximations*, in *Mathematics in Science and Engineering* **53–1**, Academic Press (1969).
- [37] L.C.B. Crispino, A. Higuchi and G.E.A. Matsas, *The Unruh effect and its applications*, *Rev. Mod. Phys.* **80** (2008) 787 [[arXiv:0710.5373](#)] [[INSPIRE](#)].
- [38] I.S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products*, seventh edition, Academic Press, Amsterdam The Netherlands (2007) [<https://doi.org/10.1016/C2009-0-22516-5>].