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On the Interiors of Production Sets in Infinite Dimensional Spaces

M. Ali Khan
N. T. Peck

College of Commerce and Business Administration
Bureau of Economic and Business Research
University of Illinois, Urbana-Champaign

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M. Ali Khan, Professor
Department of Economics

N. T. Peck
Department of Mathematics

On the Interiors of Production Sets in
Infinite Dimensional Spaces†

by

M. Ali Khan* and N. T. Peck**

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Abstract. We show that "bounded marginal rates of substitution" as formalized by Khan-Vohra imply that a closed, convex production set with "free disposal" has a nonempty interior. This result is true for Banach spaces but false for more general locally convex spaces.

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*Department of Economics, University of Illinois, 1206 South Sixth Street, Champaign, IL 61820.

**Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801.

1. Introduction

The importance of the Hahn-Banach Theorem for the second fundamental theorem of welfare economics on the decentralization of Pareto optimal allocations as price equilibria is, by now, well understood. In the setting of economies with a finite dimensional commodity space, the essential hypothesis on preferences and production sets is that of convexity, as in Arrow [2] and Debreu [5]. However, in the infinite dimensional case, this needs to be supplemented by an interiority condition on the set to be supported. If the commodity space is equipped with an order structure and is one whose positive cone has a nonempty interior, the interiority condition follows from economically innocuous assumptions such as "free disposal" or "desirability"; see Debreu [6] and Bewley [3]. If the positive cone does not have a nonempty interior, as in $L^p(\mu)$, $\infty > p > 1$, or in the space of regular measures on a compact Hausdorff space, additional assumptions have to be made on preferences and production sets if one is to avoid assuming the interiority condition at the outset. For economies with production, as is our concern here, a variety of such assumptions have been made on production sets; see the work of Aliprantis-Brown-Burkinshaw [1], Mas-Colell [12], Khan-Vohra [9] and Zame [14]. These assumptions have been seen as formalizations of "bounded marginal rates of substitution" in production and in the case of [1], [12] and [14], have been directly inspired by Mas-Colell's [11] concept of "properness."

In this note, we focus on the condition proposed in [9]. This is an obvious formalization of bounded marginal rates of substitution and simply requires that the set of supporting functionals to a production set be bounded below, in terms of the induced order, by a nonzero positive functional. Note that the condition does not require the production set to have any support points and it is vacuously fulfilled for sets without any such points. Under this condition on a single production set, Khan-Vohra [9] present an infinite dimensional version of the Arrow-Debreu second fundamental theorem. However, Khan-Vohra do not present any example of a production set with "free disposal," which satisfies their condition but does not possess an empty interior. Our first result is that in an ordered Banach space such sets do not exist. Our result is a consequence of the Bishop-Phelps Theorem and can be restated to say that in an ordered Banach space, order boundedness from below of supporting functionals forces closed, convex sets containing the negative orthant to have a nonempty interior. We also present an extension of our result to weak * closed convex sets in a dual Banach space. This extension prompted us to ask whether our result itself generalizes to locally convex spaces. We present three examples in spaces of interest in mathematical economics for which this conjecture is false. These three counterexamples are the second contribution of this note.

It is worth stating that, in terms of the economics, our result can be more constructively viewed as providing a sufficient condition for the validity of Debreu's [6] theorem in ordered Banach spaces whose positive orthant has an empty interior. Furthermore, our examples bring out that the second welfare theorem presented in [9] has to be evaluated in the context of ordered locally convex spaces which are not Banach spaces. Finally, it is of some interest that our result allows us to make the observation that the technology in Zame's [14] fourth example has a nonempty interior.

The next section presents the result, Section 3 the examples and Section 4 concludes the paper by relating our work to the examples presented in Zame [14].

2. The Result

Let X be a Banach space over the reals and with X^* its dual. The norm in X and in X^* will be denoted by $\| \;\|$. We shall assume that X is ordered by $\geq$ and denote the positive cone by X_+. X^*_+ will denote the positive cone of X^* with the order on X^* induced by $\geq$. For any $x \in X$ and any $f \in X^*$, we shall denote the value by $f(x)$.

We shall denote the weak topology on X by $\sigma(X,X^*)$ and the weak * topology on X^* by $\sigma(X^*,X)$. For any nonempty subset C of X (or of X^*) and for any $f \in X^*$ (or of X), let

$$\alpha(f,C) = \sup_{x \in C} f(x).$$

For any $C \subset X$, C^0 will denote the norm-interior of C.

We can now present

<u>Theorem 1</u>. <u>Let</u> X <u>be an ordered Banach space with</u> X_+ <u>its positive cone</u>. <u>Let</u> C <u>be a closed convex subset of</u> X <u>such that</u>

(i) $-X_+ \subset C$,

(ii) $\exists\, g \in X_+$, $g \neq 0$ <u>such that</u> $f \in X^*$, f <u>supports</u> C, $\| f \| = 1$ <u>and</u> $\alpha(f,C) < \infty$ implies $f \geq g$.

Then $C^0 \neq \phi$.

Proof. By the Bishop-Phelps Theorem [4, Corollary 2]

$$C = \bigcap_{f \in S} H_f, \text{ where}$$

$$S = \{f \in X^*: \ \alpha(f,C) < \infty, \ \| f \| = 1 \text{ and } f \text{ supports } C\}$$

$$H_f = \{x \in X: \ f(x) \leq \alpha(f,C)\}.$$

Since $g \neq 0$, there exists $x_0 \in X_+$ such that $g(x_0) > 0$. Let $\varepsilon = g(x_0)/2$. We shall show that $\{y \in X: \ \| -x_0 - y \| < \varepsilon \} \subset C$. It suffices to show that $y \in H_f$ for any $f \in S$. Thus pick any $f \in S$. Then $f(-y) = f(x_0) + f(-y-x_0) \geq g(x_0) + f(-y-x_0)$. Since $|f(-y-x_0)| \leq \| f \| \| -y-x_0 \| \leq \varepsilon$, certainly $f(-y-x_0) \geq -\varepsilon$. Hence $f(-y) \geq \varepsilon$. By linearity $f(y) \leq -\varepsilon$. Since $0 \in C$, $\alpha(f,C) \geq 0$. Hence $f(y) < \alpha(f,C)$ or $y \in H_f$. We are done. ||

Remark. Note that $f \in S \Rightarrow f \geq 0$. (Suppose there exists $x_0 \in X_+$ such that $f(x_0) < 0$. Then $f(-x_0) > 0$. Since $-X_+ \subset C$, we can contradict $\alpha(f,C) < \infty$.) However, we do not use this fact in the proof.

Theorem 1 admits of the following extension.

Theorem 2. Theorem 1 is true with X and X_+ interchanged with X^* and X_+^* and weak * closed substituted for closed.

Proof. Simply use Phelps' Theorem [13, Corollary 2] instead of the Bishop-Phelps Theorem. ||

Theorem 2 leads us to the following conjecture.

<u>Conjecture</u>. <u>Let</u> X^* <u>be a dual Banach space with</u> X <u>its predual and with</u> X_+^* <u>and</u> X_+ <u>their respective positive cones. Let</u> C <u>be a weak * closed, convex set in</u> X^* <u>such that</u>

(i) $-X_+^* \subset C$

(ii) $\exists$ $g \in X_+$, $g \neq 0$ <u>such that</u> $f \in X$, $\| f \| = 1$ <u>and</u> $\alpha(f,C) < \infty \Rightarrow f \geq g$. <u>Then weak * interior</u> $C \neq \phi$.

In the next section we shall exhibit three examples for which this is false. Note that unlike the theorem, the conjecture does not even require that f support C.

3. <u>Three Examples</u>

We now present examples of three production sets for each of which the conjecture is false. Furthermore, the sets also satisfy the economic assumption of "irreversibility," i.e., $Y \cap \{-Y\} = \{0\}$.

<u>Counterexample 1</u>

For our first counterexample we work in the dual pair $[\ell^1, c_0]$. Let

$$Y_i = \{y \in \ell^1: \; y_1 + y_i \leq 0\} \qquad i = 2, 3, \ldots,$$

$$Y_1 = \{y \in \ell^1: \; y_1 \leq 0\},$$

$$Y = \bigcap_{i \in N} Y_i$$

(For any $y \in \ell^1$, y_i denotes the i^{th} coordinate).

Claim 1. Y has an empty $\sigma(\ell^1, c_0)$-interior.

Proof. Suppose to the contrary that e is such an interior point. Then there exists a weak * open set containing e and contained in Y. Hence there exist a postive integer k, positive numbers ε_α and $f^\alpha \in c_0$ $(\alpha=1,\ldots,k)$ such that

$$\{y \in \ell^1 : \ |f^\alpha(y-e)| \leq \varepsilon_\alpha, \ \alpha = 1, \ldots k\} \subset Y.$$

Certainly, $e_1 < 0$. Indeed, if $e_1 = 0$ pick $\hat{y} \in \ell^1$ such that

$$\hat{y}_1 = \varepsilon/2M$$

$$y_i = e_i \quad (i \neq 1)$$

where

$$\varepsilon = \underset{\alpha}{\mathrm{Min}}\ \varepsilon_\alpha, \ M = \underset{\alpha}{\mathrm{Max}} \| f^\alpha \|.$$

Then $\| \hat{y} - e \| = \varepsilon/2M$. Hence for any α, $|f^\alpha(y-e)| \leq \| f^\alpha \| \varepsilon/2M \leq \varepsilon/2$. But $\hat{y} \notin Y_1$, and hence $\hat{y} \notin Y$.

Now pick $\hat{n} \in N$ such that for any α, $|f^\alpha_{\hat{n}}| < (-\varepsilon/4e_1)$. Since f^α_j converge to zero and since there are only a finite number of f^α, such an $\hat{n}$ can be found.

Since $e \in \ell_1$, $\Sigma|e_i| < \infty$. Hence we can find $\bar{n} \in N$ such that $|e_{\bar{n}}| < (-2e_1)$.

Let $n = \mathrm{Max}\ (\hat{n}, \bar{n})$ and

$$y^*_i = e_i \qquad (i \neq n)$$

$$y_n^* = -2e_1.$$

Since $e \in \ell^1$, $y^* \in \ell^1$. Also $y^* \notin Y_n$ and therefore $y^* \notin Y$ since $y_1^* + y_n^* = -e_1 > 0$.

For any α,

$$\begin{aligned} |f^\alpha(y^*-e)| &= |f_n^\alpha(-2e_1-e_n)| \\ &\leq |f_n^\alpha(-2e_1)| + |f_n^\alpha(e_n)| \\ &\leq \varepsilon/2 + \varepsilon/2 \leq \varepsilon_\alpha. \end{aligned}$$

We have a contradiction. ||

Next we have

<u>Claim 2</u>. Y is a $\sigma(\ell^1, c_0)$-closed, convex set such that $(-\ell_+^1) \subset Y$ and $Y \cap (-Y) = \{0\}$.

<u>Proof</u>. Since the intersection of convex sets is convex, convexity is obvious. Each $Y_i (i \neq 1)$ is $\sigma(\ell^1, c_0)$-closed since it is a closed half-space defined by an element of c_0. Again, as intersection of weak * closed sets, Y is weak * closed. The fact that $(-\ell_+^1) \subset Y$ is obvious. Finally, pick $y \in Y$. Then $y \in Y_i$ for all i. Hence $y_1 + y_i \leq 0$. If $y \in (-Y)$, then $-(y_1+y_i) \leq 0$. Hence $y_i \leq -y_1$ and $y_i \geq -y_1$ and therefore $y_i = y_1$ for all i. But $y \in Y$ and $y \in (-Y)$ implies $y_1 = 0$. Hence $y = 0$.

<u>Claim 3</u>. $f \in c_0$, $\| f \| = 1$, $\alpha(f,Y) < \infty$ implies $f \geq (1, 0, 0, \ldots 0)$.

Proof. Suppose $f_1 = 0$. Then there exists $j \in N$ such that $f_j > 0$ where N denotes the set of positive integers. Let

$$\hat{y}_i^n = 0 \quad (i \neq j, 1)$$

$$\hat{y}_1^n = -n \text{ and } \hat{y}_j^n = n \text{ for } n \in N.$$

Certainly $\hat{y}^n \in Y$ for all $n \in N$. $f(\hat{y}^n) = nf_j$. By choosing n large enough we can show $f(\hat{y}^n) > \alpha(f,Y)$.

Now suppose $f_1 < 1$. Pick $\varepsilon > 0$ such that $f_1 < 1-\varepsilon$. Then there exists $j \in N$ such that $f_j > 1-\varepsilon$. Now let

$$y_1 = \frac{-f_j}{f_1}, \; y_j = \frac{1-\varepsilon}{f_1}, \; y_i = 0 \qquad (i \neq 1, j).$$

Since $y_1 + y_j = \frac{1}{f_1}(1-\varepsilon-f_j) < 0$, $y \in Y_j$. Since $y \in Y_1$, and also $y \in Y_i (i \neq 1, j)$, $y \in Y$. Now $f(y) = -f_j + \frac{(1-\varepsilon)f_j}{f_1} = \frac{f_j}{f_1}(1-\varepsilon-f_1) > 0$. Since Y is a cone, $\alpha(f,Y) = \infty$, a contradiction. ||

Counterexample 2

We shall work in the dual pair $[\ell^\infty,\ell^1]$. Bewley [3] was the first to work with the pair $[L^\infty(\mu),L^1(\mu)]$. Let Y be as defined in Counterexample 1 but regarded as a subset of ℓ^∞.

Claim 1. Y has an empty $\sigma(\ell^\infty,\ell^1)$-interior.

Proof. Suppose to the contrary that e is an interior point of Y. Then there exist a positive integer k, positive numbers ε_α and $f^\alpha \in \ell^1$ $(\alpha=1,\ldots,k)$ such that

$$\{y \in \ell^\infty : |f^\alpha(y-e)| < \varepsilon_\alpha, \alpha = 1, \ldots k\} \subset Y.$$

Certainly $e_1 \leq 0$. If $e_1 = 0$, pick $\hat{y} \in \ell^\infty$ such that $\hat{y}_1 = \varepsilon/2M$ and $\hat{y}_i = e_i (i \neq 1)$ where $\varepsilon = \underset{\alpha}{\text{Min}}\ \varepsilon_\alpha$ and $M = \underset{\alpha}{\text{Max}} \| f^\alpha \|$. Then for any α, $|f^\alpha(\hat{y}-e)| = \frac{\varepsilon}{2M} |f_1^\alpha| \leq \varepsilon/2 < \varepsilon_\alpha$. But $\hat{y}_1 > 0$ and hence $\hat{y} \notin Y_1$.

Since $f^\alpha \in \ell^1$, there exists $n \in N$ such that $\sum_n^\infty |f_i^\alpha| < \varepsilon/3 \| e \|_\infty$ for all α. Hence, for all α, certainly $|f_n^\alpha| < \varepsilon/3 \| e \|_\infty$.

Now let $y_i = e_i$ $(i \neq n)$ and $y_n = -2e_1$. Since $y_1 + y_n = e_1 - 2e_1 = -e_1 > 0$, $y \notin Y_n$ and hence $y \notin Y$. For any α,

$$\begin{aligned} |f^\alpha(y-e)| &= |f_n^\alpha(-2e_1 - e_n)| \\ &\leq |f_n^\alpha|(-2e_1)| + |f_n^\alpha|\ |e_n| \\ &\leq \varepsilon \leq \varepsilon_\alpha . \end{aligned}$$

We are done.

Claim 2. Y is weak * closed, convex and contains $(-\ell_+^\infty)$ and $Y \cap (-Y) = 0$.

Proof. As easy as the proof of Claim 2 of Counterexample 1.

Claim 3. $f \in \ell_1$, $\| f \| = 1$, $\underset{x \in Y}{\text{Sup}}\ f(x) = \alpha_f < \infty$ implies $f \geq (1/2, 0, \ldots, 0)$.

Proof. Pick an f as in Claim 3. Suppose $f_1 = 0$. Follow the argument as in Counterexample 1.

Suppose $0 < f_1 < 1/2$. Since $\| f \| = 1$, $\sum_{i=2}^{\infty} |f_i| = 1 - |f_1| = 1 - f_1$. Since $-\ell_+^{\infty} \subset Y$, certainly $f_i \geq 0$. Now define $\hat{y}_i = 1/f_1$ $(i \neq 1)$ and $\hat{y}_1 = \frac{-1}{f_1}$. Certainly $\hat{y} \in \ell_\infty$ and $\hat{y} \in Y$.

$$f(\hat{y}) = -1 + \sum_2^{\infty} f_i/f_1 = \frac{1}{f_1}(\sum_2^{\infty} f_i - f_1) = \frac{1-2f_1}{f_1} > 0.$$

Since $y \in Y \Rightarrow \lambda y \in Y$ for all $\lambda \geq 0$, $f(\lambda\hat{y}) = \lambda f(\hat{y})$ and hence $\alpha(f,Y) < \infty$ can be contradicted. ‖

Counterexample 3

Let $\mathcal{M}$ denote the space of signed regular Borel measures on [0,1] and $\mathcal{C}$ the space of (bounded) continuous real-valued functions on [0,1]. We shall work in the dual pair $[\mathcal{M}, \mathcal{C}]$. For any $f \in \mathcal{C}$ and $t \in [0,1]$, we shall denote the value of f at t by f(t) and use the notation $f[\mu]$, $\mu \in \mathcal{M}$ for the canonical pairing. $\delta_{\{x\}}$ will denote the Dirac measure at $\{x\}$. It is worth pointing out that Mas-Colell [10] was the first to work with a commodity space modelled on the space of signed regular Borel measures on a compact Hausdorff space and endowed with the weak * topology.

We consider the following set:

$$Y = \{\mu \in \mathcal{M} : f[\mu] \leq 0 \text{ whenever } \| f \| = 1,\ f \geq 0,\ f(1) = 1\}.$$

Claim 1. Y has an empty $\sigma(\mathcal{M}, \mathcal{C})$-interior.

Proof. Suppose to the contrary that e is a weak * interior point of Y. Then there exist a positive integer k, positive numbers ε_α and $f^\alpha \in \mathcal{C}$ $(\alpha=1,\ldots,k)$ such that

$$\rho \in W \equiv \{\mu \in \mathcal{M} : f^{\alpha}[\mu] < \varepsilon_{\alpha},\ \alpha = 1, \ldots, k\}$$

implies $(\rho+e) \in Y$. As above, we shall denote $\operatorname{Min}_{\alpha} \varepsilon_{\alpha}$ by ε and $M = \operatorname{Max}_{\alpha} \| f^{\alpha} \|$. Certainly $M > 0$. If $M = 0$, then $f^{\alpha}(t) = 0$ for all $t \in [0,1]$ and for all α. But in this case $(n\delta_{\{1\}}) \in W$ for all positive integers n. Since $(n\delta_{\{1\}}+e) \in Y$, we obtain $e([0,1]) + n \leq 0$ for all n which is an absurdity.

Observe that $e([0,1]) < 0$. Since $e \in Y$, $\int_{[0,1]} de \leq 0$. Hence $e([0,1]) \leq 0$. Suppose $e([0,1]) = 0$. Let $\sigma = (\varepsilon/M)\delta_{\{1\}}$. Then $|f^{\alpha}[\sigma]| = |(\varepsilon/M)f^{\alpha}(1)| < \varepsilon \leq \varepsilon_{\alpha}$. But $\int_{[0,1]} d(\sigma+e) = e[0,1] + (\varepsilon/M) = (\varepsilon/M) > 0$, which is a contradiction to the fact that $(e+\sigma) \in Y$.

Now pick any $t \in [0,1]$, $t \neq 0, 1$ and let

$$\beta = 1 + e^{-}(t) + e^{-}(1)$$

where e^{+} and e^{-} are respectively the positive and negative parts of e, i.e., $e = e^{+} - e^{-}$. Since f^{α} is a continuous function, we can find an open set V^{α} in $[0,1]$ which contains t and is such that $f^{\alpha}(V^{\alpha}) \subset f^{\alpha}(t) + B_{\varepsilon/\beta}(0)$. ($B_{\varepsilon}(0)$ is the open ε-ball around 0.) Let $V = \bigcap V^{\alpha}$. Certainly V is open and contains t. Pick $s \in V$, $s \neq t$ and let $\rho = \beta(\delta_{\{t\}}-\delta_{\{s\}})$. Certainly $\rho \in \mathcal{M}$. Furthermore, for any α,

$$|f^{\alpha}[\rho]| = |f^{\alpha}[\beta(\delta_{\{t\}}-\delta_{\{s\}})]| = |\beta|\ |f^{a}(t) - f^{\alpha}(s)| < \varepsilon < \varepsilon_{\alpha}$$

which shows that $\rho \in W$. Let $T_n = B_{1/n}(t) \cup (B_{1/n}(1) \cap [0,1])$. Certainly there exists a positive integer $\bar{n}$ such that $s \notin T_n$ for all $n \geq \bar{n}$. Henceforth, n will refer to integers greater than $\bar{n}$. For any $A \subset [0,1]$, let A^c denote the complement of A in $[0,1]$. Then $T_n^c =$

$(B_{1/n}(t))^c \cap [0,1-(1/n)]$. Certainly T_n^c is a closed set disjoint from the closed set $\{t\} \cup \{1\}$. We can construct the following "broken-line" continuous non-negative function f_n such that $\|f_n\| = 1$ and $f_n(1) = 1$.

$$\begin{aligned} f_n(x) &= 0 && x \in T_n^c \\ &= nx + (1-n) && (1-(1/n)) \leq x \leq 1 \\ &= nx + (1-nt) && (t-(1/n)) \leq x \leq t \\ &= -nx + (1+nt) && t \leq x \leq (t+(1/n)). \end{aligned}$$

We can now show that for large enough n the set Y does not contain $(\rho+e)$. Towards this end, note to begin with, $T_n \supset T_{n+1} \supset \ldots\ldots (\{t\} \cup \{1\})$ and hence $e^-(T_n) \to e^-(\{t\} \cup \{1\}) = e^-(t) + e^-(1)$. Now,

$$\begin{aligned} f_n[\rho+e] &= f_n[\rho] + f_n[e] \\ &= f_n[\beta\delta_{\{t\}}] - f_n[\beta\delta_{\{s\}}] + f_n[e] \\ &= \beta + \int_{T_n} f_n(x)de \\ &= 1 + e^-(t) + e^-(1) + \int_{T_n} f_n(x)de^+ - \int_{T_n} f_n(x)de^- \\ &\geq 1 + \int_{T_n} f_n(x)de^+ + e^-(t) + e^-(1) - e^-(T_n). \end{aligned}$$

For large enough n, we obtain $f_n[\rho+e] > 0$ and hence a contradiction to the fact that $(\rho+e) \in Y$.

Next we have

Claim 2. Y is a $\sigma(\mathcal{M}, \mathcal{C})$-closed, convex set such that $(-\mathcal{M}_+) \subset Y$ and $Y \cap (-Y) = \{0\}$.

Proof. We only prove the third assertion. Suppose μ and $(-\mu)$ are in Y. Then for all f such that $\| f \| = 1$, $f \geq 0$, $f(1) = 1$, $f[\mu] \leq 0$ and $f[-\mu] \leq 0$ which implies $f[\mu] = 0$.

We first claim that $\mu(\{1\}) = 0$. Indeed, let $f_n \in \mathcal{C}$, $0 \leq f_n \leq 1$, $f(1) = 1$, and f_n converges pointwise to the characteristic function of $\{1\}$. Then for each n, $f_n[\mu] = 0$; on passage to the limit, $\mu\{1\} = 0$.

Now let A be any closed subset of [0,1] not containing 1. Using Urysohn's Lemma [8, p. 146], construct a sequenc (f_n) in $\mathcal{C}$, $0 \leq f_n \leq 1$, $f_n \equiv 1$ on $A \cup \{1\}$, and f_n converges pointwise to the characteristic function of $A \cup \{1\}$. Then for each n, $f_n[\mu] = 0$; on passage to the limit, $\mu(A) = 0$ (using that $\mu\{1\} = 0$). This proves that $\mu \equiv 0$.

Finally we have

Claim 3. $f \in \mathcal{C}$, $\| f \| = 1$, $\alpha(f,Y) < \infty$ implies $f(1) = 1$.

Proof. Suppose there exists $f \in \mathcal{C}$ such that $\| f \| = 1$, $\alpha(f,Y) < \infty$ and $f(1) < 1$. As in the Remark in Section 2, certainly $f \geq 0$. Since $\| f \| = 1$, there exists t in [0,1] such that $f(t) = 1$. Let $\mu = -\delta_{\{1\}} + \delta_{\{t\}}$. Certainly $\mu \in Y$. But $f[\mu] = (f(t) - f(1)) \equiv \beta > 0$. Since $\mu \in Y$ implies $k\mu \in Y$ for all $k > 0$, $f[k\mu] = k\beta$. This contradicts the fact that $\alpha(f,Y) < \infty$ and completes the proof of the claim.

||

4. The Examples of Zame

In [14], Zame has presented a theorem on the existence of competitive equilibrium under a condition on each of the production sets that can be seen as an alternative formalization of "bounded marginal rates of substitution." Zame also presents four examples, the first three of which do not satisfy his condition and in which there does not exist any equilibrium. It is of some interest that the production sets in these examples also do not satisfy the condition studied in this paper. The fourth example of Zame, in which an equilibrium does exist, has a production set that also satisfies our condition and has a nonempty interior by virtue of our Theorem 1. We devote this section to the establishment of these assertions.

Let N denote the set of positive integers.

Example 1. This is set in the dual pair $[\ell^1, \ell^\infty]$. Let $e^i \in \ell^1$ be such that

$$e^i_i = -1$$

$$e^i_{i+1} = 2$$

$$e^i_j = 0 \qquad (j \neq i,\ i + 1)$$

Let

$$Z = \Big(\bigcup_{n \in N} e^{2n-1}\Big) \cup (-\ell^1_+)$$

$$\tilde{Y} = \overline{\mathrm{co}Z}$$

$$Y = \{\lambda y:\ y \in \tilde{Y},\ \lambda \geq 0\}.$$

Claim. For any $j \in N$, there exists $f \in \ell^\infty$, $f \geq 0$ $\| f \| = 1$, $\alpha(f,Y) < \infty$ and $f_j = 0$.

Proof. In the first case, pick $j \in N$ not a power of 2. Pick $\tilde{f} \in \ell^{\infty}$ such that

$$\tilde{f}_j = 0$$

$$\tilde{f}_{j+2} = 1$$

$$\tilde{f}_i = 0 \qquad (i \neq j,\ j+2).$$

We shall show that $\alpha(\tilde{f},Y) = 0$.

Since $0 \in Y$, certainly $\alpha(\tilde{f},Y) \geq 0$. Suppose $\alpha(\tilde{f},Y) > 0$. Then certainly there exists $y \in \tilde{Y}$ such that $y_j > 0$. There exists $z^{\nu} \in \overline{\text{co}Z}$ such that $z^{\nu} \to y$, i.e., for any $\varepsilon > 0$, there exists $\overline{\nu}$ such that $\nu \geq \overline{\nu}$ implies $\sum_{i \in N} |z_i^{\nu} - y_i| < \varepsilon$. This implies that for large enough ν, $z_j^{\nu} > 0$. But then this implies the existence of an element in Z with its j^{th} coordinate positive. Since j is not a power of 2, this is impossible.

Now suppose that j is a power of 2. Pick $f \in \ell^{\infty}$ such that

$$\tilde{f}_j = 0$$

$$\tilde{f}_{j+1} = 1$$

$$\tilde{f}_i = 0 \qquad (i \neq j,\ j+2).$$

We can show that $\alpha(\tilde{f},Y) = 0$ exactly as above. ||

Example 2. In this example there are two production sets in $[\ell^1,\ell^{\infty}]$. The first is identical to that in the first example. The second is a slight variant. Let $e^i \in \ell^1$ be such that

$$e^i_i = -1$$

$$e^i_{i+1} = +1$$

$$e^i_j = 0 \qquad (j \neq i,\ i+1).$$

Now substitute

$$Z = (\bigcup_{n \in N} e^{2n}) \cup (-\ell^1_+)$$

in the definition of $\tilde{Y}$ and Y. We can again claim

<u>Claim</u>. For any $j \in N$, there exists $f \in \ell^\infty$, $\| f \| = 1$, $\alpha(f,Y) < \infty$ and $f_j = 0$.

<u>Proof</u>. Define $\tilde{f} \in \ell^\infty$ by

$$\tilde{f}_j = 0$$

$$\tilde{f}_{j+1} = \begin{cases} 1 & \text{if } j \text{ is not a power of } 2 \\ 0 & \text{if } j \text{ is a power of } 2 \end{cases}$$

$$\tilde{f}_{j+2} = \begin{cases} 0 & \text{if } j \text{ is not a power of } 2 \\ 1 & \text{if } j \text{ is a power of } 2 \end{cases}$$

$$\tilde{f}_i = 0 \qquad (i \neq j,\ j+1,\ j+2).$$

The argument is now identical to Example 1. ||

<u>Example 3</u>. The example is set in $[\ell^1, \ell^\infty]$. Let

$$e^i_i = 2^{-i+1}$$

$$e^i_{i+1} = -2^{-i}$$

$$e^i_j = 0 \qquad (j \neq i,\ i+1)$$

$$Z = (\bigcup_{n \varepsilon N} e^n) \cup (-\ell^1_+)$$

$$Y = \overline{co}Z.$$

Claim. For any $j \varepsilon N$, there exists $f \varepsilon \ell^\infty$ such that $\| f \| = 1$, $f \geq 0$, $\alpha(f,Y) < \infty$ and $f_j = 0$.

Proof. Choose any $\tilde{f} \varepsilon \ell^\infty$ such that $\tilde{f}_{j+1} = 1$ and $\tilde{f}_i = 0$ $(i \neq j + 1)$. Pick any $y^* \varepsilon Y$. Then $\exists\ w^\nu \varepsilon$ coZ such that $w^\nu \to y^*$. But each $w^\nu = \sum_{i=1}^{k^\nu} \lambda_i^\nu z_i^\nu$ with $\lambda_i^\nu \geq 0$, $\sum_{i=1}^{k^\nu} \lambda_i^\nu = 1$, and $z^\nu(i) \varepsilon Z$. Now $\tilde{f}(w^\nu) = \sum_{i=1}^{k^\nu} \lambda_i^\nu f(z^\nu(i))$. But only the $(j+1)^{th}$ coordinate of $\tilde{f}$ is positive and hence $\tilde{f}(w^\nu) = w_{j+1} = \sum_{i=1}^{k^\nu} \lambda_i^\nu z_{j+1}^\nu (i) \leq \frac{1}{2^j}$ since $w^\nu \to y^*$, $\tilde{f}(w^\nu) \to \tilde{f}(y^*)$. Hence $\tilde{f}(y^*) \leq \frac{1}{2^j}$. Since y^* was arbitrary, $\alpha(\tilde{f},Y) \leq \frac{1}{2^j}$. ‖

Example 4. This example is also set in $[\ell^1, \ell^\infty]$. Here

$$Y = \{y \varepsilon \ell^1 :\ y^{T+1} \leq \sum_{t=1}^{T} y(t)\, \gamma^{T-t} \text{ for all } T,\ 0 < \gamma < 1.\}$$

Claim. Let $f \varepsilon \ell^\infty$, $f \geq 0$, $\| f \| = 1$ and $\alpha(f,Y) < \infty$. Then $f_1 = 1$.

Proof. Suppose $0 \leq f_1 < 1$. Then there exist $f_i = 1$ $(i \neq 1)$. Let $\bar{y} \varepsilon \ell^1$, $\bar{y}_i = 1$, $y_j = 0$ $(j \neq i)$. Certainly $\bar{y} \varepsilon Y$ and $f(\bar{y}) = 1$. Since Y is a cone, $\alpha(f,Y) < \infty$ is contradicted. ‖

Remark. By Theorem 1, Y has a nonempty norm interior.

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