

# Second Stage Short Run $(\bar{X}, v_c)$ and $(\bar{X}, s_c)$ Control Charts

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## Abstract

In their 1970 paper titled "Mean and Variance Control Chart Limits Based on a Small Number of Subgroups" (*Journal of Quality Technology*, Volume 2, Number 1, pp. 9-16), Yang and Hillier originally derived equations for calculating the factors required to determine second stage short run control limits for  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  charts. Two issues have restricted the applicability of this particular control chart methodology. These are the limited tabulated values of factors Yang and Hillier present and no example to illustrate the use of the methodology. This paper addresses the first issue by presenting a computer program that accurately calculates the factors regardless of the values of the required inputs. An example shows how to incorporate the methodology into a two stage short run control charting procedure. The computer program is available at <http://program.20m.com>.

**Key Words:** Control Chart; Short Run Statistical Process Control; Two Stage Control Charting; Distribution of the Variance; Distribution of the Studentized Variance

# 1 Introduction

Several choices exist for estimating a process variance or standard deviation from subgrouped data (i.e., data collected from a process as  $m$  subgroups, each of size  $n$ ) for the purpose of constructing control charts to monitor the process centering and/or spread. One of these choices is based on combining the subgrouped data into a single sample of  $m \cdot n$  data values. This paper denotes the variance and standard deviation calculated from this single sample as  $v_c$  and  $s_c$ , respectively.

Yang and Hillier (1970) originally derived equations for calculating the factors required to construct control charts for centering and spread based on  $\bar{X}$ ,  $v_c$ , and  $s_c$  (Yang and Hillier (1970) denote  $v_c$  and  $s_c$  as  $s^2$  and  $s$ , respectively) in short run situations. A short run situation is one in which little or no historical information is available about a process in order to estimate process parameters to begin control charting. Consequently, the initial data obtained from the early run of the process must be used for this purpose. According to Hillier (1969), short run control charting is necessary in the initiation of a new process, during the startup of a process just brought into statistical control again, and for a process whose total output is not large enough to use conventional control chart constants.

Integrated into Yang and Hillier's (1970) derivations was Hillier's (1969) two stage theory of control charting. In the first stage of the two stage procedure, the initial subgroups drawn from the process are used to determine the control limits. The initial subgroups are plotted against the control limits to retrospectively test if the process was in control while the initial subgroups were being drawn. Once control is established, the procedure moves to the second stage, where the subgroups that were not deleted in the first stage are used to determine the control limits for testing if the process remains in control while future subgroups are drawn. Each stage uses a

different set of control chart factors called first stage short run control chart factors and second stage short run control chart factors.

Yang and Hillier (1970) derived only second stage short run control chart factor equations for  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  charts. They gave three reasons for doing this. The main issue is that if the process mean is shifting between subgroups while the initial subgroups are being drawn in the first stage, then  $v_c$  and  $s_c$  calculated from these initial subgroups would have inflated values. This would decrease the ability of the first stage short run control limits calculated using  $v_c$  and  $s_c$  to delineate common cause from special cause signals. Also, when constructing second stage short run control limits for  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  charts, Yang and Hillier (1970) state that the data used must be from an in control process.

## 1.1 Problem

Two issues exist with Yang and Hillier's (1970) results for  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  control charts (see their Tables 7, 8, and 9) that have restricted their applicability. One is that the results in these tables are limited, showing second stage short run control chart factors for  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  charts for  $n=5$  only, 16 values for  $m$  (1-10, 15, 20, 25, 50, 100,  $\infty$ ),  $\alpha$  values of 0.001, 0.002, 0.01, and 0.05 for the  $\bar{X}$  chart, and  $\alpha$  values of 0.001, 0.005, and 0.025 for the  $v_c$  and  $s_c$  charts both above the upper control limit and below the lower control limit ( $\alpha$  is the probability of a false alarm). The other issue is that there is no example to illustrate the use of second stage short run  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  charts.

## 1.2 Solution and Outline

This paper addresses both of these issues. It first mentions the distributions of the variance and the studentized variance and the roles these two distributions play in the construction of second stage short run  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  control charts. It then presents a computer program that runs in the software *Mathcad 8.03 Professional* (1998) (or later versions) with the *Numerical Recipes Extension Pack* (1997). The program uses the previously mentioned distributions, the appropriate equations from Yang and Hillier (1970), and numerical routines provided by the software to accurately calculate the factors for second stage short run  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  control charts. The program accepts values for  $n$ ,  $m$ ,  $\alpha$  for the  $\bar{X}$  chart, and  $\alpha$  for the  $v_c$  and  $s_c$  charts both above the upper control limit and below the lower control limit. Consequently, results may be obtained for a wide range of applications. The computer program is available at <http://program.20m.com>.

This paper also presents tables (Tables 1, 2, and 3) showing factors calculated using the computer program for the same values of  $n$ ,  $m$ , and  $\alpha$  as in Yang and Hillier's (1970) Tables 7, 8, and 9, respectively. Implications of these tabulated results are also discussed. An example shows how to use second stage short run  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  control charts in a two stage short run control charting procedure. This paper then concludes with its contributions.

## 2 Probability Distributions

### 2.1 The Distribution of the Variance

The distribution of the variance for subgroups of size  $n$  sampled from a Normal population with mean  $\mu$  and variance  $\sigma^2$  is given by Pearson and Hartley (1962) as equation (1a) (with some modifications in notation):

$$p(v) = \left(\frac{v1}{2}\right)^{\frac{v1}{2}} \cdot \left(\Gamma\left(\frac{v1}{2}\right)\right)^{-1} \cdot \sigma^{-v1} \cdot v^{\frac{v1}{2}-1} \cdot e^{-\frac{v1 \cdot v}{2 \cdot \sigma^2}} \quad (1a)$$

The value  $v$  (the variance) is an independent estimate of  $\sigma^2$  based on  $v1 = (n - 1)$  degrees of freedom. Equation (1a) may also be represented as equation (1b) (see Appendix A):

$$p(v) = \left(\frac{1}{\sigma^{v1}}\right) \cdot \left[ e^{\left(\frac{v1}{2}\right) \cdot \ln\left(\frac{v1}{2}\right) - \text{gammln}\left(\frac{v1}{2}\right) + \left(\frac{v1}{2} - 1\right) \cdot \ln(v) - \frac{v1 \cdot v}{2 \cdot \sigma^2}} \right] \quad (1b)$$

Equation (1b) is the form used in the computer program because it allows for large values of  $v1$  (hence large values for  $n$ ) in the program. The function `gammln` is a numerical recipe in the *Numerical Recipes Extension Pack* (1997) that calculates the natural logarithm of the gamma ( $\Gamma$ ) function.

The distribution of the variance  $v$  with  $v1$  degrees of freedom is equivalent to a second distribution as shown in equation (2):

$$p(v) = c \left(\frac{v1 \cdot v}{\sigma^2}\right) \cdot \frac{v1}{\sigma^2} \quad (2)$$

where  $c$  is the  $\chi^2$  distribution with  $v_1$  degrees of freedom (this equivalency is shown in Appendix A). Also, percentage points of the distribution of the variance  $v$  with  $v_1$  degrees of freedom are equivalent to percentage points of the  $\chi^2$  distribution with  $v_1$  degrees of freedom divided by  $v_1$ .

The cumulative distribution function (cdf) of the variance  $v$  with  $v_1$  degrees of freedom is equation (3):

$$P(V) = \int_0^v p(v) dv \quad (3)$$

The computer program uses equation (3) (with  $\sigma=1.0$ ) to determine  $\alpha$ -based conventional control chart constants for the  $v_c$  and  $s_c$  charts.

## 2.2 The Distribution of the Studentized Variance

The distribution of the studentized variance (i.e., the F distribution) for subgroups of size  $n$  sampled from a Normal population with mean  $\mu$  and variance  $\sigma^2$  is given by Bain and Engelhardt (1992) as equation (4a) (with some modifications in notation):

$$p_3(f) = \frac{\Gamma\left(\frac{v_1 + v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right) \cdot \Gamma\left(\frac{v_2}{2}\right)} \cdot \left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} \cdot f^{\frac{v_1}{2}-1} \cdot \left(1 + \frac{v_1}{v_2} \cdot f\right)^{-\frac{v_1+v_2}{2}} \quad (4a)$$

The value  $f$  (the studentized variance) is equal to  $v/v'$ , where  $v'$  is a second independent estimate of  $\sigma^2$  based on  $v_2 = m \cdot n - 1$  degrees of freedom ( $m$  is the number of subgroups).

Equation (4a) may also be represented as equation (4b) (see Appendix A):

$$p_3(f) = e^{p_1 + p_2(f)} \quad (4b)$$

where

$$p_1 = \text{gammln}\left(\frac{v_1 + v_2}{2}\right) - \text{gammln}\left(\frac{v_1}{2}\right) - \text{gammln}\left(\frac{v_2}{2}\right) \quad (4c)$$

$$p_2(f) = \left(\frac{v_1}{2}\right) \cdot (\ln(v_1) - \ln(v_2)) + \left(\frac{v_1}{2} - 1\right) \cdot \ln(f) - \left(\frac{v_1 + v_2}{2}\right) \cdot \ln\left(1 + \frac{v_1}{v_2} \cdot f\right) \quad (4d)$$

Equations (4b)-(4d) are used in the computer program because they allow for large values of  $v_1$  (hence large values for  $n$ ) and large values of  $v_2$  (hence large values for  $m$  and  $n$ ) in the program.

The cdf of the studentized variance  $f = (v/v')$  with  $v_1$  degrees of freedom for  $v$  and  $v_2$  degrees of freedom for  $v'$  is equation (5):

$$P_3(F) = \int_0^F p_3(f) df \quad (5)$$

The computer program uses equation (5) to determine second stage short run control chart factors for the  $v_c$  and  $s_c$  charts.

As  $v \rightarrow \infty$  (i.e., as  $m \rightarrow \infty$ ) for any  $n$ , the distribution of the studentized variance  $f = (v/v')$  converges to the distribution of the variance  $v$  (when  $\sigma=1.0$ ). This fact is used to derive equations to calculate  $\alpha$ -based conventional control chart constants for the  $v_c$  and  $s_c$  charts.

### 3 The Computer Program

The computer program is in Appendix B. It has five pages, three of which are further divided into sections. It should be noted that results from the program are for processes generating parts with independent measurements that follow a Normal distribution.

#### 3.1 *Mathcad* (1998) Note

It is possible for a section of code in the program to turn red and have the error message "Unknown Error". To correct this, delete one character in the red code and type it back in. Click the mouse arrow outside of the code. The code should turn black, indicating that the error has been eliminated. If not, repeat the procedure (it will eventually correct the problem).

#### 3.2 Page 1

The first page of the program has the data entry section. The program requires the user to enter the following values:  $\alpha_{\bar{X}}$  ( $\alpha$  for the  $\bar{X}$  chart),  $\alpha_{vcu}$  ( $\alpha$  for the  $v_c$  or  $s_c$  chart above the UCL),  $\alpha_{vcl}$  ( $\alpha$  for the  $v_c$  and  $s_c$  chart below the LCL),  $m$  (number of subgroups), and  $n$  (subgroup size for the  $(\bar{X}, v_c)$  or  $(\bar{X}, s_c)$  charts). If no lower control limit on the  $v_c$  or  $s_c$  chart is desired, the entry for  $\alpha_{vcl}$  should be left blank (do not enter zero). Before a value can be entered, the cursor must be moved to the right side of the appropriate equal sign. This may be



done using the arrow keys on the keyboard or by moving the mouse arrow to the right side of the equal sign and clicking once with the left mouse button.

The program is activated by paging down once the last entry is made. When using *Mathcad 8.03 Professional* (1998), paging down is not allowed while a calculation is taking place. However, starting with *Mathcad 2000 Professional* (1999), the user is allowed to page down to the output section of the program (explained later) after the last entry is made.

### 3.3 Page 2

Page 2 of the program begins with section 2.1. The value TOL is the tolerance. The calculations that use this value will be accurate to twelve places to the right of the decimal. The population standard deviation  $\sigma$  is set equal to one to achieve the convergence of the distribution of the studentized variance  $f = (v/v')$  with  $v_1$  degrees of freedom for  $v$  and  $v_2$  degrees of freedom for  $v'$  (see equation (4b)) to the distribution of the variance  $v$  with  $v_1$  degrees of freedom (see equation (1b)) as  $v_2 \rightarrow \infty$  (i.e., as  $m \rightarrow \infty$ ) for any  $n$ . The equations for  $v_1$  and  $v_2$  were given earlier in relation to equations (1a) and (4a), respectively.

Page 2, section 2.2 of the program has the equations for the distribution of the variance  $v$  with  $v_1$  degrees of freedom and its cdf, both given earlier as equations (1b) and (3), respectively. The last part of page 2 is section 2.3 of the program. The code in this section determines  $v_{B10}$  and  $v_{B9}$ , the  $(1-\alpha)_{vcu}$  and  $\alpha_{vcl}$  percentage points, respectively, of the distribution of the variance  $v$  with  $v_1$  degrees of freedom. The values  $v_{B10}$  and  $v_{B9}$  are used to determine the  $\alpha$ -based conventional upper and lower control chart constants, respectively, for the  $v_c$  and  $s_c$  charts. The roots of the equations  $DUCL(V)$  and  $DLCL(V)$  are  $v_{B10}$  and  $v_{B9}$ , respectively, and are determined using `zbrent` (a numerical recipe in the *Numerical Recipes Extension Pack* (1997))

that uses Brent's method to find the roots of an equation). The subprograms Vseed1 and Vseed2 generate seed values seedB10 and seedB9, respectively, for Brent's method.

The subprogram Vseed1 works as follows. Initially,  $V_0$  and  $V_1$  are set equal to 0.01 and 0.02, respectively.  $A_0$  and  $A_1$  result from evaluating  $DUCL(V)$  at  $V_0$  and  $V_1$ , respectively. The while loop begins by checking if the product of  $A_0$  and  $A_1$  is negative. If so, the root for  $DUCL(V)$  lies between 0.01 and 0.02. If not,  $V_0$  and  $V_1$  are incremented by 0.01.  $A_0$  and  $A_1$  are recalculated and if their product is negative, the root for  $DUCL(V)$  lies between 0.02 and 0.03. Otherwise, the while loop repeats. Once a root for  $DUCL(V)$  is bracketed, the bracketing values are passed out of the subprogram into the  $2 \times 1$  vector seedB10 to be used by Brent's method to determine  $vB10$ . The subprogram Vseed2 works similarly to construct the  $2 \times 1$  vector seedB9 to be used by Brent's method to determine  $vB9$ , except the starting value is 0.000001.

### 3.4 Page 3

Page 3 of the program begins with section 3.1. It has the equations for the distribution of the studentized variance  $f = (v/v')$  with  $v_1$  degrees of freedom for  $v$  and  $v_2$  degrees of freedom for  $v'$  and its cdf, both given earlier as equations (4b)-(4d) and (5), respectively.

Section 3.2 contains the calculations required to determine  $fB10$ , the  $(1-\alpha_{vcu})$  percentage point of the distribution of the studentized variance  $f = (v/v')$  with  $v_1$  degrees of freedom for  $v$  and  $v_2$  degrees of freedom for  $v'$ . The value  $fB10$  is used to determine the second stage short run upper control chart factor for the  $v_c$  and  $s_c$  charts. The subprogram Fseed1 generates the seed value seed1 for Brent's method or for root (root is a numerical routine in *Mathcad* (1998) that uses the Secant method to determine the roots of an equation). Either root-finding method

determines the root fB10 of D1(x). Both Brent's method and the Secant method are given because one may not work when the other one does. If Brent's method fails (which is signified in *Mathcad* (1998) by the code turning red), type fB10 on the left side of the equal sign in equation (6):

$$= \text{root}\left[|P3(\text{seed1}) - (1 - \alpha\_vcu)|, \text{seed1}\right] \quad (6)$$

The subprogram Fseed1 begins by generating values for  $F_0$  and  $F_1$ .  $A_0$  and  $A_1$  result from evaluating  $P3(F)$  at  $F_0$  and  $F_1$ , respectively. The while loop continually increments  $F_0$  and  $F_1$  by delta1 and evaluates  $P3(F)$  at these two values until  $A_1$  becomes greater than  $(1 - \alpha\_vcu)$  for the first time, at which point  $A_0$  will be less than  $(1 - \alpha\_vcu)$ . When this occurs,  $P3(F)$  is equal to  $(1 - \alpha\_vcu)$  for some value  $F$  between  $F_0$  and  $F_1$ . An initial guess for this value is determined using linterp (a numerical routine in *Mathcad* (1998) that performs linear interpolation) and stored in Fguess. The initial guess is passed out of the subprogram as seed1.

### 3.5 Page 4

Page 4 of the program begins with section 4.1. The code in this section is used to determine fB9, the  $\alpha\_vcl$  percentage point of the distribution of the studentized variance  $f = (v/v')$  with  $v_1$  degrees of freedom for  $v$  and  $v_2$  degrees of freedom for  $v'$ . The value fB9 is used to determine the second stage short run lower control chart factor for the  $v_c$  and  $s_c$  charts. The subprogram Fseed2 generates the seed value seed2 for Brent's method or for root. Either root-finding method determines the root fB9 of D2(x). Both Brent's method and the Secant method are given because

one may not work when the other one does. If Brent's method fails, type fB9 on the left side of the equal sign in equation (7):

$$= \text{root}(|P3(\text{seed2}) - \alpha\_vcl|, \text{seed2}) \quad (7)$$

The subprogram Fseed2 begins by generating values for  $F_0$  and  $F_1$ .  $A_0$  and  $A_1$  result from evaluating  $P3(F)$  at  $F_0$  and  $F_1$ , respectively. The while loop continually increments  $F_0$  and  $F_1$  by  $\text{delta2}$  and evaluates  $P3(F)$  at these two values until  $A_1$  becomes greater than  $\alpha\_vcl$  for the first time, at which point  $A_0$  will be less than  $\alpha\_vcl$ . When this occurs,  $P3(F)$  is equal to  $\alpha\_vcl$  for some value  $F$  between  $F_0$  and  $F_1$ . An initial guess for this value is determined using `linterp` and stored in `Fguess`. The initial guess is passed out of the subprogram as `seed2`.

On page 4, section 4.2 of the program, the function `qt(adj_α, v2)` in *Mathcad* (1998) determines the critical value `crit_t` for a cumulative area of `adj_α` under the Student's  $t$  curve with `v2` degrees of freedom. The value `crit_t` is used to determine first and second stage short run control chart factors for the  $\bar{X}$  chart. The function `qnorm(adj_α, 0, 1)` in *Mathcad* (1998) determines the critical value `crit_z` for a cumulative area of `adj_α` under the standard Normal curve. The value `crit_z` is used to determine the conventional control chart constant for the  $\bar{X}$  chart.

The last part of page 4 is section 4.3 of the program. This section contains the equations for calculating the factors required to determine second stage short run control limits for  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  charts from Yang and Hillier (1970). It also has the equations for calculating

conventional control chart constants for  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  charts. The notation used is defined below.

- A52: the second stage short run control chart factor for the  $\bar{X}$  chart
- A5: the conventional control chart constant for the  $\bar{X}$  chart (the equation for A5 may be obtained by taking the limit of A52 as  $m \rightarrow \infty$  (i.e., as  $v_2 \rightarrow \infty$ ) for any  $n$ )
- B102: the second stage short run upper control chart factor for the  $v_c$  chart
- B10: the  $\alpha$ -based conventional upper control chart constant for the  $v_c$  chart (the equation for B10 may be obtained by taking the limit of B102 as  $m \rightarrow \infty$  (i.e., as  $v_2 \rightarrow \infty$ ) for any  $n$ )
- B92: the second stage short run lower control chart factor for the  $v_c$  chart
- B9: the  $\alpha$ -based conventional lower control chart constant for the  $v_c$  chart (the equation for B9 may be obtained by taking the limit of B92 as  $m \rightarrow \infty$  (i.e., as  $v_2 \rightarrow \infty$ ) for any  $n$ )
- B102sqrt: the second stage short run upper control chart factor for the  $s_c$  chart
- B10sqrt: the  $\alpha$ -based conventional upper control chart constant for the  $s_c$  chart (the equation for B10sqrt may be obtained by taking the limit of B102sqrt as  $m \rightarrow \infty$  (i.e., as  $v_2 \rightarrow \infty$ ) for any  $n$ )
- B92sqrt: the second stage short run lower control chart factor for the  $s_c$  chart
- B9sqrt: the  $\alpha$ -based conventional lower control chart constant for the  $s_c$  chart (the equation for B9sqrt may be obtained by taking the limit of B92sqrt as  $m \rightarrow \infty$  (i.e., as  $v_2 \rightarrow \infty$ ) for any  $n$ )

### 3.6 Page 5

The last page of the program has the output. The five values entered at the beginning of the program are given, as well as the values for  $v_1$  and  $v_2$ . The control chart factors are broken down into second stage and conventional. To copy results into another software package (like Excel), follow the directions from *Mathcad's* (1998) help menu or highlight a value and copy and paste it into the other software package. When highlighting a value with the mouse arrow, place the arrow in the middle of the value, depress the left mouse button, and drag the arrow to the right. This will ensure just the numerical value of the result is copied and pasted.

## 4 Tables of Factors

Tables 1, 2, and 3 were generated using the computer program. These tables replicate the results given by Yang and Hillier (1970) in their Tables 7, 8, and 9, respectively, thus validating the

**Table 1.** Values of  $A_{52}$  and  $A_5$  for  $n=5$

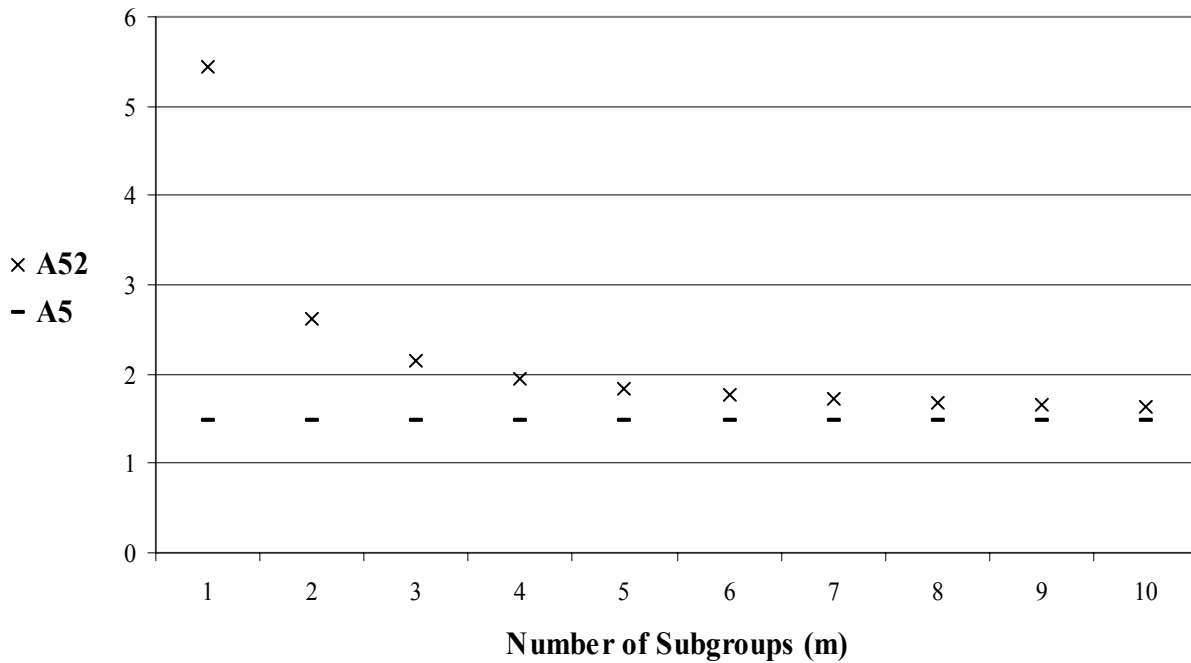
<b>m</b>	<b><math>\alpha_{\bar{X}}</math></b>			
	<b>0.001</b>	<b>0.002</b>	<b>0.01</b>	<b>0.05</b>
<b>1</b>	5.44563	4.53672	2.91189	1.75598
<b>2</b>	2.61861	2.35346	1.78001	1.23903
<b>3</b>	2.13812	1.95580	1.53723	1.10756
<b>4</b>	1.94170	1.78970	1.43047	1.04651
<b>5</b>	1.83486	1.69837	1.37021	1.01110
<b>6</b>	1.76766	1.64054	1.33146	0.98794
<b>7</b>	1.72147	1.60062	1.30442	0.97160
<b>8</b>	1.68776	1.57139	1.28448	0.95945
<b>9</b>	1.66208	1.54907	1.26915	0.95005
<b>10</b>	1.64185	1.53146	1.25701	0.94257
<b>15</b>	1.58282	1.47989	1.22117	0.92032
<b>20</b>	1.55419	1.45479	1.20357	0.90928
<b>25</b>	1.53728	1.43993	1.19311	0.90269
<b>50</b>	1.50405	1.41067	1.17239	0.88957
<b>100</b>	1.48772	1.39626	1.16214	0.88304
$\infty$	1.47157	1.38199	1.15195	0.87652

**Table 2.** Values of B92, B92sqrt, B9, and B9sqrt for n=5

m	$\alpha_{vcl}$					
	0.001		0.005		0.025	
	B92	B92sqrt	B92	B92sqrt	B92	B92sqrt
1	0.01871	0.1368	0.04319	0.20782	0.10412	0.32267
2	0.02063	0.14363	0.04731	0.21750	0.11230	0.33511
3	0.02130	0.14595	0.04875	0.22078	0.11516	0.33935
4	0.02164	0.14711	0.04948	0.22244	0.11661	0.34149
5	0.02185	0.14782	0.04993	0.22344	0.11750	0.34278
6	0.02199	0.14829	0.05022	0.22411	0.11809	0.34364
7	0.02209	0.14863	0.05044	0.22459	0.11852	0.34426
8	0.02217	0.14888	0.05060	0.22495	0.11884	0.34472
9	0.02222	0.14908	0.05073	0.22523	0.11909	0.34509
10	0.02227	0.14924	0.05083	0.22545	0.11929	0.34538
15	0.02241	0.14971	0.05113	0.22613	0.11989	0.34625
20	0.02249	0.14995	0.05129	0.22646	0.12019	0.34669
25	0.02253	0.15009	0.05138	0.22667	0.12037	0.34695
50	0.02261	0.15038	0.05156	0.22707	0.12074	0.34747
100	0.02266	0.15052	0.05165	0.22728	0.12092	0.34774
$\infty$	0.02270	0.15067	0.05175	0.22748	0.12110	0.34800

**Table 3.** Values of B102, B102sqrt, B10, and B10sqrt for n=5

m	$\alpha_{vcu}$					
	0.001		0.005		0.025	
	B102	B102sqrt	B102	B102sqrt	B102	B102sqrt
1	53.43583	7.30998	23.15450	4.81191	9.60453	3.09912
2	12.56032	3.54405	7.95589	2.82062	4.71808	2.17211
3	8.62232	2.93638	5.99841	2.44916	3.89191	1.97279
4	7.26546	2.69545	5.26809	2.29523	3.55871	1.88645
5	6.58924	2.56695	4.88978	2.21129	3.37936	1.83830
6	6.18626	2.48722	4.65908	2.15849	3.26744	1.80761
7	5.91931	2.43296	4.50388	2.12223	3.19098	1.78633
8	5.72965	2.39367	4.39240	2.09581	3.13544	1.77072
9	5.58805	2.36391	4.30848	2.07569	3.09329	1.75877
10	5.47833	2.34058	4.24303	2.05986	3.06020	1.74934
15	5.16706	2.27312	4.05527	2.01377	2.96418	1.72168
20	5.02090	2.24074	3.96598	1.99148	2.91794	1.70820
25	4.93605	2.22172	3.91381	1.97834	2.89073	1.70022
50	4.77249	2.18460	3.81250	1.95256	2.83751	1.68449
100	4.69365	2.16648	3.76331	1.93992	2.81148	1.67675
$\infty$	4.61671	2.14865	3.71506	1.92745	2.78582	1.66908



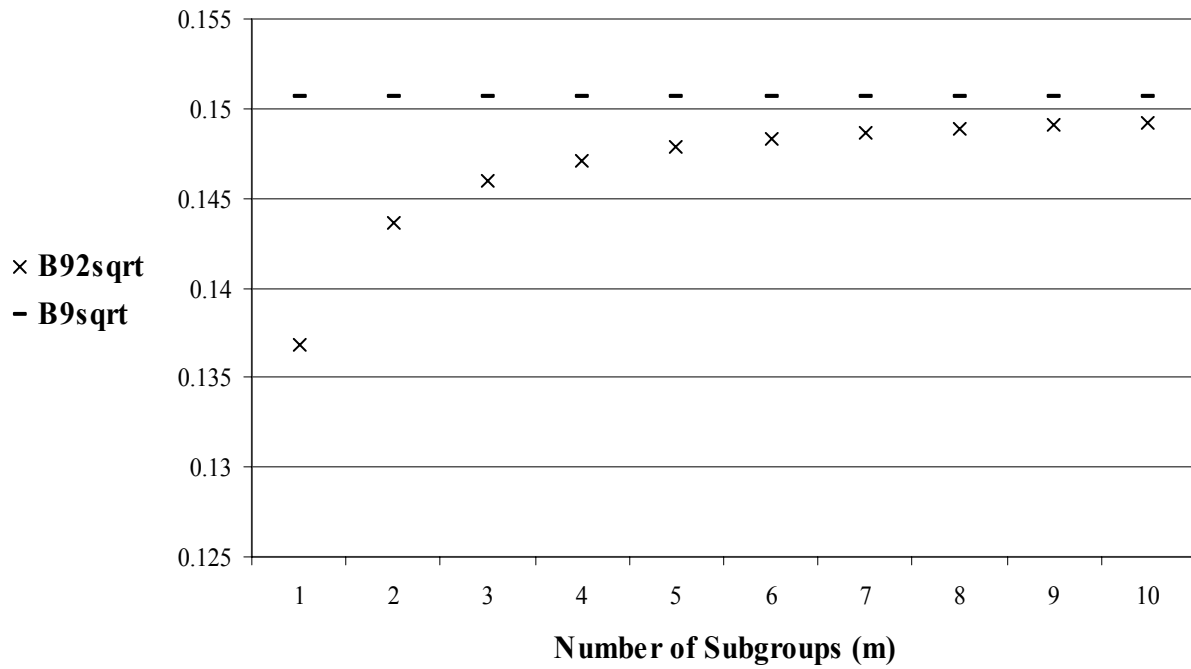
**Figure 1.** A52 and A5 for  $\alpha_{\bar{X}}=0.001$  and  $n=5$

accuracy of the program. However, when similarly rounded, some of the results in Tables 1, 2, and 3 differ from their respective counterparts in Yang and Hillier's (1970) Tables 7, 8, and 9 in the last decimal place shown by one and, in a few cases, two digits. This may be due to the programming style and/or software used to generate Tables 1, 2, and 3.

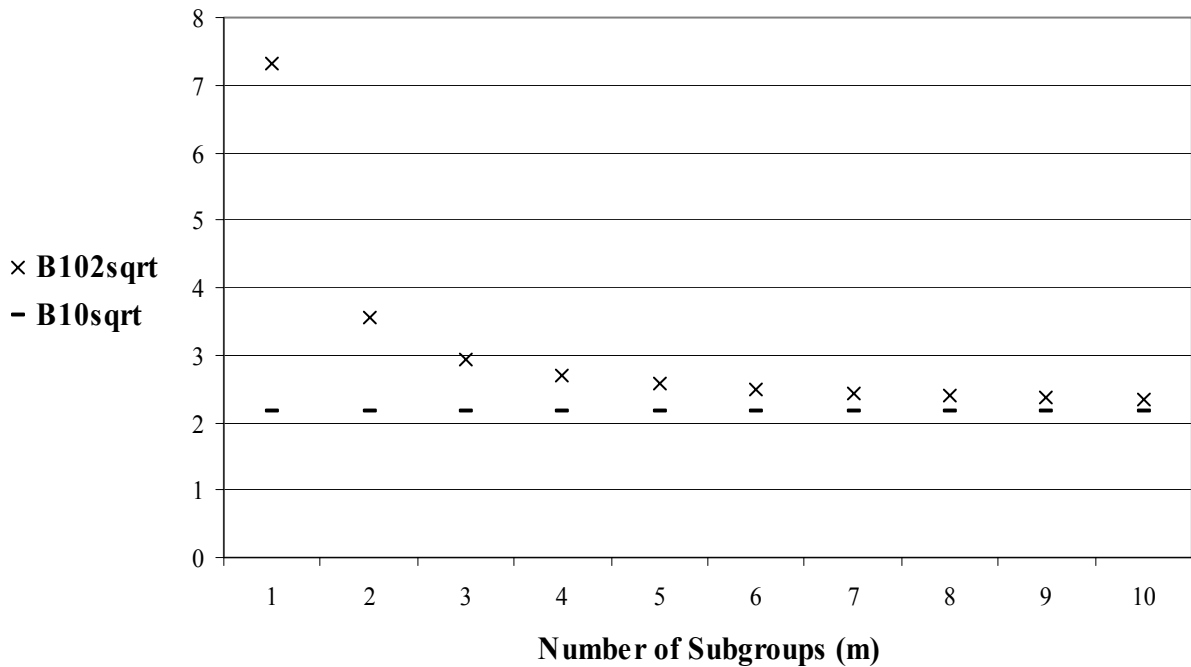
Figure 1 is a plot of A52 and A5 for  $\alpha_{\bar{X}}=0.001$ ,  $n=5$ , and  $m: 1, 2, \dots, 10$ . It indicates that if one were to construct  $\bar{X}$  charts using conventional control chart constants when ten or less subgroups of size five (i.e., a combined sample of 50 or less data values) are available to estimate the process mean and standard deviation, the upper and lower control limits would not be wide enough, resulting in a higher probability of a false alarm.

Figure 2 is a plot of B92sqrt and B9sqrt for  $\alpha_{s_c}=0.001$ ,  $n=5$ , and  $m: 1, 2, \dots, 10$ . It indicates that if one were to construct  $s_c$  charts using conventional control chart constants when ten or less





**Figure 2.** B92sqrt and B9sqrt for  $\alpha_{vcl}=0.001$  and  $n=5$



**Figure 3.** B102sqrt and B10sqrt for  $\alpha_{vcu}=0.001$  and  $n=5$

subgroups of size five are available to estimate the process standard deviation, the lower control limit would not be wide enough, resulting in a higher probability of a false alarm.

Figure 3 is a plot of  $B_{102\text{sqrt}}$  and  $B_{10\text{sqrt}}$  for  $\alpha_{\text{vcu}}=0.001$ ,  $n=5$ , and  $m: 1, 2, \dots, 10$ . It indicates that if one were to construct  $s_c$  charts using conventional control chart constants when ten or less subgroups of size five are available to estimate the process standard deviation, the upper control limit would not be wide enough, resulting in a higher probability of a false alarm.

A common rule of thumb is that 20 to 30 subgroups of size four or five are necessary to use conventional control chart constants for constructing control limits. The results in Tables 1, 2, and 3 and the interpretations of Figures 1, 2, and 3 indicate that this rule may not be applicable to  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  control charts. Quesenberry (1993) also investigated the validity of the common rule of thumb and concluded that  $400/(n-1)$  subgroups are needed for the  $\bar{X}$  chart before conventional control chart constants may be used. However, for all practical purposes, the computer program eliminates the need for these and similar rules.

## 5 A Numerical Example

Consider the data in Table 4 obtained from a process requiring short run control charting techniques (assume  $\alpha_{\text{Xbar}}=0.0027$ ,  $\alpha_{\text{vcu}}=0.005$ , and  $\alpha_{\text{vcl}}=0.001$ ). This example will be worked two ways, the first with  $(\bar{X}, v_c)$  control charts and the second with  $(\bar{X}, s_c)$  control charts.

### 5.1 $(\bar{X}, v_c)$ Control Charts

**Table 4.** A Numerical Example

Subgroup	$X_1$	$X_2$	$X_3$	$X_4$	$\bar{X}$	R
1	1.17	1.14	1.20	1.18	1.17250	0.06000
2	1.38	1.29	1.36	1.44	1.36750	0.15000
3	1.20	1.21	1.30	1.14	1.21250	0.16000
4	1.40	1.40	1.21	1.43	1.36000	0.22000
5	1.12	1.20	1.61	1.34	1.31750	0.49000
<b>Averages</b>					1.28600	0.21600
<b>Revised Averages</b>					1.27813	0.14750
$v_c$					0.01184	
$s_c$					0.10881	

For first stage short run control charting, we use the appropriate factors for  $(\bar{X}, R)$  charts from Elam and Case (2001) (R is the range of a subgroup). For  $m=5$  and  $n=4$ , the following first stage short run control chart factors for  $(\bar{X}, R)$  charts are obtained from Table A4 in Appendix III of Elam and Case (2001):  $A_{21}=0.77660$ ,  $D_{41}=2.11840$ , and  $D_{31}=0.11338$ .  $UCL(R)$ ,  $LCL(R)$ ,  $UCL(\bar{X})$ , and  $LCL(\bar{X})$  are calculated as follows:

$$UCL(R) = D_{41} \cdot \bar{R} = 2.11840 \cdot 0.21600 = 0.45757$$

$$LCL(R) = D_{31} \cdot \bar{R} = 0.11338 \cdot 0.21600 = 0.02449$$

$$UCL(\bar{X}) = \bar{\bar{X}} + A_{21} \cdot \bar{R} = 1.28600 + 0.77660 \cdot 0.21600 = 1.45375$$

$$LCL(\bar{X}) = \bar{\bar{X}} - A_{21} \cdot \bar{R} = 1.28600 - 0.77660 \cdot 0.21600 = 1.11825$$

The range for subgroup five ( $R=0.49000$ ) is above  $UCL(R)$ . Find, investigate, and remove from the process the special cause (or causes) that created this out of control point, delete subgroup five, recalculate averages (shown as the Revised Averages in Table 4), and reconstruct first stage control limits (this approach is from Hillier's (1969) example). For  $m=4$  and  $n=4$ , the following

first stage short run control chart factors for  $(\bar{X}, R)$  charts are obtained from Table A4 in Appendix III of Elam and Case (2001):  $A_{21}=0.78832$ ,  $D_{41}=2.07041$ , and  $D_{31}=0.11848$ . Revised  $UCL(R)$ ,  $LCL(R)$ ,  $UCL(\bar{X})$ , and  $LCL(\bar{X})$  are calculated as follows:

$$UCL(R) = D_{41} \cdot \bar{R} = 2.07041 \cdot 0.14750 = 0.30539$$

$$LCL(R) = D_{31} \cdot \bar{R} = 0.11848 \cdot 0.14750 = 0.01748$$

$$UCL(\bar{X}) = \bar{\bar{X}} + A_{21} \cdot \bar{R} = 1.27813 + 0.78832 \cdot 0.14750 = 1.39441$$

$$LCL(\bar{X}) = \bar{\bar{X}} - A_{21} \cdot \bar{R} = 1.27813 - 0.78832 \cdot 0.14750 = 1.16185$$

Since none of the remaining values plot out of control (i.e., control has been established), the next step is to construct second stage control limits using the following second stage short run control chart factors for  $(\bar{X}, v_c)$  charts for  $m=4$  and  $n=4$  (these were obtained using the computer program):  $A_{52}=2.00485$ ,  $B_{102}=6.47604$ , and  $B_{92}=0.00785$ .  $UCL(v_c)$ ,  $LCL(v_c)$ ,  $UCL(\bar{X})$ , and  $LCL(\bar{X})$  are calculated as follows:

$$UCL(v_c) = B_{102} \cdot v_c = 6.47604 \cdot 0.01184 = 0.07668$$

$$LCL(v_c) = B_{92} \cdot v_c = 0.00785 \cdot 0.01184 = 0.000093$$

$$UCL(\bar{X}) = \bar{\bar{X}} + A_{52} \cdot s_c = 1.27813 + 2.00485 \cdot 0.10881 = 1.49628$$

$$LCL(\bar{X}) = \bar{\bar{X}} - A_{52} \cdot s_c = 1.27813 - 2.00485 \cdot 0.10881 = 1.05998$$

These control limits may be used to monitor the future performance of the process.

## 5.2 $(\bar{X}, s_c)$ Control Charts

First stage control charting may be performed in the same manner as it was for the  $(\bar{X}, v_c)$  control chart example. To construct second stage control limits, we use the following second stage short run control chart factors for  $(\bar{X}, s_c)$  charts for  $m=4$  and  $n=4$  (these were obtained using the computer program):  $A52=2.00485$ ,  $B102\text{sqrt}=2.54481$ , and  $B92\text{sqrt}=0.08861$ .

$UCL(s_c)$ ,  $LCL(s_c)$ ,  $UCL(\bar{X})$ , and  $LCL(\bar{X})$  are calculated as follows:

$$UCL(s_c) = B102\text{sqrt} \cdot s_c = 2.54481 \cdot 0.10881 = 0.27690$$

$$LCL(s_c) = B92\text{sqrt} \cdot s_c = 0.08861 \cdot 0.10881 = 0.00964$$

$$UCL(\bar{X}) = \bar{\bar{X}} + A52 \cdot s_c = 1.27813 + 2.00485 \cdot 0.10881 = 1.49628$$

$$LCL(\bar{X}) = \bar{\bar{X}} - A52 \cdot s_c = 1.27813 - 2.00485 \cdot 0.10881 = 1.05998$$

These control limits may be used to monitor the future performance of the process.

## 6 Conclusions

This paper makes important contributions to the area of short run control charting. By using the computer program, those involved with quality control in industry will, for the first time, be able to obtain theoretically precise control chart factors to determine control limits for  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  charts regardless of the subgroup size, number of subgroups, and  $\alpha$  values. This flexibility is valuable in that process monitoring will no longer have to be adjusted to use the

limited results previously available in the literature. Furthermore, as already mentioned, the computer program eliminates the need for rules stating how many subgroups are enough before conventional control chart constants may be used. Also, it provides a valuable reference for anyone interested in anything having to do with  $(\bar{X}, v_c)$  and  $(\bar{X}, s_c)$  control charts.

Show: The distribution of the variance  $v$  with  $v_1$  degrees of freedom may be represented as

follows:

$$p(v) = \left( \frac{1}{\sigma^{v_1}} \right) \cdot \left[ e^{\left( \frac{v_1}{2} \right) \ln \left( \frac{v_1}{2} \right) - \text{gammln} \left( \frac{v_1}{2} \right) + \left( \frac{v_1-1}{2} \right) \cdot \ln(v) - \frac{v_1 \cdot v}{2 \cdot \sigma^2}} \right]$$

From Pearson and Hartley (1962),

$$p(v) = \left( \frac{v_1}{2} \right)^{\frac{v_1}{2}} \cdot \left( \Gamma \left( \frac{v_1}{2} \right) \right)^{-1} \cdot \sigma^{-v_1} \cdot v^{\frac{v_1-1}{2}} \cdot e^{-\frac{v_1 \cdot v}{2 \cdot \sigma^2}}$$

$$\Rightarrow p(v) = e^{\ln \left[ \left( \frac{v_1}{2} \right)^{\frac{v_1}{2}} \cdot \left( \Gamma \left( \frac{v_1}{2} \right) \right)^{-1} \cdot \sigma^{-v_1} \cdot v^{\frac{v_1-1}{2}} \cdot e^{-\frac{v_1 \cdot v}{2 \cdot \sigma^2}} \right]}$$

$$= \left( \frac{1}{\sigma^{v_1}} \right) \cdot \left[ e^{\left( \frac{v_1}{2} \right) \ln \left( \frac{v_1}{2} \right) - \ln \left( \Gamma \left( \frac{v_1}{2} \right) \right) + \left( \frac{v_1-1}{2} \right) \cdot \ln(v) - \frac{v_1 \cdot v}{2 \cdot \sigma^2}} \right]$$

$$= \left( \frac{1}{\sigma^{v_1}} \right) \cdot \left[ e^{\left( \frac{v_1}{2} \right) \ln \left( \frac{v_1}{2} \right) - \text{gammln} \left( \frac{v_1}{2} \right) + \left( \frac{v_1-1}{2} \right) \cdot \ln(v) - \frac{v_1 \cdot v}{2 \cdot \sigma^2}} \right]$$

Show:  $p(v) = c\left(\frac{v1 \cdot v}{\sigma^2}\right) \cdot \frac{v1}{\sigma^2}$ , where  $p(v)$  is the distribution of the variance  $v$  with  $v1$  degrees of

freedom and  $c$  is the  $\chi^2$  distribution with  $v1$  degrees of freedom.

Bain and Engelhardt (1992) give the  $\chi^2$  distribution as follows:

$$c(x) = \frac{1}{2^{\frac{v1}{2}} \cdot \Gamma\left(\frac{v1}{2}\right)} \cdot x^{\frac{v1}{2}-1} \cdot e^{-\frac{x}{2}}$$

$$\text{Let } x = \frac{v1 \cdot v}{\sigma^2}$$

$$\Rightarrow dx = \frac{v1}{\sigma^2} dv \Rightarrow c(x) dx = c\left(\frac{v1 \cdot v}{\sigma^2}\right) \cdot \frac{v1}{\sigma^2} dv$$

$$\Rightarrow c\left(\frac{v1 \cdot v}{\sigma^2}\right) \cdot \frac{v1}{\sigma^2} dv = \frac{1}{2^{\frac{v1}{2}} \cdot \Gamma\left(\frac{v1}{2}\right)} \cdot \left(\frac{v1 \cdot v}{\sigma^2}\right)^{\frac{v1}{2}-1} \cdot e^{-\frac{(v1 \cdot v)}{2\sigma^2}} \cdot \frac{v1}{\sigma^2} dv$$

$$= \frac{v1^{\frac{v1}{2}-1} \cdot v1}{2^{\frac{v1}{2}} \cdot \Gamma\left(\frac{v1}{2}\right) \cdot (\sigma^2)^{\frac{v1}{2}-1} \cdot \sigma^2} \cdot v^{\frac{v1}{2}-1} \cdot e^{-\frac{v1 \cdot v}{2\sigma^2}} dv$$

$$= \frac{v1^{\frac{v1}{2}} \cdot 2^{-\frac{v1}{2}} \cdot \left(\Gamma\left(\frac{v1}{2}\right)\right)^{-1}}{\sigma^{v1}} \cdot v^{\frac{v1}{2}-1} \cdot e^{-\frac{v1 \cdot v}{2\sigma^2}} dv$$

$$= v1^{\frac{v1}{2}} \cdot \left(\frac{1}{2}\right)^{\frac{v1}{2}} \cdot \left(\Gamma\left(\frac{v1}{2}\right)\right)^{-1} \cdot \sigma^{-v1} \cdot v^{\frac{v1}{2}-1} \cdot e^{-\frac{v1 \cdot v}{2\sigma^2}} dv$$

$$= \left(\frac{v1}{2}\right)^{\frac{v1}{2}} \cdot \left(\Gamma\left(\frac{v1}{2}\right)\right)^{-1} \cdot \sigma^{-v1} \cdot v^{\frac{v1}{2}-1} \cdot e^{-\frac{v1 \cdot v}{2\sigma^2}} dv$$

$$= p(v) dv$$



Show: The distribution of the studentized variance  $f = (v/v')$  with  $v_1$  degrees of freedom for  $v$  and  $v_2$  degrees of freedom for  $v'$  may be represented as follows:

$$p_3(f) = e^{p_1 + p_2(f)}$$

where

$$p_1 = \text{gammln}\left(\frac{v_1 + v_2}{2}\right) - \text{gammln}\left(\frac{v_1}{2}\right) - \text{gammln}\left(\frac{v_2}{2}\right)$$

$$p_2(f) = \left(\frac{v_1}{2}\right) \cdot (\ln(v_1) - \ln(v_2)) + \left(\frac{v_1}{2} - 1\right) \cdot \ln(f) - \left(\frac{v_1 + v_2}{2}\right) \cdot \ln\left(1 + \frac{v_1}{v_2} \cdot f\right)$$

From Bain and Engelhardt (1992),

$$p_3(f) = \frac{\Gamma\left(\frac{v_1 + v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right) \cdot \Gamma\left(\frac{v_2}{2}\right)} \cdot \left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} \cdot f^{\frac{v_1}{2} - 1} \cdot \left(1 + \frac{v_1}{v_2} \cdot f\right)^{-\frac{v_1 + v_2}{2}}$$

$$\Rightarrow p_3(f) = e^{\left[ \ln\left(\frac{\Gamma\left(\frac{v_1 + v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right) \Gamma\left(\frac{v_2}{2}\right)}\right) + \frac{v_1}{2} \ln\left(\frac{v_1}{v_2}\right) + \left(\frac{v_1}{2} - 1\right) \ln(f) - \frac{v_1 + v_2}{2} \ln\left(1 + \frac{v_1}{v_2} \cdot f\right) \right]}$$

$$= e^{\ln\left(\Gamma\left(\frac{v_1 + v_2}{2}\right)\right) - \ln\left(\Gamma\left(\frac{v_1}{2}\right)\right) - \ln\left(\Gamma\left(\frac{v_2}{2}\right)\right) + \left(\frac{v_1}{2}\right) \cdot \ln\left(\frac{v_1}{v_2}\right) + \left(\frac{v_1}{2} - 1\right) \cdot \ln(f) - \left(\frac{v_1 + v_2}{2}\right) \cdot \ln\left(1 + \frac{v_1}{v_2} \cdot f\right)}$$

$$= e^{\text{gammln}\left(\frac{v_1 + v_2}{2}\right) - \text{gammln}\left(\frac{v_1}{2}\right) - \text{gammln}\left(\frac{v_2}{2}\right) + \left(\frac{v_1}{2}\right) (\ln(v_1) - \ln(v_2)) + \left(\frac{v_1}{2} - 1\right) \cdot \ln(f) - \left(\frac{v_1 + v_2}{2}\right) \cdot \ln\left(1 + \frac{v_1}{v_2} \cdot f\right)}$$

$$\text{Let } p_1 = \text{gammln}\left(\frac{v_1 + v_2}{2}\right) - \text{gammln}\left(\frac{v_1}{2}\right) - \text{gammln}\left(\frac{v_2}{2}\right)$$

$$p_2(f) = \left(\frac{v_1}{2}\right) \cdot (\ln(v_1) - \ln(v_2)) + \left(\frac{v_1}{2} - 1\right) \cdot \ln(f) - \left(\frac{v_1 + v_2}{2}\right) \cdot \ln\left(1 + \frac{v_1}{v_2} \cdot f\right)$$

$$\Rightarrow p_3(f) = e^{p_1 + p_2(f)}$$

**Page 1 of program: ccfsvc.mcd**

**ENTER the following 5 values:**

- (1)  $\alpha_{\bar{X}} := 0.0027$                        $\alpha_{\bar{X}}$  -  $\alpha$  for the  $\bar{X}$  chart.
- (2)  $\alpha_{v_{cu}} := 0.005$                        $\alpha_{v_{cu}}$  -  $\alpha$  for the  $v_c$  or  $s_c$  chart above the UCL.
- (3)  $\alpha_{v_{cl}} := 0.001$                        $\alpha_{v_{cl}}$  -  $\alpha$  for the  $v_c$  or  $s_c$  chart below the LCL \*.
- (4)  $m := 5$                                        $m$  - number of subgroups.
- (5)  $n := 5$                                        $n$  - subgroup size for the  $(\bar{X}, v_c)$  or  $(\bar{X}, s_c)$  charts.

\* Note - If no LCL is desired, leave  $\alpha_{v_{cl}}$  blank (do not enter zero).

**Please PAGE DOWN to begin the program.**

**Page 2 of program: ccfsvc.mcd**

$$(2.1) \quad \text{TOL} := 10^{-12} \quad \sigma := 1.0 \quad v1 := n - 1 \quad v2 := m \cdot n - 1$$

$$(2.2) \quad p(v) := \left( \frac{1}{\sigma^{v1}} \right) \cdot e^{\left[ \left( \frac{v1}{2} \right) \cdot \ln \left( \frac{v1}{2} \right) - \text{gammln} \left( \frac{v1}{2} \right) + \left( \frac{v1}{2} - 1 \right) \cdot \ln(v) - \frac{v1 \cdot v}{2 \cdot \sigma^2} \right]}$$

$$P(V) := \int_0^V p(v) \, dv$$

$$(2.3) \quad \text{DUCL}(V) := P(V) - (1 - \alpha_{\text{vcu}})$$

$$\text{DLCL}(V) := P(V) - \alpha_{\text{vcl}}$$

$$\text{Vseed1}(\text{start}) := \left| \begin{array}{l} V_0 \leftarrow \text{start} \\ V_1 \leftarrow \text{start} + 0.01 \\ A_0 \leftarrow \text{DUCL}(V_0) \\ A_1 \leftarrow \text{DUCL}(V_1) \\ \text{while } A_0 \cdot A_1 > 0 \\ \quad \left| \begin{array}{l} V_0 \leftarrow V_1 \\ V_1 \leftarrow V_1 + 0.01 \\ A_0 \leftarrow A_1 \\ A_1 \leftarrow \text{DUCL}(V_1) \end{array} \right. \\ \quad V \end{array} \right.$$

$$\text{seedB10} := \text{Vseed1}(0.01)$$

$$vB10 := \text{zbrent}(\text{DUCL}, \text{seedB10}_0, \text{seedB10}_1, \text{TOL})$$

$$\text{Vseed2}(\text{start}) := \left| \begin{array}{l} V_0 \leftarrow \text{start} \\ V_1 \leftarrow \text{start} + 0.01 \\ A_0 \leftarrow \text{DLCL}(V_0) \\ A_1 \leftarrow \text{DLCL}(V_1) \\ \text{while } A_0 \cdot A_1 > 0 \\ \quad \left| \begin{array}{l} V_0 \leftarrow V_1 \\ V_1 \leftarrow V_1 + 0.01 \\ A_0 \leftarrow A_1 \\ A_1 \leftarrow \text{DLCL}(V_1) \end{array} \right. \\ \quad V \end{array} \right.$$

$$\text{seedB9} := \text{Vseed2}(0.000001)$$

$$vB9 := \text{zbrent}(\text{DLCL}, \text{seedB9}_0, \text{seedB9}_1, \text{TOL})$$

**Page 3 of program: ccfsvc.mcd**

$$(3.1) \quad p1 := \text{gammln}\left(\frac{v1 + v2}{2}\right) - \text{gammln}\left(\frac{v1}{2}\right) - \text{gammln}\left(\frac{v2}{2}\right)$$

$$p2(f) := \left(\frac{v1}{2}\right) \cdot (\ln(v1) - \ln(v2)) + \left(\frac{v1}{2} - 1\right) \cdot \ln(f) - \left(\frac{v1 + v2}{2}\right) \cdot \ln\left(1 + \frac{v1}{v2} \cdot f\right)$$

$$p3(f) := e^{p1+p2(f)}$$

$$P3(F) := \int_0^F p3(f) df$$

```
(3.2) Fseed1(start, delta) :=
  F0 ← start
  F1 ← start + delta
  A0 ← P3(F0)
  A1 ← P3(F1)
  while A1 < (1 - α_vcu)
    F0 ← F1
    F1 ← F1 + delta
    A0 ← A1
    A1 ← P3(F1)
  Fguess ← linterp(A, F, 1 - α_vcu)
  Fguess

seed1 := Fseed1(0.1, delta)
delta := 100.0 if (n = 2) · (m = 1)
        0.1 otherwise

D1(x) := P3(x) - (1 - α_vcu)

fb10 := zbrent(D1, seed1 - delta, seed1 + delta, TOL)

r := root[|P3(seed1) - (1 - α_vcu)|, seed1]
```

**Page 4 of program: ccfsvc.mcd**

```

(4.1) Fseed2(start, delta2) :=
  F0 ← start
  F1 ← start + delta2
  A0 ← P3(F0)
  A1 ← P3(F1)
  while A1 < α_vcl
    F0 ← F1
    F1 ← F1 + delta2
    A0 ← A1
    A1 ← P3(F1)
  Fguess ← linterp(A, F, α_vcl)
  Fguess

seed2 := Fseed2(0.000001, delta2)
delta2 :=
  0.0000001 if (n = 2)
  0.001 otherwise

D2(x) := P3(x) - α_vcl

fB9 := zbrent(D2, seed2 - delta2, seed2 + delta2, TOL)

■ := root(|P3(seed2) - α_vcl|, seed2)

(4.2) adj_α := 1 -  $\frac{\alpha \cdot \bar{X}}{2}$ 
crit_t := qt(adj_α, v2)
crit_z := qnorm(adj_α, 0, 1)

(4.3) A52 := crit_t  $\left(\frac{m+1}{n \cdot m}\right)^{0.5}$ 
A5 :=  $\frac{\text{crit}_z}{n^{0.5}}$ 

B102 := fB10
B10 := vB10
B92 := fB9
B9 := vB9

B102sqrt := B1020.5
B10sqrt := B100.5
B92sqrt := B920.5
B9sqrt := B90.5

```

**Page 5 of program: ccfsvc.mcd****FINAL RESULTS:**

- (1)  $\alpha_{\bar{X}} = 0.0027$                        **$\alpha_{\bar{X}}$  -  $\alpha$  for the  $\bar{X}$  chart.**
- (2)  $\alpha_{v_{cu}} = 0.005$                        **$\alpha_{v_{cu}}$  -  $\alpha$  for the  $v_c$  or  $s_c$  chart above the UCL.**
- (3)  $\alpha_{v_{cl}} = 0.001$                        **$\alpha_{v_{cl}}$  -  $\alpha$  for the  $v_c$  or  $s_c$  chart below the LCL \*.**
- (4)  $m = 5$                                        **$m$  - number of subgroups.**
- (5)  $n = 5$                                        **$n$  - subgroup size for the  $(\bar{X}, v_c)$  or  $(\bar{X}, s_c)$  charts.**

$v_1 = 4$                        $v_2 = 24$

**Control Chart Factors****Second Stage**

$A_{52} = 1.63857$        $B_{102} = 4.88978$        $B_{92} = 0.02185$        $B_{102\text{sqrt}} = 2.21129$        $B_{92\text{sqrt}} = 0.14782$

**Conventional**

$A_5 = 1.3416304973$        $B_{10} = 3.7150647501$        $B_9 = 0.0227010089$        $B_{10\text{sqrt}} = 1.9274503237$        $B_{9\text{sqrt}} = 0.1506685398$

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