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On the Optimality of the Modified Generalized Goal Decomposition Model: A Numerical Example

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Abstract

The purpose of this note is to demonstrate that Freeland's [6] modification of Ruefli's Generalized Goal Decomposition model [10] can generate nonoptimal solutions. Although it is theoretically possible to overcome these difficulties, degeneracy makes an optimal implementation of the modified GGD model difficult to achieve.

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1. INTRODUCTION

Several years ago, Freeland [6] suggested that Benders' decomposition algorithm [1] could be used to overcome deficiencies in Ruefli's three-level Generalized Goal Decomposition (GGD) model [10]. In developing and testing a set of computer codes that could implement a group of two-level and three-level "composition" (see [11] for an elaboration of this term) models [4, 6, 7, 10, 12], it was discovered that Freeland's formulation still has computational deficiencies.

2. THE MODIFIED GGD FORMULATION

The three-level organization modeled by the GGD is given in Figure 1. The organization consists of a central unit, M management units (k = 1,...,M) and N operating units. Define a set of integers r_0 through r_M, with r_0 equal to zero and r_M equal to N such that operating units $r_{k-1}+1$ through r_k are subordinate to management unit k.

The mathematical structure of Freeland's modified GGD is given in equations (1) through (9).[1] Equations (1) through (4) specify the central unit's problem, while equations (5) through (7) and (8) through (9) provide the formulation of management unit k and operating unit i, respectively. Nonnegativity requirements although not provided are of course applicable.

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[1] The notation used in equations (1)-(9) differs slightly from that found in either [6] or [10]. Instead, it follows the more general structure seen in Lasdon [9, pp. 150-151].

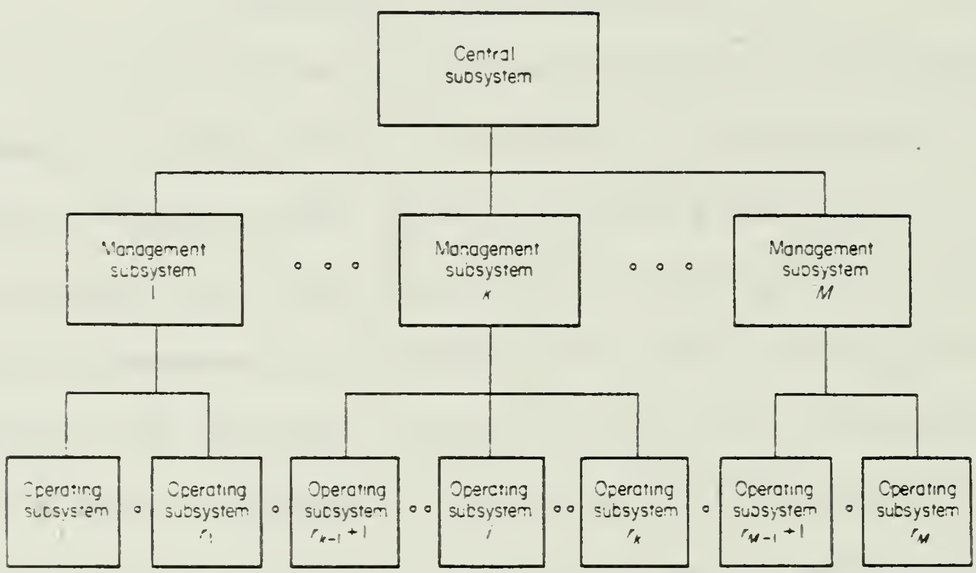

Figure 1

Hierarchical Structure of the Model

Central Unit

$$\text{Min} \sum_{k=1}^{M} \sigma_k(t+1) \qquad (1)$$

$$\text{s.t. } \sigma_k(t+1) - {}_k(\tau)G_k(t+1) \geq Z_k^*(\tau) - {}_k(\tau)G_k(\tau) \qquad (2)$$

for $k = 1,\ldots,M$ and $\tau = 1,\ldots,t$

$$\sum_{k=1}^{M} P_k G_k(t+1) \lesseqgtr G_0 \qquad (3)$$

$$G_k(t+1) \geq 0 \qquad \text{for } k = 1,\ldots,M \qquad (4)$$

In this subproblem, $G_k(\tau)$ is a vector of goals assigned to management unit k on iteration τ, and G_0 is a vector of stipulations for the central unit. P_k is a matrix relating management unit k's goals, $G_k(\tau)$, to G_0. ${}_k(\tau)$ is a vector of dual variables at iteration τ associated with constraint (6) in the k-th management unit's problem. $Z_k^*(\tau)$ is the optimal value of the k-th management unit's objective function at iteration τ.

Management Unit k (k=1,...,M)

$$\text{Min } Z_k(t) = W_k^+ Y_k^+(t) + W_k^- Y_k^-(t) \qquad (5)$$

$$\text{s.t.} \sum_{i=r_{k-1}+1}^{r_k} \sum_{\tau=1}^{t} B_i X_i(\tau)\lambda_i(\tau) - I_{m_k} Y_k^+(t) + I_{m_k} Y_k^-(t) = G_k(t) \qquad (6)$$

$$\sum_{\tau=1}^{t} \lambda_i(\tau) = 1 \qquad \text{for } i = r_{k-1}+1,\ldots,r_k \qquad (7)$$

where I_{m_k} is the $(m_k x m_k)$-identity matrix.

This problem is a standard goal programming formulation that minimizes weighted deviations $W_k^+Y_k^+(t)$ and $W_k^-Y_k^-(t)$ from the goals specified for management unit k on iteration t, $G_k(t)$. B_i is a matrix that relates the i-th operating unit's vector of proposals on iteration t, $X_i(t)$, to $G_k(t)$. Finally $\lambda_i(\tau)$ is a scalar included to generate the convex combination requirement in the Dantzig-Wolfe decomposition procedure [2].

Operating Unit i $(i=r_{k-1}+1,\ldots,r_k)$

$$\text{Min} \quad -\Pi_k(t)\ B_i\ X_i(t+1) \tag{8}$$

$$\text{s.t.}\ D_iX_i(t+1) \lesseqgtr F_i \tag{9}$$

Here D_i is a matrix of technological coefficients relating the vector of operating decisions or proposals for operating unit i, $X_i(t+1)$, at iteration t+1, to a vector of stipulations, F_i. A discussion of the modified GGD iterative solution procedure is given in [6].

3. DISCUSSION

The principal motivation for using Benders' partitioning constraints within the GGD model can be seen in a paper by Freeland and Baker [7]. This two-level model consisting of a central unit and a group of M subordinate managers (k = 1,...,M) is a straightforward application of Benders' algorithm, which was originally defined as a two-level decomposition model. Hence convergence and optimality of the Freeland and Baker two-level model can be demonstrated via Benders' proofs. Unfortunately extension of the proofs of convergence and optimality to the three-level model proposed by Freeland is not straightforward.

A Breakdown in Coordination

In order to construct the partitioning constraints used in the highest level problem of his decomposition procedure, Benders required the optimal solution for each subproblem (SP_k, $k=1,\ldots,M$) at the second level of the decomposed problem on every iteration. In the three-level model, however, subproblem SP_k is an overall problem obtained via an ensemble of subproblems belonging to management k and its subordinate operating units ($i = r_{k-1}+1,\ldots,r_k$). In a three-level GGD context, subproblem SP_k is "solved" via the Dantzig-Wolfe decomposition algorithm. On a given iteration the solution of the generalized linear programming problem normally requires several interactions between a given management unit and its subordinates. Unfortunately Freeland and Ruefli advocate a single interaction between each manager and its operating units on each iteration of the GGD.

Actually Freeland did not directly address himself to this issue. Instead in footnote 1 on page 101 of [6], he states:

> the interaction between management unit k and operating unit j,k is not discussed in this note because Ruefli's procedure handles this with no shortcomings.

Ruefli's position is given in his Figure 2 (see [10], p. B-509). Here Ruefli clearly advocates a single interaction between operating unit j,k [operating unit i ($i = r_{k-1} + 1, \ldots, r_k$) in this paper] and manager k on each iteration t. In this figure on iteration t, the vector, $\Pi_k^{(t)}$, flows simultaneously to both the central unit and operating unit j,k. This flow of information is described on page B-509, lines 3 and 4 of [10]. Using $\Pi_k^{(t)}$ and equation (2.j.k) ([10], p. B-506), operating unit j,k then generates a proposal, $A_{j,k}^{(t+1)}$ [$X_i(t+1)$ in this presentation]. Simultaneously the central unit solves its problems using this same $\Pi_k^{(t)}$ vector to generate $G_k^{(t+1)}$.

If one were to assume more than one interaction per iteration, manager k would initially pass $\Pi_k^{(t,1)}$ [note that it is necessary to include an interaction counter, (t,1)] to its operating units. A series of interactions would continue between manager k and its operating units until the optimal solution to the overall problem for manager k and its operating units is ascertained, given the current value for $G_k^{(t)}$. Let $\Pi_k^{(t,\tau*)}$ be the vector of simplex multipliers for manager k's optimal decision at iteration t. For Benders' algorithm to be implemented correctly, only $\Pi_k^{(t,\tau*)}$ should be sent to the central unit. The central unit must not receive the simplex multiplier vectors, $\Pi_k^{(t,1)}$, ..., $\Pi_k^{(t,\tau*-1)}$.

As a result of a breakdown in coordination within the GGD, the solution of the Dantzig-Wolfe restricted master for subproblem SP_k (i.e., the k-th manager's problem) will probably not generate the optimal solution for subproblem SP_k given $G_k(t)$. Accordingly the simplex multiplier vector, $\Pi_k(t)$, does not necessarily represent the optimum dual solution to subproblem SP_k. In addition, objective function value for subproblem SP_k is probably suboptimal. From an implementation viewpoint nonoptimal (dual infeasible) simplex multipliers and objective function values can have a dramatic impact upon the solution process. These nonoptimal inputs introduce cutting planes into the central unit's problem, and once introduced they remain for all subsequent iterations. [See equation (2).]

A Numerical Example of Nonoptimality

The following example shows the potential for nonoptimality in Benders' algorithm when subproblems at the lower level decision-making units are not solved optimally. This example was originally formulated by Burton, Damon, and Loughridge [2]. Freeland [5] later showed that Benders' partioning procedure could be applied to this problem. In his application, the problem has two divisions whose subproblems are given below.

Division I (on iteration t)

$$\max \phi_1^t = x_{11}^t + x_{12}^t$$

$$\begin{aligned} \text{s.t.} \quad x_{11}^t + x_{12}^t &\leq b_{11}^t \quad (v_1^t) \\ 2x_{11}^t + 3x_{12}^t &\leq b_{12}^t \quad (v_2^t) \\ x_{11}^t + 3x_{12}^t &\leq 30 \quad (v_3^t) \\ 2x_{11}^t + x_{12}^t &\leq 20 \quad (v_4^t) \\ x_{11}^t,\ x_{12}^t &\geq 0 \end{aligned}$$

Division II (on iteration t)

$$\max \phi_1^t = 2x_{21}^t + x_{22}^t$$

$$\begin{aligned} \text{s.t.} \quad 3x_{21}^t + x_{22}^t &\leq b_{21}^t \quad (z_1^t) \\ 3x_{21}^t + 2x_{22}^t &\leq b_{22}^t \quad (z_2^t) \\ x_{21}^t &\leq 10 \quad (z_3^t) \\ x_{22}^t &\leq 10 \quad (z_4^t) \\ x_{21}^t + x_{22}^t &\leq 15 \quad (z_5^t) \\ x_{21}^t,\ x_{22}^t &\geq 0 \end{aligned}$$

For these problems $v_i^t(i=1, \ldots, 4)$ and $z_j^t(j=1, \ldots, 5)$ are the dual variables of Division I and II on iteration t, respectively. The coordinator's problem in this two-level hierarchy on iteration t is

$$\max \quad \sigma_1^t + \sigma_2^t$$

$$\begin{array}{lllll}
\text{s.t.} & \sigma_1^t - v_1^\tau b_{11}^k - v_2^\tau b_{12}^k & & \leq \phi_1^\tau & \\
 & \sigma_2^t - z_1^\tau b_{21}^t - z_2^\tau b_{22}^t & & \leq \phi_2^\tau & \Big\} \ (\tau=1, \ldots, t-1) \\
 & b_{11}^t + b_{21}^t & & \leq 41 & \\
 & b_{12}^t + b_{22}^t & & \leq 62 & \\
 & b_{11}^t, \ b_{12}^t, \ b_{21}^t, \ b_{22}^t & & \geq 0 &
\end{array}$$

Freeland [5] gives the solutions of the coordinator's problem for the first three iterations. On the third iteration, an optimum solution to this problem (multiple optimum solutions do exist) is given as

$$\sigma_1^3 = 10 \qquad b_{11}^3 = 15 \qquad b_{12}^3 = 20$$

$$\sigma_2^3 = 26 \qquad b_{21}^3 = 26 \qquad b_{22}^3 = 42 \quad .$$

On iteration 4, the optimum solution values for Division I's problem are:

$$\phi_1^4 = 10 \qquad x_{11}^4 = 10 \qquad v_1^4 = 0$$

$$x_{12}^4 = 0 \qquad v_2^4 = .5 \quad .$$

The solution of Division II's problem on iteration 4 requires three tableaus. The solution values for each tableau are given in Table 1, where Π_1^4 and Π_2^4 are the first two components of the simplex multiplier vector for iteration 4.

Table 1

Values for Division II's Subproblem on Iteration Four

Tableau	ϕ_2^4	x_{21}^4	x_{22}^4	Π_1^4	Π_2^4
1	0	0	0	0	0
2	20	10	0	0	0
3	25	10	5	0	0

At the optimum, the simplex multiplier equals the dual solution implying that $(z_1^4, z_2^4) = (0,0)$. Note for this example the first two components of the simplex multiplier vector equal the first two components of the dual solution vector at each tableau.

The coordinator's problem on iteration 4 is:

$$\text{Max } \sigma_1^4 + \sigma_2^4$$

$$\begin{aligned}
\text{s.t. } \sigma_1^4 \quad - 0b_{11}^4 - .25b_{12}^4 \quad &\leq 5 \\
\sigma_2^4 \quad - 0b_{21}^4 - .5b_{22}^4 &\leq 5 \\
\sigma_1^4 \quad - 0b_{11}^4 - .5b_{12}^4 \quad &\leq 0 \\
\sigma_2^4 \quad - 0b_{21}^4 - b_{22}^4 &\leq 0 \\
\sigma_1^4 \quad - 0b_{11}^4 - .5b_{12}^4 \quad &\leq 10 \\
\sigma_2^4 \quad - 0b_{21}^4 - 0b_{22}^4 &\leq \phi_2^4 \\
b_{11}^4 + b_{21}^4 &\leq 41 \\
b_{12}^4 + b_{22}^4 &\leq 62 \\
b_{11}^4, \; b_{12}^4, \; b_{21}^4, \; b_{22}^4 &\geq 0
\end{aligned}$$

If ϕ_2^4 is set to the optimum value of 25 then one of the multiple optimum solutions to this problem is given by Freeland as:

$\sigma_1^4 = 10.5$ $\qquad b_{11}^4 = 16$ $\qquad b_{12}^4 = 22$

$\sigma_2^4 = 25$ $\qquad b_{21}^4 = 25$ $\qquad b_{22}^4 = 40$.

Thus for iteration four the optimum value of the coordinator's objective function is 35.5, and this is the optimum solution to the overall problem.

Suppose that a feasible but nonoptimum solution for Division II was passed to the coordinator on iteration four. In particular, assume that the solution corresponding to the second tableau was transmitted. In this case, the sixth constraint of the coordinator's problem on iteration four,

$$\sigma_2^4 \leqslant \phi_2^4 = 25,$$

becomes:

$$\sigma_2^4 \leqslant 20$$

The optimum solution to this revised problem is:

$$\sigma_1^4 = 10.5 \qquad \sigma_2^4 = 20,$$

which implies that

$$\sigma_1^4 + \sigma_2^4 = 30.5 \leqslant 35.5.$$

Because the constraints placed in the coordinator's problem on iteration 4 remain for all subsequent iterations, the solution process is thus destined to nonoptimality.

It is important to note that in this example, only a nonoptimal objective function value was introduced. In most cases, an infeasible dual solution would also be passed if subproblem SP_k were not optimally solved. This situation will further exacerbate the ability of the Benders' partioning procedure to achieve the optimal solution.

4. CONCLUSION

Based upon the preceding example it is clear that if the optimality of manager k's solution to subproblem SP_k cannot be demonstrated at every iteration, the optimality of Freeland's three-level model cannot be demonstrated. Theoretically, these difficulties could be a eliminated by allowing manager k and its subordinates to interact via the Dantzig-Wolfe algorithm until the optimal solution to SP_k is obtained. Unfortunately the goal programming structure of this problem often makes the Dantzig-Wolfe procedure difficult to implement because the solution of the restrictive master for SP_k tends to be highly degenerate [8]. This degeneracy mathematically implies the existence of multiple optimum solutions for the dual of this problem. When multiple optimal dual solutions exist for the Dantzig-Wolfe restricted master, a viable convergence criterion is often difficult to implement. When the Dantzig-Wolfe algorithm was applied to several test problems with a structure similar to that of subproblem SP_k, it converged to and remained at the same nonoptimal objective function value for several iterations (sometimes ten or more) before displaying further progress toward the optimum solution. No foolproof stopping criterion has been developed that does not compromise to some degree, the basic mathematical intent of the decomposition procedure. Thus an optimal implementation of Freeland's three-level model, though theoretically possible with the proposed modifications, may be computationally difficult to achieve.

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