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New results on the existence of ground state solutions for generalized quasilinear Schrödinger equations coupled with the Chern–Simons gauge theory

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Abstract. In this paper, we study the following quasilinear Schrödinger equation

$$\begin{aligned}-\Delta u + V(x)u - \kappa u\Delta(u^2) + \mu\frac{h^2(|x|)}{|x|^2}(1+\kappa u^2)u\\ + \mu\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}(2+\kappa u^2(s))u^2(s)\mathrm{d}s\right)u = f(u) \quad \text{in } \mathbb{R}^2,\end{aligned}$$

where $\kappa > 0$, $\mu > 0$, $V \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. By using a constraint minimization of Pohožaev–Nehari type and analytic techniques, we obtain the existence of ground state solutions.

Keywords: gauged Schrödinger equation, Pohožaev identity, ground state solutions.

2020 Mathematics Subject Classification: 35J60, 35J20.

1 Introduction

In this paper, we are interested in the existence of ground state solutions for the following nonlocal quasilinear Schrödinger equation

$$\begin{aligned}-\Delta u + V(x)u - \kappa u\Delta(u^2) + \mu\frac{h^2(|x|)}{|x|^2}(1+\kappa u^2)u\\ + \mu\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}(2+\kappa u^2(s))u^2(s)\mathrm{d}s\right)u = f(u) \quad \text{in } \mathbb{R}^2,\end{aligned} \tag{1.1}$$

where $u : \mathbb{R}^2 \to \mathbb{R}$ is a radially symmetric function, κ, μ are positive constants, $h(s) = \int_0^s u^2(l)l\mathrm{d}l$ $(s \geq 0)$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfies the following suitable assumptions:

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(f_1) $\lim_{|s|\to 0}\frac{f(s)}{s}=0$ and there exist constants $C>0$ and $q\in(2,+\infty)$ such that

$$|f(s)|\le C(1+|s|^{q-1}),\quad \forall s\in\mathbb{R};$$

(f_2) there exists a constant $p\in(6,8)$ such that $\lim_{|s|\to+\infty}\frac{F(s)}{|s|^p}=+\infty$, where $F(s)=\int_0^s f(t)\mathrm{d}t$;

(f_3) $\frac{[f(s)s-(8-p)F(s)]}{|s|^{p-1}s}$ is nondecreasing on both $(-\infty,0)$ and $(0,+\infty)$.

Moreover, we assume that potential $V:\mathbb{R}^2\to\mathbb{R}$ verifies:

(V_1) $V\in\mathcal{C}^1(\mathbb{R}^2,\mathbb{R})$ and $V_\infty:=\lim_{|y|\to+\infty}V(y)>V_0:=\min_{x\in\mathbb{R}^2}V(x)>0$ for all $x\in\mathbb{R}^2$;

(V_2) $t\to t^{6\alpha-2}\big[(2\alpha-2)V(tx)-\nabla V(tx)\cdot(tx)\big]$ is nondecreasing on $(0,+\infty)$ for any $x\in\mathbb{R}^2$, where $\alpha:=\frac{2}{8-p}>1$, which is inspired by [6] where Kirchhoff-type problems were studied.

If $\kappa=0$, (1.1) turns into the following nonlocal elliptic problem

$$-\Delta u+V(x)u+\mu\frac{h^2(|x|)}{|x|^2}u+2\mu\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}u^2(s)\mathrm{d}s\right)u=f(u)\quad\text{in }\mathbb{R}^2. \tag{1.2}$$

(1.2) appears in the study of the following Chern–Simons–Schrödinger system

$$\begin{cases} iD_0\phi+(D_1D_1+D_2D_2)\phi+f(\phi)=0,\\ \partial_0A_1-\partial_1A_0=-\mathrm{Im}(\overline{\phi}D_2\phi),\\ \partial_0A_2-\partial_2A_0=-\mathrm{Im}(\overline{\phi}D_1\phi),\\ \partial_1A_2-\partial_2A_1=-\frac{1}{2}|\phi|^2,\end{cases} \tag{1.3}$$

where i denotes the imaginary unit, $\partial_0=\frac{\partial}{\partial t}$, $\partial_1=\frac{\partial}{\partial x_1}$, $\partial_2=\frac{\partial}{\partial x_2}$ for $(t,x_1,x_2)\in\mathbb{R}^{1+2}$, $\phi:\mathbb{R}^{1+2}\to\mathbb{C}$ is the complex scalar field, $A_\mu:\mathbb{R}^{1+2}\to\mathbb{R}$ is the gauge field, $D_\mu=\partial_\mu+iA_\mu$ is the covariant derivative for $\mu=0,1,2$. Model (1.3) was first proposed and studied in [12,13], which described the non-relativistic thermodynamic behavior of large number of particles in an electromagnetic field. In [1], the authors considered the standing waves of system (1.3) with power type nonlinearity, that is, $f(u)=\lambda|u|^{p-1}u$, and established the existence and nonexistence of positeve solutions for (1.3) of type

$$\begin{aligned} \phi(t,x)&=u(|x|)e^{iwt}, & A_0(t,x)&=k(|x|),\\ A_1(t,x)&=\frac{x_2}{|x|^2}h(|x|), & A_2(t,x)&=-\frac{x_1}{|x|^2}h(|x|),\end{aligned} \tag{1.4}$$

where $w>0$ is a given frequency, $\lambda>0$ and $p>1$, u,k,h are real valued functions depending only on $|x|$. The ansatz (1.4) satisfies the Coulomb gauge condition $\partial_1A_1+\partial_2A_2=0$. Byeon et al. [1] got the following nonlocal semi-linear elliptic equation

$$-\Delta u+wu+\frac{h^2(|x|)}{|x|^2}u+\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}u^2(s)\mathrm{d}s\right)u=\lambda|u|^{p-1}u\quad\text{in }\mathbb{R}^2. \tag{1.5}$$

Later, based on the work of [1], the results for the case $p\in(1,3)$ have been extended by Pomponio and Ruiz in [20]. They investigated the geometry of the functional associated with (1.5) and obtained an explicit threshold value for w. The existence and properties of ground state solutions of (1.5) have also been studied widely by many researchers, see, e.g., [2,7,10,11, 14,19,21,29,31,33,35] and references therein. If we replace $w > 0$ with the radially symmetric potential V and more general nonlinearity f, then (1.5) will turns into (1.2). Very recently, by using variational methods, Chen et al. in [4] studied the existence of sign-changing multi-bump solutions for (1.2) with deepening potential. In [25], when f satisfied more general 6-superlinear conditions, Tang et al. proved the existence and multiplicity results of (1.2). For more related work about the problem (1.2), we refer to [9,15,28,35] and references therein.

If $\mu = 0$, (1.1) reduces to the following quasilinear elliptic problem

$$-\Delta u + V(x)u - \kappa u\Delta(u^2) = f(u) \quad \text{in } \mathbb{R}^2. \tag{1.6}$$

(1.6) is obtained from the quasilinear Schrödinger equation

$$i\hat{\phi}_t + \Delta\hat{\phi} - W(x)\hat{\phi} + \kappa\hat{\phi}\Delta(|\hat{\phi}|^2) + \hat{h}(|\hat{\phi}|^2)\hat{\phi} = 0 \quad \text{in } \mathbb{R}^2,$$

by setting $\hat{\phi} = e^{-iwt}u(x)$, $V(x) = W(x) - w$, where $w \in \mathbb{R}$, W is a given potential, $\hat{h}$ is a suitable function. The existence and properties of ground state solutions of (1.6) as well as the stability of standing wave solutions have also been studied widely in [16,32] and references therein.

Motivated by [3,8], we try to establish the existence of positive ground state solutions for (1.1) involving radially symmetric variable potential V and more general nonlinearity f than [8]. Compared to [3], the equation (1.1) has appearance the Chern–Simons terms

$$\left(\int_{|x|}^{+\infty} \frac{h(s)}{s}u^2(s)\mathrm{d}s + \frac{h^2(|x|)}{|x|^2}\right)u,$$

so that the equation (1.1) is no longer a pointwise identity. This nonlocal term causes some mathematical difficulties that make the study of it is rough and particularly interesting. To overcome these difficulties, we adopted a constraint minimization of the Pohožaev–Nehari type as in [5,8] and establish some new inequalities.

In order to state our main theorem, let us define the metric space

$$\chi = \left\{u \in H_r^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} u^2|\nabla u^2|\mathrm{d}x < +\infty\right\} = \left\{u \in H_r^1(\mathbb{R}^2) : u^2 \in H_r^1(\mathbb{R}^2)\right\},$$

endowed with the distance

$$d_\chi(u,v) = \|u - v\| + \|\nabla(u^2) - \nabla(v^2)\|_{L^2}.$$

We will show that (1.1) can obtain the following energy functional: $I : \chi \to \mathbb{R}$,

$$\begin{aligned} I(u) = \frac{1}{2}\int_{\mathbb{R}^2} [(1 + 2\kappa u^2)|\nabla u|^2 + V(x)u^2]\mathrm{d}x + \frac{\mu}{2}\int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} su^2(s)\mathrm{d}s\right)^2 \mathrm{d}x \\ + \frac{\mu}{4}\kappa\int_{\mathbb{R}^2} \frac{u^4(x)}{|x|^2}\left(\int_0^{|x|} su^2(s)\mathrm{d}s\right)^2 dx - \int_{\mathbb{R}^2} F(u)\mathrm{d}x, \qquad \forall u \in \chi. \end{aligned} \tag{1.7}$$

Similarly to [1,8,16,22,29], any weak solution u of (1.1) satisfies the Pohožaev identity, that is, $P(u) = 0$. For the nice properties of the generalized Nehari manifold, we refer to previous works in [17, 18, 34] and references therein. Inspired by this fact, we define the following Pohožaev–Nehari functional $\Gamma(u) = \alpha N(u) - P(u)$ and the Pohožaev–Nehari manifold of I

$$\mathcal{M} := \left\{u \in \chi\backslash\{0\} : \Gamma(u) = 0\right\}.$$

Although χ is not a vector space (it is not close with the respect to the sum), it is easy to check that I is well-defined and continuous on χ. For any $\varphi \in C_{0,r}^{\infty}(\mathbb{R}^2)$, $u \in \chi$ and $u+\varphi \in \chi$, we can compute the Gateaux derivative

$$\langle I'(u),\varphi\rangle = \int_{\mathbb{R}^2}\left\{(1+2\kappa u^2)\nabla u\cdot\nabla\varphi + 2\kappa u|\nabla u|^2\varphi + V(x)u\varphi + \mu\frac{h^2(|x|)}{|x|^2}(1+\kappa u^2)u\varphi\right\}\mathrm{d}x$$
$$+\mu\int_{\mathbb{R}^2}\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}(2+\kappa u^2(s))u^2(s)\mathrm{d}s\right)u\varphi\mathrm{d}x - \int_{\mathbb{R}^2} f(u)\varphi\mathrm{d}x. \quad (1.8)$$

Then $u \in \chi$ is a weak solution of (1.1) if and only if the Gateaux derivative of I along any direction $\varphi \in C_{0,r}^{\infty}(\mathbb{R}^2)$ vanishes (see Proposition 2.2 below). A radial weak solution is called a radial ground state solution if it has the least energy among all nontrivial radial weak solutions.

Our main result is the following theorem.

Theorem 1.1. *Assume that* (V_1)–(V_2) *and* (f_1)–(f_3) *are satisfied. Then* (1.1) *has a positive ground state solution* $\bar{u} \in \chi\backslash\{0\} \cap C^2(\mathbb{R}^2)$*, such that* $I(\bar{u}) = \inf_{u\in\mathcal{M}} I(u) = \inf_{u\in\chi\backslash\{0\}}\max_{t>0} I(u_t)$ *where* $u_t = (u)_t := t^{\alpha}u(tx)$.

Remark 1.2. Theorem 1.1 can be viewed as a partial extension to the counterpart of the result and method in [8]. The assumptions on f in this paper are from the reference [5]. Furthermore, by [5, Remark 1.4],

$$f(u) = (|u|^{p-2} - a|u|^{q-2})u,$$

satisfies (f_1)–(f_3) when $a > 0$ and $2 < q < p \in (6, 8]$.

To prove the Theorem 1.1, by using some new techniques and inequalities related to $I(u)$, $I(u_t)$ and $\Gamma(u)$, as performed in [3, 5, 24], we prove that a minimizing sequence $\{u_n\} \subset \chi$ of $\inf_{u\in\mathcal{M}} I(u)$ weakly converges to some nontrivial $\bar{u}$ in χ (after a translation and extraction of a subsequence) and $\bar{u} \in \mathcal{M}$ is a minimizer of $\inf_{u\in\mathcal{M}} I(u)$.

Notations. Throughout this paper, we make use of the following notations:

- V_∞ is a positive constant;
- C, C_0, C_1, C_2,... denote positive constants, not necessarily the same one;
- $L^r(\mathbb{R}^2)$ denotes the Lebesgue space with norm $\|u\|_{L^r} = \left(\int_{\mathbb{R}^2}|u|^r\mathrm{d}x\right)^{1/r}$, where $1 \le r < +\infty$;
- $H^1(\mathbb{R}^2)$ denotes a Sobolev space with norm $\|u\| = \left(\int_{\mathbb{R}^2} u^2 + |\nabla u|^2\mathrm{d}x\right)^{1/2}$;
- $H_r^1(\mathbb{R}^2) := \{u \in H^1(\mathbb{R}^2) : u \text{ is radially symmetric}\}$;
- $C_{0,r}^{\infty}(\mathbb{R}^2) := \{u \in C_0^{\infty}(\mathbb{R}^2) : u \text{ is radially symmetric}\}$;
- For any $x \in \mathbb{R}^2$ and $r > 0$, $B_r(x) = \{y \in \mathbb{R}^2 : |y - x| < r\}$;
- " $\rightharpoonup$ " and " $\to$ " denote weak and strong convergence, respectively.

2 Variational framework and preliminaries

In this section, we will give the variational framework of (1.1) and some preliminaries. Now we find that if $u \in \chi$ is a solution of (1.1), then it solves $Q(u) = 0$, where

$$Q(u) = \operatorname{div} A(u, \nabla u) + B(x, u, \nabla u),$$

with

$$\begin{aligned} A(u, \nabla u) &= (1 + 2\kappa u^2)\nabla u, \\ B(x, u, \nabla u) &= -\Big(2\kappa|\nabla u|^2 + V(x) + \mu K_1(x)(1 + \kappa u^2) + \mu K_2(x)\Big)u + f(u), \end{aligned} \tag{2.1}$$

and

$$K_1(x) = \begin{cases} \frac{h^2(|x|)}{|x|^2}, & x \neq 0, \\ 0, & x = 0, \end{cases} \qquad K_2(x) = \int_{|x|}^{+\infty} \frac{h(s)}{s}\Big(2 + \kappa u^2(s)\Big)u^2(s)\mathrm{d}s.$$

We observe from (2.1) that (1.1) is a quasilinear elliptic equation with principal part in divergence form and it satisfies all the structure conditions in [19] or [26].

In order to show that any weak solutions of (1.1) are classical ones, we introduce the following lemma.

Lemma 2.1 ([8]). *Let us fix $u \in \chi$. We have:*

(i) K_1, K_2 are nonnegative and bounded;

(ii) if we suppose further that $u \in C(\mathbb{R}^2)$, then $K_1, K_2 \in C^1(\mathbb{R}^2)$.

Arguing as in [1,8], standard computations show that

Proposition 2.2. *The functional I in (1.7) is well-defined and continuous in χ and if the Gateaux derivative of I evaluated in $u \in \chi$ is zero in every direction $\varphi \in C^\infty_{0,r}(\mathbb{R}^2)$, then u is a weak solution of (1.1). Furthermore, the weak solution of (1.1) belongs to $C^2(\mathbb{R}^2)$, so the weak solution u is a classical solution of (1.1).*

Lemma 2.3. *Any weak solution u of (1.1) satisfies the Nehari identity $N(u) = 0$ and the Pohožaev identity $P(u) = 0$, where*

$$N(u) = \int_{\mathbb{R}^2} \left[(1 + 4\kappa u^2)|\nabla u|^2 + V(x)u^2 + \mu\frac{h^2(|x|)}{|x|^2}(3 + 2\kappa u^2)u^2\right] dx - \int_{\mathbb{R}^2} f(u)u dx, \tag{2.2}$$

$$P(u) = \int_{\mathbb{R}^2} \left[V(x)u^2 + \frac{1}{2}\nabla V(x)\cdot x|u|^2 + \mu\frac{h^2(|x|)}{|x|^2}(2 + \kappa u^2)u^2\right] dx - 2\int_{\mathbb{R}^2} F(u)dx. \tag{2.3}$$

Proof. By a density argument, we can use $u \in \chi$ as a test function in (1.8), we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \Big[(1 + 2\kappa u^2)|\nabla u|^2 + 2\kappa u^2|\nabla u|^2 + V(x)u^2 - f(u)u\Big]\mathrm{d}x \\ &\quad + \mu\int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2}(1 + \kappa u^2)u^2 + \mu\int_{\mathbb{R}^2}\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}(2 + \kappa u^2(s))u^2(s)\mathrm{d}s\right)u^2\mathrm{d}x = 0. \end{aligned} \tag{2.4}$$

We claim that: for $\beta = 2$ or $\beta = 4$, we have

$$\int_{\mathbb{R}^2} \frac{h^2(|x|)}{|x|^2} u^\beta \mathrm{d}x = \int_{\mathbb{R}^2} \left(\int_{|x|}^{+\infty} \frac{u^\beta(s)h(s)}{s} \mathrm{d}s \right) u^2 \mathrm{d}x.$$

Now we using the integration by parts to prove the claim. A simple computation yields that

$$\begin{aligned} \int_{\mathbb{R}^2} \left[\frac{u^\beta h(|x|)}{|x|^2} \left(\int_0^{|x|} s u^2(s) \mathrm{d}s \right) \right] \mathrm{d}x &= \int_0^{2\pi} \left[\int_0^{+\infty} \frac{u^\beta h(r)}{r^2} \left(\int_0^r s u^2(s) \mathrm{d}s \right) r \mathrm{d}r \right] \mathrm{d}\theta \\ &= \int_0^{2\pi} \int_0^{+\infty} \left(\int_r^{+\infty} \frac{u^\beta(s)h(s)}{s} \mathrm{d}s \right) u^2 r \mathrm{d}r \mathrm{d}\theta \\ &= \int_{\mathbb{R}^2} \left(\int_{|x|}^{+\infty} \frac{u^\beta(s)h(s)}{s} \mathrm{d}s \right) u^2 \mathrm{d}x. \end{aligned}$$

Then, we conclude that the identity $N(u) = 0$ holds.

Next, let $u \in \chi \cap C^2(\mathbb{R}^2)$ be a solution of (1.1). Then multiplying by $\nabla u \cdot x$ and integrating by parts on B_R. Arguing as in [1,8], we get the following identities:

$$\begin{aligned} \int_{B_R} \Delta u (\nabla u \cdot x) \mathrm{d}x &= \int_{\partial B_R} \frac{\partial u}{\partial \overrightarrow{n}} (\nabla u \cdot x) \mathrm{d}S_x - \int_{B_R} \nabla u \cdot \nabla (\nabla u \cdot x) \mathrm{d}x \\ &= R \int_{\partial B_R} \left(\frac{\partial u}{\partial \overrightarrow{n}} \right)^2 \mathrm{d}S_x - \frac{R}{2} \int_{\partial B_R} |\nabla u|^2 \mathrm{d}S_x \\ &= \frac{R}{2} \int_{\partial B_R} |\nabla u|^2 \mathrm{d}S_x =: \mathrm{I}, \end{aligned}$$

$$\begin{aligned} \int_{B_R} u \Delta(u^2) (\nabla u \cdot x) \mathrm{d}x &= \int_{\partial B_R} \frac{\partial u^2}{\partial \overrightarrow{n}} u (\nabla u \cdot x) \mathrm{d}S_x - \int_{B_R} \nabla u^2 \cdot \nabla (u (\nabla u \cdot x)) \mathrm{d}x \\ &= \frac{R}{2} \int_{\partial B_R} \left(\frac{\partial u^2}{\partial \overrightarrow{n}} \right)^2 \mathrm{d}S_x - \frac{1}{2} \int_{B_R} \nabla u^2 \cdot \nabla (\nabla u^2 \cdot x) \mathrm{d}x \\ &= \frac{R}{4} \int_{\partial B_R} |\nabla u^2|^2 \mathrm{d}S_x =: \mathrm{II}, \end{aligned}$$

$$\begin{aligned} \int_{B_R} V(x) u (\nabla u \cdot x) \mathrm{d}x &= \int_{B_R} V(x) \left(\nabla \left(\frac{1}{2} u^2 \right) \cdot x \right) \mathrm{d}x \\ &= - \int_{B_R} V(x) u^2 \mathrm{d}x - \frac{1}{2} \int_{B_R} \Big(\nabla V(x) \cdot x \Big) u^2 \mathrm{d}x + \frac{R}{2} \int_{\partial B_R} V(x) u^2 \mathrm{d}S_x \\ &=: - \int_{B_R} V(x) u^2 \mathrm{d}x - \frac{1}{2} \int_{B_R} \Big(\nabla V(x) \cdot x \Big) u^2 \mathrm{d}x + \mathrm{III}, \end{aligned}$$

$$\begin{aligned} \int_{B_R} f(u) (\nabla u \cdot x) \mathrm{d}x &= \int_{B_R} \nabla (F(u)) \cdot x \mathrm{d}x \\ &= -2 \int_{B_R} F(u) \mathrm{d}x + R \int_{\partial B_R} F(u) \mathrm{d}S_x \\ &=: -2 \int_{B_R} F(u) \mathrm{d}x + \mathrm{IV}. \end{aligned}$$

We note that if $f(x) \geq 0$ is integrable on $\mathbb{R}^2$, then $\liminf_{R \to +\infty} R \int_{\partial B_R} f \mathrm{d}S = 0$. Since $u \in \chi$, then $u^2 \in H^1(\mathbb{R}^2)$ and the integrands in the terms I, II, III and IV are all nonnegative and contained in $L^1(\mathbb{R}^2)$, one can take a sequence $\{R_j\}$ such that the terms I, II, III and IV with R_j

replacing R converge to 0 as $j \to +\infty$. Moreover, for $\beta = 2$ or $\beta = 4$, we have

$$\begin{aligned}
&\frac{4}{\beta}\int_{B_{R_j}}\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}u^{\beta}(s)\mathrm{d}s\right)u(\nabla u\cdot x)\mathrm{d}x+\int_{B_{R_j}}\frac{h^2(|x|)}{|x|^2}u^{\beta-1}(\nabla u\cdot x)\mathrm{d}x\\
&=\int_{B_{R_j}}\frac{h^2(|x|)}{|x|^2}u^{\beta-1}(\nabla u\cdot x)\mathrm{d}x+\frac{4}{\beta}\int_{B_{R_j}}\frac{u^{\beta}(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\mathrm{d}s\right)\left(\int_0^{|x|}s^2u(s)u'(s)\mathrm{d}s\right)\mathrm{d}x\\
&\quad-\frac{4}{\beta}\int_{B_{R_j}}\frac{u^{\beta}(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\mathrm{d}s\right)\left(\int_0^{|x|}s^2u(s)u'(s)\mathrm{d}s\right)\mathrm{d}x\\
&\quad+\frac{4}{\beta}\int_{B_{R_j}}\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}u^{\beta}(s)\mathrm{d}s\right)u(\nabla u\cdot x)\mathrm{d}x\\
&=\frac{1}{\beta}\frac{d}{dt}\bigg|_{t=1}\int_{B_{R_j}}\frac{u^{\beta}(tx)}{|x|^2}\left(\int_0^{|x|}su^2(ts)\mathrm{d}s\right)^2\mathrm{d}x\\
&\quad-\frac{4}{\beta}\int_{B_{R_j}}\frac{u^{\beta}(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\mathrm{d}s\right)\left(\int_0^{|x|}s^2u(s)u'(s)\mathrm{d}s\right)\mathrm{d}x\\
&\quad+\frac{4}{\beta}\int_{B_{R_j}}\left(\int_{|x|}^{+\infty}\frac{h(s)}{s}u^{\beta}(s)\mathrm{d}s\right)u(\nabla u\cdot x)\mathrm{d}x\\
&=-\frac{4}{\beta}\int_{B_{R_j}}\frac{u^{\beta}(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\mathrm{d}s\right)^2\mathrm{d}x+\frac{R_j}{\beta}\int_{\partial B_{R_j}}\frac{u^{\beta}(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\mathrm{d}s\right)^2\mathrm{d}S_x\\
&\quad+\frac{4}{\beta}\left(\int_{(\mathbb{R}^2\setminus B_{R_j})}\frac{u^{\beta}(x)h(|x|)}{|x|^2}\mathrm{d}x\right)\int_0^{R_j}s^2u(s)u'(s)\mathrm{d}s\\
&=-\frac{4}{\beta}\int_{B_{R_j}}\frac{u^{\beta}(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\mathrm{d}s\right)^2\mathrm{d}x+o_n(1).
\end{aligned}$$

Then, from (1.1), we get

$$\int_{B_{R_j}}\left[V(x)u^2+\frac{1}{2}\nabla V(x)\cdot x|u|^2+\mu\frac{h^2(|x|)}{|x|^2}(2+\kappa u^2)u^2\right]\mathrm{d}x-2\int_{B_{R_j}}F(u)\mathrm{d}x+o_n(1)=0.$$

This implies that $P(u) = 0$ holds. The proof is completed. □

Remark 2.4. From (2.2) and (2.3), by Lemma 2.3, any weak solution of (1.1) belongs to $\mathcal{M}$.

For functionals $D(u)$, $E(u)$ (see Section 3 below), we have the following compactness lemma:

Lemma 2.5 ([8]). *Suppose that a sequence $\{u_n\}$ converges weakly to a function u in $H_r^1(\mathbb{R}^2)$ as $n \to +\infty$. Then for each $\psi \in H_r^1(\mathbb{R}^2)$, $D(u_n)$, $D'(u_n)\psi$ and $D'(u_n)u_n$, $E(u_n)$, $E'(u_n)\psi$ and $E'(u_n)u_n$ converges up to a subsequence to $D(u)$, $D'(u)\psi$ and $D'(u)u$, $E(u)$, $E'(u)\psi$, and $E'(u)u$, respectively, as $n \to +\infty$.*

3 Existence of ground state solutions

Throughout this section, for any $u \in \chi$, we denote

$$A(u)=\int_{\mathbb{R}^2}|\nabla u|^2\mathrm{d}x,\qquad B(u)=\int_{\mathbb{R}^2}V(x)u^2\mathrm{d}x,\qquad C(u)=\int_{\mathbb{R}^2}u^2|\nabla u|^2\mathrm{d}x,$$

$$D(u)=\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\mathrm{d}s\right)^2\mathrm{d}x,$$
$$E(u)=\int_{\mathbb{R}^2}\frac{u^4(x)}{|x|^2}\left(\int_0^{|x|}su^2(s)\mathrm{d}s\right)^2\mathrm{d}x.$$

To complete the proof of Theorem 1.1, we prepare several lemmas.

Lemma 3.1. *Assume that* (f_1) *and* (f_3) *hold. Then*

$$g_1(t,\varrho):=t^{-2}F(t^\alpha\varrho)-F(\varrho)+\frac{1-t^{8\alpha-4}}{4(2\alpha-1)}\left[\alpha f(\varrho)\varrho-2F(\varrho)\right]\geq 0,\qquad \forall t>0,\ \varrho\in\mathbb{R},\tag{3.1}$$

and

$$f(\varrho)\varrho-\frac{(8\alpha-2)}{\alpha}F(\varrho)\geq 0,\qquad \forall\varrho\in\mathbb{R}.\tag{3.2}$$

Proof. It is easy to see that $g_1(t,0)\geq 0$. For $\varrho\neq 0$, by (f_3), we have

$$\begin{aligned}\frac{d}{dt}g_1(t,\varrho)&=t^{8\alpha-5}|\varrho|^{\frac{8\alpha-2}{\alpha}}\left[\frac{\alpha f(t^\alpha\varrho)t^\alpha\varrho-2F(t^\alpha\varrho)}{|t^\alpha\varrho|^{\frac{8\alpha-2}{\alpha}}}-\frac{\alpha f(\varrho)\varrho-2F(\varrho)}{|\varrho|^{\frac{8\alpha-2}{\alpha}}}\right]\\&=\frac{2t^{\frac{5p-24}{8-p}}|\varrho|^p}{8-p}\left[\frac{f(t^{\frac{2}{8-p}}\varrho)t^{\frac{2}{8-p}}\varrho-(8-p)F(t^{\frac{2}{8-p}}\varrho)}{|t^{\frac{2}{8-p}}\varrho|^p}-\frac{f(\varrho)\varrho-(8-p)F(\varrho)}{|\varrho|^p}\right],\end{aligned}$$

and this expression is greater than or equal to zero for $t\geq 1$ and less than or equal to zero for $0<t<1$. Together with the continuity of $g_1(\cdot,\varrho)$, this implies that $g_1(t,\varrho)\geq g_1(1,\varrho)=0$ for all $t\geq 0$ and $\varrho\in\mathbb{R}\backslash\{0\}$. This shows that (3.1) holds. By (f_1) and (3.1), we have

$$\lim_{t\to 0}g_1(t,\varrho)=\frac{1}{4(2\alpha-1)}\left[\alpha f(\varrho)\varrho-(8\alpha-2)F(\varrho)\right]\geq 0,\qquad\forall\varrho\in\mathbb{R},$$

which implies that (3.2) holds. □

Lemma 3.2. *Assume that* (V_1)–(V_2) *hold. Then*

$$\begin{aligned}g_2(t,x):=&\,V(x)-t^{2\alpha-2}V(t^{-1}x)-\frac{1-t^{8\alpha-4}}{4(2\alpha-1)}\left[(2\alpha-2)V(x)-\nabla V(x)\cdot x\right]\\&\geq 0,\qquad\forall\, t\geq 0,\ x\in\mathbb{R}^2\setminus\{0\},\end{aligned}\tag{3.3}$$

and

$$(6\alpha-2)V(x)+\nabla V(x)\cdot x\geq 0,\qquad\forall x\in\mathbb{R}^2.\tag{3.4}$$

Proof. For any $x\in\mathbb{R}^2$, by (V_1) and (V_2), we have

$$\begin{aligned}\frac{d}{dt}g_2(t,x)=t^{8\alpha-5}\Big\{&(2\alpha-2)V(x)-\nabla V(x)\cdot x\\&-t^{-(6\alpha-2)}\left[(2\alpha-2)V(t^{-1}x)-\nabla V(t^{-1}x)\cdot(t^{-1}x)\right]\Big\},\end{aligned}$$

and this expression is greater than or equal to zero for $t\geq 1$ and less than or equal to zero for $0<t<1$. Together with the continuity of $g_2(\cdot,x)$, this implies that $g_2(t,x)\geq g_2(1,x)$ for all $t\geq 0$ and $x\in\mathbb{R}^2$. This shows that (3.3) holds. By (3.3), one has

$$\lim_{t\to 0}g_2(t,x)=\frac{(6\alpha-2)V(x)+\nabla V(x)\cdot x}{4(2\alpha-1)}\geq 0,$$

which implies that (3.4) holds. □

For $t \geq 0$, let

$$\tau_1(t) = \alpha t^{8\alpha-4} - (4\alpha-2)t^{2\alpha} + 3\alpha - 2, \tag{3.5}$$

$$\tau_2(t) = \alpha t^{8\alpha-4} - (2\alpha-1)t^{4\alpha} + \alpha - 1\,, \tag{3.6}$$

$$\tau_3(t) = (3\alpha-2)t^{8\alpha-4} - (4\alpha-2)t^{6\alpha-4} + \alpha. \tag{3.7}$$

Since $\alpha > 1$, for all $t \in (0,1) \cup (1,+\infty)$,

$$\tau_1(t) > \tau_1(1) = 0, \qquad \tau_2(t) > \tau_2(1) = 0, \qquad \tau_3(t) > \tau_3(1) = 0. \tag{3.8}$$

Lemma 3.3. *Assume that* (V_1)–(V_2), (f_1) *and* (f_3) *hold. Then for all* $u \in H^1(\mathbb{R}^2)$ *and* $t > 0$,

$$I(u) \geq I(u_t) + \frac{1-t^{8\alpha-4}}{4(2\alpha-1)}\Gamma(u) + \frac{\tau_1(t)}{4(2\alpha-1)}A(u) + \frac{\tau_2(t)}{(2\alpha-1)}C(u). \tag{3.9}$$

Proof. Note that

$$\begin{aligned} I(u_t) = &\frac{t^{2\alpha}}{2}A(u) + \frac{t^{2\alpha-2}}{2}\int_{\mathbb{R}^2} V(t^{-1}x)u^2 \mathrm{d}x + t^{4\alpha}\kappa C(u) \\ &+ \frac{t^{6\alpha-4}}{2}\mu D(u) + \frac{t^{8\alpha-4}}{4}\mu\kappa E(u) - \frac{1}{t^2}\int_{\mathbb{R}^2} F(t^\alpha u)\mathrm{d}x, \qquad \forall u \in H^1(\mathbb{R}^2). \end{aligned} \tag{3.10}$$

Since $\Gamma(u) = \alpha N(u) - P(u)$ for $u \in \chi$, then (1.7) and (1.8) imply that

$$\begin{aligned} \Gamma(u) = &\alpha A(u) + \frac{1}{2}\int_{\mathbb{R}^2} \left[(2\alpha-2)V(x) - \nabla V(x)\cdot x\right]u^2\mathrm{d}x \\ &+ 4\alpha\kappa C(u) + (3\alpha-2)\mu D(u) + (2\alpha-1)\mu\kappa E(u) + \int_{\mathbb{R}^2}\left[2F(u) - \alpha f(u)u\right]\mathrm{d}x. \end{aligned} \tag{3.11}$$

Then, it follows from (1.7), (3.1)–(3.7), (3.10)–(3.11) that

$$\begin{aligned} &I(u) - I(u_t) \\ &= \frac{1-t^{2\alpha}}{2}A(u) + \frac{1}{2}\int_{\mathbb{R}^2}\left[V(x) - t^{2\alpha-2}V(t^{-1}x)\right]u^2\mathrm{d}x + (1-t^{4\alpha})\kappa C(u) \\ &\quad + \left(\frac{1-t^{6\alpha-4}}{2}\right)\mu D(u) + \left(\frac{1-t^{8\alpha-4}}{4}\right)\mu\kappa E(u) + \int_{\mathbb{R}^2}\left[t^{-2}F(t^\alpha u) - F(u)\right]\mathrm{d}x \\ &= \frac{1-t^{8\alpha-4}}{4(2\alpha-1)}\left\{\alpha A(u) + \frac{1}{2}\int_{\mathbb{R}^2}\left[(2\alpha-2)V(x) - \nabla V(x)\cdot x\right]u^2\mathrm{d}x\right. \\ &\quad \left. + 4\alpha\kappa C(u) + (3\alpha-2)\mu D(u) + (2\alpha-1)\mu\kappa E(u) + \int_{\mathbb{R}^2}\left[2F(u) - \alpha f(u)u\right]\mathrm{d}x\right\} \\ &\quad + \left[\frac{1-t^{2\alpha}}{2} - \frac{\alpha(1-t^{8\alpha-4})}{4(2\alpha-1)}\right]A(u) + \left[\left(\frac{1-t^{6\alpha-4}}{2}\right) - \frac{(1-t^{8\alpha-4})(3\alpha-2)}{4(2\alpha-1)}\right]\mu D(u) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2}\left\{V(x) - t^{2\alpha-2}V(t^{-1}x) - \frac{1-t^{8\alpha-4}}{4(2\alpha-1)}\left[(2\alpha-2)V(x) - \nabla V(x)\cdot x\right]\right\}u^2\mathrm{d}x \\ &\quad + \left[1 - t^{4\alpha} - \frac{4\alpha(1-t^{8\alpha-4})}{4(2\alpha-1)}\right]\kappa C(u) \\ &\quad + \int_{\mathbb{R}^2}\left\{t^{-2}F(t^\alpha u) - F(u) + \frac{1-t^{8\alpha-4}}{4(2\alpha-1)}\left[\alpha f(u)u - 2F(u)\right]\right\}\mathrm{d}x \\ &\geq \frac{1-t^{8\alpha-4}}{4(2\alpha-1)}\Gamma(u) + \frac{\tau_1(t)}{4(2\alpha-1)}A(u) + \frac{\tau_2(t)}{(2\alpha-1)}C(u), \end{aligned}$$

for all $u \in H^1(\mathbb{R}^2)$ and $t > 0$. This implies that (3.9) holds. □

From Lemma 3.3, we have the following corollary.

Corollary 3.4. *Assume that* (V_1)–(V_2), (f_1) *and* (f_3) *hold. Then for all* $u \in \mathcal{M}$,

$$I(u) = \max_{t>0} I(u_t).$$

Lemma 3.5. *Assume that* (V_1)–(V_2), (f_1)–(f_3) *hold. Then for any* $\chi\backslash\{0\}$, *there exists a unique* $t_u > 0$, *such that* $(u)_{t_u} \in \mathcal{M}$.

Proof. Inspired by [3,5], we let $u \in \chi\backslash\{0\}$ be fixed and define the function $\gamma(t) := I(u_t)$ on $(0, +\infty)$. Clearly by (3.10), (3.11), we have

$$\begin{aligned}\gamma'(t) = 0 \Longleftrightarrow\ & \alpha A(u)t^{2\alpha-1} + \frac{t^{2\alpha-3}}{2}\int_{\mathbb{R}^2}\left[2(\alpha-1)V(t^{-1}x) - \nabla V(t^{-1}x)\cdot(t^{-1}x)\right]u^2\mathrm{d}x \\ & + 4\alpha\kappa C(u)t^{4\alpha-1} + (3\alpha-2)\mu D(u)t^{6\alpha-5} + (2\alpha-1)\mu\kappa E(u)t^{8\alpha-5} \\ & + t^{-3}\int_{\mathbb{R}^2}\left[2F(t^\alpha u) - \alpha f(t^\alpha u)t^\alpha u\right]\mathrm{d}x = 0 \\ \Longleftrightarrow\ & \Gamma(u_t) = 0 \Longleftrightarrow u_t \in \mathcal{M}.\end{aligned}$$

From (V_1) and (V_2), (f_1) and (3.10), it follows that $\lim_{t\to 0}\gamma(t) = 0$, $\gamma(t) > 0$ for $t > 0$ small. Moreover, from (f_1) and (f_2), for every $\theta > 0$, there exists $C_\theta > 0$ such that

$$F(\varrho) \geq \theta|\varrho|^p - C_\theta\varrho^2, \qquad \forall \varrho \in \mathbb{R}. \tag{3.12}$$

We note from Lemma 2.1 and Hölder inequality that for some $C_0 > 0$,

$$h(s) = \int_0^s u^2(r)r\mathrm{d}r = \int_{B_s}\frac{1}{2\pi}u^2(y)\mathrm{d}y \leq C_0 s\|u\|_{L^4}^2, \tag{3.13}$$

then

$$D(u) = \int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} su^2(s)\mathrm{d}s\right)^2\mathrm{d}x \leq C_0\|u\|_{L^4}^4\|u\|_{L^2}^2, \tag{3.14}$$

$$E(u) = \int_{\mathbb{R}^2}\frac{u^4(x)}{|x|^2}\left(\int_0^{|x|} su^2(s)\mathrm{d}s\right)^2\mathrm{d}x \leq C_0\|u\|_{L^4}^8. \tag{3.15}$$

By (V_1), we have $V_{\max} := \max_{x\in\mathbb{R}^2} V(x) > 0$ and by (3.10), (3.12) and (3.14), (3.15), we have

$$\begin{aligned}I(u_t) \leq\ & \frac{t^{2\alpha}}{2}A(u) + \frac{t^{2\alpha-2}}{2}V_{\max}\|u\|^2 + t^{4\alpha}\kappa C(u) \\ & + \frac{t^{6\alpha-4}}{2}\mu C_0\|u\|_{L^4}^4\|u\|_{L^2}^2 + \frac{t^{8\alpha-4}}{4}\mu\kappa\|u\|_{L^4}^8 - \theta t^{8\alpha-4}\|u\|_{L^p}^p \\ & + t^{2\alpha-2}C_\theta\|u\|_{L^2}^2.\end{aligned} \tag{3.16}$$

Let θ be large enough in (3.16), then $\gamma(t) < 0$ for t large. Therefore, $\max_{t>0}\gamma(t)$ is achieved at some $t_u > 0$, so that $\gamma'(t_u) = 0$ and $(u)_{t_u} \in \mathcal{M}$.

Next, we claim that $t_u > 0$ is unique for any $u \in \chi\backslash\{0\}$. If there exist two positive constants $t_1 \neq t_2$, such that both $u_{t_1}, u_{t_2} \in \mathcal{M}$, that is, $\Gamma(u_{t_1}) = \Gamma(u_{t_2}) = 0$, then (3.5)–(3.7), (3.10) imply

$$\begin{aligned}I(u_{t_1}) &> I(u_{t_2}) + \frac{t_1^{6\alpha-4} - t_2^{6\alpha-4}}{4(2\alpha-1)t_1^{6\alpha-4}}\Gamma(u_{t_1}) = I(u_{t_2}) \\ &> I(u_{t_1}) + \frac{t_2^{6\alpha-4} - t_1^{6\alpha-4}}{4(2\alpha-1)t_2^{6\alpha-4}}\Gamma(u_{t_2}) = I(u_{t_1}).\end{aligned}$$

This contradiction shows that $t_u > 0$ is unique for any $u \in \chi\backslash\{0\}$. □

Arguing as in [5], standard computations show that

Lemma 3.6. *Assume that* (V_1)–(V_2) *hold. Then there exist constants* $C_1, C_2 > 0$*, such that*

$$(2\alpha - 2)V(x) - \nabla V(x) \cdot x \geq C_1, \qquad \forall x \in \mathbb{R}^2. \tag{3.17}$$

and

$$(6\alpha - 2)V(x) + \nabla V(x) \cdot x \geq C_2, \qquad \forall x \in \mathbb{R}^2. \tag{3.18}$$

Lemma 3.7. *Assume that* (V_1) *and* (V_2)*,* (f_1)–(f_3) *hold. Then*

(i) there exists $\rho_0 > 0$ *such that* $\|u\| \geq \rho_0, \ \ \forall u \in \mathcal{M}$*;*

(ii) $m := \inf_{u \in \mathcal{M}} I(u) = \inf_{u \in \mathcal{X} \setminus \{0\}} \max I(u_t) > 0.$

Proof. (i) Since $\Gamma(u) = 0$ for $u \in \mathcal{M}$, it follows from (f_1), (3.11), (3.17) and Sobolev embedding inequality, there exists a constant $C_3 > 0$, such that

$$\begin{aligned}
&\alpha A(u) + 4\alpha\kappa C(u) + \frac{1}{2}C_1\|u\|_{L^2}^2 \\
&\quad \leq \alpha A(u) + 4\alpha\kappa C(u) + \frac{1}{2}\int_{\mathbb{R}^2}\left[(2\alpha-2)V(x) - \nabla V(x)\cdot x\right]u^2 \mathrm{d}x \\
&\quad \leq \int_{\mathbb{R}^2}\left[\alpha f(u)u - 2F(u)\right]\mathrm{d}x \\
&\quad \leq \frac{1}{4}C_1\|u\|_{L^2}^2 + C_3\|u\|^p,
\end{aligned}$$

for all $u \in \mathcal{M}$. This implies that there exists $\rho_0 > 0$ such that

$$\|u\| \geq \rho_0 := \left(\frac{\min\{4\alpha, C_1\}}{4C_3}\right)^{\frac{1}{p-2}}, \qquad \forall u \in \mathcal{M}. \tag{3.19}$$

(ii) From Corollary 3.4 and Lemma 3.5, we have

$$\mathcal{M} \neq \varnothing \quad \text{and} \quad m = \inf_{u \in \mathcal{X}\setminus\{0\}} \max I(u_t).$$

Next, we prove that $m > 0$. Let

$$\begin{aligned}
\Psi(u) &:= I(u) - \frac{1}{4(2\alpha-1)}\Gamma(u) \\
&= \frac{3\alpha-2}{4(2\alpha-1)}A(u) + \frac{1}{8(2\alpha-1)}\int_{\mathbb{R}^2}\left[(6\alpha-2)V(x) + \nabla V(x)\cdot x\right]u^2\mathrm{d}x \\
&\quad + \frac{\alpha-1}{(2\alpha-1)}\kappa C(u) + \frac{\alpha}{4(2\alpha-1)}\mu D(u) \\
&\quad + \frac{1}{4(2\alpha-1)}\int_{\mathbb{R}^2}\left[\alpha f(u)u - (8\alpha-2)F(u)\right]\mathrm{d}x, \qquad \forall u \in H^1(\mathbb{R}^2).
\end{aligned} \tag{3.20}$$

Since $\Gamma(u) = 0$ for all $u \in \mathcal{M}$, then it follows from (3.2), (3.4), (3.18) and (3.19), (3.20) that

$$\begin{aligned}
I(u) &\geq \frac{3\alpha-2}{4(2\alpha-1)}A(u) + \frac{1}{8(2\alpha-1)}\int_{\mathbb{R}^2}\left[(6\alpha-2)V(x) + \nabla V(x)\cdot x\right]u^2\mathrm{d}x \\
&\geq \frac{\min\{2(3\alpha-2), C_2\}}{8(2\alpha-1)}\|u\|^2 \geq \frac{\min\{2(3\alpha-2), C_2\}}{8(2\alpha-1)}\rho_0^2 := \rho_1 > 0, \qquad \forall u \in \mathcal{M}.
\end{aligned}$$

This shows that $m = \inf_{u\in\mathcal{M}} I(u) \geq \rho_1 > 0.$ □

Next, we establish the following lemma.

Lemma 3.8. *Assume that* (V_1)–(V_2) *and* (f_1)–(f_3) *hold. If* $u \in \mathcal{M}$ *and* $I(u) = m$*, then* u *is a radial ground state solution of* (1.1)*. Moreover, it is positive (up to a change of sign).*

Proof. We argue as in [8, 22]. Suppose by contradiction that u is not a weak solution of (1.2). Then, we can choose $\varphi \in C^\infty_{0,r}(\mathbb{R}^2)$ such that

$$\langle I'(u), \varphi \rangle < -1.$$

Hence, we fix $\varepsilon > 0$ sufficiently small such that

$$\langle I'(u_t + \vartheta\varphi), \varphi \rangle \leq -\frac{1}{2}, \quad \text{for } |t-1|, |\vartheta| \leq \varepsilon, \tag{3.21}$$

and introduce $\zeta \in C^\infty_0(\mathbb{R})$ be a cut-off function $0 \leq \zeta \leq 1$ such that $\zeta(t)$=1 for $|t-1| \leq \frac{\varepsilon}{2}$ and $\zeta(t) = 0$ for $|t-1| \geq \varepsilon$. For $t \geq 0$, we construct a path $\sigma : \mathbb{R}^+ \to \chi$ defined by

$$\sigma(t) = \begin{cases} u_t, & \text{if } |t-1| \geq \varepsilon, \\ u_t + \varepsilon\zeta(t)\varphi, & \text{if } |t-1| < \varepsilon. \end{cases}$$

Note that η is continuous on the metric space (χ, d_χ) and eventually, choosing a smaller ε, if necessary, we obtain that $d_\chi(\sigma(t), 0) > 0$ for $|t-1| < \varepsilon$.

We claim that

$$\sup_{t \geq 0} I(\sigma(t)) < m. \tag{3.22}$$

Indeed, if $|t-1| \geq \varepsilon$, from Corollary 3.4, we have $I(\sigma(t)) = I(u_t) < I(u) = m$. If $|t-1| < \varepsilon$, by using the mean value theorem, we get

$$\begin{aligned} I(\sigma(t)) = I(u_t + \varepsilon\zeta(t)\varphi) &= I(u_t) + \int_0^\varepsilon \langle I'(u_t + \vartheta\zeta(t)\varphi), \zeta(t)\varphi \rangle \mathrm{d}\tau \\ &\leq I(u_t) - \frac{1}{2}\varepsilon\zeta(t) < m, \end{aligned}$$

where in the first inequality we have used (3.21).

To conclude that $\Gamma(\sigma(1+\varepsilon)) < 0$ and $\Gamma(\sigma(1-\varepsilon)) > 0$. By the continuity of the map $t \to \Gamma(\sigma(t))$, there exists $t_0 \in (1-\varepsilon, 1+\varepsilon) < 0$ such that $\Gamma(\sigma(t_0)) = 0$. This implies that $\sigma(t_0) = u_{t_0} + \varepsilon\zeta(t_0)\varphi \in \mathcal{M}$ and $I(\sigma(t_0)) < m$. By Lemma 3.7, this gives the desired contradiction, hence u is a weak solution of (1.2). By Remark 2.4, we conclude that u is a radial ground state solution. Moreover, if $u \in \mathcal{M}$ is a minimizer of $I|_\mathcal{M}$, then $|u|$ is also a minimizer and a solution. So we can assume that u is nonnegative. By Proposition 2.2, we know that $u \in \mathcal{C}^2(\mathbb{R}^2)$ and by the Harnack inequality [27], we know that $u > 0$. This completes the proof. □

Lemma 3.9. *Assume that* (V_1)–(V_2) *and* (f_1)–(f_3) *hold. Then* m *is achieved.*

Proof. Let $\{u_n\} \subset \mathcal{M}$ be such that $I(u_n) \to m$, then by (3.20),

$$m + o(1) = I(u_n) \geq \frac{3\alpha - 2}{4(2\alpha - 1)} A(u_n) + \frac{C_2}{8(2\alpha - 1)} \|u_n\|^2_{L^2} + \frac{\alpha - 1}{(2\alpha - 1)} \kappa C(u_n),$$

which implies that $\{u_n\}$ and $\{u_n^2\}$ are bounded in $H^1_r(\mathbb{R}^2)$. Therefore, by the compactness result due to [23], there exists $\overline{u} \in \chi$ such that, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup \overline{u} \quad \text{in } H_r^1(\mathbb{R}^2), \\ u_n^2 &\rightharpoonup \overline{u}^2 \quad \text{in } H_r^1(\mathbb{R}^2), \\ u_n &\to \overline{u} \quad \text{in } L^q(\mathbb{R}^2) \text{ for any q} > 2, \\ u_n &\to \overline{u} \quad \text{a.e. in } \mathbb{R}^2. \end{aligned}$$

There are two possible cases (i) $\overline{u} = 0$ and (ii) $\overline{u} \neq 0$. Next, we prove that $\overline{u} \neq 0$.

Arguing by contradiction, suppose that $\overline{u} = 0$, that is $u_n \rightharpoonup 0$ in $H_r^1(\mathbb{R}^2)$ and $u_n^2 \rightharpoonup 0$ in $H_r^1(\mathbb{R}^2)$. Then $u_n \to 0$ in $L^q(\mathbb{R}^2)$ for $q > 2$ and $u_n \to 0$ a.e. in $\mathbb{R}^2$. From $\Gamma(u_n) = 0$, (3.17) and (3.19), one has

$$\begin{aligned} \min\{\alpha, \frac{1}{2}C_1\}{\rho_0}^2 &\leq \min\left\{\alpha, \frac{1}{2}C_1\right\}\|u_n\|^2 \\ &\leq \alpha A(u) + \frac{1}{2}C_1\|u_n\|_{L^2}^2 \\ &\leq \alpha A(u_n) + \frac{1}{2}\int_{\mathbb{R}^2}\left[(2\alpha-2)V(x) - \nabla V(x)\cdot x\right]u_n^2 \mathrm{d}x \\ &\quad + 4\alpha\kappa C(u_n) + (3\alpha-2)\mu D(u_n) + (2\alpha-1)\mu\kappa E(u_n) \\ &= \int_{\mathbb{R}^2}\left[\alpha f(u_n)u_n - 2F(u_n)\right]\mathrm{d}x + o(1). \end{aligned} \tag{3.23}$$

Using (f_1), (f_2), clearly, (3.23) contradicts with $u_n \to 0$ in $L^q(\mathbb{R}^2)$ for $q > 2$, therefore $\overline{u} \neq 0$.

Let $v_n = u_n - \overline{u}$. Then by Lemma 2.5 and the Brezis–Lieb Lemma (see [22, 24, 30]), yield

$$I(u_n) = I(\overline{u}) + I(v_n) + o(1), \tag{3.24}$$

and

$$\Gamma(u_n) = \Gamma(\overline{u}) + \Gamma(v_n) + o(1). \tag{3.25}$$

Since $I(u_n) \to m$, $\Gamma(u_n) = 0$, then it follows from (3.20), (3.24) and (3.25), we have

$$\begin{aligned} \Psi(v_n) := I(v_n) - \frac{1}{4(2\alpha-1)}\Gamma(v_n) \\ = m - \Psi(\overline{u}) + o(1) \\ = m - \left[I(\overline{u}) - \frac{1}{4(2\alpha-1)}\Gamma(\overline{u})\right] + o(1), \end{aligned} \tag{3.26}$$

and

$$\Gamma(v_n) = -\Gamma(\overline{u}) + o(1). \tag{3.27}$$

If there eixsts a subsequence $\{v_{n_i}\}$ of $\{v_n\}$ such that $v_{n_i} = 0$, then

$$I(\overline{u}) = m, \qquad \Gamma(\overline{u}) = 0, \tag{3.28}$$

which implies that the conclusion of Lemma 3.9 holds. Next, we assume that $v_n \neq 0$. In view of Lemma 3.5, there exists $t_n > 0$ such that $(v_n)_{t_n} \in \mathcal{M}$ for large n, we claim that $\Gamma(\overline{u}) \leq 0$, otherwise, if $\Gamma(\overline{u}) > 0$, then (3.27) implies that $\Gamma(v_n) < 0$ for large n. From (1.7), (3.9) and (3.26), we obtain

$$\begin{aligned} m - \Psi(\overline{u}) + o(1) = \Psi(v_n) &= I(v_n) - \frac{1}{4(2\alpha-1)}\Gamma(v_n) \\ &\geq I((v_n)_{t_n}) - \frac{t_n^{8\alpha-4}}{4(2\alpha-1)}\Gamma(v_n) + \frac{\tau_1(t_n)}{4(2\alpha-1)}A(v_n) + \frac{\tau_2(t_n)}{(2\alpha-1)}C(v_n) \\ &\geq I((v_n)_{t_n}) - \frac{t_n^{8\alpha-4}}{4(2\alpha-1)}\Gamma(v_n) \geq m \quad \text{for large } n \in \mathbb{N}, \end{aligned}$$

which implies that $\Gamma(\bar{u}) \leq 0$ due to $\Psi(\bar{u}) > 0$. Applying Lemma 3.5, there exists $\bar{t} > 0$ such that $\bar{u}_{\bar{t}} \in \mathcal{M}$. From (1.7), (3.5), (3.6) and (3.9), the weak semicontinuity of norm and Fatou's Lemma, one has

$$\begin{aligned} m &= \lim_{n\to\infty} \Psi(u_n) \\ &= \lim_{n\to\infty} \left[I(u_n) - \frac{1}{4(2\alpha-1)}\Gamma(u_n) \right] \\ &\geq I(\bar{u}) - \frac{1}{4(2\alpha-1)}\Gamma(\bar{u}) \\ &\geq I(\bar{u}_{\bar{t}}) - \frac{\bar{t}^{8\alpha-4}}{4(2\alpha-1)}\Gamma(\bar{u}) + \frac{\tau_1(\bar{t})}{4(2\alpha-1)}A(\bar{u}) + \frac{\tau_2(\bar{t})}{(2\alpha-1)}C(\bar{u}) \\ &\geq m - \frac{\bar{t}^{8\alpha-4}}{4(2\alpha-1)}\Gamma(\bar{u}) \geq m, \end{aligned}$$

which implies that (3.28) holds. □

Proof of Theorem 1.1. In view of Lemmas 3.7, 3.8, 3.9, there exists $\bar{u} \in \mathcal{M}$ such that $I'(\bar{u}) = 0$, $I(\bar{u}) = m = \inf_{u\in\chi\setminus\{0\}} \max I(u_t)$, we can conclude that, actually, $\bar{u}$ is a positive radial ground state solution of (1.1). This completes the proof. □

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