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On nonlocal problems for semilinear second order differential inclusions without compactness

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Abstract. Existence of mild solutions for a nonlocal abstract problem driven by a semilinear second order differential inclusion is studied in Banach spaces in the lack of compactness both on the fundamental system generated by the linear part and on the nonlinear multivalued term. The method used for proving our existence theorems is based on the combination of a fixed point theorem and a selection theorem developed by ourselves with an approach that uses De Blasi measure of noncompactness and the weak topology. As application of our existence result we present the study of the controllability of a problem guided by a wave equation.

Keywords: nonlocal abstract problem, semilinear second order differential inclusion, fundamental system, De Blasi measure of noncompactness, controllability problem, wave equation.

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1 Introduction

Let us consider the nonlocal abstract problem controlled by a semilinear second order differential inclusion

$$\begin{cases} x''(t) \in A(t)x(t) + F(t, x(t)), \ t \in J = [0, 1] \\ x(0) = g(x) \\ x'(0) = h(x). \end{cases} \tag{P}$$

where $g, h : \mathcal{C}(J; X) \to X$ are suitable functions, without compactness conditions both on the multimap F and on the fundamental system generated by the family $\{A(t)\}_{t \in J}$.

The concept of *nonlocal initial condition* was introduced to extend the classical theory of initial value problems by Byszewski in [3]. This notion is more appropriate then the classical one to describe natural phenomena because it allows us to consider additional informations. Nonlocal problems has been widely studied because of their applications in different fields to

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applied science (see [8,10,33] and the reference cited therein). For instance, in [10] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using a first order differential equation and the following

$$g(u) = \sum_{i=0}^{p} c_i u(t_i),$$

where c_i is given constant and t_i is a fixed instant of time, $i = 0, 1, \ldots, p$.

On the other hand, there exists an extensive literature concerning abstract second order equations in the autonomous case starting with the initial research works of Kato [19], [20] and [21] (see, e.g. [12,23,27,28,30]), while the theory dealing with non-autonomous second order abstract equations/inclusions has only recently been studied by using a concept of fundamental Cauchy operator generated by the family $\{A(t)\}_{t\in J}$, introduced by Kozak in [24].

On this subject we recall Henríquez [15], Henríquez, Poblete and Pozo [16] for second order differential equations; Cardinali and Gentili [5], Cardinali and De Angelis [4] for second order differential inclusions. In all these papers the existence of mild solutions is studied with topological techniques based on fixed point theorems for a suitable solution operator and requesting strong compactness conditions, which are usually not satisfied in an infinite dimensional framework.

Our purpose is to obtain existence results in the lack of this compactness both on the semigroup generated by the linear part and on the nonlinear multivalued term. To achieve this goal we use De Blasi measure of noncompactness and the weak topology. This approach is present in [2], but with the aim of studying the existence of mild solutions for a problem controlled by a semilinear *first order* differential inclusion.

Moreover the techniques for non-autonomous second order differential equations/inclusions developed in [24] and [5] play a key role in the proof of our existence results.

This paper is organized as follows. After introducing in Section 2 some notations and some preliminary results, in Section 3 we present the problem setting. Section 4 is devoted to obtain some properties of the fundamental Cauchy operator, a new version of a selection theorem proved in [2] (see Theorem 4.2) and, by using the classic Glicksberg Theorem, a variant of the fixed point theorem introduced in [2] for x_0-unpreserving multimaps (see Theorem 4.3) and its version in Banach spaces (see Corollary 4.4).

In Section 5 we deal with the existence of mild solutions for the nonlocal abstract problem controlled by a semilinear second order differential inclusion in Banach not necessarily reflexive spaces; we end this section by presenting also an new existence theorem in the context of reflexive spaces, omitting some assumption required in the previous result on the multimap F and on the functions g and h (the reflexivity doesn't imply these hypotheses removed). Finally, in Section 6, we apply our abstract existence theorem in reflexive Banach spaces to study controllability of a Cauchy problem guided by the following wave equation

$$\frac{\partial^2 w}{\partial t^2}(t,\xi) = \frac{\partial^2 w}{\partial \xi^2}(t,\xi) + b(t)\frac{\partial w}{\partial \xi}(t,\xi) + T(t)w(t,\cdot)(\xi) + u(t,\xi).$$

(see Theorem 6.1).

2 Preliminaries

In this paper X is a Banach space with the norm $\|\cdot\|_X$ and $\mathcal{P}(X)$ is the family of nonempty subsets of X. Moreover we will use the following notations:

$$\mathcal{P}_b(X) = \{H \in \mathcal{P}(X) : \ H \text{ bounded}\},$$
$$\mathcal{P}_c(X) = \{H \in \mathcal{P}(X) : \ H \text{ convex}\},$$
$$\mathcal{P}_{wk}(X) = \{H \in \mathcal{P}(X) : \ H \text{ weakly compact}\}, \dots$$

Further, we recall that a Banach space X is said to be *weakly compactly generated* (WCG, for short) if there exists a weakly compact subset K of X such that $X = \overline{\text{span}}\{K\}$ (see [14])

Remark 2.1. Let us note that every separable space is weakly compact generated as well as the reflexive ones (see [14]).

Moreover, we recall that (see [26, Theorem 1.12.15]) a Banach space X is separable if and only if it is compactly generated.

Moreover, we denote as X^* the dual space of X.

Now, if τ_w is the weak topology on X and $(A_n)_n$, $A_n \in \mathcal{P}(X)$, we set (see [17, Definition 7.1.3])

$$w - \limsup_{n\to+\infty} A_n = \{x \in X : \ \exists (x_{n_k})_k, \ x_{n_k} \in A_{n_k}, \ n_k < n_{k+1}, \ x_{n_k} \rightharpoonup x\} \tag{2.1}$$

Then, we denote by $\overline{B}_X(0,n)$ the closed ball centered at the origin and of radius n of X, and for a set $A \subset X$, the symbol $\overline{A}^w$ denotes the weak closure of A. We take for granted that a bounded subset A of a reflexive space X is relatively weakly compact. Moreover we recall that a subset C of a Banach space X is called relatively weakly sequentially compact if any sequences of points in C has a subsequence weakly convergent to a point in X (see [26]).

In the sequel, on the interval J we consider the usual Lebesgue measure μ and we denote by $\mathcal{C}(J;X)$ the space consisting of all continuous functions from J to X provided with the norm $\|\cdot\|_\infty$ of uniform convergence.

A function $f : J \to X$ is said *weakly sequentially continuous* if for every sequence $(x_n)_n$, $x_n \rightharpoonup x$, then $f(x_n) \rightharpoonup f(x)$. Moreover f is said to be *B-measurable* if there is a sequence of simple functions $(s_n)_n$ which converges to f almost everywhere in J (see [11, Definition 3.10.1 (a)]).

It easy to see that Theorem 4 of [22] can be rewritten in the following way.

Theorem 2.2. *Let $(f_n)_n$ and g be respectively a sequence and a function in $\mathcal{C}(J;X)$. Then $f_n \rightharpoonup g$ if and only if $(f_n - g)_n$ is uniformly bounded and $f_n(t) \rightharpoonup g(t)$, for every $t \in J$.*

Moreover, we call by $L^1(J;X)$ the space of all X- valued Bochner integrable functions on J with norm $\|u\|_{L^1(J;X)} = \int_0^a \|u(t)\|_X \, dt$ and $L^1_+(J) = \{f \in L^1(J;\mathbb{R}) : \ f(t) \geq 0, \ a.e. \ t \in J\}$. If $X = \mathbb{R}$ we put $\|\cdot\|_1 = \|\cdot\|_{L^1(J;\mathbb{R})}$.

A set $A \subset L^1(J;X)$ has the property of *equi-absolute continuity of the integral* if for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that, for every $E \in \mathcal{M}(J)$, $\mu(E) < \delta_\varepsilon$, we have

$$\int_E \|f(t)\|_X \, dt < \varepsilon$$

whenever $f \in A$.

Remark 2.3. We observe that if $A \subset L^1(J;X)$ is integrably bounded, i.e. there exists $\nu \in L^1_+(J)$ such that
$$\|f(t)\|_X \leq \nu(t), \quad a.e.\ t \in J,\ \forall f \in A,$$
then the set A has the property of equi-absolute continuity of the integral.

Now we give Theorem 4.4.2 of [31] that we will use in Section 5 for the suitable pre-ideal regular Lebesgue–Bochner space $L^2(\mathbb{T},\mathbb{C})$ (see [31, pp. 8,9,48]).

Theorem 2.4. *An abstract function $x : J \to X$, where X is a pre-ideal regular space on $\mathbb{R}$, is B-measurable if and only if there exists a measurable function $y : J \times \mathbb{R} \to X$, such that $x(t) = y(t,\cdot)$.*

A multimap $F : X \to \mathcal{P}(Y)$, where Y is a topological space:

- is *upper semicontinuous* at point $\overline{x} \in X$ if, for every open $W \subset Y$ such that $F(\overline{x}) \subset W$, there exists a neighborhood $V(\overline{x})$ of $\overline{x}$ with the property that $F(V(\overline{x})) \subset W$,
- is *upper semicontinuous* (u.s.c. for short) if it is upper semicontinuous at every point $x \in X$,
- is *compact* if its range $F(X)$ is relatively compact in Y, i.e. $\overline{F(X)}$ is compact in Y,
- is *locally compact* if every point $x \in X$ there exists a neighborhood $V(x)$ such that the restriction of F to $V(x)$ is compact,
- has *closed graph* if the set $graphF = \{(x,y) \in X \times Y :\ y \in F(x)\}$ is closed in $X \times Y$,
- if Y is a linear topological space, F has *(s-w)sequentially closed graph* [*weakly sequentially closed graph*] if for every $(x_n)_n$, $x_n \in X$, $x_n \to x$ [$x_n \rightharpoonup x$] and for every $(y_n)_n$, $y_n \in F(x_n)$, $y_n \rightharpoonup y$, we have $y \in F(x)$.

Next, we recall that, if K is a subset of X, $F : K \to \mathcal{P}(X)$ is a multimap and $x_0 \in K$, a closed convex set $M_0 \subset K$ is (x_0,F)-*fundamental*, if $x_0 \in M_0$ and $F(M_0) \subset M_0$ (see [2, p. 620]).

In this setting we recall the following result which allows to characterize the smallest (x_0,F)-fundamental set (see [2, Theorem 3.1])

Proposition 2.5. *Let X be a locally convex Hausdorff space, $K \subset X$, $x_0 \in K$. Let $F : K \to \mathcal{P}(X)$ be a multimap such that*

i) $\overline{co}(F(K) \cup \{x_0\}) \subset K$.

Then

1) $\mathcal{F} = \{H :\ H \text{ is } (x_0,F) - \text{fundamental set}\} \neq \varnothing$;

2) put $M_0 = \bigcap_{H\in\mathcal{F}} H$, *we have* $M_0 \in \mathcal{F}$ *and* $M_0 = \overline{co}(F(M_0) \cup \{x_0\})$.

Theorem 2.6 ([2, Theorem 4.4] (Containment Theorem))**.** *Let X a Banach space and $G_n, G : J \to \mathcal{P}(X)$ be such that*

α) a.e. $t \in J$, for every $(u_n)_n$, $u_n \in G_n(t)$, there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ and $u \in G(t)$ such that $u_{n_k} \rightharpoonup u$;

αα) there exists a sequence $(y_n)_n$, $y_n : J \to X$, having the property of equi-absolute continuity of the integral, such that $y_n \in G_n(t)$, a.e. $t \in J$, for all $n \in \mathbb{N}$.

Then there exists a subsequence $(y_{n_k})_k$ of $(y_n)_n$ such that $y_{n_k} \rightharpoonup y$ in $L^1(J;X)$ and, moreover, $y(t) \in \overline{co}G(t)$, a.e. $t \in J$.

Now, a function $\varphi : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ is said to be a *Sadovskij functional* in X if it satisfies $\varphi(\overline{co}(\Omega)) = \varphi(\Omega)$, for every $\Omega \in \mathcal{P}_b(X)$ (see [1]).

Definition 2.7 ([6, Definition 4.1])**.** A function $\omega : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ is said to be a *measure of weak noncompactness* (MwNC, for short) if the following properties are satisfied:

ω_1) ω is a Sadowskii functional;

ω_2) $\omega(\Omega) = 0_n$ if and only if $\overline{\Omega}^w$ is weakly compact (i.e. ω is regular).

Further, a MwNC $\omega : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ is said to be:

monotone if $\Omega_1, \Omega_2 \in \mathcal{P}_b(X)$: $\Omega_1 \subset \Omega_2$ implies $\omega(\Omega_1) \leq \omega(\Omega_2)$;

nonsingular if $\omega(\{x\} \cup \Omega) = \omega(\Omega)$, for every $x \in X$, $\Omega \in \mathcal{P}_b(X)$;

x_0-stable if, fixed $x_0 \in X$, $\omega(\{x_0\} \cup \Omega) = \omega(\Omega)$, $\Omega \in \mathcal{P}_b(X)$;

invariant under closure if $\omega(\overline{\Omega}) = \omega(\Omega)$, $\Omega \in \mathcal{P}_b(X)$;

invariant with respect to the union with compact set if $\omega(\Omega \cup C) = \omega(\Omega)$, for every relatively compact set $C \subset X$ and $\Omega \in \mathcal{P}_b(X)$.

Remark 2.8. In particular in [9] De Blasi introduces the function $\beta : \mathcal{P}_b(X) \to \mathbb{R}_0^+$ so defined

$$\beta(\Omega) = \inf\{\varepsilon \in [0,\infty[: \text{there exists } C \subset X \text{ weakly compact} : \ \Omega \subseteq C + B_X(0,\varepsilon)\},$$

and he proves that β is a regular Sadowskii functional. Then β is MwNC, named in literature *De Blasi measure of weak noncompactness*.

We recall that β has all the properties mentioned before and it is also algebraically subadditive, i.e. $\beta\left(\sum_{k=1}^{n} M_k\right) \leq \sum_{k=1}^{n} \beta(M_k)$, where $M_k \in \mathcal{P}_b(X)$, $k = 1, \ldots, n$. Moreover, for every bounded linear operator $L : X \to X$ the following property holds ([18], p.35)

$$\beta(L(\Omega)) \leq \|L\| \ \beta(\Omega), \quad \text{for every } \Omega \in \mathcal{P}_b(X),$$

where $\|L\|$ denotes the norm of the operator L.

We recall the following interesting result for MwNC.

Proposition 2.9 ([25, Theorem 2.8 and Remark 2.7 (b)] or [2, Theorem 2.7])**.** *Let (Ω, Σ, μ) be a finite positive measure space and X be a weakly compactly generated Banach space. Then for every countable family C having the property of equi-absolute continuity of the integral of functions $x : \Omega \to X$, the function $\beta(C(\cdot))$ is measurable and*

$$\beta\left(\left\{\int_\Omega x(s)\,ds : \ x \in C\right\}\right) \leq \int_\Omega \beta(C(s))\,ds,$$

where β is a MwNC.

We recall a Sadowskii functional that we will use in the following.

Definition 2.10 ([2, Definition 3.9]). Let X a Banach space, $N \in \mathbb{R}$, and M a bounded subspace of $\mathcal{C}([a,b];X)$.

We use the notation $M(t) = \{x(t) : \ x \in M\}$ and define

$$\beta_N(M) = \sup_{\substack{C \subset M \\ \text{countable}}} \sup_{t \in [a,b]} \beta(C(t)) e^{-Nt}, \tag{2.2}$$

where β is the De Blasi MwNC.

Remark 2.11. We recall that the Sadowskii functional β_N is x_0-stable and monotone (see [2, Proposition 3.10]) and β_N has the two following properties

(I) β_N is algebraically subadditive;

(II) $M \subset \mathcal{C}([a,b];X)$ is relatively weakly compact $\Rightarrow \beta_N(M) = 0$.

We note that (I) holds since β is algebraically subadditive while (II) is true taking into account of the regularity of β.

3 Problem setting

First of all, on the linear part of the second order differential inclusion, presented in the nonlocal problem (**P**), we assume the following property:

(A) $\{A(t)\}_{t \in J}$ is a family of bounded linear operators $A(t) : D(A) \to X$, where $D(A)$, independent on $t \in J$, is a subset dense in X, such that, for each $x \in D(A)$, the function $t \mapsto A(t)x$ is continuous on J and generating a fundamental system $\{S(t,s)\}_{t,s \in J}$, and F is a suitable X-valued multimap defined in $J \times X$.

In the following we recall the concept of fundamental system introduced by Kozak in [24] and recently used in [4], [5] and [16].

Definition 3.1. A family $\{S(t,s)\}_{t,s \in J}$ of bounded linear operators $S(t,s) : X \to X$ is called a *fundamental system* generated by the family $\{A(t)\}_{t \in J}$ if

S1. for each $x \in X$, $S(\cdot,\cdot)x : J \times J \to X$ is a $\mathcal{C}^1$-function and

 a. for each $t \in J$, $S(t,t)x = 0$, for every $x \in X$;

 b. for each $t,s \in J$ and for each $x \in X$, $\frac{\partial}{\partial t} S(t,s)\big|_{t=s} x = x$ and $\frac{\partial}{\partial s} S(t,s)\big|_{t=s} x = -x$;

S2. for all $t,s \in J$, $x \in D(A)$, then $S(t,s)x \in D(A)$, the map $S(\cdot,\cdot)x : J \times J \to X$ is of class $\mathcal{C}^2$ and

 a′. $\frac{\partial^2}{\partial t^2} S(t,s)x = A(t)S(t,s)x$;

 b′. $\frac{\partial^2}{\partial s^2} S(t,s)x = S(t,s)A(s)x$;

 c′. $\frac{\partial^2}{\partial t \partial s} S(t,s)\big|_{t=s} x = 0$;

S3. for all $t,s \in J$, $x \in D(A)$, then $\frac{\partial}{\partial s} S(t,s)x \in D(A)$. Moreover, there exist $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x$, $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x$ and a″. $\frac{\partial^3}{\partial t^2 \partial s} S(t,s)x = A(t)\frac{\partial}{\partial s} S(t,s)x$;

b″. $\frac{\partial^3}{\partial s^2 \partial t} S(t,s)x = \frac{\partial}{\partial t} S(t,s)A(s)x$;

and, for all $x \in D(A)$, the function $(t,s) \mapsto A(t)\frac{\partial}{\partial s} S(t,s)x$ is continuous in $J \times J$.

Moreover, a map $S : J \times J \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is the space of all bounded linear operators in X with the norm $\| \cdot \|_{\mathcal{L}(X)}$, is said to be a *fundamental operator* if the family $\{S(t,s)\}_{t,s \in J}$ is a fundamental system.

To abbreviate the notation we use, for each $(t,s) \in J \times J$, the linear cosine operator

$$C(t,s) = -\frac{\partial}{\partial s} S(t,s) : X \to X.$$

Remark 3.2. We recall that, by using Banach–Steinhaus Theorem, the fundamental system $\{S(t,s)\}_{t,s \in J}$ satisfies the following properties (see [5]): there exist $K, K^* > 0$ such that

p1. $\|C(t,s)\|_{\mathcal{L}(X)} \le K$, $(t,s) \in J \times J$;

p2. $\|S(t,s)\|_{\mathcal{L}(X)} \le K|t-s|$, $(t,s) \in J \times J$;

p3. $\|S(t,s)\|_{\mathcal{L}(X)} \le Ka$, $(t,s) \in J \times J$;

p4. $\|S(t_2,s) - S(t_1,s)\|_{\mathcal{L}(X)} \le K^*|t_2 - t_1|$, $t_1, t_2, s \in J$.

Further we denote with $G_S : L^1(J;X) \to C(J;X)$ the *fundamental Cauchy operator*, introduced in [5], defined by

$$G_S f(t) = \int_0^t S(t,s) f(s)\, ds, \qquad t \in J,\ f \in L^1(J;X).$$

It is easy to see that, by using Theorem 1.3.5 of [18] and the properties p3., p4. and S1., the operator G_S is well posed.

We investigate the existence of mild solutions for the nonlocal problem (**P**) (see [5, Definition 2.2])

Definition 3.3. A continuous function $u : J \to X$ is a *mild solution* for (**P**) if

$$u(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi) f(\xi)\, d\xi, \qquad t \in J,$$

where $f \in S^1_{F(\cdot,u(\cdot))} = \{f \in L^1(J;X) :\ f(t) \in F(t,u(t)),\ a.e.\ t \in J\}$.

4 Auxiliary results

First of all we describe some properties of the fundamental Cauchy operator by the following

Proposition 4.1. *If $\{S(t,s)\}_{(t,s) \in J \times J}$ is the fundamental system, then the fundamental Cauchy operator $G_S : L^1(J;X) \to C(J;X)$ is linear, bounded, weakly continuous and weakly sequentially continuous.*

Proof. Clearly G_S is a bounded and linear operator. Hence we can deduce that G_S is weakly continuous.

Now we prove that G_S is also weakly sequentially continuous.

Fixed $t \in J$ and $e' \in X^*$, let us consider the map $H_t : L^1(J;X) \to \mathbb{R}$, where $H_t(g) = e'(G_S g(t))$, for every $g \in L^1(J;X)$.

Obviously H_t is a linear and continuous functional. Fixed a sequence $(f_n)_n$, $f_n \in L^1(J;X)$ such that $f_n \rightharpoonup f$, by using the properties of the weak convergence, we have $e'(G_S f_n(t)) \to e'(G_S f(t))$. Then, by the arbitrariness of $e' \in X^*$, we have

$$G_S f_n(t) \rightharpoonup G_S f(t), \ \forall t \in J.$$

Moreover we can say that the sequence $(G_S(f_n - f))_n$ is uniformly bounded in $\mathcal{C}(J;X)$. Indeed, by using p3. and the weak convergence of $(f_n)_n$, we can write

$$\begin{aligned}\|G_S f_n - G_S f\|_{\mathcal{C}(J;X)} &= \sup_{t\in J} \left\| \int_0^t S(t,\xi)(f_n(\xi) - f(\xi))\, d\xi \right\|_X \\ &\leq K(\|f_n\|_{L^1(J;X)} + \|f\|_{L^1(J;X)}) \leq K(Q + \|f\|_{L^1(J;X)}),\end{aligned}$$

where Q is a positive constant such that $\|f_n\|_{L^1(J;X)} \leq Q$, for every $n \in \mathbb{N}$. Therefore $(G_S f_n - G_S f)_n$ satisfies all the hypotheses of Theorem 2.2, so we have

$$G_s f_n \rightharpoonup G_S f.$$

□

Now, let us introduce the following result, that will play a key role in the proof of our existence theorem. Let us note that the analogous Proposition 4.5 of [2] is not able to work in the proof of our existence theorem because the hypothesis d) is weaker of the assumption **(d)** required in Proposition 4.5 of [2].

Theorem 4.2. *Let M be a metric space, X a Banach space and $F : J \times M \to \mathcal{P}(X)$ a multimap having the following properties:*

a) for a.e. $t \in J$, for every $x \in M$, the set $F(t,x)$ is closed and convex ;

b) for every $x \in M$, the multimap $F(\cdot, x)$ has a B-measurable selection;

c) for a.e. $t \in J$ the multimap $F(t,\cdot) : M \to \mathcal{P}(X)$ has a (s-w)sequentially closed graph in $M \times X$;

d) for almost all $t \in J$ and every convergent sequence $(x_n)_n$ in M the set $\bigcup_n F(t,x_n)$ is relatively weakly compact;

e) there exists $\varphi : J \to [0,\infty)$: $\varphi \in L^1_+(J)$ such that

$$\sup_{z\in F(t,M)} \| z \| \leq \varphi(t), \quad a.e.\ t \in J.$$

Then, for every B-measurable $u : J \to M$, there is a B-measurable $y : J \to X$ with $y(t) \in F(t,u(t))$ for a.e. $t \in J$.

Proof. First of all we note that hypothesis *b)* implies

b)$_w$ for every $s : J \to M$ simple function, the multimap $F(\cdot, s(\cdot))$ has a B-measurable selection.

Next fix $u : J \to M$ a B-measurable function, then there exists a sequence $(u_p)_p$, $u_p : J \to M$ simple function, such that

$$u_p(t) \to u(t), \quad a.e.\ t \in J. \tag{4.1}$$

Using $b)_w$, for every $p \in \mathbb{N}$, in correspondence of the simple function u_p, there exists a B-measurable function $y_p : J \to X$ such that

$$y_p(t) \in F(t, u_p(t)), \quad a.e.\ t \in J. \tag{4.2}$$

Now, let us consider $A = \{y_p,\ p \in \mathbb{N}\}$, subset of $L^1(J;X)$ (see e)).

First of all we note that, if N is the null measure set for which *a), c), d), e)*, (4.1) and (4.2) hold, we can write (see (4.2))

$$A(t) = \{y_p(t),\ p \in \mathbb{N}\} \subset \overline{\bigcup_{p\in\mathbb{N}} F(t, u_p(t))}^{w}, \qquad t \in J \setminus N \tag{4.3}$$

where the set $\overline{\bigcup_{p\in\mathbb{N}} F(t, u_p(t))}^{w}$ is weakly compact. Therefore the set $A(t)$ is relatively weakly compact.

Now, by using hypothesis e) we can say that A is bounded in $L^1(J;X)$. Indeed, put $r = \|\varphi\|_1$, we have

$$\|y_p\|_{L^1(J;X)} \le r, \qquad \forall p \in \mathbb{N}.$$

Moreover, by recalling that $\varphi \in L^1_+(J)$, we can say that, for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0 :$ for every $H \in \mathcal{M}(J)$, $\mu(H) < \delta(\varepsilon)$ then

$$\left| \int_H y_p(t)\, dt \right| \le \int_H \|y_p(t)\|_X\, dt \le \int_H \varphi(t)\, dt \le \varepsilon, \qquad \forall p \in \mathbb{N},$$

i.e., A has the property of equi-absolute continuity of the integral.

Since, as we have showed, the set A satisfies all the hypotheses of [29, Corollary 9], we can conclude that A is relatively weakly compact in $L^1(J;X)$. Therefore there exists $(y_{p_k})_k \subset (y_p)_p$ such that $y_{p_k} \rightharpoonup y$, $y \in L^1(J;X)$.

Now, we can apply [[17], Proposition 7.3.9] to the multimap $G : J \to \mathcal{P}_{wk}(X)$, defined by $G(s) = B_s$, $\forall s \in J$, where $B_s = \overline{\bigcup_{p\in\mathbb{N}} F(s, u_p(s))}^{w}$, and to the sequence $(y_{p_k})_k$ of $L^1(J;X)$. It is possible since (see (4.3)) $y_{p_k}(t) \in B_t$, $t \in J \setminus N$, $\forall p_k$. Hence we can conclude that, for the fixed $t \in J \setminus N$, we have (see (2.1))

$$y(t) \in \overline{co}\ w - \limsup_{k\to\infty} \{y_{p_k}(t)\}_k. \tag{4.4}$$

Then, by (4.2), we can say

$$\overline{co}\ w - \limsup_{k\to\infty} \{y_{p_k}(t)\}_k \subset \overline{co}\ w - \limsup_{k\to\infty} F(t, u_{p_k}(t)). \tag{4.5}$$

Finally, we will prove that (see hypothesis a) and (4.1))

$$\overline{co}\ w - \limsup_{k\to\infty} F(t, u_{p_k}(t)) \subset F(t, u(t)). \tag{4.6}$$

Let us fix $z \in \overline{co}\ w - \limsup_{p_k\to\infty} F(t, u_{p_k}(t))$, then there exists $z_{p_{k_q}} \in F(t, u_{p_{k_q}}(t))$ such that

$$z_{p_{k_q}} \rightharpoonup z$$

in X, where $(p_{k_q})_{q\in\mathbb{N}}$ is an increasing sequence. Moreover, by (4.1) we know that

$$u_{p_{k_q}}(t) \to u(t).$$

Therefore, since $t \notin N$, hypothesis *c)* implies that $z \in F(t, u(t))$. For the arbitrariness of z we can conclude that (4.6) is true.

Thanks to (4.4), (4.5), (4.6), finally we can say that the map $y \in L^1(J; X)$ satisfies $y(t) \in F(t, u(t))$ a.e. $t \in J$, so the thesis holds. □

Now, by using the concept of smallest (x_0, T)-fundamental set (see *2)* of Proposition 2.5), taking into account of Proposition 2.5 and the classical Glicksberg Fixed Point Theorem of [13] we deduce a variant of Theorem 3.7 of [2] proved by Benedetti–Väth for x_0-unpreserving multimaps T.

Theorem 4.3. *Let X be a locally convex Hausdorff space, $K \subset X$, $x_0 \in K$ and $T : K \to \mathcal{P}(X)$ a multimap such that*

i) $\overline{co}(T(K) \cup \{x_0\}) \subset K$;

ii) $T(x)$ *is convex, for every* $x \in M_0$;

iii) M_0 *is compact;*

iv) $T_{|M_0}$ *has closed graph,*

where M_0 is the smallest (x_0, T)-fundamental set.

Then there exists at least one fixed point for T, i.e. there exists $\overline{x} \in M_0$: $\overline{x} \in T(\overline{x})$.

Proof. First of all, since M_0 is a (x_0, T)-fundamental set, we know that M_0 is convex and $T(M_0) \subset M_0$. Moreover by *iii)* M_0 is also compact.

Therefore, taking into account of *ii)* and *iv)*, the multimap $T_{|M_0} : M_0 \to \mathcal{P}(M_0)$ has convex values and closed graph. So we are in a position to apply the Glicksberg Theorem to the multimap $T_{|M_0}$, then there exists $\overline{x} \in M_0$ such that $\overline{x} \in T(\overline{x})$. □

When we deal with the weak topology in a Banach space, we can replace equivalently the hypothesis about closed graph by a sequentially closed graph (see [2], Corollary 3.2), so we have the following:

Corollary 4.4. *Let X be a Banach space, $K \subset X$, $x_0 \in K$ and $T : K \to \mathcal{P}(X)$ be a multimap such that*

i) $\overline{co}(T(K) \cup \{x_0\}) \subset K$;

ii) $T(x)$ *convex, for every* $x \in M_0$;

iii) M_0 *is weakly compact;*

iv) $T_{|M_0}$ *has weakly sequentially closed graph,*

where M_0 is the smallest (x_0, T)-fundamental set.

Then there exists at least one fixed point for T.

5 Existence result

In this section we assume the following hypotheses on the multimap $F : J \times X \to \mathcal{P}(X)$

F1. for every $(t, x) \in J \times X$, the set $F(t, x)$ is convex;

F2. for every $x \in X$, $F(\cdot, x) : J \to X$ admits a B-measurable selection;

F3. for a.e. $t \in J$, $F(t, \cdot) : X \to X$ has a weakly sequentially closed graph;

F4. there exists $(\varphi_n)_n$, $\varphi_n \in L^1_+(J)$, such that

$$\limsup_{n\to\infty} \frac{\int_0^1 \varphi_n(\xi)\, d\xi}{n} < \frac{1}{K} \tag{5.1}$$

and

$$\|F(t, \overline{B}_X(0, n))\| \leq \varphi_n(t),\ a.e.\ t \in J,\ n \in \mathbb{N}, \tag{5.2}$$

where K is the constant presented in Remark 3.2;

and the two properties related to functions $g, h : \mathcal{C}(J; X) \to X$

gh1. g, h are weakly sequentially continuous;

gh2. for every countable, bounded $H \subset \mathcal{C}(J; X)$, the sets $g(H)$ and $h(H)$ are relatively compact.

Now we state the main result of the paper.

Theorem 5.1. *Let X be a weakly compactly generated Banach space and $\{A(t)\}_{t\in J}$ a family of operators which satisfies the property **(A)**.*

Let $F : J \times X \to \mathcal{P}(X)$ be a multimap satisfying F1, F2, F3, F4 and the following hypothesis

F5. there exists $H \subset J$, $\mu(H) = 0$, such that, for all $n \in \mathbb{N}$, there exists $\nu_n \in L^1_+(J)$ with the property

$$\beta(C_1) \leq \nu_n(t)\beta(C_0),\ t \in J \setminus H$$

for all countable $C_0 \subseteq \overline{B}_X(0, n)$, $C_1 \subseteq F(t, C_0)$, where β is the De Blasi measure of weak noncompactness.

Let $g, h : \mathcal{C}(J; X) \to X$ be two functions satisfying gh1, gh2 and having the following properties

gh3. g, h are bounded;

gh4. for every bounded and closed subset M of $\mathcal{C}(J; X)$, the sets

$$C(\cdot, 0)g(M) \text{ and } S(\cdot, 0)h(M)$$

are relatively weakly compact in $\mathcal{C}(J; X)$.

Then there exists at least one mild solution for the nonlocal problem (**P**).

Proof. First of all we prove that

$$F(t,x) \text{ is closed, for a.e. } t \in J \text{ and for every } x \in X. \tag{5.3}$$

Denoted by N a null measure set such that *F3.* and *F5.* hold in $J \setminus N$, we fix $t \in J \setminus N$ and $x \in X$. Put $C_0 = \{x\}$ and $C_1 = \{y_n : n \in \mathbb{N}\}$, where $y_n \in F(t,x)$, $\forall n \in \mathbb{N}$. Being $C_0 \subset \overline{B}_X(0,p)$, for a suitable $p \in \mathbb{N}$, and $C_1 \subset F(t,C_0)$, by *F5.* we have $\beta(C_1) \le \nu_p(t)\beta(C_0) = 0$. Therefore C_1 is relatively w-compact and so by Eberlein–Šmulian Theorem we can say that there exists $(y_{n_k})_k$, $y_{n_k} \rightharpoonup y$. Then *F3.* implies that $y \in F(t,x)$. So we have that $F(t,x)$ is w-sequentially compact and, invoking again the Eberlein–Šmulian Theorem, $F(t,x)$ is w-compact. Therefore, by using Theorem 3 of [32], in order to establish the closeness of the convex set $F(t,x)$ it is sufficient to observe that $F(t,x)$ is w-sequentially closed by virtue of hypothesis *F3.* too.

Now, we consider the integral multioperator $T : \mathcal{C}(J;X) \to \mathcal{P}_c(\mathcal{C}(J;X))$ defined, for every $u \in \mathcal{C}(J;X)$, as

$$Tu = \Big\{ y \in \mathcal{C}(J;X) : y(t) = C(t,0)g(u) + S(t,0)h(u) + \int_0^t S(t,\xi)f(\xi)\,d\xi, t \in J,\ f \in S^1_{F(\cdot,u(\cdot))} \Big\} \tag{5.4}$$

where

$$S^1_{F(\cdot,u(\cdot))} = \{f \in L^1(J;X) : f(t) \in F(t,u(t)) \ a.e.\ t \in J\}. \tag{5.5}$$

Note that, for all $u \in \mathcal{C}(J;X)$, $Tu \neq \varnothing$. Indeed, put

$$M_u = \overline{B}_X(0,n_u), \tag{5.6}$$

where $n_u \in \mathbb{N}$: $\|u(t)\|_X \le n_u$, for all $t \in J$, we note that the multimap $F_{|J\times M_u}$ satisfies all the hypotheses of Theorem 4.2, by considering on M_u the metric d induced by that on X.

First of all *F1.*, (5.3) and *F2.* imply respectively *a)* and *b)* of Theorem 4.2 for the restriction $F_{|J\times M_u}$.

By *F3.* we have that $F_{|J\times M_u}$ has the property *c)* of Theorem 4.2.

Moreover, fixed $t \in J \setminus H$ (where H is presented in *F5.*), if $(u_n)_n$, $u_n \in M_u$, $u_n \to v$ in (M_u,d), we can consider the countable set $\tilde{C}_0 = \{u_n : n \in \mathbb{N}\} \subset M_u$ and the set $\tilde{C}_1 = \bigcup_n F(t,u_n) \subset F(t,\tilde{C}_0)$ and by *F5.* we can write

$$\beta(\tilde{C}_1) \le \nu_n(t)\beta(\tilde{C}_0) = 0,$$

hence $\beta(\tilde{C}_1) = 0$, i.e. the set $\tilde{C}_1$ is relatively w-compact for the regularity of the De Blasi MwNC. Therefore also *d)* of Theorem 4.2 holds.

Finally, for $n_u \in \mathbb{N}$ presented in (5.6), by *F4.* we can say that there exists $\varphi_{n_u} \in L^1_+(J)$ such that

$$\|F(t,M_u)\| \le \varphi_{n_u}(t), \quad \text{a.e. } t \in J$$

and so also *e)* of Theorem 4.2 is satisfied. Therefore we can conclude that there exists a B-selection f_u of the multimap $F(\cdot,u(\cdot))$, i.e. $S^1_{F(\cdot,u(\cdot))}$ is nonempty. Then the map y_u defined by

$$y_u(t) = C(t,0)g(u) + S(t,0)h(u) + G_S f_u, \qquad t \in J,$$

is such that $y_u \in Tu$, i.e. $Tu \neq \varnothing$.

Moreover T takes convex values thanks the convexity of the values of F.

From now on we proceed by steps.

Step 1. The multioperator T has a weakly sequentially closed graph.

Let $(q_n)_n$ and $(x_n)_n$ be two sequences in $\mathcal{C}(J;X)$ such that

$$x_n \in Tq_n, \qquad \forall n \in \mathbb{N} \tag{5.7}$$

and there exist $q, x \in \mathcal{C}(J;X)$ such that

$$q_n \rightharpoonup q, \qquad x_n \rightharpoonup x; \tag{5.8}$$

we have to show that $x \in Tq$.

First of all we recall that, by the properties of the convergence $q_n \rightharpoonup q$, there exists $\overline{n} \in \mathbb{N}$ such that

$$\|q_n\|_{\mathcal{C}(J;X)} \leq \overline{n}, \qquad \forall n \in \mathbb{N}. \tag{5.9}$$

Moreover, for every $t \in J$, the weak convergence of the sequence $(q_n)_n$ to q implies also that

$$q_n(t) \rightharpoonup q(t). \tag{5.10}$$

Then by (5.7), for every $n \in \mathbb{N}$, there exists (see (5.5))

$$f_n \in S^1_{F(\cdot, q_n(\cdot))} \tag{5.11}$$

such that (see (5.4))

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + \int_0^t S(t,\xi)f_n(\xi)\,d\xi, \qquad t \in J.$$

Now we want to prove that the multimaps $G_n : J \to \mathcal{P}(X)$, $n \in \mathbb{N}$ and $G : J \to \mathcal{P}(X)$ respectively defined by

$$G_n(t) = F(t, q_n(t)), \qquad t \in J, \tag{5.12}$$
$$G(t) = F(t, q(t)), \qquad t \in J \tag{5.13}$$

satisfy all the hypotheses of the Containment Theorem. To this aim we consider the null measure set N for which *F3.* and *F5.* hold. Let us fix $t \in J \setminus N$, we consider a sequence $(u_n)_n$ such that

$$u_n \in G_n(t), \ \forall n \in \mathbb{N}. \tag{5.14}$$

Now, we define a countable set of X

$$C_0 = \{q_n(t) : n \in \mathbb{N}\}. \tag{5.15}$$

It is evident that $C_0 \subset \overline{B}_X(0, \overline{n})$ (see (5.9)). Then, put $C_1 = \{u_n : \ n \in \mathbb{N}\}$ we have that (see (5.14), (5.12) and (5.15))

$$C_1 \subset F(t, \{q_n(t)\}_n) = F(t, C_0).$$

Now, in correspondence of $\overline{n} \in \mathbb{N}$ chosen in (5.9), by virtue of *F5.* there exists $\nu_{\overline{n}} \in L^1_+(J)$ such that

$$\beta(C_1) \leq \nu_{\overline{n}}(t)\beta(C_0). \tag{5.16}$$

Taking account of (5.10) we can say that the set C_0 is relatively weakly compact and so, for the regularity of β, $\beta(C_0) = 0$. By virtue of the Eberlein–Šmulian Theorem, by (5.16) we deduce that C_1 is relatively weakly sequentially compact, i.e. there exist $(u_{n_k})_k \subset (u_n)_n$ and $u \in X$ such that $u_{n_k} \rightharpoonup u$. Now by (5.10), (5.14) and (5.12), thanks to *F3.*, we have $u \in G(t)$. Moreover, being the sequence $(f_n)_n$ integrably bounded (see (5.11) and (5.9)), it has the property of equi-absolute continuity of the integral (also named uniformly integrability) and, obviously $f_n(t) \in G_n(t)$, a.e. $t \in J$ (see (5.12)).

Therefore, applying the Containment Theorem to the multimaps $G_n, G : J \to \mathcal{P}(X)$, $n \in \mathbb{N}$, (see (5.12) and (5.13)), we can say that there exists $(f_{n_k})_k \subset (f_n)_n$ such that

$$f_{n_k} \rightharpoonup f \text{ in } L^1(J;X),$$

where (see (5.13), *F1.* and (5.3))

$$f(t) \in \overline{co}G(t) = \overline{co}F(t,q(t)) = F(t,q(t)), \quad a.e.\ t \in J.$$

Hence, we can conclude that

$$f \in S^1_{F(\cdot,q(\cdot))}. \tag{5.17}$$

By using the weak continuity of the Cauchy operator G_S (see Proposition 4.1) we have $G_S f_{n_k} \rightharpoonup G_S f$. Then, for every fixed $t \in J$ we have

$$G_S f_{n_k}(t) \rightharpoonup G_S f(t), \tag{5.18}$$

and by hypothesis *gh1.* and taking into account of the linearity and continuity of $S(t,0)$ and $C(t,0)$ we have

$$C(t,0)g(q_{n_k}) \rightharpoonup C(t,0)g(q) \quad \text{and} \quad S(t,0)h(q_{n_k}) \rightharpoonup S(t,0)h(q).$$

So, by using (5.7) and (5.18), we can write

$$x_{n_k}(t) \rightharpoonup C(t,0)g(q) + S(t,0)h(q) + \int_0^t S(t,\xi)f(\xi)\,d\xi =: \tilde{x}(t). \tag{5.19}$$

On the other hand, by (5.8)), we know that $x_{n_k} \rightharpoonup x$ in $\mathcal{C}(J;X)$, hence $x_{n_k}(t) \rightharpoonup x(t)$, for all $t \in J$. From the uniqueness of the limit we have

$$x(t) = \tilde{x}(t), \qquad t \in J. \tag{5.20}$$

Finally, from (5.20), (5.19), (5.17) and (5.4) we deduce that $x \in Tq$. Therefore we can conclude that T has a weakly sequentially closed graph.

Step 2. There exists a subset of $\mathcal{C}(J;X)$ which is invariant under the action of the operator T.

We will show that exists $\overline{p} \in \mathbb{N}$ such that the operator T maps the ball $\overline{B}_{\mathcal{C}(J;X)}(0,\overline{p})$ into itself.

Assume by contradiction that, for every $n \in \mathbb{N}$, there exists $q_n \in \mathcal{C}(J;X)$, with $\|q_n\|_{\mathcal{C}(J;X)} \leq n$, such that there exists $x_{q_n} \in Tq_n$, $\|x_{q_n}\|_{\mathcal{C}(J;X)} > n$.

Since $\|x_{q_n}\|_{\mathcal{C}(J;X)} > n$, there exists $t_n \in J$ such that $\|x_{q_n}(t_n)\|_X \geq n$. Now, taking into account the p1. and p3. of Remark 3.2 we can write

$$\begin{aligned}
n \leq \|x_{q_n}(t_n)\|_X &\leq \|C(t_n,0)g(q_n)\|_X + \|S(t_n,0)h(q_n)\|_X + \int_0^{t_n} \|S(t_n,\xi)f_{q_n}(\xi)\|_X\,d\xi \\
&\leq \|C(t_n,0)\|_{\mathcal{L}(X)}\|g(q_n)\|_X + \|S(t_n,0)\|_{\mathcal{L}(X)}\|h(q_n)\|_X + \int_0^{t_n} \|S(t_n,\xi)\|_{\mathcal{L}(X)}\|f_{q_n}(\xi)\|_X\,d\xi \\
&\leq KQ + KQ + K\int_0^1 \|f_{q_n}(\xi)\|_X\,d\xi,
\end{aligned}$$

where $Q > 0$ is such that $\|g(u)\|_X \leq Q$, $\|h(u)\|_X \leq Q$, for every $u \in \mathcal{C}(J;X)$ (see *gh3.*) and $f_{q_n} \in S^1_{F(\cdot,q_n(\cdot))}$. Next, since $\|q_n\|_{\mathcal{C}(J;X)} = \sup_{t\in J}\|q_n(t)\|_X \leq n$, there exists (see (5.2) of hypothesis *F4.*) a function $\varphi_n \in L^1_+(J)$ such that

$$\|f_{q_n}(t)\|_X \leq \varphi_n(t), \quad a.e.\ t \in J,$$

then we deduce

$$n \leq \|x_{q_n}(t_n)\|_X \leq 2KQ + K\int_0^1 \varphi_n(\xi)\,d\xi. \tag{5.21}$$

Therefore, since (5.21) is true for every $n \in \mathbb{N}$, we have

$$1 \leq \frac{2KQ}{n} + \frac{K\int_0^1 \varphi_n(\xi)\,d\xi}{n}, \qquad \forall n \in \mathbb{N}.$$

Hence, passing to the superior limit, by (5.1) we obtain the following contradiction

$$1 \leq \limsup_{n\to\infty}\left(\frac{2KQ}{n} + \frac{K\int_0^1 \varphi_n(\xi)\,d\xi}{n}\right) \leq \limsup_{n\to\infty}\frac{K\int_0^1 \varphi_n(\xi)\,d\xi}{n} < 1.$$

Therefore we can conclude that there exists $\overline{p} \in \mathbb{N}$ such that $\overline{B}_{\mathcal{C}(J;X)}(0,\overline{p})$ is invariant under the action of the operator T.

Step 3. There exists the smallest $(0,T)$-*fundamental set* which is weakly compact.

First of all, fixed $\overline{p}$ as in **Step2.**, put $x_0 = 0$ and $K = \overline{B}_{\mathcal{C}(J;X)}(0,\overline{p})$. We know that K is a subset of the locally convex Hausdorff space $\mathcal{C}(J;X)$ equipped with the weak topology. Since $T(K) \subset K$, we have $\overline{co}(T(K)\cup\{0\}) \subset K$.

Therefore by Proposition 2.5, we can say that there exists the smallest $(0,T)$-fundamental set M_0 such that

$$M_0 \subset \overline{B}_{\mathcal{C}(J;X)}(0,\overline{p}) = K, \tag{5.22}$$

and

$$M_0 = \overline{co}(T(M_0)\cup\{0\}). \tag{5.23}$$

Now, we will prove that M_0 is weakly compact.

We consider the Sadovskij functional β_N, defined in (2.2), where $N \in \mathbb{R}^+$. Being β_N 0-stable (where 0 denotes the null function), we can write (see (5.23))

$$\beta_N(T(M_0)) = \beta_N(M_0), \tag{5.24}$$

hence, since β_N satisfies (I) and (II) of Remark 2.11, (5.24), (5.4) and *gh4.* imply

$$\begin{aligned}\beta_N(M_0) &= \beta_N\left(\{C(\cdot,0)g(u) + S(\cdot,0)h(u) + G_S f :\ f \in S^1_{F(\cdot,u(\cdot))},\ u \in M_0\}\right)\\ &\leq \beta_N(C(\cdot,0)g(M_0)) + \beta_N(S(\cdot,0)h(M_0)) + \beta_N(\{G_S f :\ f \in S^1_{F(\cdot,u(\cdot))},\ u \in M_0\})\\ &= \beta_N(\{G_S f :\ f \in S^1_{F(\cdot,u(\cdot))},\ u \in M_0\})\\ &= \sup_{\substack{C\subset S^1_{F(\cdot,M_0(\cdot))}\\ C\text{ countable}}}\ \sup_{t\in J}\ \beta\left(\left\{\int_0^t S(t,\xi)f(\xi)\,d\xi :\ f \in C\right\}\right)e^{-Nt}.\end{aligned} \tag{5.25}$$

Now, fixed $t \in J$ and a countable set $C \subset S^1_{F(\cdot,M_0(\cdot))}$, we define

$$C_t^C = \{S(t,\cdot)f(\cdot) :\ f \in C\}.$$

By using *F4.* and p3. of Remark 3.2 we can say that the countable set is integrably bounded, so it has the property of equi-absolute continuity of the integral. Now, since X is a weakly compact generated Banach space, we are in the position to apply Proposition 2.9 of the Preliminaries to the countable set C_t^C, so we have

$$\beta\left(\left\{\int_0^t S(t,\xi)f(\xi)\,d\xi:\ f\in C\right\}\right)\le\int_0^t\beta(C_t^C)\,d\xi,\qquad t\in J,\tag{5.26}$$

so, by using (5.25), (5.26) and p3. of Remark 3.2 ($a=1$), we can write

$$\begin{aligned}\beta_N(M_0)&\le\sup_{\substack{C\subset S^1_{F(\cdot,M_0(\cdot))}\\ C\text{ countable}}}\sup_{t\in J}\left(\int_0^t\beta(C_t^C)\,d\xi\right)e^{-Nt}\\&\le\sup_{\substack{C\subset S^1_{F(\cdot,M_0(\cdot))}\\ C\text{ countable}}}\sup_{t\in J}\left(\int_0^t\|S(t,\xi)\|_{\mathcal{L}(X)}\beta(C(\xi))\,d\xi\right)e^{-Nt}\\&\le\sup_{\substack{C\subset S^1_{F(\cdot,M_0(\cdot))}\\ C\text{ countable}}}\sup_{t\in J}\left(K\int_0^t\beta(C(\xi))\,d\xi\right)e^{-Nt}.\end{aligned}\tag{5.27}$$

Further let us note that for every $f\in S^1_{F(\cdot,M_0(\cdot))}$ we can consider, by the Axiom of Choice, a continuous map $q_f\in M_0$ such that $f(\xi)\in F(\xi,q_f(\xi))$ a.e. $\xi\in J$. So the set $C_0^C=\{q_f\in M_0:\ f\in C\}$ is countable too. Now, taking into account of the numerability of C, there exists a null measure set $V\subset J$: $H\subset V$, where H is the null measure set defined in *F5.*, such that

$$f(\xi)\in F(\xi,q_f(\xi)),\ \text{for every }\xi\in J\setminus V,\ f\in C,$$

where $q_f\in C_0^C$.

Hence, fixed $\xi\in J\setminus V$, we observe that $C_0^C(\xi)\subset M_0(\xi)\subset\overline{B}_X(0,\overline{p})$ (see (5.22)) and $C(\xi)\subset F(\xi,C_0^C(\xi))$. By hypothesis *F5.* we can write

$$\beta(C(\xi))\le\nu_{\overline{p}}(\xi)\beta(C_0^C(\xi)).$$

The above considerations allow to claim that, for every countable set $C\subset S^1_{F(\cdot,M_0(\cdot))}$, there exists a countable subset $C_0^C\subset M_0\subset\overline{B}_X(0,\overline{p})$ such that

$$\beta(C(\xi))\le\nu_{\overline{p}}(\xi)\beta(C_0^C(\xi))\le\nu_{\overline{p}}(\xi)\sup_{\substack{C_0\subset M_0\\ C_0\text{ countable}}}\beta(C_0(\xi)),\quad\text{a.e. }\xi\in J.\tag{5.28}$$

Therefore, taking into account of (5.28), by (5.27) we deduce

$$\begin{aligned}\beta_N(M_0)&\le\sup_{\substack{C\subset S^1_{F(\cdot,M_0(\cdot))}\\ C\text{ countable}}}\sup_{t\in J}\left(K\int_0^t\beta(C(\xi))\,d\xi\right)e^{-Nt}\\&\le\sup_{\substack{C\subset S^1_{F(\cdot,M_0(\cdot))}\\ C\text{ countable}}}\sup_{t\in J}\left(K\int_0^t\nu_{\overline{p}}(\xi)\sup_{\substack{C_0\subset M_0\\ C_0\text{ countable}}}\beta(C_0(\xi))\,d\xi\right)e^{-Nt}\\&\le\sup_{t\in J}\left(K\int_0^t e^{-N(t-\xi)}\nu_{\overline{p}}(\xi)\sup_{\substack{C_0\subset M_0\\ C_0\text{ countable}}}\sup_{\xi\in J}e^{-N\xi}\beta(C_0(\xi))\,d\xi\right)\\&=\beta_N(M_0)\sup_{t\in J}\int_0^t Ke^{-N(t-\xi)}\nu_{\overline{p}}(\xi)\,d\xi\end{aligned}\tag{5.29}$$

By virtue of [7, Lemma 3.1] we can say that there exists $H \in \mathbb{N}$ such that

$$\sup_{t\in J} \int_0^t K e^{-H(t-\xi)} \nu_{\overline{p}}(\xi)\, d\xi < 1. \tag{5.30}$$

Now, if we assume that $\beta_H(M_0) > 0$, and we consider in (5.29) the constant H characterized as in (5.30), we have the following contradiction

$$\beta_H(M_0) \le \beta_H(M_0) \sup_{t\in J} \int_0^t K e^{-H(t-\xi)} \nu_{\overline{p}}(\xi)\, d\xi < \beta_H(M_0).$$

Therefore we conclude that this fact

$$\beta_H(M_0) = 0 \tag{5.31}$$

is true.

By definition of $\beta_H(M_0)$, first of all, we have that, for every $t \in J$, the set $M_0(t)$ is relatively weakly sequentially compact. Indeed, fixed $t \in J$ and a sequence $(q_n(t))_n$ in $M_0(t)$, we consider the countable set $\tilde{C}(t) = \{q_n(t) : \ n \in \mathbb{N}\}$. By (5.31) we can say that $\beta(\tilde{C}(t)) = 0$, so we deduce that $\tilde{C}(t)$ is relatively weakly compact. By the Eberlein–Šmulian Theorem we have that the set $\tilde{C}(t)$ is relatively weakly sequentially compact, i.e. there exists a subsequence $(q_{n_k}(t))_k$ of $(q_n(t))_n$ such that $q_{n_k}(t) \rightharpoonup q(t) \in X$. Therefore, by the arbitrariness of $(q_n(t))_n$ we can conclude that $M_0(t)$ is relatively weakly sequentially compact.

Now, we show that also the set $S^1_{F(\cdot,M_0(\cdot))}$ is relatively weakly compact in $L^1(J;X)$.

To this aim we note that $S^1_{F(\cdot,M_0(\cdot))}$ is integrably bounded. Indeed, for every $f \in S^1_{F(\cdot,M_0(\cdot))}$, taking into account of $M_0 \subset \overline{B}_{C(J;X)}(0,\overline{p})$ (see (5.22)), we can write

$$f(t) \in F(t, M_0(t)) \subset F(t, \overline{B}_X(0,\overline{p})), \quad \text{a.e. } t \in J.$$

So, by *F4.* there exists $\varphi_{\overline{p}} \in L^1_+(J)$ such that

$$\|f(t)\|_X \le \varphi_{\overline{p}}(t), \quad \textit{a.e. } t \in J, \text{ for every } f \in S^1_{F(\cdot,M_0(\cdot))}. \tag{5.32}$$

Therefore $S^1_{F(\cdot,M_0(\cdot))}$ is integrably bounded and then $S^1_{F(\cdot,M_0(\cdot))}$ has the property of equi-absolute continuity of the integral (see Remark 2.3).

Moreover, by (5.32) we also deduce that $S^1_{F(\cdot,M_0(\cdot))}$ is bounded in $L^1(J;X)$.

Now, we show that $S^1_{F(t,M_0(t))}$ is relatively weakly compact in X, for a.e. $t \in J$.

Let us fix $t \in J \setminus H^*$, where H^* is the null measure set for which *F4.* and *F5.* hold. First of all, we note that $S^1_{F(t,M_0(t))}$ is norm bounded in X by the constant $\varphi_{\overline{p}}(t)$. Indeed we have

$$\|x\|_X \le \|F(t, M_0(t))\| \le \|F(t, \overline{B}_X(0,\overline{p}))\| \le \varphi_{\overline{p}}(t),$$

for every $x \in S^1_{F(t,M_0(t))}$.

Next, let us fix a sequence $(y_n)_n$, where $y_n \in S^1_{F(t,M_0(t))}$, $n \in \mathbb{N}$. Then there exists a sequence $(f_n)_n \subset S^1_{F(\cdot,M_0(\cdot))}$ such that

$$y_n = f_n(t) \in F(t, M_0(t));$$

let us note that, for every $n \in \mathbb{N}$, there exists $q_n \in M_0(t)$ such that

$$y_n \in F(t, q_n). \tag{5.33}$$

Now, by considering the two countable sets $C_0 = \{q_n : n \in \mathbb{N}\}$ and $C_1 = \{y_n : n \in \mathbb{N}\}$ we have (see (5.22) and (5.33))

$$C_0 \subset \overline{B}_X(0,\overline{p}) \quad \text{and} \quad C_1 \subset F(t,C_0).$$

So, by *F5.* and recalling that $M_0(t)$ is relatively weakly compact we can write

$$0 \leq \beta(C_1) \leq \nu_{\overline{p}}(t)\beta(C_0) \leq \nu_{\overline{p}}(t)\beta(M_0(t)) = 0,$$

so $\beta(C_1) = 0$, i.e. C_1 is relatively weakly compact. Hence there exists a subsequence $(y_{n_k})_k \subset (y_n)_n$ such that $(y_{n_k})_k$ is weakly convergent.

By the arbitrariness of $(y_n)_n$ in $S^1_{F(t,M_0(t))}$ and taking into account the Eberlein–Šmulian Theorem, we can claim that $S^1_{F(t,M_0(t))}$ is relatively weakly compact.

So, we are in the position to apply [29, Corollary 9], hence $S^1_{F(\cdot,M_0(\cdot))}$ is relatively weakly compact in $L^1(J;X)$.

Next we are able to prove that $T(M_0)$ is relatively weakly compact in $\mathcal{C}(J;X)$.

To this aim we fix a sequence $(x_n)_n$, $x_n \in T(M_0)$. Then there exists $(p_n)_n$, $p_n \in M_0$ such that, for every $n \in \mathbb{N}$, $x_n \in Tp_n$, hence

$$x_n(t) = C(t,0)g(p_n) + S(t,0)h(p_n) + \int_0^t S(t,\xi)f_n(\xi)\,d\xi,\ t \in J, \tag{5.34}$$

where $f_n \in S^1_{F(\cdot,p_n(\cdot))} \subset S^1_{F(\cdot,M_0(\cdot))}$.

By the relative weak sequential compactness of $S^1_{F(\cdot,M_0(\cdot))}$ in $L^1(J;X)$ we can find a subsequence $(f_{n_k})_{n_k}$ of $(f_n)_n$ such that $f_n \rightharpoonup f$ in $L^1(J;X)$. By using Proposition 4.1, we have

$$G_S f_{n_k} \rightharpoonup G_S f. \tag{5.35}$$

Moreover, thanks to hypothesis *gh2.*, since $\{p_{n_k} : k \in \mathbb{N}\} \subset M_0$ is countable and bounded in $\mathcal{C}(J;X)$ (see (5.22)), there exists a subsequence of $(p_{n_k})_k$, w.l.o.g we name also $(p_{n_k})_k$, such that

$$g(p_{n_k}) \to x \quad \text{and} \quad h(p_{n_k}) \to y \quad \text{in } X. \tag{5.36}$$

Now, let us consider the subsequence $(x_{n_k})_k$ of $(x_n)_n$ (see (5.34)).

For every linear and continuous functional $e' : \mathcal{C}(J;X) \to \mathbb{R}$, we can write

$$e'(x_{n_k}) = e'(C(\cdot,0)g(p_{n_k})) + e'(S(\cdot,0)h(p_{n_k})) + e'(G_S f_{n_k}), \qquad \forall n_k.$$

Taking into account (5.35) and (5.36), passing to the limit for $k \to +\infty$, we obtain

$$\lim_{k\to\infty} e'(x_{n_k}) = e'(C(\cdot,0)x) + e'(S(\cdot,0)y) + e'(G_S f) = e'(C(\cdot,0)x + S(\cdot,0)y + G_S f).$$

By definition of weak convergence we have

$$x_{n_k} \rightharpoonup C(\cdot,0)x + S(\cdot,0)y + G_S f =: \overline{x},$$

where $\overline{x} \in \mathcal{C}(J;X)$, which means that $T(M_0)$ is relatively weakly sequentially compact and so, using again the Eberlein–Šmulian Theorem we can claim that $T(M_0)$ is relatively weakly compact.

Finally, recalling (5.23), we can conclude M_0 is weakly compact.

Step 4. Existence of a fixed point for T.

Finally we are in the position to apply Corollary 4.4 to the multimap $T_{|M_0}$. Hence the multioperator T has a fixed point in M_0, i.e. there exists $x \in M_0$ such that

$$x(t) = C(t,0)g(x) + S(t,0)h(x) + \int_0^t S(t,\xi)f(\xi)\,d\xi, \qquad t \in J$$

where $f \in S^1_{F(\cdot,x(\cdot))}$. Of course, x is a mild solution for (**P**). □

An immediate consequence of Theorem 5.1 is the following existence result for Cauchy problems.

Corollary 5.2. *Let X be a weakly compactly generated Banach space and $x_0, x_1 \in X$. Under the assumptions (A), F1.–F5. of Theorem 5.1, there exists at least one mild solution for the Cauchy problem*

$$\begin{cases} x''(t) \in A(t)x(t) + F(t,x(t)), \ t \in J \\ x(0) = x_0 \\ x'(0) = x_1. \end{cases} \tag{PC}$$

Now, we propose the following existence result for reflexive Banach spaces. We note that in this proposition the assumptions *F5. gh3.* and *gh4.* of Theorem 5.1 are omitted. Let us note that the lack of these hypotheses implies that this result is a new one with respect to Theorem 5.1 since the reflexivity doesn't imply that these assumptions hold. For this reason it is necessary to modify in some points the proof of the previous existence result.

Theorem 5.3. *Let X be a reflexive Banach space and $\{A(t)\}_{t\in J}$ a family of operators which satisfies the property* (**A**).

Let $F : J \times X \to \mathcal{P}(X)$ be a multimap satisfying F1, F2, F3, F4 and $g,h : \mathcal{C}(J;X) \to X$ be two functions having the properties gh1 and gh2.

Then there exists at least one mild solution for the nonlocal problem (**P**).

Proof. First we note that if $N \subset J$ is null measure set such that (5.2) and *F3.* hold, fixed $t \in J \setminus N$ and $x \in X$, by (5.2) we deduce the boundedness of the set $F(t,x)$, therefore the reflexivity of the space X imply the relative weak compactness of $F(t,x)$. Moreover, by *F3.* we have that the set $F(t,x)$ is weakly sequentially closed, so invoking Theorem 3 of [32] and *F1.* we can claim that

$$F(t,x) \text{ is closed, for a.e. } t \in J \text{ and for every } x \in X. \tag{5.37}$$

Let us consider the integral multioperator $T : \mathcal{C}(J;X) \to \mathcal{P}_c(\mathcal{C}(J;X))$ defined in (5.4) and (5.5).

First of all we have to prove that the multioperator T is well defined, i.e. it has nonempty and convex values.

Let $\overline{u} \in \mathcal{C}(J;X)$, by using the uniform continuity of $\overline{u}$ in J we can construct a sequence $(u_n)_n$, $u_n : J \to X$, of step functions such that

$$\sup_{t\in J} \|u_n(t) - \overline{u}(t)\|_X \to 0, \quad \text{for } n \to \infty, \tag{5.38}$$

then, for every $n \in \mathbb{N}$, by virtue of *F2.*, there exists a B-measurable function $f_n : J \to X$ such that

$$f_n(t) \in F(t,u_n(t)), \quad a.e.\ t \in J. \tag{5.39}$$

Moreover, by (5.38), there exists $N_{\overline{u}} \in \mathbb{R}^+$ such that

$$u_n(t), \overline{u}(t) \in \overline{B}_X(0, N_{\overline{u}}) := M_{\overline{u}}, \quad \text{for every } t \in J,\ n \in \mathbb{N}, \tag{5.40}$$

so, by hypothesis *F4.* and (5.39), we can claim

$$\|f_n(t)\|_X \leq \|F(t, \overline{B}_X(0, N_{\overline{u}}))\| \leq \varphi_{N_{\overline{u}}}(t), \quad a.e.\ t \in J, \forall n \in \mathbb{N}, \tag{5.41}$$

where $\varphi_{N_{\overline{u}}} \in L^1_+(J)$.

Therefore, since f_n is B-measurable, by (5.41) we can deduce that $f_n \in L^1(J; X)$, for every $n \in \mathbb{N}$.

Now, taking into account of (5.41), the set $A_{\overline{u}} = \{f_n :\ n \in \mathbb{N}\}$ is bounded in $L^1(J; X)$ and it has the property of equi-absolute continuity of the integral. Moreover, by (5.41), $A_{\overline{u}}(t) \subset \overline{B}_X(0, \varphi_{N_{\overline{u}}}(t))$, a.e. $t \in J$. According to the reflexivity of the space and by [29, Corollary 9] we can conclude that the set $A_{\overline{u}}$ is relatively weakly compact in $L^1(J; X)$. Therefore, there exists $(f_{n_k})_k$, subsequence of $(f_n)_n$, such that

$$f_{n_k} \rightharpoonup f_{\overline{u}} \in L^1(J; X).$$

Now, in order to obtain that $f_{\overline{u}} \in S^1_{F(\cdot, \overline{u}(\cdot))}$, we want to prove that $f_{\overline{u}}(t) \in F(t, \overline{u}(t))$, a.e. $t \in J$.

Since $f_{n_k} \rightharpoonup f_{\overline{u}}$, by using Mazur's convexity theorem, there exists a sequence $(\tilde{f}_{n_k})_k$ made up of convex combinations of the f_{n_k}'s such that $\tilde{f}_{n_k} \to f_{\overline{u}}$ in $L^1(J; X)$ and, up to a subsequence,

$$\tilde{f}_{n_{k_p}}(t) \to f_{\overline{u}}(t), \quad a.e.\ t \in J. \tag{5.42}$$

Now, put H^* the null measure set for which hypothesis *F3.*, (5.37), (5.39), (5.41) and (5.42) hold, by using respectively (5.37) and (5.41) we have

$$F(t, x) \text{ is weakly closed}, \quad \text{for every } x \in X,\ t \in J \setminus H^*, \tag{5.43}$$

$$\sup_{x \in \overline{B}_X(0, N_{\overline{u}})} \|F(t, x)\| \leq \varphi_{N_{\overline{u}}}(t), \text{ for every } t \in J \setminus H^*. \tag{5.44}$$

Next, we want to prove that fixed $\overline{t} \in J \setminus H^*$, the multimap $F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, \cdot)$ is weakly upper semicontinuous and, in order to do that, we will show that all the hypotheses of [18, Theorem 1.1.5] are satisfied.

For every $x \in \overline{B}_X(0, N_{\overline{u}})$ from (5.44), we can write $F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, x) \subset \overline{B}_X(0, \varphi_{N_{\overline{u}}}(\overline{t}))$, therefore, by the reflexivity of X, we can say (see (5.43)) that the set $F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, x)$ is weakly compact and the multimap $F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, \cdot)$ is weakly compact. Hence, recalling hypothesis *F3.*, by [2, Corollary 3.2] we have that $F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, \cdot)$ is a weakly closed multimap. Since all the hypotheses of [18, Theorem 1.1.5] are satisfied, $F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, \cdot)$ is also weakly upper semicontinuous. Hence we can conclude that $F_{|\overline{B}_X(0, N_{\overline{u}})}(t, \cdot)$ is weakly upper semicontinuous, for every $t \in J \setminus H^*$.

Now, let us fix again $\overline{t} \in J \setminus H^*$ and assume that absurdly $f_{\overline{u}}(\overline{t}) \notin F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, \overline{u}(\overline{t}))$.
We note that, thanks to *F1.* and (5.37), all the hypotheses of the Hahn–Banach Theorem are satisfied, so there exists a weakly open convex set $V \supset F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, \overline{u}(\overline{t}))$ satisfying

$$f_{\overline{u}}(\overline{t}) \notin \overline{V} = \overline{V}^w. \tag{5.45}$$

Taking into account of the weak upper semicontinuity of $F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, \cdot)$, there exists a weak neighborhood $W_{\overline{u}(\overline{t})}$ of $\overline{u}(\overline{t})$ such that $F_{|\overline{B}_X(0, N_{\overline{u}})}(\overline{t}, W_{\overline{u}(\overline{t})})) \subset V$. Therefore

$$F(\overline{t}, x) \subset V, \quad \text{for every } x \in W_{\overline{u}(\overline{t})} \cap \overline{B}_X(0, N_{\overline{u}}). \tag{5.46}$$

Now, by (5.38) the subsequence $(u_{n_{k_p}}(\bar{t}))_p$, indexed as in (5.42), satisfies $u_{n_{k_p}}(\bar{t}) \rightharpoonup \overline{u}(\bar{t})$, so there exists $\overline{n} \in \mathbb{N}$ such that, for every $n_{k_p} > \overline{n}$ we have $u_{n_{k_p}}(\bar{t}) \in W_{\overline{u}(\bar{t})}$, hence by (5.40) $u_{n_{k_p}}(\bar{t}) \in W_{\overline{u}(\bar{t})} \cap \overline{B}_X(0, N_{\overline{u}})$.

Further, by (5.39) and (5.46) we deduce that $f_{n_{k_p}}(\bar{t}) \in V$, for every $n_{k_p} > \overline{n}$.

Now, the convexity of V implies that $\tilde{f}_{n_{k_p}}(\bar{t}) \in V$, for every $n_{k_p} > \overline{n}$ and, by the convergence of $(\tilde{f}_{n_{k_p}}(\bar{t}))_k$ to $f_{\overline{u}}(\bar{t})$, we arrive to the contradictory conclusion $f_{\overline{u}}(\bar{t}) \in \overline{V}^w$ (see (5.45)). So we can conclude that $f_{\overline{u}}(t) \in F(t, \overline{u}(t))$ a.e. $t \in J$.

By recalling (5.5) and the fact that $f_{\overline{u}} \in L^1(J;X)$ we finally obtain that $f_{\overline{u}} \in S^1_{F(\cdot,\overline{u}(\cdot))}$, i.e. $S^1_{F(\cdot,\overline{u}(\cdot))} \neq \varnothing$.

Now, we consider the function $y_{\overline{u}} : J \to X$ defined by

$$y_{\overline{u}}(t) = C(t,0)g(\overline{u}) + S(t,0)h(\overline{u}) + \int_0^t S(t,\xi) f_{\overline{u}}(\xi)\, d\xi, \qquad t \in J.$$

It is easy to prove that $y_{\overline{u}}$ is well posed and continuous in J, so $y_{\overline{u}} \in T\overline{u}$, i.e. $T\overline{u} \neq \varnothing$. Clearly, $T\overline{u}$ is convex.

We can conclude that the integral multioperator T assumes values in $\mathcal{P}_c(\mathcal{C}(J;X))$.

Form now on we proceed by steps.

Step 1. The multioperator T has a weakly sequentially closed graph.

As in **Step 1** of Theorem 5.1 we fix two sequences $(q_n)_n$ and $(x_n)_n$ in $\mathcal{C}(J;X)$ with the properties (5.7) and (5.8).

Using analogous considerations of **Step 1** of Theorem 5.1 we can say that (5.9) and (5.10) hold, so for every $n \in \mathbb{N}$, by (5.7) there exists (see (5.5))

$$f_n \in S^1_{F(\cdot,q_n(\cdot))} \tag{5.47}$$

such that (see (5.4))

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + \int_0^t S(t,\xi) f_n(\xi)\, d\xi, \qquad t \in J.$$

Now we want to prove that, put $A = \{f_n : n \in \mathbb{N}\}$ (see (5.47)), A satisfies all the hypotheses of [29, Corollary 9]. Obviously A is a subset of $L^1(J;X)$.

Moreover, by (5.47) and (5.9) we deduce

$$f_n(t) \in F(t, q_n(t)) \subset F(t, \overline{B}_X(0,\overline{n})), \quad \textit{a.e. } t \in J,\ \forall n \in \mathbb{N}. \tag{5.48}$$

Now, put H the null measure set for which *F4.* and (5.48) hold, we have that there exists $\varphi_{\overline{n}} \in L^1_+(J)$ such that (see (5.2))

$$\|f_n(t)\|_X \leq \varphi_{\overline{n}}(t), \qquad \forall t \in J \setminus H,\ \forall n \in \mathbb{N} \tag{5.49}$$

that implies

$$\|f_n\|_{L^1(J;X)} \leq \|\varphi_{\overline{n}}\|_1, \qquad \forall n \in \mathbb{N}, \tag{5.50}$$

i.e. the set A is bounded in $L^1(J;X)$. Then, by (5.50) we also say that A has the property of equi-absolute continuity of the integral (see Remark 2.3).

Now, by using (5.50) and the reflexivity of X we can also say that $A(t)$ is relatively weakly compact a.e. $t \in J$. Hence, since also (5.49) is true, thanks to [29, Corollary 9], we can conclude that A is relatively weakly compact in $L^1(J;X)$. So there exists a subsequence $(f_{n_k})_k$ of $(f_n)_n$ such that $f_{n_k} \rightharpoonup f \in L^1(J;X)$, then by using again the Mazur's convexity theorem and analogous arguments presented in the previous part of the proof we can claim that $f(t) \in F(t, q(t))$, a.e. $t \in J$. Therefore we can say that (see (5.5)) $f \in S^1_{F(\cdot,q(\cdot))}$.

Now, by using *gh1.* and the same technique of the final part of **Step 1** of Theorem 5.1 we can obtain that $x \in Tq$.

Therefore we can conclude that T has a weakly sequentially closed graph.

Step 2. There exists a subset of $\mathcal{C}(J;X)$ which is invariant under the action of the operator T.

We omit this step of the proof, since it is identical to **Step 2** of the proof of Theorem 5.1. So, we can say that there exists $\overline{p} \in \mathbb{N}$ such that $\overline{B}_{\mathcal{C}(J;X)}(0,\overline{p})$ is invariant under the action of the operator T.

Step 3. There exists the smallest $(0, T)$-*fundamental set* which is weakly compact.

First of all, fixed $\overline{p}$ as in **Step 2**, by Proposition 2.5, put $x_0 = 0$ and $K = \overline{B}_{\mathcal{C}(J;X)}(0,\overline{p})$ a subset of the locally convex Hausdorff space $\mathcal{C}(J;X)$ equipped with the weak topology, we can say that there exists

$$M_0 \subset \overline{B}_{\mathcal{C}(J;X)}(0,\overline{p}) = K \tag{5.51}$$

such that

$$M_0 = \overline{co}(T(M_0) \cup \{0\}) \tag{5.52}$$

Now, we will prove that M_0 is weakly compact. To this end we establish that the set $T(M_0)$ is relatively weakly compact.

Let $(q_n)_n$ be a sequence in M_0 and $(x_n)_n$ be a sequence in $\mathcal{C}(J;X)$ such that $x_n \in Tq_n$, for every $n \in \mathbb{N}$. Now, by definition of the multioperator T, there exists a sequence $(f_n)_n$, $f_n \in S^1_{F(\cdot,q_n(\cdot))}$, such that

$$x_n(t) = C(t,0)g(q_n) + S(t,0)h(q_n) + \int_0^t S(t,\xi) f_n(\xi)\, d\xi, \qquad t \in J.$$

Next, put $A = \{f_n : \ n \in \mathbb{N}\}$, reasoning as in **Step 1** of this proof, we can show that A is bounded in $L^1(J;X)$, it has the property of equi-absolute continuity of the integral and, by using the reflexivity of X, we can say that $A(t)$ is relatively weakly compact, for a.e. $t \in J$. Therefore, thanks again to [29, Corollary 9] we can say that A is relatively weakly compact in $L^1(J;X)$, so there exists $(f_{n_k})_k$ subsequence of $(f_n)_n$ such that $f_{n_k} \rightharpoonup f \in L^1(J;X)$.

Now, by the weak sequential continuity of G_S (see Proposition 4.1), we can write

$$G_S f_{n_k} \rightharpoonup G_S f. \tag{5.53}$$

Moreover, thanks to hypothesis *gh2.*, since $\{q_{n_k} : \ k \in \mathbb{N}\} \subset M_0$ is countable and bounded (see (5.51)), there exists a subsequence of $(q_{n_k})_k$ w.l.o.g. named again $(q_{n_k})_k$, such that

$$g(q_{n_k})_k \to x \quad \text{and} \quad h(q_{n_k})_k \to y \quad \text{in } X. \tag{5.54}$$

Now, by (5.53) and (5.54) the subsequence $(x_{n_k})_k$ of $(x_n)_n$ weakly converges to $\overline{x} = C(\cdot,0)x + S(\cdot,0)y + G_S f \in \mathcal{C}(J;X)$. Therefore $T(M_0)$ is relatively weakly compact and, invoking (5.52), M_0 is weakly compact.

Finally, reasoning as in **Step 4** of Theorem 5.1 we can conclude that there exists at least one mild solution for (**P**). $\square$

We can immediately formulate the following consequence of Theorem 5.3 for Cauchy problems.

Corollary 5.4. *Let X be a reflexive Banach space and $x_0, x_1 \in X$. Under the assumptions (A), F1.–F4. of Theorem 5.3, there exists at least one mild solution for the Cauchy problem* (**PC**)*.*

Remark 5.5. Let us note that, if $J = [0, a]$, all the results of Sections 4 and 5 hold too. In particular, Theorem 5.1, Theorem 5.3 and their respectively corollaries continue to be true if we modify (5.1) by

$$\limsup_{n\to\infty} \frac{\int_0^a \varphi_n(\xi)\, d\xi}{n} < \frac{1}{Ka}.$$

6 An application

In this section we apply the theory developed in Section 5 to study the following controllability problem

$$\begin{cases} \frac{\partial^2 w}{\partial t^2}(t,\xi) = \frac{\partial^2 w}{\partial \xi^2}(t,\xi) + b(t)\frac{\partial w}{\partial \xi}(t,\xi) + T(t)w(t,\cdot)(\xi) + u(t,\xi) \\ w(t,0) = w(t,2\pi), \qquad t \in J \\ \frac{\partial w}{\partial \xi}(t,0) = \frac{\partial w}{\partial \xi}(t,2\pi), \qquad t \in J \\ w(0,\xi) = x_0, \qquad \xi \in \mathbb{R} \\ \frac{\partial w}{\partial t}(0,\xi) = x_1, \qquad \xi \in \mathbb{R} \\ \|u(t,\xi)\|_{\mathbb{C}} \in \left[f_1\left(t,\xi,\int_0^{2\pi} k_1(t,\theta)w(t,\theta)\, d\theta\right), f_2\left(t,\xi,\int_0^{2\pi} k_2(t,\theta)w(t,\theta)\, d\theta\right)\right] \end{cases} \tag{6.1}$$

where $x_0, x_1 \in \mathbb{C}$ and $b : J \to \mathbb{R}$, $k_i : J \times \mathbb{R} \to \mathbb{R}$, $f_i : J \times \mathbb{R} \times \mathbb{C} \to \mathbb{R}_0^+$, $i = 1, 2$, $\{T(t)\}_{t\in J}$ is a suitable family of operators.

First of all, as in [16], we will use the identification between functions defined on the quotient group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ with values in $\mathbb{C}$ and 2π-periodic functions from $\mathbb{R}$ to $\mathbb{C}$. In order to model the problem above in an abstract form we consider the space $X = L^2(\mathbb{T}, \mathbb{C})$, i.e. the space of all functions $x : \mathbb{R} \to \mathbb{C}$, 2π-periodic and 2-integrable in $[0, 2\pi]$, endowed with the usual norm $\|\cdot\|_{L^2(\mathbb{T},\mathbb{C})}$. Moreover we denote by $H^1(\mathbb{T}, \mathbb{C})$ and by $H^2(\mathbb{T}, \mathbb{C})$ respectively the following Sobolev subspaces of $L^2(\mathbb{T}, \mathbb{C})$

$$H^1(\mathbb{T},\mathbb{C}) = \left\{ x \in L^2(\mathbb{T},\mathbb{C}) : \frac{dx}{d\xi} \in L^2(\mathbb{T},\mathbb{C}) \right\}$$
$$H^2(\mathbb{T},\mathbb{C}) = \left\{ x \in L^2(\mathbb{T},\mathbb{C}) : \frac{dx}{d\xi}, \frac{d^2x}{d\xi^2} \in L^2(\mathbb{T},\mathbb{C}) \right\},$$

where $\frac{dx}{d\xi}, \frac{d^2x}{d\xi^2}$ denote the weak derivatives.

Further we consider the operator $A_0 : D(A_0) = H^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$ defined by

$$A_0 x = \frac{d^2x}{d\xi^2}, \qquad x \in H^2(\mathbb{T},\mathbb{C})$$

and we assume that the operator A_0 is the infinitesimal generator of a strongly continuous cosine family $\{C_0(t)\}_{t\in\mathbb{R}}$, where $C_0(t) : L^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$, for every $t \in \mathbb{R}$ (see [16]).

Then we fix the function $P : J \to \mathcal{L}(H^1(\mathbb{T}, \mathbb{C}), L^2(\mathbb{T}, \mathbb{C}))$ defined as

$$P(t)x = b(t)\frac{dx}{d\xi}, \qquad t \in J,\ x \in H^1(\mathbb{T},\mathbb{C})$$

where $b : J \to \mathbb{R}$ is of class $\mathcal{C}^1$ on J.

Now we are able to define the family $\{A(t) : t \in J\}$ where, for every $t \in J$, $A(t) : D(A) = H^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$ is an operator defined as

$$A(t) := A_0 + P(t), \qquad t \in J.$$

Let us note that, as the Authors of [16] say (see Lemma 4.1), the family $\{A(t) : t \in J\}$ generates a fundamental system $\{S(t,s)\}_{(t,s)\in J\times J}$. In the sequel, we denote with K the constant, linked to $\{S(t,s)\}_{(t,s)\in J\times J}$, satisfying the properties of Remark 3.2.

In what follows we revise functions $w, u : J \times \mathbb{R} \to \mathbb{C}$ such that $w(t,\cdot) \in H^2(\mathbb{T}, \mathbb{C})$ and $u(t,\cdot) \in L^2(\mathbb{T}, \mathbb{C})$, for every $t \in J$, as the maps $x : J \to H^2(\mathbb{T}, \mathbb{C})$, $v : J \to L^2(\mathbb{T}, \mathbb{C})$, respectively, are defined by

$$\begin{aligned} x(t)(\xi) &= w(t,\xi), \qquad t \in J,\ \xi \in \mathbb{R} \\ v(t)(\xi) &= u(t,\xi), \qquad t \in J,\ \xi \in \mathbb{R}. \end{aligned}$$

Moreover we construct, by using the family $\{T(t)\}_{t\in J}$ and the functions f_1, f_2, a suitable multimap F such that we can rewrite the problem (6.1) in the abstract form

$$\begin{cases} \frac{d^2x}{dt^2}(t) \in A_0x(t) + P(t)x(t) + F(t, x(t)) = A(t)x(t) + F(t, x(t)),\ t \in J \\ x(0) = \tilde{x_0} \\ \frac{dx}{dt}(0) = \tilde{x_1} \end{cases} \tag{6.2}$$

where $\tilde{x_0}, \tilde{x_1} : \mathbb{R} \to \mathbb{C}$ are functions of $L^2(\mathbb{T}, \mathbb{C})$ respectively defined $\tilde{x_0}(\xi) = x_0$, $\tilde{x_1}(\xi) = x_1$, for every $\xi \in \mathbb{R}$.

Let us note that, since we settle for proving the existence of a mild solution (therefore the existence of derivatives is not necessary) it is sufficient to consider that $w(t,\cdot) \in L^2(\mathbb{T}, \mathbb{C})$ instead of $w(t,\cdot) \in H^2(\mathbb{T}, \mathbb{C})$.

Hence, in order to apply our Corollary 5.4 we consider $X = L^2(\mathbb{T}, \mathbb{C})$ and we assume the following properties on the family of operators $\{T(t)\}_{t\in J}$ and the functions k_i, f_i, $i = 1, 2$

(T) for every $t \in J$, $T(t) : L^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$ is linear, bounded and, for every $y \in L^2(\mathbb{T}, \mathbb{C})$, $T(\cdot)y$ is B-measurable and $\|T(\cdot)\|_{\mathcal{L}(L^2(\mathbb{T},\mathbb{C}))} \in L^1_+(J)$;

(k) $k_i(t,\cdot) \in L^2(\mathbb{T})$, for every $t \in J$, $i = 1, 2$;

(f1) $f_1, f_2 : J \times \mathbb{R} \times \mathbb{C} \to \mathbb{R}^+_0$ are 2π- periodic functions with respect to the second variable, such that

  (1) for every $(t, \xi, w) \in J \times \mathbb{R} \times \mathbb{C}$, $f_1(t, \xi, w) \le f_2(t, \xi, w)$;

  (2) for each $y \in L^2(\mathbb{T}, \mathbb{C})$ there exists a B-measurable map $z_y : J \times \mathbb{R} \to \mathbb{C}$ such that

$$\|z_y(t,\xi)\|_{\mathbb{C}} \in \left[f_1\left(t, \xi, \int_0^{2\pi} k_1(t,\theta)y(\theta)\, d\theta\right), f_2\left(t, \xi, \int_0^{2\pi} k_2(t,\theta)y(\theta)\, d\theta\right) \right],$$

  for every $t \in J$ and for a.e. $\xi \in \mathbb{R}$;

(f2) for a.e. $t \in J$ and for a.e. $\xi \in \mathbb{R}$, $f_1(t,\xi,\cdot)$ is lower semicontinuous and $f_2(t,\xi,\cdot)$ is upper semicontinuous, i.e.

$$f_1(t,\xi,s) \le \liminf_{w\to s} f_1(t,\xi,w) \qquad f_2(t,\xi,s) \ge \limsup_{w\to s} f_2(t,\xi,w),$$

for every $s \in \mathbb{C}$;

(f3) there exists $\varphi \in L^1_+(J)$, $K\int_0^1(\varphi(\theta) + \|T(\theta)\|_{\mathcal{L}(L^2(\mathbb{T},\mathbb{C})})\,d\theta < 1$, such that, for every $t \in J$ and each $r > 0$, there exists $\psi_{t,r} \in L^2_+([0,2\pi])$, with

$$\sup_{\|s\|_{\mathbb{C}} \le r\|k_2(t,\cdot)\|_{L^2(\mathbb{T})}} f_2(t,\xi,s) \le \psi_{t,r}(\xi), \text{ a.e. } \xi \in [0,2\pi],$$

such that

$$\|\psi_{t,r}\|_{L^2([0,2\pi])} \le r\varphi(t). \tag{6.3}$$

Now we define the function $g : J \times L^2(\mathbb{T},\mathbb{C}) \to L^2(\mathbb{T},\mathbb{C})$ such that

$$g(t,y)(\xi) = (T(t)y)(\xi),\ \xi \in \mathbb{R},\ (t,y) \in J \times L^2(\mathbb{T},\mathbb{C}). \tag{6.4}$$

Recalling that $T(t)$ assumes values in $L^2(\mathbb{T},\mathbb{C})$, we have that g is obviously well posed.

Next we consider the multimap $U : J \times L^2(\mathbb{T},\mathbb{C}) \to \mathcal{P}(L^2(\mathbb{T},\mathbb{C}))$, defined for every $t \in J$ and $y \in L^2(\mathbb{T},\mathbb{C})$ by

$$\begin{aligned} U(t,y) = \Big\{ & z \in L^2(\mathbb{T},\mathbb{C}) : \\ & f_1\left(t,\xi,\int_0^{2\pi} k_1(t,\theta)y(\theta)\,d\theta\right) \le \|z(\xi)\|_{\mathbb{C}} \le f_2\left(t,\xi,\int_0^{2\pi} k_2(t,\theta)y(\theta)\,d\theta\right), \\ & \text{a.e. } \xi \in \mathbb{R}\Big\}. \end{aligned} \tag{6.5}$$

Let us show that the multimap U assumes non empty values.

First of all, fixed $y \in L^2(\mathbb{T},\mathbb{C})$, we consider the B-measurable map $z_y : J \times \mathbb{R} \to \mathbb{C}$ characterized in (2) of (f1). Fixed $t \in J$, by the B-measurability of z_y, we can claim that $z_y(t,\cdot)$ is also B-measurable. Moreover $\|z_y(t,\cdot)\|_{\mathbb{C}} \in L^2(\mathbb{T})$, indeed taking into account of (f1)(2) we have

$$\|z_y(t,\xi)\|_{\mathbb{C}} \le f_2\left(t,\xi\int_0^{2\pi} k_2(t,\theta)y(\theta)\,d\theta\right), \quad \text{a.e. } \xi \in \mathbb{R}. \tag{6.6}$$

Now, by hypothesis (k), we have $\left\|\int_0^{2\pi} k_2(t,\theta)y(\theta)\,d\theta\right\|_{\mathbb{C}} \le \|k_2(t,\cdot)\|_{L^2(\mathbb{T})}\|y\|_{L^2(\mathbb{T},\mathbb{C})}$, therefore, put $r = \|y\|_{L^2(\mathbb{T},\mathbb{C})}$, by (f3) and (6.6) there exists $\psi_{t,r} \in L^2_+([0,2\pi])$ such that $\|z_y(t,\xi)\|_{\mathbb{C}} \le \psi_{t,r}(\xi)$, a.e. $\xi \in \mathbb{R}$. Hence $\|z_y(t,\cdot)\|_{\mathbb{C}} \in L^2(\mathbb{T})$ and so

$$z_y(t,\cdot) \in L^2(\mathbb{T},\mathbb{C}). \tag{6.7}$$

Finally, using again hypothesis (f1)(2) and by (6.7) we conclude that $z_y(t,\cdot) \in U(t,y)$ (see (6.5)), so $U(t,y)$ is non empty.

We are in the position to define the multimap $F : J \times L^2(\mathbb{T},\mathbb{C}) \to \mathcal{P}(L^2(\mathbb{T},\mathbb{C}))$ as (see (6.4) and (6.5))

$$F(t,y) = \{T(t)y + v,\ v \in U(t,y)\}, \qquad t \in J,\ y \in L^2(\mathbb{T},\mathbb{C}). \tag{6.8}$$

Since the operator $T(t)$ assumes values in $L^2(\mathbb{T},\mathbb{C})$, F is obviously well defined.

Now we want to show that we can apply Corollary 5.4 to the problem (6.2).

First of all we note that $X = L^2(\mathbb{T},\mathbb{C})$ is obviously a reflexive Banach space. Moreover hypothesis (A) is clearly true because of our construction of the family $\{A(t) : t \in J\}$.

Now, let us show that hypotheses *F1.*– *F4.* are satisfied.

First of all, since U has convex values, we can say that F takes convex values too, i.e. *F1.* of our Corollary 5.4 holds.

Next, we prove that, fixed $y \in L^2(\mathbb{T},\mathbb{C})$, the multimap $F(\cdot, y)$ has a B-selection.

By the previous arguments we can say that the function $z_y : J \times \mathbb{R} \to \mathbb{C}$ characterized in (2) of (f1) is such that, for every $t \in J$,

$$z_y(t,\cdot) \in U(t,y). \tag{6.9}$$

Now, taking into account that the function $z_y : J \times \mathbb{R} \to \mathbb{C}$ is B-measurable with $z_y(t,\cdot) \in L^2(\mathbb{T},\mathbb{C})$, for every $t \in J$, from Theorem 2.4, we can say that the following abstract function

$$\hat{z}_y : J \to L^2(\mathbb{T},\mathbb{C}), \text{ defined by } \hat{z}_y(t) = z_y(t,\cdot), \text{ is B-measurable.} \tag{6.10}$$

Next, we define $p_y : J \to L^2(\mathbb{T},\mathbb{C})$ as

$$p_y(t) = T(t)y + \hat{z}_y(t), \qquad t \in J.$$

By using (6.10) and hypothesis (T) we have that p_y is obviously well posed and B-measurable.

Moreover, as a consequence of (6.9) and (6.8) we can write that $p_y(t) \in F(t,y)$, for every $t \in J$.

Therefore, for every $y \in L^2(\mathbb{T},\mathbb{C})$, p_y is a B-selection of $F(\cdot,y)$, i.e. hypothesis *F2.* of our Corollary 5.4 holds.

Now, let us show that also hypothesis *F3.* is satisfied.

Let $N \subset J$ be the null measure set for which hypothesis (f2) holds, $t \in J \setminus N$ and $(y_n)_n$ and $(q_n)_n$ be two sequences in $L^2(\mathbb{T},\mathbb{C})$ such that $y_n \rightharpoonup y$, $y \in L^2(\mathbb{T},\mathbb{C})$, $q_n \rightharpoonup q$, $q \in L^2(\mathbb{T},\mathbb{C})$ and $q_n \in F(t,y_n)$, $\forall n \in \mathbb{N}$, i.e. $q_n = T(t)y_n + v_n$, where $v_n \in U(t,y_n)$, for every $n \in \mathbb{N}$.

Now, if we consider

$$v_n = q_n - T(t)y_n, \qquad \forall n \in \mathbb{N} \tag{6.11}$$

taking into account of hypothesis (T) and the weak convergence of $(y_n)_n$ and $(q_n)_n$ we have

$$v_n \rightharpoonup q - T(t)y =: v, \tag{6.12}$$

i.e. $q = T(t)y + v$, where $v \in L^2(\mathbb{T},\mathbb{C})$.

Further, from (6.12), for every $\xi \in \mathbb{R}$, we can write

$$v_n(\xi) \rightharpoonup v(\xi). \tag{6.13}$$

In order to prove that $q \in F(t,y)$, we establish that $v \in U(t,y)$ (see (6.8)).

First of all, in the sequel we consider for $i = 1,2$ and for every $t \in J$, the linear and bounded operator $l_i^t : L^2(\mathbb{T},\mathbb{C}) \to \mathbb{C}$ defined by

$$l_i^t(y) = \int_0^{2\pi} k_i(t,\theta)y(\theta)\,d\theta,$$

for every $y \in L^2(\mathbb{T}, \mathbb{C})$.

Taking into account of the weak convergence of $(y_n)_n$ we have $\lim_{n\to\infty} l_i^t(y_n) = l_i^t(y)$, $i = 1, 2$.

Now, we consider the multimaps $G_n^t : \mathbb{R} \to \mathcal{P}(\mathbb{R})$, $n \in \mathbb{N}$, and $G^t : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ respectively defined by

$$\begin{aligned} G_n^t(\xi) &= \left[f_1\left(t, \xi, l_1^t(y_n)\right), f_2\left(t, \xi, l_2^t(y_n)\right)\right] \\ G^t(\xi) &= \left[f_1\left(t, \xi, l_1^t(y)\right), f_2\left(t, \xi, l_2^t(y)\right)\right], \end{aligned} \tag{6.14}$$

for every $\xi \in \mathbb{R}$.

Moreover, let us fix a null measure set $H_t \subset \mathbb{R}$ for which hypotheses (f2) and (f3) and (see (6.11) and (6.5)) $\|v_n(\xi)\|_{\mathbb{C}} \in G_n^t(\xi)$ hold, for every $\xi \in \mathbb{R} \setminus H_t$.

Let us note that, in order to apply the Containment Theorem (see Theorem 2.6), since f_1, f_2 are 2π-period functions with respect to the second variable, we can assume without loss of generality that G_n^t and G^t are defined on $[0, 2\pi]$.

Now, fixed $\xi \in [0, 2\pi] \setminus (H_t \cap [0, 2\pi])$, we consider an arbitrary sequence $(u_n)_n$ such that $u_n \in G_n^t(\xi)$, for all $n \in \mathbb{N}$, i.e.

$$f_1\left(t, \xi, l_1^t(y_n)\right) \le u_n \le f_2\left(t, \xi, l_2^t(y_n)\right), \qquad \forall n \in \mathbb{N}. \tag{6.15}$$

Next, by the strong convergence of $(l_2^t(y_n))_n$, there exists $\hat{r} > 0$ such that $\|l_2^t(y_n)\|_{\mathbb{C}} \le \hat{r}$, $\forall n \in \mathbb{N}$. Hence, taking into account that f_1 is a nonnegative function, fixed $r > 0$ such that $r\|k_2(t, \cdot)\|_{L^2(\mathbb{T})} = \hat{r}$, by hypothesis (f3) there exists $\psi_{t,r} \in L^2_+([0, 2\pi])$ such that

$$G_n^t(\xi) \subset [0, \psi_{t,r}(\xi)], \qquad \forall n \in \mathbb{N},$$

so, we can say that there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ such that

$$u_{n_k} \to u. \tag{6.16}$$

Now, taking into account of (6.15), by using (f2) and (6.16), passing to the limit we obtain

$$f_1\left(t, \xi, l_1^t(y)\right) \le u \le f_2\left(t, \xi, l_2^t(y)\right).$$

i.e. $u \in G^t(\xi)$ (see (6.14)). So hypothesis α) of the Containment Theorem holds.

Next, let $(\hat{y}_n)_n$ be a sequence such that, for all $n \in \mathbb{N}$, $\hat{y}_n : [0, 2\pi] \to \mathbb{R}$ is defined by

$$\hat{y}_n(\xi) = \|v_n(\xi)\|_{\mathbb{C}}, \qquad \xi \in [0, 2\pi],$$

where v_n is a function presented in (6.11).

First of all, fixed $\xi \in [0, 2\pi] \setminus (H_t \cap [0, 2\pi])$, we know that $\hat{y}_n(\xi) = \|v_n(\xi)\|_{\mathbb{C}} \in G_n^t(\xi)$, for all $n \in \mathbb{N}$. Moreover by the same arguments above presented we have

$$|\hat{y}_n(\xi)| \le \psi_{t,r}(\xi), \qquad \forall n \in \mathbb{N},$$

Being $\psi_{t,r} \in L^2_+([0, 2\pi]) \subset L^1_+([0, 2\pi])$, we can say that $(\hat{y}_n)_n$ is an integrably bounded sequence and so it has also the property of equi-absolute continuity of the integral (see Remark 2.3), i.e. hypothesis $\alpha\alpha$) of Theorem 2.6 is true.

Now, since all the hypotheses of the Containment Theorem hold, there exists a subsequence $(\hat{y}_{n_k})_k$ of $(\hat{y}_n)_n$ such that

$$\hat{y}_{n_k} \rightharpoonup \hat{y} \quad \text{in } L^1_+([0, 2\pi]) \tag{6.17}$$

and

$$\hat{y}(\xi) \in \overline{co}G^t(\xi) = G^t(\xi), \quad \text{a.e. } \xi \in [0, 2\pi]. \tag{6.18}$$

Next, since the strong and weak topologies are the same in $\mathbb{R}$ and $\mathbb{C}$, taking into account of (6.17) and (6.13) respectively, we can write

$$\hat{y}_{n_k}(\xi) = \|v_{n_k}(\xi)\|_{\mathbb{C}} \to \hat{y}(\xi), \tag{6.19}$$

and

$$\|v_{n_k}(\xi)\|_{\mathbb{C}} \to \|v(\xi)\|_{\mathbb{C}}, \tag{6.20}$$

for a.e. $\xi \in [0, 2\pi]$.

Finally, by using (6.19), (6.20) and the uniqueness of the limit we have (see (6.18))

$$\|v(\xi)\|_{\mathbb{C}} \in G^t(\xi), \quad \text{a.e. } \xi \in [0, 2\pi].$$

In the same way, by applying the Containment Theorem to the restrictions $G^t_{n|[2k\pi, 2(k+1)\pi]}$ and $G^t_{|[2k\pi, 2(k+1)\pi]}$ we have

$$\|v(\xi)\|_{\mathbb{C}} \in \left[f_1\left(t, \xi, l_1^t(y)\right), f_2\left(t, \xi, l_2^t(y)\right)\right],$$

a.e. $\xi \in \mathbb{R}$.

In conclusion, by recalling that $v \in L^2(\mathbb{T}, \mathbb{C})$ (see (6.12)), we can claim that $v \in U(t, y)$, a.e. $t \in J$. So also *F3.* of Corollary 5.4 holds.

Now we will prove that hypothesis *F4.* is true. First of all, for every $n \in \mathbb{N}$, let us fix $y \in \overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0, n)$, $t \in J$. Now, fixed $q \in F(t, y)$, there exists $v \in U(t, y)$ such that $q = T(t)y + v$ (see (6.8)) and we have

$$\|q\|_{L^2(\mathbb{T},\mathbb{C})} = \|T(t)y + v\|_{L^2(\mathbb{T},\mathbb{C})} \le \|T(t)y\|_{L^2(\mathbb{T},\mathbb{C})} + \|v\|_{L^2(\mathbb{T},\mathbb{C})}. \tag{6.21}$$

By using analogous arguments of the previous part of the proof, (k) and (f3) imply

$$\|v(\xi)\|_{\mathbb{C}} \le f_2\left(t, \xi, l_2^t(y)\right) \le \psi_{t,n}(\xi),$$

for a.e. $\xi \in [0, 2\pi]$, where $\psi_{t,n} \in L^2_+([0, 2\pi])$.

Therefore, by using (6.3) of (f3) we have

$$\|v\|_{L^2(\mathbb{T},\mathbb{C})} \le \|\psi_{t,r}\|_{L^2_+([0,2\pi])} \le n\varphi(t). \tag{6.22}$$

Then, by using (6.21) and (6.22) we are in the position to claim the following inequality

$$\|q\|_{L^2(\mathbb{T},\mathbb{C})} \le n\left(\varphi(t) + \|T(t)\|_{\mathcal{L}(L^2(\mathbb{T},\mathbb{C}))}\right).$$

Therefore, by the arbitrariness of $y \in \overline{B}_{L^2(\mathbb{T},\mathbb{C})}(0, n)$ we deduce

$$\|F(t, y)\| \le n\left(\varphi(t) + \|T(t)\|_{\mathcal{L}(L^2(\mathbb{T},\mathbb{C}))}\right) =: \varphi_n(t),$$

where $\varphi_n \in L^1_+(J)$, since $\varphi \in L^1_+(J)$ and $\|T(\cdot)\|_{\mathcal{L}(L^2(\mathbb{T},\mathbb{C}))} \in L^1_+(J)$ (see (T)).

Finally we note that (see hypothesis (f3))

$$\limsup_{n\to\infty} \frac{K\int_0^1 \varphi_n(\theta)\, d\theta}{n} = K\int_0^1 (\varphi(\theta) + \|T(\theta)\|_{\mathcal{L}(L^2(\mathbb{T},\mathbb{C}))})\, d\theta < 1,$$

so also *F4.* of Corollary 5.4 holds.

By means of the arguments above presented, we are in a position to apply the Cauchy version of our Theorem 5.3. Then we can deduce that there exists a continuous function $\hat{x} : J \to L^2(\mathbb{T}, \mathbb{C})$ that is a mild solution for (6.2), i.e.

$$\hat{x}(t) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,\theta)\hat{q}(\theta)\,d\theta, \qquad t \in J,$$

where

$$\hat{q} \in S^1_{F(\cdot,\hat{x}(\cdot))} = \{q \in L^1(J; L^2(\mathbb{T}, \mathbb{C})) : \ q(t) \in F(t, \hat{x}(t)) \text{ a.e. } t \in J\}. \tag{6.23}$$

Therefore, since a.e. $t \in J$, $\hat{q}(t) \in F(t, \hat{x}(t))$, there exists $v_{\hat{x}}(t) \in U(t, \hat{x}(t))$ (see (6.8)) such that

$$v_{\hat{x}}(t) = \hat{q}(t) - T(t)\hat{x}(t), \quad \text{a.e. } t \in J. \tag{6.24}$$

Hence we can consider the map $v_{\hat{x}} : J \to L^2(\mathbb{T}, \mathbb{C})$ defined as in (6.24) which is B-measurable, since $\hat{q}(\cdot)$ and $T(\cdot)\hat{x}(\cdot)$ are B-measurable (see respectively (6.23) and (T)).

At this point, by considering functions $w : J \times \mathbb{R} \to \mathbb{C}$ and $u : J \times \mathbb{R} \to \mathbb{C}$ respectively defined by

$$\begin{aligned} w(t,\xi) &= \hat{x}(t)(\xi), & t \in J,\ \xi \in \mathbb{R} \\ u(t,\xi) &= v_{\hat{x}}(t)(\xi), & t \in J,\ \xi \in \mathbb{R}, \end{aligned}$$

which are 2π-periodic with respect to the second variable and 2-integrable in $[0, 2\pi]$, we can conclude that $\{w, u\}$ is an admissible mild-pair for problem (6.1).

Finally we are able to enunciate the following result.

Theorem 6.1. *In the framework above described, there exists an admissible mild-pair $\{w, u\}$ for problem (6.1), i.e. $w, u : J \times \mathbb{R} \to \mathbb{C}$ satisfying the following properties*

(w1) for every $t \in J$, $w(t, \cdot)$ is 2π-periodic and 2-integrable on $[0, 2\pi]$;

(w2) for every $\xi \in \mathbb{R}$, $w(\cdot, \xi)$ is continuous on J;

(w3) $w(0, \xi) = x_0$, for every $\xi \in \mathbb{R}$;

(w4) for every $\xi \in \mathbb{R}$ such that $w(\cdot, \xi)$ is derivable at 0, we have $\frac{\partial w}{\partial t}(0, \xi) = x_1$;

(u1) for every $t \in J$, $u(t, \cdot)$ is 2π-periodic and 2-integrable on $[0, 2\pi]$;

(u2) for every $\xi \in \mathbb{R}$, $u(\cdot, \xi)$ is B-measurable and such that
$\|u(t,\xi)\|_{\mathbb{C}} \in \big[f_1\big(t, \xi, \int_0^{2\pi} k_1(t,\theta)w(t,\theta)\,d\theta\big), f_2\big(t, \xi, \int_0^{2\pi} k_2(t,\theta)w(t,\theta)\,d\theta\big)\big]$,
a.e. $t \in J$ and for every $\xi \in \mathbb{R}$.

Moreover, w, u are such that

$$w(t,\xi) = C(t,0)x_0 + S(t,0)x_1 + \int_0^t S(t,\theta)q(\theta,\xi)\,d\theta, \qquad t \in J,\ \xi \in \mathbb{R}$$

where $q : J \times \mathbb{R} \to \mathbb{C}$ is defined by

$$q(t,\xi) = T(t)w(t,\cdot)(\xi) + u(t,\xi), \qquad t \in J,\ \xi \in \mathbb{R}.$$

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