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Bistable equation with discontinuous density dependent diffusion with degenerations and singularities

Dedicated to the memory of Professor Josef Daněček, our friend and mentor

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Abstract. In this article we introduce rather general notion of the stationary solution of the bistable equation which allows to treat discontinuous density dependent diffusion term with singularities and degenerations, as well as degenerate or non-Lipschitz balanced bistable reaction term. We prove the existence of new-type solutions which do not occur in case of the "classical" setting of the bistable equation. In the case of the power-type behavior of the diffusion and bistable reaction terms near the equilibria we provide detailed asymptotic analysis of the corresponding solutions and illustrate the lack of smoothness due to the discontinuous diffusion.

Keywords: density dependent diffusion, bistable balanced nonlinearity, asymptotic behavior, discontinuous diffusion, degenerate and singular diffusion, degenerate non-Lipschitz reaction.

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1 Introduction

Let us consider the bistable equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(u) \tag{1.1}$$

in $\mathbb{R}$, where the reaction term $g : [0,1] \to \mathbb{R}$ is continuous and there exists $s_* \in (0,1)$ such that

$$g(0) = g(s_*) = g(1) = 0, \quad g(s) < 0 \text{ for } s \in (0, s_*), \quad g(s) > 0 \text{ for } s \in (s_*, 1).$$

Equation (1.1) appears in many mathematical models in population dynamics, genetics, combustion or nerve propagation, see e.g. [1,2] and references therein.

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This kind of reaction is called *bistable*, cf. [3,7–9]. We distinguish between two different cases of bistable reactions which lead to different type of solutions to (1.1). Namely, when

$$\int_0^1 g(s)\,\mathrm{d}s = 0, \tag{1.2}$$

we say that g is *balanced bistable nonlinearity* while in case

$$\int_0^1 g(s)\,\mathrm{d}s \neq 0$$

the bistable nonlinearity g is called *unbalanced*. In the former case the equation (1.1) possesses (time independent) *stationary solutions* which connect constant equilibria $u_0 \equiv 0$ and $u_1 \equiv 1$, i.e., solutions $u = u(x)$ of (1.1) satisfying

$$\lim_{x\to-\infty} u(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 1 \tag{1.3}$$

or

$$\lim_{x\to-\infty} u(x) = 1 \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 0. \tag{1.4}$$

On the other hand, the latter case leads to the (time dependent) *nonstationary travelling wave solutions* connecting u_0 and u_1, see e.g. [6,10].

The stationary solutions of (1.1) satisfying (1.3) or (1.4) can be found in the closed form for special reaction terms. For example, for

$$g(s) = s(1-s)\left(s - \frac{1}{2}\right)$$

we get stationary solution of (1.1), (1.3) in the following form

$$u(x) = \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right) + \frac{1}{2},$$

cf. [4]. Then solution $u = u(x) \in (0,1)$, $x \in \mathbb{R}$, is a strictly increasing function which approaches equilibria u_0 and u_1 at an exponential rate:

$$u(x) \sim \mathrm{e}^{x} \quad \text{as } x \to -\infty \quad \text{and} \quad 1 - u(x) \sim \mathrm{e}^{-x} \quad \text{as } x \to +\infty. \tag{1.5}$$

If we consider the quasilinear bistable equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) + g(u), \tag{1.6}$$

where $p > 1$ and g is balanced bistable nonlinearity then the structure of stationary solutions to (1.6), (1.3) or (1.6), (1.4) may be considerably different as shown in [4]. For example, if

$$g(s) = s^{\alpha}(1-s)^{\alpha}\left(s - \frac{1}{2}\right), \quad s \in (0,1), \quad \alpha > 0,$$

we distinguish between the following two qualitatively different cases:

Case 1: $\alpha + 1 \geq p$,

Case 2: $\alpha + 1 < p$.

In Case 1 solution $u = u(x)$ of (1.6), (1.3) is again a strictly increasing continuously differentiable function which assumes values in $(0, 1)$. However, (1.5) holds only in the case $\alpha + 1 = p$. In the case $\alpha + 1 > p$ we have

$$u(x) \sim |x|^{\frac{p}{p-(\alpha+1)}} \quad \text{as } x \to -\infty \quad \text{and} \quad 1 - u(x) \sim |x|^{\frac{p}{p-(\alpha+1)}} \quad \text{as } x \to +\infty\,.$$

In Case 2 there exist real numbers $x_0 < x_1$ such that for all $x \in (x_0, x_1)$ we have $u(x) \in (0, 1)$, u is strictly increasing continuously differentiable, $u(x) = 0$ for all $x \in (-\infty, x_0]$ and $u(x) = 1$ for all $x \in [x_1, +\infty)$. Moreover,

$$u(x) \sim (x - x_0)^{\frac{p}{p-(\alpha+1)}} \quad \text{as } x \to x_0+ \quad \text{and} \quad 1 - u(x) \sim (x_1 - x)^{\frac{p}{p-(\alpha+1)}} \quad \text{as } x \to x_1- \;.$$

Our ambition in this paper is to study similar properties for the quasilinear bistable equation with density dependent diffusion coefficient $d = d(s)$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(d(u) \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x} \right) + g(u), \tag{1.7}$$

where the properties of $d = d(s)$ are specified in the next section.

2 Preliminaries

Let $p > 1$, $g : [0, 1] \to \mathbb{R}$, $g \in C[0, 1]$ be such that $g(0) = g(s_*) = g(1) = 0$ for $s_* \in (0, 1)$ and

$$g(s) < 0,\ s \in (0, s_*), \quad g(s) > 0,\ s \in (s_*, 1).$$

The diffusion coefficient $d : [0, 1] \to \mathbb{R}$ is supposed to be a nonnegative lower semicontinuous function and $d > 0$ in $(0, 1)$. There exist $0 = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} = 1$ such that $d\big|_{(s_i,s_{i+1})} \in C(s_i, s_{i+1})$, $i = 0, \dots, n$, and d has discontinuity of the first kind (finite jump) at s_i, $i = 1, \dots, n$.

For $p = 2$ and $d(s) \equiv 1$ in $[0, 1]$ equation (1.7) reduces to the bistable equation (1.1) with constant diffusion coefficient and *bistable reaction* term g. In this paper we deal with diffusion which allows for *singularities* and for *degenerations* both at 0 and/or 1. We also consider d to be a *discontinuous* function. Last but not least, reaction term g can degenerate in 0 and/or in 1. In particular, we admit $g'(0) = 0$ and/or $g'(1) = 0$, as well as $g'(0) = -\infty$ and/or $g'(1) = -\infty$. This in turn yields that our solution is not a C^1-function in $\mathbb{R}$ and it does not satisfy the equation pointwise in the classical sense. For this purpose we have to employ the first integral of the second order differential equation. Since our primary interest in this paper is the investigation of stationary solutions to (1.7) which are monotone (i.e., nonincreasing or nondecreasing) between the equilibria 0 and 1, we provide rather general definition of monotone solutions to the second order ODE

$$\left(d(u)\, |u'|^{p-2}\, u'\right)' + g(u) = 0, \tag{2.1}$$

where, for the sake of simplicity, we write $(\,\cdot\,)'$ instead of $\frac{\mathrm{d}}{\mathrm{d}x}(\,\cdot\,)$.

Let $u : \mathbb{R} \to [0, 1]$ be a monotone continuous function. We denote

$$M_u := \{x \in \mathbb{R} : u(x) = s_i,\ i = 1, 2, \dots, n\}, \qquad N_u := \{x \in \mathbb{R} : u(x) = 0 \text{ or } u(x) = 1\}.$$

Then M_u and N_u are closed sets, M_u is a union of a finite number of points or intervals,

$$N_u = (-\infty, x_0] \cup [x_1, +\infty),$$

where $-\infty \le x_0 < x_1 \le +\infty$ and we use the convention $(-\infty, x_0] = \emptyset$ if $x_0 = -\infty$ and $[x_1, +\infty) = \emptyset$ if $x_1 = +\infty$.

Definition 2.1. A monotone continuous function $u : \mathbb{R} \to [0,1]$ is a *solution* of equation (2.1) if

(a) For any $x \notin M_u \cup N_u$ there exists finite derivative $u'(x)$ and for any $x \in \operatorname{int} M_u \cup \operatorname{int} N_u$ we have $u'(x) = 0$.

(b) For any $x \in \partial M_u$ there exist finite one sided derivatives $u'(x-), u'(x+)$ and

$$L(x) := \left|u'(x-)\right|^{p-2} u'(x-) \lim_{y \to x-} d(u(y)) = \left|u'(x+)\right|^{p-2} u'(x+) \lim_{y \to x+} d(u(y)).$$

(c) Function $v : \mathbb{R} \to \mathbb{R}$ defined by

$$v(x) := \begin{cases} d(u(x))\, |u'(x)|^{p-2} u'(x), & x \notin M_u \cup N_u\,, \\ 0, & x \in N_u \cup \operatorname{int} M_u\,, \\ L(x), & x \in \partial M_u \end{cases}$$

is continuous and for any $x, y \in \mathbb{R}$

$$v(y) - v(x) + \int_x^y g(u(\xi))\, \mathrm{d}\xi = 0. \tag{2.2}$$

Moreover, $\lim_{x \to \pm\infty} v(x) = 0$ if either $\lim_{x \to -\infty} u(x) = 0$ and $\lim_{x \to +\infty} u(x) = 1$ or $\lim_{x \to -\infty} u(x) = 1$ and $\lim_{x \to +\infty} u(x) = 0$.

Remark 2.2. Constant functions

$$u_0(x) = 0, \quad u_*(x) = s_*, \quad u_1(x) = 1, \quad x \in \mathbb{R},$$

are solutions of (2.1). It follows from the properties of d and g that those are the only constant solutions of (2.1) and they are called equilibria.

Remark 2.3. If we set $y = x + h, h \neq 0$ in (2.2), multiply both sides of (2.2) by $\frac{1}{h}$ and pass to the limit for $h \to 0$, we obtain that v is continuously differentiable and the equation

$$v'(x) + g(u(x)) = 0 \tag{2.3}$$

holds for all $x \in \mathbb{R}$.

Remark 2.4. Let u be a solution of (2.1) in the sense of Definition 2.1. If $M_u \neq \emptyset$, i.e., d is not continuous in $(0,1)$, then $M_u = \partial M_u$, $\operatorname{int} M_u = \emptyset$ unless $s_i = s_*$ for some $i = 1, 2, \dots, n$. In this case u can be constant on some interval (a,b), $-\infty \le a < b \le +\infty$, and equal to s_*. The equation (2.1) would then be satisfied pointwise for all $x \in (a,b)$ and $(a,b) \subset \operatorname{int} M_u$. Furthermore, it follows from the continuity of v that if $a > -\infty$ or $b < +\infty$ we have $u'(a) = u'(b) = 0$ because $d(s_*) > 0$. Also note that for $x \in \partial N_u$ one sided derivatives $u'(x-), u'(x+)$ exist but one of them can be infinite.

If u is strictly monotone between 0 and 1 then $M_u = \{\xi_1, \xi_2, \dots, \xi_n\}$ where $u(\xi_i) = s_i$, $i = 1, 2, \dots, n$.

Remark 2.5. Let $p = 2$, $d \equiv 1$ and $g \in C^1[0,1]$. Let $u = u(x)$ be a solution in the sense of Definition 2.1. Then $M_u = \emptyset$ if u is not a constant, $N_u = \emptyset$, and (2.1) holds pointwise, i.e., $u \in C^2(\mathbb{R})$ and it is a classical solution, cf. [1], [2] or [6].

3 Existence results

We are concerned with the existence of solutions of the equation (2.1) which satisfy the "boundary conditions"

$$\lim_{x\to-\infty} u(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 1. \tag{3.1}$$

Remark 3.1. Let u be a solution of the BVP (2.1), (3.1). Passing to the limit for $x \to -\infty$ in (2.2) and writing x in place of y, we derive that for arbitrary $x \in \mathbb{R}$ we have

$$v(x) + \int_{-\infty}^{x} g(u(\xi))\,\mathrm{d}\xi = 0. \tag{3.2}$$

Theorem 3.2. *Let d and g be as in Section 2 and recall that $p > 1$. Then the BVP (2.1), (3.1) has a nondecreasing solution if and only if*

$$\int_0^1 (d(s))^{\frac{1}{p-1}} g(s)\,\mathrm{d}s = 0. \tag{3.3}$$

If (3.3) holds then there is a unique solution $u = u(x)$ of (2.1), (3.1) such that the following conditions hold (see Figure 3.1):

(i) *there exist $-\infty \le x_0 < 0 < x_1 \le +\infty$ such that $u(x) = 0$ for $x \le x_0$, $u(x) = 1$ for $x \ge x_1$ and $0 < u(x) < 1$ for $x \in (x_0, x_1)$;*

(ii) *u is strictly increasing in (x_0, x_1), $u(0) = s_*$;*

(iii) *for $i = 1, 2, \ldots, n$ let $\xi_i \in \mathbb{R}$ be such that $u(\xi_i) = s_i$, $\xi_0 = x_0$ and $\xi_{n+1} = x_1$. Then u is a piecewise C^1-function in the sense that u is continuous,*

$$u\big|_{(\xi_i,\xi_{i+1})} \in C^1(\xi_i, \xi_{i+1}), \quad i = 0, 1, \ldots, n,$$

and the limits $u'(\xi_i-) := \lim_{x\to\xi_i-} u'(x)$, $u'(\xi_i+) := \lim_{x\to\xi_i+} u'(x)$ exist finite for all $i = 1, 2, \ldots, n$;

(iv) *for any $i = 1, 2, \ldots, n$, the following transition condition holds:*

$$\left(u'(\xi_i-)\right)^{p-1} \lim_{s\to s_i-} d(s) = \left(u'(\xi_i+)\right)^{p-1} \lim_{s\to s_i+} d(s).$$

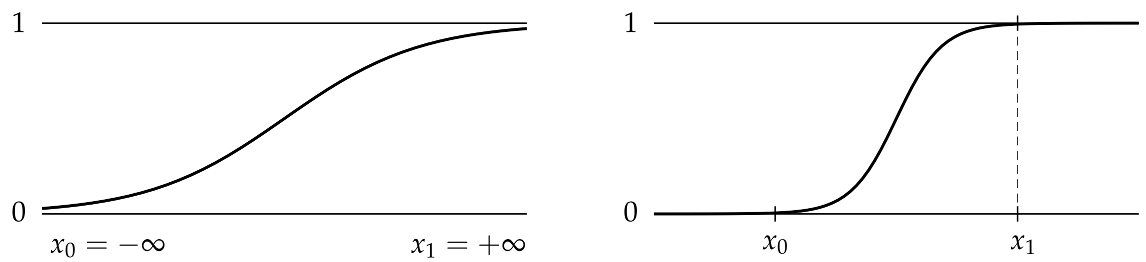

Figure 3.1: Increasing solutions

Proof. Necessity of (3.3). Let $u = u(x)$ be a nondecreasing solution of the BVP (2.1), (3.1) such that $u(0) = s_*$. Since the equation is autonomous this condition is just a normalization of a solution. It follows from (3.1) that

$$-\infty \le x_0 := \inf\{x \in \mathbb{R} : u(x) > 0\} < 0$$

is well defined. By (3.2) and continuity of v we have

$$0 < x_1 := \sup\{x \in \mathbb{R} : v(y) > 0 \text{ for all } y \in (x_0, x)\} \le +\infty\,.$$

Since $d(s) > 0, s \in (0,1)$, it follows from the definition of $v(x)$ that u is a strictly increasing function in (x_0, x_1) and therefore the following limit

$$\bar{u}(x_1) := \lim_{x \to x_1-} u(x)$$

is well defined. If $x_1 = +\infty$ then by the second condition in (3.1) it must be $\bar{u}(x_1) = 1$. On the other hand, if $x_1 < +\infty$, we have $\bar{u}(x_1) = u(x_1)$, $v(x_1) = 0$ and $s_* < u(x_1) \le 1$. We rule out the case $u(x_1) < 1$. Indeed, $v(x_1) = 0$ implies $u'(x_1-) = u'(x_1+) = u'(x_1) = 0$. From $s_* < u(x_1)$ and (2.3) we deduce $v'(x_1) = -g(u(x_1)) < 0$. Therefore, there exists $\delta > 0$ such that for all $x \in (x_1, x_1 + \delta)$ we have $v(x) < 0$ and hence also $u'(x-) < 0$ and $u'(x+) < 0$. This contradicts our assumption that u is nondecreasing.

We proved that $u(x_1) = 1$, i.e., $u = u(x)$ is strictly increasing and maps (x_0, x_1) onto $(0,1)$. Let $\xi_i \in (x_0, x_1)$ be such that

$$u(\xi_i) = s_i\,, \quad i = 1, 2, \dots, n, \quad \xi_0 = x_0\,, \quad \xi_{n+1} = x_1\,.$$

Then u is continuous in (x_0, x_1) and piecewise C^1-function in the sense that

$$u\big|_{(\xi_i, \xi_{i+1})} \in C^1(\xi_i, \xi_{i+1}), \quad u'(x) > 0, \quad x \in (\xi_i, \xi_{i+1}), \quad i = 0, 1, \dots, n,$$

and the limits $\lim_{x \to \xi_i -} u'(x)$, $\lim_{x \to \xi_i +} u'(x)$, $i = 1, 2, \dots, n$, exist finite. Hence there exists continuous strictly increasing inverse function $u^{-1} : (0,1) \to (x_0, x_1)$, $x = u^{-1}(u)$, such that

$$u^{-1}\Big|_{(s_i, s_{i+1})} \in C^1(s_i, s_{i+1}), \quad i = 0, 1, \dots, n\,,$$

and the limits

$$\lim_{u \to s_i -} \frac{\mathrm{d}}{\mathrm{d}u} u^{-1}(u), \qquad \lim_{u \to s_i +} \frac{\mathrm{d}}{\mathrm{d}u} u^{-1}(u)$$

exist finite, $i = 1, 2, \dots, n$. We employ the change of variables as indicated in [5, p. 174]. Set

$$w(u) = v(u^{-1}(u)), \quad u \in (0,1).$$

Then w is piecewise C^1-function in $(0,1)$,

$$w\big|_{(s_i, s_{i+1})} \in C^1(s_i, s_{i+1}), \quad i = 0, 1, \dots, n$$

with finite limits $\lim_{u \to s_i -} w'(u)$, $\lim_{u \to s_i +} w'(u)$, $i = 1, 2, \dots, n$. For any $x \in (\xi_i, \xi_{i+1})$ and $u \in (s_i, s_{i+1})$, $i = 0, 1, \dots, n$,

$$\frac{\mathrm{d}}{\mathrm{d}x} v(x) = \frac{\mathrm{d}}{\mathrm{d}x} w(u(x)) = \frac{\mathrm{d}w}{\mathrm{d}u}\,(u(x))\, u'(x). \tag{3.4}$$

From $v(x) = d(u(x))(u'(x))^{p-1}$ we deduce

$$u'(x) = \left(\frac{v(x)}{d(u(x))}\right)^{p'-1}, \qquad p' = \frac{p}{p-1}. \tag{3.5}$$

It follows from (3.4), (3.5) that

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}w}{\mathrm{d}u}(u(x))\left(\frac{v(x)}{d(u(x))}\right)^{p'-1} = \frac{\mathrm{d}w}{\mathrm{d}u}(u)\left(\frac{w(u)}{d(u)}\right)^{p'-1}.$$

Therefore, the equation

$$v'(x) + g(u(x)) = 0, \quad x \in (\xi_i, \xi_{i+1}),$$

transforms to

$$\frac{\mathrm{d}w}{\mathrm{d}u}\left(\frac{w(u)}{d(u)}\right)^{p'-1} + g(u) = 0, \quad u \in (s_i, s_{i+1}),$$

$i = 0, 1, \dots, n$, or equivalently,

$$(w(u))^{p'-1}\frac{\mathrm{d}w}{\mathrm{d}u} + (d(u))^{p'-1}g(u) = 0, \tag{3.6}$$

$$\frac{1}{p'}\frac{\mathrm{d}}{\mathrm{d}u}(w(u))^{p'} + (d(u))^{p'-1}g(u) = 0. \tag{3.7}$$

The last equality holds in $(0,1)$ except the points $s_1, s_2, \dots, s_n$ and w is continuous in $(0,1)$. Set

$$f(s) := -(d(s))^{\frac{1}{p-1}}g(s), \quad s \in (0,1),$$

$$F(s) := \int_0^s f(\sigma)\,\mathrm{d}\sigma.$$

Integrating (3.7) over the interval $(0,u)$ we arrive at

$$(w(u))^{p'} = p'F(u) + (w(0+))^{p'}, \quad u \in (0,1).$$

Clearly, $F(0) = 0$, and

$$\lim_{u\to 0+} w(u) = \lim_{x \to x_0+} v(x) = 0 \tag{3.8}$$

by the definition of a solution. Therefore we have

$$w(u) = \left(p'F(u)\right)^{\frac{1}{p'}}, \quad u \in (0,1). \tag{3.9}$$

By the definition of a solution we must also have

$$\lim_{u\to 1-} w(u) = \lim_{x\to x_1-} v(x) = 0. \tag{3.10}$$

But (3.9) and (3.10) imply $F(1) = 0$, i.e., (3.3) must hold. Therefore, (3.3) is a necessary condition.

Sufficiency of (3.3). Let (3.3) hold. Then $w = w(u)$ given by (3.9) satisfies (3.6)–(3.10) above. For $u \in (0,1)$ set

$$x(u) = \int_{s_*}^{u}\left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}}\mathrm{d}s.$$

The function $x = x(u)$ is strictly increasing and maps the interval $(0,1)$ onto (x_0, x_1) where $-\infty \le x_0 < 0 < x_1 \le +\infty$. Let $u : (x_0, x_1) \to (0,1)$ be an inverse function. Then $u(0) = s_*$, u is strictly increasing and

$$\lim_{x \to x_0+} u(x) = 0, \qquad \lim_{x \to x_1-} u(x) = 1.$$

Let $x \in (\xi_i, \xi_{i+1})$, $i = 0, 1, \dots, n$, where $u(\xi_i) = s_i$, $i = 0, 1, \dots, n+1$. Then

$$\frac{\mathrm{d}u(x)}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x(u)}{\mathrm{d}u}} = \left(\frac{w(u(x))}{d(u(x))}\right)^{\frac{1}{p-1}}, \quad u(x) \in (s_i, s_{i+1}),$$

i.e., $u \in C^1(\xi_i, \xi_{i+1}), u'(x) > 0$ and

$$d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1} = w(u(x)) =: v(x), \tag{3.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1}\right] = \frac{\mathrm{d}}{\mathrm{d}x} w(u(x)) = \frac{\mathrm{d}w}{\mathrm{d}u}\,\frac{\mathrm{d}u(x)}{\mathrm{d}x}. \tag{3.12}$$

From (3.6), (3.11) we deduce

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}u} &= -\,(w(u))^{-(p'-1)}\,(d(u))^{p'-1}\,g(u) \\ &= -\,(d(u(x)))^{-(p'-1)} \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{-(p-1)(p'-1)} (d(u(x)))^{p'-1}\,g(u(x)) \\ &= -\left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{-1} g(u(x)).\end{aligned}$$

Substituting this to (3.12), we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1}\right] = -g(u(x)), \quad x \in (\xi_i, \xi_{i+1}).$$

It follows from (3.8), (3.10) and (3.11) that

$$\lim_{x \to x_0+} d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1} = \lim_{x \to x_1-} d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1} = 0$$

and the following one-sided limits are finite

$$\lim_{x \to \xi_i-} d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1} = \lim_{x \to \xi_i+} d(u(x)) \left(\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right)^{p-1}, \tag{3.13}$$

$i = 1, 2, \dots, n$. Since $u = u(x)$ is monotone increasing function, we have

$$\lim_{x \to \xi_i-} d(u(x)) = \lim_{s \to s_i-} d(s) \quad \text{and} \quad \lim_{x \to \xi_i+} d(u(x)) = \lim_{s \to s_i+} d(s). \tag{3.14}$$

Transition condition (iv) now follows from (3.13), (3.14).

Therefore, if for $x_0 > -\infty$ we set $u(x) = 0, x \in (-\infty, x_0]$ and for $x_1 < +\infty$ we set $u(x) = 1, x \in [x_1, +\infty)$, then $u = u(x), x \in \mathbb{R}$, is a nondecreasing solution of the BVP (2.1), (3.1) and it has the properties listed in the statement of Theorem 3.2. This proves the sufficiency of (3.3). □

Remark 3.3. The condition (3.3) substitutes the balanced bistable nonlinearity condition (1.2) in case of density dependent diffusion. It follows from Theorem 3.2 that it is not only the reaction term but rather mutual interaction between the density dependent diffusion coefficient and reaction which decides about the existence and/or nonexistence of nonconstant stationary solutions of the generalized version of the bistable equation (1.6).

Remark 3.4. Let us replace the boundary conditions (3.1) by "opposite" ones:

$$\lim_{x\to-\infty} u(x) = 1 \quad \text{and} \quad \lim_{x\to+\infty} u(x) = 0. \tag{3.15}$$

If u is a solution of the BVP (2.1), (3.15) then passing to the limit for $y \to +\infty$ in (2.2) we arrive at

$$v(x) - \int_x^{+\infty} g(u(\xi))\,\mathrm{d}\xi = 0 \tag{3.16}$$

for arbitrary $x \in \mathbb{R}$. Modifying the proof of Theorem 3.2 and using (3.16) instead of (3.2), we show that (3.3) is a necessary and sufficient condition for the existence of nonincreasing solution of the BVP (2.1), (3.15). If (3.3) holds then there is a unique solution $u = u(x)$ of (2.1), (3.15) satisfying analogue of (i)–(iv). In particular, it is strictly decreasing in (x_0, x_1), $u(x) = 1$ for $x \in (-\infty, x_0]$ if $x_0 > -\infty$ and $u(x) = 0$ for $x \in [x_1, +\infty)$ if $x_1 < +\infty$, see Figure 3.2.

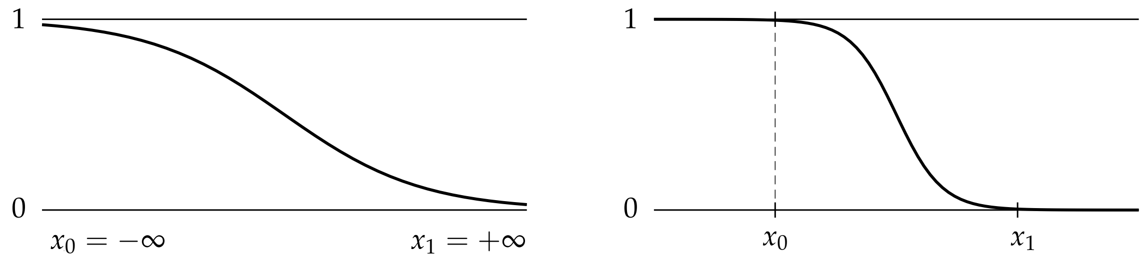

Figure 3.2: Decreasing solutions

Remark 3.5. It follows from the proof of Theorem 3.2 that

$$x_0 = x(0) = \int_{s_*}^{0} \left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}} \mathrm{d}s = \left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{0} \frac{(d(s))^{\frac{1}{p-1}}}{-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\,\mathrm{d}\sigma}\,\mathrm{d}s, \tag{3.17}$$

$$x_1 = x(1) = \int_{s_*}^{1} \left(\frac{d(s)}{w(s)}\right)^{\frac{1}{p-1}} \mathrm{d}s = \left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{1} \frac{(d(s))^{\frac{1}{p-1}}}{-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\,\mathrm{d}\sigma}\,\mathrm{d}s. \tag{3.18}$$

Therefore, the fact that x_0 and x_1 are finite or infinite depends on the asymptotic behavior of the diffusion coefficient $d = d(s)$ and reaction term $g = g(s)$ near the equilibria 0 and 1. The detailed discussion of different configurations between d and g which lead to x_0 and/or x_1 finite or infinite is presented in the next section.

Remark 3.6. Since the equation (2.1) is autonomous, if $u = u(x)$ is a solution to (2.1), (3.1) then given any $\xi \in \mathbb{R}$ fixed, $\tilde{u} = \tilde{u}(x) := u(x - \xi)$ is also a solution of (2.1), (3.1). Of course, if x_0 and/or x_1 are finite, then corresponding $\tilde{x}_0$ and $\tilde{x}_1$ associated with $\tilde{u}$ satisfy $\tilde{x}_0 = x_0 + \xi$ and $\tilde{x}_1 = x_1 + \xi$. Obviously, the same applies to (2.1), (3.15). If $x_0 = -\infty$ and $x_1 = +\infty$ and (3.3) holds, all possible solutions of (2.1), (3.1) are strictly increasing in $(-\infty, +\infty)$ and satisfy (i)–(iv) of Theorem 3.2, where $u(0) = s_*$ is replaced by $u(\xi) = s_*$, $\xi \in \mathbb{R}$. On the other hand, if $x_0 \in \mathbb{R}$ and/or $x_1 \in \mathbb{R}$, then the set of possible solutions of (2.1), (3.1) is much richer than in the previous case. Indeed, we have plenty of possibilities how to define also a nonmonotone solution of (2.1), (3.1) (or (2.1), (3.15)). For example, if both x_0 and x_1 associated with strictly increasing solution $u = u(x)$ from Theorem 3.2 are finite then the same holds for corresponding $\hat{x}_0$ and $\hat{x}_1$ associated with the strictly decreasing solution from Remark 3.4. Having in mind the translation invariance of solutions mentioned above, we may choose u_1 and $\hat{u}$ such that $x_1 < \hat{x}_0$. If we define $u(x) = 0$, $x \in (-\infty, x_0]$, $u(x) = u_1(x)$, $x \in (x_0, x_1)$, $u(x) = 1$, $x \in [x_1, \hat{x}_0]$, $u(x) = \hat{u}(x)$, $x \in (\hat{x}_0, \hat{x}_1)$, $u(x) = 0$, $x \in [\hat{x}_1, +\infty)$, we get solution of (2.1) satisfying the boundary conditions

$$\lim_{x\to-\infty} u(x) = \lim_{x\to+\infty} u(x) = 0. \tag{3.19}$$

Now, if $\tilde{u}_1 = \tilde{u}_1(x)$ is a translation of u_1 such that $\tilde{x}_0 > \hat{x}_1$, we can extend the previous function u as $u(x) = 0$, $x \in [\hat{x}_1, \tilde{x}_0]$, $u(x) = \tilde{u}(x)$, $x \in (\tilde{x}_0, \tilde{x}_1)$, $u(x) = 1$, $x \in [\tilde{x}_1, +\infty)$ to get a nonmonotone solution of (2.1), (3.1), see Figure 3.3. It is obvious that by suitably modifying the above construction we may construct continuum of solutions not only of (2.1), (3.1) but also of (2.1), (3.19). Of course, the same approach leads to the continuum of solutions of (2.1), (3.15) and of (2.1), (3.20), respectively, where

$$\lim_{x\to-\infty} u(x) = \lim_{x\to+\infty} u(x) = 1. \tag{3.20}$$

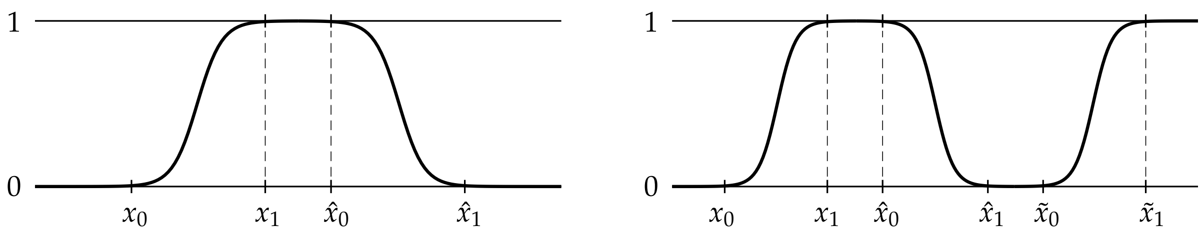

Figure 3.3: Nonmonotone solutions

4 Qualitative properties of solutions

In this section we study the qualitative properties of the solutions from Theorem 3.2. In particular, we focus on two issues. Our primary concern is to provide detailed classification of the *asymptotic behavior* of the stationary solution $u = u(x)$ as $x \to -\infty$ and $x \to +\infty$ and to show how it is affected by the behavior of the diffusion coefficient d and reaction g near the equilibria 0 and 1. However, we also want to study the impact of the *discontinuity* of $d = d(s)$ on the *lack of smoothness of the solution* $u = u(x)$. The role of the transition condition at the points where u assumes values where the discontinuity of d occurs will be illustrated.

In order to simplify the expressions arising throughout this section we will use the following notation:

$$h_1(t) \sim h_2(t) \text{ as } t \to t_0 \quad \text{if and only if} \quad \lim_{t\to t_0} \frac{h_1(t)}{h_2(t)} \in (0, +\infty).$$

We start with the asymptotic analysis of

$$x(u) = x(s_*) + \left(\frac{1}{p'}\right)^{\frac{1}{p}} \int_{s_*}^{u} \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\,\mathrm{d}\sigma\right)^{\frac{1}{p}}}\,\mathrm{d}s$$

for $u \to 0+$. Let us assume that $g(s) \sim -s^\alpha$, $d(s) \sim s^\beta$ as $s \to 0+$ for some $\alpha > 0$, $\beta \in \mathbb{R}$. Then formally we get

$$-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\,\mathrm{d}\sigma \;\sim\; \int_0^s \sigma^{\alpha+\frac{\beta}{p-1}}\,\mathrm{d}\sigma \;\sim\; s^{\alpha+\frac{\beta}{p-1}+1} \quad \text{as} \quad s \to 0+\,.$$

Since we assume that $s \mapsto (d(s))^{\frac{1}{p-1}} g(s)$ is integrable in $(0,1)$, we have to assume

$$\alpha + \frac{\beta}{p-1} > -1\,. \tag{4.1}$$

Then for $u \to 0+$ we can write

$$x(u) \;\sim\; \int_{s_*}^u \frac{(d(s))^{\frac{1}{p-1}}}{\left(-\int_0^s (d(\sigma))^{\frac{1}{p-1}} g(\sigma)\,\mathrm{d}\sigma\right)^{\frac{1}{p}}}\,\mathrm{d}s \;\sim\; \int_{s_*}^u s^{\frac{\beta}{p-1}-\frac{\alpha}{p}-\frac{\beta}{p(p-1)}-\frac{1}{p}}\,\mathrm{d}s = \int_{s_*}^u s^{\frac{\beta-\alpha-1}{p}}\,\mathrm{d}s. \tag{4.2}$$

Convergence or divergence of the integral

$$I := \int_0^{s_*} s^{\frac{\beta-\alpha-1}{p}}\,\mathrm{d}s$$

leads to the following primary distinction between two qualitatively different cases:

Case 1: $I = +\infty$ if $\alpha - \beta \geq p - 1$,

Case 2: $I < +\infty$ if $\alpha - \beta < p - 1$.

Case 1. Let $\alpha - \beta = p - 1$. Then (4.2) implies that $x(u) \sim \ln u$ as $u \to 0+$ and performing the change of variables yields the asymptotics for $u = u(x)$:

$$u(x) \sim \mathrm{e}^x \to 0+ \quad \text{for } x \to -\infty\,.$$

For $\alpha - \beta > p - 1$ we have by (4.2) that $x(u) \sim -u^{\frac{\beta-\alpha-1}{p}+1} = -u^{\frac{p-1-(\alpha-\beta)}{p}} \to -\infty$ as $u \to 0+$ and applying the inverse function we obtain

$$u(x) \sim |x|^{\frac{p}{p-1-(\alpha-\beta)}} \to 0+ \quad \text{for } x \to -\infty\,.$$

In both cases x_0 defined by (3.17) is equal to $-\infty$ and solution $u = u(x)$ approaches zero at either an exponential or power rate.

Remark 4.1. It is interesting to observe that $x_0 = -\infty$ occurs even in the case when the diffusion coefficient degenerates or has a singularity if this fact is compensated by a proper degeneration of the reaction term g.

Possible values of parameters α, β for which Case 1 occurs for different values of p are shown in Figures 4.1, 4.2, 4.3 where condition (4.1) is taken into account.

Case 2. Let $\alpha - \beta < p - 1$. Then $I < +\infty$ and hence from (3.17) we deduce $x(0) = x_0 > -\infty$. Moreover,

$$I \to +\infty \quad \text{as} \quad \frac{\beta-\alpha-1}{p} \to -1+\,,$$

i.e., we have $x_0 \to -\infty$ as $p - 1 - (\alpha - \beta) \to 0+$. More precisely, for $u \to 0+$ we have

$$x(u) - x_0 \;\sim\; u^{\frac{\beta-\alpha-1}{p}+1} = u^{\frac{p-1-(\alpha-\beta)}{p}}\,.$$

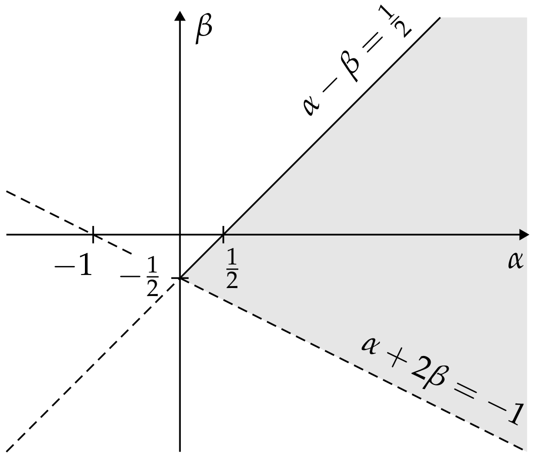

Figure 4.1: $p = \frac{3}{2}$

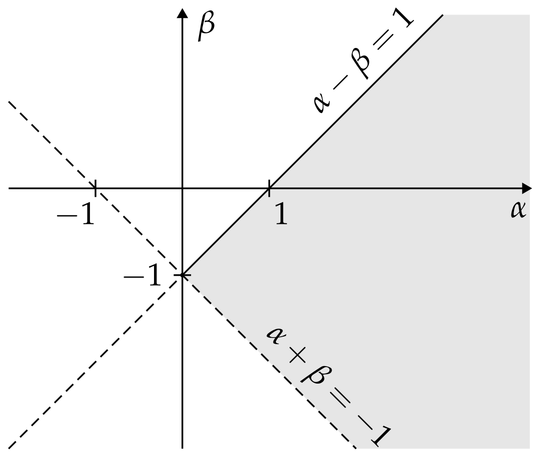

Figure 4.2: $p = 2$

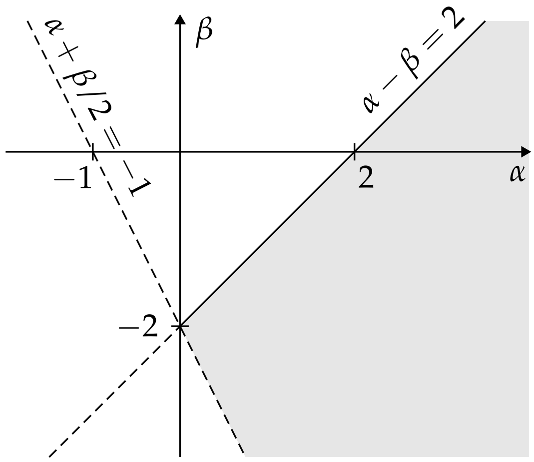

Figure 4.3: $p = 3$

Depending on the shape of $x(u)$ we further distinguish among three cases:

a) $\left.\dfrac{\mathrm{d}x}{\mathrm{d}u}\right|_{u=0+} \sim u^{\frac{\beta-\alpha-1}{p}} \to +\infty \quad \text{for } u \to 0+ \qquad \text{if } \alpha - \beta > -1,$

b) $\left.\dfrac{\mathrm{d}x}{\mathrm{d}u}\right|_{u=0+} \sim u^{0} \to k > 0 \quad \text{for } u \to 0+ \qquad \text{if } \alpha - \beta = -1,$

c) $\left.\dfrac{\mathrm{d}x}{\mathrm{d}u}\right|_{u=0+} \sim u^{\frac{\beta-\alpha-1}{p}} \to 0_{+} \quad \text{for } u \to 0+ \qquad \text{if } \alpha - \beta < -1.$

An inverse point of view gives us the asymptotics of $u = u(x)$ for $x \to x_0$:

$$u(x) \sim (x - x_0)^{\frac{p}{p-1-(\alpha-\beta)}} .$$

As for the derivatives, we have

a) $\left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim (x-x_0)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \to 0 \quad \text{for } x \to x_0+ \qquad \text{if } \alpha-\beta > -1,$

b) $\left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim (x-x_0)^0 \to k > 0 \quad \text{for } x \to x_0+ \qquad \text{if } \alpha-\beta = -1,$

c) $\left.\dfrac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_0+} \sim (x-x_0)^{\frac{\alpha-\beta+1}{p-1-(\alpha-\beta)}} \to +\infty \quad \text{for } x \to x_0+ \qquad \text{if } \alpha-\beta < -1.$

Remark 4.2. We observe that only in case a) the solution $u = u(x)$ is smooth in the neighborhood of x_0 since $u(x) = 0$ for $x \in (-\infty, x_0]$. In the other two cases we only get continuous solutions instead of smooth ones as a consequence of allowing for the diffusion term $d = d(s)$ to degenerate as $s \to 0+$. The asymptotic behavior of such solutions near the point x_0 is illustrated in Figures 4.4, 4.5, 4.6.

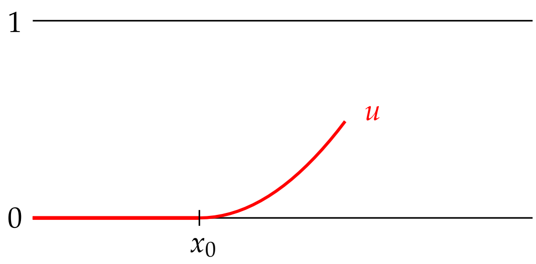

Figure 4.4: Case a)

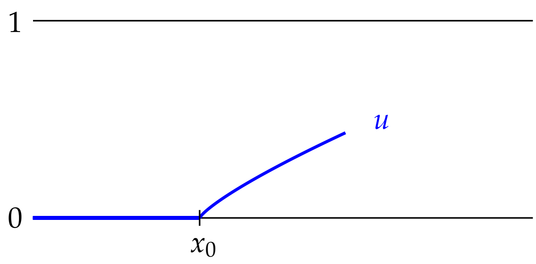

Figure 4.5: Case b)

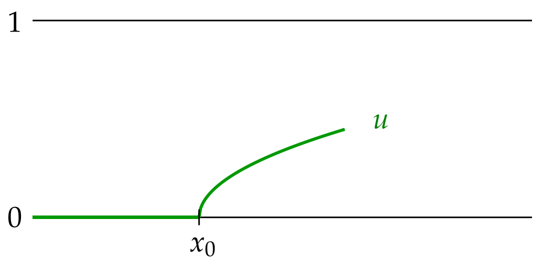

Figure 4.6: Case c)

Values of α, β for which these cases occur are for different values of p depicted in Figures 4.7, 4.8, 4.9. Areas corresponding to cases a) – c) are shown in respective colors as in Figures 4.4, 4.5, 4.6.

Proceeding similarly for $u \to 1-$ and assuming $g(s) \sim (1-s)^\gamma$, $d(s) \sim (1-s)^\delta$ as $s \to 1-$ for some $\gamma > 0$, $\delta \in \mathbb{R}$ satisfying the analogue of condition (4.1):

$$\gamma + \frac{\delta}{p-1} > -1,$$

we get the following asymptotics:

Case 1: $\gamma - \delta \geq p - 1$. Then $x_1 = +\infty$ by (3.18) and we distinguish between two cases. Either

$$u(x) \sim 1 - \mathrm{e}^{-x} \to 1- \quad \text{for } x \to +\infty$$

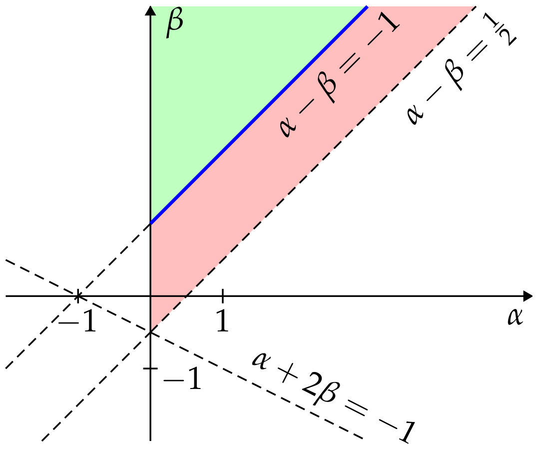

Figure 4.7: $p = \frac{3}{2}$

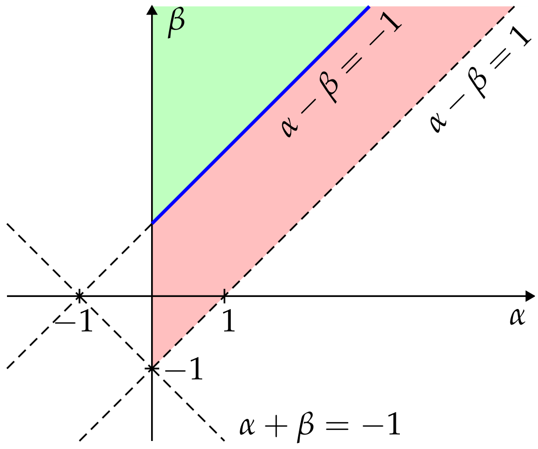

Figure 4.8: $p = 2$

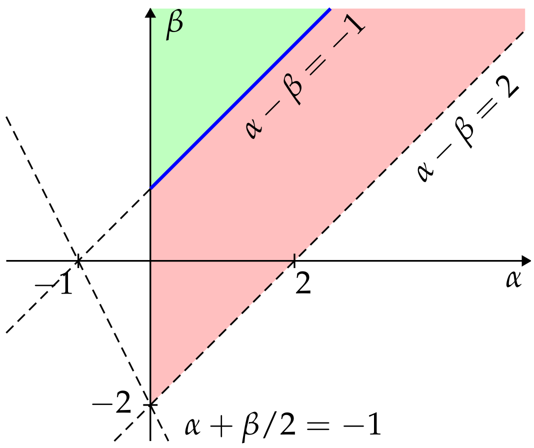

Figure 4.9: $p = 3$

if $\gamma - \delta = p - 1$, or else

$$u(x) \sim 1 - |x|^{\frac{p}{p-1-(\gamma-\delta)}} \to 1-$$

if $\gamma - \delta > p - 1$.

Case 2: $\gamma - \delta < p - 1$. Then $x_1 < +\infty$ by (3.18) and

$$u(x) \sim 1 - (x_1 - x)^{\frac{p}{p-1-(\gamma-\beta)}} \to 1- \quad \text{for } x \to x_1- \, .$$

As for the one-sided derivatives of u at x_1 we have

a) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim (x_1 - x)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to 0 \quad \text{for } x \to x_1- \qquad \text{if } \gamma - \delta > -1,$$

b) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim (x_1 - x)^0 \to k > 0 \quad \text{for } x \to x_1- \qquad \text{if } \gamma - \delta = -1,$$

c) $$\left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=x_1-} \sim (x_1 - x)^{\frac{\gamma-\delta+1}{p-1-(\gamma-\delta)}} \to +\infty \quad \text{for } x \to x_1- \qquad \text{if } \gamma - \delta < -1.$$

Remark 4.3. While all the illustrative pictures in Section 3 do not reflect the effect of the discontinuity of d, finally, we want to focus on how the solution $u = u(x)$ is affected by discontinuous diffusion coefficient $d = d(s)$. Let us assume for simplicity that d only has one point of discontinuity $s_1 \in (0,1)$ and it is smooth and bounded in $(0, s_1)$ and $(s_1, 1)$. Then $M_u = \{\xi_1\}$ and it follows from Theorem 3.2, (iv), that the jump of d at s_1 must be compensated by the proper "opposite" jump of u' at ξ_1, see Figure 4.10, namely

$$|u'(\xi_1-)|^{p-2}\, u'(\xi_1-) \lim_{s\to s_1-} d(s) = |u'(\xi_1+)|^{p-2}\, u'(\xi_1+) \lim_{s\to s_1+} d(s).$$

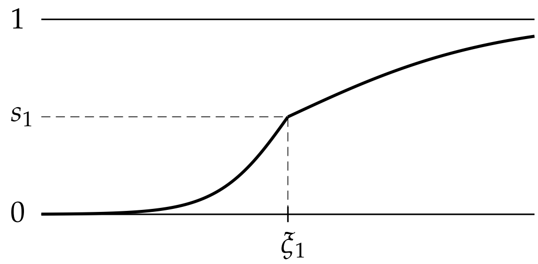

Figure 4.10: Profile of solution $u = u(x)$ for d discontinuous at s_1

5 Final discussions

Let us consider the initial value problem for the quasilinear bistable equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(d(u(x,t)) \left|\dfrac{\partial u}{\partial x}\right|^{p-2} \dfrac{\partial u}{\partial x}\right) + g(u(x,t)), \quad x \in \mathbb{R},\, t > 0, \\ u(x,0) = \varphi(x), \quad x \in \mathbb{R}. \end{cases} \tag{5.1}$$

Here, $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous function, d and g are as in Section 2 and (3.3) (balanced bistable condition) holds. If $\varphi = \varphi(x)$ satisfies the hypothesis

$$\limsup_{x\to-\infty} \varphi(x) < s_* \quad \text{and} \quad \liminf_{x\to+\infty} \varphi(x) > s_*$$

then one would expect that there exists $\xi \in \mathbb{R}$ such that the solution $u = u(x,t)$ of (5.1) satisfies

$$\lim_{t\to+\infty} u(x,t) = u(x-\xi), \quad x \in \mathbb{R},$$

where $u = u(x)$ is a solution given by Theorem 3.2, see Figure 5.1.

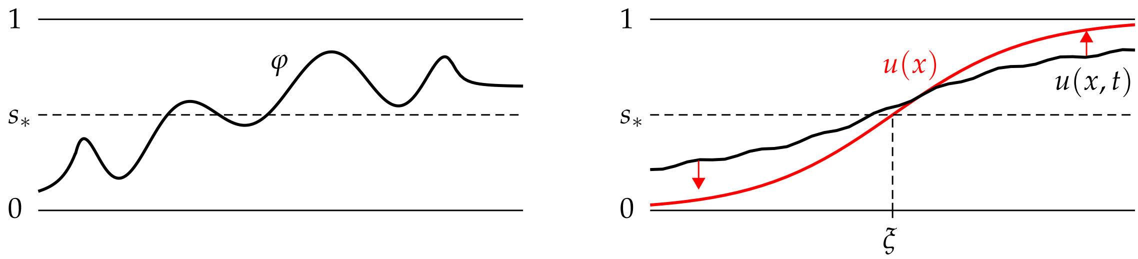

Figure 5.1: Convergence to a stationary solution

It is maybe too ambitious to prove this fact if d is a discontinuous function. However, an affirmative answer to this question, even for d continuous or smooth, would be an interesting result. Even reliable numerical simulation of the asymptotic behavior of the solution $u = u(x,t)$ of the initial value problem (5.1) for $t \to +\infty$ might be of great help.

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