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Geometry, integrability and bifurcation diagrams of a family of quadratic differential systems as application of the Darboux theory of integrability

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Abstract. During the last forty years the theory of integrability of Darboux, in terms of algebraic invariant curves of polynomial systems has been very much extended and it is now an active area of research. These developments are covered in numerous papers and several books, not always following the conceptual historical evolution of the subject and its significant connections to Poincaré's problem of the center. Our first goal is to give in a concise way, following the history of the subject, its conceptual development. Our second goal is to display the many aspects of the theory of Darboux we have today, by using it for studying the special family of planar quadratic differential systems possessing an invariant hyperbola, and having either two singular points at infinity or the infinity filled up with singularities. We prove the integrability for systems in 11 of the 13 normal forms of the family and the generic non-integrability for the other 2 normal forms. We construct phase portraits and bifurcation diagrams for 5 of the normal forms of the family, show how they impact the changes in the geometry of the systems expressed in their configurations of their invariant algebraic curves and point out some intriguing questions on the interplay between this geometry and the integrability of the systems. We also solve the problem of Poincaré of algebraic integrability for 4 of the normal forms we study.

Keywords: quadratic differential system, invariant algebraic curve, invariant hyperbola, Darboux integrability, Liouvillian integrability, configuration of invariant algebraic curves, bifurcation of configuration, singularity and bifurcation.

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1 Introduction

Let $\mathbb{R}[x,y]$ be the set of all real polynomials in the variables x and y. Consider the planar system

$$\begin{aligned} \dot{x} &= P(x,y), \\ \dot{y} &= Q(x,y), \end{aligned} \tag{1.1}$$

where $\dot{x} = dx/dt$, $\dot{y} = dy/dt$ and P, $Q \in \mathbb{R}[x,y]$. We define the degree of a system (1.1) as $\max\{\deg P, \deg Q\}$. In the case where the polynomials P and Q are relatively prime i.e. they do not have a non-constant common factor, we say that (1.1) is *non-degenerate.*

Consider

$$\chi = P(x,y)\,\frac{\partial}{\partial x} + Q(x,y)\,\frac{\partial}{\partial y} \tag{1.2}$$

the polynomial vector field associated to (1.1).

A real *quadratic differential system* is a polynomial differential system of degree 2, i.e.

$$\begin{aligned} \dot{x} &= p_0 + p_1(\tilde{a},x,y) + p_2(\tilde{a},x,y) \equiv p(\tilde{a},x,y), \\ \dot{y} &= q_0 + q_1(\tilde{a},x,y) + q_2(\tilde{a},x,y) \equiv q(\tilde{a},x,y) \end{aligned} \tag{1.3}$$

with $\max\{\deg p, \deg q\} = 2$ and

$$\begin{aligned} &p_0 = a, \qquad p_1(\tilde{a},x,y) = cx + dy, \qquad p_2(\tilde{a},x,y) = gx^2 + 2hxy + ky^2, \\ &q_0 = b, \qquad q_1(\tilde{a},x,y) = ex + fy, \qquad q_2(\tilde{a},x,y) = lx^2 + 2mxy + ny^2. \end{aligned}$$

Here we denote by $\tilde{a} = (a,c,d,g,h,k,b,e,f,l,m,n)$ the 12-tuple of the coefficients of system (1.3). Thus a quadratic system can be identified with a point $\tilde{a}$ in $\mathbb{R}^{12}$.

We denote the class of all quadratic differential systems with **QS**.

Planar polynomial differential systems occur very often in various branches of applied mathematics, in modeling natural phenomena, for example, modeling the time evolution of conflicting species, in biology, in chemical reactions, in economics, in astrophysics, in the equations of continuity describing the interactions of ions, electrons and neutral species in plasma physics (see, for example: [45], [73], [8] and [55]). Such differential systems have also theoretical importance. Several problems on polynomial differential systems, which were stated more than one hundred years ago, are still open: the second part of Hilbert's 16th problem stated by Hilbert in 1900 [32], the problem of the center stated by Poincaré in 1885 [50], the problem of algebraic integrability stated by Poincaré in 1891 [51], [52] (both problems later discussed in this work), and problems on integrability resulting from the work of Darboux [20] published in 1878. With the exception of the problem of the center for quadratic differential systems that was solved, all the other problems mentioned above, are still unsolved even in the quadratic case.

The theory of Darboux [20] (1878) was built for complex polynomial differential equations over the complex projective plane. Here we are interested in polynomial differential systems over the real affine plane. But every system (1.1) with real coefficient can be extended over the complex affine plane and it leads to a polynomial differential equation with homogeneous coefficients over the complex projective plane (see for example [40], pp. 316–317). As a consequence, the theory of Darboux can be applied to real polynomial differential systems. This is a theory of integrability of polynomial differential systems (1.1) which is based on the existence of particular solutions that are algebraic. The cases of integrable systems are rare but as Arnold said in [2, p. 405] "*... these integrable cases allow us to collect a large amount of information* *about the motion in more important systems. . . .*" Poincaré was enthusiastic about the theory of Darboux and called it "admirable" in [51] and "oeuvre magistrale" in [52]. In [52] Poincaré stated his problem of algebraic integrability on systems (1.1), which is still open today. The French Academy of Sciences proposed this problem for a prize which was won by Painlevé and Autonne received an honorable mention but although the new results were interesting they have not provided a complete solution to the problem posed by Poincaré. After the research done by Poincaré, Painlevé and Autonne at the end of the 19th century we have the work of Dulac and of Lagutinskiĭ at the beginning of the 20th century. The work of Dulac [24] will later be briefly discussed in this work. Lagutinskiĭ's work is not well known because except for one paper written in French, all of his other 16 papers, published between 1903 and 1914, were written in Russian. He died in 1915 at the age of 44. The interested reader could find information about his life and work in [21], [22]. Almost a century passed before Darboux theory began again to significantly attract researchers. It started to flower towards the end of the last century and the beginning of the 21th century when in numerous works Darboux's theory has been enriched with new notions and results. Now this is a very active field with new results scattered in many articles and several books. The various aspects of this *extended theory* appear in the literature in surveys, some incomplete as they were published earlier, some containing the latest additions to the theory such as [44]. These surveys are mainly concerned with results and not with the historical conceptual development of the subject, which is fascinating. For example we mentioned above the 1908 work of Dulac on Poincaré's problem of the center where connections with Darboux integrability are present. These connections go deep. They allowed Dulac to solve the problem of the center for complex quadratic systems with a center, the only case where the problem was solved. The method of Darboux is also powerful in unifying proofs of integrability for whole families of systems with centers or for other families of systems like the ones we consider in this paper. For other applications of the theory of Darboux see the survey article of Llibre and Zhang [44].

One of the goals of this article is to make this task easier by providing here a brief conceptual survey of this beautiful theory, which closely follows the historical evolution of the subject. We also prove here that even when trying to understand the integrability of real systems, their complex invariant curves are essential (see in Section 2, Example 40).

Definition 1.1 ([20])**.** An algebraic curve $f(x,y) = 0$ with $f(x,y) \in \mathbb{C}[x,y]$ is called an *invariant algebraic curve* of system (1.1) if it satisfies the following identity:

$$f_x P + f_y Q = Kf, \tag{1.4}$$

for some $K \in \mathbb{C}[x,y]$ where f_x and f_y are the derivatives of f with respect to x and y. K is called the *cofactor* of the curve $f = 0$.

For simplicity we write the curve f instead of the curve $f = 0$ in $\mathbb{C}^2$. Note that if system (1.1) has degree m then the cofactor of an invariant algebraic curve f of the system has degree $m - 1$.

Definition 1.2 ([20])**.** Consider a planar polynomial system (1.1). An algebraic solution of (1.1) is an algebraic invariant curve f which is irreducible over $\mathbb{C}$.

Definition 1.3. Let U be an open subset of $\mathbb{R}^2$. A real function H: $U \to \mathbb{R}$ is a *first integral* of system (1.1) if it is constant on all solution curves $(x(t), y(t))$ of system (1.1), i.e., $H(x(t), y(t)) = k$, where k is a real constant, for all values of t for which the solution $(x(t), y(t))$ is defined on U.

If H is differentiable in U then H is a first integral on U if and only if

$$H_x P + H_y Q = 0. \tag{1.5}$$

The problem of integrating a polynomial system by using its algebraic invariant curves over $\mathbb{C}$ was considered for the first time by Darboux in [20].

Theorem 1.4 (Darboux [20])**.** *Suppose that a polynomial system* (1.1) *has m invariant algebraic curves $f_i(x,y) = 0$, $i \leq m$, with $f_i \in \mathbb{C}[x,y]$ and with $m > n(n+1)/2$ where n is the degree of the system. Then we can compute complex numbers $\lambda_1, ..., \lambda_m$ such that $f_1^{\lambda_1} ... f_m^{\lambda_m}$ is a first integral of the system.*

Definition 1.5. If a system (1.1) has a first integral of the form

$$H(x,y) = {f_1}^{\lambda_1} ... {f_p}^{\lambda_p} \tag{1.6}$$

where f_i are the invariant algebraic curves of system (1.1) and $\lambda_i \in \mathbb{C}$ then we say that system (1.1) is *Darboux integrable* and we call the function H a *Darboux function*.

Remark 1.6. We stress that the theorem of Darboux gives only a sufficient condition for Darboux integrability of a system (1.1) (see example below), expressed in a relation between the number of invariant algebraic curves the system possesses and the degree of the system.

Example 1.7. Consider the system

$$\begin{cases} \dot{x} = 3 + 2x^2 + xy, \\ \dot{y} = 3 + xy + 2y^2. \end{cases}$$

This system admits the invariant line $x - y = 0$ and the invariant hyperbola $2 + xy = 0$. Then, $m = 2 < 3 = n(n+1)/2$. However we still have here a Darboux first integral $H(x,y) = (x - y)^{-3/2}(2 + xy)$. Thus the lower bound on the number of invariant curves sufficient for Darboux integrability in his theorem is in general greater than necessary. The following question arises then naturally: Could we find a necessary and sufficient condition for Darboux integrability?

Definition 1.8. Let U be an open subset of $\mathbb{R}^2$ and let $R : U \to \mathbb{R}$ be an analytic function which is not identically zero on U. The function R is an integrating factor of a polynomial system (1.1) on U if one of the following two equivalent conditions on U holds:

$$\operatorname{div}(RP, RQ) = 0, \qquad R_x P + R_y Q = -R \operatorname{div}(P,Q),$$

where $\operatorname{div}(P,Q) = P_x + Q_y$.

A first integral H of

$$\dot{x} = RP, \qquad \dot{y} = RQ$$

associated to the integrating factor R is then given by

$$H(x,y) = \int R(x,y)P(x,y)dy + h(x),$$

where $H(x,y)$ is a function satisfying $H_x = -RQ$. Then,

$$\dot{x} = H_y, \qquad \dot{y} = -H_x.$$

In order that this function H be well defined the open set U must be simply connected.

The simplest integrable systems (1.1) are the Hamiltonian ones having a polynomial first integral. Next we have the systems (1.1) which admit a rational first integral. These were called by Poincaré *algebraically integrable systems*. Such a first integral yields a foliation with singularities of the plane in algebraic phase curves. The question asked by Poincaré in [52] is the following:

Can we recognize when a system (1.1) *admits a rational first integral?*

This is Poincaré's *problem of algebraic integrability* and it is not even solved for quadratic differential systems. We say more on this question in the next section.

To advance knowledge on algebraic, Darboux or more general types of integrability it is useful to have a large number of examples to analyze. In the literature, scattered isolated examples were analyzed, among them is the family of quadratic differential systems possessing a center, i.e. a singular point surrounded by closed phase curves. There is a rather strong relationship between the problem of the center and the theory of Darboux. In particular, every quadratic system with a center possesses invariant algebraic curves and in the generic case it possesses a Darboux first integral. For non-generic cases such a system is still integrable but with a more general type of a first integral.

A more systematic approach for studying families of integrable systems was initiated in the papers of Schlomiuk and Vulpe [65], [66], [67], [68] and [64] where they classified topologically the phase portraits of quadratic systems with invariant lines of at least four total multiplicity (including the line at infinity) as well as the quadratic systems with the line at infinity filled up with singularities and proved their integrability. These results were applied by Schlomiuk and Vulpe [69,70] to the family of Lotka–Volterra differential systems (the **L-V** family), important for so many applications. Not all the systems in this family are integrable but since the **L-V** systems always have at least three invariant lines (including the line at infinity), numerous systems in this family also belong to the family of systems possessing at least four invariant lines and using this fact and the results in the papers above indicated, simplified the classification. There are thus many **L-V** systems that are integrable according to the method of Darboux. For the Liouvillian integrability of **L-V** systems see [6]. The case of quadratic systems possessing two complex invariant lines intersecting at a real finite point was completed in [71]. Systems in this family are not always integrable but as the authors show, for a large subfamily we can apply the Darboux theory of integrability. Work is in progress for completing the study of the family of systems possessing three invariant lines, including the line at infinity. In the above studies, the properties of the "configuration" of invariant lines (term we will later define) were important to distinguish the types of integrability of the systems. A natural question which arises is the following one:

What is the relation between the geometry of a "configuration" of invariant algebraic curves of a system (1.1) *and its integrability?*

In order to be able to provide responses to such a question, data involving only invariant lines is insufficient. Data involving more general curves and in particular conics and cubics, is needed. In [47] the authors classified the family **QSH** of non-degenerate quadratic differential systems possessing an invariant hyperbola according to "*configurations of invariant hyperbolas and lines*". They proved that the family **QSH** is geometrically rich as it has 205 distinct such configurations. The problem of integrability of systems in **QSH** according to the theory of Darboux was not considered in [47]. This is the problem we study in the second part of this paper. Considered from the viewpoint of integrability, the family **QSH** is also very rich displaying a vast array of systems of various kinds of integrability as we see in the examples we provide in this paper. This data will be precious in deeper exploring the Darboux theory of integrability. Here by "deeper" we mean understanding the relationship between the integrability of the systems and the *geometry of the "configurations" of invariant algebraic curves* they possess.

Since its creation by Darboux in 1878 [20], this theory has evolved and it has been significantly extended. Much of this development occurred during the past forty years. The literature on this extended theory is scattered in many articles and also some books, not necessarily following the history of the conceptual evolution of the subject with its connections with the problem of the center. These connections were important for drawing attention to the role of the theory of Darboux and its *unifying capacity for proving integrability of families of polynomial differential systems* as we explain in the next Section and for classifying families of vector fields not necessarily integrable such as the family of **L-V** systems previousy discussed.

The second goal is to study the systems of the family **QSH** from the viewpoint of what we call today the Darboux theory of integrability. This adds a lot of integrability data next to the data we have from the work of Schlomiuk and Vulpe, mentioned above, on quadratic systems with invariant straight lines by allowing us to also include conics. Apart from richly illustrating the theory and pointing out some rather subtle issues, this testing ground provides us with the possibility of asking new questions relating the *geometry of the configuration of invariant algebraic curves* and the Darboux theory of integrability. It is this relationship that is our main motivation.

Our paper is organized as follows:

In Section 2 we give a short conceptual and historical overview of Darboux theory as we have it today, including all essential new notions not used in Darboux's work, as well as new results, extensions of his theory. We also recall the unifying character of the method of Darboux in proving integrability for some families of vector fields and we prove that the theory of Darboux is essentially a theory over the complex field even when we search to calculate real first integrals of real systems (see Example 2.34 in this Section).

In Section 3 we discuss the class **QSH** from the viewpoint of the relationship between integrability and the geometry of the "configuration of invariant algebraic curves" which the systems possess. In particular we are concerned here with the family $\mathbf{QSH}_{\eta=0}$ of systems in **QSH** which possess either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities. In [47] the authors calculated the invariant lines and hyperbolas of each normal form in $\mathbf{QSH}_{\eta=0}$.

In Section 4 we introduce a number of geometrical concepts which are very helpful in understanding the relation between the geometry of the configuration of invariant algebraic curves and the integrability of the systems.

In Section 5 we prove that for the 11 of the 13 normal forms for the systems in $\mathbf{QSH}_{\eta=0}$ all systems have a Liouvillian first integral. We present the invariant algebraic curves, exponential factors, integrating factors and first integrals for each one of these 11 normal forms for $\mathbf{QSH}_{\eta=0}$.

In Section 6 we prove the generic non-integrability for the remaining two normal forms for $\mathbf{QSH}_{\eta=0}$, cases where the number of invariant curves and exponential factors are not sufficient for finding a first integral or integrating factor.

In Section 7 we apply the Darboux theory of integrability to the geometric analysis of five families of systems in $\mathbf{QSH}_{\eta=0}$. We exhibit the bifurcation diagrams of the configurations of invariant algebraic curves as well as the bifurcation diagrams of the systems and raise the problem of interaction between these two kinds of bifurcations. Phase portraits for quadratic system with an invariant hyperbola and an invariant straight line were also constructed in [41]. However, we point out that the authors of [41] did not get all of the phase portraits, in particular, in Section 7 we point out some of their missing phase portraits. This is due to the fact that their normal form for this family misses some of the systems in the family. We also solve the Poincaré problem of algebraic integrability for four of the families we studied.

In Section 8 we highlight some significant points raised in this paper, explain the relation between the bifurcations of configurations of invariant curves and topological bifurcations, raise a number of questions and state some problems. Finally we mention that we also obtained, as limiting cases of the family (D), three other normal forms, i.e. (F), (G) and (I).

2 Brief conceptual and historical overview of the theory of Darboux [20] as it is understood today

After the publication of the works of Poincaré, Painlevé and Autonne in the 1890's originating in the work of Darboux [20], the first article using the method of integration of Darboux was Dulac's paper [24] (1908) in which he solved Poincaré's problem of the center [50] for quadratic differential systems (see more on this problem on page 9). After the publication of Dulac's paper, the next important result concerning the Darboux theory of integrability is Jouanolou's who in [34] (1979) gave a sufficient condition for algebraic integrability.

Theorem 2.1 (Jouanolou [34])**.** *Consider a polynomial system* (1.1) *of degree n and suppose that it admits m invariant algebraic curves $f_i(x,y) = 0$ where $1 \le i \le m$, then if $m \ge 2 + \frac{n(n+1)}{2}$, there exists integers $N_1, N_2, \ldots, N_m$ such that $I(x,y) = \prod_{i=1}^{m} f_i^{N_i}$ is a first integral of* (1.1)*.*

If a differential system (1.1) has a rational first integral $H(x,y) = f(x,y)/g(x,y)$ with $f, g \in \mathbb{C}[x,y]$, then the solution curves are located on its level curves $H(x,y) = C$ where C is a constant, i.e. on the algebraic curves $f(x,y) - Cg(x,y) = 0$. We call degree of the first integral H the number $\max(\deg(f), \deg(g))$. Then all the algebraic invariant curves of the system have a degree bounded by the degree of H.

We can argue that in case we can show that a system has invariant algebraic curves of bounded degree, in order to decide whether the system is algebraically integrable it remains to compute, by solving algebraic equations, a sufficient amount of invariant algebraic curves. This is true because we know that a finite number of steps will be sufficient. For this reason the problem of Poincaré is sometimes understood as the problem of bounding the degrees of the invariant algebraic curves the system possesses. Thus, in [7] the problem of Poincaré is stated as follows:

Let $\mathcal{F}$ be a holomorphic foliation by curves of the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$. Let C be an algebraic curve in $\mathbb{P}^2_{\mathbb{C}}$. Is it possible to bound the degree of C in terms of the degree of $\mathcal{F}$?

The problem of Poincaré is understood in this way elsewhere in the literature, see for example [33], page 242. But solving this problem is far from solving the problem as initially formulated by Poincaré. Indeed, the algebraic equations we would need to solve in order to find a sufficient amount of algebraic invariant curves of the systems, to obtain algebraic integrability, can easily surpass the present day capacity of computers. Besides, the problem of bounding the degree of an algebraic invariant curve is not even solved in the general case. For a solution of this problem under restrictive conditions see [7].

So far we mentioned only three steps in the hierarchy of first integrals: polynomial, rational and Darboux first integrals which could be rational or transcendental. What other kinds of first integrals can we have next in this hierarchy? We can have elementary first integrals. Roughly speaking these are functions which are constructed by using addition, multiplication, composition of finitely many rational functions, trigonometric and exponential functions and their inverses.

The next important result, obtained in 1983, involves elementary first integrals and is due to Prelle and Singer. It was stated for more general vector fields in $\mathbb{C}^n$ in differential algebra language. Here we consider only the case of planar differential systems (1.1).

Theorem 2.2 (Prelle–Singer [53])**.** *If a polynomial differential system* (1.1) *has an elementary first integral, then the system has a first integral of the following form:*

$$f(x,y) + c_1 \log(f_1(x,y)) + c_2 \log(f_2(x,y)) + \cdots + c_k \log(f_k(x,y))$$

where f and f_i, are algebraic functions over $\mathbb{C}(x,y)$ and $c_i \in \mathbb{C}$, $i = 1,2,\dots k$.

Taking the exponential of the above expression we obtain the following corollary.

Corollary 2.3. *If a polynomial differential system* (1.1) *possesses an elementary first integral then it also admits a first integral of the form:*

$$e^{f(x,y)} f_1(x,y)^{c_1} f_2(x,y)^{c_2} \dots f_k^{c_k}.$$

where f and f_i, are algebraic functions over $\mathbb{C}(x,y)$ and $c_i \in \mathbb{C}$, $i = 1,2,\dots k$.

In particular we can take for $f(x,y)$ a rational function and for all $f_i's$ polynomial functions over $\mathbb{C}$. This kind of expression differs from a Darboux first integral by the exponential factor $e^{f(x,y)}$ which appears though not explicitly, in Prelle–Singer's paper [53] and also f_i's are here algebraic and not just polynomials over $\mathbb{C}$.

The above expression is a more general first integral that includes the case of a Darboux first integral when f is the zero-function and f_i's are polynomials. Although this kind of expression does not appear in [20], nowadays a first integral of this more general kind, with f rational and all f_i' s polynomial functions, is still called a Darboux first integral in the literature.

In Section 3 of their paper [53] Prelle and Singer talk about "Algorithmic considerations" and they say:

The preceding work was motivated by our desire to develop a decision procedure for finding elementary first integrals. These results show that we need only look for elementary integrals of a prescribed form. In this section we shall discuss the problem of finding an elementary first integral for a two-dimensional autonomous system of differential equations and reduce this problem to that of bounding the degrees of algebraic solutions of this system.

They base their algorithm on the following two propositions.

Proposition 2.4 ([53])**.** *If the planar system* (1.1) *has an elementary first integral, then there exists an integer n and an invariant algebraic curve f such that*

$$Pf_x + Qf_y = -n\left(P_x + Q_y\right) f.$$

Proposition 2.5. *If the equations of* (1.1) *have an elementary first integral, then there exists an element R algebraic over $\mathbb{C}(x,y)$ such that $R_x P + R_y Q = -(P_x + Q_y)R$.*

We use here a version of the Prelle–Singer algorithm provided in [31].

Theorem 2.6 (The Prelle–Singer algorithm [53] (1983), as presented in [31] (2001))**.**

(1) Let $N = 1$.

(2) Find all the invariant algebraic curves $C : f(x, y) = 0$ *with*

$$Pf_x + Qf_y = Kf$$

such that $K(x, y) \in \mathbb{C}[x, y]$ *and* $\deg(f) \leq N$.

(3) Decide if there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbb{C}$*, not all zero, such as*

$$\sum_{i=0}^{m} \lambda_i K_i = 0,$$

where K_i *is cofactor of a curve* f_i *found in (2). If such* λ_i*'s exist, then* $I = \prod_{i=0}^{m} {f_i}^{\lambda_i}$ *is a first integral. Otherwise, go to (4).*

(4) Decide if there exist constants $\lambda_1, \lambda_2, \ldots., \lambda_m \in \mathbb{C}$*, not all zero, such as*

$$\sum_{i=0}^{m} \lambda_i K_i = -(P_x + Q_y),$$

where K_i *is cofactor of a curve* f_i *found in (2).*

If such λ_i*'s exist, then* $R = \prod_{i=0}^{m} {f_i}^{\lambda_i}$ *is an integrating factor and a first integral can be obtained by integrating the equations:*

$$I_x = RQ,$$
$$I_y = -RP.$$

If such λ_i*'s do not exist, return to (1) increasing* N *by* 1 *and continue the process.*

In further exploring the evolution of ideas and development of the theory of Darboux it is important to mention the connections between this theory and the problem of the center stated by Poincaré in [50] in 1885. These connections have done much to draw attention to the theory of Darboux and its unifying power in proving integrability of polynomial systems. We indicate here some of these connections as well as the story of the solution of the problem of the center for quadratic systems and in proving their integrability in a unified way by the method of Darboux.

For quadratic systems the problem of the center as already mentioned at the beginning of this section was solved by Dulac. Unlike Poincaré, Dulac considered differential systems defined over $\mathbb{C}$. In [23] he defined the following notion of center: *A singular point of a planar holomorphic differential system with non-zero eigenvalues is a center if and only if the quotient of its eigenvalues is negative and rational and the system has a local analytic first integral.* In his paper [24], Dulac mentions that the general case is more difficult to treat, he supposes that the quotient of the eigenvalues is -1. Placing the singular point at the origin, he used the following normal form for quadratic systems:

$$\dot{x} = x + a_{20}x^2 + a_{11}xy + a_{02}y^2,$$
$$\dot{y} = -y + b_{20}x^2 + b_{11}xy + b_{02}y^2.$$

To solve the problem of the center for quadratic systems means to find necessary and sufficient conditions in terms the coefficients a_{ij} and b_{ij} so that the origin be a center. He solved this problem in 1908 [24] and used the method of integration of Darboux in one case.

This work of Dulac could not be readily applied for real system. Indeed, in the normal form considered by Dulac, if we assume that the coefficients of the equations are real than this real system has a saddle at the origin and we cannot pass from this normal form to the normal form used by Poincaré (where the linear terms of the two equations are respectively $-y$, x) by a real linear transformation. Thus the conditions for the center obtained by Dulac cannot be readily used in the case of real systems for centers as defined by Poincaré.

In 1985, working on perturbations of quadratic Hamiltonian systems with a center, Guckenheimer, Rand and Schlomiuk needed the conditions on a real quadratic differential system to have a center. Exploring the literature they found that it is very messy, containing many errors. In [58] (1990), after making a historical survey pointing out the errors, they proved by diverse ad hoc methods that each real quadratic system with a center is integrable. The correct conditions for a center were obtained by Kapteyn and Bautin (see [35,36]) thus solving the problem of the center for a real quadratic differential systems. At the suggestion of Guckenheimer, Schlomiuk then tried to give a geometric interpretation of the Kapteyn–Bautin conditions for a center. This geometric interpretation was revealed by studying the bifurcation diagram of the family **QSC** of quadratic systems with a center (see [61]). The conditions for the center can be interpreted in terms of the types of invariant algebraic curves the systems possess.

These results were presented for the first time by Schlomiuk at the Luminy conference in France on differential equations in 1989 and later in 1992 at the NATO Advanced Study Institute in Montreal where she also presented the work of Prelle–Singer (see [60]). Meetings are always very useful for disseminating information. Thus, it was in 1989 at the Luminy conference that Moussu, present at that meeting, told Schlomiuk about the work of Darboux. Specialists in integrability in the audience at the Montreal meeting in 1992, not previously aware of this work of Prelle and Singer, found out about this work from Schlomiuk's lectures. A *unified proof of integrability based on the theory of Darboux*, for all systems in **QSC** was obtained (see [59,60]) (1993). While the proof in [58] was done by using diverse ad hoc methods, in the new proof all the cases were treated in the same way, by the method of Darboux in terms of invariant algebraic curves. These and other articles mentioned further below played a role in drawing attention to the unifying role the method of integration of Darboux played in proving integrability for entire families of certain planar polynomial differential systems. The articles [59,61] were read by a number of people, in particular they were cited in [10] (1997), which contains an extension of the theory of Darboux to be later discussed, and also in [5].

In his PhD Thesis (1990) entitled *Invariant algebraic curves in polynomial differential systems* as well as later in his paper [11] (1994) Christopher had independently explored the relationship between the presence of invariant algebraic curves and conditions for the center in quadratic and also some cubic differential systems such as the cubic Kukles systems without a term in y^3 in the equation for dy/dt, or the cubic system of Dolov. He showed that the conditions for the center given by Kukles were incomplete and proved that the system of Dolov was integrable by using four invariant lines and a circle.

Work on these connections between the problem of the center and the Darboux theory of integrability continued to be published. We only mention here a few of the earliest papers such as [38] (1992) of Cozma and Șubă on cubic differential systems and of Żołądek [76] (1994) on quadratic systems and their perturbations. More work on cubic systems done by Cozma and Şubă and also by Żołądek alone or together with some of his students, can be accessed through MathSciNet. The cubic symmetric systems were proven to be integrable using the method of Darboux by Rousseau and Schlomiuk in [57] (1995) and they also had integrability results on the reduced cubic Kukles systems [56] (1995).

To get to a higher echelon in the hierarchy of first integrals, we need to consider *Liouvillian first integrals*. In [72] Singer describes Liouvillian functions as follows:

Liouvillian functions are functions that are built up from rational functions using exponentiation, integration, and algebraic functions.

Thus, the logarithm as a function of one variable is a Liouvillian function being defined as the integral from 0 to x of $1/x$. In general, Liouvillian functions are defined in the context of differential algebra.

The following result was proved by Singer in 1992.

Theorem 2.7 ([72]). *If the system* (1.1) *has a Liouvillian first integral, then it has an integrating factor of the form*

$$e^{\int Udx+Vdy}, \qquad U_y = V_x,$$

where U and V are rational functions over $\mathbb{C}[x,y]$.

A consequence of Singer's theorem is the following.

Corollary 2.8 ([72]). *A system of differential equations* (1.1) *has a Liouvillian first integral if and only if it has an integrating factor of the form*

$$R(x,y) = e^{\int Udx+Vdy}, \;\; U_y = V_x \;\; (U, V \text{ are rational function over } \mathbb{C}[x,y])$$

in which case

$$F(x,y) = \int R(x,y)Q(x,y)dx - R(x,y)P(x,y)dy$$

is a Liouvillian first integral.

It is important to mention that a Liouvillian integrable system does not necessarily have an affine invariant algebraic curve. An example of such a polynomial differential system is presented in [30].

The following notion was defined by Christopher in 1994 (see [11]) where he called it "degenerate invariant algebraic curve".

Definition 2.9. Let $F(x,y) = \exp\left(\frac{G(x,y)}{H(x,y)}\right)$ with G, $H \in \mathbb{C}[x,y]$ coprime. We say that F is an *exponential factor* of system (1.1) if it satisfies the equality

$$F_x P + F_y Q = LF, \tag{2.1}$$

for some $L \in \mathbb{C}[x,y]$. The polynomial L is called the *cofactor* of the exponential factor F.

Proposition 2.10 ([11]). *If $F = \exp(G/H)$ is an exponential factor of system* (1.1) *with cofactor L then $H = 0$ is an invariant algebraic curve of the system* (1.1) *with cofactor K_H and G satisfies the equation*

$$PG_x + QG_y = K_H G + LH, \quad \text{where } G, H, L, K_H \in \mathbb{C}[x,y]. \tag{2.2}$$

See [15] for a detailed proof.

A theorem of Darboux was rephrased by Chavarriga, Llibre and Sotomayor [10] (1997) by introducing in [10] the notion of independent points.

If $S(x,y) = \sum_{i+j=0}^{m-1} a_{ij}x^iy^j$ is a polynomial of degree at most $m-1$ with $m(m+1)/2$ coefficients in $\mathbb{C}$, then we write $S \in \mathbb{C}_{m-1}[x,y]$. We identify the linear space $\mathbb{C}_{m-1}[x,y]$ with $\mathbb{C}^{m(m+1)/2}$ through the isomorphism

$$S \to (a_{00}, a_{10}, a_{01}, \ldots, a_{m-1,0}, a_{m-2,1}, \ldots, a_{0,m-1}).$$

Definition 2.11 ([10])**.** We say that r singular points $(x_k, y_k) \in \mathbb{C}^2$, $k = 1, \ldots, r$ of a differential system (1.1) of degree m are independent with respect to $\mathbb{C}_{m-1}[x,y]$ if the intersection of the r hyperplanes

$$\sum_{i+j=0}^{m-1} x_k^i y_k^j a_{ij} = 0, \qquad k = 1, \ldots, r,$$

in $\mathbb{C}^{m(m+1)/2}$ is a linear subspace of dimension $[m(m+1)/2] - r$.

We remark that the maximum number of isolated singular points of the polynomial system (1.1) of degree m is m^2 (by Bézout's Theorem), that the maximum number of independent isolated singular points of the system is $m(m+1)/2$, and that $m(m+1)/2 < m^2$ for $m \geq 2$.

The following is a theorem of Darboux as stated by Chavarriga, Llibre and Sotomayor proved in [10].

Theorem 2.12 ([20])**.** *Assume that a real (complex) polynomial system of degree m admits $q = m(m+1)/2 + 1 - p$ algebraic solutions $f_i = 0$, $i = 1, 2, \ldots, q$, not passing through p real (complex) independent singular points (x_k, y_k), $k = 1, 2, \ldots, p$, then the system has a first integral of the form $f_1^{\lambda_1} f_2^{\lambda_2} \ldots f_q^{\lambda_q}$ with $\lambda_i \in \mathbb{R}$ ($\mathbb{C}$).*

Remark 2.13. The above theorem is interesting because it reduces the number of invariant algebraic curves we need to have, that according to Darboux's theorem is $m(m+1)/2 + 1$, to just $m(m+1)/2 + 1 - p$.

Let us consider again Example 1.7:

$$\begin{cases} \dot{x} = 3 + 2x^2 + xy, \\ \dot{y} = 3 + xy + 2y^2. \end{cases}$$

The line $f_1(x,y) = x - y = 0$ and the hyperbola $f_2(x,y) = 2 + xy = 0$ are invariant for this system with co-factors $K_1(x,y) = 2x + 2y$ and $K_2(x,y) = 3x + 3y$. Here $m = 2 = n$ and hence $m < n(n+1)/2$. Still, the number of curves suffices to compute the first integral $H(x,y) = (x-y)^{-3/2}(2 + xy)$ although the condition in the theorem of Darboux is not satisfied by this number. But here we have that the singular points $P_{1,2} = \pm(-i\sqrt{3}, i\sqrt{3})$ of the system are independent. Indeed, solving the system $H_1 = a_{00} - i\sqrt{3}a_{10} + i\sqrt{3}a_{01} = 0$, $H_2 = a_{00} + i\sqrt{3}a_{10} - i\sqrt{3}a_{01} = 0$, we get $a_{00} = 0$ and $a_{10} = a_{01}$ and hence $\dim(H_1 \bigcap H_2) = 1$. Also $f_1(P_i) \neq 0$ and $f_2(P_i) \neq 0$. So the points P_i's are independent. Applying the above theorem we have $q = 2$, $p = 2$, $n = 2$ and we have $q = n(n+1)/2 + 1 - p$.

Definition 2.14. A singular point (x_0, y_0) of system (1.1) is called weak if the divergence of system (1.1) at (x_0, y_0) is zero.

In what follows we state a generalization of Darboux's theorem taking into account exponential factors, independent points and invariants. The result was stated and proved by Christopher and Llibre in 2000 in [15]. An earlier version appeared in [5] (1999).

Theorem 2.15 ([15]). *Suppose that a $\mathbb{C}-$polynomial system (1.1) of degree m admits p algebraic solutions $f_i = 0$ with cofactors K_i for $i = 1, \dots, p$, q exponential factors $F_j = \exp(g_j/h_j)$ with cofactors L_j for $j = 1, \dots, q$, and r independent singular points $(x_k, y_k) \in \mathbb{C}^2$ such that $f_i(x_k, y_k) \neq 0$ for $i = 1, \dots, p$ and for $k = 1, \dots, r$.*

(i) There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0,$$

if and only if the (multi-valued) function

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} \tag{2.3}$$

is a first integral of system (1.1).

(ii) If $p + q + r \geq [m(m+1)/2] + 1$, then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{i=1}^{q} \mu_j L_j = 0.$$

(iii) If $p+q+r \geq [m(m+1)/2]+2$, then system (1.1) has a rational first integral, and consequently all trajectories of the system are contained in invariant algebraic curves.

(iv) There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\operatorname{div}(P, Q),$$

if and only if function (2.3) is an integrating factor of system (1.1).

(v) If $p + q + r = m(m+1)/2$ and the r independent singular points are weak, then function (2.3) is a first integral if

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{i=1}^{q} \mu_j L_j = 0,$$

or an integrating factor if

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\operatorname{div}(P, Q),$$

under the condition that not all $\lambda_i, \mu_j \in \mathbb{C}$ are zero.

(vi) If there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -s$$

for some $s \in \mathbb{C} \backslash \{0\}$, then the (multi-valued) function

$$f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} \exp(st) \tag{2.4}$$

is an invariant of system (1.1).

Of course, each irreducible factors of each h_j is one of the f_i's.

Definition 2.16. If system (1.1) has a first integral of the form

$$H(x,y) = f_1^{\lambda_1} \dots f_p^{\lambda_p} F_1^{\mu_1} \dots F_q^{\mu_q} \tag{2.5}$$

where f_i and F_j are respectively the invariant algebraic curves and exponential factors of a system (1.1) and $\lambda_i, \mu_j \in \mathbb{C}$, then we say that the system is *generalized Darboux integrable*. We call the function H a *generalized Darboux function*.

Remark 2.17. In [20] Darboux considered functions of the type (1.6), not of type (2.5). In recent works functions of type (2.5) were called Darboux functions. Since in this work we need to pay attention to the distinctions among the various kinds of first integral we call (1.6) a Darboux and (2.5) a generalized Darboux first integral.

Proposition 2.18 ([25])**.** *For a real polynomial system* (1.1) *the function* $\exp(G/H)$ *is an exponential factor with cofactor K if and only if the function* $\exp(\overline{G}/\overline{H})$ *is an exponential factor with cofactor* $\overline{K}$.

Remark 2.19 ([25])**.** If among exponential factors of the real system (1.1) a complex pair $F = \exp(G/H)$ and $\overline{F} = \exp(\overline{G}/\overline{H})$ occurs, then the first integral (2.5) has a real factor of the form

$$(\exp(G/H))^{\mu} \left(\exp(\overline{G}/\overline{H})\right)^{\overline{\mu}} = \exp\left(2\,\mathrm{Re}(\mu(G/H))\right),$$

where $\mu \in \mathbb{C}$ and $\mathrm{Im}(\mu)\,\mathrm{Im}(F) \neq 0$. This means that function (2.5) is real when system (1.1) is real.

Considering the definition of generalized Darboux function we can rewrite Corollary 2.8 as follows.

Theorem 2.20 ([11,72])**.** *A planar polynomial differential system* (1.1) *has a Liouvillian first integral if and only if it has a generalized Darboux integrating factor.*

For a proof see [75], page 134.

We can also state easily the following result of Preller–Singer.

Theorem 2.21 ([9, 53])**.** *If a planar polynomial vector field* (1.2) *has a generalized Darboux first integral, then it has a rational integrating factor.*

In 2019, a converse of the previous result was proved in [16] as a consequence of [54].

Theorem 2.22 ([16])**.** *If a planar polynomial vector field* (1.2) *has a rational integrating factor, then it has a generalized Darboux first integral.*

We have the following table summing up these results.

First integral		Integrating factor
Generalized Darboux	$\Leftrightarrow$	Rational
Liouvillian	$\Leftrightarrow$	Generalized Darboux

To study the way integrable systems vary within families of polynomial differential systems (1.1) using the theory of Darboux, one needs to consider perturbations of a system within such a family. An algebraic invariant curve $f(x,y) = 0$ of such a system could split in several algebraic invariant curves occurring in nearby systems. In [11] (1994) C. Christopher considered in an example the coalescence of two such curves and its relationship with exponential factors but in [11] he did not yet talk about *multiplicity of an invariant algebraic curve*.

In [62] (1997) Schlomiuk introduced a general notion of multiplicity of an invariant algebraic curve $f = 0$ of a polynomial differential system (1.1). This definition was given in terms of the multiplicities of singularities of the system located on the projective completion of the curve (Definition 4.1 in [62]).

A notion of multiplicity was defined by Schlomiuk and Vulpe in 2004 for invariant lines of quadratic differential systems and in [64] they classified the family of quadratic systems with invariant lines of total multiplicity at least five, including the line at infinity, according to configurations of straight lines of such systems. Around the same time this study was in progress, Christopher, Llibre and Pereira were working on their important paper [18] (2007) and produced a preprint, earlier version of their work, containing several notions of multiplicity of an invariant algebraic curve. In [18] they gave a condition for these notions to coincide. In this work, as we see later, we use three of the notions introduced in [18].

Suppose that a polynomial differential system has an algebraic solution $f(x,y) = 0$ where $f(x,y) \in \mathbb{C}[x,y]$ is of degree n given by

$$f(x,y) = c_0 + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + \cdots + c_{n0}x^n + c_{n-1,1}x^{n-1}y + \cdots + c_{0n}y^n,$$

with $\hat{c} = (c_0, c_{10}, \ldots, c_{0n}) \in \mathbb{C}^N$ where $N = (n+1)(n+2)/2$. We note that the equation

$$\lambda f(x,y) = 0, \qquad \lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$$

yields the same locus of complex points in the plane as the locus induced by $f(x,y) = 0$. Therefore, a curve of degree n is defined by $\hat{c}$ where

$$[\hat{c}] = [c_0 : c_{10} : \cdots : c_{0n}] \in P_{N-1}(\mathbb{C}).$$

We say that a sequence of curves $f_i(x,y) = 0$, each one of degree n, *converges* to a curve $f(x,y) = 0$ if and only if the sequence of points $[c_i] = [c_{i0} : c_{i10} : \cdots : c_{i0n}]$ converges to $[\hat{c}] = [c_0 : c_{10} : \cdots : c_{0n}]$ in the topology of $P_{N-1}(\mathbb{C})$.

We observe that if we rescale the time $t' = \lambda t$ by a positive constant λ the geometry of the systems (1.1) (phase curves) does not change. So for our purposes we can identify a system (1.1) of degree n with a point

$$[a_0 : a_{10} : \cdots : a_{0n} : b_0 : b_{10} : \cdots : b_{0n}] \in \mathbb{S}^{N-1}(\mathbb{R})$$

where $N = (n+1)(n+2)$.

Definition 2.23 ([64]).

(1) We say that an invariant curve

$$\mathcal{L}: f(x,y)=0, \qquad f\in\mathbb{C}[x,y]$$

for a polynomial system (S) of degree n has geometric multiplicity m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}(\mathbb{R})$ where $N=(n+1)(n+2)$ such that each (S_k) has m distinct invariant curves

$$\mathcal{L}_{1,k}: f_{1,k}(x,y)=0,\ldots,\mathcal{L}_{m,k}: f_{m,k}(x,y)=0$$

over $\mathbb{C}$, $\deg(f)=\deg(f_{i,k})=r$, converging to $\mathcal{L}$ as $k\to\infty$, in the topology of $P_{R-1}(\mathbb{C})$, with $R=(r+1)(r+2)/2$ and this does not occur for $m+1$.

(2) We say that the line at infinity

$$\mathcal{L}_\infty: Z=0$$

of a polynomial system (S) of degree n has geometric multiplicity m if there exists a sequence of real polynomial systems (S_k) of degree n converging to (S) in the topology of $\mathbb{S}^{N-1}(\mathbb{R})$ where $N=(n+1)(n+2)$ such that each (S_k) has $m-1$ distinct invariant lines

$$\mathcal{L}_{1,k}: f_{1,k}(x,y)=0,\ldots,\mathcal{L}_{m-1,k}: f_{m-1,k}(x,y)=0$$

over $\mathbb{C}$, converging to the line at infinity $\mathcal{L}_\infty$ as $k\to\infty$, in the topology of $P_2(\mathbb{C})$ and this does not occur for m.

In 2007 the authors of [18] introduced the following notion of geometric multiplicity:

Definition 2.24 ([18]). Consider χ a polynomial vector field of degree d. An invariant algebraic curve $f=0$ of degree n of the vector field χ has geometric multiplicity m if m is the largest integer for which there exists a sequence of vector fields $(\chi_i)_{i>0}$ of bounded degree, converging to $h\chi$, for some polynomial h, not divisible by f, such that each χ_r has m distinct invariant algebraic curves, $f_{r,1}=0, f_{r,2}=0,\ldots,f_{r,m}=0$, of degree at most n, which converge to $f=0$ as r goes to infinity. If $h=1$, then we say that the curve has strong geometric multiplicity m.

Definition 2.25 ([18,49]). Let $\mathbb{C}_m[x,y]$ be the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x,y]$ of degree at most m and of dimension $R=(m+1)(m+2)/2$. Let $\{v_1,v_2,\ldots,v_R\}$ be a base of $\mathbb{C}_m[x,y]$. We denote by $M_R(m)$ the $R\times R$ matrix

$$M_R(m)=\begin{pmatrix} v_1 & v_2 & \cdots & v_R\\ \chi(v_1) & \chi(v_2) & \cdots & \chi(v_R)\\ \vdots & \vdots & \ddots & \vdots\\ \chi^{R-1}(v_1) & \chi^{R-1}(v_2) & \cdots & \chi^{R-1}(v_R)\end{pmatrix}, \tag{2.6}$$

where $\chi^{k+1}(v_i)=\chi(\chi^k(v_i))$. The mth extactic curve of χ, $\mathcal{E}_m(\chi)$, is given by the equation $\det M_R(m)=0$. We also call $\mathcal{E}_m(\chi)$ the mth extactic polynomial.

From the properties of the determinant we note that the extactic curve is independent of the choice of the base of $\mathbb{C}_m[x,y]$.

Theorem 2.26 ([49]). *Consider a planar vector field* (1.2). *We have* $\mathcal{E}_m(\chi)=0$ *and* $\mathcal{E}_{m-1}(\chi)\neq 0$ *if and only if* χ *admits a rational first integral of exact degree* m.

Observe that if $f = 0$ is an invariant algebraic curve of degree m of χ, then f divides $\mathcal{E}_m(\chi)$. This is due to the fact that if f is a member of a base of $\mathbb{C}_m[x, y]$, then f divides the whole column in which f is located.

Definition 2.27 ([18])**.** We say that an invariant algebraic curve $f = 0$ of degree $m \geq 1$ has algebraic multiplicity k if $\det M_R(m) \neq 0$ and k is the maximum positive integer such that f^k divides $\det M_R(m)$; and it has no defined algebraic multiplicity if $\det M_R(m) \equiv 0$.

Definition 2.28 ([18])**.** We say that an invariant algebraic curve $f = 0$ of degree $m \geq 1$ has integrable multiplicity k with respect to χ if k is the largest integer for which the following is true: there are $k - 1$ exponential factors $\exp(g_j/f^j)$, $j = 1, \ldots, k-1$, with $\deg(g_j) \leq j_m$, such that each g_j is not a multiple of f.

In the next result we see that the algebraic and integrable multiplicity coincide if $f = 0$ is an irreducible invariant algebraic curve.

Theorem 2.29 ([18])**.** *Consider an algebraic solution $f = 0$ of degree $m \geq 1$ of χ. Then f has algebraic multiplicity k if and only if the vector field* (1.2) *has $k - 1$ exponential factors* $\exp(g_j/f^j)$, *where $(g_j, f) = 1$ and g_j is a polynomial of degree at most jm, for $j = 1, \ldots, k-1$.*

In 2007 Christopher, Llibre and Pereira showed in [18] that the definitions of geometric (see Definition 2.24), algebraic and integrable multiplicity are equivalent when $f = 0$ is an algebraic solution of the vector field (1.2). The algebraic multiplicity has the advantage that we have the possibility of calculating it via the extactic curve and if the curve is irreducible then this coincides with either the integrable (reflected in the exponential factors) or the geometric one. Christopher, Llibre and Pereira also stated and proved the following theorem about Darboux theory of integrability that takes into account the multiplicity of the invariant algebraic curves.

Theorem 2.30 ([18], see Theorem 8.3.)**.** *Consider a planar vector field* (1.2)*. Assume that* (1.2) *has p distinct irreducible invariant algebraic curves $f_i = 0$, $i = 1, \ldots, p$ of multiplicity m_i, and let $N = \sum_{i=1}^{p} m_i$. Suppose, furthermore, that there are q critical points $p_1, \ldots, p_q$ which are independent with respect to $\mathbb{C}_{m-1}[x, y]$, and $f_j(p_k) \neq 0$ for $j = 1, \ldots, p$ and $k = 1, \ldots, q$. We have:*

(a) If $N + q \geq [m(m+1)/2] + 2$, then χ has a rational first integral.

(b) If $N + q \geq [m(m+1)/2] + 1$, then χ has a Darboux first integral.

(c) If $N + q \geq [m(m+1)/2]$ and p_i's are weak, then χ has either a Darboux first integral or a Darboux integrating factor.

This theorem was generalized by Llibre and Zhang in [42] for invariant hypersurfaces in $\mathbb{C}^n$. In the same paper they also generalized the theorem of Jouanolou and gave a simplified, elementary proof.

The term of *total multiplicity of invariant curves, finite and infinite*, of a polynomial differential system was used for the first time in the theory of Darboux by Schlomiuk and Vulpe in [64], in the specific context of invariant straight lines of quadratic differential systems. In [18] the total multiplicity N of the finite number of *affine (finite) invariant algebraic curves* appeared for the first time in the general context of the theory of Darboux in the above quoted theorem. This number is clearly not the total multiplicity of invariant algebraic curves of the system as the line at infinity is invariant and could have multiplicity (for examples see [64, 68]).

The total multiplicity of the invariant algebraic curves finite and infinite occurs for the first time in the general setting in the work of Llibre and Zhang (2009) but only for invariant hyper-surfaces of polynomial vector fields in $\mathbb{R}^n$ and to this day we do not have the analog of this theorem for multiple invariant hypersurfaces, both finite and the hyper plane at infinity.

We consider now the result of Llibre and Zhang in [43]. To state it the authors generalized the Poincaré compactification on the sphere for planar differential systems to the Poincaré compactification of polynomial differential systems in $\mathbb{R}^n$ which they constructed in the Appendix of [43].

To talk about multiplicity of the hyperplane at infinity they only needed to pass by central projection from the systems in $\mathbb{R}^n$, considered as the hyperplane $Z = 1$ in $\mathbb{R}^{n+1}$ tangent to the n-sphere with radius 1 centered at the origin of $\mathbb{R}^{n+1}$, and then further into the chart $x_1 = 1$ and obtain $(x_1, \ldots, x_n, 1) = \lambda(1, y_2, \ldots, y_n, Z)$ for some non-zero real λ. Hence we must have $\lambda = x_1$ and therefore $y_2 = x_2/x_1, \ldots, y_n = x_n/x_1$, $Z = 1/x_1$ and $x_1 = 1/Z$, $x_2 = y_2/Z, \ldots, x_n = y_n/Z$. Transferring the vector field in this chart we obtain that it has a pole on $Z = 0$. In complete analogy with the compactification of the plane we can obtain an analytic vector field on the n-sphere which is conjugate to the vector field thus obtained. In this way our initial hyper-surface at infinity, becomes just an affine hypersurface in the chart $x_1 = 1$ and hence we can apply to it our notions of multiplicity. Let $\overline{\chi} = (P_1(x), P_2(x), \ldots, P_n(x))$ be the expression of the compactified vector field χ. We say that the infinity of χ has algebraic multiplicity k if $Z = 0$ has algebraic multiplicity k for the vector field $\overline{\chi}$; and that it has no defined algebraic multiplicity if $Z = 0$ has no defined algebraic multiplicity for χ. One thing the authors did not say is that this definition of the multiplicity of the infinite hypersurface does not depend on the chart x_1 we chose, and that it leads to the same value if we replace this chart by any other chart $x_i = 1$ with $i \neq 1$.

Theorem 2.31 ([43]). *Let $\overline{\chi}$ be the expression of the compactified vector field χ. Assume that χ restricted to $Z = 0$ has no rational first integral. Then $Z = 0$ has algebraic multiplicity k for $\overline{\chi}$ if and only if $\overline{\chi}$ has $k-1$ exponential factors $\exp(\overline{g_j}/Z^j)$ where $j = 1, \ldots, k-1$ with $\overline{g_j} \in \mathbb{C}_j[Z, y_2, \ldots, y_n]$ having no factor Z.*

The next result provides a relation between the exponential factors of χ and those of $\overline{\chi}$ associated with $Z = 0$.

Proposition 2.32 ([43]). *For the exponential factors associated with the hyperplane at infinity the following statements hold.*

(a) *If $E = \exp(g(x))$ with g a polynomial of degree k is an exponential factor of χ with cofactor $L_E(x)$, then $\overline{E} = \exp\left(\frac{\overline{g}}{Z^k}\right)$ with $\overline{g} = Z^k g\left(\frac{1}{Z}, \frac{y_2}{Z}, \ldots, \frac{y_n}{Z}\right)$ is an exponential factor of $\overline{\chi}$ with cofactor $L_{\overline{E}} = Z^{d-1} L_E\left(\frac{1}{Z}, \frac{y_2}{Z}, \ldots, \frac{y_n}{Z}\right)$.*

(b) *Conversely if $\overline{F} = \exp\left(\frac{\overline{h}}{Z^k}\right)$ with $\overline{h} \in \mathbb{R}_k[Z, y_2, \ldots, y_n]$ is an exponential factor of $\overline{\chi}$ with cofactor $L_{\overline{F}}$, then $F = \exp(h(x))$ with $h(x) = x^k \overline{h}\left(\frac{1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)$ is an exponential factor of χ with cofactor $L_F = x^{d-1} L_{\overline{F}}\left(\frac{1}{x_1}, \frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1}\right)$.*

The following result was proved in 2009 by Llibre and Zhang.

Theorem 2.33 ([43]). *Assume that the polynomial vector field χ in $\mathbb{R}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ for $i = 1, \ldots, p$ and the invariant hyperplane at infinity.*

(i) *If one of these irreducible invariant algebraic hypersurfaces or the invariant hyperplane at infinity has no defined algebraic multiplicity, then the vector field χ has a rational first integral.*

(ii) Suppose that each irreducible invariant algebraic hypersurfaces $f_i = 0$ has algebraic multiplicity m_i for $i = 1, \dots, p$ and that the invariant hyperplane at infinity has algebraic multiplicity k. If the vector field restricted to the hyperplane at infinity or to any invariant hypersurface with multiplicity larger than 1 has no rational first integral, then the following hold

(a) If $\sum_{i=1}^{p} m_i + k = N + 2$, then the vector field χ has a real Darboux first integral, where $N = \binom{n+d-1}{n}$.

(b) If $\sum_{i=1}^{p} m_i + k = N + n + 1$, then (1.2) has a real rational first integral.

We remark that by "real Darboux first integral" in Theorem 2.33 the authors mean generalized Darboux first integral. For two-dimensional polynomial vector fields, the additional condition in Theorem 2.33 on the nonexistence of rational first integrals of the vector field restricted to the invariant algebraic curves including the line at infinity is not necessary.

We end our conceptual and historical survey with some comments about this result over the reals. Darboux constructed his theory over the complex projective space which we think is the natural field and natural space for this theory. Firstly the complex numbers form an algebraically closed field. So an essential ingredient in his theory, the theory of algebraic curves, can be properly done. Indeed, Bézout's theorem cannot be proved over the reals. Secondly the complex projective plane is a compact space and in particular "the line at infinity" of the affine plane completely looses its special status in the projective plane. It is like any other line.

On the other hand it is important to observe that when we consider the theory of Darboux for real systems, we can go to their complexification and these systems could have complex invariant algebraic curves $f(x, y) = 0$ with $f \in \mathbb{C}[x, y]$. We can therefore end up with more invariant curves than those with real coefficients. Let us consider an example.

Example 2.34.

$$\begin{cases} \dot{x} = x^2 + 1, \\ \dot{y} = x + y. \end{cases}$$

This system clearly has two invariant lines which are complex $x \pm i = 0$ with respective co-factors $x \mp i$. It can easily be checked that the line at infinity has the multiplicity two. So the total multiplicity of invariant lines over $\mathbb{C}$ is four. This system was proved to be integrable in [66] having the inverse Darboux integrating factor $(x + i)^{1+i/2}(x - i)^{1-i/2}$.

Let us now consider this real system without taking into consideration its complexification. Suppose now that we want to prove just by using real curves that the system is integrable. The lines $x \pm iy = 0$ are defined over $\mathbb{C}$ and it is only their union, the conic $x^2 + 1 = 0$ which is defined over $\mathbb{R}$. This is an invariant curve of the real system with the cofactor $2x$. We also have an exponential factor e^{1+2y} with co-factor $2(x + y)$. However this is insufficient for proving integrability as we can check by trying to apply the usual algorithm for computing an integrating factor. Indeed, given $\lambda_1, \lambda_2 \in \mathbb{C}$ we have $2\lambda_1 x + 2\lambda_2(x + y) = -\mathrm{div}(P, Q) = -2x - 1$, which has no solution. So although the real system is integrable and has a real first integral, we cannot compute this real first integral without considering the two invariant lines. So this supports the idea that the full real extension of the Darboux theory that also covers the line at infinity with its own multiplicity cannot produce all the real integrable systems.

In conclusion we really need an extension of the Darboux theory over $\mathbb{C}$ that includes the multiplicity of the line at infinity and work in this direction is in progress.

3 Applications of Darboux's theory to the family QSH of quadratic differential systems with an invariant hyperbola, case $\eta = 0$

The notion of *configuration of invariant curves* of a polynomial differential system appears in several works, see for instance [64].

Definition 3.1 ([64]). Consider a real planar polynomial system (1.1) with a finite number of singular points. By *configuration of algebraic solutions* of the system we mean a set of algebraic solutions over $\mathbb{C}$ of the system, each one of these curves endowed with its own multiplicity and together with all the real singular points of this system located on these curves, each one of these singularities endowed with its own multiplicity.

The notion of equivalence of configurations was used in many papers (see for instance, [47, 64–68]) to classify systems in **QS** possessing invariant algebraic curves according to the kind of configurations these systems.

In particular, in [47], **QSH** was classified according to the configuration of invariant hyperbolas and lines the systems possess. The equivalence of configurations in the class **QSH** depends on whether the systems admits a finite or an infinite number of invariant hyperbolas. See Definition 1.9 of [47] in case the system has a finite number of invariant hyperbolas and Definition 1.10 of [47] for the case the system has an infinite family of invariant hyperbolas. The classification of **QSH** led to 205 distinct configurations.

Here we introduce some invariant polynomials that play an important role in the study of polynomial vector fields. Considering $C_2(\tilde{a}, x, y) = yp_2(\tilde{a}, x, y) - xq_2(\tilde{a}, x, y)$ as a cubic binary form of x and y we calculate

$$\eta(\tilde{a}) = \text{Discrim}[C_2, \xi], \qquad M(\tilde{a}, x, y) = \text{Hessian}[C_2],$$

where $\xi = y/x$ or $\xi = x/y$. It is known that the singular points at infinity of quadratic systems are given by the solutions in x and y of $C_2(\tilde{a}, x, y) = 0$. If $\eta < 0$ then this means we have one real singular point at infinity and two complex ones.

Remark 3.2. We note that since a system in **QSH** *always has an invariant hyperbola then clearly we always have at least* 2 *real singular points at infinity. So we must have* $\eta \geq 0$.

The family **QSH** can be split as follows: $\mathbf{QSH}_{\eta=0}$ of systems which possess either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities and $\mathbf{QSH}_{\eta>0}$ of systems which possess three distinct real singularities at infinity in $P_2(\mathbb{C})$. In this paper we present a study of $\mathbf{QSH}_{\eta=0}$.

In [47] the authors gave necessary and sufficient conditions for a quadratic system to have an invariant hyperbola. These conditions were given in terms of 40 affine invariant polynomials and hence these conditions are independent of the normal forms in which the systems may be presented. For the sake of completeness, we give below in the following tables these conditions for $\mathbf{QSH}_{\eta=0}$.

In the next table we present in the first column the number associated to the equations in [47], which are the normal forms for the systems in **QSH**. In the second column are the necessary and sufficient conditions. For a proof see [47].

Equations in [47]	Invariants
(4.4)	$\eta=0, M\neq 0, \theta\neq 0, \beta_2\neq 0, \beta_1\neq 0, \mathcal{R}_1\neq 0, B_1\neq 0$
(4.10)	$\eta=0, M\neq 0, \theta\neq 0, \beta_2\neq 0, \beta_1\neq 0, \mathcal{R}_1\neq 0, B_1=0$
(4.11)	$\eta=0, M\neq 0, \theta\neq 0, \beta_2\neq 0, \beta_1=0, \gamma_1=0, \mathcal{R}_3\neq 0$
(4.13)	$\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathcal{R}_{10}\neq 0$
(4.13) $g=1/4$	$\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathcal{R}_{10}\neq 0, \beta_7\beta_8=0, \mathcal{R}_{10}<0, \beta_8=0$ $\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathcal{R}_{10}\neq 0, \beta_7\beta_8=0, \mathcal{R}_{10}>0, \beta_8=0$
(4.13) $g=1/2$	$\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathcal{R}_{10}\neq 0, \beta_7\beta_8=0, \mathcal{R}_{10}<0, \beta_7=0$ $\eta=0, M\neq 0, \theta\neq 0, \beta_2=0, \beta_1=\gamma_{14}=0, \mathcal{R}_{10}\neq 0, \beta_7\beta_8=0, \mathcal{R}_{10}>0, \beta_7=0$
(4.16)	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0\neq 0, \beta_3=\gamma_8=0, \mathcal{R}_7\neq 0, \chi_A^{(7)}<0$ $\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0\neq 0, \beta_3=\gamma_8=0, \mathcal{R}_7\neq 0, \chi_A^{(7)}>0$
(4.16) $c^2=a$	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0\neq 0, \beta_3=\gamma_8=0, \mathcal{R}_7\neq 0, \chi_A^{(7)}=0$
(4.18)	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}\neq 0, \mu_2\neq 0$
(4.18) $g=0$	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}\neq 0, \mu_2=0, \gamma_{16}\neq 0$
(4.18) $c=0$	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}\neq 0, \mu_2=0, \gamma_{16}=0$
(4.22)	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}=0, \gamma_{17}<0$ $\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}=0, \gamma_{17}>0$
(4.22) $\epsilon=0$	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N\neq 0, (\mathfrak{C}_1), \beta_{12}=0, \gamma_{17}=0$
(4.25) $c\neq 0$	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N=0, \beta_{13}\neq 0, \gamma_{10}=\gamma_{17}=0, \mathcal{R}_{11}\neq 0, \gamma_{16}\neq 0$
(4.25) $c=0$	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N=0, \beta_{13}\neq 0, \gamma_{10}=\gamma_{17}=0, \mathcal{R}_{11}\neq 0, \gamma_{16}=0$
(4.27)	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N=0, \beta_{13}=0, \tilde{\gamma}_{18}=\tilde{\gamma}_{19}=0, \mu_2\neq 0$
(4.28)	$\eta=0, \mathcal{A}_1\ [M\neq 0, \theta=0], \mu_0=0, N=0, \beta_{13}=0, \tilde{\gamma}_{18}=\tilde{\gamma}_{19}=0, \mu_2=0$
(4.30)	$\eta=0, \mathcal{A}_2\ [C_2=0, M=0], N_7=0, H_{10}\neq 0, H_9<0$ $\eta=0, \mathcal{A}_2\ [C_2=0, M=0], N_7=0, H_{10}\neq 0, H_9>0$
(4.31)	$\eta=0, \mathcal{A}_2\ [C_2=0, M=0], N_7=0, H_{10}\neq 0, H_9=0$
(4.34)	$\eta=0, \mathcal{A}_2\ [C_2=0, M=0], N_7=0, H_{10}=0, H_{12}\neq 0, H_2\neq 0$
(4.36)	$\eta=0, \mathcal{A}_2\ [C_2=0, M=0], N_7=0, H_{10}=0, H_{12}\neq 0, H_2=0$
(4.38)	$\eta=0, \mathcal{A}_2\ [C_2=0, M=0], N_7=0, H_{10}=0, H_{12}=0$

In this table we denote by (4.25) the following system that appears in [47] without number

$$\dot{x}=-\tfrac{3c^2}{16}+cx-x^2, \qquad \dot{y}=1-2xy.$$

If $c\neq 0$ we may assume $c=4$ by the rescaling $(x,y,t)\mapsto(cx/4, 4y/c, 4t/c)$. So we obtain the system denoted by (4.25) in [47] which we denote here by (4.25) $c\neq 0$.

The normal forms numbered in the table from (4.4) and up to (4.38), that were obtained in [47], appear below in a condensed table in the following proposition.

Proposition 3.3 ([47]). *Every system in $\boldsymbol{QSH}_{\eta=0}$ can be brought by an affine transformation and time rescaling to one of the following 13 normal forms, where a, g, c, ϵ are real parameters. Next to each normal forms we present the respective invariant hyperbola.*

$$\begin{cases}\dot{x}=2a+x+gx^2+xy,\\ \dot{y}=a(2g-1)-y+(g-1)xy+y^2,\end{cases} \qquad \Phi(x,y)=a+xy \qquad \text{(A)}$$
where $a(g-1)\neq 0$

$$\begin{cases} \dot{x} = 2a + gx^2 + xy, \\ \dot{y} = a(2g-1) + (g-1)xy + y^2, \end{cases} \qquad \Phi(x,y) = a + xy \tag{B}$$
where $a(g-1) \neq 0$

$$\begin{cases} \dot{x} = 2a + 3cx + x^2 + xy \\ \dot{y} = a - c^2 + y^2, \end{cases} \qquad \Phi(x,y) = a + cx + xy \tag{C}$$
where $a \neq 0$

$$\begin{cases} \dot{x} = (c+x)(c(2g-1) + gx) \\ \dot{y} = 1 + (g-1)xy, \end{cases} \qquad \Phi(x,y) = \frac{1}{(-1+2g)} + cy + xy \tag{D}$$
where $(g \pm 1)(3g-1)(2g-1) \neq 0$

$$\begin{cases} \dot{x} = x^2 + \epsilon \\ \dot{y} = 1 - 2xy \end{cases} \qquad \Phi_{1,2}(x,y) = -1 \pm i\sqrt{\epsilon}y + xy \tag{E}$$

$$\begin{cases} \dot{x} = (x-1)(3-x) \\ \dot{y} = 1 - 2xy \end{cases} \qquad \Phi(x,y) = \tfrac{1}{3} + y - xy \tag{F}$$

$$\begin{cases} \dot{x} = -x^2 \\ \dot{y} = 1 - 2xy \end{cases} \qquad \Phi(x,y) = -1 + 3xy \tag{G}$$

$$\begin{cases} \dot{x} = (2x-1)(2x+1)/4 \\ \dot{y} = y \end{cases} \qquad \Phi(x,y) = -\tfrac{q}{2} + qx + \tfrac{y}{2} + 2xy,\ q \neq 0 \tag{H}$$

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = 1 \end{cases} \qquad \Phi(x,y) = 1 + rx + xy \tag{I}$$

$$\begin{cases} \dot{x} = a + y + x^2 \\ \dot{y} = xy \end{cases} \qquad \Phi(x,y) = a + 2y + x^2 - m^2y^2 \tag{J}$$

$$\begin{cases} \dot{x} = (1+3x)(2+3x)/9 \\ \dot{y} = xy \end{cases} \qquad \Phi(x,y) = 4 + 12x + 9x^2 + my + 3mxy \tag{K}$$

$$\begin{cases} \dot{x} = a + x^2 \\ \dot{y} = xy, \end{cases} \qquad \Phi(x,y) = a + x^2 - m^2xy \tag{L}$$
where $a \neq 0$

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = 1 + xy \end{cases} \qquad \Phi(x,y) = 1 + mx^2 + 2xy. \tag{M}$$

Using the invariants described previously which are powerful so as to give the necessary and sufficient conditions for systems (1.3) to have an invariant hyperbola, in [47] the authors considered the two possibilities: $M(\tilde{a}, x, y) \neq 0$ (i.e. at infinity we have two distinct real singularities) and $M = 0 = C_2$ (when we have an infinite number of singularities at infinity).

(i) $M(\tilde{a}, x, y) \neq 0$: This brings systems (1.3) to the systems

$$\begin{cases} \dot{x} = a + cx + dy + gx^2 + hxy, \\ \dot{y} = b + ex + fy + (g-1)xy + hy^2. \end{cases} \tag{3.1}$$

with invariants $C_2(x, y) = x^2 y$ and $\theta = -h^2(g-1)/2$.

(i.1) The case $\theta \neq 0$ gives the condition $h(g-1) \neq 0$ for (3.1) and via the bifurcation diagram in [47] we arrive at the normal forms

- (A) where $a(g-1) \neq 0$,
- (B) where $a(g-1) \neq 0$.

(i.2) The case $\theta = 0$ gives the condition $h(g-1) = 0$ for (3.1) and via the bifurcation diagram in [47] we arrive at another invariant $\mu_0 = gh^2$.

(i.2.1) For $\mu_0 \neq 0$ we have the normal form

- (C) where $a \neq 0$.

(i.2.2) For $\mu_0 = 0$ they calculated the invariant $N = 9(g-1)(g+1)x^2$ and we need to consider two possibilities.

(i.2.2.1) For the case $N \neq 0$ we have the normal forms

- (D) where $(g-1)(g+1)(2g-1)(3g-1) \neq 0$,
- (E) where $\epsilon \neq 0$.

(i.2.2.2) The case $N = 0$ gives the condition $(g-1)(g+1) = 0$ and we have the normal forms

- (F),
- (G),
- (H),
- (I).

(ii) $M(\tilde{a}, x, y) = 0 = C_2$: We arrive at the normal forms

- (J),
- (K),
- (L) where $a \neq 0$,
- (M).

Remark 3.4. The invariant hyperbolas involve:

(i) sometimes all the parameters of the system (such as (C));

(ii) sometimes only some parameters (such as (A)) and

(iii) sometimes additional parameters (such as (J)).

The next theorem is the main result of this paper.
Consider the following sets:

$$\begin{aligned}
&L_1 = \cup_{k\in\mathbb{N}} L_{1,k}, \text{ where } L_{1,k} = \left\{(a,g) \in \mathbb{R}^2 : g = k/2 \text{ and } a \neq 0\right\}, \ k \in \mathbb{N}, \\
&L_2 = \cup_{k\in\mathbb{N}} L_{2,k}, \text{ where } L_{2,k} = \left\{(a,g) \in \mathbb{R}^2 : g = k/3 \text{ and } a \neq 0\right\}, \ k \in \mathbb{N}, \\
&L_3 = \left\{(a,g) \in \mathbb{R}^2 : g = 1/4 \text{ and } a \neq 0\right\}, \\
&C' = \cup_{k\in\mathbb{N}} C_k, \text{ where } C_k = \left\{(a,g) \in \mathbb{R}^2 : g = (2 + a - 2ak)/4a \text{ and } a \neq 0\right\}, \ k \in \mathbb{N}.
\end{aligned}$$

Main Theorem. *Consider the polynomial systems in* $\boldsymbol{QSH}_{\eta=0}$.

(a) The 11 normal forms (C)–(M) *are all Liouvillian integrable. The following table sums up the results regarding the types of integrability:*

Systems	*Parameters*	*Type of first integral*
(C)	$a = 8c^2/9$ *and* $c \neq 0$	*Generalized Darboux*
(C)	$a(a - 8c^2/9) \neq 0$	*Liouvillian*
(D)	$g(g \pm 1)(2g - 1)(3g - 1) \neq 0$	*Darboux*
(D)	$g = 0$ *and* $c \neq 0$	*Generalized Darboux*
(E)	$\epsilon \in \mathbb{R}$	*Polynomial (hamiltonian)*
(F)	-	*Rational*
(G)	-	*Rational*
(H)	-	*Rational*
(I)	-	*Rational*
(J)	$a \in \mathbb{R}$	*Rational*
(K)	-	*Rational*
(L)	$a \neq 0$	*Rational*
(M)	-	*Rational*

(b) For the normal forms (A) *and* (B) *we have the following:*

(i) If $(a, g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$, *then systems* (A) *are not Liouvillian integrable.*

(i.1) If $(a, g) \in \mathrm{L}_{1,1}$ *then systems* (A) *are not Liouvillian integrable.*

(ii) If $(a, g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_3)$, *then systems* (B) *are not Liouvillian integrable.*

(ii.1) If $(a, g) \in \mathrm{L}_{1,1}$ *then systems* (B) *are generalized Darboux integrable.*
(ii.2) If $(a, g) \in \mathrm{L}_3$ *then systems* (B) *are Liouvillian integrable.*

The following table sums up the results regarding the types of integrability:

Systems	*Parameters*	*Type of first integral*
(A)	$(a, g) \in \mathbb{R} - (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$	*Not Liouvillian integrable*
(B)	$g = 1/2$ *and* $a \neq 0$	*Generalized Darboux*
(B)	$g = 1/4$ *and* $a \neq 0$	*Liouvillian*
(B)	$(a, g) \in \mathbb{R} - (\mathrm{L}_1 \cup \mathrm{L}_3)$	*Not Liouvillian integrable*

Observation 3.5. *The Liouvillian integrability of any system in class* (A) *with* $(a, g) \in (\mathrm{L}_1 - \mathrm{L}_{1,1}) \cup \mathrm{L}_2 \cup \mathrm{C}'$ *or in class* (B) *with* $(a, g) \in (\mathrm{L}_1 - \mathrm{L}_{1,1})$ *is still open. The reason is that the methods applied in this paper for proving the existence or non-existence of a Liouvillian first integral do not work in these cases and so new ideas are needed for proving or disproving their Liouvillian integrability.*

For a proof, see Section 5 (integrable cases) and Section 6 (non integrable cases).

4 Geometrical concepts and results useful for studying the geometry of the configurations of invariant curves and their bifurcations

Remark 4.1. In the theory of Darboux presented in the preceding section what counts is mainly the number of invariant curves, their multiplicities, the number of independent points. When certain inequalities involving these numbers are satisfied then we have integrability of the system.

However in 1993 Christopher and Kooij stated a theorem in [13] where, if we reformulate the theorem in geometric terms, we see a beautiful relation between the geometry of the "configuration of invariant curves" and their Darboux integrability. This theorem was proved in [17].

Theorem 4.2 ([13])**.** *Consider a polynomial system* (1.1) *that has k algebraic solutions* $C_i = 0$ *such that*

(a) all curves $C_i = 0$ *are non-singular and have no repeated factor in their highest order terms,*

(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,

(c) no two curves have a common factor in their highest order terms,

(d) the sum of the degrees of the curves is $n + 1$*, where n is the degree of system* (1.1)*.*

Then system (1.1) *has an integrating factor*

$$\mu(x, y) = 1/(C_1 C_2 \dots C_k).$$

This theorem has a geometric content which is not completely explicit in the algebraic way they stated the result. We rewrite the theorem above in geometric terms as follows:

Theorem 4.3. *Consider a polynomial system* (1.1) *that has k algebraic solutions* $C_i = 0$ *such that*

(a) all curves $C_i = 0$ *are non-singular and they intersect transversally the line at infinity* $Z = 0$*,*

(b) no more than two curves meet at any point in the finite plane and are not tangent at these points,

(c) no two curves intersect at a point on the line at infinity $Z = 0$*,*

(d) the sum of the degrees of the curves is $n + 1$*, where n is the degree of system* (1.1)*.*

Then system (1.1) *has an integrating factor*

$$\mu(x, y) = 1/(C_1 C_2 \dots C_k).$$

In the hypotheses of this theorem the way the curves are placed with respect to one another in the totality of the curves, in other words the "geometry of the configuration of invariant algebraic curves" has an impact of the kind of integrating factor we could have.

We are interested in relating the geometry of the invariant algebraic curves curves taken in their totality with the various kinds of integrability. To begin doing this we need to recall some concepts and in particular those introduced by Poincaré in [52]. Among them we have the following.

Let $H = f/g$ be a rational first integral of the polynomial vector field (1.2). We say that H has degree n if n is the maximum of the degrees of f and g. We say that the degree of H

is *minimal* among all the degrees of the rational first integrals of χ if any other rational first integral of χ has a degree greater than or equal to n. Let $H = f/g$ be a rational first integral of χ. According to Poincaré [52] we say that $c \in \mathbb{C} \cup \{\infty\}$ is a *remarkable value* of H if $f + cg$ is a reducible polynomial in $\mathbb{C}[x, y]$. Here, if $c = \infty$, then $f + cg$ denotes g. Note that for all $c \in \mathbb{C}$ the algebraic curve $f + cg = 0$ is invariant. The curves in the factorization of $f + cg$, when c is a remarkable value, are called *remarkable curves*.

Now suppose that c is a remarkable value of a rational first integral H and that $u_1^{\alpha_1} \dots u_r^{\alpha}$ is the factorization of the polynomial $f + cg$ into reducible factors in $\mathbb{C}[x, y]$. If at least one of the α_i is larger than 1 then we say, following again Poincaré (see for instance [28]), that c is a *critical remarkable value* of H, and that $u_i = 0$ having $\alpha_i > 1$ is a *critical remarkable curve* of the vector field (1.2) with exponent α_i.

Since we can think of $\mathbb{C} \cup \{\infty\}$ as the projective line $P_1(\mathbb{R})$ we can also use the following definition.

Definition 4.4. Consider $\mathcal{F}_{(c_1,c_2)} : c_1 f - c_2 g = 0$ where f/g is a rational first integral of (1.2). We say that $[c_1 : c_2]$ is a remarkable value of the curve $\mathcal{F}_{(c_1,c_2)}$ if $\mathcal{F}_{(c_1,c_2)}$ is reducible over $\mathbb{C}$.

It was proved in [9] that there are finitely many remarkable values for a given rational first integral H and if (1.2) has a rational first integral and has no polynomial first integrals, then it has a polynomial inverse integrating factor if and only if the first integral has at most two critical remarkable values.

Given $H = f/g$ a rational first integral, consider $F_{(c_1,c_2)} = c_1 f - c_2 g$ where $\deg F_{(c_1,c_2)} = n$. If $F_{(c_1,c_2)} = f_1 f_2$ where $f_1, f_2 \in \mathbb{C}[x, y]$ and $\deg f_i = n_i < n$ then necessarily the points on the intersection of $f_1 = 0$ and $f_2 = 0$ must be singular points of the curve $F_{(c_1,c_2)}$.

Lemma 4.5 ([11])**.** *Assume that system* (1.1) *with degree m has an invariant algebraic curve f of degree n. Let f_n, P_m and Q_m be the homogeneous parts of f with degree n, P and Q with degree m. Then each one of the irreducible factors of f_n divides $yP_m - xQ_m$.*

In geometric terms, this lemma means that the points at infinity of any invariant algebraic curve $f = 0$ of a system (1.1) are singularities of this system.

Let us recall the algebraic-geometric definition of an r-cycle on an irreducible algebraic variety of dimension n.

Definition 4.6. Let V be an irreducible algebraic variety of dimension n over a field $\mathbb{K}$. A cycle of dimension r or r-cycle on V is a formal sum

$$\sum_W n_W W$$

where W is a subvariety of V of dimension r which is not contained in the singular locus of V, $n_W \in \mathbb{Z}$, and only a finite number of n_W's are non-zero. We call degree of an r-cycle the sum

$$\sum_W n_W.$$

An $(n-1)$-cycle is called a divisor.

Definition 4.7. For a non-degenerate polynomial differential system (S) possessing a finite number of algebraic solutions

$$\mathcal{F} = \{f_i\}_{i=1}^m, \quad f_i(x, y) = 0, \quad f_i(x, y) \in \mathbb{C}[x, y],$$

each with multiplicity n_i and a finite number of singularities at infinity, we define the algebraic solutions divisor (also called the invariant curves divisor) on the projective plane attached to the family $\mathcal{F}$,

$$ICD_{\mathcal{F}} = \sum_{n_i} n_i \mathcal{C}_i + n_\infty \mathcal{L}_\infty$$

where $C_i : F_i(X, Y, Z) = 0$ are the projective completions of $f_i(x, y) = 0$, n_i is the multiplicity of the curve $C_i = 0$ and n_∞ is the multiplicity of the line at infinity $\mathcal{L}_\infty : Z = 0$.

Proposition 4.8 ([2]). *Every polynomial differential system of degree n and with a finite number of invariant lines has at most $3n$ invariant straight lines, including the line at infinity.*

In particular the maximum number of invariant lines for a quadratic system with a finite number of invariant lines is six. In the case we consider here, we have a particular instance of the divisor ICD because the invariant curves we consider are invariant hyperbolas and invariant lines of a quadratic differential system, in case these are in finite number. In case we have an infinite number of hyperbolas we can construct the divisor of the invariant straight lines which are always in finite number.

Another ingredient of the configuration of algebraic solutions are the real singularities situated on these curves. We also need to use here the notion of multiplicity divisor of real singularities of a system, located on the algebraic solutions of the system.

Definition 4.9.

1. Suppose a real quadratic system (1.3) has a non-empty finite set of invariant hyperbolas $\mathcal{H}_i$ and a finite number of affine invariant lines $\mathcal{L}_j$, where $\mathcal{H}_i : h_i(x, y) = 0$, $i = 1, 2, \ldots, k$, $\mathcal{L}_j : f_j(x, y) = 0$, $j = 1, 2, \ldots, l$ and $h_i, f_j \in \mathbb{C}[x, y]$.

   We denote the line at infinity $\mathcal{L}_\infty : Z = 0$ and suppose that on this line we have a finite number of singularities. The divisor of invariant hyperbolas and invariant lines on the complex projective plane of the system is the following

   $$ICD = n_1\mathcal{H}_1 + \cdots + n_k\mathcal{H}_k + m_1\mathcal{L}_1 + \cdots + m_l\mathcal{L}_l + m_\infty\mathcal{L}_\infty$$

   where n_i (respectively m_j) is the multiplicity of the hyperbola $\mathcal{H}_i$ (respectively m_j of the line $\mathcal{L}_j$), and m_∞ is the multiplicity of $\mathcal{L}_\infty$. We mark the complex (non-real) invariant hyperbolas (respectively lines) denoting them by $\mathcal{H}_i^C$ (respectively $\mathcal{L}_i^C$). We define the total multiplicity TM of the divisor as the sum $\sum_i n_i + \sum_j m_j + m_\infty$.

2. The zero-cycle on the real projective plane, of singularities of a quadratic system (1.3) located on a configuration of invariant lines and invariant hyperbolas, is given by

   $$M_{0CS} = r_1P_1 + \cdots + r_lP_l + v_1P_1^\infty + \cdots + v_nP_n^\infty$$

   where P_i (respectively P_j^∞) are all the finite (respectively infinite) real singularities of the system and r_i (respectively v_j) are their corresponding multiplicities. We mark the complex singular points denoting them by P_i^C. We define the total multiplicity TM of zero-cycles as the sum $\sum_i r_i + \sum_j v_j$.

Definition 4.10.

(1) In case we have an infinite number of hyperbolas and just two or three singular points at infinity but we have a finite number of invariant straight lines we define the invariant lines divisor as

$$ILD = m_1\mathcal{L}_1 + \cdots + m_l\mathcal{L}_l + m_\infty\mathcal{L}_\infty,$$

where m_i denotes the multiplicity of the line $\mathcal{L}_i$ and m_∞ the multiplicity of $\mathcal{L}_\infty$.

(2) In case we have an infinite number of hyperbolas, the line at infinity is filled up with singularities and we have a finite number of affine lines, we define the invariant lines divisor
$$ILD = m_1\mathcal{L}_1 + \cdots + m_l\mathcal{L}_l.$$

Definition 4.11.

(1) Suppose we have a finite number of invariant hyperbolas and invariant straight lines of a system (S) and that they are given by equations
$$f_i(x,y) = 0, \quad i \in \{1,2,\ldots,k\}, \quad f_i \in \mathbb{C}[x,y].$$
Set $F_i(X,Y,Z) = 0$ the projection completion of the invariant curves $f_i = 0$ in $P_2(\mathbb{C})$. The total invariant algebraic curve of the system (S) in QSH, on $P_2(\mathbb{R})$, is the curve
$$T(S) = \prod_i F_i(X,Y,Z)^{m_i} Z^{m_\infty} = 0,$$
where m_i is the multiplicity of $f_i = 0$, $i = 1,\ldots,k$ and m_∞ is the multiplicity of the line at infinity.

(2) Suppose that a system (S) has an infinite number of invariant hyperbola. Then the system (S) has a finite number of invariant affine straight lines (see [47]). Set $L_i(X,Y,Z) = 0$ the projective completions of the invariant lines $l_i(x,y) = 0$, $i \in \{1,2,\ldots,k\}$ in $P_2(\mathbb{C})$.

(i) If there are a finite number of singular points at infinity, the total invariant curve of system (S) is
$$T(S) = \prod_i L_i(X,Y,Z)^{m_i} Z^{m_\infty} = 0,$$
where m_i is the multiplicity of the line $l_i = 0$, $i = 1,\ldots,k$ and m_∞ is the multiplicity of the line at infinity.

(ii) If the line at infinity is filled up with singularities, the total invariant curve of system (S) is
$$T(S) = \prod_i L_i(X,Y,Z)^{m_i} = 0,$$
where m_i is the multiplicity of the line $l_i = 0$, $i = 1,\ldots,k$.

The singular points of the system (S) situated on $T(S)$ are of two kinds: those which are simple (or smooth) points of $T(S)$ and those which are multiple points of $T(S)$.

Remark 4.12. To each singular point of the system we have its associated multiplicity as a singular point of the system. In addition, when these singular points are situated on the total curve, we also have the multiplicity of these points as points on the total curve $T(S)$. Through a singular point of the systems there may pass several of the curves $F_i = 0$ and $Z = 0$. Also we may have the case when this point is a singular point of one or even of several of the curves in case we work with invariant curves with singularities. This leads to the multiplicity of the point as point of the curve $T(S)$. The simple points of the curve $T(S)$ are those of multiplicity one. They are also the smooth points of this curve.

Definition 4.13.

(i) Suppose a quadratic system (S) has a finite number of singularities finite or infinite. The zero-cycle of singularities of the total curve $T(S)$ of system (S) is given by

$$M_{0CT} = r_1P_1 + \cdots + r_lP_l + v_1P_1^\infty + \cdots + v_nP_n^\infty$$

where P_i (respectively P_j^∞) are all the finite (respectively infinite) singularities situated on $T(S)$ and r_i (respectively v_j) are their corresponding multiplicities as points on the total curve $T(S)$. We mark the complex singular points denoting them by P_i^C. We define the total multiplicity TM of the zero-cycle M_{0CT} as the sum $\sum_i r_i + \sum_j v_j$.

(ii) Suppose a system (S) possessess the line at infinity filled up with singularities. The zero-cycle of the total curve $T(S)$ of system (S) is given by

$$M_{0CT} = r_1P_1 + \cdots + r_lP_l$$

where P_i are all the finite singularities situated on $T(S)$ and r_i are their corresponding multiplicities as points on the total curve $T(S)$. We mark the complex singular points denoting them by P_i^C. The total multiplicity TM of the zero-cycle M_{0CT} as the sum $\sum_i r_i$.

Definition 4.14. If the intersection multiplicity [29] of two curves is one then we say that the curves intersect transversally or that this point is a simple point of intersection.

If at a point two curves are tangent, we have an intersection multiplicity higher than or equal to two.

Definition 4.15 ([63])**.** Two polynomial differential systems S_1 and S_2 are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of S_1 to the oriented phase curves of S_2 and preserving the orientation.

To cut the number of non equivalent phase portraits in half we use here another equivalence relation.

Definition 4.16. Two polynomial differential systems S_1 and S_2 are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of S_1 to the oriented phase curves of S_2, preserving or reversing the orientation.

Notation: $\cong_{top}$.

In [4] the authors provide a complete classification of **QS** according to the geometric equivalence relation of topological configurations of singularities, finite or infinite. Here we use the same terminology and notation for singularities introduced in [4].

We say that a singular point is *elemental* if it possesses two non-zero eigenvalues; *semi-elemental* if it possess exactly one eigenvalue equal to zero and *nilpotent* if it possesses two zero eigenvalues and the linear part is not zero. We call *intricate* a singular point with its Jacobian matrix identically zero.

We place first the finite singular points denoted with lower case letters and secondly the infinite singular points denoted by capital letters, separating them by a semicolon ';'.

In our study we have real and complex finite singular points for real systems and from the topological viewpoint only the real ones are interesting. When we have a complex finite singular point we use the notation ©. For the elemental singular points we use the notation

's', 'S' for saddles, 'n', 'N' for nodes, 'f' for foci and 'c' for centers. We also denote by 'a' (anti-saddle) for either a focus or any type of node when the local phase portraits are topologically equivalent.

Non-elemental singular points are multiple points. We denote by $\binom{a}{b}$ the maximum number a (respectively b) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point at infinity. For example, $\binom{1}{1}SN$ and $\binom{0}{2}SN$ correspond to two saddle-nodes at infinity which are locally topologically distinct since the first arises from the coalescence of a finite with an infinite singularity and the second from the coalescence of two infinite singularities.

The semi-elemental singular points can either be nodes, saddles or saddle-nodes (finite or infinite). If they are finite singular points we denote them by '$n_{(3)}$', '$s_{(3)}$' and '$sn_{(2)}$', respectively and if they are infinite singular points by '$\binom{a}{b}N$', '$\binom{a}{b}S$' and '$\binom{a}{b}SN$', where $\binom{a}{b}$ indicates their multiplicity. We note that semi-elemental nodes and saddles are respectively topologically equivalent with elemental nodes and saddles.

The nilpotent singular points can either be saddles, nodes, saddle-nodes, elliptic-saddles, cusps, foci or centers. The only finite nilpotent points for which we need to introduce notation are the elliptic-saddles and cusps which we denote respectively by 'es' and 'cp'.

In the case of nilpotent infinite points, the relative positions of the sectors with respect to the line at infinity, can produce topologically different phase portraits. Then we use a notation for these points similar to the notation which we will use for the intricate points.

The intricate singular points are degenerate singular points. It is known that the neighbourhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic (p), hyperbolic (h) and elliptic (e) (see [25]). Then, a reasonable way to describe intricate and nilpotent points at infinity is to use a sequence formed by the types of their sectors. From the topological view point, any two adjacent parabolic geometrical sectors merge into one and any elliptic sector, in a small vicinity of the singularity, always has two parabolic sectors one of each side. We make the convention to eliminate the parabolic sectors adjacent to the elliptic sectors, according to the notation in [4].

In quadratic systems, we have just four topological possibilities for finite intricate singular points of multiplicity four:

- *phph*;
- *hh*;
- *hhhhhh*;
- *ee*.

For intricate and nilpotent singular points at infinity, we insert a dash (hyphen) between the sectors to split those which appear on one side or the other of the equator of the sphere. When describing a single finite nilpotent or intricate singular point, one can always apply an affine change of coordinates to the system, so it does not really matter which sector starts the sequence, or the direction (clockwise or counter-clockwise) we choose. If it is an infinite nilpotent or intricate singular point, then we always start with a sector bordering the infinity (to avoid using two dashes).

If the line at infinity is filled up with singularities, then it is known that any such system has in a sufficiently small neighbourhood of infinity one of 7 topological distinct phase portraits (see [67]). The way to determine these portraits is by studying the reduced systems on the infinite local charts after removing the degeneracy of the systems within these charts. Following [3] we use the notation $[\infty;\emptyset]$, $[\infty;N]$, $[\infty;N^d]$ (one-direction node, that is a node with two identical eigenvalues whose Jacobian matrix cannot be diagonal), $[\infty;S]$, $[\infty;C]$, $[\infty;\binom{0}{2}SN]$, $[\infty;\binom{0}{3}ES]$ indicating the kinds of singularities obtained after removing the line filled with singularities.

The degenerate systems are systems with a common factor in the polynomials defining the system. We denote this case with the symbol $\ominus$. The degeneracy can be produced by a non-constant common factor of degree one which defines a straight line or a common quadratic factor which defines a conic. In this paper we have just the second case happening.

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If the reduced system has no finite singularity on this curve, we use the symbol $\varnothing$ to describe this situation. If some singular points remain on this curve we use the corresponding notation of their various kinds. In this situation, the geometrical properties of the singularity that remains after the removal of the degeneracy, may produce topologically different phenomena, even if they are topologically equivalent singularities. So, we need to keep the geometrical information associated to that singularity. In this paper we use the notation $(\ominus[)(];\varnothing)$ which denotes the presence of a hyperbola filled up with singular points in the system such that the reduced system has no finite singularity on this curve.

The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points. We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non-isolated singularity. There is a detailed description of this notation in [3]. In case that after the removal of the finite degeneracy, a singular point at infinity remains at the same place, we must denote it with all its geometrical properties since they may influence the local topological phase portrait. In this paper we use the notation $(\ominus[)(];N,\emptyset)$ that means that the system has at infinity a node, and one non-isolated singular point which is part of a real hyperbola filled up with singularities and that the reduced linear system has no infinite singular point in that position.

See [4] for more details on the notation for singularities.

In order to distinguish topologically the phase portraits of the systems we obtained, we also use some invariants introduced in [66]. Let SC be the total number of separatrix connections, i.e. of phase curves connecting two singularities which are local separatrices of the two singular points. We denote by

- SC_f^f the total number of SC connecting two finite singularities,
- SC_f^∞ the total number of SC connecting a finite with an infinite singularity,
- SC_∞^∞ the total number of SC connecting two infinite.

A *graphic* as defined in [26] is formed by a finite sequence of singular points $p_1, p_2, \ldots, p_n$, $p_{n+1} = p_1$ and oriented regular orbits $s_1, \ldots, s_n$ connecting them such that s_j has p_j as α-limit set and p_{j+1} as ω-limit set for $j < n$ and s_n has p_n as α-limit set and p_1 as ω-limit set. Graphics may or may not have a return map. Particular graphics are given special names. A *loop* is a graphic through a unique singular point and with a return map. A *polycycle* is a graphic through several singular points and with a return map. A *degenerate graphic* as defined in [26] is formed by singular points $p_1, p_2, \ldots, p_n, p_{n+1} = p_1$, oriented regular orbits and segments

$s_1, \ldots, s_n$ of curves of singular points (which are also oriented) such that either s_j is a orbit that has p_j as α-limit and p_{j+1} as ω-limit for $j < n$ and s_n has p_n as α-limit set and p_1 as ω-limit set or an open segment of a curve of singular points with end points p_j and p_{j+1}, for each $j < n$. Moreover, the regular orbits and the curves of singular points have coherent orientations in the sense that if s_{j-1} has left hand orientation then so does s_j. For more details, see [26].

In what follows we present an example of the notation used in paper to describe the global configuration of singularities of **QSH**.

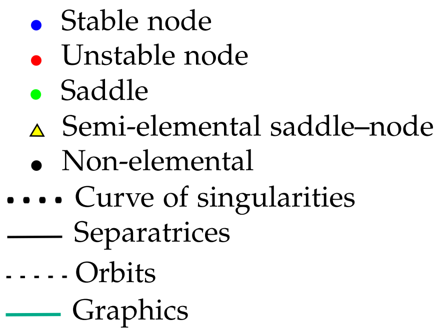

Figure 4.1: Notations used on the phase portraits.

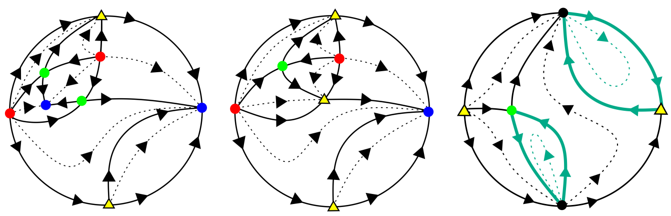

Figure 4.2: Some examples of phase portraits.

The notation used to describe the topological type of the singularities in Figure 4.2 is

$$(a, s, a, s; \binom{0}{2}SN, N)$$
$$(s, sn, a; \binom{0}{2}SN, N)$$
$$(s; \binom{2}{2}E - E, \binom{1}{1}SN)$$

for each phase portrait appearing in the respective order. The first letters appearing with lower case represents the topological type of the finite singularities. Here 'sn' denotes a saddle-node which arises from the coalescence of a finite saddle with a finite node so this is a singularity of multiplicity two, 'a' denotes an elemental anti-saddle and 's' denotes an elemental saddle. The capital letters give the topological type of the singularities at infinity: '$\binom{0}{2}SN$' denotes a saddle-node which arises from the coalescence of two infinite singularities (saddle and node) so this is a double singularity, '$\binom{1}{1}SN$' also denotes a saddle-node but here this multiplicity arises from the coalescence of a finite with an infinite singularity, '$\binom{2}{2}E - E$' denotes an intricate singularity arising from the coalesce of two finite singularities with two infinite singularities and the neighbourhood of this singularity is formed by an elliptic sector which has, in a small vicinity of the singularity, two parabolic sectors one of each side. The cases where we do no indicate the multiplicity means the singularity is simple, which is the case of '*S*' (elemental saddle) and '*N*' (elemental node).

5 Proof of the Main theorem for the integrable cases

The data described in Table 5.2 led us to the proof of Main theorem for the integrable case.

We begin by using the Prelle–Singer algorithm (including the exponential factors, when they exist) in order to prove integrability.

The result of our calculations are given in Table 5.2 where we have the invariant algebraic curves, exponential factors and their cofactors, first integrals or integrating factors for each normal form of Proposition 3.3 obtained using the software *Mathematica*.

In the first column are the normal forms for $\mathbf{QSH}_{\eta=0}$.

In the second column are the invariant algebraic curves, the exponential factors and the respective cofactors.

In the third column are the expressions of the first integrals or the expressions of the integrating factors. If we give the expression for the first integral then it is not necessary to give the integrating factor to guarantee the integrability. When we give the expression for the integrating factor instead of the first integral this means that we could not compute the expression for the first integral using *Mathematica* and we use the notation "—". When "—" appears in both the first integral and integrating factor this means that we could find neither of them applying the Prelle–Singer algorithm.

In the fourth and fifth columns are the normal forms and their possible configurations as in [47]. The notation "—" appears when we do not have them appearing in [47].

In the sixth column are indicated the types of integrability of each normal form using the notations in Table 5.1.

The precise integrating factor, first integral that did not fit in the table will be given in the text following the table.

Thereby, the proof of the Main theorem follows except for the non integrable cases, that will be done in Section 6.

Table 5.1: Notations used in Table 5.2.

Notation
N-I : Systems admit neither a Darboux nor a Liouvillian first integral;
D: Systems are Darboux integrable;
GD: Systems are generalized Darboux integrable;
L: Systems are Liouvillian integrable;
P: Systems admit a polynomial first integral;
R: Systems admit a rational first integral;
HAM: Systems are Hamiltonian.
open case : We could prove neither the integrability nor the non-integrability;
$\mathcal{R}$: Represents an integrating factor;
$\mathcal{F}$: Represents a first integral;

Table 5.2: Proof of the Main theorem for the integrable cases.

Orbit representative $a, g, c, \epsilon \in \mathbb{R} : a \neq 0$	Invariant curves/ ExpFac Respective cofactors	Integrating Factor $\mathcal{R}_i$ First integral $\mathcal{F}_i$	Eq. [47]	Config. $\tilde{H}$	Integ.
(A) $\begin{cases} \dot{x} = 2a + x + gx^2 + xy, \\ \dot{y} = a(2g-1) - y + (g-1)xy + y^2, \end{cases}$	$a + xy$	—	(4.4)	1, 3, 4, 5, 6, 7, 8, 9, 10, 11	N-I
where $a(g-1)(2g-1)(3g-1)(g-2)(25a-3) \neq 0$	$(-1+2g)x + 2y$	—			
(A) where $g = 1/2$	$y, \ a + xy$	—	(4.10)	12, 13, 14, 15, 16, 17	N-I
	$-1 - \frac{x}{2} + y, \ 2y$	—			
(A) where $g = 2$ and $a = 3/25$	$xy + \frac{3}{25}, \ x + \frac{5y^2}{9} - y + \frac{3}{5}$	—	—	—	open case
	$3x + 2y, \ 2x + 2y - \frac{1}{5}$	—			
(A) where $g = 1/3$ and $a = \bar{a}/2$	$\bar{a} + 2xy$	—	(4.11)	1, 4, 5, 6, 10	open case
	$2y - \frac{x}{3}$	—			
(B) $\begin{cases} \dot{x} = 2a + gx^2 + xy, \\ \dot{y} = a(2g-1) + (g-1)xy + y^2, \end{cases}$	$a + xy$	—	(4.13)	1, 2, 6	N-I
where $a(g-1)(2g-1)(4g-1) \neq 0$	$(-1+2g)x + 2y$	—			
(B) where $g = 1/2$	$y, \ a + xy, \ e^{-\frac{a+2y^2}{2(a+xy)}}$		—	31, 32	GD
	$-\frac{x}{2} + y, \ 2y, \ y$	$\mathcal{F}_{B,1} = e^{\frac{a+2y^2}{a+xy}}(a + xy)$			
(B) where $g = 1/4$	$a + xy, \ e^{\frac{y^2}{a+xy}}$	$\mathcal{R}_{B,3} = \frac{e^{\frac{2y^2}{a+xy}}}{\sqrt{a+xy}}$	—	29, 30	L
	$-\frac{x}{2} + 2y, \ -y$	—			

(C) $\begin{cases}\dot{x} = 2a+3cx+x^2+xy,\\ \dot{y} = a-c^2+y^2,\end{cases}$ where $a(c^2-a)(9a-8c^2)\neq 0$	$y+\sqrt{c^2-a}$, $y-\sqrt{c^2-a}$, $a+cx+xy$	$\mathcal{R}_C$	(4.16)	18, 20, 21, 22, 33	L
	$y-\sqrt{c^2-a}$, $y+\sqrt{c^2-a}$, $2c+x+2y$	—			
(C) where $a=c^2$	y, $c^2+cx+xy$, $e^{\frac{1}{y}}$	$\mathcal{R}_{C,1}$	—	23	L
	y, $2c+x+2y$, -1	—			
(C) where $a=8c^2/9$	$3y-c$, $3y+c$, $8c^2+9cx+9xy$, $e^{\frac{-cx+48cy+63xy-24y^2}{48c(8c^2+9cx+9xy)}}$		—	33	GD
	$\frac{c}{3}+y$, $y-\frac{c}{3}$, $2c+x+2y$, $\frac{y}{18c}-\frac{1}{54}$	$\mathcal{F}_{C,2}$			
(D) $\begin{cases}\dot{x} = (c+x)(c(2g-1)+gx),\\ \dot{y} = 1+(g-1)xy,\end{cases}$ where $cg(g\pm 1)(2g-1)(3g-1)\neq 0$	$x+c$, $c(2g-1)+gx$, $\frac{1}{(-1+2g)}+cy+xy$		(4.18)	19	D
	$c(-1+2g)+gx$, $cg+gx$, $c(-1+2g)+(-1+2g)x$	$\mathcal{F}_D$			
(D) where $g=0$ and $c\neq 0$	$c+x$, $-1+cy+xy$, e^{x+1}		—	24	GD
	$-c$, $-c-x$, $-c^2-cx$	$\mathcal{F}_{D,1}=e^{x+1}(y(c+x)-1)^{-c}$			
(D) where $c=0$ and $g\neq 0,-1/2$	x, $\frac{1}{-1+2g}+xy$, $e^{\frac{1}{x}}$, $e^{\frac{2gxy+1}{x^2}}$			25, 34	D
	gx, $(-1+2g)x$, $-g$, $-2gy$	$\mathcal{F}_{D,2}=\frac{x^{\frac{1}{g}-2}(2gxy-xy+1)}{2g-1}$			
(E) $\begin{cases}\dot{x} = x^2+\epsilon,\\ \dot{y} = 1-2xy,\end{cases}$ where $\epsilon\neq 0$	$x+i\sqrt{\epsilon}$, $x-i\sqrt{\epsilon}$, $-1+i\sqrt{\epsilon}y+xy$, $-1-i\sqrt{\epsilon}y+xy$		(4.22)	27, 28	P/HAM
	$x-i\sqrt{\epsilon}$, $x+i\sqrt{\epsilon}$, $-x-i\sqrt{\epsilon}$, $-x+i\sqrt{\epsilon}$	$\mathcal{F}_E=(x^2+\epsilon)\left((xy-1)^2+y^2\epsilon\right)$			

(E) where $\epsilon = 0$	$x,\ -1+xy,\ e^{\frac{1}{x}},$ $e^{\frac{2xy+1}{x^2}}, e^{\frac{y}{xy-1}},\ e^{\frac{y^2(2xy-3)}{(xy-1)^2}}$		—	34	P/HAM
	$x,\ -x,\ -1,$ $-6y,\ -1,\ -6y$	$\mathcal{F}_{E,1} = x(-1+xy)$			
(F) $\begin{cases} \dot{x} = (x-1)(3-x), \\ \dot{y} = 1-2xy \end{cases}$	$1-x,\ 3-x,$ $-\frac{1}{3}-y+xy,$ $-\frac{19}{8}+x+3y-\frac{x^2}{8}$		(4.25)	19	R
	$3-x,\ 1-x,\ 3-3x,$ $-2x$	$\mathcal{F}_F = -\frac{(x-3)^2}{3(x-1)y-1}$			
(G) $\begin{cases} \dot{x} = -x^2, \\ \dot{y} = 1-2xy \end{cases}$	$x,\ -1+3xy,\ e^{\frac{1}{x}},$ $e^{\frac{1-2xy}{x^2}},\ e^{\frac{1-3xy-2x^2y+x}{x^3}}$		—	26	R
	$-x,\ -3x,\ 1,\ 2y,$ $2y$	$\mathcal{F}_G = \frac{x^3}{3xy-1}$			
(H) $\begin{cases} \dot{x} = (2x-1)(2x+1)/4, \\ \dot{y} = y \end{cases}$	$1+2x,\ 1-2x,\ y,$ $-\frac{q}{2}+qx+y+2xy,$ $e^y,\ e^{\frac{-2x+y+1}{1-2x}}$		(4.27)	35	R
	$-\frac{1}{2}+x,\ \frac{1}{2}+x,\ 1,$ $\frac{1}{2}+x,\ y,\ \frac{y}{2}$	$\mathcal{F}_H = \frac{(2x+1)y}{q\left(x-\frac{1}{2}\right)+2xy+y}$			
(I) $\begin{cases} \dot{x} = x^2, \\ \dot{y} = 1 \end{cases}$	$x,\ 1+rx+xy,\ e^{\frac{x+1}{x}}$ $e^{\frac{x^2+2xy+x+1}{x^2}},\ e^{y^2+y+1}$		(4.28)	36	R
	$x,\ x,\ -1,$ $-1-2y,\ 2y+1$	$\mathcal{F}_I = \dfrac{x}{1+rx+xy}$			
(J) $\begin{cases} \dot{x} = a+y+x^2, \\ \dot{y} = xy, \end{cases}$	$y,\ -i\sqrt{a}+x-\frac{iy}{\sqrt{a}},$ $i\sqrt{a}+x+\frac{iy}{\sqrt{a}},$ $a+2y+x^2-m^2y^2$		(4.30)	39, 41	R
where $a \neq 0$	$x,\ i\sqrt{a}+x,$ $-i\sqrt{a}+x,\ 2x$	$\mathcal{F}_J = \frac{y^2}{a+2y+x^2-m^2y^2}$			

(J) where $a=0$	$y,\ 2y+x^2-m^2y^2,\ e^{\frac{x}{y}},\ e^{\frac{x^2+2xy+2y^2}{2y^2}}$		—	43	R
	$x,\ 2x,\ 1,\ 1$	$\mathcal{F}_{J,1}=\frac{y^2}{2y+x^2-m^2y^2}$			
(K) $\begin{cases}\dot{x}=(1+3x)(2+3x)/9,\\ \dot{y}=xy\end{cases}$	$y,\ 2+3x,\ 1+3x,\ 4+12x+9x^2+my+3mxy$		(4.34)	37	R
	$x,\ \frac{1}{3}+x,\ \frac{2}{3}+x, \frac{2}{3}+2x$	$\mathcal{F}_K=\frac{(3x+1)y}{(3x+2)^2}$			
(L) $\begin{cases}\dot{x}=a+x^2,\\ \dot{y}=xy,\end{cases}$	$y,\ 1-\frac{ix}{\sqrt{a}},\ 1+\frac{ix}{\sqrt{a}},\ a+x^2-m^2y^2$		(4.36)	38, 40	R
where $a\neq 0$	$x,\ -i\sqrt{a}+x,\ i\sqrt{a}+x, 2x$	$\mathcal{F}_L=\dfrac{x^2+a}{ay^2}$			
(M) $\begin{cases}\dot{x}=x^2,\\ \dot{y}=1+xy\end{cases}$	$x,\ 1+mx^2+2xy,\ e^{1/x},\ e^{\frac{x^2+2xy+x+1}{x^2}}$		(4.38)	42	R
	$x,\ 2x,\ -1,\ -1$	$\mathcal{F}_M=\dfrac{x^2}{1+mx^2+2xy}$			

$$
\begin{aligned}
\mathcal{R}_C &= (y+\sqrt{c^2-a})^{\frac{1}{2}\left(1+\frac{c}{\sqrt{c^2-a}}\right)}(y-\sqrt{c^2-a})^{\frac{1}{2}\left(1-\frac{c}{\sqrt{c^2-a}}\right)}(a+cx+xy)^{-2};\\
\mathcal{R}_{C,1} &= y\left(c+x+\frac{xy}{c}\right)^{-2} e^{\frac{-c}{y}};\\
\mathcal{F}_{C,2} &= (c+3y)\left(e^{\frac{-cx+48cy+63xy-24y^2}{48c(8c^2+9cx+9xy)}}\right)^{-18c};\\
\mathcal{F}_D &= (c(2g-1)+gx)\left(y(c+x)+\frac{1}{2g-1}\right)^{-\frac{g}{2g-1}}.
\end{aligned}
$$

6 Proof of the Main theorem for the non integrable cases

Consider the sets:

$$
\begin{aligned}
&\mathrm{L}_1 = \cup_{k\in\mathbb{N}} \mathrm{L}_{1,k}, \text{ where } \mathrm{L}_{1,k} = \left\{(a,g)\in\mathbb{R}^2 : g = k/2 \text{ and } a\neq 0\right\},\ k\in\mathbb{N},\\
&\mathrm{L}_2 = \cup_{k\in\mathbb{N}} \mathrm{L}_{2,k}, \text{ where } \mathrm{L}_{2,k} = \left\{(a,g)\in\mathbb{R}^2 : g = k/3 \text{ and } a\neq 0\right\},\ k\in\mathbb{N},\\
&\mathrm{L}_3 = \left\{(a,g)\in\mathbb{R}^2 : g = 1/4 \text{ and } a\neq 0\right\},\\
&\mathrm{C}' = \cup_{k\in\mathbb{N}} \mathrm{C}_k, \text{ where } \mathrm{C}_k = \left\{(a,g)\in\mathbb{R}^2 : g = (2+a-2ak)/4a,\ a\neq 0\right\},\ k\in\mathbb{N}.
\end{aligned}
$$

6.1 The systems (A)

$$
\begin{cases}
\dot{x} = 2a + x + gx^2 + xy,\\
\dot{y} = a(2g-1) - y + (g-1)xy + y^2,
\end{cases}
$$

where $a(g-1)\neq 0$.

Theorem 6.1.

(a) If $(a,g)\notin \mathrm{L}_1$ then the only invariant algebraic curves of a system in the family (A) are of the form $J_1^m = 0$ where $J_1(x,y) = a+xy$ and m is a positive integer.

(b) If $(a,g)\notin (\mathrm{L}_1\cup\mathrm{L}_2\cup\mathrm{C}')$ then any system in the family (A) has no exponential factors.

(c) If $(a,g)\notin (\mathrm{L}_1\cup\mathrm{L}_2\cup\mathrm{C}')$ then any system in the family (A) is not Liouvillian integrable.

Remark: *When $g = 1/2$ the systems posses the invariant line $y = 0$ but this invariant curve is still not enough to prove the integrability. The non integrability in this case can be done just by adapting $g = 1/2$ in the proof below. In (a) we find $C = y^{m-l}(a+xy)^l$.*

Proof. (a) By a straightforward computation, we can verify that $J_1(x,y) = a+xy$ is an invariant hyperbola with cofactor $\alpha_1(x,y) = (-1+2g)x+2y$. Assume that

$$
C = \sum_{i=0}^{n} C_i(x,y) = 0
$$

is an invariant algebraic curve of the system (A) with cofactor $K = K_0 + K_1x + K_2y$, where C_i are homogeneous polynomial of degree i where $0\leq i\leq n$. From the definition of invariant algebraic curve (1.4), we have:

$$(2a + x + gx^2 + xy) \sum_{i=0}^{n} C_{i,x} + (a(2g-1) - y + (g-1)xy + y^2) \sum_{i=0}^{n} C_{i,y}$$
$$= (K_0 + K_1 x + K_2 y) \sum_{i=0}^{n} C_i. \quad (6.1)$$

Taking from (6.1) the terms of degree $n+1$ we have:

$$(gx^2 + xy)\ C_{n,x} + ((g-1)xy + y^2)\ C_{n,y} = (K_1 x + K_2 y)\ C_n. \quad (6.2)$$

For this system we have

$$yP_2 - xQ_2 = x^2 y.$$

Then, from Lemma 4.5 we can assume that

$$C_n = x^m y^l, \quad \text{where } n = m + l.$$

Substituting C_n in (6.2) and doing some computations we terminate that

$$K_1 = gm + (g-1)l; \qquad K_2 = m + l.$$

Now, taking from (6.1) the terms of degree n we have:

$$x\ C_{n,x} + (gx^2 + xy)\ C_{n-1,x} - y\ C_{n,y} + ((g-1)xy + y^2)\ C_{n-1,y}$$
$$= K_0\ C_n + [(gm + (g-1)l)\ x + (m+l)\ y]\ C_{n-1}. \quad (6.3)$$

Set $C_{n-1} = \sum_{i=0}^{n-1} c_{n-1-i} x^{n-1-i} y^i$. Replacing C_n, C_{n-1} in (6.3) and doing some calculations, we obtain

$$\sum_{i=0}^{m+l-1} (l - i - g)\ c_{m+l-1-i}\ x^{m+l-i}\ y^i - \sum_{i=0}^{m+l-1} c_{m+l-1-i}\ x^{m+l-1-i}\ y^{i+1} = (K_0 - m + l)\ x^m y^l.$$

Note that this equation can be written as

$$\sum_{i=0}^{m+l} [(l - i - g)\ c_{m+l-1-i} - c_{m+l-i}]\ x^{m+l-i}\ y^i = (K_0 - m + l)\ x^m y^l,$$

where $c_i = 0$ for $i < 0$ and $i > m + l - 1$. Equating the coefficients of $x^i y^j$ in the above equation, we get:

$$\begin{cases} (l - i - g)\ c_{m+l-1-i} - \ c_{m+l-i} = 0, \text{ where } \ i = 0, 1, \ldots, l-1, l+1, \ldots, m+l \\ (-g)\ c_{m-1} - \ c_m = K_0 - m + l. \end{cases} \quad (6.4)$$

For $i = m + l, m + l - 1, \ldots, l + 1$ we have

$$c_0 = c_1 = \cdots = c_{m-1} = 0.$$

Then $c_m = -K_0 + m - l$. Working recursively we have

$$\begin{aligned} c_{m+1} &= (-g+1)(-K_0 + m - l), \\ c_{m+2} &= (-g+2)(-g+1)(-K_0 + m - l), \ldots \\ c_{m+l-1} &= (-g+l-1)\ldots(-g+2)(-g+1)(-K_0 + m - l). \end{aligned}$$

Replacing c_{m+l-1} in (6.4) where $i = 0$, we get

$$(-g+l)(-g+l-1)\dots(-g+2)(-g+1)(-K_0+m-l) = 0.$$

Note that

$$(-g+l)(-g+l-1)\dots(-g+2)(-g+1)$$

is a polynomial of degree l in the variable g, which has at most l real roots. Denote by S_1^l the set of roots. If $g \notin S_1^l$ then $K_0 = m - l$. Therefore, we can conclude that

$$K = (m-l) + (gm + (g-1)l)x + (m+l)y,$$

since $g \notin S_1^l$. This is

$$C_{n-1} \equiv 0.$$

Now, taking from (6.1) the terms of degree $n-1$ we have:

$$\begin{aligned} 2a\, C_{n,x} + (gx^2+xy)\, C_{n-2,x} + a(2g-1)\, C_{n,y} + ((g-1)xy + y^2)\, C_{n-2,y} \\ = [(gm+(g-1)l)\, x + (m+l)\, y]\, C_{n-2}. \end{aligned} \tag{6.5}$$

Setting $C_{n-2} = \sum_{i=0}^{n-2} c_{n-2-i} x^{n-2-i} y^i$ and replacing C_n, C_{n-2} in (6.5) we obtain

$$\begin{aligned} \sum_{i=0}^{m+l-2} (l-i-2g)\, c_{m+l-2-i}\, x^{m+l-1-i}\, y^i + \sum_{i=0}^{m+l-2} (-2)\, c_{m+l-2-i}\, x^{m+l-2-i}\, y^{i+1} \\ = -a(2g-1)l\, x^m y^{l-1} - 2am\, x^{m-1} y^l. \end{aligned}$$

This equation can be written as

$$\begin{aligned} \sum_{i=0}^{m+l-2} [(l-i-2g)\, c_{m+l-2-i} + (-2)\, c_{m+l-1-i}]\, x^{m+l-1-i}\, y^i \\ = -a(2g-1)l\, x^m y^{l-1} - 2am\, x^{m-1} y^l \end{aligned}$$

where $c_i = 0$ for $i < 0$ and $i > m+l-2$. Equating the coefficients of $x^i y^j$ in the above equation, we get

$$\begin{cases} (l-i-2g)\, c_{m+l-2-i} - 2\, c_{m+l-1-i} = 0, \text{ where } i = 0, \dots, l-2, l+1, \dots, m+l-1, \\ (-2g+1)c_{m-1} - 2c_m = -a(2g-1)l, \\ (-2g)c_{m-2} - 2c_{m-1} = -2am. \end{cases} \tag{6.6}$$

For $i = m+l-1, m+l-2, \dots, l+1$ we have

$$c_0 = c_1 = \dots = c_{m-2} = 0.$$

Then $c_{m-1} = am$. Working recursively we have

$$c_m = \frac{a(-2g+1)(m-l)}{2}, \quad c_{m+1} = \frac{a(-2g+1)(-2g+2)(m-l)}{4}, \dots$$

$$c_{m+l-2} = \frac{a(-2g+1)(-2g+2)\dots(-2g+l-1)(m-l)}{2^{l-1}}.$$

Replacing c_{m+l-2} in (6.6) where $i=0$, we get

$$\frac{a(-2g+1)(-2g+2)\dots(-2g+l-1)(-2g+l)(m-l)}{2^{l-1}}=0.$$

Note that

$$(-2g+1)(-2g+2)\dots(-2g+l-1)(-2g+l)$$

is a polynomial of degree l in the variable g, which has at most l real roots. Denote by S_2^l the set of roots. If $g \notin S_1^l \cup S_2^l$ then $a=0$ or $m=l$.

The hyperbola is not an invariant algebraic curve when $a=0$ and this cases does not matter for us. We assume $m=l$. Then, we have the following:

$$\begin{aligned} &K=(2g-1)m\,x+2m\,y,\\ &C_n=x^m y^m, \quad C_{n-1}\equiv 0, \quad C_{n-2}=amx^{m-1}y^{m-1}, \end{aligned} \tag{6.7}$$

for $g \notin S_1^m \cup S_2^m$ which is a numerable set.

Following similar arguments for terms of degree $n-2, n-3, \dots$ in (6.1) we conjecture that $C=(a+xy)^m$. Now we prove this statement by induction:

Suppose that for $k=1,2\dots,L$ we have

$$C_{n-(2k-1)}\equiv 0, \qquad C_{n-2k}=\frac{a^k(m-(k-1))!}{k!}x^{m-k}y^{m-k}. \tag{6.8}$$

We shall prove that:

$$C_{n-2L-1}\equiv 0, \qquad C_{n-2L-2}=\frac{a^{L+1}(m-L)!}{(L+1)!}x^{m-L-1}y^{m-L-1}.$$

Considering in (6.1) the terms of degree $n-2L$ we have:

$$\begin{aligned} 2a\,C_{n-2L+1,x}+x\,C_{n-2L,x}+(gx^2+xy)\,C_{n-2L-1,x}+a(2g-1)\,C_{n-2L+1,y}&\\ -y\,C_{n-2L,y}+\left((g-1)xy+y^2\right)\,C_{n-2L-1,y}=\left((2g-1)m\,x+2m\,y\right)\,C_{n-2L-1}.& \end{aligned}$$

By the induction hypothesis $C_{n-2L+1}\equiv 0$ then:

$$\begin{aligned} x\,C_{n-2L,x}+(gx^2+xy)\,C_{n-2L-1,x}-y\,C_{n-2L,y}+\left((g-1)xy+y^2\right)C_{n-2L-1,y}&\\ =[(2g-1)m\,x+2m\,y]\,C_{n-2L-1}.& \end{aligned} \tag{6.9}$$

Setting $C_{n-2L-1}=\sum_{i=0}^{2m-2L-1}c_{2m-2L-1-i}x^{2m-2L-1-i}y^i$ and replacing C_{n-2L}, C_{n-2L-1} in (6.9) we obtain

$$\begin{aligned} &\sum_{i=0}^{2m-2L-1}(m-i-g(2L+1))\,c_{2m-2L-1-i}\,x^{2m-2L-i}\,y^i\\ &\qquad\qquad+\sum_{i=0}^{2m-2L-1}(-2L-1)\,c_{2m-2L-1-i}\,x^{2m-2L-1-i}\,y^{i+1}=0. \end{aligned}$$

This equation can be written as

$$\sum_{i=0}^{2m-2L}\left[(m-i-g(2L+1))\,c_{2m-2L-1-i}+(-2L-1)\,c_{2m-2L-i}\right]\,x^{2m-2L-i}\,y^i=0,$$

where $c_i = 0$ for $i < 0$ and $i > 2m - 2L - 1$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$(m - i - g(2L+1))\ c_{2m-2L-1-i} + (-2L-1)\ c_{2m-2L-i} = 0,$$

for $i = 0, 1, \ldots, 2m - 2L$. As $L \in \mathbb{N}$ then $L \neq -1/2$ and:

$$c_{2m-2L-1} = c_{2m-2L-2} = \cdots = c_1 = c_0 = 0.$$

Therefore,

$$C_{n-2L-1} \equiv 0.$$

Now, considering in (6.1) the terms of degree $n - 2L - 1$ we have:

$$\begin{aligned} 2a\ C_{n-2L,x} + x\ C_{n-2L-1,x} + (gx^2 + xy)\ C_{n-2L-2,x} + a(2g-1)\ C_{n-2L,y} \\ - y\ C_{n-2L-1,y} + ((g-1)xy + y^2)\ C_{n-2L-2,y} = [(2g-1)m\ x + 2m\ y]\ C_{n-2L-2}. \end{aligned}$$

We just proved that $C_{n-2L-1} \equiv 0$, then we have:

$$\begin{aligned} 2a\ C_{n-2L,x} + (gx^2 + xy)\ C_{n-2L-2,x} + a(2g-1)\ C_{n-2L,y} \\ + ((g-1)xy + y^2)\ C_{n-2L-2,y} = [(2g-1)m\ x + 2m\ y]\ C_{n-2L-2} \end{aligned} \tag{6.10}$$

By the induction hypothesis it follows that $C_{n-2L} = \frac{a^L(m-(L-1))!}{L!} x^{m-L} y^{m-L}$. Setting $C_{n-2L-2} = \sum_{i=0}^{2m-2L-2} c_{2m-2L-2-i} x^{2m-2L-2-i} y^i$ and replacing C_{n-2L}, C_{n-2L-2} in (6.10) we have:

$$\begin{aligned} &\sum_{i=0}^{2m-2L-2} (m - i - g(2(L+1))\ c_{2m-2L-2-i}\ x^{2m-2L-1-i}\ y^i \\ &\quad + \sum_{i=0}^{2m-2L-2} (-2L-2)\ c_{2m-2L-2-i}\ x^{2m-2L-2-i}\ y^{i+1} \\ &= -(2g-1)\frac{a^{L+1}(m-L)!}{L!} x^{m-L} y^{m-L-1} - \frac{2a^{L+1}(m-L)!}{L!} x^{m-L-1} y^{m-L}. \end{aligned}$$

This equation can be rewritten as

$$\begin{aligned} \sum_{i=0}^{2m-2L-1} [(m - i - g(2(L+1))\ c_{2m-2L-2-i} + (-2L-2)\ c_{2m-2L-1-i}]\ x^{2m-2L-1-i}\ y^i \\ = -(2g-1)\frac{a^{L+1}(m-L)!}{L!} x^{m-L} y^{m-L-1} - \frac{2a^{L+1}(m-L)!}{L!} x^{m-L-1} y^{m-L} \end{aligned}$$

where $c_i = 0$ for $i < 0$ and $i > 2m - 2L - 2$. Equating the coefficients of $x^i y^j$ in the above equation, we get the following equations

$$\begin{cases} (m - i - g(2(L+1)))\ c_{2m-2L-2-i} + (-2L-2)\ c_{2m-2L-1-i} = 0, \\ (L + 1 - g(2(L+1)))c_{m-L-1} + (-2L-2)c_{m-L} = -(2g-1)\dfrac{a^{L+1}(m-L)!}{L!}, \\ (L - g(2(L+1)))c_{m-L-2} + (-2L-2)c_{m-L-1} = -\dfrac{2a^{L+1}(m-L)!}{L!}, \end{cases}$$

for $i = 0, 1, \ldots, m - L - 2, m - L + 1, \ldots, 2m - 2L - 1$. As $L \in \mathbb{N}$ then $L \neq -1$ and

$$c_{m-L-2} = \cdots = c_1 = c_0 = 0.$$

Then,

$$c_{m-L-1} = \frac{a^{L+1}(m-L)!}{(L+1)!}, \qquad c_{m-L} = 0.$$

When $i = m-L-2,\dots,0$, we obtain

$$c_{m-L+1} = c_{m-L+2} = \cdots = c_{2m-2L-2} = 0.$$

Therefore,

$$C_{n-2L-2} = \frac{a^{L+1}(m-L)!}{(L+1)!} x^{m-L-1} y^{m-L-1}.$$

This finishes the induction proof. It follows that

$$C = J_1^m, \quad m \in \mathbb{N}$$

for all $(a,g) \notin \mathrm{L}_1$, where $\mathrm{L}_1 = \cup_{k\in\mathbb{N}} \mathrm{L}_{1,k} = \cup_{k\in\mathbb{N}} \{(a,g) \in \mathbb{R}^2 : g = \frac{k}{2} \text{ and } a \neq 0\}$.

(b) From (a) systems (A) have only the algebraic solution $J_1(x,y) = a + xy$ for $(a,g) \in \mathbb{R}^2 - \mathrm{L}_1$. Then by Proposition 2.10, if systems (A) have an exponential factor, it must have the form:

$$F = \exp\left(G/J_1^l\right)$$

with cofactor $L = \overline{L}_0 + \overline{L}_1 x + \overline{L}_2 y$ and where l is non-negative integer. Since the invariant algebraic curve $J_1^l = 0$ has the cofactor

$$K = l\alpha_1 = l(-1+2g)x + 2ly,$$

it follows by (2.2) that G satisfies the following equation:

$$\begin{aligned}\left[2a + x + gx^2 + xy\right] G_x + \left[a(2g-1) - y + (g-1)xy + y^2\right] G_y \\ + \left[l(1-2g)x - 2ly\right] G = \left[\overline{L}_0 + \overline{L}_1 x + \overline{L}_2 y\right] \sum_{k=0}^{l} \binom{l}{k} a^k x^{l-k} y^{l-k}. \quad (6.11)\end{aligned}$$

From now on we assume $G(x,y) = \sum_{i=0}^{n} G_i(x,y)$, where G_i is a homogeneous polynomial of degree i and split the study in cases.

Case 1: $n+1 < 2l$.

By equating the homogeneous terms of highest degree in (6.11) we obtain that

$$\overline{L}_1 = \overline{L}_2 = 0 \quad \text{and} \quad \overline{L}_0 = 0.$$

Thus, G is an invariant algebraic curve. Then, $G = c\, J_1^l$ where c is a constant. Therefore, F is constant and it cannot be an exponential factor of system (A).

Case 2: $n+1 = 2l$.

By equating the homogeneous terms of highest degree in (6.11) we obtain that

$$\overline{L}_1 = \overline{L}_2 = 0.$$

Set $G_n = \sum_{i=0}^{n} c_{n-i} x^{n-i} y^i$ where c_{n-i} are constants. Equating the terms of degree $n+1$ in (6.11) and using that $n+1 = 2l$, we have:

$$\textstyle\sum_{i=0}^{2l} \left[(-g-i+l)c_{2l-i-1} + (-1)c_{2l-i}\right] x^{2l-i} y^i = \overline{L}_0 x^l y^l,$$

where $c_i = 0$ for $i < 0$ and $i > n$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-g-i+l)\ c_{2l-i-1} -\ c_{2l-i} = 0, \text{ where } i = 0,1,\dots,l-1,l+1,\dots,2l \\ (-g)\ c_{l-1} -\ c_l = \overline{L}_0. \end{cases}$$

For $i = 2l, 2l-1, \dots, l+1$ we obtain: $c_0 = c_1 = \cdots = c_{l-1} = 0$. Then $c_l = -L_0$. Working recursively,

$$c_{l+1} = (-g+1)(-\overline{L}_0), \qquad c_{l+2} = (-g+2)(-g+1)(-\overline{L}_0), \dots$$
$$c_{2l-1} = (-g+l)(-g+l-1)\dots(-g+1)(-\overline{L}_0).$$

Therefore, if $g \notin \{l, l-1, \dots, 1\}$, this is $(a,g) \notin \mathrm{L}_1$ we have $\overline{L}_0 = 0$ then

$$L = 0.$$

Consequently, system (A) has no exponential factors for $(a,g) \in \mathbb{R}^2 - \mathrm{L}_1$.

Case 3: $n = 2l$.

Consider the notation for G_n introduced in the study of Case 2. Equating the terms of degree $n+1$ in (6.11) we have

$$\textstyle\sum_{i=0}^{2l+1} (l-i)\ c_{2l-i} x^{2l-i+1} y^i = \overline{L}_1\ x^{l+1} y^l + \overline{L}_2\ x^l y^{l+1},$$

where $c_i = 0$ for $i < 0$ and $i > n$. These equations are equivalent to

$$\begin{cases} (l-i)\ c_{2l-i} = 0, \text{ where } i = 0,1,\dots,l-1,l+2,l+3,\dots,2l+1 \\ 0\ c_l = \overline{L}_1, \\ (-1)\ c_{l-1} = \overline{L}_2, \end{cases} \tag{6.12}$$

For $i = 2l, 2l-1, \dots, l+2$ we obtain: $c_0 = c_1 = \cdots = c_{l-2} = 0$. Then,

$$c_{l-1} = -\overline{L}_2, \ \ c_l \text{ is free}, \ \ \overline{L}_1 = 0 \text{ and } c_{l+1} = c_{l+2} = \cdots = c_{2l} = 0.$$

Therefore,

$$G_n = c_l\ x^l y^l - \overline{L}_2 x^{l-1} y^{l+1}.$$

Since $c_l \neq 0$, without loss of generality, we assume that $c_l = 1$.

Equating the terms of degree n in (6.11) we have

$$x\ G_{n,x} + \left[g x^2 + xy\right] G_{n-1,x} - y\ G_{n,y} + \left[(g-1)xy + y^2\right] G_{n-1,y} + \left[l(1-2g)x - 2ly\right] G_{n-1} = \overline{L}_0 x^l y^l.$$

Set $G_{n-1} = \sum_{i=0}^{n-1} c_{n-i-1} x^{n-i-1} y^i$. Replacing G_{n-1} in the above equation and using that $n = 2l$ we obtain:

$$\textstyle\sum_{i=0}^{2l} \left[(-g-i+l)c_{2l-i-1} - c_{2l-i}\right] x^{2l-i} y^i = \overline{L}_0\ x^l y^l + 2\overline{L}_2\ x^{l-1} y^{l+1},$$

where $c_i = 0$ for $i < 0$ and $i > 2l-1$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-g-i+l)c_{2l-i-1} - c_{2l-i} = 0, \text{ where } i = 0,1,\dots,l-1,l+2,\dots,2l \\ (-g)c_{l-1} - c_l = \overline{L}_0, \\ (-g-1)c_{l-2} - c_{l-1} = 2\overline{L}_2. \end{cases} \tag{6.13}$$

For $i = 2l, \dots, l+2$ we have: $c_0 = c_1 = \cdots = c_{l-2} = 0$. Then, $c_{l-1} = -2\overline{L}_2$, $c_l = 2g\overline{L}_2 - \overline{L}_0$. Working recursively,

$$c_{l+1} = (-g+1)(2g\overline{L}_2 - \overline{L}_0), \qquad c_{l+2} = (-g+1)(-g+2)(2g\overline{L}_2 - \overline{L}_0), \dots$$
$$c_{2l-1} = (-g+1)(-g+2)\dots(-g+l-1)(2g\overline{L}_2 - \overline{L}_0).$$

Replacing c_{2l-1} in (6.13) where $i = 0$, we have:

$$(-g+l)(-g+l-1)\dots(-g+1)(2g\overline{L}_2 - \overline{L}_0) = 0.$$

Then, if $g \notin \{l, l-1, \dots, 1\}$ we must have $\overline{L}_0 = 2g\overline{L}_2$ and

$$G_{n-1} = -2\overline{L}_2 x^{l-1} y^l.$$

Therefore, $L = 2g\overline{L}_2 + \overline{L}_2 y$.

Equating the terms of degree $n-1$ in (6.11) we have:

$$2aG_{n,x} + x\, G_{n-1,x} + \left[gx^2 + xy\right] G_{n-2,x} + a(2g-1)G_{n,y} - y\, G_{n-1,y}$$
$$+ \left[(g-1)xy + y^2\right] G_{n-2,y} + [l(1-2g)x - 2ly]\, G_{n-2} = al\overline{L}_2 x^{l-1} y^l.$$

Set $G_{n-2} = \sum_{i=0}^{n-2} c_{n-i-2} x^{n-i-2} y^i$. Replacing G_{n-2} in the above equation and using that $n = 2l$ we obtain:

$$\sum_{i=0}^{2l} \left[(-2g-i+l)c_{2l-i-2} - 2c_{2l-i-1}\right] x^{2l-i-1} y^i = al(-2g+1)\overline{L}_2\, x^l y^{l-1}$$
$$+ \left(\overline{L}_2(al-2) - 2al + a(2g-1)(l+1)\overline{L}_2\right) x^{l-1} y^l + 2a(l-1)\overline{L}_2\, x^{l-2} y^{l+1},$$

where $c_i = 0$ for $i < 0$ and $i > 2l-2$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-2g-i+l)c_{2l-i-2} - 2c_{2l-i-1} = 0, \ \text{where } i = 0, 1, \dots, l-2, l+2, \dots, 2l-1 \\ (-2g+1)c_{l-1} - 2c_l = al(-2g+1)\overline{L}_2, \\ (-2g)c_{l-2} - 2c_{l-1} = \overline{L}_2(al-2) - 2al + a(2g-1)(l+1)\overline{L}_2, \\ (-2g-1)c_{l-3} - 2c_{l-2} = 2a(l-1)\overline{L}_2. \end{cases} \tag{6.14}$$

For $i = 2l-1, \dots, l+2$, we obtain: $c_0 = c_1 = \cdots = c_{l-3} = 0$. Then,

$$c_{l-2} = -a(l-1)\overline{L}_2, \qquad c_{l-1} = al + \overline{L}_2\left(1 + \frac{a}{2} - 2ag\right),$$

$$c_l = \frac{al}{2}(2g-1)\overline{L}_2 + \frac{(-2g+1)}{2}\left(\overline{L}_2 + al + \frac{a\overline{L}_2}{2} - 2ag\overline{L}_2\right) \doteq A$$

Working recursively, we have:

$$c_{l+1} = \frac{(-2g+2)A}{2}, \qquad c_{l+2} = \frac{(-2g+2)(-2g+3)A}{2^2}, \dots$$

$$c_{2l-2} = \frac{(-2g+l-1)(-2g+l-2)\dots(-2g+2)A}{2^{l-2}}.$$

Replacing c_{2l-2} in (6.14) where $i=0$ we obtain:

$$\frac{(-2g+l)(-2g+l-1)\dots(-2g+2)A}{2^{l-2}}=0, \text{ or}$$

$$\frac{(-2g+l)(-2g+l-1)\dots(-2g+2)(-2g+1)}{2^{l-1}}\left(-al\overline{L}_2+\overline{L}_2+al+\frac{a\overline{L}_2}{2}-2ag\overline{L}_2\right)=0$$

If $g\notin\left\{\frac{1}{2},\frac{2}{2},\frac{3}{2},\dots\right\}$, this is $(a,g)\notin \mathrm{L}_1$ then we must have:

$$-al\overline{L}_2+\overline{L}_2+al+\frac{a\overline{L}_2}{2}-2ag\overline{L}_2=0,$$

that happens if, and only if,

$$\overline{L}_2=\frac{-2al}{-2al+2+a-4ag}, \quad \text{for } -2al+2+a-4ag\neq 0, \quad \text{or} \quad g=\frac{2+a-2al}{4a}.$$

Suppose that $(a,g)\notin \mathrm{L}_1\cup \mathrm{C}'$ where

$$\mathrm{C}'=\cup_{k\in\mathbb{N}}\mathrm{C}_k=\cup_{k\in\mathbb{N}}\left\{(a,g): g=\frac{2+a-2ak}{4a} \text{ and } a\neq 0\right\}.$$

Therefore, we have:

$$\begin{aligned}
G_n&=x^ly^l+\frac{2al}{(-2al+2+a-4ag)}x^{l-1}y^{l+1}, \qquad G_{n-1}=\frac{4al}{(-2al+2+a-4ag)}x^{l-1}y^l,\\
G_{n-2}&=\frac{2a^2l(l-1)}{(-2al+2+a-4ag)}x^{l-2}y^l-\frac{2a^2l^2}{(-2al+2+a-4ag)}x^{l-1}y^{l-1},\\
L&=\frac{-4agl}{(-2al+2+a-4ag)}-\frac{2al}{(-2al+2+a-4ag)}y.
\end{aligned}$$

Equating the terms of degree $n-2$ in (6.11) we have:

$$\begin{aligned}
&2aG_{n-1,x}+x\,G_{n-2,x}+\left[gx^2+xy\right]G_{n-3,x}+a(2g-1)G_{n-1,y}-y\,G_{n-2,y}\\
&\quad+\left[(g-1)xy+y^2\right]G_{n-3,y}+\left[l(1-2g)x-2ly\right]G_{n-3}=\frac{-4a^2gl^2}{(-2al+2+a-4ag)}x^{l-1}y^{l-1}.
\end{aligned}$$

Set $G_{n-3}=\sum_{i=0}^{n-3}c_{n-i-3}x^{n-i-3}y^i$. Replacing G_{n-3} in the above equation and using that $n=2l$ we obtain:

$$\begin{aligned}
&\sum_{i=0}^{2l-2}\left[(-3g-i+l)c_{2l-i-3}-3c_{2l-i-2}\right]x^{2l-i-2}y^i\\
&\qquad=\frac{4a^2l^2(1-3g)}{(-2al+2+a-4ag)}x^{l-1}y^{l-1}-\frac{4a^2l(l-1)}{(-2al+2+a-4ag)}x^{l-2}y^l,
\end{aligned}$$

where $c_i=0$ for $i<0$ and $i>2l-3$. Equating the coefficients of x^iy^j in the above equation, we have:

$$\begin{cases}
(-3g-i+l)c_{2l-i-3}-3c_{2l-i-2}=0, \text{ where } i=0,1,\dots,l-2,l+1,\dots,2l-2\\
(-3g+1)c_{l-2}-3c_{l-1}=\dfrac{4a^2l^2(1-3g)}{(-2al+2+a-4ag)},\\
(-3g)c_{l-3}-3c_{l-2}=-\dfrac{4a^2l(l-1)}{(-2al+2+a-4ag)}.
\end{cases} \tag{6.15}$$

For $i = 2l-2, \dots, l+1 : c_0 = c_1 = \cdots = c_{l-3} = 0$. Then,

$$c_{l-2} = \frac{4a^2 l(l-1)}{3(-2al+2+a-4ag)}, \qquad c_{l-1} = \frac{(4a^2 l(2l-1))(1-3g)}{(-9)(-2al+2+a-4ag)} \doteq B.$$

Working recursively, we obtain:

$$c_l = \frac{(-3g+2)B}{3}, \qquad c_{l+1} = \frac{(-3g+2)(-3g+3)B}{3^2}, \dots$$
$$c_{2l-3} = \frac{(-3g+2)(-3g+3)\dots(-3g+l-1)B}{3^{l-3}}.$$

Replacing c_{2l-3} in (6.15) where $i = 0$:

$$\frac{(-3g+l)(-3g+l-1)\dots(-3g+2)B}{3^{l-2}} = 0, \quad \text{or}$$

$$\frac{(-3g+l)(-3g+l-1)\dots(-3g+2)(-3g+1)}{3^l}\left(\frac{4a^2 l(2l-1)}{2al-2-a+4ag}\right) = 0.$$

Consider $\mathrm{L}_2 = \cup_{k\in\mathbb{N}} \mathrm{L}_{2,k} = \cup_{k\in\mathbb{N}} \left\{(a,g) : g = \frac{k}{3}\right\}$. Then, if $(a,g) \notin (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$ we must have

$$4a^2 l(2l-1) = 0.$$

What happens if, and only if $a = 0$ or $l = 0$ or $l = 1/2$. Therefore, systems (A) have no exponential factors for $(a,g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$

Case 4: $n > 2l$.

Consider the notation for G_n introduced in the study of Case 2. Equating the terms of degree $n+1$ in (6.11) we have:

$$\left[gx^2 + xy\right]\sum_{i=0}^{n}(n-i)c_{n-i}x^{n-i-1}y^i + \left[(g-1)xy + y^2\right]\sum_{i=0}^{n} ic_{n-i}x^{n-i}y^{i-1}$$
$$+ [l(1-2g)x - 2ly]\sum_{i=0}^{n} c_{n-i}x^{n-i}y^i = 0.$$

Working in a similar way to the previous cases, we obtain:

$$\sum_{i=0}^{n+1}\left[(gn - i + l(1-2g))c_{n-i} + (n-2l)c_{n-i+1}\right]x^{n-i+1}y^i = 0,$$

when $c_i = 0$ for $i < 0$ and $i > n$. Therefore,

$$(gn - i + l(1-2g))c_{n-i} + (n-2l)c_{n-i+1} = 0,$$

for $i = 0, 1, \dots, n+1$. As $n \neq 2l$ we have

$$c_0 = c_1 = \cdots = c_n = 0.$$

Then, $G_n = 0$.

Summing up these four cases the proof follows.

(c) Suppose $(a, g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_2 \cup \mathrm{C}')$. Then, by (a) and (b) we get that the systems (A) have only the algebraic solution

$$J_1(x, y) = a + xy$$

with cofactor $\alpha_1 = (-1 + 2g)x + 2y$ and they have no exponential factor. Under these assumptions

$$\begin{cases} \lambda_1 \alpha_1 = 0 \Leftrightarrow \lambda_1 = 0 \quad \text{and} \\ \lambda_1 \alpha_1 = -\operatorname{div}(P, Q) = -(1 + 2gx + y) - (-1 + (g-1)x + 2y) \text{ has no solution.} \end{cases}$$

Hence, from the Darboux theory of integrability it follows that systems (A) are not Liouvillian integrable. □

6.2 The systems (B)

$$\begin{cases} \dot{x} = 2a + gx^2 + xy, \\ \dot{y} = a(2g-1) + (g-1)xy + y^2, \end{cases}$$

where $a(g-1) \neq 0$.

Theorem 6.2.

(a) If $(a, g) \notin \mathrm{L}_1$ then the only invariant algebraic curves of a system in the family (B) are of the form $J_1^m = 0$ where $J_1(x, y) = a + xy$ and m is a positive integer.

(b) If $(a, g) \notin (\mathrm{L}_1 \cup \mathrm{L}_3)$ then any system in the family (B) has no exponential factors.

(c) If $(a, g) \notin (\mathrm{L}_1 \cup \mathrm{L}_3)$ then any system in the family (B) is not Liouvillian integrable.

Proof. (a) By a straightforward computation, we can verify that $J_1(x, y) = a + xy$ is an invariant hyperbola with cofactor $\alpha_1(x, y) = (-1 + 2g)x + 2y$. Assume that

$$C = \sum_{i=0}^{n} C_i(x, y) = 0$$

is an invariant algebraic curve of the systems (B) with cofactor $K = K_0 + K_1 x + K_2 y$, where C_i are homogeneous polynomial of degree i where $0 \leq i \leq n$. From the definition of the invariant algebraic curve (1.4), we have:

$$\left(2a + gx^2 + xy\right) \sum_{i=0}^{n} C_{i,x} + \left(a(2g-1) + (g-1)xy + y^2\right) \sum_{i=0}^{n} C_{i,y} = (K_0 + K_1 x + K_2 y) \sum_{i=0}^{n} C_i. \quad (6.16)$$

The step where we take the terms of degree $n+1$ in (6.16) is exactly the same made for system (A). Then we have:

$$C_n = x^m\, y^l, \qquad \text{where } n = m + l,$$

$$K_1 = gm + (g-1)l; \quad K_2 = m + l.$$

Now, taking from (6.16) the terms of degree n we have:

$$
\begin{aligned}
(gx^2+xy)\ C_{n-1,x} + ((g-1)xy+y^2)\ C_{n-1,y} \\
= K_0\ C_n + [(gm+(g-1)l)\ x + (m+l)\ y]\ C_{n-1}.
\end{aligned} \tag{6.17}
$$

Set $C_{n-1} = \sum_{i=0}^{n-1} c_{n-1-i} x^{n-1-i} y^i$. Replacing C_n, C_{n-1} in (6.17) and doing some calculations, we obtain:

$$
\sum_{i=0}^{m+l-1} (l-i-g)\ c_{m+l-1-i}\ x^{m+l-i}\ y^i - \sum_{i=0}^{m+l-1} c_{m+l-1-i}\ x^{m+l-1-i}\ y^{i+1} = K_0\ x^m y^l.
$$

This equation can be written as

$$
\sum_{i=0}^{m+l} [(l-i-g)\ c_{m+l-1-i} - \ c_{m+l-i}]\ x^{m+l-i}\ y^i = K_0\ x^m y^l,
$$

where $c_i = 0$ for $i < 0$ and $i > m+l-1$. Equating the coefficients of $x^i y^j$ in the above equation, we get:

$$
\begin{cases}
(l-i-g)\ c_{m+l-1-i} - \ c_{m+l-i} = 0, \text{ where } i = 0,1,\ldots,l-1,l+1,\ldots,m+l \\
(-g)\ c_{m-1} - \ c_m = K_0.
\end{cases} \tag{6.18}
$$

For $i = m+l, m+l-1, \ldots, l+1$ we have:

$$
c_0 = c_1 = \cdots = c_{m-1} = 0.
$$

Then $c_m = -K_0$. Working recursively we have

$$
\begin{aligned}
c_{m+1} &= (-g+1)(-K_0), \\
c_{m+2} &= (-g+1)(-g+2)(-K_0), \ldots \\
c_{m+l-1} &= (-g+1)(-g+2)\ldots(-g+l-1)(-K_0).
\end{aligned}
$$

Replacing c_{m+l-1} in (6.18) where $i = 0$, we get

$$
(-g+l)(-g+l-1)\ldots(-g+2)(-g+1)(-K_0) = 0.
$$

Note that

$$
(-g+l)(-g+l-1)\ldots(-g+2)(-g+1)
$$

is a polynomial of degree l in the variable g, which has at most l real roots. Denote by S_1^l the set of roots. If $g \notin S_1^l$ we have that $K_0 = 0$. Therefore, we can conclude that

$$
K = (gm+(g-1)l)x + (m+l)y,
$$

since $g \notin S_1^l$. This is $C_{n-1} \equiv 0$.

The step where we take the terms of degree $n-1$ in (6.16) is exactly the same made for system (A). Then we have $m = l$ which leads us to

$$
\begin{aligned}
&K = (2g-1)mx + 2my, \\
&C_n = x^m y^m, \qquad C_{n-1} \equiv 0, \qquad C_{n-2} = amx^{m-1}y^{m-1},
\end{aligned} \tag{6.19}
$$

for $g \notin S_1^m \cup S_2^m$ numerable set, where S_2^m is the set of roots of the polynomial

$$(-2g+1)(-2g+2)\dots(-2g+m-1)(-2g+m).$$

Following similar arguments for terms of degree $n-2, n-3, \dots$ in (6.16) we can prove that $C = (a+xy)^m$. It follows that $C = J_1^m$, $m \in \mathbb{N}$ for all $(a,g) \notin \mathrm{L}_1$, where $\mathrm{L}_1 = \cup_{k\in\mathbb{N}}\mathrm{L}_{1,k} = \cup_{k\in\mathbb{N}}\left\{(a,g) \in \mathbb{R}^2 : g = \frac{k}{2} \text{ and } a \neq 0\right\}$.

(b) From (a) systems (B) have only the algebraic solution $J_1(x,y) = a + xy$ for $(a,g) \in \mathbb{R}^2 - \mathrm{L}_1$. Then, by Proposition 2.10, if systems (B) have an exponential factor, it must have the form

$$F = \exp\left(G/J_1^l\right)$$

with a cofactor $L = \overline{L}_0 + \overline{L}_1 x + \overline{L}_2 y$ and where l is non-negative integers. Since the invariant algebraic curve $J_1^l = 0$ has the cofactor

$$K = l\alpha_1 = l(-1+2g)x + 2ly,$$

it follows by (2.2) that G satisfies the following equation:

$$\begin{aligned}\left[2a+gx^2+xy\right]G_x + \left[a(2g-1)+(g-1)xy+y^2\right]G_y + [l(1-2g)x - 2ly]\,G \\ = [\overline{L}_0 + \overline{L}_1 x + \overline{L}_2 y]\sum_{k=0}^{l}\binom{l}{k} a^k x^{l-k} y^{l-k}\end{aligned} \tag{6.20}$$

From now on we assume $G(x,y) = \sum_{i=0}^n G_i(x,y)$, where G_i is a homogeneous polynomial of degree i and split the study in cases.

Case 1: $n+1 < 2l$.

This case is exactly the same proved for systems (A). We have that F is constant, what cannot happen.

Case 2: $n+1 = 2l$.

This case is also the same proved for systems (A). We have that if $g \notin \{l, l-1, \dots, 1\}$, this is $(a,g) \notin \mathrm{L}_1$ then $L = 0$. Consequently, system (B) has no exponential factors for $(a,g) \in \mathbb{R}^2 - \mathrm{L}_1$.

Case 3: $n = 2l$.

Set $G_n = \sum_{i=0}^n c_{n-i} x^{n-i} y^i$, where c_{n-i} are constants. Equating the terms of (6.20) with degree $n+1$ we have

$$\textstyle\sum_{i=0}^{2l+1}(l-i)c_{2l-i}x^{2l-i+1}y^i = \overline{L}_1 x^{l+1}y^l + \overline{L}_2 x^l y^{l+1},$$

where $c_i = 0$ for $i < 0$ and $i > n$. This is the same equation solved for systems (A) in case 3. Then we have:

$$G_n = x^l y^l - \overline{L}_2 x^{l-1} y^{l+1} \quad \text{and} \quad \overline{L}_1 = 0.$$

Equating the terms of degree n in (6.20) we have

$$\left[gx^2+xy\right]G_{n-1,x} + \left[(g-1)xy+y^2\right]G_{n-1,y} + [l(1-2g)x - 2ly]\,G_{n-1} = \overline{L}_0\, x^l y^l.$$

Set $G_{n-1} = \sum_{i=0}^{n-1} c_{n-i-1} x^{n-i-1} y^i$. Replacing G_{n-1} in the above equation and using that $n = 2l$ we obtain:

$$\sum_{i=0}^{2l} \left[(-g-i+l)c_{2l-i-1} - c_{2l-i}\right] x^{2l-i} y^i = \overline{L}_0\, x^l y^l,$$

where $c_i = 0$ for $i < 0$ and $i > 2l-1$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-g-i+l)c_{2l-i-1} - c_{2l-i} = 0, \text{ where } i = 0, 1, \dots, l-1, l+1, \dots, 2l \\ (-g)c_{l-1} - c_l = \overline{L}_0. \end{cases} \tag{6.21}$$

Then

$$\begin{aligned} c_0 = c_1 = \cdots = c_{l-1} = 0, &\quad c_l = -\overline{L}_0, \\ c_{l+1} = (-g+1)(-\overline{L}_0), &\quad c_{l+2} = (-g+1)(-g+2)(-\overline{L}_0), \dots \\ c_{2l-1} = (-g+1)(-g+2)\dots(-g+l-1)(-\overline{L}_0). \end{aligned}$$

Replacing c_{2l-1} in (6.21) where $i = 0$, we have:

$$(g+l)(-g+l-1)\dots(-g+1)(-\overline{L}_0) = 0.$$

Then, if $g \notin \{l, l-1, \dots, 1\}$ we must have $\overline{L}_0 = 0$. Therefore,

$$G_{n-1} \equiv 0 \quad \text{and} \quad L = \overline{L}_2 y.$$

Equating the terms of degree $n-1$ in (6.20) we have:

$$\begin{aligned} 2aG_{n,x} + \left[gx^2 + xy\right] G_{n-2,x} + a(2g-1)G_{n,y} + \left[(g-1)xy + y^2\right] G_{n-2,y} \\ + \left[l(1-2g)x - 2ly\right] G_{n-2} = al\overline{L}_2 x^{l-1} y^l. \end{aligned}$$

Set $G_{n-2} = \sum_{i=0}^{n-2} c_{n-i-2} x^{n-i-2} y^i$. Replacing G_{n-2} in the above equation and using that $n = 2l$ we obtain:

$$\begin{aligned} \sum_{i=0}^{2l} \left[(-2g-i+l)c_{2l-i-2} - 2c_{2l-i-1}\right] x^{2l-i-1} y^i = (al(1-2g))x^l y^{l-1} \\ + (\overline{L}_2 al - 2al + a(2g-1)(l+1)\overline{L}_2)x^{l-1}y^l + 2a(l-1)\overline{L}_2 x^{l-2} y^{l+1}, \end{aligned}$$

where $c_i = 0$ for $i < 0$ and $i > 2l-2$. Equating the coefficients of $x^i y^j$ in the above equation, we have:

$$\begin{cases} (-2g-i+l)c_{2l-i-1} + (-2)c_{2l-i} = 0, \text{ where } i = 0, 1, \dots, l-2, l+2, \dots, 2l-1 \\ (-2g+1)c_{l-1} + (-2)c_l = al(1-2g), \\ (-2g)c_{l-2} + (-2)c_{l-1} = \overline{L}_2 al - 2al + a(2g-1)(l+1)\overline{L}_2, \\ (-2g-1)c_{l-3} + (-2)c_{l-2} = 2a(l-1)\overline{L}_2. \end{cases} \tag{6.22}$$

Therefore,

$$\begin{aligned} &c_0 = c_1 = \cdots = c_{l-3} = 0, \quad c_{l-2} = -a(l-1)\overline{L}_2, \quad c_{l-1} = \frac{a}{2}(2l + \overline{L}_2) - 2ag\overline{L}_2, \\ &c_l = \frac{\overline{L}_2 a}{2}\left(-2g + \frac{1}{2}\right)(-2g+1) \doteq A, \quad c_{l+1} = \frac{(-2g+2)A}{2}, \\ &c_{l+2} = \frac{(-2g+2)(-2g+3)A}{4}, \dots, c_{2l-2} = \frac{(-2g+l-1)(-2g+l-2)\dots(-2g+2)}{2^{l-2}}\, A. \end{aligned}$$

Replacing c_{2l-2} in (6.22) where $i = 0$ we have:

$$\frac{(-2g+l)(-2g+l-1)\dots(-2g+2)}{2^{l-2}}A = 0, \quad \text{or}$$

$$\frac{(-2g+l)(-2g+l-1)\dots(-2g+2)(-2g+1)}{2^{l-2}}\left[\frac{\overline{L}_2 a}{2}\left(-2g+\frac{1}{2}\right)\right] = 0.$$

Then, if $g \notin \{1,2,\dots,1\} \cup \{\frac{1}{2},\frac{2}{2},\dots,\frac{l}{2}\} \cup \{1/4\}$, this is $(a,g) \notin (\mathrm{L}_1 \cup \mathrm{L}_3)$ we must have

$$\overline{L}_2 = 0.$$

Therefore, $L = 0$ and systems (B) have no exponential factors for $(a,g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_3)$.

Case 4: $n > 2l$.

This case is the same proved for systems (A). Then, $G_n = 0$ that cannot happen.

Summing up these four cases the proof follows.

(c) Suppose $(a,g) \in \mathbb{R}^2 - (\mathrm{L}_1 \cup \mathrm{L}_3)$. Then, by (a) and (b) we get that the systems (B) have only the algebraic solution

$$J_1(x,y) = a + xy$$

with cofactor $\alpha_1 = (-1+2g)x + 2y$ and they have no exponential factor. Under these assumptions

$$\begin{cases} \lambda_1\alpha_1 = 0 \Leftrightarrow \lambda_1 = 0 \text{ and} \\ \lambda_1\alpha_1 = -\operatorname{div}(P,Q) = -(2gx+y) - ((g-1)x+2y) \text{ has no solution.} \end{cases}$$

Hence, from the Darboux theory of integrability it follows that systems (B) are not Liouvillian integrable. □

7 Geometric study of the families (C) and (D)

In this section we present a detailed study of two of the normal forms (C) and (D) for the family $\mathbf{QSH}_{\eta=0}$. We note that we obtained, as limiting cases of the family (D), three other normal forms, i.e. (F), (G) and (I). This is part of the more ample project of gathering data on the geometry of polynomial systems as expressed in the configurations of invariant algebraic curves and their impact on integrability. This data is useful in gaining more insight into the Darboux theory of integrability in order to enable us to answer some questions and in particular to give an answer to the problem of Poincaré for specific families of polynomial differential systems. We are also interested in the topological phase portraits of systems in $\mathbf{QSH}_{\eta=0}$ and their bifurcation diagrams. Is there any relationship between the two kinds of bifurcation diagrams? Can we determine when we have algebraically integrable systems? All these motivate our study in this section and we answer some of these questions here or in the last section.

We first present the results of our calculations of the geometric features of the configurations as well as the information on the singularities. Afterwards we sum up these in a proposition and in pictures of the two bifurcation diagrams: one of the changes in the configurations, the other on the topological bifurcation of the systems.

7.1 Geometric analysis of family (C)

Consider the family

$$(\mathrm{C}) \quad \begin{cases} \dot{x} = 2a + 3cx + x^2 + xy \\ \dot{y} = a - c^2 + y^2, \end{cases}$$

where $a \neq 0$.

For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (C) we study here also the limit case $a = 0$ where the equations are still defined.

In the **generic case**

$$a(a - c^2)(a - 8c^2/9) \neq 0$$

the systems have two invariant lines J_1 and J_2 and only one invariant hyperbolas J_3 with respective cofactors α_i, $1 \leq i \leq 3$ where

$$\begin{aligned} J_1 &= y - \sqrt{c^2 - a}, & \alpha_1 &= y + \sqrt{c^2 - a}, \\ J_2 &= y + \sqrt{c^2 - a}, & \alpha_2 &= y - \sqrt{c^2 - a}, \\ J_3 &= a + cx + xy, & \alpha_3 &= 2c + x + 2y. \end{aligned}$$

We note that if $a = c^2$ the two lines coincide and we get a double line. Also if $a = 8v^2/9$ we get a double hyperbola as we later prove.

The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the lines and the 2nd extactic polynomial for the hyperbola.

(i) The generic case: $a(a-c^2)(a-8c^2/9) \neq 0$.

Table 7.1: Invariant curves, cofactors, singularities and intersection points of family (C) for the generic case.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = y - \sqrt{c^2-a}$ $J_2 = y + \sqrt{c^2-a}$ $J_3 = a + cx + xy$ $\alpha_1 = y + \sqrt{c^2-a}$ $\alpha_2 = y - \sqrt{c^2-a}$ $\alpha_3 = 2c + x + 2y$	$P_1 = (-\sqrt{c^2-a}-c, -\sqrt{c^2-a})$ $P_2 = (-2(\sqrt{c^2-a}+c), \sqrt{c^2-a})$ $P_3 = (\sqrt{c^2-a}-c, \sqrt{c^2-a})$ $P_4 = (2(\sqrt{c^2-a}-c), -\sqrt{c^2-a})$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $a < \frac{8c^2}{9}$ we have $a, s, a, s; \binom{0}{2}SN, N$ For $\frac{8c^2}{9} < a < c^2$ we have $s, s, a, a; \binom{0}{2}SN, N$ if $c < 0$ $a, a, s, s; \binom{0}{2}SN, N$ if $c > 0$ For $c^2 < a$ we have $©, ©, ©, ©; \binom{0}{2}SN, N$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.2: Divisor and zero-cycles of family (C) for the generic.

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + \mathcal{L}_\infty \text{ if } a < c^2 \\ J_1^C + J_2^C + J_3 + \mathcal{L}_\infty \text{ if } a > c^2 \end{cases}$	4 4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty \text{ if } a < c^2 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty \text{ if } a > c^2 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = \begin{cases} 2P_1 + P_2 + 2P_3 + P_4 + 2P_1^\infty + 4P_2^\infty \text{ if } a < c^2 \\ 2P_1^C + P_2^C + 2P_3^C + P_4^C + 2P_1^\infty + 4P_2^\infty \text{ if } a < c^2 \end{cases}$	12 12

where the total curve T has four distinct tangents at P_2^∞.

Remark 7.1. Mathematica could not give a response for the computation of the first integral of family (C) in this generic case.

Table 7.3: Integrating factor of family (C) for the generic case.

	Integrating Factor
General	$R = J_1^{\frac{c+\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_2^{\frac{-c+\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_3^{-2}$
Simple example	$\mathcal{R} = J_1^{\frac{c+\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_2^{\frac{-c+\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_3^{-2}$

(ii) The non-generic cases: $a(a-c^2)(a-8c^2/9)=0$.

(ii.1) $a = c^2$ and $c \neq 0$.

Table 7.4: Invariant curves, exponential factor, cofactors, singularities and intersection points of family (C) for $a = c^2$ and $c \neq 0$.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = \frac{xy}{c} + c + x$ $E_3 = e^{\frac{g_0+g_1 y}{y}}$	$P_1 = (-2c, 0)$ $P_2 = (-c, 0)$	$\bar{J}_1 \cap \bar{J}_2 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$
	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$	$\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple
$\alpha_1 = y$ $\alpha_2 = 2c + x + 2y$ $\alpha_3 = -g_0$	$sn, sn; \binom{0}{2}SN, N$	$\bar{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.5: Divisor and zero-cycles of family (C) for $a = c^2$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + J_2 + \mathcal{L}_\infty$	4
$M_{0CS} = 2P_1 + 2P_2 + 2P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^2 \bar{J}_2 = 0$	5
$M_{0CT} = 2P_1 + 3P_2 + 2P_1^\infty + 4P_2^\infty$	11

Table 7.6: First integral and integrating factor of family (C) for $a = c^2$ and $c \neq 0$.

	First integral	Integrating Factor
General	I	$R = J_1 J_2^{-2} E_3^{-\frac{c}{g_0}}$
Simple example	$\mathcal{I}$	$\mathcal{R} = J_1 J_2^{-2} E_3^{-c}$

$$I = \mathcal{I} = \frac{c^2\left(e^{c/y} E_i\left(-\frac{c}{y}\right)(c^2+cx+xy) + y(c+x-y)\right)\left(e^{\frac{g_0}{y}+g_1}\right)^{-\frac{c}{g_0}}}{c^2+cx+xy}$$

where $E_i(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t}\, dt$ is the exponential integral function that has a branch cut discontinuity in the complex z plane running from $-\infty$ to 0.

(ii.2) $a = 8c^2/9$ and $c \neq 0$.

Table 7.7: Invariant curves, exponential factor, cofactors, singularities and intersection points of family (C) for $a = 8c^2/9$ and $c \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -c + 3y$ $J_2 = c + 3y$ $J_3 = 8c^2 + 9cx + 9xy$ $E_4 = e^{\frac{c^2(48g_0 - g_1 x + 48 g_1 y) + 54 c g_0 x + 3 c g_1 y(21x - 8y) + 54 g_0 xy}{48c^2(8c^2+9cx+9xy)}}$ $\alpha_1 = y + \frac{c}{3}$ $\alpha_2 = y - \frac{c}{3}$ $\alpha_3 = 2c + x + 2y$ $\alpha_4 = -\frac{g_1(c-3y)}{54c}$	$P_1 = \left(-\frac{8c}{3}, \frac{c}{3}\right)$ $P_2 = \left(-\frac{4c}{3}, -\frac{c}{3}\right)$ $P_3 = \left(-\frac{2c}{3}, \frac{c}{3}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, sn, a; \binom{0}{2}SN, N$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.8: Divisor and zero-cycles of family (C) for $a = 8c^2/9$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$	5
$M_{0CS} = P_1 + 2P_2 + P_3 + 2P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0$	7
$M_{0CT} = P_1 + 3P_2 + 3P_3 + 3P_1^\infty + 5P_2^\infty$	15

Table 7.9: First integral and integrating factor of family (C) for $a = 8c^2/9$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^0 E_4^{-\frac{18c\lambda_2}{g_1}}$	$R = J_1^2 J_2^{\lambda_2'} J_3^{-2} E_4^{-\frac{18(c\lambda_2'+c)}{g_1}}$
Simple example	$\mathcal{I} = J_2 E_4^{-18c}$	$\mathcal{R} = \frac{J_1^2}{J_2 J_3^2}$

(ii.3) $a = 0$ and $c \neq 0$.

Under this condition, systems defined by (C) do not belong to **QSH**. All the invariant lines are $x = 0$ and $\pm c + y = 0$ that are simple. By perturbing the reducible conic $x(c + y) = 0$ we can produce the hyperbola $a + cx + xy = 0$. Furthermore, the conic $x(c + y) = 0$ has integrable multiplicity two.

Table 7.10: Invariant curves, exponential factor, cofactors, singularities and intersection points of family (C) for $a = 0$ and $c \neq 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -c + y$ $J_2 = c + y$ $J_3 = x$ $E_4 = e^{\left(-\frac{c^2(g_0 - 3g_1 x) + 2cg_0(x-y) - 3cg_1 xy + g_0 y^2}{3cx(c+y)}\right)}$ $\alpha_1 = c + y$ $\alpha_2 = -c + y$ $\alpha_3 = 3c + x + y$ $\alpha_4 = \frac{g_0(y-c)}{3c}$	$P_1 = (-4c, c)$ $P_2 = (-2c, -c)$ $P_3 = (0, -c)$ $P_4 = (0, c)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, a, s, a; \binom{0}{2}SN, N$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Table 7.11: Divisor and zero-cycles of family (C) for $a = 0$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$	4
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	4
$M_{0CT} = P_1 + P_2 + 2P_3 + 2P_4 + 2P_1^\infty + 3P_2^\infty$	11

Table 7.12: First integral and integrating factor of family (C) for $a = 0$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^0 E_4^{-\frac{3c\lambda_2}{g_0}}$	$R = J_1 J_2^{\lambda_2'} J_3^{-2} E_4^{-\frac{3(2c+c\lambda_2')}{g_0}}$
Simple example	$\mathcal{I} = J_2 E_4^{-3c}$	$\mathcal{R} = \frac{J_1}{J_2^2 J_3^2}$

(ii.4) $a = c = 0$.

Under this condition, systems defined by (C) do not belong to **QSH**. Here we have a single system which has a generalized Darboux first integral. The affine invariant lines $x = 0$ and $y = 0$ are both double. By perturbing the reducible conic $xy = 0$ we can produce the hyperbola $a + cx + xy = 0$.

Table 7.13: Invariant curves, exponential factor, cofactors, singularities and intersection points of family (C) for $a = c = 0$.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = y$ $E_3 = e^{g_0+\frac{g_1 y}{x}}$ $E_4 = e^{h_0+\frac{h_1}{y}}$ $\alpha_1 = c + y$ $\alpha_2 = -c + y$ $\alpha_3 = -g_1 y$ $\alpha_4 = -h_1$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $phph; \binom{0}{2}SN, N$	$\overline{J}_1 \cap \overline{J}_2 = P_1$ simple $\overline{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\overline{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple

Table 7.14: Divisor and zero-cycles of family (C) for $a = c = 0$.

Divisor and zero-cycles	Degree
$ICD = 2J_1 + 2J_2 + \mathcal{L}_\infty$	5
$M_{0CS} = 4P_1 + 2P_1^\infty + P_2^\infty$	7
$T = Z\overline{J}_1^2\overline{J}_2^2 = 0$	5
$M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$	10

Table 7.15: First integral and integrating factor of family (C) for $a = c = 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{g_1\lambda_3} E_3^{\lambda_3} E_4^0$	$R = J_1^{-2} J_2^{-1+g_1\lambda_3'} E_3^{\lambda_3'} E_4^0$
Simple example	$\mathcal{I} = J_2 E_3$	$\mathcal{R} = \frac{1}{J_1^2 J_2}$

We sum up the topological, dynamical and algebraic geometric features of family (C) and also confront our results with previous results in literature in the following proposition. We show that all the phase portraits for family (C) are missing in [41].

Proposition 7.2.

(a) For the family (C) we obtained six distinct configurations $C_1^{(C)}$–$C_6^{(C)}$ of invariant hyperbolas and lines (see Figure 7.1 for the complete bifurcation diagram of configurations of this family). The bifurcation set of configurations in the full parameter space is is $a(a - c^2)(a - 8c^2/9) = 0$ and it is made of points of bifurcation due to change in the multiplicities of the invariant algebraic invariant curves: On $a = c^2$ and $c \neq 0$ the invariant lines coalesce into a double line. On $a = 8c^2/9$ and $c \neq 0$ the hyperbola becomes double. Outside the parameter space, i.e. on $a = 0$ the invariant hyperbola becomes reducible producing the lines $x = 0$ and $c + y = 0$ and when also $c = 0$ then $x = 0$ and $y = 0$ become double lines.

(b) The family (C) is Liouvillian integrable for $a(a - 8c^2/9) \neq 0$ and generalized Darboux integrable for $a = 8c^2/9$. All systems in family (C) do not have a polynomial inverse integrating factor. Outside the parameter space, i.e. on $a = 0$ we have a polynomial inverse integrating factor only when $c = 0$.

(c) For the family (C) we have five topologically distinct phase portraits $P_1^{(C)}$–$P_5^{(C)}$. The topological bifurcation set is the same as the one for configurations, i.e. it is $a(a - c^2)(a - 8c^2/9) = 0$ (see Figure 7.2 for the complete topological bifurcation diagram). The parabolas $a = c^2$ and $a = 8c^2/9$ are bifurcation sets of singularities and the line $a = 0$ is a bifurcation of separatrices connection. The phase portraits $P_1^{(C)} - P_5^{(C)}$ are not topologically equivalent with anyone of the phase portraits in [41].

Proof of Proposition 7.2. (a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (C):

Table 7.16: Configurations for family (C).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(C)}$	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + 2P_3 + P_4 + 2P_1^\infty + 4P_2^\infty$
$C_2^{(C)}$	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + 2P_3 + P_4 + 2P_1^\infty + 4P_2^\infty$
$C_3^{(C)}$	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + 2P_3 + P_4 + 2P_1^\infty + 4P_2^\infty$
$C_4^{(C)}$	$ICD = J_1^C + J_2^C + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1^C + P_2^C + 2P_3^C + P_4^C + 2P_1^\infty + 4P_2^\infty$
$C_5^{(C)}$	$ICD = 2J_1 + J_2 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 3P_2 + 2P_1^\infty + 4P_2^\infty$
$C_6^{(C)}$	$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$ $M_{0CT} = P_1 + 3P_2 + 3P_3 + 3P_1^\infty + 5P_2^\infty$

Note that $C_1^{(C)}$, $C_2^{(C)}$ and $C_3^{(C)}$ admit the same type of divisor and zero-cycles but the configurations are non equivalent. In fact, consider the convex quadrilateral in Figure 7.1 formed by the four finite singularities in these configurations. In $C_1^{(C)}$ any two consecutive or opposite points of this quadrilateral are not joined by anyone of the two branches of the hyperbola, in $C_2^{(C)}$, two opposite points are joined by a branch of the hyperbola and in $C_3^{(C)}$ two consecutive points of this quadrilateral is joined by a branch of the hyperbola.

Therefore, the configurations $C_1^{(C)}$ up to $C_6^{(C)}$ are all distinct. For the limit cases of family (C) we have the following configurations:

Table 7.17: Configurations for the limit cases of family (C).

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = P_1 + P_2 + 2P_3 + 2P_4 + 2P_1^\infty + 3P_2^\infty$
c_2	$ICD = 2J_1 + 2J_2 + \mathcal{L}_\infty$ $M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$

Therefore, we have two distinct configurations for the limit cases.

(b) In the generic case $a(a-c^2)(a-8c^2/9) \neq 0$ the three cofactors α_1, α_2, α_3 of J_1, J_2, J_3 are linearly independent. Hence we cannot get a Darboux first integral by using these curves. Furthermore the curves are each of multiplicity 1 and hence we cannot have exponential factors attached to them. However we obtained an integrating factor for (C) in the generic case. Using Mathematica we could not obtain an expression for the first integral of these systems but we know that it exists and it is Liouvillian. For the non-generic cases we obtained first integrals and they were given in previously exhibited tables.

Let us show that the family does not admit a polynomial inverse integrating factor.

(i) The **generic case**: $a(a-c^2)(a-8c^2/9) \neq 0$.

We have the following integrating factor

$$R = J_1^{\frac{c+\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_2^{\frac{-c+\sqrt{c^2-a}}{2\sqrt{c^2-a}}} J_3^{-2}.$$

In order to R^{-1} to be polynomial we must have that

$$\begin{cases} \frac{c+\sqrt{c^2-a}}{2\sqrt{c^2-a}} = \frac{c}{2\sqrt{c^2-a}} + \frac{1}{2} & = -m_1,\ m_1 \in \mathbb{N} \\ \frac{-c+\sqrt{c^2-a}}{2\sqrt{c^2-a}} = \frac{-c}{2\sqrt{c^2-a}} + \frac{1}{2} & = -m_2,\ m_2 \in \mathbb{N}. \end{cases}$$

Adding up these two expressions we have

$$1 = -(m_1 + m_2), \qquad m_1, m_2 \in \mathbb{N}$$

and this equation does not have a solution. Therefore, R^{-1} cannot be polynomial.

(ii) The **non-generic case**: $a(a-c^2)(a-8c^2/9) = 0$.

(ii.1) $a = c^2$: We have the integrating factor

$$R = J_1 J_2^{-2} E_3^{-c/g_0}$$

and it is clear that R^{-1} cannot be polynomial.

(ii.2) $a = 8c^2/9$: We have the integrating factor

$$R = J_1^2 J_2^{\lambda_2'} J_3^{-2} E_4^{-18(c\lambda_2'+c)/g_1}$$

again it is clear that R^{-1} cannot be polynomial.

(ii.3) $a = 0$ and $c \neq 0$. We have the integrating factor

$$R = J_1 J_2^{\lambda_2'} J_3^{-2} E_4^{-3(2c+c\lambda_2')/g_0}.$$

again it is clear that R^{-1} cannot be polynomial.

(ii.4) $a = 0$ and $c = 0$: We have the integrating factor

$$R = J_1^{-2} J_2^{-1+g_1\lambda_3'} E_3^{\lambda_3'} E_4^0.$$

Taking $\lambda_3' = 0$ we have that $R^{-1} = J_1^2 J_2$ which is polynomial.

(c) We have:

Table 7.18: Phase portraits for family (C).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(C)}$	$(\binom{0}{2}SN, N)$	(a, s, a, s)	$4SC_f^f\ \ 6SC_f^\infty\ \ 0SC_\infty^\infty$
$P_2^{(C)}$	$(\binom{0}{2}SN, N)$	(a, s, a, s)	$4SC_f^f\ \ 5SC_f^\infty\ \ 1SC_\infty^\infty$
$P_3^{(C)}$	$(\binom{0}{2}SN, N)$	$(©, ©, ©, ©)$	$0SC_f^f\ \ 0SC_f^\infty\ \ 2SC_\infty^\infty$
$P_4^{(C)}$	$(\binom{0}{2}SN, N)$	(sn, sn)	$1SC_f^f\ \ 5SC_f^\infty\ \ 1SC_\infty^\infty$
$P_5^{(C)}$	$(\binom{0}{2}SN, N)$	(s, sn, a)	$3SC_f^f\ \ 5SC_f^\infty\ \ 1SC_\infty^\infty$

Therefore, we have five distinct phase portraits for systems (C). For the limit case of family (C) we have the following phase portraits:

Table 7.19: Phase portraits for the limit case of family (C).

Phase portraits	Sing. at ∞	Finite sing.	Separatrix connections
p_1	$(\binom{0}{2}SN, N)$	(s, a, s, a)	$4SC_f^f\ \ 5SC_f^\infty\ \ 0SC_\infty^\infty$
p_2	$(\binom{0}{2}SN, N)$	$phph$	$0SC_f^f\ \ 4SC_f^\infty\ \ 0SC_\infty^\infty$

Therefore, we have two topologically distinct phase portraits for the limit cases.

Table 7.20: Phase portraits in [41] that admit 2 singular points at infinity with the type (SN, N).

Phase Portrait	Sing. at ∞	Real finite sing.	Separatrix connections
L_{01}	$(\binom{0}{2}SN, N)$	$\emptyset$	$0SC_f^f\ \ 0SC_f^\infty\ \ 3SC_\infty^\infty$
L_{03}	$(\binom{0}{2}SN, N)$	$\emptyset$	$0SC_f^f\ \ 0SC_f^\infty\ \ 3SC_\infty^\infty$
ω_1	$(\binom{0}{2}SN, N)$	(s, a)	$1SC_f^f\ \ 6SC_f^\infty\ \ 0SC_\infty^\infty$

Therefore, the phase portraits $P_1^{(C)}$–$P_5^{(C)}$ are not topologically equivalent with anyone of the phase portraits in [41]. □

Remark 7.3. Note that c_1 (for example, for $c > 0$) has three distinct lines, each line is an irreducible curve and for these lines the algebraic, integrable and geometric multiplicities coincide and this multiplicity is one. Hence in perturbations the line $y + c = 0$ can produce at most one line and in this case, it produces the line $y + \sqrt{c^2 - a} = 0$.

Remark 7.4. Note that the necessary and sufficient condition for systems defined by the equations (C) to have a double hyperbola or a double line is that it has two singularities of the system of multiplicity two or just one singularity of multiplicity four.

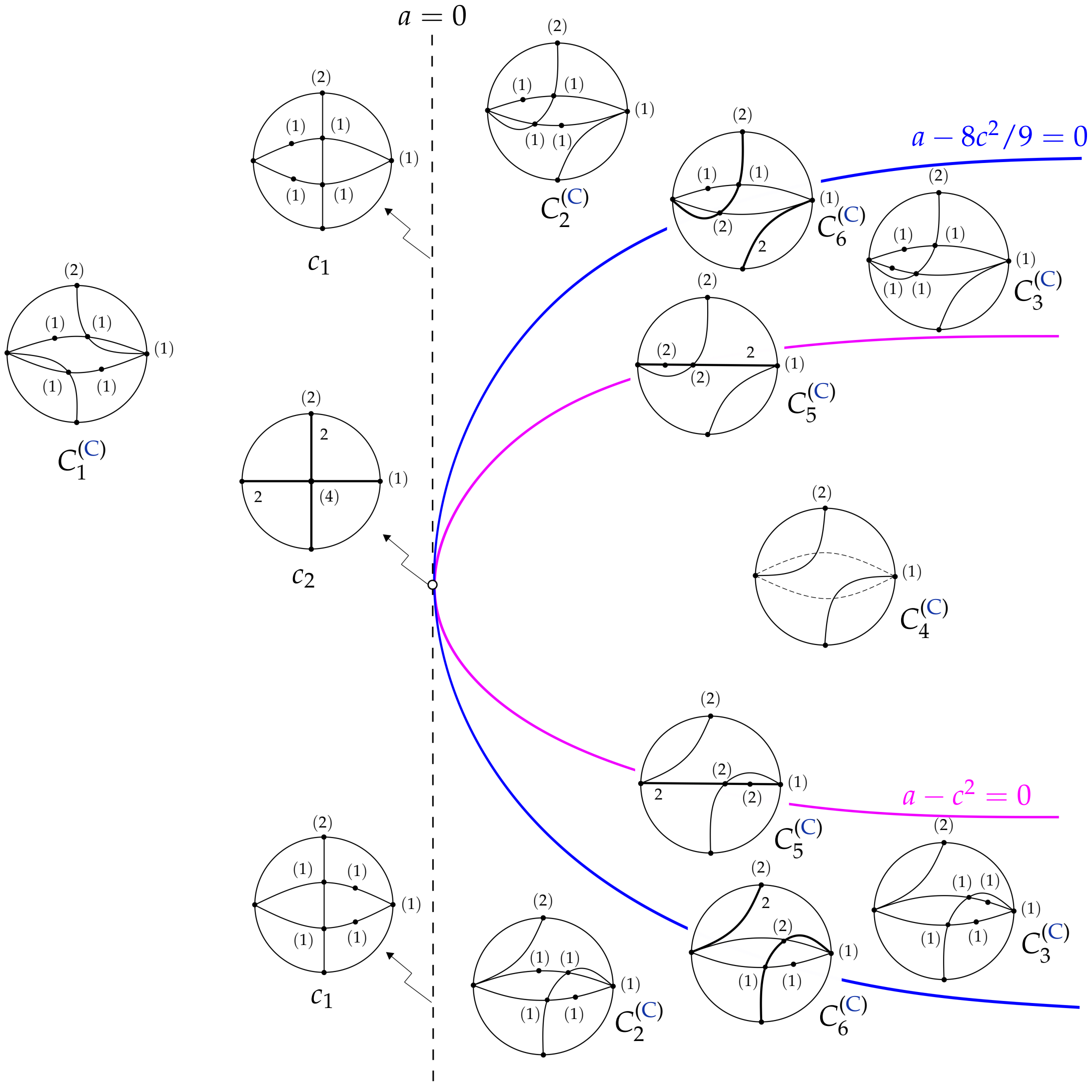

Figure 7.1: Bifurcation diagram of configurations for family (C). The dashed line $a = 0$ is a limit case of this family. The multiple invariant curves are emphasized and the complex curves are drawn as dashed in the drawings of the configurations.

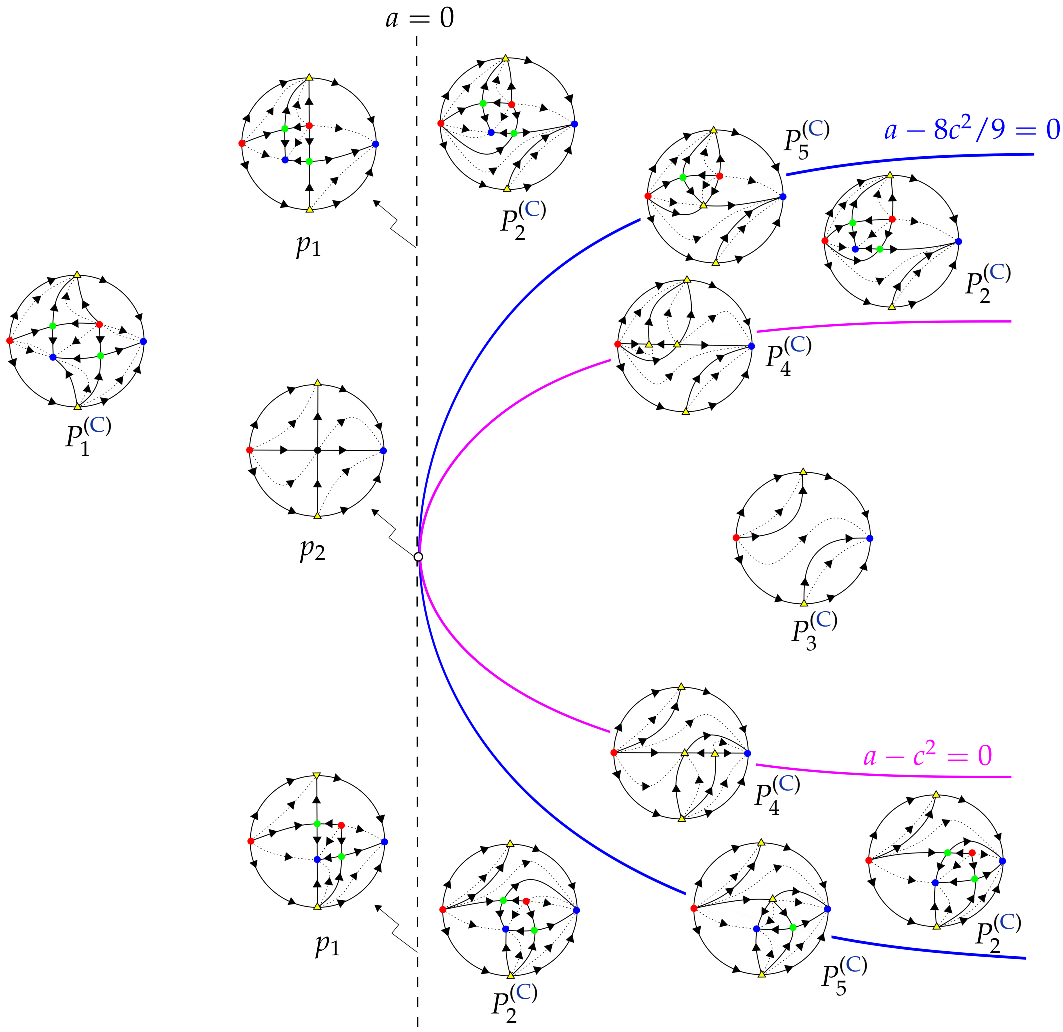

Figure 7.2: Topological bifurcation diagram for family (C). The continuous curves in the phase portraits are separatrices. The dashed curves are the orbits given in each region of the phase portraits. The green bullet represents an elemental saddle, the red bullet an elemental unstable node and the blue an elemental stable node. The yellow triangle represents a saddle-node (semi-elemental) and the black bullet is an intricate singularity.

7.2 Geometric analysis of family (D)

Consider the family:

$$(D)\quad \begin{cases} \dot{x} = (c+x)(c(2g-1)+gx) \\ \dot{y} = 1+(g-1)xy, \end{cases}$$

where $(g \pm 1)(3g-1)(2g-1) \neq 0$ and $c^2 + g^2 \neq 0$.

For a complete understanding of the bifurcation diagram of the systems in the full family defined by the equations (D) we study here also the limit cases $(g \pm 1)(3g-1)(2g-1) = 0$ where the equations are still defined.

In the **generic case**

$$cg(g \pm 1)(3g-1)(2g-1) \neq 0$$

the systems have two invariant lines J_1 and J_2 and only one invariant hyperbola J_3 with respective cofactors α_i, $1 \leq i \leq 3$ where

$$\begin{aligned} J_1 &= c + x, & \alpha_1 &= c(-1+2g) + gx, \\ J_2 &= c(-1+2g) + gx, & \alpha_2 &= -cg + gx, \\ J_3 &= \tfrac{1}{2g-1} + y(c+x), & \alpha_3 &= c(-1+2g) + (-1+2g)x. \end{aligned}$$

We note that if $g = 1$ or $c = 0$ then the two lines coincide and we get a multiple line.

The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using the 1st extactic polynomial for the line and the 2nd extactic polynomial for the hyperbola.

(i) The generic case: $cg(g \pm 1)(3g-1)(2g-1) \neq 0$.

Table 7.21: Invariant curves, cofactors, singularities and intersection points of family (D) for the generic case.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = c + x$ $J_2 = c(-1+2g) + gx$ $J_3 = \frac{1}{2g-1} + y(c+x)$ $\alpha_1 = c(-1+2g) + gx$ $\alpha_2 = -cg + gx$ $\alpha_3 = c(-1+2g) + (-1+2g)x$	$P_1 = \left(-c, \frac{1}{c(g-1)}\right)$ $P_2 = \left(c\left(\frac{1}{g}-2\right), \frac{g}{2cg^2-3cg+c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $c<0$ we have $s, a; \binom{2}{2}E-E, S$ if $g<0$ $s, s; \binom{2}{2}E-E, N$ if $0<g<1/2$ $s, a; \binom{2}{2}PH-PH, N$ if $g>1/2$ For $c>0$ we have $s, a; \binom{2}{2}E-E, S$ if $g<0$ $s, s; \binom{2}{2}E-E, N$ if $0<g<1/2$ $s, a; \binom{2}{2}PH-PH, N$ if $g>1/2$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1^\infty$ double $\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.22: Divisor and zero-cycles of family (D) for the generic.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$	4
$M_{0CS} = P_1 + P_2 + 4P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = P_1 + 2P_2 + 4P_1^\infty + 2P_2^\infty$	9

where the total curve T has only three distinct tangents at P_1^∞, but one of them is double.

Table 7.23: First integral and integrating factor of family (D) for the generic case.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^{-\frac{g\lambda_2}{2g-1}}$	$R = J_1^0 J_2^{\lambda_2'} J_3^{\frac{1-g(\lambda_2'+3)}{2g-1}}$
Simple example	$\mathcal{I} = J_2 J_3^{-\frac{g}{2g-1}}$	$\mathcal{R} = \frac{1}{J_1 J_2}$

(ii) The non-generic case: $cg(g \pm 1)(3g-1)(2g-1) = 0$

(ii.1) $c = 0$ and $g(g \pm 1)(3g-1)(2g-1) \neq 0$.

Here the invariant line $x = 0$ has multiplicity 3 so we compute the exponential factors E_3 and E_4. The line at infinity and the hyperbolas are both simple. Thus the total multiplicity of the hyperbolas and lines is 5 and by the Theorem 2.33 we must have a Darboux first integral. We do the calculations to effectively compute the first integrals and the geometric features of this family.

Table 7.24: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) for $c = 0$ and $g \neq 0, \pm 1, 1/3, 1/2$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = \frac{1}{2g-1} + xy$ $E_3 = e^{\frac{g_0+g_1x}{x}}$ $E_4 = e^{\frac{2gh_0xy+h_0+x(h_1+h_2x)}{x^2}}$ $\alpha_1 = gx$ $\alpha_2 = (2g-1)x$ $\alpha_3 = -gg_0$ $\alpha_4 = -g(h_1 + 2h_0y)$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $g < 0$ we have $\emptyset; \binom{4}{2} E - E, S$ For $g > 0$ we have $\emptyset; \binom{4}{2} PHP - PHP, N$ if $g < 1/2$ $\emptyset; \binom{4}{2} PH - HP, N$ if $g > 1/2$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ double $\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.25: Divisor and zero-cycles of family (D) for $c = 0$ and $g \neq 0, \pm 1, 1/3, 1/2$.

Divisor and zero-cycles	Degree
$ICD = 3J_1 + J_2 + \mathcal{L}_\infty$	5
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^3\bar{J}_2 = 0.$	6
$M_{0CT} = 5P_1^\infty + 2P_2^\infty$	7

where the total curve T has only two distinct tangents at P_1^∞, but one of them is quadruple.

Table 7.26: First integral and integrating factor of family (D) when $c = 0$ and $g \neq 0, 1/2$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{g\lambda_1}{2g-1}} E_3^0 E_4^0$	$R = J_1^{\lambda_1'} J_2^{\frac{1-g(\lambda_1'+3)}{2g-1}} E_3^0 E_4^0$
Simple example	$\mathcal{I} = J_1 J_2^{-\frac{g}{2g-1}}$	$\mathcal{R} = \frac{1}{J_1 J_2}$

(ii.2) $c = 0$ and $g(g \pm 1)(3g - 1)(2g - 1) = 0$.

(ii.2.1) $c = 0$ and $g = -1$.

Under this condition, systems defined by the equations (D) do not belong to family (D). We note that system defined by the equation (D) when $c = 0$ and $g = -1$ is exactly the family (G). Here we have a single system that possess an invariant line which has multiplicity four. The line at infinity and the hyperbolas are both simple. Thus the total multiplicity of hyperbolas and lines is 6 and by the Theorem 2.33 we must have a rational first integral. We do the calculations to effectively compute the first integrals and the geometric features of this family.

Table 7.27: Invariant curves, cofactors, singularities and intersection points of system (D) when $c = 0$ and $g = -1$.

Inv.curves and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = -1 + 3xy$ $E_3 = e^{\frac{g_0+g_1x}{x}}$ $E_4 = e^{\frac{-2h_0xy+h_0+h_1x+h_2x^2}{x^2}}$ $E_5 = e^{\left(\frac{-3l_0xy+l_0+x(-2l_1xy+l_1+x(l_2+l_3x))}{x^3}\right)}$ $\alpha_1 = -x$ $\alpha_2 = -3x$ $\alpha_3 = g_0$ $\alpha_4 = h_1 + 2h_0y$ $\alpha_5 = l_2 + 2l_1y$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $\emptyset; \binom{4}{2}E - E, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ double $\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.28: Divisor and zero-cycles of system (D) when $c = 0$ and $g = -1$.

Divisor and zero-cycles	Degree
$ICD = 4J_1 + J_2 + \mathcal{L}_\infty$	6
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\overline{J_1^4}\overline{J_2} = 0.$	7
$M_{0CT} = 6P_1^\infty + 2P_2^\infty$	8

where the total curve T has only two distinct tangents at P_1^∞, but one of them is quintuple.

Table 7.29: First integral and integrating factor of system(D) when $c = 0$ and $g = -1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{\lambda_1}{3}} E_3^{-\frac{(h_1 l_1 - h_0 l_2)\lambda_4}{g_0 l_1}} E_4^{\lambda_4} E_5^{-\frac{h_0 \lambda_4}{l_1}}$	$R = J_1^{\lambda_1'} J_2^{-\frac{4}{3}-\frac{\lambda_1'}{3}} E_3^{-\frac{(h_1 l_1 - h_0 l_2)\lambda_4'}{g_0 l_1}} E_4^{\lambda_4'} E_5^{-\frac{h_0 \lambda_4'}{l_1}}$
Simple example	$\mathcal{I} = \frac{J_1^3}{J_2}$	$\mathcal{R} = \frac{1}{J_1 J_2}$

Remark 7.5. Consider $\mathcal{F}^1_{(c_1,c_2)} = c_1 J_1^3 - c_2 J_2 = 0$, $\deg \mathcal{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathcal{F}^1_{(c_1,c_2)}$ is $[1:0]$ for which we have

$$\mathcal{F}^1_{(1,0)} = J_1^3.$$

Therefore, J_1 is a critical remarkable curves and $[1:0]$ is a critical remarkable value of $\mathcal{I}_1$.

(ii.2.2) $c = 0$ and $g = 0$.

Under this condition, systems defined by (D) do not belong to **QSH**. The hyperbola $-1 + xy = 0$ filled up with singularities and the following study is done with the reduced system.

Table 7.30: Singularities of the reduced system (D) when $c = g = 0$. $(\ominus[)(];\varnothing)$ denotes the presence of a hyperbola filled up with singular points in the system such that the reduced system has no finite singularity on this curve and $(\ominus[)(];N,\emptyset)$ denotes that the system has at infinity a node, and one non-isolated singular point which is part of a real hyperbola filled up with singularities and that the reduced linear system has no infinite singular point in that position.

Singularities
$P_1^\infty = [0:1:0]$ $(\ominus[)(];\varnothing); (\ominus[)(];N,\emptyset)$

Table 7.31: First integral and integrating factor of the reduced system (D) when $c = g = 0$.

	First integral	Integrating Factor
General	$I = -x$	$R = -1$
Simple example	$\mathcal{I} = -x$	$\mathcal{R} = -1$

(ii.2.3) $c = 0$ and $g = 1/3$.

Under this condition, systems defined by the equations (D) do not belong to family (D). Here we have one affine invariant line and one invariant hyperbola, both of them are triple so we compute the exponential factors E_3, E_4, E_5 and E_6. This system is Hamiltonian so it admits a polynomial first integral.

Table 7.32: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) when $c = 0$ and $g = 1/3$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = -3 + xy$ $E_3 = e^{\frac{g_0+g_1x}{x}}$ $E_4 = e^{\frac{h_0+h_1x+h_2x^2+\frac{2h_0xy}{3}}{x^2}}$ $E_5 = e^{\frac{l_0y}{-3+xy}}$ $E_6 = e^{\frac{m_0}{9}+\frac{y(m_1(6-2xy)+3m_2y(2xy-9))}{6(xy-3)^2}}$ $\alpha_1 = \frac{x}{3}$ $\alpha_2 = -\frac{x}{3}$ $\alpha_3 = -\frac{g_0}{3}$ $\alpha_4 = -\frac{h_1}{3} - \frac{2h_0y}{3}$ $\alpha_5 = -t\frac{l_0}{3}$ $\alpha_6 = \frac{m_1}{9} - m_2y$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $\emptyset; \binom{4}{2}PHP-PHP, N$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ double $\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.33: Divisor and zero-cycles of family (D) when $c = 0$ and $g = 1/3$.

Divisor and zero-cycles	Degree
$ICD = 3J_1 + 3J_2 + \mathcal{L}_\infty$	7
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^3\bar{J}_2^3 = 0$	10
$M_{0CT} = 7P_1^\infty + 4P_2^\infty$	11

where the total curve T has
1) only three distinct tangents at P_1^∞, but two of them are triple;
2) only two distinct tangents at P_2^∞, but one of them is triple.

Table 7.34: First integral and integrating factor of family (D) when $c = 0$ and $g = 1/3$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} E_3^{\lambda_3} E_4^{\lambda_4} E_5^{-\frac{g_0\lambda_3}{l_0} - \frac{\lambda_4(2h_0m_1+9h_1m_1)}{9l_0m_2}} E_6^{-\frac{2h_0\lambda_4}{3m_2}}$	R
Simple example	$\mathcal{I}_1 = J_1J_2$	$\mathcal{R}_1 = \frac{1}{J_1J_2}$

.

where $R = J_1^{\lambda_1'} J_2^{\lambda_1'} E_3^{\lambda_3'} E_4^{\lambda_4'} E_5^{-\frac{g_0\lambda_3'}{l_0} - \frac{\lambda_4'(2h_0m_1+9h_1m_1)}{9l_0m_2}} E_6^{-\frac{2h_0\lambda_4'}{3m_2}}$.

(ii.2.4) $c = 0$ and $g = 1/2$.

Under this condition, systems defined by (D) do not belong to **QSH**. Here we have only one triple invariant line so we compute the exponential factors E_2 and E_3. We have that the line at infinity $Z = 0$ also is triple.

Table 7.35: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) when $c = 0$ and $g = 1/2$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $E_2 = e^{\frac{g_0+g_1x}{x}}$ $E_3 = e^{\frac{h_0xy+h_0+x(h_1+h_2x)}{x^2}}$ $E_4 = e^{l_0+l_1xy}$ $\alpha_1 = \frac{x}{2}$ $\alpha_2 = -\frac{g_0}{2}$ $\alpha_3 = -h_0y - \frac{h_1}{2}$ $\alpha_4 = l_1x$	$P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $\emptyset; \binom{4}{2}PH - HP, N$	$\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Table 7.36: Divisor and zero-cycles of family (D) when $c = 0$ and $g = 1/2$.

Divisor and zero-cycles	Degree
$ICD = 3J_1 + \mathcal{L}_\infty$	4
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^3 = 0.$	4
$M_{0CT} = 4P_1^\infty + P_2^\infty$	5

where the total curve T has only two distinct tangents at P_1^∞, one of them triple.

Table 7.37: First integral and integrating factor of family (D) when $c = 0$ and $g = 1/2$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} E_2^0 E_3^0 E_4^{-\frac{\lambda_1}{2l_1}}$	$R = J_1^{\lambda_1'} E_2^0 E_3^0 E_4^{-\frac{(1+\lambda_1')}{2l_1}}$
Simple example	$\mathcal{I} = J_1^2 E_4^{-1}$	$\mathcal{R} = \frac{1}{J_1}$

(ii.2.5) $c = 0$ and $g = 1$.

Under this condition, systems defined by the equations (D) do not belong to family (D). The system defined by the equations (D) when $c = 0$ and $g = 1$ is exactly the family (I). Here the systems possess an invariant line with multiplicity three and a family of invariant hyperbolas

$$1 + rx + xy,$$

where $r \in \mathbb{R}$. The line at infinity $\mathcal{L}_\infty : Z = 0$ has multiplicity 3.

Table 7.38: Invariant curves, exponential factors, cofactors, singularities and intersection points of system (D) when $c = 0$ and $g = 1$.

Inv.cur./exp.fac and cofactors	Singularities	Intersection points
$J_1 = x$ $J_{2,r} = 1 + rx + xy$ $E_3 = e^{g_0+g_1y+g_2y^2}$ $E_4 = e^{\frac{h_0+h_1x}{c+x}}$ $E_5 = e^{\frac{l_0+l_1x+l_2x^2+2h_0xy}{x^2}}$ $\alpha_1 = x$ $\alpha_2 = x$ $\alpha_3 = g_1 + 2g_2x$ $\alpha_4 = -h_0$ $\alpha_5 = -l_1 - 2l_0y$	$P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 0 : 0]$ $\emptyset; \binom{4}{2} PH - HP, N$	$\bar{J}_1 \cap \bar{J}_{2,r} = P_1^\infty$ double $\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_{2,r} \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.39: Divisor and zero-cycles of system (D) when $c = 0$ and $g = 1$.

Divisor and zero-cycles	Degree
$ILD = 3J_1 + 3\mathcal{L}_\infty$	6
$M_{0CS} = 6P_1^\infty + P_2^\infty$	7
$T = Z^3 \bar{J}_1^3 = 0$	6
$M_{0CT} = 6P_1^\infty + 3P_2^\infty$	9

where the total curve T has

1) only two distinct tangents at P_1^∞, both of them triple and

2) only one triple tangent at P_2^∞.

Table 7.40: First integral and integrating factor of system (D) when $c = 0$ and $g = 1$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_{2,r}^{-\lambda_1} E_3^{\lambda_3} E_4^{-\frac{\lambda_3(g_2 l_1 - g_1 l_0)}{h_0 l_0}} E_5^{\frac{g_2\lambda_3}{l_0}}$	$R = J_1^{\lambda_1'} J_{2,r}^{-2-\lambda_1'} E_3^{\lambda_3'} E_4^{-\frac{\lambda_3(g_2 l_1 - g1 l_0)}{h_0 l_0}} E_5^{\frac{g_2\lambda_3}{l_0}}$
Simple example	$\mathcal{I}_1 = \frac{J_1}{J_{2,1}}$	$\mathcal{R}_1 = \frac{1}{J_1^2}$

Remark 7.6. Consider $\mathcal{F}^1_{(c_1,c_2)} = c_1 J_1 - c_2 J_{2,1} = 0$, $\deg \mathcal{F}^1_{(c_1,c_2)} = 2$. We do not have any remarkable values and remarkable curves for $\mathcal{I}_1$.

(ii.3) $c \neq 0$ and $g(g \pm 1)(3g-1)(2g-1) = 0$.

(ii.3.1) $g = -1$ and $c \neq 0$.

Under this condition, systems defined by the equations (D) do not belong to family (D). Here we have two invariant lines, one invariant hyperbola and one invariant parabola. We note that in the case $c = g = -1$ this system is exactly family (F).

Table 7.41: Invariant curves, cofactors, singularities and intersection points of system (D) when $g = -1$ and $c \neq 0$.

Inv.curves and cofactors	Singularities	Intersection points
$J_1 = c + x$ $J_2 = 3c + x$ $J_3 = -1 + 3cy + 3xy$ $J_4 = 3c^2 y + \frac{x^2}{8c} + \frac{19c}{8} + x$ $\alpha_1 = -3c - x$ $\alpha_2 = -c - x$ $\alpha_3 = -3c - 3x$ $\alpha_4 = -2x$	$P_1 = \left(-c, -\frac{1}{2c}\right)$ $P_2 = \left(-3c, -\frac{1}{6c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, a; \binom{2}{2}E - E, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1^\infty$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ triple} \end{cases}$ $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathcal{L}_\infty = P_1^\infty$ double

Table 7.42: Divisor and zero-cycles of system (D) when $g = -1$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$	5
$M_{0CS} = P_1 + P_2 + 4P_1^\infty + P_2^\infty$	7
$T = Z\overline{J}_1\overline{J}_2\overline{J}_3\overline{J}_4 = 0.$	7
$M_{0CT} = 2P_1 + 3P_2 + 5P_1^\infty + 2P_2^\infty$	12

where the total curve T has

1) only three distinct tangents at P_1^∞, two of them double and one simple,

2) two distinct tangents at P_2, but one of them is double.

Table 7.43: First integral and integrating factor of system (D) when $g = -1$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\lambda_1 - \frac{\lambda_2}{3}} J_4^{\lambda_1}$	$R = J_1^{\lambda_1'} J_2^{\lambda_2'} J_3^{-\frac{4}{3} - \lambda_1' - \frac{\lambda_2'}{3}} J_4^{\lambda_1'}$
Simple example	$\mathcal{I}_1 = \frac{J_2^3}{J_3}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_4}$

Remark 7.7. Consider $\mathcal{F}^1_{(c_1,c_2)} = c_1 J_2^3 - c_2 J_3 = 0$, $\deg \mathcal{F}^1_{(c_1,c_2)} = 3$. The remarkable value of $\mathcal{F}^1_{(c_1,c_2)}$ are $[1 : -8c^3]$ and $[1 : 0]$ for which we have

$$\mathcal{F}^1_{(1,-8c^3)} = 8cJ_1J_4, \qquad \mathcal{F}^1_{(1,0)} = J_2^3.$$

Therefore, J_1, J_2, J_4 are remarkable curves and $[1 : -8c^3]$, $[1 : 0]$ are remarkable values of $\mathcal{I}_1$. Moreover, $[1 : 0]$ is a critical remarkable values and J_2 is critical remarkable curve of $\mathcal{I}_1$. The singular points are P_1 for $\mathcal{F}^1_{(1,-8c^3)}$ and P_2 for $\mathcal{F}^1_{(1,0)}$.

(ii.3.2) $g = 0$ and $c \neq 0$.

Here we have only one affine invariant line and one invariant hyperbola both of them are simple. The line at infinity $Z = 0$ is double so we compute the exponential factor E_3.

Table 7.44: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) for $g = 0$ and $c \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = c + x$ $J_2 = -1 + cy + xy$ $E_3 = e^{g_0 + g_1 x}$ $\alpha_1 = -c$ $\alpha_2 = -c - x$ $\alpha_3 = -c^2 g_1 - c g_1 x$	$P_1 = \left(-c, -\frac{1}{c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ For $c < 0$ we have $s; \binom{2}{2}E - E, \binom{1}{1}SN$ For $c > 0$ we have $s; \binom{2}{2}E - E, \binom{1}{1}SN$	$\overline{J}_1 \cap \overline{J}_2 = P_1^\infty$ double $\overline{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\overline{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.45: Divisor and zero-cycles of family (D) for $g = 0$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2\mathcal{L}_\infty$	4
$M_{0CS} = P_1 + 4P_1^\infty + 2P_2^\infty$	7
$T = Z^2 \overline{J}_1 \overline{J}_2 = 0.$	4
$M_{0CT} = P_1 + 4P_1^\infty + 3P_2^\infty$	8

where the total curve T has

1) only three distinct tangents at P_1^∞, but one of them is double and

1) only two distinct tangents at P_2^∞, but one of them is double.

Table 7.46: First integral and integrating factor of family (D) when $g = 0$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} E_3^{-\frac{\lambda_2}{c g_1}}$	$R = J_1^0 J_2^{\lambda_2'} E_3^{-\frac{1+\lambda_2'}{c g_1}}$
Simple example	$\mathcal{I} = J_2^c E_3^{-1}$	$\mathcal{R} = \frac{1}{J_2}$

(ii.3.3) $g = 1/3$ and $c \neq 0$.

Under this condition, systems defined by the equations (D) do not belong to family (D).

Here we have two invariant lines and two hyperbolas. These systems are Hamiltonian so they admit a polynomial first integral.

Table 7.47: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) when $g = 1/3$ and $c \neq 0$.

Inv.curves and cofactors	Singularities	Intersection points
$J_1 = -c + x$ $J_2 = c + x$ $J_3 = -3 - cy + xy$ $J_4 = -3 + cy + xy$ $\alpha_1 = \frac{c}{3} + \frac{x}{3}$ $\alpha_2 = \frac{x}{3} - \frac{c}{3}$ $\alpha_3 = \frac{c}{3} - \frac{x}{3}$ $\alpha_4 = -\frac{c}{3} - \frac{x}{3}$	$P_1 = \left(-c, -\frac{3}{2c}\right)$ $P_2 = \left(c, \frac{3}{2c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s, s; \binom{2}{2}E - E, N$	$\overline{J}_1 \cap \overline{J}_2 = P_1^\infty$ simple $\overline{J}_1 \cap \overline{J}_3 = P_1^\infty$ double $\overline{J}_1 \cap \overline{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\overline{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\overline{J}_2 \cap \overline{J}_3 = \begin{cases} P_1^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\overline{J}_2 \cap \overline{J}_4 = P_1^\infty$ double $\overline{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\overline{J}_3 \cap \overline{J}_4 = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ triple} \end{cases}$ $\overline{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\overline{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Table 7.48: Divisor and zero-cycles of family (D) when $g = 1/3$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$	5
$M_{0CS} = P_1 + P_2 + 4P_1^\infty + P_2^\infty$	7
$T = Z\overline{J}_1\overline{J}_2\overline{J}_3\overline{J}_4 = 0.$	7
$M_{0CT} = 2P_1 + 2P_2 + 5P_1^\infty + 3P_2^\infty$	12

where the total curve T has

1) only three distinct tangents at P_1^∞, but two of them are double;
2) three distinct tangents at P_2^∞.

Table 7.49: First integral and integrating factor of family (D) when $g = 1/3$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{\lambda_2} J_4^{\lambda_1}$	$R = J_1^{\lambda_1'} J_2^{\lambda_2'} J_3^{\lambda_2'} J_4^{\lambda_1'}$
Simple example	$\mathcal{I}_1 = J_1 J_4$	$\mathcal{R}_1 = \frac{1}{J_1 J_4}$

(ii.3.4) $g = 1/2$ and $c \neq 0$.

Under this condition, systems defined by (D) do not belong to **QSH**. Here we have two invariant lines. We also could find an exponential factor but it did not arise from multiple curves since by calculating the 1st extactic polynomial we checked that the multiplicity of the affine invariant lines is one. We also checked the multiplicity of the line at infinity and it is also simple.

Table 7.50: Invariant curves, exponential factors, cofactors, singularities and intersection points of family (D) for $g = 1/2$ and $c \neq 0$.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $J_2 = \frac{x}{c} + 1$ $E_3 = e^{cg_1 y + g_0 + g_1 xy}$ $\alpha_1 = \frac{c}{2} + \frac{x}{2}$ $\alpha_2 = \frac{x}{2}$ $\alpha_3 = cg_1 + g_1 x$	$P_1 = \left(-c, -\frac{2}{c}\right)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $s; \binom{3}{2}E - PH, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1^\infty$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Table 7.51: Divisor and zero-cycles of family (D) for $g = 1/2$ and $c \neq 0$.

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + \mathcal{L}_\infty$	3
$M_{0CS} = P_1 + 5P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1\bar{J}_2 = 0.$	3
$M_{0CT} = P_1 + 3P_1^\infty + P_2^\infty$	5

where the total curve T has three distinct tangents at P_1^∞.

Table 7.52: First integral and integrating factor of family (D) when $g = 1/2$ and $c \neq 0$.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^0 E_3^{-\frac{\lambda_1}{2g_1}}$	$R = J_1^{\lambda_1'} J_2^0 E_3^{-\frac{\lambda_1'}{2g_1} - \frac{1}{2g_1}}$
Simple example	$\mathcal{I} = J_1^2 E_3^{-1}$	$\mathcal{R} = \frac{1}{J_1}$

(ii.3.5) $g = 1$ and $c \neq 0$.

Note that this case can be reduced to the case $c = 0$ and $g = 1$ due to the translation $(x, y) \to (x - c, y)$. Therefore, on the line $g = 1$ and $c \neq 0$ we have the same configuration and the phase portrait as at the point $(g, c) = (1, 0)$.

We sum up the topological, dynamical and algebraic geometric features of family (D) and also confront our results with previous results in literature in the following proposition.

Proposition 7.8.

(a) *For the family* (D) *we obtained three distinct configurations* $C_1^{(D)}$–$C_3^{(D)}$ *of invariant hyperbolas and lines (see Figure 7.3 for the complete bifurcation diagram of configurations of this family). The parameter space for this family is* $(g \pm 1)(2g-1)(3g-1) \neq 0$, $c^2 + g^2 \neq 0$ *and the bifurcation set for the full family is* $cg = 0$. *On* $c = 0$ *and* $g \neq 0$ *the invariant lines coalesce and two finite singularities coalesced with an infinite singularity. On* $g = 0$ *and* $c \neq 0$ *the line at infinity has multiplicity two and we have just one invariant line. The bifurcation set of configurations in the full parameter space is* $cg(g \pm 1)(2g-1)(3g-1) = 0$. *Outside the parameter space occurs the following: On* $g = 1/3$ *and* $c \neq 0$ *we have an additional invariant hyperbola. On* $g = 1/3$ *and* $c = 0$ *we have one triple line and one triple hyperbola. On* $g = -1$ *and* $c \neq 0$ *we have an invariant parabola. On* $g = -1$ *and* $c = 0$ *we have one quadruple line. On* $g = 1$ *we have a family of invariant hyperbolas and the invariant lines coalesce. On* $g = 1/2$ *the invariant hyperbola becomes reducible and we have two invariant lines* $c + x = 0$ *and* $x = 0$ *when* $c \neq 0$ *and only one triple line* $x = 0$ *when* $c = 0$. *On* $c = g = 0$ *the hyperbola is filled up with singularities.*

(b) *The family* (D) *is Darboux integrable in the generic case* $cg(g \pm 1)(3g-1)(2g-1) \neq 0$ *and also when* $c = 0$ *and* $g \neq 0, \pm 1, 1/3, 1/2$. *The family* (D) *is generalized Darboux integrable when* $g = 0$ *and* $c \neq 0$. *All systems in family* (D) *have an inverse integrating factor which is polynomial.*

(c) *For the family* (D) *we have seven topologically distinct phase portraits* $P_1^{(D)}$–$P_7^{(D)}$. *The topological bifurcation diagram of family* (D) *is done in Figure 7.4. The bifurcation set is* $cg(2g-1)(g-1) = 0$ *and it is the bifurcation set of singularities. The phase portrait* $P_7^{(D)}$ *is not topologically equivalent with anyone of the phase portraits in [41].*

Proof of Proposition 7.8. (a) We have the following type of divisors and zero-cycles of the total invariant curve T for the configurations of family (D):

Table 7.53: Configurations for family (D).

Configurations	Divisors and zero-cycles of the total inv. curve T
$C_1^{(D)}$	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = P_1 + 2P_2 + 4P_1^\infty + 2P_2^\infty$
$C_2^{(D)}$	$ICD = 3J_1 + J_2 + \mathcal{L}_\infty$ $M_{0CT} = 5P_1^\infty + 2P_2^\infty$
$C_3^{(D)}$	$ICD = J_1 + J_2 + 2\mathcal{L}_\infty$ $M_{0CT} = P_1 + 4P_1^\infty + 3P_2^\infty$

Therefore, the configurations $C_1^{(D)}$ up to $C_3^{(D)}$ are all distinct. For the limit cases of family (D) we have the following configurations:

Table 7.54: Configurations for the limit cases of family (D).

Configurations	Divisors and zero-cycles of the total inv. curve T
c_1	$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 5P_1^\infty + 3P_2^\infty$
c_2	$ICD = 3J_1 + 3J_2 + \mathcal{L}_\infty$ $M_{0CT} = 7P_1^\infty + 4P_2^\infty$
c_3	$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 3P_2 + 6P_1^\infty + 2P_2^\infty$
c_4	$ICD = 4J_1 + J_2 + \mathcal{L}_\infty$ $M_{0CT} = 6P_1^\infty + 2P_2^\infty$
c_5	$ILD = 3J_1 + 3\mathcal{L}_\infty$ $M_{0CT} = 6P_1^\infty + 3P_2^\infty$
c_6	$ICD = J_1 + J_2 + \mathcal{L}_\infty$ $M_{0CT} = P_1 + 3P_1^\infty + P_2^\infty$
c_7	$ICD = 3J_1 + 3\mathcal{L}_\infty$ $M_{0CT} = 6P_1^\infty + 3P_2^\infty$
c_8	$ICD = \mathcal{L}_\infty$ $M_{0CT} = P_1^\infty$

The other statements on (a) follows from the study done previously.

(b) It follows directly from the tables.

(c) We have:

Table 7.55: Phase portraits for family (D).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(D)}$	$(\binom{2}{2}E - E, S)$	(s, a)	$1SC_f^f$ $4SC_f^\infty$ $1SC_\infty^\infty$
$P_2^{(D)}$	$(\binom{2}{2}E - E, N)$	(s, s)	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
$P_3^{(D)}$	$(\binom{2}{2}PH - PH, N)$	(s, a)	$1SC_f^f$ $5SC_f^\infty$ $0SC_\infty^\infty$
$P_4^{(D)}$	$(\binom{4}{2}E - E, S)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_5^{(D)}$	$(\binom{4}{2}PHP - PHP, N)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $3SC_\infty^\infty$
$P_6^{(D)}$	$(\binom{4}{2}PH - HP, N)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
$P_7^{(D)}$	$(\binom{2}{2}E - E, \binom{1}{1}SN)$	s	$0SC_f^f$ $4SC_f^\infty$ $1SC_\infty^\infty$

Therefore, we have seven distinct phase portraits for systems (D). For the limit case of family (D) we have the following phase portraits:

Table 7.56: Phase portraits for the limit case of family (D).

Phase Portraits	Sing. at ∞	Finite sing.	Separatrix connections
$P_1^{(D)}$	$(\binom{2}{2}E-E,S)$	(s,a)	$1SC_f^f$ $4SC_f^\infty$ $1SC_\infty^\infty$
$P_2^{(D)}$	$(\binom{2}{2}E-E,N)$	(s,s)	$0SC_f^f$ $8SC_f^\infty$ $0SC_\infty^\infty$
$P_4^{(D)}$	$(\binom{4}{2}E-E,S)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$
$P_5^{(D)}$	$(\binom{4}{2}PHP-PHP,N)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $3SC_\infty^\infty$
$P_6^{(D)}$	$(\binom{4}{2}PH-HP,N)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_\infty^\infty$
p_1	$(\binom{3}{2}E-PH,N)$	s	$0SC_f^f$ $4SC_f^\infty$ $1SC_\infty^\infty$
p_2	$(\ominus[)(];N,\emptyset)$	$(\ominus[)(];\emptyset)$	$0SC_f^f$ $0SC_f^\infty$ $0SC_\infty^\infty$

In [41] the authors do not have any phase portrait with 2 singular points at infinity and with 1 singular point in the finite plane. Therefore, the phase portrait $P_7^{(D)}$ is missing. $\square$

7.2.1 The solution of the Poincaré problem for the family (D)

The following theorem solves the problem of Poincaré for the family defined by the equations (D) with $(c,g)\in\mathbb{R}^2$.

Theorem 7.9. *The necessary and sufficient condition for a system (S) defined by the equations (D) with $(c,g)\in\mathbb{R}^2$ to have a rational first integral given by invariant algebraic curves of degree at most two, is that $g\in\mathbb{Q}$ and either $g(2g-1)\neq 0$ or $g=0=c$.*

Proof. The proof of this result is based on all the formulas for the first integrals contained in the Tables calculated for the family of systems defined by the equations (D) with $(c,g)\in\mathbb{R}^2$.

Case 1. This is the generic case, $cg(g\pm 1)(2g-1)(3g-1)\neq 0$. We first show the necessity of the condition, so suppose that the system has a rational first integral after fixing the corresponding value for the free parameter λ_2. Then according to Table 7.23 the exponents of J_2 and J_3 in this Table need to be integers. This implies that $g/(2g-1)=r\in\mathbb{Q}$ and hence $g=r/(2r-1)\in\mathbb{Q}$ (here we have $2r-1=2g/(2g-1)-1=1/(2g-1)\neq 0$). By hypothesis we have that $cg\neq 0$. To prove necessity we need to show that $g(2g-1)\neq 0$ but this we have by the hypothesis of Case 1.

To prove the sufficiency we now suppose that $g\in\mathbb{Q}$, that is $g=m/n$ where $m,n\in\mathbb{Z}$ with $n\neq 0$ and m and n are relatively prime. We also suppose that (c,g) satisfies the above generic condition. Then the general first integral is given in Table 7.23. The exponents of J_2, respectively J_3 are λ_2 and $-g\lambda_2/(2g-1)$. But $g/(2g-1)=m/(2m-n)$ where $2m-n\neq 0$ since otherwise the fraction m/n would be reducible. So by taking $\lambda_2=2m-n$ we get the first integral $J_2^{2m-n}J_3^{-m}$ which is a rational first integral. Hence the condition is also sufficient.

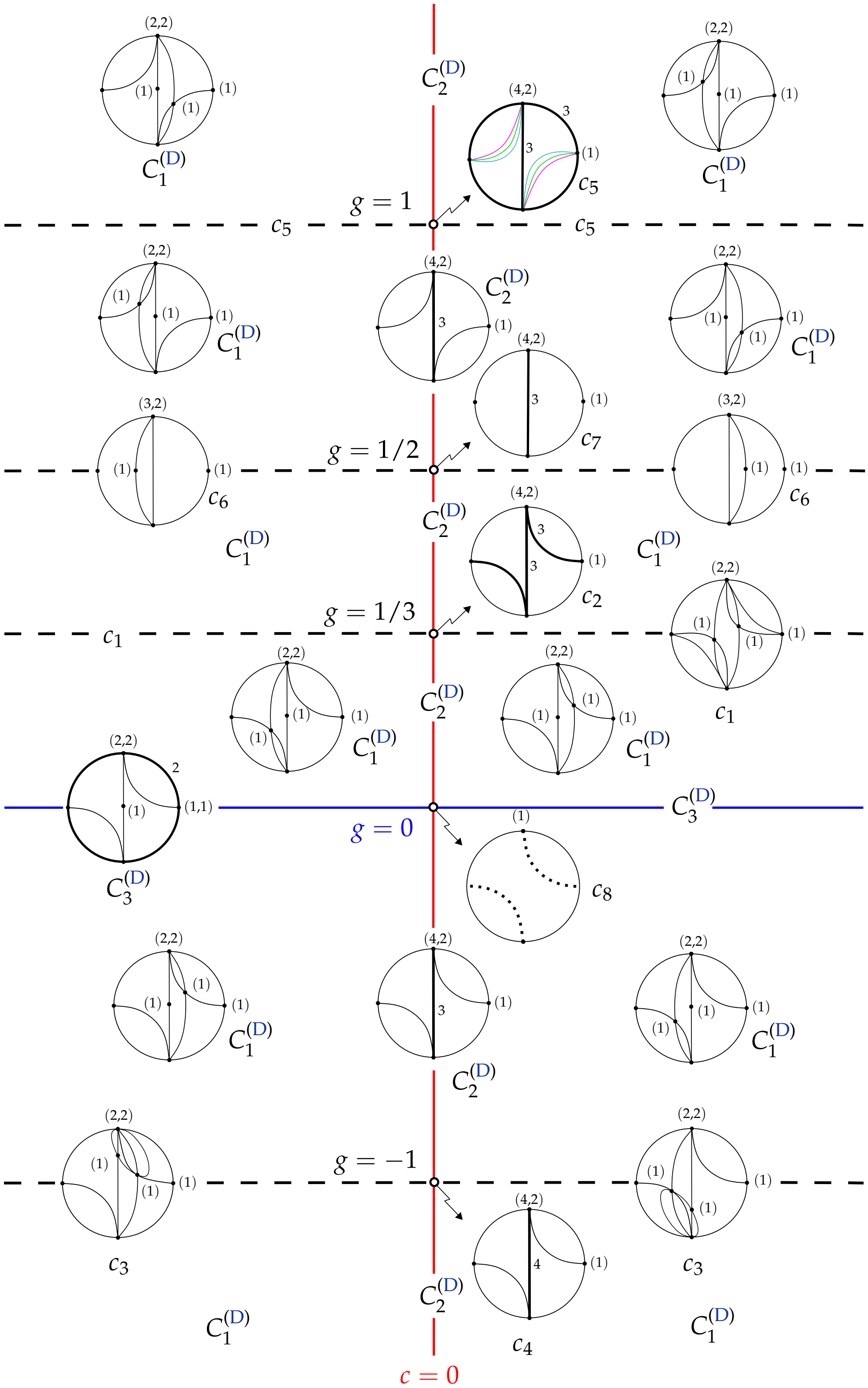

Figure 7.3: Bifurcation diagram of configurations for family (D). The dashed lines $g = \pm 1$, $g = 1/2$ and $g = 1/3$ are limit cases of this family. The multiple invariant curves are emphasized in the drawings of the configurations. When $g = 1$ we have a family of invariant hyperbolas that are drawn in colors. The dotted hyperbola on $c = g = 0$ represents an invariant hyperbola filled up with singularities.

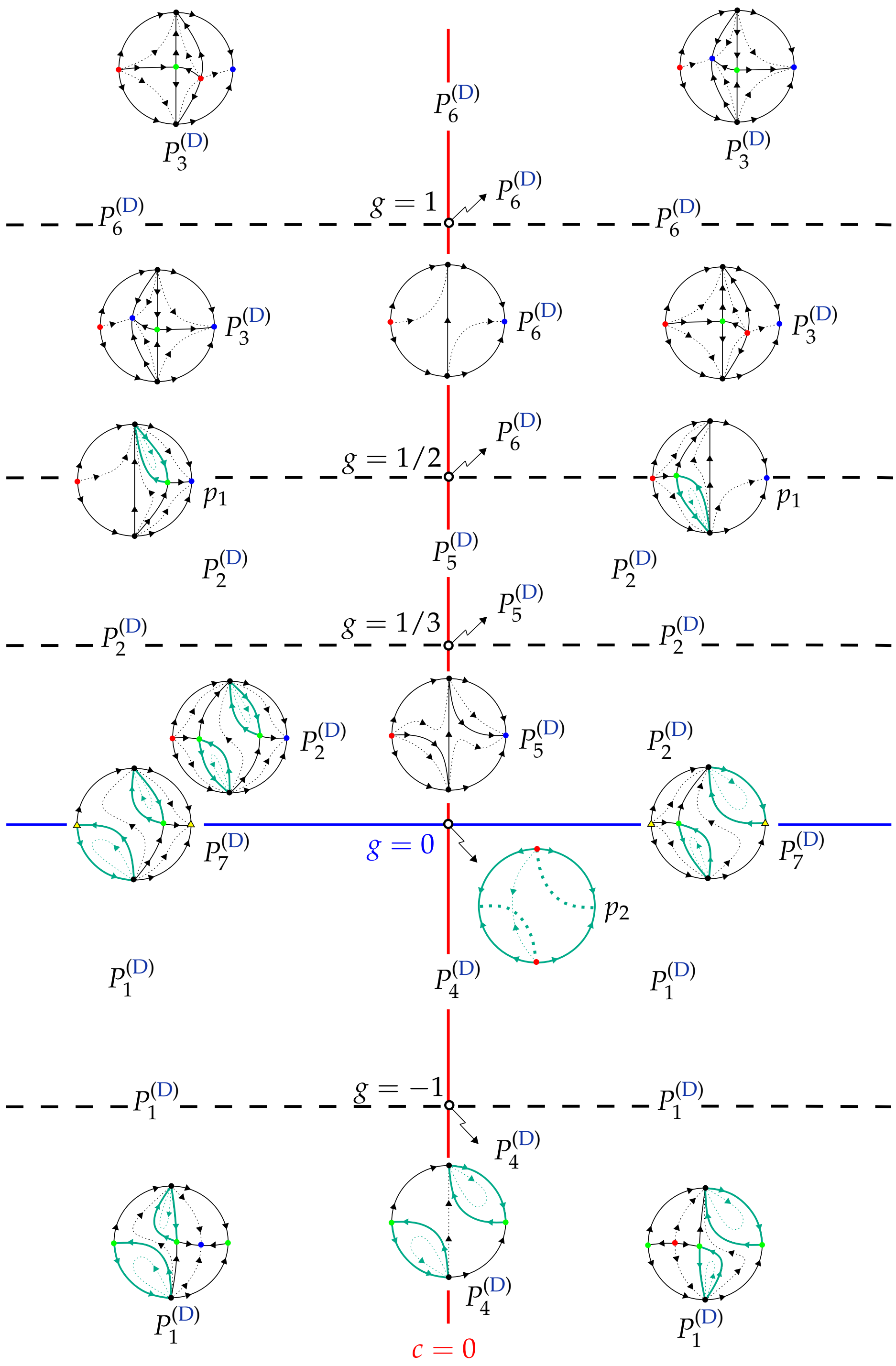

Figure 7.4: Topological bifurcation diagram for family (D). Note that the phase portraits $P_1^{(D)}$, $P_2^{(D)}$, $P_4^{(D)}$, $P_7^{(D)}$, p_1 and p_2 possess graphics (drew in green). The continuous curves in the phase portraits are separatrices. The dashed curves are the orbits given in each region of the phase portraits. The green bullet represents an elemental saddle, the red bullet an elemental unstable node and the blue an elemental stable node. The yellow triangle represents a saddle-node (semi-elemental) and the black bullet is an intricate singularity. In $c = g = 0$ we have an hyperbola filled up with singularities represented by a dotted hyperbola.

Case 2. This is the non-generic case, $cg(g \pm 1)(2g-1)(3g-1) = 0$.

Case 2.1. Consider $c = 0 \neq g(g \pm 1)(2g-1)(3g-1)$. We look for the general expression of first integrals for this case which is $J_1^{\lambda_1} J_2^{-g\lambda_1/(2g-1)}$. Suppose that we have a rational first integral. Then in this integral the exponents $\lambda_1, -g\lambda_1/(2g-1)$ must be integers. But this implies that $g/(2g-1) = r \in \mathbb{Q}$. Then $g = r/(2r-1) \in \mathbb{Q}$. By hypothesis we have that $g(2g-1) \neq 0$. Hence the necessity is proved. For the sufficiency we suppose that $g \in \mathbb{Q}$. Then again $g = m/n$ with $m, n \in \mathbb{Q}$, $n \neq 0$ and m, n relatively prime. The general form of the first integral is $J_1^{\lambda_1} J_2^{-g\lambda_1/(2g-1)}$. Replacing here g by m/n we obtain $J_1^{\lambda_1} J_2^{-m\lambda_1/(2m-n)}$. Hence by taking $\lambda_1 = 2m - n$ the first integral becomes $J_1^{2m-n} J_2^{-m}$ which is rational.

Case 2.2. Consider $c = 0$ and $g(g \pm 1)(2g-1)(3g-1) = 0$. This is a set of five points all with a rational coordinate g. In this case a necessary and sufficient condition to have a rational first integral is that $g \neq 1/2$. Indeed, by checking the Tables for first integrals we see that except for the point $g = 1/2$ at all four other points, we have a rational first integral. So the condition to have a rational first integral is satisfied in this case too.

Case 2.3. Consider $c \neq 0$ and $g(g \pm 1)(2g-1)(3g-1) = 0$. This is a collection of five lines out of which we exclude their intersection with $c = 0$.

Case 2.3.1. Suppose $c \neq 0$ and $g(2g-1) = 0$ then we see that the general first integrals for either $g = 0$ or $g = 1/2$ must contain an exponential factor and hence they are never rational. Therefore if $c \neq 0$ then to have a rational first integral it is necessary and sufficient to have $g \neq 0, 1/2$.

Case 2.3.2. Suppose $c \neq 0$ and $(g \pm 1)(3g-1) = 0$.

Case 2.3.2.1. We consider first $c \neq 0$ and $g = -1$. In this case two half lines. For each c the system $(c, g) = (c, -1)$ has a rational first integral as the Table for this case indicates and the condition $g \in \mathbb{Q}$ and $g \neq 0, 1/2$ is clearly satisfied.

Case 2.3.2.2. The case $c \neq 0$ and $(g-1)(3g-1)$ is treated in analogous manner as the **Case 2.3.2.1**. □

We note that the set of systems defined by the equations (D) for $(c, g) \in \mathbb{R}^2$ that are algebraically integrable is dense in $\mathbb{R}^2$.

8 Concluding comments and problems

There are many papers on the Darboux theory of integrability for planar polynomial vector fields but it is actually impossible to find a single source summing up this theory with all its extended features, as we know it today. The literature also often overlooks some significant moments in its developments as well as its most useful consequences such as its unifying character in proofs of integrability for whole classes of polynomial differential systems as well as its help in topologically classifying some families of systems. In this paper we covered all these aspects and proved that its complex character is essential.

In this paper the study of integrability of the family $\mathbf{QSH}_{\eta=0}$ displayed all the types from algebraic to Liouvillian integrability as well as non-integrability, proved for the generic case.

We then pursued the geometric analysis of two of the four 2-parameter sub-families of $\mathbf{QSH}_{\eta=0}$ by applying the method of Darboux and obtained all their phase portraits as well as their bifurcation diagrams and the bifurcation diagrams of the configurations of invariant hyperbolas and lines.

In this Section we are interested in the relationship between these two bifurcation diagrams, more precisely we show how the dynamics of the systems expressed in their topological bifurcations impacts the bifurcations of the geometry of the configurations and the resulting bifurcations in integrability.

8.1 Family (C)

The parameter space for this family is $\{(a,c) \in \mathbb{R}^2 : a \neq 0\}$, its topological bifurcation set is $(a - 8c^2/9)(a - c^2) = 0$ and it is formed of bifurcation points of finite singularities. On $a - 8c^2/9 = 0$ we see coalescence of two finite singularities, both situated *on the same one of the two invariant lines, yielding a double singular point on this line*. On $a - c^2 = 0$ we have two coalesces, but each one of them being a *coalescence of two points situated on distinct lines yielding a double singular point situated on a double line*.

The bifurcation set for configurations of invariant lines and hyperbolas is also $(a - 8c^2/9) \cdot (a - c^2) = 0$. On $a - 8c^2/9 = 0$ the coalescence of the two singularities on the same line yielding a double singular point generates a distinct configuration than the one in the generic case surrounding points on this parabola where none of the singularities is double. As already mentioned above, on the parabola $a - c^2 = 0$ we get a double line due to the coalescence of the four singularities located on the two lines into two double singularities on the double line. Can we explain in a similar way the appearance of a double hyperbola on the parabola $a - 8c^2/9 = 0$? Within this family this is however not possible. Indeed moving in all directions from a point on this parabola, we always get just one hyperbola, no two hyperbolas coalesce when on $a - 8c^2/9 = 0$ in the parameter space of this family.

We claim however that the same kind of phenomenon occurs as on $a - c^2 = 0$, namely that a bifurcation of singularities does occur on $a - 8c^2/9 = 0$ but when we unfold these systems in a larger family that includes systems with three distinct singular points at infinity and hence for these systems we have $\eta > 0$. Looking at the families of systems in the set of systems with $\eta > 0$ in [47] we find the configuration denoted by Config. $H.139$ (see Figure 8.1) with three singular points at infinity in the real projective plane and with 4 singular points in the affine plane. This configuration has two hyperbolas that coalesce when two of the three singular points at infinity collide and we also have collision of two finite singular points located on distinct hyperbolas. To prove this, consider the systems:

$$\begin{cases} \dot{x} = \frac{72c^2(1-\epsilon)(2+\epsilon)}{(-9+\epsilon^2)^2} + 3cx + x^2 + (1+\epsilon)xy \\ \dot{y} = -\frac{9c^2(1+\epsilon^2)}{(-9+\epsilon^2)^2} + y^2, \end{cases} \tag{8.1}$$

where ϵ is sufficiently small. These systems possess the configuration Config. $H.139$ (see Figure 8.1) of [47] for any value of $\epsilon > 0$ as we can show that it satisfies the required conditions on the polynomial invariants. On the other hand, the systems (8.1) form a perturbation of the system obtained by setting $\epsilon = 0$ which has the configuration Config. $\tilde{H}.33$ (see Figure 8.1) of [47] (here family (C) when $a = 8v^2/9$ with configuration $C_6^{(C)}$).

We conclude that on both parabolas $a - c^2 = 0$ and $a - 8c^2/9 = 0$ bifurcation of multiple singular points produce bifurcation points of configurations corresponding to multiple invariant curves but this time we have apart from coalescence of finite singularities, also coalescence of two infinite singularities.

In the article [47] the classification of **QSH** according to the configurations of invariant hyperbolas and lines was done separately for the two subfamilies corresponding to $\eta > 0$ and $\eta = 0$ leading to two (non-integrated) bifurcation diagrams in terms of invariant polynomials.

As the above example clearly illustrates there is the need of obtaining an *integrated bifurcation diagram* of **QSH**. We thus propose the following problem:

Problem: *Obtain an integrated bifurcation diagram for the family* **QSH** *of the configurations of invariant hyperbolas and lines that systems in* **QSH** *have, by finding a common set of invariant polynomials to be applied jointly to both subfamilies $\eta > 0$ and $\eta = 0$.*

Finally, in the full (extended) parameter space we observe that on $a = 0$ the hyperbola becomes reducible. For $c \neq 0$ the hyperbola splits into the lines $x = 0$ and $c + y = 0$. On $a = 0 = c$, the two lines coincide yielding a double line $x = 0$ and in addition the hyperbola splits into the lines $x = 0$ and y=0.

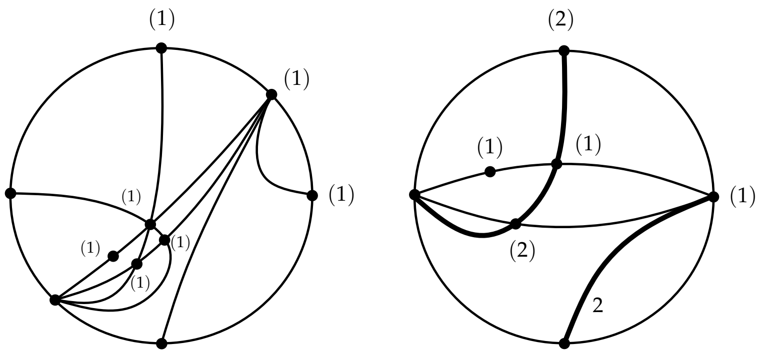

Figure 8.1: *Config.* $H.139$ and *Config.* $\tilde{H}.33$ (respectively) from [47]. The left configuration becomes the right one when the hyperbola with infinite points point $[1 : 1 : 0]$ and $[1 : 0 : 0]$ is identified with the other hyperbola by moving the point here at $[1 : 0 : 0]$ to coincide with $[0 : 1 : 0]$ in $P_2(\mathbb{C})$.

8.2 The family (D)

The parameter space for this family is $\{(c, g) \in \mathbb{R}^2 : (g \pm 1)(3g - 1)(2g - 1) \neq 0$ and $c^2 + g^2 \neq 0\}$. The topological bifurcation set for this family is the set $cg = 0$, with $c^2 + g^2 \neq 0$. On $c = 0$ and $g \neq 0$ we have that all the singularities of the systems are at infinity and this occurs nowhere else. Moreover, on $c = 0$ and $g \neq 0$ we have that $[0 : 1 : 0]$ is of multiplicity $\binom{2}{4}$ while $[1 : 0 : 1]$ is of multiplicity one. On $g = 0$ and $c \neq 0$ the singular point $[1 : 0 : 1]$ is of multiplicity $\binom{1}{1}$ while for neighbouring parameters this point is of multiplicity 1.

The bifurcation set of the configurations is again $cg = 0$, with $c^2 + g^2 \neq 0$. On $c = 0$ the line $x = 0$ is a triple line, except for the value $(c, g) = (0, -1)$ where $x = 0$ is a quadruple line. This phenomenon is forced by the topological bifurcation of singularities. Indeed, on this line two of the finite singularities, one on a line and one at the intersection of the hyperbola with the line coalesced with $[0 : 1 : 0]$ producing the a line of multiplicity at least two. In fact calculation indicates that the multiplicity of $x = 0$ is actually 3 for $g \neq 0$. Everywhere else in the parameter space of (D) we either have just one simple invariant line (this occurs on $g = 0$) or two simple invariant lines. This proves that $g = 0$ is a bifurcation line of configurations.

Thus for both families of systems (C) and (D) the bifurcation of configurations is produced by coalescence of singularities either finite or infinite or coalescence of a finite with an infinite singularities.

The following problem was stated in the article [48].

Problem: *Generalize the Theorem 4.2 so as to cover more cases than the ones imposed by the hypotheses of this theorem.*

The study of the families (C) and (D) give more motivation for solving this problem. For example the systems in the family (D) with $c = 0 \neq (g \pm 1)(2g-1)(3g-1)$ have the invariant line $J_1 = x$ and the invariant hyperbola $J_2 = 1/(2g-1) + xy$ but these curves do not satisfy the (C-K) conditions because the line intersects the hyperbola at infinity but we still have the inverse integrating factor $J_1 J_2$. We also have other examples.

Remark 8.1. Finally we observe that if we take in the family of systems with equations (D) $c = 0$ and $g = -1$ we obtain exactly the system denoted by (G) in the list of normal forms. The normal form (F) is also for just one system. This system coincides with the system in the family (D) when $g = c = -1$. If we take $c = 0$ and $g = 1$ in the systems defined by equations (D) we obtain exactly (I). Hence in this paper we covered five of the normal forms listed in Proposition 3.3: (C), (D), (F) and (G), (I).

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