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λ-lemma for nonhyperbolic point in intersection

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Abstract. The well known λ-Lemma has been proved by J. Palis for a hyperbolic fixed point of a C^1-diffeomorphism. In this paper we show that the result is true for some cases of nonhyperbolic point.

Keywords: λ-lemma, nonhyperbolic point.

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1 Introduction

The well known λ-Lemma [9] gives an important description of chaotic dynamics. A basic assumption of this theorem is hyperbolicity.

Theorem 1.1 (Palis)**.** *Let f be a C^1 diffeomorphism of $\mathbb{R}^n$ with a hyperbolic fixed point at 0 and m- and p-dimensional stable and unstable manifolds W^S and W^U $(m+p=n)$. Let D be a p-disk in W^U, and w be another p-disk in W^U meeting W^S at some point A* transversely*. Then $\bigcup_{n\geq 0} f^n(w)$ contains p-disks arbitrarily C^1-close to D.*

Generally, for C^1 diffeomorphism f of compact manifold M periodic point z is called hyperbolic if there exists a splitting $T_z(M) = E^s \oplus E^u$ with constants $k > 0$ and $0 < \lambda < 1$ such that

$$\|(Df^n)|_{E^s}\| \leq k\lambda^n \qquad (n>0),$$
$$\|(Df^{-n})|_{E^u}\| \leq k\lambda^n \qquad (n>0).$$

Here E^s and E^u are called stable and unstable subspaces of f, respectively. If z is nonhyperbolic this splitting can be written as $T_z(M) = E^s \oplus E^u \oplus E^c$, where E^s and E^u are the same as above and E^c is called the center subspace of f.

Some extensions of this lemma can be found in the [1–4, 11]. One question that arises is whether it is possible to put weaker conditions in this lemma instead of being hyperbolic. In this paper we append some new cases in which we have affirmative answer. We think these cases can be used in extending the connecting lemma of Hayashi[5]. Our result can help to generalize [7, 8] to some new cases.

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Definition 1.2. We say that a nonhyperbolic periodic point z satisfies the invariant conditions (IC) if there is a local chart (U, ϕ) at z such that in $\phi(U)$ one of the following is true:

I_1) $E^s \oplus E^u$ is invariant under f;

I_2) $E^c \oplus E^s$ is invariant under f;

I_3) $E^c \oplus E^u$ is invariant under f.

Notice that f in I_1, I_2 and I_3 is in fact $\tilde{f} = \phi f \phi^{-1}$.
Let us give an example of a system which satisfies in IC.

Example 1.3. Let $\mathbb{R}^3_\infty$ be the compactification* of $\mathbb{R}^3$. As known this is a C^∞ manifold with two charts, one at the origin and the other at ∞. We define the diffeomorphism

$$f(x,y,z) = \left(2x, \frac{y}{2}, z\right)$$

It is easy to see that the axes are the three invariant manifolds of the origin and the whole of coordinate surface are invariant under f. But, origin is not hyperbolic.

2 Preliminaries

Let $A = Df(0)$ and Let p be a nonhyperbolic fixed point of f satisfying IC, i.e. f satisfies either I_1, I_2 or I_3.

First assume I_1 is true. Since f is locally invariant on $E^s \oplus E^u$, if $W^s_{loc}(0)$ and $W^u_{loc}(0)$ are the graphs of ϕ^s and ϕ^u respectively, then locally we can write

$$\phi^s : B^s \to E^u \quad \text{and} \quad \phi^u : B^u \to E^s.$$

Here ϕ^s and ϕ^u are C^r, $D\phi^s(0) = 0$, $D\phi^u(0) = 0$, $\phi^s(0) = 0$ and $\phi^u(0) = 0$. Consider the map

$$\phi : B^s \oplus B^u \oplus E^c \to E^s \oplus E^u \oplus E^c$$

$$(x_s, x_u, x_c) \mapsto (x_s - \phi^u(x_u), x_u - \phi^s(x_s), x_c).$$

It is clear that ϕ is C^r and $D\phi(0)$ is a the identity and ϕ is diffeomorphism when restricted to some neighborhood of 0. Let $\tilde{f} = \phi f \phi^{-1}$ then $\tilde{f}$ is a diffeomorphism on a neighborhood of 0 and $\tilde{f}(0) = 0$, $D\tilde{f}(0) = A$ and E^s, E^u are local stable and unstable manifold of $\tilde{f}$. It is clear that $E^s \oplus E^u$ is still invariant. This shows that in this case we can always assume that local stable and unstable manifolds of f are discs in E^s and E^u, respectively.

Let $B^s \subseteq E^s$ and $B^u \subseteq E^u$ be such that $B^s \subseteq W^s_{loc}(0)$ and $B^u \subseteq W^u_{loc}(0)$. Let B^c be the intersection of local chart containing z, with E^c.

Now we can rewrite the proof of λ-lemma in [10] as the following lemmas.

Lemma 2.1. *Let z be a nonhyperbolic fixed point of f which satisfies I_1. Let $V = B^s \times B^u \times B^c$, and let D be a disc transversal to B^s at q with $\dim(D) = \dim(E^u)$. If D_n is the connected component of $f^n(D) \cap V$ to which $f^n(q)$ belongs, then for any given small positive ϵ we can find n such that D_n is ϵ-C^1 close to B^u.*

*We have to suppose compactification because in the definition of a hyperbolic fixed point that was mentioned above we need a compact manifold.

The proof is very similar to the proof of λ-lemma in [10]. Notice that the existence of E^c does not change the the main flow of the original proof, since $E^s \oplus E^u$ is invariant under f.

Let I_2 be true. We get the C^r map $\phi^u : B^u \to E^s \oplus E^c$ that its graph is $W^s_{loc}(0)$. Thus, $D\phi^u(0) = 0$ and $\phi^u(0) = 0$. Assume that $\phi^u(x_u) = (\phi^{us}(x_u), \phi^{uc}(x_u))$. Consider the map

$$\phi : B^s \oplus E^u \oplus E^c \to E^s \oplus E^u \oplus E^c,$$

$$(x_s, x_u, x_c) \mapsto (x_s - \phi^{us}(x_u), x_u, x_c - \phi^{uc}(x_c)),$$

where ϕ is C^r and $D\phi(0)$ is identity. Thus, ϕ is a diffeomorphism defined on a neighborhood of 0. Let $\tilde{f} = \phi f \phi^{-1}$, then $\tilde{f}$ is a diffeomorphism of a neighborhood of 0 with $\tilde{f}(0) = 0$, and, $D\tilde{f}(0) = A$. Moreover, E^u and $E^c \oplus E^s$ are invariant under $\tilde{f}$. This implies that for every f which satisfies I_2 for a nonhyperbolic fixed point, we can find a local chart such that E^u and $E^s \oplus E^c$ are invariant with respect to f.

Lemma 2.2. *Let z be a nonhyperbolic fixed point of f which satisfies I_2 and D be a transversal disc to $E^c \oplus E^s$ at $q \in E^s$ and $D^u \subseteq E^u$ a disc containing 0, then for an arbitrary small positive ϵ, there exists n such that a section of $f^n(D)$ is ϵ-C^1 close to D^u.*

Proof. Let $A = Df(0)$ and A^{cs} and A^u be respectively restriction of A to subspaces $E^{cs} = E^c \oplus E^s$ and E^u, thus f on a neighborhood V of origin becomes:

$$f(x_{cs}, x_u) = (A^{cs}x_{cs} + \phi_{cs}(x_{cs}, x_u), A^u x_u + \phi_u(x_{cs}, x_u)),$$

whence

$$(Df)_0 = (A^{cs}, A^u), \qquad x_{cs} \in B^{cs} = V \cap E^{cs}, \qquad x_u \in B^u = V \cap E^u,$$
$$\|A^{cs}\| \leq 1, \qquad \|A^u\| \geq a > 1,$$

$$\left.\frac{\partial \phi_{cs}}{\partial x_u}\right|_{B^u} = \left.\frac{\partial \phi_u}{\partial x_{cs}}\right|_{B^{cs}} = 0.$$

From above and continuity of partial differential we can find $0 < k < 1$ such that $k < \frac{a-1}{8}$ and for $V' \subset V$,

$$\max_{V'} = \left\| \frac{\partial \phi_i}{\partial x_j} \right\| \leq k, \qquad i, j = cs, u.$$

Let $q \in V'$, $B^u \subset V'$ take arbitrary unit vector v_0 in $(TD)_q$. Because $V = B^{cs} \times B^u$ then $v_0 = (v_0^{cs}, v_0^u)$. If λ_0 is the slope of v_0 then $\lambda_0 = \frac{\|v_0^{cs}\|}{\|v_0^u\|}$. In this fraction $\|v_0^u\| \neq 0$ because D is transversal disc to B^{cs}.

$$\begin{aligned}
q_1 &= f(q), & v_1 &= Df_q(v_0) \\
q_2 &= f(q_1), & v_2 &= Df_{q_1}(v_1) \\
&\vdots & &\vdots \\
q_n &= f(q_{n-1}), & v_n &= Df_{q_{n-1}}(v_{n-1}).
\end{aligned} \tag{2.1}$$

for $q \in \partial B^{cs}$

$$\begin{aligned}
Df_q(v_0) &= \begin{pmatrix} A^{cs} + \frac{\partial \phi_{cs}}{\partial x_{cs}}(q) & \frac{\partial \phi_{cs}}{\partial x_u}(q) \\ 0 & A^u + \frac{\partial \phi_u}{\partial x_u}(q) \end{pmatrix} \begin{pmatrix} v_0^{cs} \\ v_0^u \end{pmatrix} \\
&= \begin{pmatrix} A^{cs}v_0^{cs} + \frac{\partial \phi_{cs}}{\partial x_{cs}}(q)v_0^{cs} + \frac{\partial \phi_{cs}}{\partial x_u}(q)v_0^u \\ A^u v_0^u + \frac{\partial \phi_u}{\partial x_u}(q)v_0^u \end{pmatrix}.
\end{aligned}$$

Thus

$$\lambda_1 = \frac{\|v_1^{cs}\|}{\|v_1^u\|} = \frac{\left\|A^{cs}v_0^{cs} + \frac{\partial\phi_{cs}}{\partial x_{cs}}(q)v_0^{cs} + \frac{\partial\phi_{cs}}{\partial x_u}(q)v_0^u\right\|}{\left\|A^u v_0^u + \frac{\partial\phi_u}{\partial x_u}(q)v_0^u\right\|}.$$

The numerator of above fraction is less than

$$\|A^{cs}v_0^{cs}\| + \left\|\frac{\partial\phi_{cs}}{\partial x_{cs}}(q)v_0^{cs}\right\| + \left\|\frac{\partial\phi_{cs}}{\partial x_u}(q)v_0^u\right\| \leq (1+k)\|v_0^{cs}\| + k\|v_0^s\|$$

and its denominator is greater than

$$\|A^u v_0^u\| - \left\|\frac{\partial\phi_u}{\partial x_u}(q)v_0^u\right\| \geq (a-k)\|v_0^u\|,$$

then

$$\begin{aligned}
\lambda_1 &\leq \frac{(1+k)\lambda_0 + k}{a-k} \leq \frac{1+k}{a-k}\lambda_0 + \frac{k}{a-k},\\
\lambda_2 &= \frac{\|v_2^{cs}\|}{\|v_2^u\|} \leq \frac{(1+k)\lambda_1 + k}{a-k} \leq \left(\frac{1+k}{a-k}\right)^2\lambda_0 + \frac{k}{1+k}\sum_{i=1}^{2}\left(\frac{1+k}{a-k}\right)^i\\
&\vdots\\
\lambda_n &= \frac{\|v_n^{cs}\|}{\|v_n^u\|} \leq \left(\frac{1+k}{a-k}\right)^n\lambda_0 + \frac{k}{1+k}\sum_{i=1}^{n}\left(\frac{1+k}{a-k}\right)^i \leq \left(\frac{1+k}{a-k}\right)^n\lambda_0 + \frac{a-k}{a-1-2k}.
\end{aligned}$$

Because $\left(\frac{1+k}{a-k}\right)^n\lambda_0 \to 0$, then there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ we have $\lambda_n < \frac{a-k}{a-1-2k}$.

Consider the number k_1 such that $0 < k_1 < \min(\epsilon, k)$. Because $\frac{\partial\phi_{cs}}{\partial x_u}\big|_{B^u} = 0$ and B^u is compact, there exists $\delta < \epsilon$ such that $V_1 = \delta B^{cs} \times B^u \subset V$ so

$$\max_{V_1}\left\|\frac{\partial\phi_{cs}}{\partial x_u}\right\| \leq k_1.$$

Let δB^{cs} be a ball with radius δ times radius of B^{cs} . We can assume that v_0 is a vector in $(TD)_q$ that has maximal slope, so for $n \geq n_0$ the slope of all unit vectors in $(TD_n)_{q_n}$ is less than $\frac{a-k}{a-1-2k}$. For a properly chosen n_0 we have $q_{n_0} \in V_1$. From the continuity of the tangent space D_{n_0}, we can find a disk $\tilde{D}$ embedded in D_{n_0} with center q_{n_0} such that for all $p \in \tilde{D}$ the slope of all unit vectors in $(T\tilde{D})_p$ is less than $\frac{2(a-k)}{a-1-2k}$.

Let $v \in (T\tilde{D})_p$ be a unit vector. If $v = (v^{cs}, v^u)$ and its slope is $\lambda_{n_0} = \frac{\|v^{cs}\|}{\|v^u\|}$ then

$$Df_p = \begin{pmatrix} A^{cs}v^{cs} + \frac{\partial\phi_{cs}}{\partial x_{cs}}(p)v^{cs} + \frac{\partial\phi_{cs}}{\partial x_u}(p)v^u \\ \frac{\partial\phi_u}{\partial x_{cs}}(p)v^{cs} + A^u v^u + \frac{\partial\phi_u}{\partial x_u}(p)v^u \end{pmatrix}.$$

Thus

$$\lambda_{n_0+1} = \frac{\left\|A^{cs}v^{cs} + \frac{\partial\phi_{cs}}{\partial x_{cs}}(p)v^{cs} + \frac{\partial\phi_{cs}}{\partial x_u}(p)v^u\right\|}{\left\|\frac{\partial\phi_u}{\partial x_{cs}}(p)v^{cs} + A^u v^u + \frac{\partial\phi_u}{\partial x_u}(p)v^u\right\|}.$$

The numerator of above fraction is less that $(1+k)\|v^{cs}\| + k_1\|v^u\|$ and its denominator is greater than

$$\|A^u v^u\| - \left\|\frac{\partial\phi_u}{\partial x_u}(p)v^u\right\| - \left\|\frac{\partial\phi_u}{\partial x_{cs}}(p)v^{cs}\right\| \geq (a-k)\|v^u\| - k\|v^{cs}\|.$$

Thus

$$\lambda_{n_0+1} \leq \frac{(1+k)\lambda_{n_0}+k_1}{a-k-k\lambda_{n_0}} \leq \frac{(1+k)\lambda_{n_0}+k_1}{a-k-k\frac{2(a-k)}{a-1-2k}}$$
$$\leq \frac{(1+k)\lambda_{n_0}+k_1}{\frac{(a-k)(a-1-4k)}{a-1-2k}}.$$

Let $b = \frac{(a-k)(a-1-4k)}{a-1-2k}$. It is easy to see that $k+1 < b$. Therefore we have

$$\lambda_{n+n_0} \leq \left(\frac{1+k}{b}\right)^n \lambda_{n_0} + k_1 \frac{b}{(b-1-k)(k+1)}.$$

Then there exits $\tilde{n}$ such that for $n \geq \tilde{n}$

$$\lambda_{n+n_0} \leq \epsilon \left(1 + \frac{b}{(b-1-k)(k+1)}\right).$$

This shows that for $n \geq \tilde{n}$ the slope of nonzero tangent vectors to $f^n(\tilde{D}) \cap V_1$ is less than given ϵ.

Now we show that the length of any tangent vector to $f^n(\tilde{D}) \cap V_1$ is growing as n is increasing. We denote the image of (v_n^{cs}, v_n^u) under Df as $(v_{n+1}^{cs}, v_{n+1}^u)$, thus

$$\frac{\sqrt{\|v_{n+1}^{cs}\|^2 + \|v_{n+1}^u\|^2}}{\sqrt{\|v_n^{cs}\|^2 + \|v_n^u\|^2}} = \frac{\|v_{n+1}^u\|}{\|v_n^u\|}\sqrt{\frac{1+\lambda_{n+1}^2}{1+\lambda_n^2}}.$$

But

$$\frac{\|v_{n+1}^u\|}{\|v_n^u\|} \geq a - k - \lambda_n.$$

As n is growing, λ_n and λ_{n+1} become small enough; then the length of the tangent vectors to $f^n(\tilde{D}) \cap V_1$ are increasing with ratio $a-k > 1$. This fact and tendency to zero of the slope of the tangent vectors imply that for $n > \tilde{n}$ the $f^n(\tilde{D}) \cap V_1$ are approaching in C^1 topology to B^u. □

Finally suppose that condition I_3 is true, we replace f by f^{-1}, then condition I_2 is true for f^{-1} and using the above lemma we have:

Lemma 2.3. *Let z be a nonhyperbolic fixed point of f that satisfies I_3 and D be a disc transversal to $E^u \oplus E^c$ at $q \in E^u$ and $D^s \subseteq E^s$ a disc containing 0, then for an arbitrary small positive ϵ there exists n that $f^{-n}(d)$ is ϵ-C^1 close to D^s.*

As a consequence of the above lemmas, the following proposition can be obtained. We first need the definition of *forwardly related* from [6].

For any C^1 diffeomorphism f of compact manifold M and $p \in M$ the forward orbit of p is

$$\mathcal{O}_f^+ = \{x \in M : \exists n \in \mathbb{Z} \ s.t. \ f^n(p) = x\}.$$

Definition 2.4. A point $p \in M$ is called forwardly related to $q \in M$ if $q \notin \mathcal{O}_f^+(p)$ and there exists a sequence diffeomorphisms $\{f_n\}$ such that $f_n \to f$ and a sequence of strings $\gamma_n = \{f_n^k(p_n) : k = 0, \ldots, s_n\}$ such that $p_n \to p$ and $f_n^{s_n}(p_n) \to q$.

Proposition 2.5. *Let z be a nonhyperbolic fixed point satisfying IC, let $p \in W_{loc}^s(z)$, and, $q \in W_{loc}^u(z)$. Then p is forwardly related to q.*

All the above results are true for periodic point p. It is sufficient to replace f by f^n where n is the period of p.

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