Electronic Journal of Qualitative Theory of Differential Equations
2021, No. 17, 1–7; https://doi.org/10.14232/ejqtde.2021.1.17 www.math.u-szeged.hu/ejqtde/

The decay rates of solutions to a chemotaxis-shallow water system

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Received 12 October 2020, appeared 24 March 2021
Communicated by Bo Zhang

Abstract. In this paper, we consider the large time behavior of solution for the chemotaxis-shallow water system in $\mathbb{R}^2$. The lower bound for time decay rates of the bacterial density and the chemoattractant concentration are proved by the method of energy estimates, which implies these two variables tend to zero at the L^2-rate $(1+t)^{-\frac{1}{2}}$. Furthermore, by the Fourier splitting method, we also show the first order spatial derivatives of the bacterial density tends to zero at the L^2-rate $(1+t)^{-1}$.

Keywords: chemotaxis, shallow water system, decay rates.

2020 Mathematics Subject Classification: 35B40, 35Q35, 35Q92, 76N10, 92C17.

1 Introduction

In this paper, we are interested in two-dimensional chemotaxis-shallow water system

$$\begin{cases} n_t + \operatorname{div}(nu) = D_n \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t + \operatorname{div}(cu) = D_c \Delta c - nf(c), \\ h_t + \operatorname{div}(hu) = 0, \\ hu_t + hu \cdot \nabla u + h^2 \nabla n + \dfrac{1}{2}(1+n)\nabla h^2 = \mu \Delta u + (\mu + \lambda)\nabla(\operatorname{div} u), \end{cases} \tag{1.1}$$

which was proposed in [2] to describe the dynamics of the oxygen and aerobic bacteria in the incompressible fluids with free surface. Here n, c, h, u denote the bacterial density, the chemoattractant concentration, the fluid height and the fluid velocity field, respectively. The constants D_n and D_c are the corresponding diffusion coefficients for the cells and substrate. The chemotactic sensitivity $\chi(c)$ and the consumption rate of the substrate by the cells $f(c)$ are supposed to be given smooth functions. The constants μ and λ are the shear viscosity and the bulk viscosity coefficients respectively with the following physical restrictions: $\mu > 0, \mu + \lambda \geq 0$. In order to complete system (1.1), the initial conditions are given by

$$(n, c, h, u)(x,t)|_{t=0} = (n_0(x), c_0(x), h_0(x), u_0(x)), \quad \text{for } x \in \mathbb{R}^2. \tag{1.2}$$

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As the space variable tends to infinity, we assume

$$\lim_{|x|\to\infty}(n_0, c_0, h_0 - 1, u_0)(x) = 0. \tag{1.3}$$

Chemotaxis exists widely in the nature. The bacteria or microorganisms often live in a viscous fluid with chemical stimulation and like to move towards a chemically more advantageous circumstance for better survival known as chemotaxis. To describe the dynamics of swimming bacteria, Tuval et al. [16] proposed a coupled system of the chemotaxis model and the viscous incompressible fluid. Since then, there has been many results in literature on the solvability and stability of this chemotaxis-fluid system. The local weak solution was proved by Lorz [9] and the local smooth solution was showed by Chae–Kang–Lee [1]. Liu–Lorz [8] and Winkler [22] established the global weak solutions. The global classical and strong solution was proved by Winkler [19] and Duan–Lorz–Markowich [4], respectively. The stability problem was studied in [3,11,20,23] and the small-convection limit was investigated by Wang et al.[18]. We also would like refer to [5–7,12,13,15,21,24] and the references therein for more related works on the chemotaxis-fluid system with nonlinear diffusion.

Considering the fact that the surface of the fluid is a free boundary, the modified shallow water type chemotactic model (1.1) is derived in [2]. For large initial data allowing vacuum, i.e. the bacterial density n is allowed to vanish, the authors in [2] established the local existence of strong solutions and the blow-up criterion. In [14], we proved the global well-posedness of strong solution and studied the upper bound decay rates of the global solution with the initial data far from vacuum. Recently, Wang–Wang [17] showed the upper bound decay estimates of the global solutions in L^p space with the initial bacterial density allowing vacuum.

In this paper, based on the previous works [14, 17], we are interested in the large time behavior of the global solution for the chemotaxis-shallow water system with the bacterial density n being allowed to vanish. The lower bound decay rates for the chemoattractant concentration c, the bacterial density n and its one order spatial derivatives will be given.

In what follows, for simplicity, let $D_n = D_c = 1, \chi(c) \equiv 1, f(c) = c$. Furthermore, throughout this paper, we use $H^k(\mathbb{R}^2)(k \in \mathbb{R})$ to denote the usual Sobolev spaces with norm $\|\cdot\|_{H^k}$ and $L^p(\mathbb{R}^2)(1 \le p \le \infty)$ to denote the usual L^p spaces with norm $\|\cdot\|_{L^p}$. C denotes constant independent of time t. For the sake of simplicity, $\|(A, B)\|_X := \|A\|_X + \|B\|_X$.

Now, we first recall the following result obtained in [17].

Theorem 1.1. *Assume that the initial data $(n_0, c_0, h_0 - 1, u_0) \in H^4 \cap L^1$ satisfies $n_0, c_0 \ge 0$ and $h_0 > 0$ and there exists a small positive constant δ_0 such that $\|(n_0, c_0, h_0 - 1, u_0)\|_{H^4\cap L^1} \le \delta_0$, then the system (1.1)–(1.3) has a unique global classical solution which satisfies*

$$\|\nabla^k(n, c, h - 1, u)(t)\|_{L^2} \le C(1 + t)^{-\frac{1+k}{2}}, \quad \textit{for} \ \ k = 0, 1, 2. \tag{1.4}$$

The main result in this paper can be stated as follows.

Theorem 1.2. *Assume that the assumptions of Theorem 1.1 hold and the Fourier transform $\mathcal{F}(n_0) = \widehat{n}_0$ and $\mathcal{F}(c_0) = \widehat{c}_0$ satisfy $|\widehat{n}_0| \ge \bar{n} > 0$ and $|\widehat{c}_0| \ge \bar{c} > 0$ for $0 \le |\xi| \ll 1$, with $\bar{n}$ and $\bar{c}$ are small constants. Then, the bacterial density n and the chemoattractant concentration c of global solution to the system (1.1)–(1.3) has the lower bound for time decay rates for all $t \ge T_1$*

$$\|(n, c)(t)\|_{L^2} \ge C(1 + t)^{-\frac{1}{2}} \quad \textit{and} \quad \|\nabla n(t)\|_{L^2} \ge C(1 + t)^{-1},$$

where T_1 is a positive large time.

Remark 1.3. By combining the results in Theorem 1.1 and Theorem 1.2, one can find that the bacterial density and the chemoattractant concentration tend to zero at the L^2-rate $(1+t)^{-\frac{1}{2}}$ and the first order spatial derivatives of the bacterial density tends to zero at the L^2-rate $(1+t)^{-1}$.

Remark 1.4. From the structure of the system (1.1), we can find the fluid height and the fluid velocity field satisfy the hyperbolic and parabolic coupled system with linear term ∇n. This means that the method in this paper will no longer be valid for the lower bound decay rates of the fluid height and the fluid velocity field.

Remark 1.5. It is worth mentioning that many functions, for example $\delta_0 e^{-|x|}$ or $\delta_0 e^{-|x|^2}$, can fulfill the hypotheses in Theorem 1.1 and Theorem 1.2 simultaneously.

2 The lower bound for time decay rates

Let us first consider the following linearized system of $(1.1)_1$ and $(1.1)_2$.

$$\begin{cases} \partial_t n_l - \Delta n_l = 0, \\ \partial_t c_l - \Delta c_l = 0, \end{cases} \tag{2.1}$$

with the initial data $(n_l, c_l)(x,0) = (n_0, c_0)(x)$.

Lemma 2.1. *Assume that the Fourier transform $\mathcal{F}(n_0) = \widehat{n}_0$ and $\mathcal{F}(c_0) = \widehat{c}_0$ satisfy $|\widehat{n}_0| \geq \bar{n} > 0$ and $|\widehat{c}_0| \geq \bar{c} > 0$ for $0 \leq |\xi| \ll 1$, with $\bar{n}$ and $\bar{c}$ are small constants. Then, n_l and c_l in (2.1) have the decay rates*

$$\|(n_l, c_l)(t)\|_{L^2} \geq C(1+t)^{-\frac{1}{2}} \quad \text{and} \quad \|\nabla(n_l, c_l)(t)\|_{L^2} \geq C(1+t)^{-1}. \tag{2.2}$$

Proof. Since n_l satisfies a heat equation, with the help of semigroup method, we have $n_l(x,t) = e^{-\Delta t} n_0(x)$. Thus, using the Fourier transform, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |n_l|^2 dx &= \int_{\mathbb{R}^2} |\widehat{n}_0|^2 e^{-2|\xi|^2 t} d\xi \geq \bar{n}^2 \int_{|\xi| \ll 1} e^{-2|\xi|^2 t} d\xi \geq C(1+t)^{-1}, \\ \int_{\mathbb{R}^2} |\nabla n_l|^2 dx &= \int_{\mathbb{R}^2} |\widehat{n}_0|^2 \xi^2 e^{-2|\xi|^2 t} d\xi \geq C(1+t)^{-2}. \end{aligned}$$

Similarly, we can also obtain the lower bounds for c_l. Therefore, we complete the proof of this lemma. □

Next, we recall a known result which will be used later (see [3,17]).

Lemma 2.2. *Assume that the assumptions of Theorem 1.1 hold. Then the global strong solution (n, c, h, u) to the Cauchy problem of system (1.1)–(1.3) satisfies*

$$n(t,x) \geq 0, \qquad c(t,x) \geq 0 \qquad \text{a.e. in } (0,+\infty) \times \mathbb{R}^2. \tag{2.3}$$

Now, we are ready to deal with the nonlinear part of $(1.1)_1$ and $(1.1)_2$. Set $n_r = n - n_l$ and $c_r = c - c_l$, then n_r and c_r satisfy

$$\begin{cases} \partial_t n_r - \Delta n_r = -\mathrm{div}(nu) - \nabla \cdot (n \nabla c), \\ \partial_t c_r - \Delta c_r = -\mathrm{div}(cu) - nc, \end{cases} \tag{2.4}$$

with the initial data $(n_r, c_r)(x,0) = (0,0)$. Here, (2.4) is a non-homogeneous linear heat equations.

Remark 2.3. It is worth mentioning that the method in our paper can be extended to parabolic equations with other different types of nonlinear sources to get the lower bound for time decay rates. However, these nonlinear sources can not contain linear part in it. More precisely, taking the logistic source term in the equation for n as an example, we consider

$$\partial_t n_r - \Delta n_r = -\mathrm{div}(nu) - \nabla\cdot(n\nabla c) + \rho n - \mu n^2,$$

where ρ and μ are constants. It follows from (1.4), the linear term $\|\rho n(t)\|_{L^2}$ only gives us $(1+t)^{-\frac{1}{2}}$ decay rate. Thus, we can not get the the lower bound for time decay rates with $\|n_r(t)\|_{L^2} \leq C(1+t)^{-\frac{1}{2}}$.

Lemma 2.4. *Assume that the assumptions of Theorem 1.1 hold. Then, n_r and c_r in (2.4) have the decay rates*

$$\|(n_r, c_r)(t)\|_{L^2} \leq C(1+t)^{-1} \quad \textit{and} \quad \|\nabla n_r(t)\|_{L^2} \leq C(1+t)^{-\frac{3}{2}}. \tag{2.5}$$

Proof. Define $S_1 = \mathrm{div}(nu) + \nabla\cdot(n\nabla c)$ and $S_2 = \mathrm{div}(cu)$. By virtue of the semigroup method, Duhamel's principle and Lemma 2.2, from (2.4) we have

$$\begin{aligned}
&\|(n_r, c_r)(t)\|_{L^2} \\
&\leq \int_0^t \left(\int_{\mathbb{R}^2} e^{-2|\xi|^2(t-\tau)} (|(\widehat{S}_1, \widehat{S}_2)|^2) d\xi \right)^{\frac{1}{2}} d\tau \\
&\leq \int_0^t \left(\int_{|\xi|\leq 1} e^{-2|\xi|^2(t-\tau)} (|(\widehat{S}_1, \widehat{S}_2)|^2) d\xi + \int_{|\xi|\geq 1} e^{-2|\xi|^2(t-\tau)} (|(\widehat{S}_1, \widehat{S}_2)|^2) d\xi \right)^{\frac{1}{2}} d\tau \\
&\leq C \int_0^t (1+t-\tau)^{-1} (\||\xi|^{-1}(\widehat{S}_1, \widehat{S}_2)\|_{L^\infty} + \|(S_1, S_2)\|_{L^2}) d\tau \\
&\leq C \int_0^t (1+t-\tau)^{-1} (\|(n, c, u, \nabla c)\|_{L^2}^2 + \|(S_1, S_2)\|_{L^2}) d\tau.
\end{aligned} \tag{2.6}$$

It follows from the Sobolev inequality and (1.4) that

$$\begin{aligned}
\|(S_1, S_2)\|_{L^2} &\leq \|\nabla u\|_{L^4} \|(n, c)\|_{L^4} + \|u\|_{L^4} \|\nabla(n, c)\|_{L^4} + \|\nabla n\|_{L^4} \|\nabla c\|_{L^4} + \|n\|_{L^\infty} \|\nabla^2 c\|_{L^2} \\
&\leq C(1+t)^{-2}.
\end{aligned} \tag{2.7}$$

Thus, using (1.4) again, we obtain

$$\begin{aligned}
\int_0^t (1+t-\tau^{-1} \|(n, c, u, \nabla c)\|_{L^2}^2 \|_{L^2} d\tau &\leq \int_0^t (1+t-\tau)^{-1} (1+\tau)^{-1} d\tau \leq (1+t)^{-1}, \\
\int_0^t (1+t-\tau)^{-1} \|(S_1, S_2)\|_{L^2} d\tau &\leq \int_0^t (1+t-\tau)^{-1} (1+\tau)^{-2} d\tau \leq (1+t)^{-1}.
\end{aligned}$$

This, together with (2.6), implies

$$\|(n_r, c_r)(t)\|_{L^2} \leq C(1+t)^{-1}. \tag{2.8}$$

Next, applying ∇ to $(2.4)_1$, then multiplying by ∇n, integrating over $\mathbb{R}^2$, after integration by parts and using (2.7), it infers that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla n_r|^2 dx + \int_{\mathbb{R}^2} |\nabla^2 n_r|^2 dx = \int_{\mathbb{R}^2} S_1 \cdot \nabla^2 n_r dx \leq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla^2 n_r|^2 dx + C(1+t)^{-4},$$

which gives

$$\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla n_r|^2dx+\int_{\mathbb{R}^2}|\nabla^2 n_r|^2dx\le C(1+t)^{-4}. \tag{2.9}$$

Denoting the time sphere S_0 (see [10]) as follows

$$S_0:=\left\{\xi\in\mathbb{R}^2 \mid |\xi|\le\left(\frac{R}{1+t}\right)^{\frac{1}{2}}\right\},$$

where R is a constant defined below. Then, we can get

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla^2 n_r|^2dx&\ge\int_{\mathbb{R}^2\setminus S_0}|\xi|^4|\widehat{n}_r|^2d\xi\\&\ge\frac{R}{1+t}\int_{\mathbb{R}^2\setminus S_0}|\xi|^2|\widehat{n}_r|^2d\xi\\&\ge\frac{R}{1+t}\int_{\mathbb{R}^2}|\xi|^2|\widehat{n}_r|^2d\xi-\frac{R^2}{(1+t)^2}\int_{S_0}|\widehat{n}_r|^2d\xi.\end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.9) and then applying (2.8), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^2}|\nabla n_r|^2dx+\frac{R}{1+t}\int_{\mathbb{R}^2}|\nabla n_r|^2dx\\&\qquad\le\frac{R^2}{(1+t)^2}\int_{\mathbb{R}^2}|n_r|^2dx+C(1+t)^{-4}\le CR^2(1+t)^{-4}.\end{aligned} \tag{2.11}$$

Choosing $R=\frac{7}{2}$, multiplying (2.11) by $(1+t)^{\frac{7}{2}}$ and integrating over $[0,t]$, it holds that

$$\|\nabla n_r(t)\|_{L^2}^2\le C(1+t)^{-3},$$

which, together with (2.8) completes the proof of this lemma. □

Proof of Theorem 1.2. It follows from Lemma 2.1 and Lemma 2.4 that

$$\begin{aligned}\|(n,c)\|_{L^2}&\ge\|(n_l,c_l)\|_{L^2}-\|(n_r,c_r)\|_{L^2}\\&\ge C(1+t)^{-\frac{1}{2}}-C(1+t)^{-1}\\&\ge C(1+t)^{-\frac{1}{2}}-\frac{C}{(1+t)^{\frac{1}{2}}}(1+t)^{-\frac{1}{2}},\\\|\nabla n\|_{L^2}&\ge\|\nabla n_l\|_{L^2}-\|\nabla n_r\|_{L^2}\\&\ge C(1+t)^{-1}-C(1+t)^{-\frac{3}{2}}\\&\ge C(1+t)^{-1}-\frac{C}{(1+t)^{\frac{1}{2}}}(1+t)^{-1}.\end{aligned}$$

Obviously, we can choose a $T_1>0$ large enough such that for $t\ge T_1$, we have the lower bound for time decay rates

$$\|(n,c)(t)\|_{L^2}\ge C(1+t)^{-\frac{1}{2}}\quad\text{and}\quad\|\nabla n(t)\|_{L^2}\ge C(1+t)^{-1}.$$

Therefore, we complete the proof of Theorem 1.2. □

Acknowledgements

The authors would like to thank the anonymous referees for the valuable comments in improving the presentation of the paper. This work was supported in part by Guangdong Basic and Applied Basic Research Foundation (2020A1515010530), the National Science Foundation of China (11971320).

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