



# Geometry and integrability of quadratic systems with invariant hyperbolas

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**Abstract.** Let **QSH** be the family of non-degenerate planar quadratic differential systems possessing an invariant hyperbola. We study this class from the viewpoint of integrability. This is a rich family with a variety of integrable systems with either polynomial, rational, Darboux or more general Liouvillian first integrals as well as non-integrable systems. We are interested in studying the integrable systems in this family from the topological, dynamical and algebraic geometric viewpoints. In this work we perform this study for three of the normal forms of **QSH**, construct their topological bifurcation diagrams as well as the bifurcation diagrams of their configurations of invariant hyperbolas and lines and point out the relationship between them. We show that all systems in one of the three families have a rational first integral. For another one of the three families, we give a global answer to the problem of Poincaré by producing a geometric necessary and sufficient condition for a system in this family to have a rational first integral. Our analysis led us to raise some questions in the last section, relating the geometry of the invariant algebraic curves (lines and hyperbolas) in the systems and the expression of the corresponding integrating factors.

**Keywords:** quadratic differential systems, invariant algebraic curves, invariant hyperbola, Darboux integrability, Liouvillian integrability.


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## 1 Introduction

Let  $\mathbb{R}[x, y]$  be the set of all real polynomials in the variables  $x$  and  $y$ . Consider the planar system

$$\begin{aligned}\dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y),\end{aligned}\tag{1.1}$$

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where  $\dot{x} = dx/dt$ ,  $\dot{y} = dy/dt$  and  $P, Q \in \mathbb{R}[x, y]$ . We call the degree of system (1.1) the integer  $\max\{\deg P, \deg Q\}$ . In the case when the polynomial  $P$  and  $Q$  are relatively prime i. e. they do not have a non-constant common factor, we say that (1.1) is *non-degenerate*.

Consider

$$\chi = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y} \quad (1.2)$$

the polynomial vector field associated to (1.1).

A real *quadratic differential system* is a polynomial differential system of degree 2, i.e.

$$\begin{aligned} \dot{x} &= p_0 + p_1(\tilde{a}, x, y) + p_2(\tilde{a}, x, y) \equiv p(\tilde{a}, x, y), \\ \dot{y} &= q_0 + q_1(\tilde{a}, x, y) + q_2(\tilde{a}, x, y) \equiv q(\tilde{a}, x, y) \end{aligned} \quad (1.3)$$

where

$$\begin{aligned} p_0 &= a, & p_1(\tilde{a}, x, y) &= cx + dy, & p_2(\tilde{a}, x, y) &= gx^2 + 2hxy + ky^2, \\ q_0 &= b, & q_1(\tilde{a}, x, y) &= ex + fy, & q_2(\tilde{a}, x, y) &= lx^2 + 2mxy + ny^2. \end{aligned}$$

Here we denote by  $\tilde{a} = (a, c, d, g, h, k, b, e, f, l, m, n)$  the 12-tuple of the coefficients of system (1.3). Thus a quadratic system can be identified with a point  $\tilde{a}$  in  $\mathbb{R}^{12}$ .

We denote the class of all real quadratic differential systems with **QS**.

In this work we are interested in polynomial differential equations (1.1) which are endowed with an algebraic geometric structure, i.e. which posses invariant algebraic curves under the flow. We are interested both in their geometry and also in the impact this geometry has on the integrability of the systems.

**Definition 1.1** ([11]). An algebraic curve  $C(x, y) = 0$  with  $C(x, y) \in \mathbb{C}[x, y]$  is called an *invariant algebraic curve* of system (1.1) if it satisfies the following identity:

$$C_x P + C_y Q = KC, \quad (1.4)$$

for some  $K \in \mathbb{C}[x, y]$  where  $C_x$  and  $C_y$  are the derivative of  $C$  with respect to  $x$  and  $y$ .  $K$  is called the *cofactor* of the curve  $C = 0$ .

For simplicity we write the curve  $C$  instead of the curve  $C = 0$  in  $\mathbb{C}^2$ . Note that if system (1.1) has degree  $m$  then the cofactor of an invariant algebraic curve  $C$  of the system has degree  $m - 1$ .

**Definition 1.2.** Let  $U$  be an open subset of  $\mathbb{R}^2$ . A real function  $H: U \rightarrow \mathbb{R}$  is a *first integral* of system (1.1) if it is constant on all solution curves  $(x(t), y(t))$  of system (1.1), i.e.,  $H(x(t), y(t)) = k$ , where  $k$  is a real constant, for all values of  $t$  for which the solution  $(x(t), y(t))$  is defined on  $U$ .

If  $H$  is differentiable in  $U$  then  $H$  is a first integral on  $U$  if and only if

$$H_x P + H_y Q = 0. \quad (1.5)$$

**Definition 1.3.** If a system (1.1) has a first integral of the form

$$H(x, y) = C_1^{\lambda_1} \dots C_p^{\lambda_p} \quad (1.6)$$

where  $C_i$  are invariant algebraic curves of system (1.1) and  $\lambda_i \in \mathbb{C}$  then we say that system (1.1) is *Darboux integrable* and we call the function  $H$  a *Darboux function*.

**Theorem 1.4** ([11]). *Suppose that a polynomial system (1.1) has  $m$  invariant algebraic curves  $C_i(x, y) = 0$ ,  $i \leq m$ , with  $C_i \in \mathbb{C}[x, y]$  and with  $m > n(n+1)/2$  where  $n$  is the degree of the system. Then there exist complex numbers  $\lambda_1, \dots, \lambda_m$  such that  $C_1^{\lambda_1} \dots C_m^{\lambda_m}$  is a first integral of the system.*

If a system (1.1) admits a rational first integral we say that (1.1) is *algebraically integrable*. Poincaré was enthusiastic about the work of Darboux [11] which he called “oeuvre magistrale” in [22] and stated *the problem of algebraic integrability* which asks to recognize when a polynomial vector field has a rational first integral. Jouanolou gave a sufficient condition for recognizing that a polynomial system has a rational first integral.

**Theorem 1.5** ([15]). *Consider a polynomial system (1.1) of degree  $n$  and suppose that it admits  $m$  invariant algebraic curves  $C_i(x, y) = 0$  where  $1 \leq i \leq m$ , then if  $m \geq 2 + \frac{n(n+1)}{2}$ , there exists integers  $N_1, N_2, \dots, N_m$  such that  $I(x, y) = \prod_{i=1}^m C_i^{N_i}$  is a first integral of (1.1).*

In connection to this problem Poincaré stated a number of definitions among them the following definitions below.

Let  $H = f/g$  be a rational first integral of the polynomial vector field (1.2). We say that  $H$  has degree  $n$  if  $n$  is the maximum of the degrees of  $f$  and  $g$ . We say that the degree of  $H$  is *minimal* among all the degrees of the rational first integrals of  $\chi$  if any other rational first integral of  $\chi$  has a degree greater than or equal to  $n$ . Let  $H = f/g$  be a rational first integral of  $\chi$ . According to Poincaré [22] we say that  $c \in \mathbb{C} \cup \{\infty\}$  is a *remarkable value* of  $H$  if  $f + cg$  is a reducible polynomial in  $\mathbb{C}[x, y]$ . Here, if  $c = \infty$ , then  $f + cg$  denotes  $g$ . Note that for all  $c \in \mathbb{C}$  the algebraic curve  $f + cg = 0$  is invariant. The curves in the factorization of  $f + cg$ , when  $c$  is a remarkable value, are called *remarkable curves*.

Now suppose that  $c$  is a remarkable value of a rational first integral  $H$  and that  $u_1^{\alpha_1} \dots u_r^{\alpha_r}$  is the factorization of the polynomial  $f + cg$  into reducible factors in  $\mathbb{C}[x, y]$ . If at least one of the  $\alpha_i$  is larger than 1 then we say, following again Poincaré (see for instance [14]), that  $c$  is a *critical remarkable value* of  $H$ , and that  $u_i = 0$  having  $\alpha_i > 1$  is a *critical remarkable curve* of the vector field (1.2) with exponent  $\alpha_i$ .

Since we can think of  $c \in \mathbb{C} \cup \{\infty\}$  as the projective line  $P_1(\mathbb{R})$  we can also use the following definition.

**Definition 1.6.** Consider  $\mathcal{F}_{(c_1, c_2)} : c_1 f - c_2 g = 0$  where  $f/g$  is a rational first integral of (1.2). We say that  $[c_1 : c_2]$  is a remarkable value of the curve  $\mathcal{F}_{(c_1, c_2)}$  if  $\mathcal{F}_{(c_1, c_2)}$  is reducible over  $\mathbb{C}$ .

It is proved in [4] that there are finitely many remarkable values for a given rational first integral  $H$  and if (1.2) has a rational first integral and has no polynomial first integrals, then it has a polynomial inverse integrating factor if and only if the first integral has at most two critical remarkable values.

Given  $H = f/g$  a rational first integral, consider  $F_{(c_1, c_2)} = c_1 f - c_2 g$  where  $\deg F_{(c_1, c_2)} = n$ . If  $F_{(c_1, c_2)} = f_1 f_2$  where  $\deg f_i = n_i < n$  then necessarily the points on the intersection of  $f_1 = 0$  and  $f_2 = 0$  must be singular points of the curve  $F_{(c_1, c_2)}$ . So to find the irreducible factors of  $F_{(c_1, c_2)}$  we start by finding the singularities of  $F_{(c_1, c_2)}$ , i.e., the points on the curve which annihilate both first derivatives in  $x$  and  $y$ .

The following notion was defined by Christopher in [5] where he called it “degenerate invariant algebraic curve”.

**Definition 1.7.** Let  $F(x, y) = \exp\left(\frac{G(x, y)}{H(x, y)}\right)$  with  $G, H \in \mathbb{C}[x, y]$  coprime. We say that  $F$  is an *exponential factor* of system (1.1) if it satisfies the equality

$$F_x P + F_y Q = LF, \quad (1.7)$$

for some  $L \in \mathbb{C}[x, y]$ . The polynomial  $L$  is called the *cofactor* of the exponential factor  $F$ .

**Definition 1.8.** If system (1.1) has a first integral of the form

$$H(x, y) = C_1^{\lambda_1} \cdots C_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \quad (1.8)$$

where  $C_i$  and  $F_j$  are the invariant algebraic curves and exponential factors of system (1.1) respectively and  $\lambda_i, \mu_j \in \mathbb{C}$ , then we say that the system is *generalized Darboux integrable*. We call the function  $H$  a *generalized Darboux function*.

**Remark 1.9.** In [11] Darboux considered functions of the type (1.6), not of type (1.8). In recent works functions of type (1.8) were called Darboux functions. Since in this work we need to pay attention to the distinctions among the various kinds of first integral we call (1.6) a Darboux and (1.8) a generalized Darboux first integral.

**Definition 1.10.** Let  $U$  be an open subset of  $\mathbb{R}^2$  and let  $R : U \rightarrow \mathbb{R}$  be an analytic function which is not identically zero on  $U$ . The function  $R$  is an integrating factor of a polynomial system (1.1) on  $U$  if one of the following two equivalent conditions holds:

$$\operatorname{div}(RP, RQ) = 0, \quad R_x P + R_y Q = -R \operatorname{div}(P, Q), \quad (1.9)$$

on  $U$ .

A first integral  $H$  of

$$\dot{x} = RP, \quad \dot{y} = RQ$$

associated to the integrating factor  $R$  is given by

$$H(x, y) = \int R(x, y)P(x, y)dy + h(x),$$

where  $H(x, y)$  is a function satisfying  $H_x = -RQ$ . Then,

$$\dot{x} = H_y, \quad \dot{y} = -H_x.$$

In order that this function  $H$  be well defined the open set  $U$  must be simply connected.

Liouvillian functions are functions that are built up from rational functions using exponentiation, integration, and algebraic functions. For more details on Liouvillian functions, see [7].

**Theorem 1.11** ([4, 21]). *If a planar polynomial vector field (1.2) has a generalized Darboux first integral, then it has a rational integrating factor.*

As for a converse, we have the following result which easily follows from [23].

**Theorem 1.12** ([8]). *If a planar polynomial vector field (1.2) has a rational integrating factor, then it has a generalized Darboux first integral.*

An important consequence of Singer's theorem (see [27]) is the following.

**Theorem 1.13** ([5, 27]). *A planar polynomial differential system (1.1) has a Liouvillian first integral if and only if it has a generalized Darboux integrating factor.*

For a proof see [28, p. 134].

We have the following table summing up these results.

First integral		Integrating factor
Generalized Darboux	$\Leftrightarrow$	Rational
Liouvillian	$\Leftrightarrow$	Generalized Darboux

**Definition 1.14** ([11]). Consider a planar polynomial system (1.1). An algebraic solution  $f = 0$  of (1.1) is an algebraic invariant curve which is irreducible over  $\mathbb{C}$ .

**Theorem 1.15** ([6]). *Consider a polynomial system (1.1) that has  $k$  algebraic solutions  $C_i = 0$  such that*

- (a) *all curves  $C_i = 0$  are non-singular and have no repeated factor in their highest order terms,*
- (b) *no more than two curves meet at any point in the finite plane and are not tangent at these points,*
- (c) *no two curves have a common factor in their highest order terms,*
- (d) *the sum of the degrees of the curves is  $n + 1$ , where  $n$  is the degree of system (1.1).*

*Then system (1.1) has an integrating factor*

$$\mu(x, y) = 1/(C_1 C_2 \cdots C_k).$$

This result of Christopher–Kooij (C–K) is interesting because it relates the geometry of the configuration of invariant algebraic curves of the systems with the expression of the integrating factors involving the polynomials defining the curves. In fact this theorem has a geometric content which is however not completely explicit in the algebraic way their theorem is stated. We restate the above result in geometric terms as follows:

**Theorem 1.16.** *Consider a polynomial system (1.1) that has  $k$  algebraic solutions  $C_i = 0$  such that*

- (a) *all curves  $C_i = 0$  are non-singular and they intersect transversally the line at infinity  $Z = 0$ ,*
- (b) *no more than two curves meet at any point in the finite plane and are not tangent at these points,*
- (c) *no two curves intersect at a point on the line at infinity  $Z = 0$ ,*
- (d) *the sum of the degrees of the curves is  $n + 1$ , where  $n$  is the degree of system (1.1).*

*Then system (1.1) has an integrating factor*

$$\mu(x, y) = 1/(C_1 C_2 \cdots C_k).$$

In the hypotheses of this theorem the way the curves are placed with respect to one another in the totality of the curves, in other words the “geometry of the configuration of invariant algebraic curves” has an impact of the kind of integrating factor we could have. One of our goals is to collect data so as to extend this theorem beyond these limiting geometric conditions.

There are some important invariant polynomials in the study of polynomial vector fields. Considering  $C_2(\tilde{a}, x, y) = yp_2(\tilde{a}, x, y) - xq_2(\tilde{a}, x, y)$  as a cubic binary form of  $x$  and  $y$  we calculate

$$\eta(\tilde{a}) = \text{Discrim}[C_2, \xi], \quad M(\tilde{a}, x, y) = \text{Hessian}[C_2],$$

where  $\xi = y/x$  or  $\xi = x/y$ . It is known that the singular points at infinity of quadratic systems are given by the solutions in  $x$  and  $y$  of  $C_2(\tilde{a}, x, y) = 0$ . If  $\eta < 0$  then this means we have one real singular point at infinity and two complex.

**Remark 1.17.** We note that since a system in **QSH** always has an invariant hyperbola then clearly we always have at least 2 real singular points at infinity. So we must have  $\eta \geq 0$ .

The family **QSH** can be split as follows: **QSH**<sub>( $\eta=0$ )</sub> of systems which possess either exactly two distinct real singularities at infinity or the line at infinity filled up with singularities and **QSH**<sub>( $\eta>0$ )</sub> of systems which possess three distinct real singularities at infinity in  $P_2(\mathbb{C})$ .

In [18] the authors proved that there are 162 distinct configurations and provided necessary and sufficient conditions for a non-degenerate quadratic differential system to have at least one invariant hyperbola and for the realization of each one of the configurations. These conditions are expressed in terms of the coefficients of the systems. They obtained the normal forms for family **QSH** and in this paper we study the following 3 normal forms:

$$\begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 4a - 3v^2 - \frac{4xy}{3} + \frac{y^2}{3}, \quad \text{where } a \neq 0. \end{cases} \quad (1.10)$$

$$\begin{cases} \dot{x} = -\frac{x^2}{2} - \frac{xy}{2} \\ \dot{y} = b - \frac{3xy}{2} + \frac{y^2}{2}, \quad \text{where } b \neq 0. \end{cases} \quad (1.11)$$

$$\begin{cases} \dot{x} = 2a + gx^2 + xy, \\ \dot{y} = a(2g - 1) + (g - 1)xy + y^2, \quad \text{where } a(g - 1) \neq 0. \end{cases} \quad (1.12)$$

Our first goal in this paper is to do a complete study of these three families of quadratic systems which possess an invariant hyperbola. Our interest is in the geometry of these systems, as expressed in terms of their invariant algebraic curves, in the impact of this geometry on the integrability of these systems, on their phase portraits and in the dynamics of the systems expressed in the bifurcation diagrams of the families we study. Our third goal is to confront our results with the existing results in the literature and bring to light some missing cases in these other studies which we point out here. Our geometric analysis is done in detail as this is part of a program of collecting data in order to obtain more global results on the family **QSH** and its Darboux theory.

Our paper is organized as follows: in Section 2 we give a number of definitions and propositions useful for the other sections. In Sections 3, 4, 5 we present a complete study of families (1.10), (1.11) and (1.12). The choice of the first two families is motivated by the fact that they do not satisfy all the conditions in the hypothesis of the Christopher–Kooij theorem, here stated in theorem 1.15, but the conclusion of the theorem still holds, while the last family does not always possess a first integral and it will provide a counterpoint. In Section 6 we raise some questions, consider the problem of Poincaré for the family **QSH**, and make some concluding comments.



## 2 Preliminaries

The notion of *configuration of invariant curves* of a polynomial differential system appears in several works, see for instance [26].

**Definition 2.1.** Consider a real planar polynomial system (1.1) with a finite number of singular points. By a *configuration of algebraic solutions* of the system we mean a set of algebraic solutions over  $\mathbb{C}$  of the system, each one of these curves endowed with its own multiplicity and together with all the real singular points of this system located on these curves, each one of these singularities endowed with its own multiplicity.

**Definition 2.2.** Suppose we have two systems  $(S_1), (S_2)$  in **QSH** with a finite number of singularities, finite or infinite, a finite set of invariant hyperbolas  $\mathcal{H}_i^1 : h_i^1(x, y) = 0, i = 1, \dots, k$  of  $(S_1)$  (respectively  $\mathcal{H}_i^2 : h_i^2(x, y) = 0, i = 1, \dots, k$  of  $(S_2)$ ) and a finite set (which could also be empty) of invariant straight lines  $L_j^1 : g_j^1(x, y) = 0, j = 1, \dots, k'$  of  $(S_1)$  (respectively  $L_j^2 : g_j^2(x, y) = 0, j = 1, \dots, k'$  of  $(S_2)$ ). We say that the two configurations  $C_1, C_2$  of hyperbolas and lines of these systems are *equivalent* if there is a one-to-one correspondence  $\Phi_h$  between the hyperbolas of  $C_1$  and  $C_2$  and a one-to-one correspondence  $\Phi_l$  between the lines of  $C_1$  and  $C_2$  such that:

- (i) the correspondences conserve the multiplicities of the hyperbolas and lines (in case there are any) and also send a real invariant curve to a real invariant curve and a complex invariant curve to a complex invariant curve;
- (ii) for each hyperbola  $\mathcal{H} : h(x, y) = 0$  of  $C_1$  (respectively each line  $L : g(x, y) = 0$ ) we have a one-to-one correspondence between the real singular points on  $\mathcal{H}$  (respectively on  $L$ ) and the real singular points on  $\Phi_h(\mathcal{H})$  (respectively  $\Phi_l(L)$ ) conserving their multiplicities, their location on branches of hyperbolas and their order on these branches (respectively on the lines);
- (iii) Furthermore, consider the total curves  $\mathcal{F}^1 : \prod H_i^1(X, Y, Z) \prod G_j^1(X, Y, Z)Z = 0$  (respectively  $\mathcal{F}^2 : \prod H_i^2(X, Y, Z) \prod G_j^2(X, Y, Z)Z = 0$ ) where  $H_i^1(X, Y, Z) = 0, G_j^1(X, Y, Z) = 0$  (respectively  $H_i^2(X, Y, Z) = 0, G_j^2(X, Y, Z) = 0$ ) are the projective completions of  $\mathcal{H}_i^1, L_j^1$  (respectively  $\mathcal{H}_i^2, L_j^2$ ). Then, there is a one-to-one correspondence  $\psi$  between the singularities of the curves  $\mathcal{F}^1$  and  $\mathcal{F}^2$  conserving their multiplicities as singular points of these (total) curves.

It is important to assume that systems (1.3) are non-degenerate because otherwise doing a time rescaling, they can be reduced to linear or constant systems. Under this assumption all the systems in **QSH** have a finite number of finite singular points.

In the family **QSH** we also have cases where we have an infinite number of hyperbolas. In these cases, by a Jouanolou result (see Theorem 1.5 on page 3), we have a rational first integral.

In [18] the authors classified the family **QSH**, according to their geometric properties encoded in the configurations of invariant hyperbolas and invariant straight lines which these systems possess. If a quadratic system has an infinite number of hyperbolas then the system has a finite number of invariant affine straight lines (see [1]). Therefore, we can talk about *equivalence of configurations of the invariant affine lines associated to the system*. Given two such

configurations  $C_{1l}$  and  $C_{2l}$  associated to systems  $(S_1)$  and  $(S_2)$  of (1.1), we say they are *equivalent* if and only if there is a one-to-one correspondence  $\Phi$  between the lines of  $C_{1l}$  and  $C_{2l}$  such that:

- (i) the correspondence preserve the multiplicities of the lines and also sends a real (respectively complex) invariant line to a real (respectively complex) invariant line;
- (ii) for each line  $L : g(x, y) = 0$  we have a one-to-one correspondence between the real singularities on  $L$  and the real singularities on  $\Phi$  preserving their multiplicities and their order on the lines.

**Definition 2.3** ([18]). Consider two systems  $(S_1)$  and  $(S_2)$  in **QSH** each one with an infinite number of invariant hyperbolas. Consider the configurations  $C_{1l}$  and  $C_{2l}$  of invariant affine straight lines  $L_j^1 : g_j^1(x, y) = 0$  where  $j = 1, 2, \dots, k$  of system  $(S_1)$  and respectively  $L_j^2 : g_j^2(x, y) = 0$  where  $j = 1, 2, \dots, k$  of system  $(S_2)$ . We say that the two configurations  $C_{1l}$  and  $C_{2l}$  are equivalent with respect to the hyperbolas of the systems if and only if:

- (i) they are equivalent as configurations of invariant lines, and
- (ii) taking any hyperbola  $\mathcal{H}_1 : h_1(x, y) = 0$  of  $(S_1)$  and any hyperbola  $\mathcal{H}_2 : h_2(x, y) = 0$  of  $(S_2)$ , then we must have a one-to-one correspondence between the real singularities of system  $(S_1)$  located on  $\mathcal{H}_1$  and of real singularities of system  $(S_2)$  located on  $\mathcal{H}_2$ , preserving their multiplicities, their location and order on branches.

Furthermore, consider the curves  $\mathcal{F}_1 : \prod h_1(x, y) \prod g_j^1 = 0$  and  $\mathcal{F}_2 : \prod h_2(x, y) \prod g_j^2 = 0$ . Then, we have a one-to-one correspondence between the singularities of the curve  $\mathcal{F}_1$  with those in the curve  $\mathcal{F}_2$  preserving their multiplicities as singularities of these curves.

The definition above is independent of the choice of the two hyperbolas  $\mathcal{H}_1 : h_1(x, y) = 0$  of  $(S_1)$  and  $\mathcal{H}_2 : h_2(x, y) = 0$  of  $(S_2)$ .

Suppose that a polynomial differential system has an algebraic solution  $f(x, y) = 0$  where  $f(x, y) \in \mathbb{C}[x, y]$  is of degree  $n$  given by

$$f(x, y) = c_0 + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + \dots + c_{n0}x^n + c_{n-1,1}x^{n-1}y + \dots + c_{0n}y^n,$$

with  $\hat{c} = (c_0, c_{10}, \dots, c_{0n}) \in \mathbb{C}^N$  where  $N = (n+1)(n+2)/2$ . We note that the equation

$$\lambda f(x, y) = 0, \lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}$$

yields the same locus of complex points in the plane as the locus induced by  $f(x, y) = 0$ . Therefore, a curve of degree  $n$  is defined by  $\hat{c}$  where

$$[\hat{c}] = [c_0 : c_{10} : \dots : c_{0n}] \in P_{N-1}(\mathbb{C}).$$

We say that a sequence of curves  $f_i(x, y) = 0$ , each one of degree  $n$ , converges to a curve  $f(x, y) = 0$  if and only if the sequence of points  $[c_i] = [c_{i0} : c_{i10} : \dots : c_{i0n}]$  converges to  $[\hat{c}] = [c_0 : c_{10} : \dots : c_{0n}]$  in the topology of  $P_{N-1}(\mathbb{C})$ .

We observe that if we rescale the time  $t' = \lambda t$  by a positive constant  $\lambda$  the geometry of the systems (1.1) (phase curves) does not change. So for our purposes we can identify a system (1.1) of degree  $n$  with a point

$$[a_0 : a_{10} : \dots : a_{0n} : b_0 : b_{10} : \dots : b_{0n}] \in \mathbb{S}^{N-1}(\mathbb{R})$$

where  $N = (n+1)(n+2)$ .



**Definition 2.4.**

(1) We say that an invariant curve

$$\mathcal{L} : f(x, y) = 0, \quad f \in \mathbb{C}[x, y]$$

for a polynomial system  $(S)$  of degree  $n$  has geometric multiplicity  $m$  if there exists a sequence of real polynomial systems  $(S_k)$  of degree  $n$  converging to  $(S)$  in the topology of  $\mathbb{S}^{N-1}(\mathbb{R})$  where  $N = (n+1)(n+2)$  such that each  $(S_k)$  has  $m$  distinct invariant curves

$$\mathcal{L}_{1,k} : f_{1,k}(x, y) = 0, \dots, \mathcal{L}_{m,k} : f_{m,k}(x, y) = 0$$

over  $\mathbb{C}$ ,  $\deg(f) = \deg(f_{i,k}) = r$ , converging to  $\mathcal{L}$  as  $k \rightarrow \infty$ , in the topology of  $P_{R-1}(\mathbb{C})$ , with  $R = (r+1)(r+2)/2$  and this does not occur for  $m+1$ .

(2) We say that the line at infinity

$$\mathcal{L}_\infty : Z = 0$$

of a polynomial system  $(S)$  of degree  $n$  has geometric multiplicity  $m$  if there exists a sequence of real polynomial systems  $(S_k)$  of degree  $n$  converging to  $(S)$  in the topology of  $\mathbb{S}^{N-1}(\mathbb{R})$  where  $N = (n+1)(n+2)$  such that each  $(S_k)$  has  $m-1$  distinct invariant lines

$$\mathcal{L}_{1,k} : f_{1,k}(x, y) = 0, \dots, \mathcal{L}_{m-1,k} : f_{m-1,k}(x, y) = 0$$

over  $\mathbb{C}$ , converging to the line at infinity  $\mathcal{L}_\infty$  as  $k \rightarrow \infty$ , in the topology of  $P_2(\mathbb{C})$  and this does not occur for  $m$ .

**Definition 2.5** ([9]). Let  $\mathbb{C}_m[x, y]$  be the  $\mathbb{C}$ -vector space of polynomials in  $\mathbb{C}[x, y]$  of degree at most  $m$  and of dimension  $R = \binom{2+m}{2}$ . Let  $\{v_1, v_2, \dots, v_R\}$  be a base of  $\mathbb{C}_m[x, y]$ . We denote by  $M_R(m)$  the  $R \times R$  matrix

$$M_R(m) = \begin{pmatrix} v_1 & v_2 & \dots & v_R \\ \chi(v_1) & \chi(v_2) & \dots & \chi(v_R) \\ \vdots & \vdots & \ddots & \vdots \\ \chi^{R-1}(v_1) & \chi^{R-1}(v_2) & \dots & \chi^{R-1}(v_R) \end{pmatrix}, \quad (2.1)$$

where  $\chi^{k+1}(v_i) = \chi(\chi^k(v_i))$ . The  $m$ th extactic curve of  $\chi$ ,  $\mathcal{E}_m(\chi)$ , is given by the equation  $\det M_R(m) = 0$ . We also call  $\mathcal{E}_m(\chi)$  the  $m$ th extactic polynomial.

From the properties of the determinant we note that the extactic curve is independent of the choice of the base of  $\mathbb{C}_m[x, y]$ .

**Theorem 2.6** ([20]). Consider a planar vector field (1.2). We have  $\mathcal{E}_m(\chi) = 0$  and  $\mathcal{E}_{m-1}(\chi) \neq 0$  if and only if  $\chi$  admits a rational first integral of exact degree  $m$ .

Observe that if  $f = 0$  is an invariant algebraic curve of degree  $m$  of  $\chi$ , then  $f$  divides  $\mathcal{E}_m(\chi)$ . This is due to the fact that if  $f$  is a member of a base of  $\mathbb{C}_m[x, y]$ , then  $f$  divides the whole column in which  $f$  is located.

**Definition 2.7** ([9]). We say that an invariant algebraic curve  $f = 0$  of degree  $m \geq 1$  has algebraic multiplicity  $k$  if  $\det M_R(m) \neq 0$  and  $k$  is the maximum positive integer such that  $f^k$  divides  $\det M_R(m)$ ; and it has no defined algebraic multiplicity if  $\det M_R(m) \equiv 0$ .

**Definition 2.8** ([9]). We say that an invariant algebraic curve  $f = 0$  of degree  $m \geq 1$  has integrable multiplicity  $k$  with respect to  $\chi$  if  $k$  is the largest integer for which the following is true: there are  $k - 1$  exponential factors  $\exp(g_j/f^j)$ ,  $j = 1, \dots, k - 1$ , with  $\deg g_j \leq jm$ , such that each  $g_j$  is not a multiple of  $f$ .

In the next result we see that the algebraic and integrable multiplicity coincide if  $f = 0$  is an irreducible invariant algebraic curve.

**Theorem 2.9** ([16]). Consider an irreducible invariant algebraic curve  $f = 0$  of degree  $m \geq 1$  of  $\chi$ . Then  $f$  has algebraic multiplicity  $k$  if and only if the vector field (1.2) has  $k - 1$  exponential factors  $\exp(g_j/f^j)$ , where  $(g_j, f) = 1$  and  $g_j$  is a polynomial of degree at most  $jm$ , for  $j = 1, \dots, k - 1$ .

In [9] the authors showed that the definitions of geometric, algebraic and integrable multiplicity are equivalent when  $f = 0$  is an irreducible invariant algebraic curve of vector field (1.2).

In order to use the infinity of  $\mathbb{R}^2$  as an additional invariant curve for studying the integrability of the vector field  $\chi$ , we need the Poincaré compactification of the vector field  $\chi$ . For  $Z \neq 0$  consider the change of variables

$$x = \frac{1}{Z}, \quad y = \frac{Y}{Z}$$

the vector field  $\chi$  is transformed to

$$\bar{\chi} = -Z \bar{P}(Z, Y) \frac{\partial}{\partial Z} + (\bar{Q}(Z, Y) - Y \bar{P}(Z, Y)) \frac{\partial}{\partial Y}$$

where  $\bar{P}(Z, Y) = Z^2 P(\frac{1}{Z}, \frac{Y}{Z})$  and  $\bar{Q}(Z, Y) = Z^2 Q(\frac{1}{Z}, \frac{Y}{Z})$ .

We note that  $Z = 0$  is an invariant line of the vector field  $\bar{\chi}$  and that the infinity of  $\mathbb{R}^2$  corresponds to  $Z = 0$  of the vector field  $\bar{\chi}$ . So we can define the algebraic multiplicity of  $Z = 0$  for the vector field  $\bar{\chi}$ .

**Definition 2.10.** We say that the infinity of  $\chi$  has algebraic multiplicity  $k$  if  $Z = 0$  has algebraic multiplicity  $k$  for the vector field  $\bar{\chi}$ ; and that it has no defined algebraic multiplicity if  $Z = 0$  has no defined algebraic multiplicity for  $\bar{\chi}$ .

Let's recall the algebraic-geometric definition of an  $r$ -cycle on an irreducible algebraic variety of dimension  $n$ .

**Definition 2.11.** Let  $V$  be an irreducible algebraic variety of dimension  $n$  over a field  $\mathbb{K}$ . A cycle of dimension  $r$  or  $r$ -cycle on  $V$  is a formal sum

$$\sum_W n_W W$$

where  $W$  is a subvariety of  $V$  of dimension  $r$  which is not contained in the singular locus of  $V$ ,  $n_W \in \mathbb{Z}$ , and only a finite number of  $n_W$ 's are non-zero. We call degree of an  $r$ -cycle the sum

$$\sum_W n_W.$$

An  $(n - 1)$ -cycle is called a divisor.

**Definition 2.12.** For a non-degenerate polynomial differential systems (S) possessing a finite number of algebraic solutions

$$\mathcal{F} = \{f_i\}_{i=1}^m, f_i(x, y) = 0, f_i(x, y) \in \mathbb{C},$$

each with multiplicity  $n_i$  and a finite number of singularities at infinity, we define the algebraic solutions divisor (also called the invariant curves divisor) on the projective plane,

$$ICD_{\mathcal{F}} = \sum_{n_i} n_i \mathcal{C}_i + n_{\infty} \mathcal{L}_{\infty}$$

where  $\mathcal{C}_i : F_i(X, Y, Z) = 0$  are the projective completions of  $f_i(x, y) = 0$ ,  $n_i$  is the multiplicity of the curve  $\mathcal{C}_i = 0$  and  $n_{\infty}$  is the multiplicity of the line at infinity  $\mathcal{L}_{\infty} : Z = 0$ .

It is well known (see [1]) that the maximum number of invariant straight lines, including the line at infinity, for polynomial systems of degree  $n \geq 2$  is  $3n$ .

**Proposition 2.13** ([1]). *Every quadratic differential system has at most six invariant straight lines, including the line at infinity.*

In the case we consider here, we have a particular instance of the divisor  $ICD$  because the invariant curves will be invariant hyperbolas and invariant lines of a quadratic differential system, in case these are in finite number. In case we have an infinite number of hyperbolas we can construct the divisor of the invariant straight lines which are always in finite number.

Another ingredient of the configuration of algebraic solutions are the real singularities situated on these curves. We also need to use here the notion of multiplicity divisor of real singularities of a system, located on the algebraic solutions of the system.

**Definition 2.14.**

1. Suppose a real quadratic system (1.3) has a non-zero finite number of invariant hyperbolas

$$\mathcal{H}_i : h_i(x, y) = 0, i = 1, 2, \dots, k$$

and a finite number of affine invariant lines

$$\mathcal{L}_j : f_j(x, y) = 0, j = 1, 2, \dots, l.$$

We denote the line at infinity  $\mathcal{L}_{\infty} : Z = 0$ . Let us assume that on the line at infinity we have a finite number of singularities. The divisor of invariant hyperbolas and invariant lines on the complex projective plane of the system is the following

$$ICD = n_1 \mathcal{H}_1 + \dots + n_k \mathcal{H}_k + m_1 \mathcal{L}_1 + \dots + m_l \mathcal{L}_l + m_{\infty} \mathcal{L}_{\infty}$$

where  $n_i$  (respectively  $m_j$ ) is the multiplicity of the hyperbola  $\mathcal{H}_i$  (respectively  $m_j$  of the line  $\mathcal{L}_j$ ), and  $m_{\infty}$  is the multiplicity of  $\mathcal{L}_{\infty}$ . We also mark the complex (non-real) invariant hyperbolas (respectively lines) denoting them by  $\mathcal{H}_i^{\mathbb{C}}$  (respectively  $\mathcal{L}_i^{\mathbb{C}}$ ). We define the total multiplicity  $TM$  of the divisor as the sum  $\sum_i n_i + \sum_j m_j + m_{\infty}$ .

2. The zero-cycle on the real projective plane, of singularities of a quadratic system (1.3) located on the configuration of invariant lines and invariant hyperbolas, is given by

$$M_{0CS} = r_1 P_1 + \dots + r_l P_l + v_1 P_1^{\infty} + \dots + v_n P_n^{\infty}$$

where  $P_i$  (respectively  $P_j^\infty$ ) are all the finite (respectively infinite) such singularities of the system and  $r_i$  (respectively  $v_j$ ) are their corresponding multiplicities. We mark the complex singular points denoting them by  $P_i^C$ . We define the total multiplicity  $TM$  of zero-cycles as the sum  $\sum_i r_i + \sum_j v_j$ .

In the family  $QSH$  we have configurations which have an infinite number of hyperbolas. These are of two kinds: those with a finite number of singular points at infinity, and those with the line at infinity filled up with singularities. To distinguish these two cases we define  $|\text{Sing}_\infty|$  to be the cardinality of the set of singular points at infinity of the systems. In the first case we have  $|\text{Sing}_\infty| = 2$  or  $3$ , and in the second case  $|\text{Sing}_\infty|$  is the continuum and we simply write  $|\text{Sing}_\infty| = \infty$ . Since in both cases the systems admit a finite number of affine invariant straight lines we can use them to distinguish the configurations.

**Definition 2.15.**

- (1) In case we have an infinite number of hyperbolas and just two or three singular points at infinity but we have a finite number of invariant straight lines we define

$$ILD = m_1 \mathcal{L}_1 + \cdots + m_l \mathcal{L}_l + m_\infty \mathcal{L}_\infty.$$

- (2) In case we have an infinite number of hyperbolas, the line at infinity is filled up with singularities and we have a finite number of affine lines, we define

$$ILD = m_1 \mathcal{L}_1 + \cdots + m_l \mathcal{L}_l.$$

Suppose we have a finite number of invariant hyperbolas and invariant straight lines of a system  $(S)$  and that they are given by equations

$$f_i(x, y) = 0, \quad i \in \{1, 2, \dots, k\}, \quad f_i \in \mathbb{C}[x, y].$$

Let us denote by  $F_i(X, Y, Z) = 0$  the projection completion of the invariant curves  $f_i = 0$  in  $P_2(\mathbb{C})$ .

**Definition 2.16.** The total invariant curve of the system  $(S)$  in  $QSH$ , on  $P_2(\mathbb{R})$ , is the curve

$$T(S) = \prod_i F_i(X, Y, Z)Z = 0.$$

In case one of the curves is multiple then it will appear with its multiplicity.

For example, if a system  $(S)$  admits an invariant hyperbola  $h(x, y)$  with multiplicity two and the line at infinity  $Z = 0$  has multiplicity one, then the total invariant curve of this system is

$$T(S) = H(X, Y, Z)^2 Z = 0$$

where  $H(X, Y, Z) = 0$  is the projection completion of  $h = 0$ . The degree of  $T(S)$  is 5.

The singular points of the system  $(S)$  situated on  $T(S)$  are of two kinds: those which are simple (or smooth) points of  $T(S)$  and those which are multiple points of  $T(S)$ .

**Remark 2.17.** To each singular point of the system we have its associated multiplicity as a singular point of the system. In addition, when these singular points are situated on the total curve, we also have the multiplicity of these points as points on the total curve  $T(S)$ . Through

a singular point of the systems there may pass several of the curves  $F_i = 0$  and  $Z = 0$ . Also we may have the case when this point is a singular point of one or even of several of the curves in case we work with invariant curves with singularities. This leads to the multiplicity of the point as point of the curve  $T(S)$ . The simple points of the curve  $T(S)$  are those of multiplicity one. They are also the smooth points of this curve.

**Definition 2.18.** The zero-cycle of the total curve  $T(S)$  of system  $(S)$  is given by

$$M_{0CT} = r_1P_1 + \cdots + r_lP_l + v_1P_1^\infty + \cdots + v_nP_n^\infty$$

where  $P_i$  (respectively  $P_j^\infty$ ) are all the finite (respectively infinite) singularities situated on  $T(S)$  and  $r_i$  (respectively  $v_j$ ) are their corresponding multiplicities as points on the total curve  $T(S)$ . We define the total multiplicity  $TM$  of zero-cycles of the total invariant curve as the sum  $\sum_i r_i + \sum_j v_j$ .

**Remark 2.19.** If two curves intersects transversally, this point will be a simple point of intersection. If they are tangent, we would have an intersection multiplicity higher than or equal to two.

**Definition 2.20** ([24]). Two polynomial differential systems  $S_1$  and  $S_2$  are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of  $S_1$  to the oriented phase curves of  $S_2$  and preserving the orientation.

To cut the number of non equivalent phase portraits in half we use here another equivalence relation.

**Definition 2.21.** Two polynomial differential systems  $S_1$  and  $S_2$  are topologically equivalent if and only if there exists a homeomorphism of the plane carrying the oriented phase curves of  $S_1$  to the oriented phase curves of  $S_2$ , preserving or reversing the orientation.

We use the notation for singularities as introduced in [2] and [3]. We say that a singular point is *elemental* if it possess two eigenvalues not zero; *semi-elemental* if it possess exactly one eigenvalue equal to zero and *nilpotent* if it posses two eigenvalues zero. We call *intricate* a singular point with its Jacobian matrix identically zero.

We will place first the finite singular points which will be denoted with lower case letters and secondly we will place the infinite singular points which will be denoted by capital letters, separating them by a semicolon ‘;’.

In our study we will have real and complex finite singular points and from the topological viewpoint only the real ones are interesting. When we have a simple (respectively double) complex finite singular point we use the notation  $\odot$  (respectively  $\odot_{(2)}$ ).

For the elemental singular points we use the notation ‘ $s$ ’, ‘ $S$ ’ for saddles, ‘ $n$ ’, ‘ $N$ ’ for nodes, ‘ $f$ ’ for foci and ‘ $c$ ’ for centers.

Non-elemental singular points are multiple points. Here we introduce a special notation for the infinity non-elemental singular point. We denote by  $\binom{a}{b}$  the maximum number  $a$  (respectively  $b$ ) of finite (respectively infinite) singularities which can be obtained by perturbation of the multiple point. For example, when we have a non-elemental point at infinity obtained by the coalescence from a node at infinite with a saddle at infinite we will denoted it by  $\binom{0}{2}SN$ .

The semi-elemental singular points can either be nodes, saddles or saddle-nodes (finite or infinite). If they are finite singular points we will denote them by ‘ $n_{(2)}$ ’, ‘ $s_{(2)}$ ’ and ‘ $sn_{(2)}$ ’,

respectively and if they are infinite singular points by  $'\binom{a}{b}N'$ ,  $'\binom{a}{b}S'$  and  $'\binom{a}{b}SN'$ , where  $\binom{a}{b}$  indicates their multiplicity. We note that semi-elemental nodes and saddles are respectively topologically equivalent with elemental nodes and saddles.

The nilpotent singular points can either be saddles, nodes, saddle-nodes, elliptic-saddles, cusps, foci or centers. The only finite nilpotent points for which we need to introduce notation are the elliptic-saddles and cusps which we denote respectively by  $'es'$  and  $'cp'$ .

The intricate singular points are degenerate singular points. It is known that the neighbourhood of any singular point of a polynomial vector field (except for foci and centers) is formed by a finite number of sectors which could only be of three types: parabolic ( $p$ ), hyperbolic ( $h$ ) and elliptic ( $e$ ) (see [12]). In this work we have the following finite intricate singular points of multiplicity four described according their sectoral decomposition:

- $hpphpp_{(4)}$
- $phph_{(4)}$
- $epep_{(4)}$

The degenerate systems are systems with a common factor in the polynomials defining the system. We will denote this case with the symbol  $\ominus$ . The degeneracy can be produced by a common factor of degree one which defines a straight line or a common quadratic factor which defines a conic. In this paper we have just the first case happening. Following [2] we use the symbol  $\ominus[\ ]$  for a real straight line.

Moreover, we also want to determine whether after removing the common factor of the polynomials, singular points remain on the curve defined by this common factor. If some singular points remain on this curve we will use the corresponding notation of their various kinds. In this situation, the geometrical properties of the singularity that remain after the removal of the degeneracy, may produce topologically different phenomena, even if they are topologically equivalent singularities. So, we will need to keep the geometrical information associated to that singularity.

In this study we use the notation  $(\ominus[\ ]; n^d)$  which denotes the presence of a real straight line filled up with singular points in the system such that the reduced system has a node  $n^d$  on this line where  $n^d$  is a one-direction node, that is, a node with two identical eigenvalues whose Jacobian matrix cannot be diagonal.

The existence of a common factor of the polynomials defining the differential system also affects the infinite singular points.

We point out that the projective completion of a real affine line filled up with singular points has a point on the line at infinity which will then be also a non-isolated singularity. There is a detailed description of this notation in [2]. In case that after the removal of the finite degeneracy, a singular point at infinity remains at the same place, we must denote it with all its geometrical properties since they may influence the local topological phase portrait. In this study we use the notation  $\binom{a}{b}SN, (\ominus[\ ]; \emptyset)$  that means that the system has at infinity a saddle-node, and one non-isolated singular point which is part of a real straight line filled up with singularities (other than the line at infinity), and that the reduced linear system has no infinite singular point in that position. See [2] and [3] for more details.

In order to distinguish topologically the phase portraits of the systems we obtained, we also use some invariants introduced in [25]. Let  $SC$  be the total number of separatrix connections, i.e. of phase curves connecting two singularities which are local separatrices of the two singular points. We denote by



- $SC_f^f$  the total number of SC connecting two finite singularities,
- $SC_f^\infty$  the total number of SC connecting a finite with an infinite singularity,
- $SC_\infty^\infty$  the total number of SC connecting two infinite.

A *graphic* as defined in [13] is formed by a finite sequence of singular points  $r_1, r_2, \dots, r_n$  (with possible repetitions) and non-trivial connecting orbits  $\gamma_i$  for  $i = 1, \dots, n$  such that  $\gamma_i$  has  $r_i$  as  $\alpha$ -limit set and  $r_{i+1}$  as  $\omega$ -limit set for  $i < n$  and  $\gamma_n$  has  $r_n$  as  $\alpha$ -limit set and  $r_1$  as  $\omega$ -limit set. Also normal orientations  $n_j$  of the non-trivial orbits must be coherent in the sense that if  $\gamma_{j-1}$  has left-hand orientation then so does  $\gamma_j$ . A *polycycle* is a graphic which has a Poincaré return map.

A *degenerate graphic* is a graphic where it is also allowed that one or several (even all) connecting orbits  $\gamma_i$  can be formed by an infinite number of singular points. For more details, see [13].

### 3 Geometric analysis of family (1.10)

Consider the family

$$(1.10) \quad \begin{cases} \dot{x} = a - \frac{x^2}{3} - \frac{2xy}{3} \\ \dot{y} = 4a - 3v^2 - \frac{4xy}{3} + \frac{y^2}{3}, \end{cases} \quad \text{where } a \neq 0.$$

This is a two parameter family depending on  $(a, v) \in (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$ . We display below the full geometric analysis of the systems in this family, which is endowed with at least three invariant algebraic curves. In the generic situation

$$av(a - v^2)(a - 3v^2/4)(a + 3v^2)(a - 8v^2/9) \neq 0 \quad (3.1)$$

the systems have only two invariant lines  $J_1$  and  $J_2$  and only two invariant hyperbolas  $J_3$  and  $J_4$  with respective cofactors  $\alpha_i$ ,  $1 \leq i \leq 4$  where

$$\begin{aligned} J_1 &= -3\sqrt{-a + v^2} - x + y, & \alpha_1 &= \sqrt{-a + v^2} - \frac{x}{3} + \frac{y}{3}, \\ J_2 &= 3\sqrt{-a + v^2} - x + y, & \alpha_2 &= -\sqrt{-a + v^2} - \frac{x}{3} + \frac{y}{3}, \\ J_3 &= -3a + 3vx - x^2 + xy, & \alpha_3 &= -v - \frac{2x}{3} - \frac{y}{3}, \\ J_4 &= -3a - 3vx - x^2 + xy, & \alpha_4 &= v - \frac{2x}{3} - \frac{y}{3}. \end{aligned}$$

We see that since the number of invariant curve is four, these systems are Darboux integrable. We note that if  $v = 0$  then the two hyperbolas coincide and we get a double hyperbola. Also if  $a = v^2$  the two lines coincide and we get a double line. So to have four distinct curves we need to put  $v(a - v^2) \neq 0$ . We inquire when we could have an additional line. Calculations yield that this happens when  $a - 3v^2/4 = 0$ . We also inquire when we could have an additional hyperbola. Calculations yield that this happens when  $(a + 3v^2)(a - 8v^2/9) = 0$ .

Straightforward calculations lead us to the tables listed below. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using for lines the 1st and for hyperbola the 2nd extactic polynomial, respectively.

(i)  $av(a - v^2)(a - 3v^2/4)(a + 3v^2)(a - 8v^2/9) \neq 0.$

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -3\sqrt{-a+v^2} - x + y$ $J_2 = 3\sqrt{-a+v^2} - x + y$ $J_3 = -3a + 3vx - x^2 + xy$ $J_4 = -3a - 3vx - x^2 + xy$  $\alpha_1 = \sqrt{-a+v^2} - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = \sqrt{-a+v^2} - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_4 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = (-v - \sqrt{v^2 - a}, -v + 2\sqrt{v^2 - a})$ $P_2 = (v - \sqrt{v^2 - a}, v + 2\sqrt{v^2 - a})$ $P_3 = (-v + \sqrt{v^2 - a}, -v - 2\sqrt{v^2 - a})$ $P_4 = (v + \sqrt{v^2 - a}, v - 2\sqrt{v^2 - a})$  $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$  For $v^2 > a$ we have  $n, s, s, n; N, N, S$ if $v > 0$ $s, n, n, s; N, N, S$ if $v < 0$  For $v^2 < a$ we have  $\odot, \odot, \odot, \odot; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty & \text{if } v^2 > a \\ J_1^C + J_2^C + J_3 + J_4 + \mathcal{L}_\infty & \text{if } v^2 < a \end{cases}$	5 5
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty & \text{if } v^2 > a \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty & \text{if } v^2 < a \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4 = 0$	7
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 5P_2^\infty + P_3^\infty & \text{if } v^2 > a \\ 3P_1^\infty + 5P_2^\infty + P_3^\infty & \text{if } v^2 < a \end{cases}$	17 9

where the total curve  $T$  has

- 1) only two distinct tangents at  $P_1^\infty$ , but one of them is double and
- 2) five distinct tangents at  $P_2^\infty$ .

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{\frac{\lambda_1 \sqrt{v^2 - a}}{v}} J_4^{-\frac{\lambda_1 \sqrt{v^2 - a}}{v}}$	$R = J_1^{\lambda_1} J_2^{-\lambda_1 - 2} J_3^{\frac{(\lambda_1 + 1) \sqrt{v^2 - a}}{v} - 1} J_4^{-\frac{(\lambda_1 + 1) \sqrt{v^2 - a}}{v} - 1}$
Simple example	$\mathcal{I} = \frac{I_1}{I_2} \left( \frac{I_3}{I_4} \right)^{\frac{\sqrt{v^2 - a}}{v}}$	$\mathcal{R} = \frac{1}{I_1 I_2 I_3 I_4}$

$$(ii) \quad av(a - v^2)(a - 3v^2/4)(a + 3v^2)(a - 8v^2/9) = 0.$$

$$(ii.1) \quad v = 0 \text{ and } a \neq 0.$$

Here the two hyperbolas coalesce yielding a double hyperbola so we compute the exponential factor  $E_4$ .

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = -3i\sqrt{a} + x - y$ $J_2 = 3i\sqrt{a} + x - y$ $J_3 = -3a + x(y - x)$ $E_4 = e^{-\frac{81x}{-3a+x(y-x)}}$  $\alpha_1 = -i\sqrt{a} - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = i\sqrt{a} - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -\frac{2x}{3} - \frac{y}{3}$ $\alpha_4 = -\frac{81}{3}$	$P_1 = (-i\sqrt{a}, 2i\sqrt{a})$ $P_2 = (i\sqrt{a}, -2i\sqrt{a})$  $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$  For $a < 0$ we have  $sn_{(2)}, sn_{(2)}; N, N, S$  For $a > 0$ we have  $\odot_{(2)}, \odot_{(2)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + 2J_3 + \mathcal{L}_\infty & \text{if } a < 0 \\ J_1^C + J_2^C + 2J_3 + \mathcal{L}_\infty & \text{if } a > 0 \end{cases}$	5 5
$M_{0CS} = \begin{cases} 2P_1 + 2P_2 + P_1^\infty + P_2^\infty + P_3^\infty & \text{if } a < 0 \\ 2P_1^C + 2P_2^C + P_1^\infty + P_2^\infty + P_3^\infty & \text{if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	7
$M_{0CT} = \begin{cases} 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty & \text{if } a < 0 \\ 3P_1^\infty + 5P_2^\infty + P_3^\infty & \text{if } a > 0 \end{cases}$	15 9

where the total curve  $T$  has

- 1) only two distinct tangents at  $P_1$  (and at  $P_2$ ), but one of them is double;
- 2) only two distinct tangents at  $P_1^\infty$ , but one of them is double and
- 3) only four distinct tangents at  $P_2^\infty$ , but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^0 E_4^{-\frac{6i\sqrt{a}\lambda_1}{s_1}}$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-2} E_4^{-\frac{6i\sqrt{a}(1+\lambda_1)}{s_1}}$
Simple example	$\mathcal{I} = \frac{J_1}{J_2 E_4^{6i\sqrt{a}}}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_3^2}$

(ii.2)  $a = v^2$ .

Here the two lines coalesce yielding a double line so we compute the exponential factor  $E_4$ .

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x - y$ $J_2 = -3v^2 + 3vx - x^2 + xy$ $J_3 = -3v^2 - 3vx - x^2 + xy$ $E_4 = e^{\frac{g_0 + g_1(x-y)}{x-y}}$ $\alpha_1 = -\frac{x}{3} + \frac{y}{3}$ $\alpha_2 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_3 = v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_4 = \frac{g_0}{3}$	$P_1 = (-v, -v)$ $P_2 = (v, v)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$ $sn_{(2)}, sn_{(2)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty \text{ simple}$ $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ triple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = 2J_1 + J_2 + J_3 + \mathcal{L}_\infty$	5
$M_{0CS} = 2P_1 + 2P_2 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1^2\bar{J}_2\bar{J}_3 = 0$	7
$M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$	15

where the total curve  $T$  has

- 1) only two distinct tangents at  $P_1$  and at  $P_2$ , but one of them is double;
- 2) only two distinct tangents at  $P_1^\infty$ , but one of them is double and
- 3) only four distinct tangents at  $P_2^\infty$ , but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} J_3^{-\lambda_2} E_4^{\frac{6v\lambda_2}{80}}$	$R = J_1^{-2} J_2^{\lambda_2} J_3^{-2-\lambda_2} E_4^{\frac{6v(1+\lambda_2)}{80}}$
Simple example	$\mathcal{I} = \frac{J_2 E_4^{6v}}{J_3}$	$\mathcal{R} = \frac{1}{J_1^2 J_2 J_3}$

(ii.3)  $a = 3v^2/4$ .

Here we have, apart from two lines and two hyperbolas, a third invariant line. Then, we have five invariant algebraic curves and hence according to Jouanolou's theorem the corresponding system has a rational first integral.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -\frac{3v}{2} + x - y$ $J_2 = \frac{3v}{2} + x - y$ $J_3 = y$ $J_4 = -\frac{x^2}{3v} + \frac{xy}{3v} - \frac{3v}{4} + x$ $J_5 = \frac{x^2}{3v} - \frac{xy}{3v} + \frac{3v}{4} + x$ $\alpha_1 = \frac{1}{6}(-3v - 2x + 2y)$ $\alpha_2 = \frac{1}{6}(3v - 2x + 2y)$ $\alpha_3 = \frac{y}{3} - \frac{4x}{3}$ $\alpha_4 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = (-\frac{3v}{2}, 0)$ $P_2 = (-\frac{v}{2}, -2v)$ $P_3 = (\frac{v}{2}, 2v)$ $P_4 = (\frac{3v}{2}, 0)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$ $n, s, s, n; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_4$ double $\bar{J}_3 \cap \bar{J}_5 = P_1$ double $\bar{J}_3 \cap \mathcal{L}_\infty = P_3^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 5P_2^\infty + 2P_3^\infty$	20

where the total curve  $T$  has

- 1) only two distinct tangents at  $P_1$  (and at  $P_4$ ), but one of them is double,
- 2) only two distinct tangents at  $P_1^\infty$ , but one of them is double and
- 3) five distinct tangents at  $P_2^\infty$ .

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{2} - \frac{\lambda_2}{2}} J_4^{\frac{\lambda_2}{2}} J_5^{\frac{\lambda_1}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-1 - \frac{\lambda_1}{2} - \frac{\lambda_2}{2}} J_4^{-\frac{1}{2} + \frac{\lambda_2}{2}} J_5^{-\frac{1}{2} + \frac{\lambda_1}{2}}$
Simple example	$\mathcal{I}_1 = \frac{J_1^2 J_5}{J_3}, \quad \mathcal{I}_2 = \frac{J_2^2 J_4}{J_3}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_4 J_5}$

**Remark 3.1.** Consider  $\mathcal{F}_{(c_1, c_2)}^1 = c_1 J_1^2 J_5 - c_2 J_3 = 0$ ,  $\deg \mathcal{F}_{(c_1, c_2)}^1 = 4$ . The remarkable

values of  $\mathcal{F}_{(c_1, c_2)}^1$  are  $[1 : 9v^2/2]$  and  $[1 : 0]$  for which we have

$$\mathcal{F}_{(1, 9v^2/2)}^1 = -J_2^2 J_4, \quad \mathcal{F}_{(1, 0)}^1 = J_1^2 J_5.$$

Therefore,  $J_1, J_2, J_4, J_5$  are remarkable curves of  $\mathcal{I}_1$ ,  $[1 : 9v^2/2]$  and  $[1 : 0]$  are the only two critical remarkable values of  $\mathcal{I}_1$  and  $J_1, J_2$  are critical remarkable curves of  $\mathcal{I}_1$ . The singular points are  $P_1, P_3$  for  $\mathcal{F}_{(1, 9v^2/2)}^1$  and  $P_2, P_4$  for  $\mathcal{F}_{(1, 0)}^1$ .

Considering the first integral  $\mathcal{I}_2$  with its associated curve  $\mathcal{F}_{(c_1, c_2)}^2 = c_1 J_2^2 J_4 - c_2 J_3$  we have the same remarkable values  $[1 : 9v^2/2]$  and  $[1 : 0]$  and the same remarkable curves  $J_1, J_2, J_4, J_5$ . However, the singular points are  $P_1, P_3$  for  $\mathcal{F}_{(1, 0)}^2$  and  $P_2, P_4$  for  $\mathcal{F}_{(1, 9v^2/2)}^2$ .

(ii.4)  $a = -3v^2$ .

Here we have, apart from two lines and two hyperbolas, a third invariant hyperbola. Then, we have five invariant algebraic curves and hence according to Jouanolou's theorem the corresponding system has a rational first integral.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -6v + x - y$ $J_2 = 6v + x - y$ $J_3 = 9v^2 + xy$ $J_4 = 9v^2 + 3vx - x^2 + xy$ $J_5 = 9v^2 - 3vx - x^2 + xy$  $\alpha_1 = -2v - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = 2v - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -\frac{5x}{3} - \frac{y}{3}$ $\alpha_4 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = (-3v, 3v)$ $P_2 = (-v, 5v)$ $P_3 = (v, -5v)$ $P_4 = (3v, -3v)$  $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$  $n, s, s, n; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ double $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ double $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_1^\infty \text{ triple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ double} \\ P_2^\infty \text{ double} \end{cases}$ $\bar{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$



Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	9
$M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 4P_1^\infty + 5P_2^\infty + 2P_3^\infty$	21

where the total curve  $T$  has

- 1) only two distinct tangents at  $P_1$  (and at  $P_4$ ), but one of them is double,
- 2) only two distinct tangents at  $P_1^\infty$ , but one of them is triple,
- 3) only four tangents at  $P_2^\infty$ , but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\lambda_1-\lambda_2} J_4^{2\lambda_2} J_5^{2\lambda_1}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-2-\lambda_1-\lambda_2} J_4^{1+2\lambda_2} J_5^{1+2\lambda_1}$
Simple example	$\mathcal{I}_1 = \frac{J_1 J_5^2}{J_3}, \quad \mathcal{I}_2 = \frac{J_2 J_4^2}{J_3}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_4 J_5}$

**Remark 3.2.** Consider  $\mathcal{F}_{(c_1, c_2)}^1 = c_1 J_1 J_5^2 - c_2 J_3 = 0$ ,  $\deg \mathcal{F}_{(c_1, c_2)}^1 = 5$ . The remarkable values of  $\mathcal{F}_{(c_1, c_2)}^1$  are  $[1 : -108v^3]$  and  $[1 : 0]$  for which we have

$$\mathcal{F}_{(1, -108v^3)}^1 = J_2 J_4^2, \quad \mathcal{F}_{(1, 0)}^1 = J_1 J_5^2.$$

Therefore,  $J_1, J_2, J_4, J_5$  are remarkable curves of  $\mathcal{I}_1$ ,  $[1 : -108v^3]$  and  $[1 : 0]$  are the only two critical remarkable values of  $\mathcal{I}_1$  and  $J_4, J_5$  are critical remarkable curves of  $\mathcal{I}_1$ . The singular points are  $P_2, P_4$  for  $\mathcal{F}_{(1, -108v^3)}^1$  and  $P_1, P_3$  for  $\mathcal{F}_{(1, 0)}^1$ .

Considering the first integral  $\mathcal{I}_2$  with its associated curve  $\mathcal{F}_{(c_1, c_2)}^2 = c_1 J_2 J_4^2 - c_2 J_3$  we have the remarkable values  $[1 : 108v^3]$  and  $[1 : 0]$  and the same remarkable curves  $J_1, J_2, J_4, J_5$ . The singular point are  $P_1, P_3$  for  $\mathcal{F}_{(1, 108v^3)}^2$  and  $P_2, P_4$  for  $\mathcal{F}_{(1, 0)}^2$ .

(ii.5)  $a = 8v^2/9$ .

Here we have, apart from two lines and two hyperbolas, a third invariant hyperbola. Then, we have five invariant algebraic curves and hence according to Jouanolou's theorem the corresponding system has a rational first integral.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -v + x - y$ $J_2 = v + x - y$ $J_3 = y(x - y) - \frac{v^2}{3}$ $J_4 = -\frac{8v^2}{3} + 3vx + x(y - x)$ $J_5 = -\frac{8v^2}{3} - 3vx + x(y - x)$  $\alpha_1 = \frac{1}{3}(-v - x + y)$ $\alpha_2 = \frac{1}{3}(v - x + y)$ $\alpha_3 = \frac{2y}{3} - \frac{5x}{3}$ $\alpha_4 = -v - \frac{2x}{3} - \frac{y}{3}$ $\alpha_5 = v - \frac{2x}{3} - \frac{y}{3}$	$P_1 = \left(-\frac{4v}{3}, -\frac{v}{3}\right)$ $P_2 = \left(-\frac{2v}{3}, -\frac{5v}{3}\right)$ $P_3 = \left(\frac{2v}{3}, \frac{5v}{3}\right)$ $P_4 = \left(\frac{4v}{3}, \frac{v}{3}\right)$  $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$  $n, s, s, n; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_3 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_4 \text{ triple} \end{cases}$ $\bar{J}_3 \cap \bar{J}_5 = \begin{cases} P_2^\infty \text{ simple} \\ P_1 \text{ triple} \end{cases}$ $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_2^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1^\infty \text{ triple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$	6
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	9
$M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 6P_2^\infty + 2P_3^\infty$	21

where the total curve  $T$  has

- 1) only two distinct tangents at  $P_1$  (and at  $P_4$ ), but one of them is double,
- 2) only two distinct tangents at  $P_1^\infty$ , but one of them is double and
- 3) six distinct tangents at  $P_2^\infty$ .

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{3} - \frac{\lambda_2}{3}} J_4^{\frac{\lambda_2}{3}} J_5^{\frac{\lambda_1}{3}}$	$R = J_1^{\lambda_1} J_2^{\lambda_2} J_3^{-\frac{\lambda_1}{3} - \frac{\lambda_2}{3} - \frac{2}{3}} J_4^{\frac{\lambda_2}{3} - \frac{2}{3}} J_5^{\frac{\lambda_1}{3} - \frac{2}{3}}$
Simple example	$\mathcal{I}_1 = \frac{J_1^3 J_5}{J_3}, \quad \mathcal{I}_2 = \frac{J_2^3 J_4}{J_3}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_4 J_5}$

**Remark 3.3.** Consider  $\mathcal{F}_{(c_1, c_2)}^1 = c_1 J_1^3 J_5 - c_2 J_3 = 0$ ,  $\deg \mathcal{F}_{(c_1, c_2)}^1 = 5$ . The remarkable values of  $\mathcal{F}_{(c_1, c_2)}^1$  are  $[1 : -16v^3]$  and  $[1 : 0]$  for which we have

$$\mathcal{F}_{(1, -16v^3)}^1 = J_2^3 J_4, \quad \mathcal{F}_{(1, 0)}^1 = J_1^3 J_5.$$

Therefore,  $J_1, J_2, J_4, J_5$  are remarkable curves of  $\mathcal{I}_1$ ,  $[1 : -16v^3]$  and  $[1 : 0]$  are the only two critical remarkable values of  $\mathcal{I}_1$  and  $J_1, J_2$  are critical remarkable curves of  $\mathcal{I}_1$ . The singular points are  $P_1, P_3$  for  $\mathcal{F}_{(1, -16v^3)}^1$  and  $P_2, P_4$  for  $\mathcal{F}_{(1, 0)}^1$ .

Considering the first integral  $\mathcal{I}_2$  with its associated curves  $\mathcal{F}_{(c_1, c_2)}^2 = c_1 J_2^3 J_4 - c_2 J_3$  we have the remarkable values  $[1 : 16v^3]$  and  $[1 : 0]$  and the same remarkable curves  $J_1, J_2, J_4, J_5$ . The singular points are  $P_1, P_3$  for  $\mathcal{F}_{(1, 0)}^2$  and  $P_2, P_4$  for  $\mathcal{F}_{(1, 16v^3)}^2$ .

(ii.6)  $a = 0$  and  $v \neq 0$ .

Under this condition, systems (1.10) do not belong to QSH, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.10). All the invariant lines are  $x = 0$  and  $\pm 3v - x + y = 0$  that are simple. Perturbing this system in the family (1.10) we can obtain two distinct configurations of lines and hyperbolas. By perturbing the reducible conics  $x(-3v - x + y) = 0$  and  $x(3v - x + y) = 0$  we can produce two distinct hyperbolas  $-3a - 3vx - x^2 + xy = 0$  and  $-3a + 3vx - x^2 + xy = 0$ , respectively. Furthermore, the cubic  $x(3v - x + y)(-3v - x + y) = 0$  has integrable multiplicity two.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = -3v - x + y$ $J_2 = 3v - x + y$ $J_3 = x$ $E_4 = e^{-\frac{6g_0(6v^2+x(y-x))+g_1x((x-y)^2-9v^2)}{2x(-3v+x-y)(3v+x-y)}}$ $\alpha_1 = v - \frac{x}{3} + \frac{y}{3}$ $\alpha_2 = -v - \frac{x}{3} + \frac{y}{3}$ $\alpha_3 = -\frac{x}{3} - \frac{2y}{3}$ $\alpha_4 = g_0$	$P_1 = (0, -3v)$ $P_2 = (2v, -v)$ $P_3 = (-2v, v)$ $P_4 = (0, 3v)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$  For $v \neq 0$ we have  $s, n, n, s; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ if $v \neq 0$	4
$M_{0CS} = P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty$ if $v \neq 0$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0.$	4
$M_{0CT} = 2P_1 + P_2 + P_3 + 2P_4 + 2P_1^\infty + 3P_2^\infty + P_3^\infty$ if $v \neq 0$	12

where the total curve  $T$  has three distinct tangents at  $P_2^\infty$ .

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^0 E_4^{-\frac{2\lambda_1 v}{s_0}}$	$R = J_1^{\lambda_1} J_2^{-4-\lambda_1} J_3^{-2} E_4^{-\frac{2v(\lambda_1+2)}{s_0}}$
Simple example	$\mathcal{I} = \frac{J_1}{J_2 E_4^{2v}}$	$\mathcal{R} = \frac{1}{J_1^2 J_2^2 J_3}$

(ii.7)  $a = v = 0.$

Under this condition, systems (1.10) do not belong to **QSH**, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.10). Here we have a single system which has a rational first integral that foliates the plane into quartic curves. All the invariant affine lines are  $x = 0$ ,  $y = 0$  that are simple and  $x - y = 0$  that is double. The lines  $x = 0$  and  $x - y = 0$  are remarkable curves. Perturbing this system in the full family (1.10) we can obtain up to ten distinct configurations of lines and hyperbolas. By perturbing the reducible conic  $x(x - y) = 0$  we can produce 2 distinct hyperbolas  $-3a + 3vx - x^2 + xy = 0$  and  $-3a - 3vx - x^2 + xy = 0$ . Perturbing the reducible conic  $y(x - y) = 0$  we can produce a third hyperbola  $y(x - y) - \frac{v^2}{3} = 0$  and by perturbing  $xy = 0$  we can produce the hyperbola  $9v^2 + xy = 0$ . We get a double hyperbola  $-3a + x(y - x) = 0$  by perturbing the double reducible conic  $x^2(x - y)^2 = 0$ .

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $E_4 = e^{\frac{s_0 + s_1(x-y)}{x-y}}$ $\alpha_1 = \frac{y}{3} - \frac{4x}{3}$ $\alpha_2 = -\frac{x}{3} - \frac{2y}{3}$ $\alpha_3 = \frac{y}{3} - \frac{x}{3}$ $\alpha_4 = \frac{s_0}{3}$	$P_1 = (0, 0)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$ $hpphpp_{(4)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = P_2^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	10

where the total curve  $T$  has

- 1) only three distinct tangents at  $P_1$ , but one of them is double;
- 2) only two distinct tangentes at  $P_2^\infty$ , but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{-3\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-4-3\lambda_1} E_4^0$
Simple example	$\mathcal{I}_1 = \frac{J_1}{J_2 J_3^3}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_3}$

**Remark 3.4.** Consider  $\mathcal{F}_{(c_1, c_2)}^1 = c_1 J_1 - c_2 J_2 J_3^3 = 0$ ,  $\deg \mathcal{F}_{(c_1, c_2)}^1 = 4$ . The remarkable value of  $\mathcal{F}_{(c_1, c_2)}^1$  is  $[0 : 1]$  for which we have

$$\mathcal{F}_{(0,1)}^1 = -J_2 J_3^3.$$

Therefore,  $J_2, J_3$  are remarkable curves of  $\mathcal{I}_1$ ,  $[0 : 1]$  is the only critical remarkable values of  $\mathcal{I}_1$  and  $J_3$  is critical remarkable curve of  $\mathcal{I}_1$ . The singular point is  $P_1$  for  $\mathcal{F}_{(0,1)}^1$ .

We sum up the topological, dynamical and algebraic geometric features of family (1.10) and we also confront our results with previous results in the literature in the following proposition. We show that there exists two more configurations of invariant hyperbolas and lines than in [18], there are four more phase portraits than the ones appearing in [17] and there is one more phase portrait than the ones appearing in [10].

**Proposition 3.5.**

- (a) For the family (1.10) we have nine distinct configurations  $C_1^{(1.10)} - C_9^{(1.10)}$  of invariant hyperbolas and lines (see Figure 3.1 for the complete bifurcation diagram of configurations of such family). The bifurcation set of configurations in the full parameter space is  $av(a - v^2)(a + 3v^2)(a - 3v^2/4)(a - 8v^2/9) = 0$ . On  $v(a - v^2) = 0$  one of the algebraic solutions is double. On  $(a + 3v^2)(a - 3v^2/4)(a - 8v^2/8) = 0$  we have an additional line or an additional hyperbola. The configurations  $C_8^{(1.10)}$  and  $C_9^{(1.10)}$  are not equivalent with anyone of the configurations for systems (3.97) (here family (1.10)) in [18].
- (b) All systems in family (1.10) have an inverse integrating factor which is polynomial. All systems in family (1.10) satisfying the genericity condition (3.1) have a Darboux first integral. If  $a = v^2$  then the systems have a double invariant line. If  $v = 0$  then the systems have a double invariant hyperbola. In both cases, the systems have a generalized Darboux first integral. In

all the following three cases, we have a rational first integral. If  $a = 3v^2/4$  then the systems have an additional invariant line and the plane is foliated into quartic algebraic curves. If  $a = -3v^2$  the plane is foliated by quintic algebraic curves. If  $a = 8v^2/9$  then the systems have an additional invariant hyperbola and the plane is foliated in quintic algebraic curves. The remarkable curves are  $J_1, J_2, J_4, J_5$  for these three algebraically integrable cases of family (1.10) for each case correspondingly.

- (c) For the family (1.10) we have five topologically distinct phase portraits  $P_1^{(1.10)} - P_5^{(1.10)}$ . The topological bifurcation diagram of family (1.10) is done in Figure 3.2. The bifurcation set of singularities is the half line  $v = 0$  and  $a < 0$ , the parabola  $a = v^2$  and the line  $a = 0$ . The phase portraits  $P_1^{(1.10)}, P_3^{(1.10)}, P_4^{(1.10)}$  and  $P_5^{(1.10)}$  are not topologically equivalent with anyone of the phase portraits in [17]. The phase portrait  $P_1^{(1.10)}$  is not topologically equivalent with anyone of the phase portraits in [10].

**Proof.** (a) We have the following types of divisors and zero-cycles of the total invariant curve  $T$  for the configurations of family (1.10) :

Configurations	Divisors and zero-cycles of the total inv. curve $T$
$C_1^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_2^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_3^{(1.10)}$	$ICD = J_1^C + J_2^C + J_3 + J_4 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_4^{(1.10)}$	$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_5^{(1.10)}$	$ICD = J_1^C + J_2^C + 2J_3 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_6^{(1.10)}$	$ICD = 2J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 3P_1^\infty + 5P_2^\infty + P_3^\infty$
$C_7^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 5P_2^\infty + 2P_3^\infty$
$C_8^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 4P_1^\infty + 5P_2^\infty + 2P_3^\infty$
$C_9^{(1.10)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 2P_2 + 2P_3 + 3P_4 + 3P_1^\infty + 6P_2^\infty + 2P_3^\infty$

Although  $C_1^{(1.10)}$  and  $C_2^{(1.10)}$  admit the same type of divisors and zero-cycles we can see they are different because in  $C_1^{(1.10)}$  each branch of the hyperbolas intersects one line while  $C_2^{(1.10)}$  have two branches intersecting both lines and two branches intersecting no line. Therefore, the configurations  $C_1^{(1.10)}$  up to  $C_9^{(1.10)}$  are all distinct. For the limit cases of family (1.10) we have the following configurations:



Configurations	Divisors and zero-cycles of the total inv. curve $T$
$c_1$	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + P_2 + P_3 + 2P_4 + 2P_1^\infty + 3P_2^\infty + P_3^\infty$
$c_2$	$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$

The other statements in (a) follows from the study done previously.

(b) This is shown in the previously exhibited tables. The computations for the remarkable curves were done in Remarks 3.1, 3.2 and 3.3 .

(c) We have that:

Phase Portraits	Sing. at $\infty$	Finite sing.	Separatrix connections
$P_1^{(1.10)}$	$(N, N, S)$	$(n, s, s, n)$	$2SC_f^f \ 8SC_f^\infty \ 0SC_\infty^\infty$
$P_2^{(1.10)}$	$(N, N, S)$	$(n, s, s, n)$	$4SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$
$P_3^{(1.10)}$	$(N, N, S)$	$(\odot, \odot, \odot, \odot)$ $(\odot_{(2)}, \odot_{(2)})$	$0SC_f^f \ 0SC_f^\infty \ 2SC_\infty^\infty$
$P_4^{(1.10)}$	$(N, N, S)$	$(sn_{(2)}, sn_{(2)})$	$1SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$
$P_5^{(1.10)}$	$(N, N, S)$	$(sn_{(2)}, sn_{(2)})$	$0SC_f^f \ 8SC_f^\infty \ 0SC_\infty^\infty$

Therefore, we have five distinct phase portraits for systems (1.10). For the limit cases of family (1.10) we have the following phase portraits:

Phase Portraits	Sing. at $\infty$	Finite sing.	Separatrix connections
$p_1$	$(N, N, S)$	$(n, s, s, n)$	$3SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$
$p_2$	$(N, N, S)$	$hpphpp_{(4)}$	$0SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$

On the table below we list the phase portraits of Llibre–Yu in [17] that satisfy the following conditions: the phase portraits admit 3 singular points at infinity with the type  $(N, N, S)$ , and it has either 0, 1, 2 or 4 real singular points in the finite region.

Phase Portraits	Sing. at $\infty$	Real finite sing.	Separatrix connections
$R_{01}, \Omega_6$	$(N, S, N)$	$\emptyset$	$0SC_f^f \ 0SC_f^\infty \ 1SC_\infty^\infty$
$L_{11}, L_{12}$	$(N, S, N)$	$cp$	$0SC_f^f \ 2SC_f^\infty \ 1SC_\infty^\infty$
$P_2$	$(N, S, N)$	$pphpph$	$0SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$
$L_{31}, L_{32}$	$(N, S, N)$	$(s, es)$	$2SC_f^f \ 6SC_f^\infty \ 2SC_\infty^\infty$
$L_{33}$	$(N, S, N)$	$(c, es)$	$1SC_f^f \ 4SC_f^\infty \ 1SC_\infty^\infty$
$R_1, R_2$	$(N, S, N)$	$(s, c)$	$1SC_f^f \ 2SC_f^\infty \ 1SC_\infty^\infty$
$R_3, \Omega_5$	$(N, S, N)$	$(c, c)$	$2SC_f^f \ 0SC_f^\infty \ 3SC_\infty^\infty$
$R_5$	$(N, S, N)$	$(s, n, n, s)$	$4SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$
$R_8, \Omega_1$	$(N, S, N)$	$(s, n, n, s)$	$4SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$

Therefore, we can see from the two above tables that the phase portraits  $P_1^{(1.10)}$ ,  $P_3^{(1.10)}$ ,  $P_4^{(1.10)}$  and  $P_5^{(1.10)}$  are not topologically equivalent with anyone of the phase portraits in [17]. They are however phase portraits of systems possessing and invariant hyperbola and an invariant line.

On the table below we list the phase portraits of Coll–Ferragut–Llibre in [17] that admit 3 singular points at infinity with the type  $(N, N, S)$ , and it has either 0, 1, 2 or 4 real singular points in the finite region:

Phase Portrait	Sing. at $\infty$	Real finite sing.	Separatrix connections
(20)	$(N, N, S)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_f^\infty$
(42)	$(N, N, S)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $1SC_f^\infty$
(59)	$(N, N, S)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $2SC_f^\infty$
(21)	$(N, N, S)$	$cp$	$0SC_f^f$ $2SC_f^\infty$ $2SC_f^\infty$
(43)	$(N, S, N)$	$cp$	$0SC_f^f$ $2SC_f^\infty$ $1SC_f^\infty$
(57)	$(N, N, S)$	$pphpph$	$0SC_f^f$ $6SC_f^\infty$ $0SC_f^\infty$
(22)	$(N, N, S)$	$(s, c)$	$1SC_f^f$ $2SC_f^\infty$ $2SC_f^\infty$
(23)	$(N, N, S)$	$(s, c)$	$0SC_f^f$ $4SC_f^\infty$ $1SC_f^\infty$
(28)	$(N, N, S)$	$(s, c)$	$0SC_f^f$ $4SC_f^\infty$ $0SC_f^\infty$
(44)	$(N, N, S)$	$(s, c)$	$1SC_f^f$ $2SC_f^\infty$ $1SC_f^\infty$
(45)	$(N, N, S)$	$(es, s)$	$2SC_f^f$ $4SC_f^\infty$ $0SC_f^\infty$
(58)	$(N, N, S)$	$(sn, sn)$	$1SC_f^f$ $6SC_f^\infty$ $0SC_f^\infty$
(77)	$(N, N, S)$	$(sn, sn)$	$0SC_f^f$ $8SC_f^\infty$ $0SC_f^\infty$
(102)	$(N, N, S)$	$(s, es)$	$2SC_f^f$ $6SC_f^\infty$ $0SC_f^\infty$
(35)	$(N, N, S)$	$(n, s, s, n)$	$4SC_f^f$ $6SC_f^\infty$ $0SC_f^\infty$
(115)	$(N, N, S)$	$(n, s, s, n)$	$3SC_f^f$ $6SC_f^\infty$ $0SC_f^\infty$

Therefore, the phase portrait  $P_1^{(1.10)}$  is not topologically equivalent with anyone of the phase portraits in [10]. It is however a phase portrait of a systems possessing a polynomial inverse integrating factor.  $\square$

### 3.1 The solution of the Poincaré problem for the family (1.10)

We can recognize when a system in this family has a rational first integral. The following is the answer to Poincaré’s problem for the family (1.10):

#### Theorem 3.6.

- i) A necessary and sufficient condition for a system in family (1.10) to have a rational first integral is that  $v^2 - a > 0$  and that  $(a, v)$  be situated on a parabola of the form  $a = (1 - r^2)v^2$  with  $r \in \mathbb{Q}$ .
- ii) The set of all points  $(a, v)$ ’s satisfying these two conditions is dense in the set  $v^2 - a > 0$  with  $v \neq 0$ .

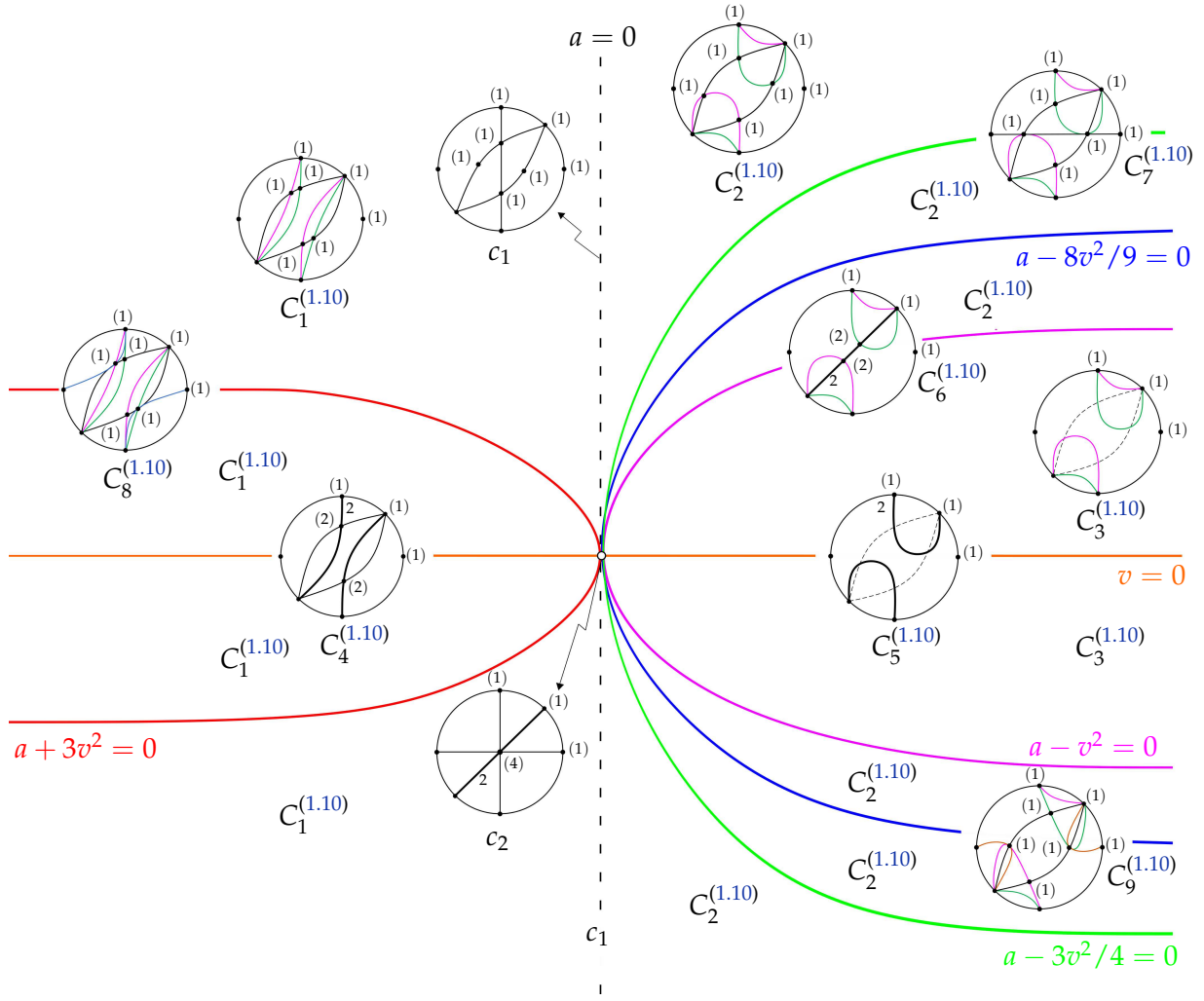


Figure 3.1: Bifurcation diagram of configurations for family (1.10): In this figure on the dashed line  $a = 0$  both hyperbolas become reducible into two lines one of them  $x = 0$ . On the bifurcation curves we either have an additional line or additional hyperbola or coalescing lines or coalescing hyperbolas or real lines becoming complex. The dashed lines represent complex lines.

*Proof.* i) We first prove that the condition is necessary. So assume that we have a system of parameters  $(a, v)$  that has a rational first integral. Assume now that  $(a, v)$  is in the generic situations  $v(a - v^2)(a + 3v^2)(a - 8v^2/9)(a - 3v^2/4) \neq 0$ . Any first integral of the system is then of the following general form:

$$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{\frac{\lambda_1 \sqrt{v^2 - a}}{v}} J_4^{-\frac{\lambda_1 \sqrt{v^2 - a}}{v}}.$$

This is a rational first integral if and only if  $\lambda_1 \in \mathbb{Z}$  and  $\frac{\lambda_1 \sqrt{v^2 - a}}{v} \in \mathbb{Z}$  in which case we must have that  $r = \sqrt{v^2 - a}/v$  must be a rational number. In view of our generic hypothesis  $r \neq 0$ . Since  $r = \sqrt{v^2 - a}/v$  is rational we have  $v^2 - a \geq 0$  and by hypothesis  $v^2 - a \neq 0$ . Therefore  $v^2 - a > 0$ . We also have  $a = (1 - r^2)v^2$  and therefore the condition is necessary in this case. Consider now the case when  $v(a - v^2)(a + 3v^2)(a - 8v^2/9)(a - 3v^2/4) = 0$ . Since on  $v(a - v^2) = 0$  we cannot have a rational first integral because as we see in the tables for

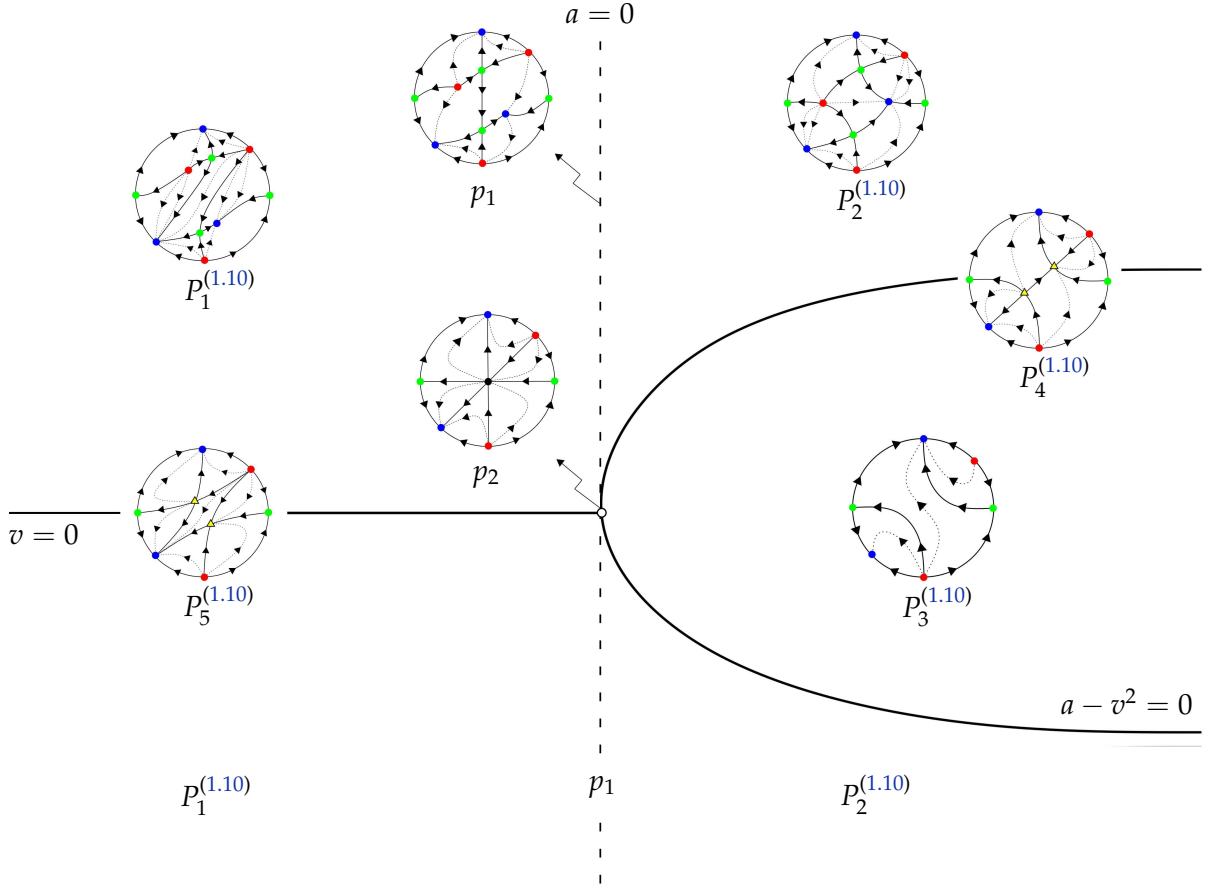


Figure 3.2: Topological bifurcation diagram for family (1.10).

these two cases, we have exponential factors in the first integrals and hence we must have  $v(a - v^2) \neq 0$ . Therefore our previous assumption is reduced to  $(a + 3v^2)(a - 8v^2/9)(a - 3v^2/4) = 0$ . Suppose first that the point  $(a, v)$  is located on the parabola  $a = -3v^2$ . Then this parabola can be written as  $a = (1 - r^2)v^2$  where  $r = 2$ . We then have  $v^2 - a = r^2v^2 = 4v^2 > 0$ . If the point  $(a, v)$  is on the parabola  $a - 8v^2/9 = 0$  then this parabola can be written as  $a = (1 - r^2)v^2$  for  $r = 1/3$ . Here again we have that  $v^2 - a = r^2v^2 = v^2/9 > 0$ . So the system situated on the parabola  $a - 8v^2/9 = 0$  satisfies  $v^2 - a > 0$  and for  $r = 1/3$  the point is located on the parabola  $a = (1 - r^2)v^2$ . So also in this case these conditions are necessary. There remains only the case when  $(a, v)$  is on the parabola  $a - 3v^2/4 = 0$ . In this case we can write this parabola as  $a = (1 - r^2)v^2$  by taking  $r = 1/2$ . Also here  $v^2 - a = r^2v^2 = v^2/4 > 0$ , i.e.  $v^2 - a > 0$ . So the necessity of the conditions is proved in this case too.

We now prove the sufficiency of the conditions. Let us assume that  $v^2 - a > 0$ ,  $v \neq 0$  and  $(a, v)$  is located on a parabola  $a = (1 - r^2)v^2$  with  $r \in \mathbb{Q}$ . Then clearly  $r \neq 0$ , otherwise  $v^2 - a = r^2v^2 = 0$  contrary to our assumption. In case  $r = 2, 1/3, 1/2$  we are on one of the three parabolas obtained from the condition  $(a + 3v^2)(a - 8v^2/9)(a - 3v^2/4) = 0$  and for these parabolas the tables give us rational first integrals. If the generic condition is satisfied, i.e.  $v(a - v^2)(a + 3v^2)(a - 8v^2/9)(a - 3v^2/4) \neq 0$ , then we know that we have the corresponding first integral indicated in the Tables for this case where the exponents for the curves  $J_i$  are  $\lambda_1$  and  $\lambda_1\sqrt{v^2 - a}/v$ . But we know by our assumption that  $(a, v)$  is located on a parabola  $a = (1 - r^2)v^2$  for some rational number  $r$ . From this equation we have that  $r^2 = (a - v^2)/v^2$ .

Hence  $r = \sqrt{v^2 - a}/v$  is rational. We may suppose  $r = m/n$  with  $m, n \in \mathbb{Z}$  and  $m, n$  coprime. Then by taking in the general expression of the first integral  $\lambda_1 = n$  and  $r = \sqrt{v^2 - a}/v$  we obtained a rational first integral in this case.

ii) Let us denote by  $\mathcal{P}_r$  the parabola corresponding to a rational number  $r$ , i.e.

$$\mathcal{P}_r := \{(a, v) \in \mathbb{R}^2 : (1 - r^2)v^2 = a\}.$$

Thus a system in the family (1.10) has a rational first integral if and only if it corresponds to a point  $(a, v)$  such that  $v^2 - a > 0$  with  $v \neq 0$  and the point is situated on a parabola  $\mathcal{P}_r$  for some rational number  $r$ . In the parameter plane  $\mathbb{R}^2$  let the  $a$ -axis be the horizontal line and the  $v$ -axis be the vertical one. The parabolas  $a = (1 - r^2)v^2$  are symmetric with respect to the  $a$ -axis. Because of this it would suffice to prove the density of points  $(a, v)$  on parabolas  $\mathcal{P}_r$  and inside  $v^2 - a > 0$  and  $v > 0$ .

Claim: The set of all points in  $A =: \cup_{r \in \mathbb{Q}} \mathcal{P}_r$  with  $v > 0$  is dense in the set  $S^+ = \{(a, v) : v^2 - a > 0, v > 0\}$ .

Take an arbitrary point  $p_0 = (a_0, v_0) \in S^+$ . So we have  $v_0^2 - a_0 > 0$  and  $v_0 > 0$ . We only need to consider  $p_0$  inside the first or second quadrant. Indeed the line  $a = 0$  is outside the parameter space of our family. So  $a_0 \neq 0$ . In view of our assumption we have that  $(v_0^2 - a_0)/v_0^2 > 0$ . So let  $r_0 = \sqrt{(v_0^2 - a_0)/v_0^2} > 0$ . Hence we have  $a_0 = (1 - r_0^2)v_0^2$ . Here  $r_0$  is not necessarily a rational number. But it can be approximated with rational numbers. So take a sequence of rational numbers  $r_n$  such that  $r_n \rightarrow r_0$ . At this point let us assume that the point  $(a_0, v_0)$  is in the second quadrant, i.e.  $a_0 < 0$ . In this case  $r_0 > 1$  and since  $r_n \rightarrow r_0$  there exists a number  $N$  such that for  $n > N$   $r_n > 1$  and hence  $r_n^2 > 1$  for all  $n > N$ . So  $\sqrt{a_0/(1 - r_n^2)} > 0$ . Denote by  $v_n = \sqrt{a_0/(1 - r_n^2)}$ . Then  $v_n \rightarrow v_0$  and hence  $(a_0, v_n) \rightarrow (a_0, v_0)$ . But each point  $(a_0, v_n)$  is located on the corresponding parabola  $a_0 = (1 - r_n^2)v_n^2$  and hence  $p_0$  is an accumulation point of points situated on such parabolas with  $r_n$  rational. Assume now that the point  $p_0$  is in the first quadrant. Then  $a_0 > 0$  and since  $(v_0^2 - a_0)/v_0^2 > 0$  we have that  $0 < r_0 = \sqrt{1 - a_0/v_0^2} < 1$  which means that there exists a natural number  $N$  such that for  $n > N$  we have  $0 < r_n < 1$  and hence  $r_n^2 < 1$  and hence we can take again  $v_n = \sqrt{a_0/(1 - r_n^2)}$ . Then clearly  $v_n \rightarrow v_0$  and we obtain a sequence of points  $(a_0, v_n)$  sitting on parabolas  $a_0 = (1 - r_n^2)v_n^2$  with  $r_n$  rational. And  $v_0^2 - a_0 = r_n^2 v_n^2 > 0$ . Since  $v_0 > 0$  then there is a natural number  $M$  such that for all  $n > M$   $v_n > 0$ .  $\square$

Considering  $r = m_1/m_2$  where  $m_1, m_2 \in \mathbb{Z}$  we can say that

$$I = \left(\frac{J_1}{J_2}\right)^{m_2} \left(\frac{J_3}{J_4}\right)^{m_1}$$

is a rational first integral of (1.10) when  $a = (1 - (m_1/m_2)^2)v^2$ . Consider

$$\mathcal{F}_{(c_1, c_2)} = c_1 J_1^{m_2} J_3^{m_1} - c_2 J_2^{m_2} J_4^{m_1} = 0.$$

We have that  $[1 : 0]$  and  $[0 : 1]$  are remarkable values for  $\mathcal{I}$ , since

$$\mathcal{F}_{(1,0)} = J_1^{m_2} J_3^{m_1}, \quad \mathcal{F}_{(0,1)} = -J_2^{m_2} J_4^{m_1}.$$

The case  $m_1 = m_2 = 1$  is when  $a = 0$  and this case was done previously. Suppose  $m_1 \neq 1$  or  $m_2 \neq 1$ . If  $m_1 > 1$  then  $[1 : 0]$  and  $[0 : 1]$  are the only two critical remarkable values for  $\mathcal{I}$  and  $J_3, J_4$  are critical remarkable curves. If we also have  $m_2 > 1$  then  $J_1, J_2$  also are critical remarkable curves.

There are some additional remarkable curves when  $a = (1 - (m_1/m_2)^2)v^2$  for especial values of  $m_1$  and  $m_2$ , see examples in the Appendix. We could find among these examples curves of degree 5, 6, 7, 8, 10, 12 etc.

## 4 Geometric analysis of family (1.11)

Consider the family

$$(1.11) \quad \begin{cases} \dot{x} = -\frac{x^2}{2} - \frac{xy}{2} \\ \dot{y} = b - \frac{3xy}{2} + \frac{y^2}{2}, \quad \text{where } b \neq 0. \end{cases}$$

This is a one parameter family depending on  $b \in \mathbb{R} \setminus \{0\}$ . Every system in the family (1.11) is endowed with five invariant algebraic curves: three lines  $J_1, J_2, J_3$  and two hyperbolas  $J_4, J_5$  with respective cofactors  $\alpha_i, 1 \leq i \leq 5$  where

$$\begin{aligned} J_1 &= -i\sqrt{2}\sqrt{b} - x + y, & \alpha_1 &= \frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}, \\ J_2 &= i\sqrt{2}\sqrt{b} - x + y, & \alpha_2 &= -\frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}, \\ J_3 &= x, & \alpha_3 &= -\frac{x}{2} - \frac{y}{2}, \\ J_4 &= x(y-x) - b, & \alpha_4 &= -x, \\ J_5 &= xy - \frac{b}{2}, & \alpha_5 &= -2x. \end{aligned}$$

Straightforward calculations lead us to the tables listed below. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using for lines the 1st and for hyperbola the 2nd extactic polynomial, respectively.

(i)  $b \neq 0$ .

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = -i\sqrt{2}\sqrt{b} - x + y$ $J_2 = i\sqrt{2}\sqrt{b} - x + y$ $J_3 = x$ $J_4 = x(y-x) - b$ $J_5 = xy - \frac{b}{2}$  $\alpha_1 = \frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}$ $\alpha_2 = -\frac{i\sqrt{b}}{\sqrt{2}} - \frac{x}{2} + \frac{y}{2}$ $\alpha_3 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_4 = -x$ $\alpha_5 = -2x$	$P_1 = \left( \frac{i\sqrt{b}}{\sqrt{2}}, -\frac{i\sqrt{b}}{\sqrt{2}} \right)$ $P_2 = \left( -\frac{i\sqrt{b}}{\sqrt{2}}, \frac{i\sqrt{b}}{\sqrt{2}} \right)$ $P_3 = \left( 0, -i\sqrt{2}\sqrt{b} \right)$ $P_4 = \left( 0, i\sqrt{2}\sqrt{b} \right)$  $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$  For $b < 0$ we have  $n, n, s, s; N, N, S$  For $b > 0$ we have  $\odot, \odot, \odot, \odot; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = P_4$ simple $\bar{J}_1 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_1 \cap \bar{J}_5 = P_2$ double $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_3$ simple $\bar{J}_2 \cap \bar{J}_4 = \begin{cases} P_2^\infty \text{ simple} \\ P_2 \text{ simple} \end{cases}$ $\bar{J}_2 \cap \bar{J}_5 = P_1$ double $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \bar{J}_4 = P_1^\infty$ double $\bar{J}_3 \cap \bar{J}_5 = P_1^\infty$ double $\bar{J}_3 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_4 \cap \bar{J}_5 = \begin{cases} P_1, P_2 \text{ simple} \\ P_1^\infty \text{ double} \end{cases}$ $\bar{J}_4 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_5 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_3^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty & \text{if } b < 0 \\ J_1^C + J_2^C + J_3 + J_4 + J_5 + \mathcal{L}_\infty & \text{if } b > 0 \end{cases}$	6 6
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + P_1^\infty + P_2^\infty + P_3^\infty & \text{if } b < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + P_1^\infty + P_2^\infty + P_3^\infty & \text{if } b > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3\bar{J}_4\bar{J}_5 = 0$	8
$M_{0CT} = \begin{cases} 3P_1 + 3P_2 + 2P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty & \text{if } b < 0 \\ 4P_1^\infty + 4P_2^\infty + 2P_3^\infty & \text{if } b > 0 \end{cases}$	20 10

where the total curve  $T$  has

- 1) only two distinct tangents at  $P_1$  (and at  $P_2$ ), but one of them is double,
- 2) only three distinct tangents at  $P_1^\infty$ , but one of them is double and
- 3) four distinct tangents at  $P_2^\infty$ .

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{2\lambda_1} J_4^{\lambda_4} J_5^{-\lambda_1 - \frac{\lambda_4}{2}}$	$R = J_1^{\lambda_1} J_2^{\lambda_1} J_3^{1+2\lambda_1} J_4^{\lambda_4} J_5^{-\lambda_1 - \frac{\lambda_4}{2} - \frac{3}{2}}$
Simple example	$\mathcal{I}_1 = \frac{J_4^2}{J_5}, \quad \mathcal{I}_2 = \frac{J_1 J_2 J_3^2}{J_5}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_3 J_4}$

**Remark 4.1.**

- Consider  $\mathcal{F}_{(c_1, c_2)}^1 = c_1 J_4^2 - c_2 J_5 = 0$ ,  $\deg \mathcal{F}_{(c_1, c_2)}^1 = 4$ . The remarkable values of  $\mathcal{F}_{(c_1, c_2)}^1$  are  $[1 : -2b]$  and  $[1 : 0]$  for which we have

$$\mathcal{F}_{(1, -2b)}^1 = J_1 J_2 J_3^2, \quad \mathcal{F}_{(1, 0)}^1 = J_4^2.$$

Therefore,  $J_1, J_2, J_3, J_4$  are remarkable curves of  $\mathcal{I}_1$ ,  $[1 : -2b]$  and  $[1 : 0]$  are the only two critical remarkable values of  $\mathcal{I}_1$  and  $J_3, J_4$  are critical remarkable curves of  $\mathcal{I}_1$ . The singular points are  $P_3, P_4$  for  $\mathcal{F}_{(1, -2b)}^1$  and  $P_1, P_2$  for  $\mathcal{F}_{(1, 0)}^1$ .

- Consider  $\mathcal{F}_{(c_1, c_2)}^2 = c_1 J_1 J_2 J_3^2 - c_2 J_5 = 0$ ,  $\deg \mathcal{F}_{(c_1, c_2)}^2 = 4$ . The remarkable values of  $\mathcal{F}_{(c_1, c_2)}^2$  are  $[1 : 2b]$  and  $[1 : 0]$  for which we have

$$\mathcal{F}_{(1, 2b)}^2 = J_4^2, \quad \mathcal{F}_{(1, 0)}^2 = J_1 J_2 J_3^2.$$

Therefore,  $J_1, J_2, J_3, J_4$  are remarkable curves of  $\mathcal{I}_2$ ,  $[1 : 2b]$  and  $[1 : 0]$  are the only two critical remarkable values of  $\mathcal{I}_2$  and  $J_3, J_4$  are critical remarkable curves of  $\mathcal{I}_2$ . The singular points are  $P_1, P_2$  for  $\mathcal{F}_{(1, 2b)}^2$  and  $P_3, P_4$  for  $\mathcal{F}_{(1, 0)}^2$ .

(ii)  $b = 0$ .

Under this condition, the system (1.11) does not belong to **QSH**, but we study it seeking a complete understanding of the bifurcation diagram of the system in the full family



(1.11). Here we have a single system which has a rational first integral that foliates the plane into cubic curves. All the affine invariant lines are  $x = 0$ ,  $y = 0$  that are simple and  $x - y = 0$  that is double. The lines  $x = 0$  and  $x - y = 0$  are remarkable curves. Perturbing this system in the family (1.11) we can obtain two distinct configurations of lines and hyperbolas. By perturbing the reducible conics  $x(x - y) = 0$  and  $xy = 0$  we obtain the hyperbolas  $x(y - x) - b = 0$  and  $xy - \frac{b}{2} = 0$ , respectively.

Inv.curves/exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $J_3 = x - y$ $E_4 = e^{\frac{g_0 + g_1(x-y)}{x-y}}$ $\alpha_1 = \frac{y}{2} - \frac{3x}{2}$ $\alpha_2 = -\frac{x}{2} - \frac{y}{2}$ $\alpha_3 = \frac{y}{2} - \frac{x}{2}$ $\alpha_4 = \frac{g_0}{2}$	$P_1 = (0, 0)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 1 : 0]$ $P_3^\infty = [1 : 0 : 0]$ $hpphpp_{(4)}; N, N, S$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \bar{J}_3 = P_1$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_3^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = P_1$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = P_2^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$	5
$M_{0CS} = 4P_1 + P_1^\infty + P_2^\infty + P_3^\infty$	7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3^2 = 0.$	5
$M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$	11

where the total curve  $T$  has

- 1) only three distinct tangents at  $P_1$ , but one of them is double;
- 2) only two distinct tangents at  $P_2^\infty$ , but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\lambda_1} J_3^{-2\lambda_1} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2-\lambda_1} J_3^{-3-2\lambda_1} E_4^0$
Simple example	$\mathcal{I}_1 = \frac{J_1}{J_2 J_3^2}, \quad \mathcal{I}_2 = \frac{J_2 J_3^2}{J_1}$	$\mathcal{R} = \frac{1}{J_1 J_2 J_3}$

**Remark 4.2.** Consider  $\mathcal{F}_{(c_1, c_2)}^1 = c_1 J_1 - c_2 J_2 J_3^2 = 0$ ,  $\deg \mathcal{F}_{(c_1, c_2)}^1 = 3$ . The remarkable value of  $\mathcal{F}_{(c_1, c_2)}^1$  is  $[0 : 1]$  for which we have

$$\mathcal{F}_{(0,1)}^1 = -J_2 J_3^2.$$

Therefore,  $J_2, J_3$  are remarkable curves of  $\mathcal{I}_1$ ,  $[1 : 0]$  is the only critical remarkable values of  $\mathcal{I}_1$  and  $J_3$  is critical remarkable curve of  $\mathcal{I}_1$ . The singular point is  $P_1$  for  $\mathcal{F}_{(0,1)}^1$ . Considering the first integral  $\mathcal{I}_2$  with its associated curves  $\mathcal{F}_{(c_1, c_2)}^2 = c_1 J_2 J_3^2 - c_2 J_1$  we have the



remarkable value  $[1 : 0]$  and the same remarkable curves  $J_2, J_3$ . The singular point is  $P_1$  for  $\mathcal{F}_{(1,0)}^2$ .

We sum up the topological, dynamical and algebraic geometric features of family (1.11) and we also confront our results with previous results in the literature in the following proposition. We show that there are two more phase portraits than the ones appearing in [17] and there is one more phase portrait than the ones appearing in [10].

**Proposition 4.3.**

- (a) For the family (1.11) we have two distinct configurations  $C_1^{(1.11)}$  and  $C_2^{(1.11)}$  of invariant hyperbolas and lines (see Figure 4.1 for the complete bifurcation diagram of configurations of such family). The bifurcation set in the full parameter space contains only the point  $b = 0$ .
- (b) All systems in family (1.11) have an inverse integrating factor which is polynomial. All systems in family (1.11) have a rational first integral and the plane is foliated into quartic algebraic curves. The remarkable curves are  $J_1, J_2, J_3, J_4$  for family (1.11).
- (c) For the family (1.11) we have two topologically distinct phase portraits  $P_1^{(1.11)}$  and  $P_2^{(1.11)}$ . The topological bifurcation diagram in the full parameter space is done in Figure 4.2. The bifurcation set of singularities is the point  $b = 0$ . The phase portraits  $P_1^{(1.11)}$  and  $P_2^{(1.11)}$  are not topologically equivalent with anyone of the phase portraits in [17].

*Proof.*

- (a) We have the following type of divisors and zero-cycles of the total invariant curve  $T$  for the configurations of family (1.11):

Configurations	Divisors and zero-cycles of the total inv. curve $T$
$C_1^{(1.11)}$	$ICD = J_1 + J_2 + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 3P_2 + 2P_3 + 2P_4 + 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$
$C_2^{(1.11)}$	$ICD = J_1^C + J_2^C + J_3 + J_4 + J_5 + \mathcal{L}_\infty$ $M_{0CT} = 4P_1^\infty + 4P_2^\infty + 2P_3^\infty$

Therefore, the configurations  $C_1^{(1.11)}$  and  $C_2^{(1.11)}$  are distinct. For the limit case of family (1.11) we have the following configuration:

Configuration	Divisors and zero-cycles of the total inv. curve $T$
$c_2$	$ICD = J_1 + J_2 + 2J_3 + \mathcal{L}_\infty$ $M_{0CT} = 4P_1 + 2P_1^\infty + 3P_2^\infty + 2P_3^\infty$

- (b) It follows directly from Jouanolou’s theorem that we always have a rational first integral for family (1.11) since we have five invariant algebraic curves. The computations for the remarkable curves were done in Remark 4.1.

(c) We have that:

Phase Portraits	Sing. at $\infty$	Finite sing.	Separatrix connections
$P_1^{(1.11)}$	$(N, N, S)$	$(n, s, s, n)$	$3SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$P_2^{(1.11)}$	$(N, N, S)$	$(\odot, \odot, \odot, \odot)$	$0SC_f^f$ $0SC_f^\infty$ $2SC_\infty^\infty$

Therefore, we have two distinct phase portraits for systems (1.11). For the limit case of family (1.11) we have the following phase portrait:

Phase Portrait	Sing. at $\infty$	Finite sing.	Separatrix connections
$p_2$	$(N, N, S)$	$hpphpp_{(4)}$	$0SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Note that  $P_1^{(1.11)} \cong_{top} p_1$  and  $P_2^{(1.11)} \cong_{top} P_3^{(1.10)}$ . We saw in the study of the previous family that  $P_3^{(1.10)}$  is not topologically equivalent with anyone if the phase portraits in [17].

On the table below we list the phase portraits of Llibre–Yu in [17] that admit 3 singular points at infinity with the type  $(N, N, S)$  and with 4 real singular points in the finite region.

Phase Portrait	Sing. at $\infty$	Real finite sing.	Separatrix connections
$R_5$	$(N, S, N)$	$(s, n, n, s)$	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$
$R_8, \Omega_1$	$(N, S, N)$	$(s, n, n, s)$	$4SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

Therefore, the phase portraits  $P_1^{(1.11)}$  is not topologically equivalent with anyone of the phase portraits in [17]. It is however a phase portrait of systems possessing an invariant line and an invariant hyperbola. □

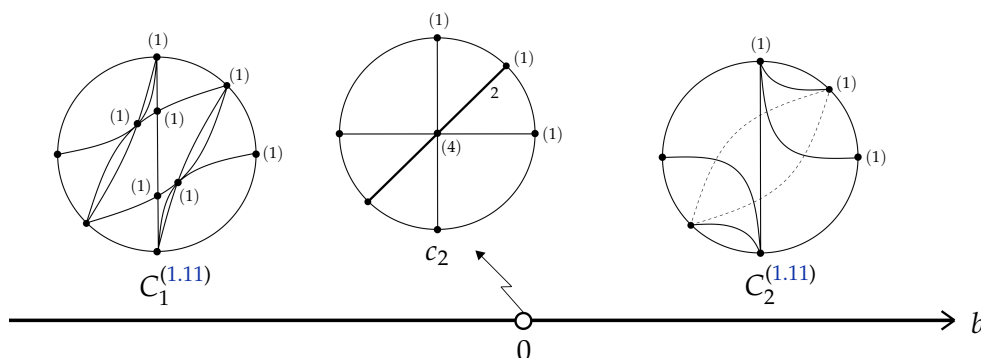


Figure 4.1: Bifurcation diagram of configurations for family (1.11). At  $b = 0$  the two hyperbolas become reducible into the lines  $x = 0$ ,  $x - y = 0$  and  $x = 0$ ,  $y = 0$ .

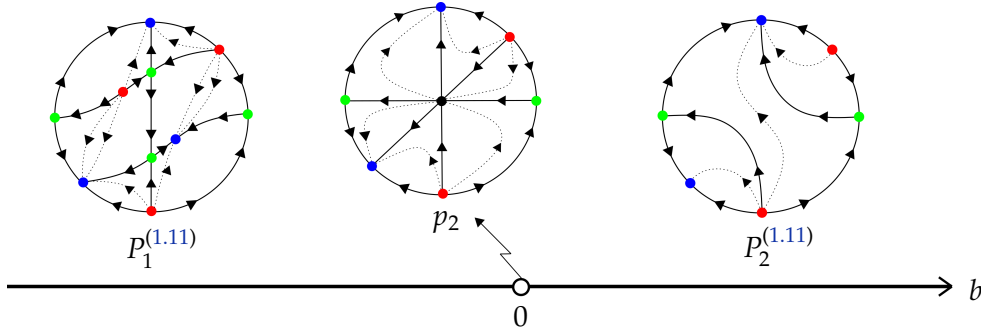


Figure 4.2: Topological bifurcation diagram for family (1.11). The only bifurcation point is at  $b = 0$  where all 4 singularities (real on the left or complex on the right) coalesce with  $(0, 0)$ .

## 5 Geometric analysis of family (1.12)

Consider the family

$$(1.12) \quad \begin{cases} \dot{x} = 2a + gx^2 + xy \\ \dot{y} = a(2g - 1) + (g - 1)xy + y^2, \quad \text{where } a(g - 1) \neq 0. \end{cases}$$

This is a two parameter family depending on  $(a, g) \in \mathbb{R}^2 \setminus \{(0, 1)\}$ . Every system in the family (1.12) is endowed with at least one invariant hyperbola  $J_1$  with cofactor  $\alpha_1$  given by

$$J_1 = a + xy, \quad \alpha_1 = (-1 + 2g)x + 2y.$$

Except for a denumerable set of lines in the parameter space, i.e. except for

$$L_k : 2g - k = 0, \quad k \in \mathbb{N} = \{0, 1, 2, \dots\} \quad \text{and} \quad L : 4g - 1 = 0,$$

systems in (1.12) are not Liouvillian integrable (see [19]). It thus remains to be shown what happens on these lines and we consider here the case  $L_1$  and  $L$ .

Straightforward calculations lead us to the tables listed below. The multiplicities of each invariant straight line and invariant hyperbola appearing in the divisor ICD of invariant algebraic curves were calculated by using for lines the 1st and for hyperbola the 2nd extactic polynomial, respectively.

- (i)  $ag(g - 1)(2g - 1)(4g - 1) \neq 0$ .

In [19] it is proved that except for the denumerable set of lines  $\cup_{k \in \mathbb{N}} L_k \cup L$ ,

$$L_k = \{(a, g) \in \mathbb{R}^2 \setminus \{(0, 1)\} : 2g - k = 0\}, \quad k \in \mathbb{N},$$

$$L = \{(a, g) \in \mathbb{R}^2 \setminus \{(0, 1)\} : 4g - 1 = 0\}$$

systems (1.12) are neither Darboux nor Liouvillian integrable. We prove below that when  $(a, g) \in L_1$  systems (1.12) are generalized Darboux integrable and when  $(a, g) \in L$  systems (1.12) are Liouvillian integrable. The cases where  $(a, g) \in \cup_{k \in \mathbb{N}} L_k - L_1$  are still open. For these cases we were not able to prove the non-integrability and we also could not find other invariant algebraic curves, which we managed to search up to degree four. Although we are unable to guarantee the existence of a first integral in  $\cup_{k \in \mathbb{N}} L_k - L_1$ , it is still possible to obtain the complete topological bifurcation diagram of this family.

Inv. curves and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $\alpha_1 = (-1 + 2g)x + 2y$	$P_1 = (-2i\sqrt{a}, i(2\sqrt{ag} - \sqrt{a}))$ $P_2 = (2i\sqrt{a}, -i(2\sqrt{ag} - \sqrt{a}))$ $P_3 = \left(-\frac{i\sqrt{a}}{\sqrt{g}}, -i\sqrt{a}\sqrt{g}\right)$ $P_4 = \left(\frac{i\sqrt{a}}{\sqrt{g}}, i\sqrt{a}\sqrt{g}\right)$  $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 0 : 0]$  For $a < 0$ we have  $f, f, \odot, \odot; \binom{0}{2}SN, S$ if $g < 0$ $f, f, s, s; \binom{0}{2}SN, N$ if $0 < g < \frac{7}{32}$ $n, n, s, s; \binom{0}{2}SN, N$ if $\frac{7}{32} \leq g < \frac{1}{4}$ $s, s, n, n; \binom{0}{2}SN, N$ if $g > \frac{1}{4}$  For $a > 0$ we have  $\odot, \odot, n, n; \binom{0}{2}SN, S$ if $g < 0$ $\odot, \odot, \odot, \odot; \binom{0}{2}SN, N$ if $g > 0$	$\bar{J}_1 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + \mathcal{L}_\infty$	2
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty & \text{if } a < 0 \text{ and } g < 0 \\ P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty & \text{if } a < 0 \text{ and } g > 0 \\ P_1^C + P_2^C + P_3 + P_4 + 2P_1^\infty + P_2^\infty & \text{if } a > 0 \text{ and } g < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty & \text{if } a > 0 \text{ and } g > 0 \end{cases}$	7 7 7 7
$T = Z\bar{J}_1 = 0$	3
$M_{0CT} = \begin{cases} 2P_1^\infty + 2P_2^\infty & \text{if } ag > 0 \\ P_3 + P_4 + 2P_1^\infty + 2P_2^\infty & \text{if } ag < 0 \end{cases}$	4 6

(ii)  $ag(g-1)(2g-1)(4g-1) = 0$ .

(ii.1)  $g = 0$  and  $a \neq 0$ .

Under this condition,  $(a, g) \in L_0$  which corresponds to an open case regarding the integrability.

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = a + xy$ $\alpha_1 = -x + 2y$	$P_1 = (2i\sqrt{a}, -i\sqrt{a})$ $P_2 = (2i\sqrt{a}, i\sqrt{a})$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 0 : 0]$ $f, f; \binom{0}{2}SN, \binom{1}{2}S$ if $a < 0$ $\odot, \odot; \binom{0}{2}SN, \binom{1}{2}N$ if $a > 0$	$\bar{J}_1 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + \mathcal{L}_\infty$	2
$M_{0CS} = \begin{cases} P_1 + P_2 + 2P_1^\infty + 3P_2^\infty & \text{if } a < 0 \\ P_1^C + P_2^C + 2P_1^\infty + 3P_2^\infty & \text{if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1 = 0$	3
$M_{0CT} = \begin{cases} P_1 + P_2 + 2P_1^\infty + 2P_2^\infty & \text{if } a < 0 \\ 2P_1^\infty + 2P_2^\infty & \text{if } a > 0 \end{cases}$	6 4

(ii.2)  $g = \frac{1}{2}$  and  $a \neq 0$ .

Here we have an additional invariant line which is simple and the invariant hyperbola becomes double so we compute the exponential factor  $E_3$ .

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = a + xy$ $E_3 = e^{-\frac{a(g_1 - g_0) + g_1 xy - 2g_0 y^2}{2(a + xy)}}$ $\alpha_1 = \frac{x}{2} + y$ $\alpha_2 = 2y$ $\alpha_3 = -g_0 y$	$P_1 = (-2i\sqrt{a}, 0)$ $P_2 = (2i\sqrt{a}, 0)$ $P_3 = \left(-i\sqrt{2}\sqrt{a}, -\frac{i\sqrt{a}}{\sqrt{2}}\right)$ $P_4 = \left(i\sqrt{2}\sqrt{a}, \frac{i\sqrt{a}}{\sqrt{2}}\right)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 0 : 0]$ $s, s, n, n; \binom{0}{2}SN, N$ if $a < 0$ $\odot, \odot, \odot, \odot; \binom{0}{2}SN, N$ if $a > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ double $\bar{J}_1 \cap \mathcal{L}_\infty = \text{simple}$ $\bar{J}_2 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$	4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty & \text{if } a < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty & \text{if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2^2 = 0$	6
$M_{0CT} = \begin{cases} P_1 + P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty & \text{if } a < 0 \\ 3P_1^\infty + 4P_2^\infty & \text{if } a > 0 \end{cases}$	13 7

where the total curve  $T$  has

- 1) only two distinct tangents at  $P_1^\infty$ , but one of them is double,
- 2) only two distinct tangents at  $P_2^\infty$ , but one of them is triple.

	First integral	Integrating Factor
General	$I = J_1^0 J_2^{\lambda_2} E_3^{\frac{2\lambda_2}{s_0}}$	$R = J_1^1 J_2^{\lambda_2} E_3^{\frac{2(2+\lambda_2)}{s_0}}$
Simple example	$\mathcal{I} = J_2 E_3^2$	$\mathcal{R} = \frac{J_1}{J_2^2}$

(ii.3)  $g = \frac{1}{4}$  and  $a \neq 0$ .

Here the hyperbola becomes double so we compute the exponential factor  $E_2$ .

Inv. cur./exp. fac. and cofac.	Singularities	Intersection points
$J_1 = a + xy$ $E_2 = e^{\frac{a s_0 + s_0 xy + s_1 y^2}{(a+xy)}}$ $\alpha_1 = -\frac{x}{2} + 2y$ $\alpha_2 = -g_1 y$	$P_1 = \left(2i\sqrt{a}, -\frac{i\sqrt{a}}{2}\right)$ $P_2 = \left(2i\sqrt{a}, \frac{i\sqrt{a}}{2}\right)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 0 : 0]$ $sn_{(2)}, sn_{(2)}; \binom{0}{2}SN, N$ if $a < 0$ $\odot_{(2)}, \odot_{(2)}; \binom{0}{2}SN, N$ if $a > 0$	$\bar{J}_1 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = 2J_1 + \mathcal{L}_\infty$	3
$M_{0CS} = \begin{cases} 2P_1 + 2P_2 + 2P_1^\infty + P_2^\infty & \text{if } a < 0 \\ 2P_1^C + 2P_2^C + 2P_1^\infty + P_2^\infty & \text{if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1^2 = 0$	5
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + 3P_1^\infty + 3P_2^\infty & \text{if } a < 0 \\ 3P_1^\infty + 3P_2^\infty & \text{if } a > 0 \end{cases}$	10 6

where the total curve  $T$  has only two distinct tangents at  $P_1^\infty$  (and  $P_2^\infty$ ), but one of them is double.

	First integral	Integrating Factor
General	$I = \frac{1}{2} \left( 2\sqrt{2}a \text{DawsonF} \left( \frac{\sqrt{2}y}{\sqrt{a+xy}} \right) + x\sqrt{a+xy} \right) \left( e^{\frac{ag_0 + g_0xy + g_1y^2}{a+xy}} \right)^{\frac{2}{g_1}}$	$R = J_1^{-\frac{1}{2}} E_2^{\frac{2}{g_1}}$
Simple example	$\mathcal{I} = \frac{1}{2} \left( 2\sqrt{2}a \text{DawsonF} \left( \frac{\sqrt{2}y}{\sqrt{a+xy}} \right) + x\sqrt{a+xy} \right) \left( e^{\frac{y^2}{a+xy}} \right)^2$	$\mathcal{R} = J_1^{-\frac{1}{2}} E_2^2$

(ii.4)  $g = 1$  and  $a \neq 0$ .

Here we have, apart from a simple hyperbola, two additional invariant lines (real or complex, depending on the sign of the parameter  $a$ ).

Invariant curves and cofactors	Singularities	Intersection points
$J_1 = 1 - \frac{iy}{\sqrt{a}}$ $J_2 = 1 + \frac{iy}{\sqrt{a}}$ $J_3 = a + xy$ $\alpha_1 = y - i\sqrt{a}$ $\alpha_2 = y + i\sqrt{a}$ $\alpha_3 = x + 2y$	$P_1 = (-i\sqrt{a}, -i\sqrt{a})$ $P_2 = (i\sqrt{a}, i\sqrt{a})$ $P_3 = (-2i\sqrt{a}, i\sqrt{a})$ $P_4 = (2i\sqrt{a}, -i\sqrt{a})$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 0 : 0]$ $n, n, s, s; \binom{0}{2}SN, N$ if $a < 0$ $\odot, \odot, \odot, \odot; \binom{0}{2}SN, N$ if $a > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_2^\infty$ simple $\bar{J}_1 \cap \bar{J}_3 = \begin{cases} P_1 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \bar{J}_3 = \begin{cases} P_2 \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$ $\bar{J}_2 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_3 \cap \mathcal{L}_\infty = \begin{cases} P_1^\infty \text{ simple} \\ P_2^\infty \text{ simple} \end{cases}$

Divisor and zero-cycles	Degree
$ICD = \begin{cases} J_1 + J_2 + J_3 + \mathcal{L}_\infty & \text{if } a < 0 \\ J_1^C + J_2^C + J_3 + \mathcal{L}_\infty & \text{if } a > 0 \end{cases}$	4 4
$M_{0CS} = \begin{cases} P_1 + P_2 + P_3 + P_4 + 2P_1^\infty + P_2^\infty & \text{if } a < 0 \\ P_1^C + P_2^C + P_3^C + P_4^C + 2P_1^\infty + P_2^\infty & \text{if } a > 0 \end{cases}$	7 7
$T = Z\bar{J}_1\bar{J}_2\bar{J}_3 = 0$	5
$M_{0CT} = \begin{cases} 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty & \text{if } a < 0 \\ 2P_1^\infty + 4P_2^\infty & \text{if } a > 0 \end{cases}$	12 6

where the total curve  $T$  has four distinct tangents at  $P_2^\infty$ .

	First integral	Integrating Factor
General	$I = \left( \sqrt{a+y^2} + y \right) e^{-\frac{\sqrt{a+y^2}}{\sqrt{a+y^2}}} e^{\frac{\sqrt{a+y^2}(x-y)}{a+xy}}$	$R = J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} J_3^{-2}$
Simple example	$\mathcal{I} = \left( \sqrt{a+y^2} + y \right) e^{-\frac{\sqrt{a+y^2}}{\sqrt{a+y^2}}} e^{\frac{\sqrt{a+y^2}(x-y)}{a+xy}}$	$\mathcal{R} = J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} J_3^{-2}$

(ii.5)  $a = 0$  and  $g \neq 0, 1$ .

Under this condition, systems (1.12) do not belong to **QSH**, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.12). All the affine invariant lines are  $y = 0$  that is simple and  $x = 0$  that is double so we compute the exponential factor  $E_3$ . By perturbing the reducible conic  $xy = 0$  we produce the hyperbola  $a + xy = 0$ .

Inv. cur./exp. fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $E_3 = e^{\frac{g_0x+g_1y}{x}}$ $\alpha_1 = (-1+g)x+y$ $\alpha_2 = gx+y$ $\alpha_3 = -g_1y$	$P_1 = (0,0)$ $P_1^\infty = [0:1:0]$ $P_2^\infty = [1:0:0]$ $epep_{(4)}; \binom{0}{2}SN, S$ if $g < 0$ $phph_{(4)}; \binom{0}{2}SN, N$ if $g > 0$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple



Divisor and zero-cycles	Degree
$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$ if $g \neq 1$	4
$M_{0CS} = 4P_1 + 2P_1^\infty + P_2^\infty$ if $g \neq 0$	7
$T = Z\bar{J}_1\bar{J}_2^2 = 0$ if $g \neq 1$	3
$M_{0CT} = 3P_1 + 3P_1^\infty + 2P_2^\infty$ if $g \neq 0$	8

where the total curve  $T$  has only two distinct tangents at  $P_1^\infty$ , but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^{-\frac{(g-1)\lambda_1}{g} - \frac{\lambda_1}{g}} E_3^{\frac{\lambda_1}{g}}$	$R = J_1^{\lambda_1} J_2^{-\frac{(g-1)\lambda_1}{g} - \frac{3g-1}{g}} E_3^{\frac{1+\lambda_1}{g}}$
Simple example	$\mathcal{I} = J_1^g J_2^{(1-g)} E_3$	$\mathcal{R} = \frac{1}{J_1 J_2^2}$

(ii.6)  $a = 0$  and  $g = 1$ .

Under this condition, systems (1.12) do not belong to QSH, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.12). All the affine invariant lines are  $y = 0$  and  $x = 0$  that are double so we compute the exponential factor  $E_3$  and  $E_4$ . By perturbing the reducible conic  $xy = 0$  we produce the hyperbola  $a + xy = 0$ .

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = y$ $J_2 = x$ $E_3 = e^{\frac{g_0x+g_1y}{x}}$ $E_4 = e^{\frac{h_0+h_1y}{y}}$ $\alpha_1 = y$ $\alpha_2 = x + y$ $\alpha_3 = -g_1y$ $\alpha_4 = -h_0$	$P_1 = (0,0)$ $P_1^\infty = [0 : 1 : 0]$ $P_2^\infty = [1 : 0 : 0]$ $phph_{(4)}; \binom{0}{2}SN, N$	$\bar{J}_1 \cap \bar{J}_2 = P_1$ simple $\bar{J}_1 \cap \mathcal{L}_\infty = P_2^\infty$ simple $\bar{J}_2 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = 2J_1 + 2J_2 + \mathcal{L}_\infty$	4
$M_{0CS} = 4P_1 + 2P_1^\infty + P_2^\infty$	7
$T = Z\bar{J}_1^2\bar{J}_2^2 = 0$	5
$M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$	10

where the total curve  $T$  has only two distinct tangents at  $P_1^\infty$  (and  $P_2^\infty$ ), but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{\lambda_1} J_2^0 E_3^{\frac{\lambda_1}{g_1}} E_4^0$	$R = J_1^{\lambda_1} J_2^{-2} E_3^{\frac{1+\lambda_1}{g_1}} E_4^0$
Simple example	$\mathcal{I} = J_1 E_3$	$\mathcal{R} = \frac{1}{J_1^2}$

(ii.7)  $a = g = 0$ .

Under this condition, systems (1.12) do not belong to **QSH**, but we study them seeking a complete understanding of the bifurcation diagram of the systems in the full family (1.12). The line  $y = 0$  is filled up with singularities, therefore this is a degenerate system. The following study is done with the reduced system. For this system the line  $x = 0$  is double so we compute the exponential factor  $E_2$ .

Inv.cur./exp.fac. and cofactors	Singularities	Intersection points
$J_1 = x$ $E_2 = e^{\frac{g_0 x + g_1 y}{x}}$ $\alpha_1 = 1$ $\alpha_2 = -g_1$	$P_1 = (0, 0)$ $P_1^\infty = [0 : 1 : 0]$ $(\ominus[[]; n^d]; \binom{0}{2}SN, (\ominus[[]; \emptyset)$	$\bar{J}_1 \cap \mathcal{L}_\infty = P_1^\infty$ simple

Divisor and zero-cycles	Degree
$ICD = 2J_1 + \mathcal{L}_\infty$	3
$M_{0CS} = P_1 + 2P_1^\infty$	3
$T = Z\bar{J}_1^2 = 0$	3
$M_{0CT} = 2P_1 + 3P_1^\infty$	5

where the total curve  $T$  has only two distinct tangents at  $P_1^\infty$ , but one of them is double.

	First integral	Integrating Factor
General	$I = J_1^{g_1 \lambda_2} E_2^{\lambda_2}$	$R = J_1^{-2+g_1 \lambda_2} E_2^{\lambda_2}$
Simple example	$\mathcal{I} = J_1 E_2$	$\mathcal{R} = \frac{1}{J_1^2}$

We sum up the topological, dynamical and algebraic geometric features of family (1.12) and also confront our results with previous results in literature in the following proposition. We show that there are two more phase portraits than the ones appearing in [17].

**Proposition 5.1.**

- (a) For the family (1.12) we obtained seven distinct configurations  $C_1^{(1.12)} - C_7^{(1.12)}$  of invariant hyperbolas and lines (see Figure 5.1 for the complete bifurcation diagram of configurations of this family). The bifurcation set of configurations in the full parameter space is  $ag(g-1)(g-1/2)(g-1/4) = 0$ . On  $(g-1/2)(g-1/4) = 0$  the invariant hyperbola is double. On  $g = 1/2$  we have an additional invariant line and on  $g = 1$  we have two additional invariant lines. On  $g = 0$  we just have a simple invariant hyperbola. On  $a = 0$  the hyperbola becomes reducible into two lines and when  $a = g = 0$  one of the lines is filled up with singularities.
- (b) The family (1.12) is generalized Darboux integrable when  $g = 1/2$  and it is Liouvillian integrable when  $g = 1/4$ .
- (c) For the family (1.12) we have seven topologically distinct phase portraits  $P_1^{(1.12)} - P_7^{(1.12)}$ . The topological bifurcation diagram of family (1.12) is done in Figure 5.2. The bifurcation set are the half lines  $g = 1/4$  and  $g = 1/2$  with  $a < 0$  and the lines  $g = 0$  and  $a = 0$ . The half line  $g = 1/4$  with  $a < 0$  and the lines  $g = 0$ ,  $a = 0$  are bifurcation sets of singularities and the half line  $g = 1/2$  with  $a < 0$  is a bifurcation of saddle to saddle connection. The phase portraits  $P_4^{(1.12)}$  and  $P_6^{(1.12)}$  are not topologically equivalent with any one of the phase portraits in [17].

**Proof.**

- (a) We have the following type of divisors and zero-cycles of the total invariant curve  $T$  for the configurations of family (1.12):

Configurations	Divisors and zero-cycles of the total inv. curve $T$
$C_1^{(1.12)}$	$ICD = J_1 + \mathcal{L}_\infty$ $M_{0CT} = P_3 + P_4 + 2P_1^\infty + P_2^\infty$
$C_2^{(1.12)}$	$ICD = J_1 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1^\infty + P_2^\infty$
$C_3^{(1.12)}$	$ICD = J_1 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1^\infty + P_2^\infty$
$C_4^{(1.12)}$	$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$ $M_{0CT} = P_1 + P_2 + 2P_3 + 2P_4 + 3P_1^\infty + 4P_2^\infty$
$C_5^{(1.12)}$	$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1^\infty + 4P_2^\infty$
$C_6^{(1.12)}$	$ICD = 2J_1 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + 3P_1^\infty + 3P_2^\infty$
$C_7^{(1.12)}$	$ICD = 2J_1 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1^\infty + 3P_2^\infty$

Therefore, the configurations  $C_1^{(1.12)}$  up to  $C_7^{(1.12)}$  are all distinct. For the limit cases of family (1.12) we have the following configurations:

Configurations	Divisors and zero-cycles of the total inv. curve $T$
$c_3$	$ICD = J_1 + 2J_2 + \mathcal{L}_\infty$ $M_{0CT} = 3P_1 + 3P_1^\infty + 2P_2^\infty$
$c_4$	$ICD = 2J_1 + 2J_2 + \mathcal{L}_\infty$ $M_{0CT} = 4P_1 + 3P_1^\infty + 3P_2^\infty$
$c_5$	$ICD = 2J_1 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 3P_1^\infty$
$c_6$	$ICD = J_1 + J_2 + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1 + 2P_2 + P_3 + P_4 + 2P_1^\infty + 4P_2^\infty$
$c_7$	$ICD = J_1^C + J_2^C + J_3 + \mathcal{L}_\infty$ $M_{0CT} = 2P_1^\infty + 4P_2^\infty$

(b) This is shown in the previously exhibited tables.

(c) We have that:

Phase Portraits	Sing. at $\infty$	Finite sing.	Separatrix connections
$P_1^{(1.12)}$	$(\binom{0}{2}SN, N)$	$(n, n, s, s)$	$2SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$
$P_2^{(1.12)}$	$(\binom{0}{2}SN, N)$	$(s, s, n, n)$	$4SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$
$P_3^{(1.12)}$	$(\binom{0}{2}SN, S)$ $(\binom{0}{2}SN, \binom{1}{2}S)$	$(f, f, \odot, \odot)$ $(f, f)$	$0SC_f^f \ 2SC_f^\infty \ 2SC_\infty^\infty$
$P_4^{(1.12)}$	$(\binom{0}{2}SN, N)$ $(\binom{0}{2}SN, N)$ $(\binom{0}{2}SN, \binom{1}{2}N)$	$(\odot, \odot, \odot, \odot)$ $(\odot_{(2)}, \odot_{(2)})$ $(\odot, \odot)$	$0SC_f^f \ 0SC_f^\infty \ 2SC_\infty^\infty$
$P_5^{(1.12)}$	$(\binom{0}{2}SN, S)$	$(\odot, \odot, n, n)$	$0SC_f^f \ 2SC_f^\infty \ 0SC_\infty^\infty$
$P_6^{(1.12)}$	$(\binom{0}{2}SN, N)$	$(s, s, n, n)$	$3SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$
$P_7^{(1.12)}$	$(\binom{0}{2}SN, N)$	$(sn_{(2)}, sn_{(2)})$	$0SC_f^f \ 6SC_f^\infty \ 0SC_\infty^\infty$

Therefore, we have seven distinct phase portraits for systems (1.12). For the limit cases of family (1.12) we have the following phase portraits:

Phase Portraits	Sing. at $\infty$	Finite sing.	Separatrix connections
$p_3$	$(\binom{0}{2}SN, N)$	$phph_{(4)}$	$0SC_f^f \ 4SC_f^\infty \ 0SC_\infty^\infty$
$p_4$	$(\binom{0}{2}SN, (\ominus[[[]]; \emptyset))$	$(\ominus[[[]]; n^d)$	$0SC_f^f \ 2SC_f^\infty \ 0SC_\infty^\infty$
$p_5$	$(\binom{0}{2}SN, S)$	$epep_{(4)}$	$0SC_f^f \ 4SC_f^\infty \ 0SC_\infty^\infty$

On the table below we list all the phase portraits of Llibre-Yu in [17] that admit 2 singular points at infinity with the type  $(SN, N)$ :

Phase Portrait	Sing. at $\infty$	Real finite sing.	Separatrix connections
$L_{01}$	$(SN, N)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $3SC_\infty^\infty$
$L_{03}$	$(SN, N)$	$\emptyset$	$0SC_f^f$ $0SC_f^\infty$ $3SC_\infty^\infty$
$\omega_1$	$(SN, N)$	$(s, n)$	$1SC_f^f$ $6SC_f^\infty$ $0SC_\infty^\infty$

The phase portraits  $P_4^{(1.12)}$  and  $P_6^{(1.12)}$  are not topologically equivalent with any of the phase portraits in [17]. They are however phase portraits of systems possessing an invariant line and an invariant hyperbola (when  $g = 1/2$ ).  $\square$

**Remark 5.2.** The family (1.12) does not have any case where the inverse integrating factor is polynomial. We just have a polynomial inverse integrating factor on the limit case  $a = 0$  of family (1.12).

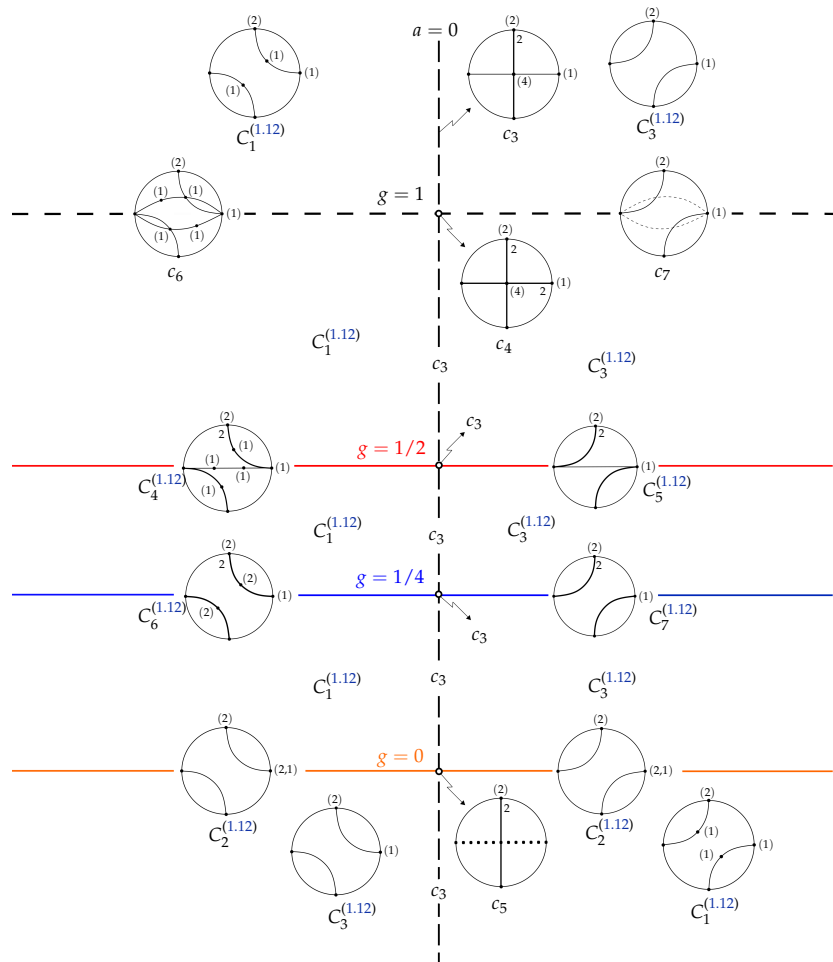


Figure 5.1: Bifurcation diagram of configurations for family (1.12): In this figure on the dashed line  $a = 0$  the hyperbola becomes reducible into two lines  $x = 0$  and  $y = 0$ . When  $a = g = 0$  the line  $y = 0$  is filled up with singularities. For the bifurcation curves we either have an additional line or coalescing hyperbolas or a change in the multiplicity of a infinity singularity. On the dashed line  $g = 1$  we have two additional lines. The dashed lines represent complex lines.

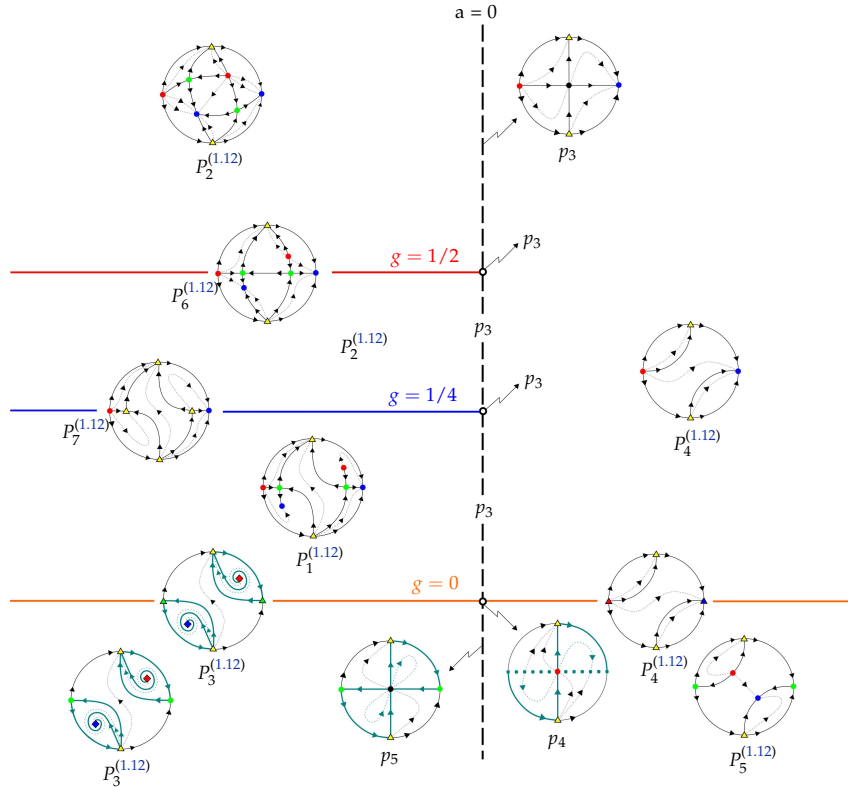


Figure 5.2: Topological bifurcation diagram for family (1.12). Note that the phase portraits  $p_4$ ,  $p_5$  and  $p_3^{(1.12)}$  possess graphics in their first and third quadrant.

We have the following number of distinct configurations and phase portraits in the normal forms (1.10), (1.11) and (1.12), denoted here by NF studied, as well as their limit points:

Config. in the NF studied	All config.	Phase port. in the NF studied	All Phase port.
18	25	12	17

## 6 Questions, the problem of Poincaré and concluding comments

We are interested both in the algebraic-geometric properties of the systems in the family **QSH**, as expressed in their global geometric configurations of algebraic solutions, and on their impact on the integrability of the systems. We are also interested in the topological phase portraits of systems in **QSH**. This family is 3-dimensional modulo the action of the affine group of transformations and time rescaling (see [18]). As we have seen in the three families we discussed in this work, the class **QSH** forms a rich testing ground for exploring integrability in terms of the global algebraic geometric features of the systems occurring in these normal forms. The geometric analysis of the systems we studied bring out a number of questions. We expect to find answers to these questions, once the full study of all the normal forms of **QSH** will be completed.

## 6.1 The problem of Poincaré

For two out of the three families discussed in this work we have an answer to Poincaré's problem of algebraic integrability for each one of the systems in these families. The answer is given entirely in geometric terms (see Theorem 3.6 in Subsection 3.1) and all systems in the family (1.11) are algebraically integrable. This raises the following question: For how many of the remaining normal forms could we solve the problem of Poincaré for all the systems in the families defined by their respective normal forms? Could this problem be solved in geometrical terms as it was possible for the normal form (A)?

## 6.2 The problem of generalizing the Christopher–Kooij Theorem 1.15

We saw that under the “generic” conditions of Christopher and Kooij (C–K), formulated algebraically on the algebraic invariant curves  $f_1(x, y), \dots, f_k(x, y)$  of a polynomial differential system, we are assured to have a polynomial inverse integrating factor of the special form

$$f_1(x, y) \dots f_k(x, y).$$

In this article we see cases where these “generic conditions” of (C–K) are not satisfied and yet we still have an integrating factor which is polynomial. Furthermore, in some cases, this polynomial inverse integrating factor is of the same form as the one in the (C–K) theorem. Here are some examples occurring in the families we considered.

(I) For the family (1.10).

(1) All the systems in family (1.10) have an inverse integrating factor which is polynomial, they are Darboux integrable and have in the generic case only two invariant lines  $J_1, J_2$  and two invariant hyperbolas  $J_3, J_4$ . An inverse polynomial factor is  $J_1 J_2 J_3 J_4$  just like in C–K theorem. The condition (a) of the C–K theorem 1.15 is satisfied since our curves are lines and hyperbolas and they are, of course, non-singular and irreducible. The condition (b) is also satisfied since the coefficients in  $M_{0ST}$  are all equal to 2. The condition (c) is not satisfied because both of the hyperbolas  $J_3$  and  $J_4$  intersect the line at infinity at  $P_1^\infty$  and they are tangent at this point. The condition (d) is not satisfied because the sum of the degrees of the curves is 6 and not 3. However, the conclusion is the same as in theorem 1.15.

(2) In the non-generic cases  $(a - 3v^2/4)(a + 3v^2)(a - 8v^2/9) = 0$  we have a similar situation and an inverse polynomial integrating factor. We have, as in the generic case, the two invariant lines  $J_1, J_2$  and we have, apart from the two invariant hyperbolas  $J_4, J_5$  and additional invariant curve  $J_3$ . We again have (a) and (b) satisfied but not (c) and (d) of (C–K) theorem 1.15. However, if we restrict our attention only to the remarkable curves  $J_1, J_2, J_4, J_5$  then we still have an inverse integrating factor of the form  $J_1 J_2 J_4 J_5$  as in the (C–K) theorem.

(II) Consider now the family (1.11).

(1) The systems in the family (1.11) have in the generic case three invariant lines  $J_1, J_2, J_3$  and two invariant hyperbolas  $J_4, J_5$ . Let us now consider for our discussion only the remarkable curves, the three lines  $J_1, J_2, J_3$  and the hyperbola  $J_4$ . These of course satisfy the conditions (a) and (b). However they do not satisfy (c) because for example  $J_1, J_2, J_4$  intersect at  $P_2$ . They also do not satisfy (d). If we limit our attention to the four curves  $J_1, J_2, J_3, J_4$  we see that we have as an inverse integrating factor the polynomial  $J_1 J_2 J_3 J_4$  which we get by taking in the general expression of the integrating factor  $\lambda_1 = \lambda_2 = \lambda_4 = -1$ .

So we can ask the following questions:

**Question 1:** How should the geometry of the configuration of algebraic solutions  $J_1, \dots, J_k$  of a polynomial system be so as to have an inverse integrating factor which is polynomial? In particular, how should this geometry be in order to have an inverse integrating factor  $J_1 \cdots J_k$ ? How could we relax, generalize, the hypotheses in the (C–K) theorem such that the same conclusion holds?

**Question 2:** If a system has a rational first integral do we always have an inverse integrating factor involving only remarkable curves?

Consider now the non-generic case  $v(a - v^2) = 0$  of the family (1.10). We have that one of the invariant curves becomes double. In the case  $v = 0$  we have two simple invariant lines  $J_1, J_2$  and one double invariant hyperbola  $J_3$ . A polynomial inverse integrating factor in this case is  $J_1 J_2 J_3^2$ . In the case  $a - v^2 = 0$  we have a double line  $J_1$  and two simple hyperbolas  $J_2, J_3$ . We have a polynomial integrating factor  $J_1^2 J_2 J_3$ . In this case we still have a polynomial inverse integrating factor.

**Question 3:** Can we generalize the (C–K) Theorem 1.15 so as to include multiplicity? In what cases there is a relation between the multiplicity  $s$  of an algebraic solution  $J_i$  and the exponent of  $J_i$  appearing in the polynomial inverse integrating factor?

### 6.3 On the bifurcation diagrams

We have two kinds of bifurcation diagrams: topological and geometrical, i.e., of geometric configurations of algebraic solutions (lines and hyperbolas).

**Question 1:** What is the relation between these two kinds of bifurcation diagrams?

In all three families the topological bifurcation set of the phase portraits is a subset of the bifurcation set of configurations of algebraic solutions. This inclusion is strict for the first and last families.

The bifurcation set  $\text{Bif}_A$  for topological phase portraits in the family (A) is formed by the half-line of  $v = 0, a < 0$  ( $(\text{Bif}_A)^{(1)}$ ); the non-zero points on the parabola  $a = v^2$  ( $(\text{Bif}_A)^{(2)}$ ).

On  $(\text{Bif}_A)^{(1)}$  and on  $(\text{Bif}_A)^{(2)}$  4 real finite singular points coalesce into 2 real finite double points. In the first case, after crossing the half-line they split again into 4 real singular points, while in the second case they split into 4 complex finite singular points which are finite points of intersection of the complexifications of each one of two real hyperbolas with the two complex invariant lines, respectively.

It is interesting to observe that these topological bifurcation points have an impact on the bifurcation set of geometrical configurations. Indeed, first we mention that above and below the half-line  $v = 0$  and  $a < 0$  we have two couples of real singularities, the points in each couple are located on distinct branches of one hyperbola. When two singular points on different hyperbolas coalesce this yields the coalescence of the respective branches and also of the two hyperbolas, producing a double hyperbola.

On the non-zero points of the parabola  $a = v^2$  the coalescence of the 4 real finite singular points into two couples of double real singular points yields the coalescence of the two lines into a double real line which afterwards splits into two complex lines. In this case again we see that the topological bifurcation points produce also bifurcations in the configurations.

We note that we have a saddle to saddle connection on the parabola  $a = v^2$  for  $(a, v) \neq (0, 0)$ .



On the bifurcation points situated on the remaining three parabolas we either have the occurrence of an additional hyperbola (on  $a - 8v^2/9 = 0$  or on  $a + 3v^2 = 0$ ) or the appearance of an additional invariant line (on  $a - 3v^2/4 = 0$ ). The presence of these additional invariant curves does not affect in any way the bifurcation diagram of the systems.

In conclusion we have:

- (i) Impact of the topological bifurcations on the bifurcations of configurations: The bifurcation points of singularities of the systems located on the algebraic solutions, when real singular points become multiple, become also bifurcation points for the multiplicity of the algebraic solutions, inducing coalescence of the respective curves and hence of their geometric configuration.
- (ii) The bifurcation points of configurations due to the appearance of additional invariant curves (three hyperbolas instead of two or three lines instead of two lines) have no consequence for the topological bifurcation diagram of this family.
- (iii) Inside the parabola  $a = v^2$  i.e. for points  $(a, v)$  such that  $v^2 - a < 0$  where we have complex singularities, we have no bifurcation points of phase portraits but we have, on the half-line  $v = 0, a > 0$  bifurcation points of configurations, the two hyperbolas coalescing into a double hyperbola. Here we need to stress the fact that on this half-line we have two *double complex singularities* and while this fact has no impact on the topological bifurcation it is important for the bifurcations of the configurations. Indeed, when the four complex singularities become two double complex singularities on this half-line, the two hyperbolas on which they are lying coalesce becoming a double hyperbola.

**Limit points of the bifurcation diagram for (A)** Let us discuss the bifurcation phenomena which occur at the limiting points of our parameter space for systems in the family (A), i.e. the points on  $a = 0$ . The topological bifurcation on this line is easy to understand. Indeed, except for the the point  $(0, 0)$  where all four singularities collide, all the other points on  $a = 0$  are bifurcation points of saddle to saddle connections. All the points on the line  $a = 0$  are also points of bifurcation of configurations of algebraic solutions. However this bifurcation is a bit harder to understand. Indeed, at these points say on  $v > 0$  we have a configuration with three *simple* affine invariant lines, the vertical line intersecting the two parallel lines at two points and forming a saddle-to saddle connection. It is clear that this configuration splits into the configuration  $C_1^{(1.10)}$  on the left which has two hyperbolas and two invariant lines. So in some sense the configuration on  $a = 0$  should be considered as a *multiple configuration* since it yields new algebraic solutions. Analyzing the bifurcation phenomenon we see that each one of the two hyperbolas splits into two lines on  $a = 0$  and  $v > 0$ . Indeed, the hyperbola  $J_4$  splits into the line  $x = 0$  and the line  $J_1$  and the hyperbola  $J_3$  splits into  $J_2$  and  $x = 0$ . So that although for  $a = 0$  each one of the lines is simple, each line contributes to the multiplicity of the configuration. Considering the composite cubic curve  $xJ_1J_2 = 0$  we may say that this configuration has (geometric) multiplicity two in this family as it splits into two cubic curves  $J_1J_4$  and  $J_2J_3$ . On the other hand we see that we have on  $a = 0$  an exponential factor involving in its exponent at the denominator of the rational function, the polynomial  $xJ_1J_2$  which turns out to be of integrable multiplicity two. The notions of integrable multiplicity and geometric multiplicity in [9] are not restricted to algebraic solutions. But the authors say there clearly that the equivalence between integrable and geometric multiplicities occurs only for integrable solutions. In the above family these two multiplicities coincide. So we have the following

**Question:** Under what condition on (finite) configurations of algebraic solutions do the two multiplicities, integrable and geometric coincide?

Finally we note that the point  $(0,0)$  produces in perturbations all nine configurations which we encounter in the extension of the family (A), apart from the fact that we are interested in producing all the phase portraits of family **QSH** as well as fully understanding the integrability of this family, the questions raised above are additional motivation for completing the study of this family.

## 7 Appendix

Considering  $r = m_1/m_2$  where  $m_1, m_2 \in \mathbb{Z}$  we can say that

$$I = \left( \frac{J_1}{J_2} \right)^{m_2} \left( \frac{J_3}{J_4} \right)^{m_1}$$

is a rational first integral of (1.10) when  $a = (1 - (m_1/m_2)^2)v^2$ . Consider

$$\mathcal{F}_{(c_1, c_2)} = c_1 J_1^{m_2} J_3^{m_1} - c_2 J_2^{m_2} J_4^{m_1} = 0.$$

We have the following:

- Taking  $m_1 = 2$  and  $m_2 = 4$  (i.e.  $a = 3v^2/4$ ) we have that

$$\mathcal{F}_{(1,1)} = -\frac{27}{16}v^3y \left( 81v^4 + 36v^2(-2x^2 + xy + y^2) + 16x(x-y)^3 \right).$$

Therefore, we have a line and a quartic as remarkable curves.

- Taking  $m_1 = 2$  and  $m_2 = 6$  (i.e.  $a = 8v^2/9$ ) we have that

$$\mathcal{F}_{(1,1)} = \frac{32}{9}v^3(v^2 + 3y(y-x)) \left( 3v^4(5x-8y) - 2v^2(x-y)^2(5x+4y) + 3x(x-y)^4 \right).$$

Therefore, we have a hyperbola and a quintic as remarkable curves.

- Taking  $m_1 = 2$  and  $m_2 = 8$  (i.e.  $a = 15v^2/16$ ) we have that

$$\begin{aligned} \mathcal{F}_{(1,1)} = & \frac{27}{262144}v^3(36v^2x - 45v^2y - 80x^2y + 160xy^2 - 80y^3) \\ & (3645v^6 + 19440v^4x^2 - 58320v^4xy + 38880v^4y^2 - 11520v^2x^4 + 23040v^2x^3y \\ & - 23040v^2xy^3 + 11520v^2y^4 + 4096x^6 - 20480x^5y + 40960x^4y^2 - 40960x^3y^3 \\ & + 20480x^2y^4 - 4096xy^5). \end{aligned}$$

Therefore, we have a cubic and a polynomial of degree 6 as remarkable curves.

- Taking  $m_1 = 3$  and  $m_2 = 6$  (i.e.  $a = 3v^2/4$ ) we have that

$$\begin{aligned} \mathcal{F}_{(1,1)} = & \frac{81}{512}v^3y(6561v^8 - 1944v^6(6x^2 - 3xy - 5y^2) \\ & + 1296v^4(x-y)^2(6x^2 + 2xy + y^2) - 1152v^2x(x-y)^4(2x+y) + 256x^2(x-y)^6). \end{aligned}$$

Therefore, we have a line and a polynomial of degree 8 as remarkable curves.

- Taking  $m_1 = 3$  and  $m_2 = 9$  (i.e.  $a = 8v^2/9$ ) we have that

$$\begin{aligned} \mathcal{F}_{(1,1)} = & \frac{16}{27}v^3 (v^2 - 3xy + 3y^2) (64v^{10} + 675v^8x^2 - 2544v^8xy + 2112v^8y^2 \\ & - 900v^6x^4 + 2520v^6x^3y - 612v^6x^2y^2 - 2736v^6xy^3 + 1728v^6y^4 \\ & + 570v^4x^6 - 2232v^4x^5y + 3420v^4x^4y^2 - 2760v^4x^3y^3 + 1530v^4x^2y^4 \\ & - 720v^4xy^5 + 192v^4y^6 - 180v^2x^8 + 936v^2x^7y - 1836v^2x^6y^2 + 1440v^2x^5y^3 \\ & + 180v^2x^4y^4 - 1080v^2x^3y^5 + 684v^2x^2y^6 - 144v^2xy^7 + 27x^{10} - 216x^9y \\ & + 756x^8y^2 - 1512x^7y^3 + 1890x^6y^4 - 1512x^5y^5 + 756x^4y^6 - 216x^3y^7 + 27x^2y^8). \end{aligned}$$

Therefore, we have a hyperbola and a polynomial of degree 10 as remarkable curves.

- Taking  $m_1 = 4$  and  $m_2 = 2$  (i.e.  $a = -3v^2$ ) we have that

$$\begin{aligned} \mathcal{F}_{(1,1)} = & 216v^3(9v^2 + xy)(405v^4x - 81v^4y - 45v^2x^3 + 63v^2x^2y - 18v^2xy^2 \\ & + x^5 - 3x^4y + 3x^3y^2 - x^2y^3). \end{aligned}$$

Therefore, we have a hyperbola and a quintic as remarkable curves.

- Taking  $m_1 = 4$  and  $m_2 = 6$  (i.e.  $a = 5v^2/9$ ) we have that

$$\begin{aligned} \mathcal{F}_{(1,1)} = & -\frac{8}{81}v^3(100v^4 - 21v^2x^2 + 270v^2xy + 75v^2y^2 - 45x^3y + 90x^2y^2 - 45xy^3) \\ & (420v^6x + 300v^6y - 385v^4x^3 + 255v^4x^2y + 105v^4xy^2 + 25v^4y^3 + 105v^2x^5 \\ & + 90x^4y^3 - 45x^3y^4 + 9x^2y^5). \end{aligned}$$

Therefore, we have a quartic and a polynomial of degree 7 as remarkable curves.

- Taking  $m_1 = 4$  and  $m_2 = 8$  (i.e.  $a = 3v^2/4$ ) we have that

$$\begin{aligned} \mathcal{F}_{(1,1)} = & -\frac{27}{2048}v^3y(81v^4 - 72v^2x^2 + 36v^2xy + 36v^2y^2 + 16x^4 - 48x^3y \\ & + 48x^2y^2 - 16xy^3)(6561v^8 - 11664v^6x^2 + 5832v^6xy + 17496v^6y^2 \\ & + 7776v^4x^4 - 12960v^4x^3y + 3888v^4x^2y^2 + 1296v^4y^4 - 2304v^2x^6 \\ & + 8064v^2x^5y - 9216v^2x^4y^2 + 2304v^2x^3y^3 + 2304v^2x^2y^4 - 1152v^2xy^5 \\ & + 256x^8 - 1536x^7y + 3840x^6y^2 - 5120x^5y^3 + 3840x^4y^4 - 1536x^3y^5 + 256x^2y^6). \end{aligned}$$

Therefore, we have a line, a quartic and a polynomial of degree 8 as remarkable curves.

- Taking  $m_1 = 4$  and  $m_2 = 12$  (i.e.  $a = 8v^2/9$ ) we have that

$$\begin{aligned} \mathcal{F}_{(1,1)} = & \frac{64}{81}v^3(v^2 - 3xy + 3y^2)(15v^4x - 24v^4y - 10v^2x^3 + 12v^2x^2y + 6v^2xy^2 \\ & - 8v^2y^3 + 3x^5 - 12x^4y + 18x^3y^2 - 12x^2y^3 + 3xy^4)(64v^{10} + 225v^8x^2 \\ & - 1104v^8xy + 960v^8y^2 - 300v^6x^4 + 840v^6x^3y + 180v^6x^2y^2 - 1680v^6xy^3 \\ & + 960v^6y^4 + 190v^4x^6 - 744v^4x^5y + 1140v^4x^4y^2 - 920v^4x^3y^3 + 510v^4x^2y^4 \\ & - 240v^4xy^5 + 64v^4y^6 - 60v^2x^8 + 312v^2x^7y - 612v^2x^6y^2 + 480v^2x^5y^3 \\ & + 60v^2x^4y^4 - 360v^2x^3y^5 + 228v^2x^2y^6 - 48v^2xy^7 + 9x^{10} - 72x^9y + 252x^8y^2 \\ & - 504x^7y^3 + 630x^6y^4 - 504x^5y^5 + 252x^4y^6 - 72x^3y^7 + 9x^2y^8). \end{aligned}$$

Therefore, we have a hyperbola, a quintic and a polynomial of degree 10 as remarkable curves.

- Taking  $m_1 = 4$  and  $m_2 = 16$  (i.e.  $a = 15v^2/16$ ) we have that

$$\begin{aligned} \mathcal{F}_{(1,1)} = & \frac{27}{219902325552} v^3 (36v^2x - 45v^2y - 80x^2y + 160xy^2 - 80y^3) \\ & (3645v^6 + 19440v^4x^2 - 58320v^4xy + 38880v^4y^2 - 11520v^2x^4 + 23040v^2x^3y \\ & - 23040v^2xy^3 + 11520v^2y^4 + 4096x^6 - 20480x^5y + 40960x^4y^2 - 40960x^3y^3 \\ & + 20480x^2y^4 - 4096xy^5)(13286025v^{12} + 383582304v^{10}x^2 - 1029814560v^{10}xy \\ & + 661348800v^{10}y^2 + 293932800v^8x^4 - 3174474240v^8x^3y + 8406478080v^8x^2y^2 \\ & - 8465264640v^8xy^3 + 2939328000v^8y^4 - 418037760v^6x^6 + 2090188800v^6x^5y \\ & - 2090188800v^6x^4y^2 - 4180377600v^6x^3y^3 + 10450944000v^6x^2y^4 \\ & - 7942717440v^6xy^5 + 2090188800v^6y^6 + 291962880v^4x^8 - 1804861440v^4x^7y \\ & + 4830658560v^4x^6y^2 - 7431782400v^4x^5y^3 + 7431782400v^4x^4y^4 \\ & - 5202247680v^4x^3y^5 + 2601123840v^4x^2y^6 - 849346560v^4xy^7 + 132710400v^4y^8 \\ & - 94371840v^2x^{10} + 660602880v^2x^9y - 1887436800v^2x^8y^2 + 2642411520v^2x^7y^3 \\ & - 1321205760v^2x^6y^4 - 1321205760v^2x^5y^5 + 2642411520v^2x^4y^6 \\ & - 1887436800v^2x^3y^7 + 660602880v^2x^2y^8 - 94371840v^2xy^9 + 16777216x^{12} \\ & - 167772160x^{11}y + 754974720x^{10}y^2 - 2013265920x^9y^3 + 3523215360x^8y^4 \\ & - 4227858432x^7y^5 + 3523215360x^6y^6 - 2013265920x^5y^7 + 754974720x^4y^8 \\ & - 167772160x^3y^9 + 16777216x^2y^{10}). \end{aligned}$$

Therefore, we have a cubic, a polynomial of degree 6 and a polynomial of degree 12 as remarkable curves.

These computations suggest that the remarkable curves of algebraically integrable systems in the family (A) have an unbounded degree.

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