On the solvability of the periodically forced relativistic pendulum equation on time scales

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Abstract. We study some properties of the range of the relativistic pendulum operator $\mathcal{P}$, that is, the set of possible continuous T-periodic forcing terms p for which the equation $\mathcal{P}x = p$ admits a T-periodic solution over a T-periodic time scale $\mathbb{T}$. Writing $p(t) = p_0(t) + \overline{p}$, we prove the existence of a nonempty compact interval $\mathcal{I}(p_0)$, depending continuously on p_0, such that the problem has a solution if and only if $\overline{p} \in \mathcal{I}(p_0)$ and at least two different solutions when $\overline{p}$ is an interior point. Furthermore, we give sufficient conditions for nondegeneracy; specifically, we prove that if T is small then $\mathcal{I}(p_0)$ is a neighbourhood of 0 for arbitrary p_0. The results in the present paper improve the smallness condition obtained in previous works for the continuous case $\mathbb{T} = \mathbb{R}$.

Keywords: relativistic pendulum, periodic solutions, time scales, degenerate equations.

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1 Introduction

The T-periodic problem for the forced relativistic pendulum equation on time scales reads

$$\mathcal{P}x(t) := (\varphi(x^{\Delta}(t)))^{\Delta} + ax^{\Delta}(t) + b \sin x(t) = p_0(t) + s, \quad t \in \mathbb{T}, \tag{1.1}$$

where $a, b > 0$ and s are real numbers, $\mathbb{T}$ is an arbitrary T-periodic nonempty closed subset of $\mathbb{R}$ for some $T > 0$, $\varphi : (-c, c) \to \mathbb{R}$ is the relativistic operator

$$\varphi(x) := \frac{x}{\sqrt{1 - \frac{x^2}{c^2}}}$$

with $c > 0$ and p_0 is continuous and T-periodic in $\mathbb{T}$, with zero average. In this work, we are concerned with the set of all possible values of s such that (1.1) admits a T-periodic solution.

The time scales theory was introduced in 1988, in the PhD thesis of Stefan Hilger [12], as an attempt to unify discrete and continuous calculus. The time scale $\mathbb{R}$ corresponds to

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the continuous case and, hence, yields results for ordinary differential equations. If the time scale is $\mathbb{Z}$, then the results apply to standard difference equations. However, the generality of the set $\mathbb{T}$ produces many different situations in which the time scales formalism is useful in several applications. For example, in the study of hybrid discrete-continuous dynamical systems, see [6].

In the past decades, periodic problems involving the relativistic forced pendulum differential equation for the continuous case $\mathbb{T} = \mathbb{R}$ were studied by many authors, see [3,4,8,14,18,19]. In particular, the works [3,19] are concerned with the so-called *solvability set*, that is, the set $\mathcal{I}(p_0)$ of values of s for which (1.1) has at least one T-periodic solution. We remark that problem (1.1) is 2π-periodic and, consequently, if x is a T-periodic solution then $x + 2k\pi$ is also a T-periodic solution for all $k \in \mathbb{Z}$. For this reason, the multiplicity results for (1.1) usually refer to the existence of *geometrically distinct* T-periodic solutions, i.e. solutions not differing by a multiple of 2π.

For the standard pendulum equation with $a = 0$, the solvability set was analyzed in the pioneering work [9], where it was proved that $\mathcal{I}(p_0) \subset [-b, b]$ is a nonempty compact interval containing 0. Moreover, $\mathcal{I}(p_0)$ depends continuously on p_0. These results were partially extended to the relativistic case in [8]; however, the method of proof in both works is variational and, consequently, cannot be applied to the case $a > 0$. This latter situation was studied in [11] for the standard pendulum and in [19] for the relativistic case. An interesting question, stated already in [9] is whether or not the equation may be degenerate, namely: is there any p_0 such that $\mathcal{I}(p_0)$ reduces to a single point? Many works are devoted to this problem and, for the classical pendulum, nondegeneracy has been proved for an open and dense subset of $\tilde{C}_T$, the space of zero-average T-periodic continuous functions. However, the question for arbitrary p_0 remains unsolved. For a survey on the pendulum equation and open problems see for example [15].

The purpose of this work is to extend the results in [3] and [19] to the context of time scales. To this end, we prove in the first place that the set $\mathcal{I}(p_0)$ is a nonempty compact interval depending continuously on p_0. The method of proof is inspired in a simple idea introduced in [11] for the standard pendulum equation, which basically employs the Schauder Theorem and the method of upper and lower solutions. Moreover, by a Leray–Schauder degree argument it shall be proved that if s is an interior point of $\mathcal{I}(p_0)$, then the problem admits at least two geometrically distinct periodic solutions.

Furthermore, sufficient conditions shall be given in order to guarantee that $0 \in \mathcal{I}(p_0)$. We recall that, when $a \neq 0$, this is not trivial even in the continuous case $\mathbb{T} = \mathbb{R}$. For the classical pendulum equation, there exist well known examples with $0 \notin \mathcal{I}(p_0)$ for arbitrary values of T; for the relativistic case, it was proved in [3] that, if $cT < \sqrt{3}\pi$, then $0 \in \mathcal{I}(p_0)^\circ$. In a very recent paper (see [10]), this bound was improved in terms of a, b and $\|p_0\|_{L^1}$, yielding the uniform condition $cT \leq 2\pi$. It is worth noticing that, however, the problem is still open for large values of T. As we shall see, a slight improvement of the previous bound can be deduced from the results in the present paper. Specifically, we shall prove the existence of T^* with $cT^* > \pi$ such that if $T \leq T^*$ then $0 \in \mathcal{I}(p_0)$ and it is an interior point when the inequality is strict. An inferior bound for T^* can be characterized as a zero of a real function; for the continuous case $\mathbb{T} = \mathbb{R}$, it is easily shown that the bound obtained in [4] is improved; furthermore, it is numerically seen that $cT^* > 6.318$, thus improving also the bound in [10]. We remark that the computation is independent of p_0: in other words, if $T < T^*$, then the range of the operator $\mathcal{P}$ contains a set of the form $\tilde{C}_T + [-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$.

We highlight that our paper is devoted to equations on time scales that involve a φ-

laplacian of relativistic type, for which the literature is scarce. For example, in [17], the existence of heteroclinic solutions for a family of equations on time scales that includes the unforced relativistic pendulum is proved. However, to our knowledge there are no papers concerned with periodic solutions and, more precisely, the solvability set for equations with a singular φ-laplacian on time scales.

This work is organized as follows. In Section 2, we establish the notation, terminology and preliminary results which will be used throughout the paper. In Section 3 we prove that the set $\mathcal{I}(p_0)$ is a nonempty compact interval depending continuously on p_0, and that two geometrically distinct T-periodic solutions exist when s is an interior point. Finally, Section 4 is devoted to find sufficient conditions in order to guarantee that $0 \in \mathcal{I}(p_0)$ and improve the condition obtained in [3] for the continuous case.

2   Notation and preliminaries

For the reader's convenience, let us firstly recall some basic definitions concerning time-scales that shall be used in this work. For a more detailed exposition, see e.g. [6,7].

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$, with the induced topology. Throughout this work, we shall assume that $\mathbb{T}$ is T-periodic for some fixed $T > 0$, namely that $\mathbb{T} + T = \mathbb{T}$. For $a, b \in \mathbb{T}$ with $a \leq b$, we shall denote $[a, b]_\mathbb{T} := [a, b] \cap \mathbb{T}$.

The *forward jump* operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}.$$

A point $t \in \mathbb{T}$ is called *right scattered* if $\sigma(t) > t$, and *right dense* otherwise. A function u is *delta differentiable* at $t \in \mathbb{T}$ if there exists a number (denoted by $u^\Delta(t)$) with the property that given any $\epsilon > 0$ there is a neighbourhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left| (u(\sigma(t)) - u(s)) - u^\Delta(t) (\sigma(t) - s) \right| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$. Thus, we call $u^\Delta(t)$ the *delta derivative* of u at t. Moreover, we say that u is *delta differentiable* on $\mathbb{T}$ provided that $u^\Delta(t)$ exists for all $t \in \mathbb{T}$. Note that for $\mathbb{T} = \mathbb{R}$, we have $u^\Delta = u'$, the usual derivative, and for $\mathbb{T} = \mathbb{Z}$ we have that $u^\Delta(t) = \Delta u(t) = u(t + 1) - u(t)$.

A function $U : \mathbb{T} \to \mathbb{R}$ is called a Δ-antiderivative of $u : \mathbb{T} \to \mathbb{R}$ provided $U^\Delta(t) = u(t)$ holds for all $t \in \mathbb{T}$. It is not difficult to prove that every continuous u has a Δ-antiderivative, which is unique up to a constant term. Thus, the Δ-integral from t_0 to t of u is well defined by

$$\int_{t_0}^t u(s)\Delta s = U(t) - U(t_0) \quad \text{for all } t \in \mathbb{T}.$$

Let $C_T = C_T(\mathbb{T}, \mathbb{R})$ be the Banach space of all continuous T-periodic real functions on $\mathbb{T}$ endowed with the uniform norm

$$\|x\|_\infty = \sup_\mathbb{T} |x(t)| = \sup_{[0,T]_\mathbb{T}} |x(t)|$$

and let $\tilde{C}_T$ be the subspace of those elements of C_T having zero average. By $C_T^1 = C_T^1(\mathbb{T}, \mathbb{R})$ we shall denote the Banach space of all continuous T-periodic functions on $\mathbb{T}$ that are Δ-differentiable functions with continuous Δ-derivatives, endowed with the standard norm

$$\|x\|_1 = \sup_{[0,T]_\mathbb{T}} |x(t)| + \sup_{[0,T]_\mathbb{T}} |x^\Delta(t)|.$$

Equation (1.1) can be written as

$$(\varphi(x^\Delta(t)))^\Delta = f(t, x(t), x^\Delta(t)), \quad t \in \mathbb{T}, \tag{2.1}$$

where $f : \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the continuous function given by $f(t, u, v) := p_0(t) + s - au - b\sin(u)$. A function $x \in C_T^1$ is said to be a solution of (2.1) if $\varphi(x^\Delta) \in C_T^1$ and verifies $(\varphi(x^\Delta(t)))^\Delta = f(t, x(t), x^\Delta(t))$ for all $t \in \mathbb{T}$. We remark that necessarily $\|x\|_\infty < c$.

For $x \in C_T$, the average, the maximum value and the minimum value of x shall be denoted respectively by $\overline{x}$, $x_{\max}$ and $x_{\min}$, namely

$$\overline{x} := \frac{1}{T} \int_0^T x(t)\Delta t, \qquad x_{\max} := \max_{t \in [0,T]_\mathbb{T}} x(t) \qquad x_{\min} := \min_{t \in [0,T]_\mathbb{T}} x(t).$$

2.1 Upper and lower solutions and degree

Let us define T-periodic lower and upper solutions for problem (2.1) as follows.

Definition 2.1. A lower T-periodic solution α (resp. upper solution β) of (2.1) is a function $\alpha \in C_T^1$ with $\|\alpha^\Delta\|_\infty < c$ such that $\varphi(\alpha^\Delta)$ is continuously Δ-differentiable and

$$\left(\varphi\left(\alpha^\Delta(t)\right)\right)^\Delta \geq f(t, \alpha(t), \alpha^\Delta(t)) \quad (resp. \ \left(\varphi\left(\beta^\Delta(t)\right)\right)^\Delta \leq f(t, \beta(t), \beta^\Delta(t))) \tag{2.2}$$

for all $t \in \mathbb{T}$. Such lower (upper) solution is called strict if the inequality (2.2) is strict for all $t \in \mathbb{T}$.

It is worth recalling the problem of finding T-periodic solutions of (2.1) over the closure of the set

$$\Omega_{\alpha,\beta} := \{x \in C_T^1 : \alpha(t) \leq x(t) \leq \beta(t) \ \text{for all} \ t\}$$

can be reduced to a fixed point equation $x = M_f(x)$, where $M_f : \overline{\Omega}_{\alpha,\beta} \to C_T^1$ is a compact operator that can be defined according to the nonlinear version of the continuation method (see e.g. [16]), namely

$$M_f(x) := \overline{x} + \overline{N_f x} + K(N_f x - \overline{N_f x}),$$

where N_f is the Nemitskii operator associated to f and $K : \tilde{C}_T \to \tilde{C}_T$ is the (nonlinear) compact operator given by $K\xi = x$, with $x \in C_T^1$ the unique solution of the problem $(\varphi(x^\Delta(t)))^\Delta = \xi(t)$ with zero average. We recall, for the reader's convenience, that the definition of K based upon the existence, easy to prove, of a (unique) completely continuous map $\phi : C_T \to \mathbb{R}$ satisfying $\int_0^T \varphi^{-1}(h + \phi(h))\Delta t = 0$ for all $h \in C_T$. For the purposes of the present paper, we shall only need the following result, which is an adaptation of Theorem 3.7 in [1]:

Theorem 2.2. *Suppose that* (2.1) *has a T-periodic lower solution α and an upper solution β such that $\alpha(t) \leq \beta(t)$ for all $t \in \mathbb{T}$. Then problem* (1.1) *has at least one T-periodic solution x with $\alpha(t) \leq x(t) \leq \beta(t)$ for all $t \in \mathbb{T}$. If furthermore α and β are strict, then $\deg_{LS}(I - M_f, \Omega_{\alpha,\beta}(0), 0) = 1$, where $\deg_{LS}$ stands for the Leray–Schauder degree.*

3 The solvability set $\mathcal{I}(p_0)$

In this section, we shall prove that the solution set $\mathcal{I}(p_0)$ is a nonempty compact set; furthermore, employing the method of upper and lower solutions it shall be verified that $\mathcal{I}(p_0)$ is an interval depending continuously on p_0. Finally, the excision property of the degree will be employed to verify that if s is an interior point of $\mathcal{I}(p_0)$, then the problem has at least 2 geometrically different T-periodic solutions.

Theorem 3.1. *Assume that $p_0 \in C_T$ has zero average. Then, there exist numbers $d(p_0)$ and $D(p_0)$, with $-b \leq d(p_0) \leq D(p_0) \leq b$, such that (1.1) has at least one T-periodic solution if and only if $s \in [d(p_0), D(p_0)]$. Moreover, the functions $d, D : \tilde{C}_T \to \mathbb{R}$ are continuous.*

Proof. For the reader's convenience, we shall proceed in several steps.

Step 1 (An associated integro-differential problem). Observe that if $x \in C_T^1$ is a solution of (1.1), then, Δ-integration over $[0, T]_\mathbb{T}$ yields $s = \frac{b}{T} \int_0^T \sin(x(t)) \Delta t$. Therefore, it proves convenient to consider the integro-differential Dirichlet problem

$$\begin{cases} (\varphi(x^\Delta(t)))^\Delta + ax^\Delta(t) + b\sin x(t) = p_0(t) + s(x), & t \in (0, T)_\mathbb{T} \\ x(0) = x(T), \end{cases} \tag{3.1}$$

with $s(x) := \frac{b}{T} \int_0^T \sin(x(t)) \Delta t$. By Schauder's fixed point theorem, it is straightforward to prove that for each $r \in \mathbb{R}$ there exists at least one solution $x \in C([0, T]_\mathbb{T})$ of (3.1) such that $x(0) = x(T) = r$.

Step 2 ($\mathcal{I}(p_0)$ is is nonempty and bounded). Let x be a solution of (3.1) such that $x(0) = x(T) = r$, then integration over $[0, T]_\mathbb{T}$ yields

$$\varphi(x^\Delta(T)) - \varphi(x^\Delta(0)) + b\int_0^T \sin x(t) \Delta t = Ts(x),$$

and hence $\varphi(x^\Delta(T)) = \varphi(x^\Delta(0))$. It follows that x may be extended in a T-periodic fashion to a solution of (1.1) with $s = s(x)$. In other words,

$$\mathcal{I}(p_0) = \{s(x) : x \text{ is a solution of (3.1) for some } r \in [0, 2\pi]\} \neq \emptyset.$$

Moreover, it is clear from definition that $|s(x)| \leq b$, so $\mathcal{I}(p_0) \subset [-b, b]$.

Step 3 ($\mathcal{I}(p_0)$ is connected). Assume that $s_1, s_2 \in \mathcal{I}(p_0)$ are such that $s_1 < s_2$, and let x_1 and x_2 be T-periodic solutions of (1.1) for s_1 and s_2, respectively. Then for any $s \in (s_1, s_2)$ it is verified that x_1 and x_2 are strict upper and a lower solutions of (1.1), respectively. Replacing x_1 by $x_1 + 2k\pi$, with k the first integer such that $x_2 < x_1 + 2k\pi$ and applying Theorem 2.2 with $\alpha = x_2$ and $\beta = x_1 + 2k\pi$, we conclude that problem (1.1) has at least one T-periodic solution, whence $s \in \mathcal{I}(p_0)$.

Step 4 ($\mathcal{I}(p_0)$ is closed). Let $\{s_n\} \subset \mathcal{I}(p_0)$ converge to some s, and let $x_n \in C_T^1$ be a solution of (1.1) for s_n. Without loss of generality, we may assume that $x_n(0) \in [0, 2\pi]$. Because $\|x_n^\Delta\|_\infty < c$, by Arzelà–Ascoli theorem there exists a subsequence (still denoted $\{x_n\}$) that converges uniformly to some x. Furthermore, from (1.1) we deduce the existence of a constant C independent of n such that $|(\varphi(x_n^\Delta(t)))^\Delta| \leq C$ for all t. We claim that $\varphi(x_n^\Delta)$ is also uniformly bounded, that is, $\|x_n^\Delta\|_\infty$ is bounded away from c. Indeed, otherwise passing to a subsequence we may suppose for example that $\varphi(x_n^\Delta)_{\max} \to +\infty$. Because $\varphi(x_n^\Delta(t_1)) - \varphi(x_n^\Delta(t_0)) \leq C(t_1 - t_0)$ for all $t_1 > t_0$, we deduce from periodicity that $\varphi(x_n^\Delta)_{\max} - \varphi(x_n^\Delta)_{\min} \leq CT$ and, consequently, $\varphi(x_n^\Delta)_{\min} \to +\infty$. This implies that $(x_n^\Delta)_{\min} \to c$, which contradicts the fact that x_n^Δ has zero average. Using Arzelà-Ascoli again, we may assume that $\varphi(x_n^\Delta)$ converges uniformly to some function v and, from the identity $x_n(t) = x_n(0) + \int_0^t x_n^\Delta(\xi) \Delta\xi$ we deduce that $x \in C_T^1$ and $x^\Delta = \varphi^{-1}(v)$. Now integrate the equation for each n and take limit for $n \to \infty$ to obtain

$$\varphi(x^\Delta(t)) = \varphi(x^\Delta(0)) + \int_0^t [s + p_0(\xi) - b\sin(x(\xi))] \Delta\xi - a[x(t) - x(0)].$$

In turn, this implies that x is a solution of (3.1) with $s(x) = s$; hence, $\mathcal{I}(p_0)$ is closed and the proof is complete.

Step 5 (continuous dependence on p_0). Let $\{p_0^n\}_{n \in \mathbb{N}} \subset \tilde{C}_T$ be a sequence that converges to some p_0. We shall prove that $D(p_0^n) \to D(p_0)$; the proof for d is analogous. Similarly to Step 4, it is seen that if a subsequence of $\{D(p_0^n)\}$ converges to some D, then the problem for p_0 with $s = D$ admits a solution and, consequently, $D \leq D(p_0)$. Thus, it suffices to prove that $\liminf_{n \to \infty} D(p_0^n) \geq D(p_0)$. Indeed, otherwise, passing to a subsequence we may suppose that $D(p_0^n) \to D < D(p_0)$. Fix $\eta > 0$ such that $D + \eta < D(p_0)$ and let x be a T-periodic solution of (1.1) for $s = D(p_0)$. Take n large enough such that

$$p_0(t) + D(p_0) > p_0^n(t) + D + \eta > p_0^n(t) + D(p_0^n) \qquad \forall t \in [0, T]_{\mathbb{T}}$$

and let x_n be a T-periodic solution of (1.1) for p_0^n and $s = D(p_0^n)$. The previous inequalities imply that x and x_n are respectively a lower and an upper solution of the problem for p_0^n and $s = D + \eta$ and, without loss of generality, we may assume that $x < x_n$. Thus, (1.1) has a T-periodic solution for p_0^n and $s = D + \eta > D(p_0^n)$, a contradiction. $\square$

The following theorem establishes the existence of at least two geometrically different T-periodic solutions to problem (1.1) when s is an interior point.

Theorem 3.2. *Assume that $p_0 \in C_T$ has zero average. If $s \in (d(p_0), D(p_0))$, then the problem (1.1) has at least two geometrically different T-periodic solutions.*

Proof. For $s \in (d(p_0), D(p_0))$, let $s_1 := d(p_0) < s < D(p_0) =: s_2$ and let x_1, x_2 be as in Step 3 of the previous proof. Then x_1 and x_2 are strict upper and lower solutions for s, respectively. Due to the 2π-periodicity of (1.1), we may assume that $x_2 < x_1$ and $x_2 + 2\pi \not\leq x_1$ and, consequently, Ω_{x_2,x_1} and $\Omega_{x_2+2\pi,x_1+2\pi}$ are disjoint open subsets of $\Omega_{x_2,x_1+2\pi}$. From Theorem 2.2 and the excision property of the Leray–Schauder degree, we deduce the existence of three different solutions $y_1, y_2, y_3 \in C_T^1$ such that

$$x_2(t) < y_1(t) < x_1(t),$$
$$x_2(t) + 2\pi < y_2(t) < x_1(t) + 2\pi$$
$$x_2(t) < y_3(t) < x_1(t) + 2\pi$$

for all $t \in \mathbb{T}$. If $y_2 = y_1 + 2\pi$, then $y_3 \neq y_1, y_1 + 2\pi$ and the conclusion follows. $\square$

4 Sufficient conditions for $0 \in \mathcal{I}(p_0)$

In this section, we shall obtain conditions guaranteeing that 0 belongs to the solvability set. Even in the continuous case, this is not clear when $a \neq 0$ since, as it is well known, counter-examples exist for the classical pendulum equation for arbitrary periods. In the relativistic case, however, it was proved that $0 \in \mathcal{I}(p_0)$ when T is sufficiently small and counter-examples for large values of T are not yet known. Here, as mentioned in the introduction, we shall improve the bounds for T obtained in previous works for $\mathbb{T} = \mathbb{R}$. The results shall be expressed in terms of $k(\mathbb{T})$, the optimal constant of the inequality

$$\|x - \bar{x}\|_\infty \leq k\|x^\Delta\|_\infty, \qquad x \in C_T^1.$$

For instance, for arbitrary $\mathbb{T}$ it is readily seen that $k(\mathbb{T}) \leq \frac{T}{2}$, because x^Δ has zero average and hence, due to periodicity,

$$x_{\max} - x_{\min} \leq \int_{t_{\min}}^{t_{\max}} [x^\Delta(t)]^+ \Delta t \leq \int_0^T [x^\Delta(t)]^+ \Delta t = \frac{1}{2} \int_0^T |x^\Delta(t)| \Delta t.$$

We recall that, in the continuous case, the (optimal) Sobolev inequality $\|x - \bar{x}\|_\infty \leq \sqrt{\frac{T}{12}} \|x'\|_2$ implies that $k(\mathbb{R}) \leq \frac{T}{2\sqrt{3}}$.

The main result of this section reads as follows.

Theorem 4.1. *Assume that $ck(\mathbb{T}) < \pi$ and define the function*

$$\psi(\delta) := 2\delta \cos(\delta) + (cT - 2\delta) \cos(ck(\mathbb{T})).$$

If $\psi(\delta) \geq 0$ for some $\delta \in (0, \frac{\pi}{2})$, then $0 \in \mathcal{I}(p_0)$. Furthermore, if the previous inequality is strict, then $0 \in \mathcal{I}(p_0)^\circ$.

Before proceeding to the proof, it is worth to recall that, from Theorem 4.1 and Example 5.3 in [2], in order to prove the existence of T-periodic solutions for $s = 0$ it suffices to verify that the equation

$$\left(\frac{x^\Delta(t)}{\sqrt{1 - \frac{x^\Delta(t)^2}{c^2}}} \right)^\Delta = \lambda[p_0(t) - ax^\Delta(t) - b \sin x(t)] \tag{4.1}$$

has no T-periodic solutions with average $\pm\frac{\pi}{2}$. For example, if $x \in C_T^1$ is a solution of (4.1) such that $\bar{x} = \frac{\pi}{2}$, then it follows from the definition of $k(\mathbb{T})$ that, for all $t \in \mathbb{T}$,

$$\left| x(t) - \frac{\pi}{2} \right| \leq ck(\mathbb{T}).$$

In particular, if $ck(\mathbb{T}) \leq \frac{\pi}{2}$, then $x(t) \in [0, \pi]$ for all $t \in \mathbb{T}$ and, upon integration of equation (4.1), we deduce:

$$0 = b \int_0^T \sin(x(t)) \Delta t > 0.$$

The same contradiction is obtained also if $\bar{x} = -\frac{\pi}{2}$. For example, the condition $cT \leq \pi$ is sufficient for arbitrary $\mathbb{T}$ and, in the continuous case, the condition $cT \leq \sqrt{3}\pi$ is retrieved. However, the previous bound $ck(\mathbb{T}) \leq \frac{\pi}{2}$ can be improved, as we shall see in the following proof.

Proof of Theorem 4.1. From the preceding discussion, it may be assumed that $\frac{\pi}{2} < ck(\mathbb{T}) < \pi$. Suppose that x is a solution of (4.1) such that $\bar{x} = \frac{\pi}{2}$, then

$$x(t) \in \left[\frac{\pi}{2} - ck(\mathbb{T}), \frac{\pi}{2} + ck(\mathbb{T}) \right] \subset \left(-\frac{\pi}{2}, \frac{3\pi}{2} \right)$$

for all $t \in \mathbb{T}$ and hence

$$\sin x(t) \geq -\sin(A) > -1, \quad \text{where } A = ck(\mathbb{T}) - \frac{\pi}{2}.$$

Fix $\delta \in (0, \frac{\pi}{2})$ and consider the set

$$C_\delta = \left\{ t \in [0, T]_\mathbb{T} : \left| x(t) - \frac{\pi}{2} \right| \leq \delta \right\}.$$

Then

$$
\begin{aligned}
0 = \int_0^T \sin(x(t))\Delta t &\geq \int_{C_\delta} (\sin(x(t)) + \sin(A))\Delta t - T \sin(A) \\
&\geq \left[\sin(\frac{\pi}{2} - \delta) + \sin(A)\right] \mathfrak{m}(C_\delta) - T \sin(A) \\
&= \cos(\delta)\mathfrak{m}(C_\delta) - [T - \mathfrak{m}(C_\delta)] \sin A,
\end{aligned} \tag{4.2}
$$

where $\mathfrak{m}(C_\delta)$ is the measure of the set C_δ associated to the Δ-integral, namely $\mathfrak{m}(C_\delta) = \int_{C_\delta} \Delta t$. Clearly, a contradiction is obtained when the latter term of (4.2) is positive.

Moreover, notice that if $x(t_0) \leq \frac{\pi}{2}$ and $t_1 > t_0$ is such that $x(t_1) \geq \frac{\pi}{2} + \delta$, then

$$
\delta \leq x(t_1) - x(t_0) = \int_{t_0}^{t_1} x^\Delta(s)\Delta s < c(t_1 - t_0).
$$

In the same way, if $t_0 < t_1$ are such that $x(t_0) \geq \frac{\pi}{2}$ and $x(t_1) \leq \frac{\pi}{2} - \delta$, then $c(t_1 - t_0) > \delta$. Thus, by periodicity, we deduce that $\mathfrak{m}(C_\delta) > \frac{2\delta}{c}$. The same conclusions are obtained if $\bar{x} = -\frac{\pi}{2}$; hence, a sufficient condition for the existence of at least one T-periodic solution is that, for some $\delta \in (0, \frac{\pi}{2})$,

$$
\cos(\delta)\frac{2\delta}{c} \geq \left(T - \frac{2\delta}{c}\right) \sin A
$$

or, equivalently, that $\psi(\delta) \geq 0$. Note, furthermore, that if the inequality is strict, then a contradiction is still obtained as in (4.2) if we add a small parameter s to the function p_0 in (4.1). $\qquad \square$

Remark 4.2. It is seen that ψ reaches its maximum at the unique $\delta^* \in (0, \frac{\pi}{2})$ such that

$$
\cos(\delta^*) - \delta^* \sin(\delta^*) = \cos(ck(\mathbb{T})). \tag{4.3}
$$

Thus, replacing (4.3) in ψ, a somewhat explicit condition on T reads:

$$
2(\delta^*)^2 \sin(\delta^*) + cT \cos(ck(\mathbb{T})) \geq 0.
$$

An immediate corollary is the following:

Corollary 4.3. *There exists a constant T^* with $cT^* > \pi$ such that $0 \in \mathcal{I}(p_0)$ for all $p_0 \in \tilde{C}_T$ if $T \leq T^*$ and it is an interior point if $T < T^*$. For the particular case $\mathbb{T} = \mathbb{R}$, it is verified that $cT^* > \sqrt{3}\pi$.*

Proof. For arbitrary $\mathbb{T}$, we know already that $k(\mathbb{T}) \leq \frac{T}{2}$, then a sufficient condition when $cT \in (\pi, 2\pi)$ is the existence of $\delta \in (0, \frac{\pi}{2})$ such that $\Psi(\delta, T) \geq 0$, where

$$
\Psi(\delta, T) := 2\delta \cos(\delta) + (cT - 2\delta) \cos\left(\frac{cT}{2}\right).
$$

The result now follows trivially from the fact that $\Psi(\delta, \frac{\pi}{c}) = 2\delta \cos(\delta)$. The proof is similar for $\mathbb{T} = \mathbb{R}$, now taking

$$
\Psi_{cont}(\delta, T) := 2\delta \cos(\delta) + (cT - 2\delta) \cos\left(\frac{cT}{2\sqrt{3}}\right). \qquad \square
$$

Remark 4.4. A more quantitative version of the previous corollary follows from the fact that the function Ψ is strictly decreasing with respect to T when $cT \in (\pi, 2\pi)$ and arbitrary $\delta \in (0, \frac{\pi}{2})$. In particular, observe that if $\Psi(\delta, \hat{T}) \geq 0$ for some $\hat{T} \in (\frac{\pi}{c}, \frac{2\pi}{c})$ and some $\delta \in (0, \frac{\pi}{2})$, then $\Psi(\delta, T) > 0$ for $T \in (\frac{\pi}{c}, \hat{T})$. Thus, a lower bound for T^* is given by the unique value of $T \in (\frac{\pi}{c}, \frac{2\pi}{c})$ such that

$$\max_{\delta \in [0, \frac{\pi}{2}]} \Psi(\delta, T) = 0.$$

Analogous conclusions are obtained when $\mathbb{T} = \mathbb{R}$ using Ψ_{cont} instead of Ψ.

4.1 Numerical examples and final remarks

As shown in Corollary 4.3, the bound thus obtained always improves the simpler one $ck(\mathbb{T}) \leq \frac{\pi}{2}$ and, in particular, it guarantees that if the latter inequality is satisfied then 0 is in fact an interior point of $\mathcal{I}(p_0)$. In the continuous case, an easy numerical computation gives the sufficient condition $cT \leq 6.318$, slightly better than the bound $cT \leq 2\pi$ deduced from [10] (see Figure 1). For arbitrary $\mathbb{T}$, numerical experiments show that $0 \in \mathcal{I}(p_0)^\circ$ for $cT \leq 4.19$, as shown in Figure 2.
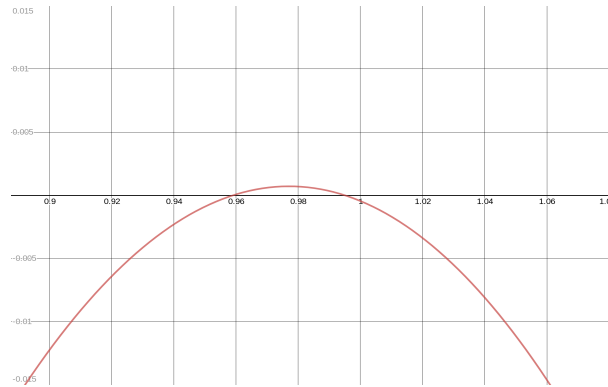

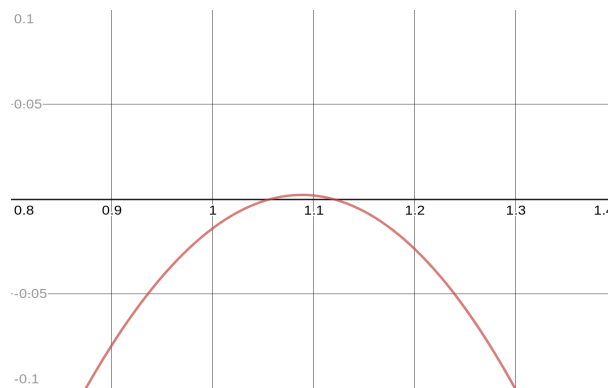Figure 4.1: Graph of ψ for $\mathbb{T} = \mathbb{R}$ with $cT = 6.318$

Figure 4.2: Graph of ψ for $k(\mathbb{T}) = \frac{T}{2}$ and $cT = 4.19$

Remark 4.5. An estimation of the constant $k(\mathbb{T})$ could be obtained analogously to the continuous case as shown for example in [13]. Let $\{e_n\}_{n \in \mathbb{Z}} \subset C_T$ be an orthonormal basis of $L^2(0, T)_{\mathbb{T}}$

with $e_0 \equiv \frac{1}{\sqrt{T}}$ and E_n be a primitive of e_n such that $\overline{E}_n = 0$. Writing $x^\Delta = \sum_{n \neq 0} a_n e_n$, it follows that

$$\|x - \overline{x}\|_\infty = \left| \sum_{n \neq 0} a_n E_n \right| \leq \|x^\Delta\|_{L^2} \sqrt{\sum_{n \neq 0} \|E_n\|_\infty^2} \leq \|x^\Delta\|_\infty \sqrt{T \sum_{n \neq 0} \|E_n\|_\infty^2}.$$

When $\mathbb{T} = \mathbb{R}$, taking the usual Fourier basis one has that $\|E_n\|_\infty = \frac{\sqrt{T}}{2\pi n}$ and the value $k(\mathbb{R}) \leq \frac{T}{2\sqrt{3}}$ is obtained from the well known equality $\sum_{n \in \mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Remark 4.6. As mentioned in the introduction, Theorem 4.1 allows to compute an inferior bound for the length of the solvability interval which does not depend on p_0, provided that T is small enough. In some obvious cases, inferior bounds are obtained for arbitrary T: for example, if $\|p_0\|_\infty < b$ then $[-\varepsilon, \varepsilon] \subset \mathcal{I}(p_0)$ for $\varepsilon = b - \|p_0\|_\infty$. This is readily verified taking $\alpha = \frac{\pi}{2}$ and $\beta = \frac{3\pi}{2}$ as lower and upper solutions.

Acknowledgements

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