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Dealing with Similarity in Argumentation

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Résumé

Le raisonnement argumentatif est basé sur la justification d'une conclusion plausible par des arguments en sa faveur. L'argumentation est un modèle prometteur pour raisonner avec des connaissances incertaines ou incohérentes, ou, plus généralement de sens communs. Ce modèle est basé sur la construction d'arguments et de contre-arguments, la comparaison de ces arguments et enfin l'évaluation de la force de chacun d'entre eux.

Dans cette thèse, nous avons abordé la notion de similarité entre arguments. Nous avons étudié deux aspects : comment la mesurer et comment la prendre en compte dans l'évaluation des forces.

Concernant le premier aspect, nous nous sommes intéressés aux arguments logiques, plus précisément à des arguments construits à partir de bases de connaissances propositionnelles. Nous avons commencé par proposer un ensemble d'axiomes qu'une mesure de similarité entre des arguments logiques doit satisfaire. Ensuite, nous avons proposé différentes mesures et étudié leurs propriétés.

La deuxième partie de la thèse a consisté à définir les fondements théoriques qui décrivent les principes et les processus impliqués dans la définition d'une méthode d'évaluation des arguments prenant en compte la similarité. Une telle méthode calcule la force d'un argument sur la base de forces de ses attaquants, des similarités entre eux, et d'un poids initial de l'argument. Formellement, une méthode d'évaluation est définie par trois fonctions dont une, nommée "fonction d'ajustement", qui s'occupe de réajuster les forces des attaquants en fonction de leur similarité. Nous avons proposé des propriétés que doivent satisfaire les trois fonctions, ensuite nous avons défini une large famille de méthodes et étudié leurs propriétés. Enfin, nous avons défini différentes fonctions d'ajustement, montrant ainsi que différentes stratégies peuvent être suivies pour contourner la redondance pouvant exister entre les attaquants d'un argument.

Mots clefs: Argumentation Abstraite, Argumentation Logique, Similarité, Sémantiques Graduelles

Abstract

ARGUMENTATIVE reasoning is based on justifying a plausible conclusion with arguments in its favour. Argumentation is a promising model for reasoning with uncertain or inconsistent knowledge, or, more generally, common sense. This model is based on the construction of arguments and counter-arguments, the comparison of these arguments and finally the evaluation of the strength of each of them.

In this thesis, we have tackled the notion of similarity between arguments. We have studied two aspects: how to measure it and how to take it into account in the evaluation of strengths.

With regards to the first aspect, we were interested in logical arguments, more precisely in arguments built from propositional knowledge bases. We started by proposing a set of axioms that a similarity measure between logical arguments must satisfy. Then, we proposed different measures and studied their properties.

The second part of the thesis was focused on defining the theoretical foundations that describe the principles and processes involved in the definition of an evaluation method for arguments, which takes similarity into account. Such a method computes the strength of an argument based on the strengths of its attackers, the similarities between them, and an initial weight of the argument. Formally, an evaluation method is defined by three functions, one of which (called the adjustment function) is concerned with readjusting the strengths of the attackers according to their similarity. We have proposed properties that the three functions must satisfy, after which we have defined a large family of methods and studied their properties. At last, we have defined different adjustment functions, showing that different strategies can be applied to avoid the redundancy that can exist between the attackers of an argument.

Keywords: Abstract Argumentation, Logical Argumentation, Similarity, Gradual Semantics

Contents

Introduction

Argumentation is a reasoning approach based on the justification of claims by arguments, i.e. reasons for accepting claims. It has received great interest from the Artificial Intelligence community since late 1980s, namely as a unifying approach for nonmonotonic reasoning (Lin and Shoham [1989]). It was later used for solving different other problems like reasoning with inconsistent information (eg. Simari and Loui [1992]; Besnard and Hunter [2001]), decision making (eg. Zhong *et al.* [2019]), classification (eg. Amgoud and Serrurier [2008]), negotiation (Sycara [1990]; Hadidi *et al.* [2010]), etc. Argumentation has also several practical applications, namely in legal and medical domains (see Atkinson *et al.* [2017] for more applications).

Whatever the problem to be solved, an argumentation process follows generally four main steps:

1. justify claims by arguments,
2. identify (attack, support) relations between arguments,
3. evaluate the strength of arguments, and
4. define an output.

The last step depends on the results of the evaluation. For instance, an inference system draws formulas that are justified by what is qualified at the evaluation step as "strong" arguments. Evaluation of arguments is thus crucial as it impacts the outcomes of argument-based systems. Consequently, a plethora of methods, called semantics, have been proposed in the literature.

The very first ones are extension-based (Dung [1995]) and the recent ones are gradual semantics (Cayrol and Lagasquie-Schiex [2005]) that quantify strength and ascribe a value (representing strength) to every argument. Both families of semantics may take into account attacks and/or supports between arguments, weights on arguments, which can represent votes (Leite and Martins [2011]) or certainty degrees (Benferhat *et al.* [1993]), weights on links between arguments, which can represent relevance (Dunne *et al.* [2011]) or again votes of users (Egilmez *et al.* [2013]). However, none of the existing semantics is able to handle similarity between arguments.

Similarity is related to commonality, in that the more commonality two arguments share, the more similar they are. In practice, existence of similarity is inevitable as arguments generally share information like the following two ones that are exchanged during a dialogue between two people who want to buy a house.

A_1: The house h1 is good since it has a big garden.

A_2: The house h1 is better than the house h2 since its garden has enough space for planting various fruit trees.

The two arguments (A_1, A_2) are quite similar since they have the same evidence (the garden being big) but different conclusions.

In this thesis, we tackled two main research questions:

- How to measure similarity between two arguments?
- What is the impact of similarity on the evaluation of arguments? And how to define semantics that are able to deal properly with similarity?

Focusing on logical arguments, i.e, arguments built from propositional knowledge bases, we defined the notion of similarity measure, as well as a set of principles that a measure should satisfy. Some principles describe rational behavior of a measure while others are about the origin of similarity between arguments. As a second contribution, we extended in various ways existing measures from the literature, namely the well-known Jaccard measure (Jaccard [1901]), Dice measure (Dice [1945]), Sorensen one (Sørensen [1948]), and their other refinements proposed in (Anderberg [1973]; Sneath *et al.* [1973]; Ochiai [1957]; Kulczynski [1927]), and studied their properties.

Regarding the second research question, we have shown that ignoring (total or partial) similarities would lead to inaccurate evaluations of arguments, and thus to wrong recommendations by argumentation systems. Hence, developing semantics that are able to take into account similarity is crucial for discarding any redundancy.

In the second part of the thesis, we discussed theoretical foundations that describe principles and processes involved in the definition of gradual semantics that deal with similarity, and we proposed a general setting for defining systematically such semantics.

A semantics computes the strength of an argument on the basis of the strengths of its attackers, similarities between those attackers, and an initial weight ascribed to the argument. It is defined using three functions:

- an adjustment function that updates the strengths of attackers on the basis of their similarities,

- an aggregation function that computes the strength of the group of attackers, and
- an influence function that evaluates the impact of the group on the argument's initial weight.

We proposed intuitive constraints for the three functions and key rationality principles for semantics, and showed how the former lead to the satisfaction of the latter. Then, we proposed a broad family of semantics whose instances satisfy the principles. Finally, we proposed various adjustment functions and analysed their properties.

The document is organised as follows:

1. In **Chapter 1 (Background)**, the required knowledge to understand our work is presented. We introduce the elements of an argumentation framework and the different families of semantics existing in the literature. Then, we explain the importance of dealing with the notion of similarity in the argumentation process. Finally, we discuss existing works on similarity.
2. In **Chapter 2 (Similarity Measures for Logical Arguments)**, we propose a set of principles that a similarity measure between logical arguments should satisfy. Furthermore, we highlight the problem of non-concise arguments which disturbs the assessment of the degree of similarity. Then, different families of similarity measures for concise and non-concise arguments are proposed and analyzed according to the principles.
3. In **Chapter 3 (Graduated Semantics dealing with similarity)**, we present a general setting composed by three functions defining gradual semantics dealing with a similarity measure. We propose also a set of principles for these three functions. Then we focus on the novel adjustment function with some instantiations and analyses.
4. In **Chapter 4 (Conclusions)**, we summarize and highlight the important contributions of this thesis, to finally give several possible directions to extend this research.
5. Finally, in **Chapter 5 (Appendix)**, there is the proofs related to the contributions in chapters 3 and 4.

Chapter 1

Background

Contents

THIS first chapter presents the different useful notions to the comprehension of this document. We will first see what an argumentation framework consists of, then how to evaluate arguments, and finally we will introduce in more details the problematic of this thesis, which is how to integrate the notion of similarity to this framework and these evaluation methods.

1.1 Argumentation Frameworks

An argumentation framework is composed of arguments, relations between these arguments, preferences between arguments and also between relations.

1.1.1 Arguments

The backbone of an argumentation framework, is the notion of argument. An argument is a reason for believing or accepting a given claim. It is made of three elements:

1. A set of premises, that are intended to support the claim.

2. A conclusion, which is the claim being justified by the premises.

3. A link that allows the conclusion to follow from the premises.

There are at least two families of arguments depending on the nature of the link relating the conclusion to the premises:

- Inductive arguments such that the truth of their premises makes the truth of their conclusion more or less likely.

- Deductive arguments such that the truth of their premises guarantees the truth of their conclusion.

Example 1. *An example of an inductive and an deductive argument is the following:*

- *Inductive: Joe will win the elections (conclusion), because 52% of the sampled voters said they will vote for Joe (premise).*

- *Deductive: Joe has DNA because Joe is human and humans have DNA.*

Note that the conclusions of deductive arguments are more likely than those of inductive arguments as their links are much stronger. However, this is not sufficient for a deductive argument to be strong. It should additionally satisfy the following criteria:

- The link should be valid in the sense that if the premises are true, the conclusion cannot be false.

- The premises should be true.
- The premises should be relevant to the conclusion.

Example 2. *Consider the arguments A,B,C below:*

A: Karl is a philosopher because he has a brain and, every philosophers has a brain.

B: Harrison is brown because he is an actor and all actors are brown.

C: 1 + 1 = 2 because grass is green.

Note that the link in A is not valid, in B the premise "all actors are brown" is false and the premise of the argument C is completely irrelevant to its conclusion. Thus, the three arguments are weak.

1.1.2 Attacks and Supports

Such flaws in the arguments give birth to attacks. Indeed, an argument may attack another argument, undermining thus one of its components (premises, conclusion, link). This attack relation is clearly negative as it may be harmful for the attacked argument.

Example 2 (Cont.) *For instance, the following argument D attacks B:*

D: All actors are not brown since Brad is an actor and is not brown.

This argument claims that the premise "all actors are brown" is false and justifies this by an evidence "Brad is an actor and is not brown".

An argument may also support another arguments. The idea is to endorse the conclusion (respectively premises, link) of the argument. This support relation is a positive interaction, which increases the confidence of the supported argument.

Example 2 (Cont.) *For instance, the following argument E supports D:*

E: Brad is an actor and is not brown since he was blond in the movie "Seven".

This argument confirms the premise of the argument D.

The support relationship is an important notion in argumentative frameworks but not always necessary. For a first study taking into account similarity, we chose to start without this relation. Thus, in the rest of the document we will ignore it.

1.1.3 Extended Argumentation Frameworks

An argument may also have an initial weight that may represent different issues. It may represent a certainty degree Benferhat *et al.* [1993], a degree of trustworthiness of the source that provided the argument da Costa Pereira *et al.* [2011], an aggregation of votes provided by users Leite and Martins [2011], an importance degree of the values promoted by the arguments Bench-Capon [2003], etc.

Like arguments, attack relations may also be weighted. In Dunne *et al.* [2011], different reasons for assigning weights to attacks have been discussed. The most prominent ones are relevance, i.e., a weight of an attack expresses to what extent the source of the attack is relevant to the attacked argument. In Egilmez *et al.* [2013], a weight of an attack represents an aggregation of votes provided by users.

An argumentation framework ($\mathbf{AF}$), called also argumentation graph, is a tuple made of a finite set of arguments, an attack relation, an initial weight of each argument, and a weight of every attack. It is represented as a graph whose nodes are the arguments, and edges are the attack relation. Before introducing formally the notion of an argumentation framework, let us first introduce the notion of Weighting.

Definition 1 (Weighting)**.** *A weighting on a set X is a function from X to [0,1].*

It is worthy to recall that other scales can be used. But, for the sake of simplicity, throughout the document we use the unit interval $[0, 1]$.

Let us now introduce the notion of weighted argumentation framework or weighted argumentation graph. Let $\texttt{Arg}$ denote the universe of all possible arguments.

Definition 2 (Weighted argumentation framework)**.** *A Weighted argumentation framework is an ordered tuple $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$, where*

- $\mathcal{A} \subseteq_f$[1] $\texttt{Arg}$ *($\mathcal{A}$ being a non-empty finite subset of* $\texttt{Arg}$*),*
- $\mathbf{w}$ *is a weighting on $\mathcal{A}$ (it assigns initial weights to arguments),*
- $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ *(it is an attack relation), and*
- σ *is a weighting on $\mathcal{R}$ (it assigns weights to attacks).*

Let $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ be a weighted argumentation framework and $A, B \in \mathcal{A}$. The notation $(B, A) \in \mathcal{R}$ means B attacks A or B is an attacker of A, and $\texttt{Att}(A)$ denotes

[1] The notation $\mathcal{A} \subseteq_f \texttt{Arg}$ stands for: $\mathcal{A}$ is a finite subset of $\texttt{Arg}$.

the set of all attackers of A, i.e., $\mathtt{Att}(A) = \{B \in \mathcal{A} \mid (B, A) \in \mathcal{R}\}$.
$\mathbf{w}(A)$ is the initial weight of A and $\sigma((A, B))$ is the weight of $(A, B) \in \mathcal{R}$.
If $\forall A \in \mathcal{A}$, $\mathbf{w}(A) = 1$, we write $\mathbf{w} \equiv 1$, and if $\forall r \in \mathcal{R}$, $\sigma(r) = 1$, we write $\sigma \equiv 1$.

Definition 3 (Semi-weighted and flat $\mathbf{AF}$). *Let $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ be a weighted argumentation framework.*

- *If $\sigma \equiv 1$, then $\mathbf{AF}$ is called semi-weighted.*
- *If $\mathbf{w} \equiv 1$ and $\sigma \equiv 1$, then $\mathbf{AF}$ is called flat.*

Let us illustrate the above notions by a simple example.

Example 3. *Consider a debate on how to reduce a country's debts.*

A: *Increasing taxes, decreasing financial market borrowing and allowing government to finance itself through money creation, reduce the country's debt.*

B_1: *For a better living standards for all, taxes must not be increased.*

B_2: *To improve the quality of life, taxes must not be increased.*

B_3: *For a better healthcare and social justice, taxes must not be increased.*

B_4: *The purpose of borrowing is to prevent the inflation caused by money creation, therefore decreasing financial market borrowing and allowing government to finance itself through money creation do not imply a reduction of the debt.*

This debate is represented by the graph $\mathbf{G}_1$ below.

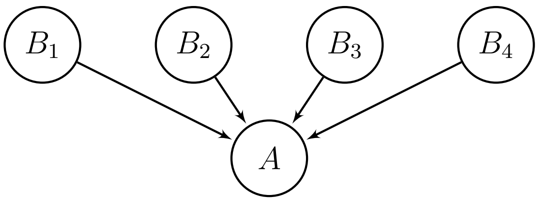


Figure 1.1: Argumentation graph $\mathbf{G}_1$

Suppose that each argument is given an initial weight based on a voting score. For instance:

$$\mathbf{w}(A) = 0.7,\ \mathbf{w}(B_1) = 0.3,\ \mathbf{w}(B_2) = 0.3,\ \mathbf{w}(B_3) = 0.4,\ \mathbf{w}(B_4) = 0.5$$

Assume also that the weight of an attack expresses a degree of relevance, and

$$\sigma(B_1, A) = 0.6,\ \sigma(B_2, A) = 0.6,\ \sigma(B_3, A) = 0.6,\ \sigma(B_4, A) = 0.9$$

Let us now introduce the useful notion of path in graph.

Definition 4 (Path)**.** *Let* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ *be a weighted argumentation framework and* $A, B \in \mathcal{A}$*. A path from* A *to* B *is a finite non-empty sequence* $\langle X_1, \cdots, X_n \rangle$ *such that* $X_1 = A$*,* $X_n = B$ *and* $\forall i < n$*,* $(X_i, X_{i+1}) \in \mathcal{R}$*.*

We also define an isomorphism between two weighted argumentation frameworks as follows.

Definition 5 (Isomorphism of weighted argumentation frameworks)**.** *Let* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ *and* $\mathbf{AF}' = \langle \mathcal{A}', \mathbf{w}', \mathcal{R}', \sigma' \rangle$ *be two weighted argumentation graphs. An isomorphism from* $\mathbf{AF}$ *to* $\mathbf{AF}'$ *is a bijective function* $\mathbf{f}$ *from* $\mathcal{A}$ *to* $\mathcal{A}'$ *such that: i)* $\forall A \in \mathcal{A}$*,* $\mathbf{w}(A) = \mathbf{w}'(\mathbf{f}(A))$*, ii)* $\forall A, B \in \mathcal{A}$*,* $(A, B) \in \mathcal{R}$ *iff* $(\mathbf{f}(A), \mathbf{f}(B)) \in \mathcal{R}'$*, and* $\sigma((A, B)) = \sigma'((\mathbf{f}(A), \mathbf{f}(B)))$*.*

Finally, we define the merging of two argumentation frameworks as follows.

Definition 6 (Merging of weighted argumentation frameworks)**.** *Let* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ *and* $\mathbf{AF}' = \langle \mathcal{A}', \mathbf{w}', \mathcal{R}', \sigma' \rangle$ *be two weighted argumentation frameworks.* $\mathbf{AF} \oplus \mathbf{AF}'$ *denotes the weighted argumentation graph* $\langle \mathcal{A} \cup \mathcal{A}', \mathbf{w}'', \mathcal{R} \cup \mathcal{R}', \sigma'' \rangle$ *such that* $\mathcal{A} \cap \mathcal{A}' = \emptyset$*,* $\forall A \in \mathcal{A}$ *(resp.* $A \in \mathcal{A}'$*),* $\mathbf{w}''(A) = \mathbf{w}(A)$ *(resp.* $\mathbf{w}''(A) = \mathbf{w}'(A)$*), and* $\forall (A, B) \in \mathcal{R}$ *(resp.* $(A, B) \in \mathcal{R}'$*),* $\sigma''((A, B)) = \sigma((A, B))$ *(resp.* $\sigma''((A, B)) = \sigma'((A, B))$*).*

1.2 Evaluation of Arguments

Evaluation of argument strength is a key step in any argument based system. Consequently, several methods, called semantics, have been proposed in the literature. In this section, we recall briefly the existing semantics.

1.2.1 Families of Semantics

In the literature, semantics can be partitioned into three families: Extension-based, Gradual and Ranking-based.
Extension-based semantics have been introduced for the first time by Dung [1995], in is seminal paper. The idea behind those semantics is to look for sets of acceptable arguments, called extensions. Then, a dialectical status is assigned to every argument.
Gradual semantics have been proposed for the first time by Cayrol and Lagasquie-Schiex [2005]. They focus on individual arguments, and ascribe to every argument a value taken from an ordered scale.
The third family of ranking-based semantics, has been proposed by Amgoud and Ben-Naim [2013]. The idea is to rank-order arguments from the strongest to the weakest.

1.2.1.1 Extension-based Semantics

Extension-based semantics have been initially defined for flat graphs, then extended to semi-weighted and then to weighted ones. In what follows, we recall the initial case. These semantics are based on two key concepts: conflict-freeness and defence.

Definition 7 (Conflict-freeness, Defence)**.** *Let* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w} \equiv 1, \mathcal{R}, \sigma \equiv 1 \rangle$ *be a flat argumentation framework and* $\mathcal{E} \subseteq \mathcal{A}$ *be a set of arguments.*

- $\mathcal{E}$ *is conflict-free iff* $\nexists A, B \in \mathcal{E}$ *such that* $(A, B) \in \mathcal{R}$*.*
- $\mathcal{E}$ *defends an argument* A *iff for all* $B \in \mathcal{A}$ *such that* $(B, A) \in \mathcal{R}$*, there exists* $C \in \mathcal{E}$ *such that* $(C, B) \in \mathcal{R}$*.*

The following definition recalls the semantics proposed by Dung [1995]. Note that other semantics refining them have been proposed in the literature. However, we do not need to recall them since they are not investigated in this thesis.

Definition 8 (Extension-based semantics)**.** *Let* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w} \equiv 1, \mathcal{R}, \sigma \equiv 1 \rangle$ *be a flat argumentation framework and* $\mathcal{E} \subseteq \mathcal{A}$ *is conflict-free.*

- $\mathcal{E}$ *is an admissible extension iff it defends all its elements.*
- $\mathcal{E}$ *is a complete extension iff it defends its elements and contains all the arguments that it defends.*
- $\mathcal{E}$ *is a grounded extension iff it is the minimal (for set inclusion[2]) complete extension.*
- $\mathcal{E}$ *is a preferred extension iff it is a maximal (for set inclusion) admissible extension.*
- $\mathcal{E}$ *is a stable extension iff it is a preferred extension that attacks any element in* $\mathcal{A} \setminus \mathcal{E}$*.*

Let $\mathtt{Ext}_x(\mathbf{AF})$ *denote the set of extensions of* $\mathbf{AF}$ *under a given semantics* x*. Let* $\mathtt{Ext}_{ad}(\mathbf{AF})$*,* $\mathtt{Ext}_{co}(\mathbf{AF})$*,* $\mathtt{Ext}_{gr}(\mathbf{AF})$*,* $\mathtt{Ext}_{pr}(\mathbf{AF})$ *and* $\mathtt{Ext}_{st}(\mathbf{AF})$ *stand respectively for the set of admissible, complete, grounded, preferred and stable extensions.*

[2] i.e. an extension included in other extensions, to not be confused with the intersection of arguments between extensions

Let us illustrate the different semantics using the following example.

Example 4. *Let us consider the flat argumentation graph depicted below.*

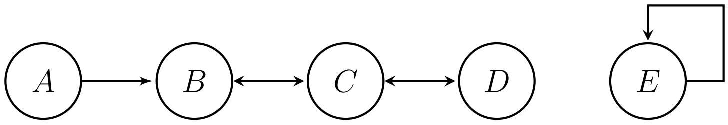


Figure 1.2: Argumentation graph $\mathbf{G}_2$

According to each definition, the following extensions are obtained:

- $\texttt{Ext}_{ad}(\mathbf{G}_2) = \{\emptyset, \{A\}, \{C\}, \{D\}, \{A, C\}, \{A, D\}\}$,
- $\texttt{Ext}_{co}(\mathbf{G}_2) = \{\{A\}, \{A, C\}, \{A, D\}\}$,
- $\texttt{Ext}_{gr}(\mathbf{G}_2) = \{\{A\}\}$,
- $\texttt{Ext}_{pr}(\mathbf{G}_2) = \{\{A, C\}, \{A, D\}\}$,
- $\texttt{Ext}_{st}(\mathbf{G}_2) = \{\ \}$.

Once the extensions have been computed, we use them to define dialectical status or acceptability status of each argument. In the literature there are different notions, we recall below the one from Cayrol and Lagasquie-Schiex [2005].

Definition 9 (Acceptability status). *Let* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w} \equiv 1, \mathcal{R}, \sigma \equiv 1 \rangle$ *be a flat argumentation framework and* $\texttt{Ext}_x(\mathbf{AF})$ *its set of extensions under the semantics* x*. Let an argument* $A \in \mathcal{A}$*:*

- *A is sceptically accepted iff $\forall\, \mathcal{E} \in \texttt{Ext}_x(\mathbf{AF})$, $A \in \mathcal{E}$.*
- *A is credulously accepted iff $\exists\, \mathcal{E}$, $\mathcal{E}$' $\in \texttt{Ext}_x(\mathbf{AF})$ such that $A \in \mathcal{E}$ and $A \notin \mathcal{E}$'.*
- *A is rejected iff $\forall\, \mathcal{E} \in \texttt{Ext}_x(\mathbf{AF})$, $A \notin \mathcal{E}$ and $\exists\, \mathcal{E}' \in \texttt{Ext}_x(\mathbf{AF})$ such that $\mathcal{E}'$ attacks A.*
- *A is undecided iff $\forall\, \mathcal{E} \in \texttt{Ext}_x(\mathbf{AF})$, $A \notin \mathcal{E}$ and $\nexists\, \mathcal{E}' \in \texttt{Ext}_x(\mathbf{AF})$ such that $\mathcal{E}'$ attacks A.*

Example 4 (Cont.) *According to each semantics, the following results are obtained:*

- *Under admissible semantics, the arguments A, C, D are credulously accepted, B is rejected and E is undecided.*

- *Under the complete and prefered semantics, A is sceptically accepted, C and D are credulously accepted, B is rejected and E is undecided.*

- *Under the grounded semantics, A is sceptically accepted, B is rejected and C, D, E are undecided.*

- *Under the stable semantics, the five arguments A, B, C, D, E are undecided.*

1.2.1.2 Gradual Semantics

Introduced for the first time by Cayrol and Lagasquie-Schiex [2005], gradual semantics provide finer-grained evaluations of arguments. They are more discriminating between arguments than extension-based semantics. Furthermore, they focus directly on individual arguments, and assign to each of them a value taken from an ordered scale. For the sake of illustration, in what follows we consider the scale [0,1].

Definition 10 (Gradual semantics)**.** *A gradual semantics is a function* $\mathbf{S}$ *transforming any weighted argumentation framework* $\mathbf{AF}\langle\mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma\rangle$ *into a weighting* $\texttt{Str}^{\mathbf{S}}_{\mathbf{AF}}$ *on* $\mathcal{A}$*. For* $A \in \mathcal{A}$, $\texttt{Str}^{\mathbf{S}}_{\mathbf{AF}}(A)$ *denotes the strength of* A*.*

The first gradual semantics that has been defined in the literature is h-Categoriser (Besnard and Hunter [2001]). It was proposed for evaluating the degree of interaction between logical arguments in an acyclic graph. It was then used by Pu *et al.* [2014], as a semantics that evaluates argument strength in flat argumentation graphs that may contain cycles.

Definition 11 (h-Categoriser)**.** *h-Categoriser is a function* $\mathbf{S}_{\mathtt{h}}$ *transforming any flat argumentation graph* $\mathbf{AF} = \langle\mathcal{A}, \mathbf{w} \equiv 1, \mathcal{R}, \sigma \equiv 1\rangle$ *into a weighting* $\texttt{Str}^{\mathbf{S}_{\mathtt{h}}}_{\mathbf{AF}}$ *such that for every* $A \in \mathcal{A}$*,*

$$\texttt{Str}^{\mathbf{S}_{\mathtt{h}}}_{\mathbf{AF}}(A) = \frac{1}{1 + \sum\limits_{B \in \texttt{Att(A)}} \texttt{Str}^{\mathbf{S}_{\mathtt{h}}}_{\mathbf{AF}}(B)}$$

When $\texttt{Att}(A) = \emptyset$, $\sum\limits_{B \in \texttt{Att(A)}} \texttt{Str}^{\mathbf{S}_{\mathtt{h}}}_{\mathbf{AF}}(B) = 0$.

Let us illustrate this semantics using Example 4.

Example 4 (Cont.) *Consider the flat argumentation graph* $\mathbf{G}_2$*. It can be checked that:*

$$\texttt{Str}^{\mathbf{S}_{\mathtt{h}}}_{\mathbf{G}_2}(A) = 1 \quad \texttt{Str}^{\mathbf{S}_{\mathtt{h}}}_{\mathbf{G}_2}(B) = 0.403 \quad \texttt{Str}^{\mathbf{S}_{\mathtt{h}}}_{\mathbf{G}_2}(C) = 0.481$$

$$\texttt{Str}^{\mathbf{S}_{\mathtt{h}}}_{\mathbf{G}_2}(D) = 0.675 \quad \texttt{Str}^{\mathbf{S}_{\mathtt{h}}}_{\mathbf{G}_2}(E) = 0.618$$

This semantics has been extended by Amgoud *et al.* [2017] for dealing with weights on arguments, and by Amgoud and Doder [2019] for dealing with weights on arguments and weights on attack relations.

Definition 12 (Weighted h-Categoriser). *Weighted h-Categoriser is a function* $\mathbf{S}_{\mathtt{wh}}$ *transforming any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ *into a weighting* $\mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{AF}}$ *on* $\mathcal{A}$ *such that for every* $A \in \mathcal{A}$*,*

$$\mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{AF}}(A) = \begin{cases} \mathbf{w}(A) & \textit{iff } \mathtt{Att}(A) = \emptyset \\ \frac{\mathbf{w}(A)}{1 + \sum\limits_{B \in \mathtt{Att(A)}} \mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{AF}}(B) \times \sigma(B,A)} & \textit{else} \end{cases}$$

To illustrate this definition, we use Example 3 with $\mathbf{G}_1$.

Example 3 (Cont.) *Consider the weighted graph* $\mathbf{G}_1$ *and assume that* $\sigma \equiv 1$*, i.e., every attack has weight 1. Recall that:*

$$\mathbf{w}(A) = 0.7, \;\; \mathbf{w}(B_1) = 0.3, \;\; \mathbf{w}(B_2) = 0.3, \;\; \mathbf{w}(B_3) = 0.4, \;\; \mathbf{w}(B_4) = 0.5$$

It can be checked that:

$$\mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(A) = 0.28 \quad \mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(B_1) = 0.3 \quad \mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(B_2) = 0.3$$
$$\mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(B_3) = 0.4 \quad \mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(B_4) = 0.5$$

Note that every B_i *keeps its initial weight since it is not attacked. Furthermore, it has a negative impact on* A*.*

Now, consider the weight on the attacks, which are recalled below:

$$\sigma(B_1, A) = 0.6, \;\; \sigma(B_2, A) = 0.6, \;\; \sigma(B_3, A) = 0.6, \;\; \sigma(B_4, A) = 0.9$$

Hence:

$$\mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(A) = 0.34 \quad \mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(B_1) = 0.3 \quad \mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(B_2) = 0.3$$
$$\mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(B_3) = 0.4 \quad \mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}_1}(B_4) = 0.5$$

Obviously, the decrease in the weight of the attacks, increases the strength of A*.*

There are different other gradual semantics in the literature, some of them like Trust-based semantics (TB) from da Costa Pereira *et al.* [2011] considers only one attacker when evaluating the strength of an argument.

Definition 13 (Trust-based semantics)**.** *Trust-based semantics is a function* TB *transforming any semi-weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \equiv 1 \rangle$*, into a weighting* $\mathtt{Str}^{\mathtt{TB}}_{\mathbf{AF}}$ *on* $\mathcal{A}$ *such that*

$$\mathtt{Str}^{\mathtt{TB}}_{\mathbf{AF}}(A) = \lim_{n \to +\infty} \alpha_n(A)$$

where $\alpha_0(A) = \mathbf{w}(A)$ *and* $\alpha_{n+1}(A) = \frac{1}{2}\alpha_n(A) + \frac{1}{2}\mathtt{min}\{\mathbf{w}(A), 1 - \max_{(B,A)\in\mathcal{R}} \alpha_n(B)\}$.

The Trust-based semantics is guided by two principles. First, the strength $\alpha(A)$ of an argument A must not be greater than the strength to which the arguments attacking it are unacceptable. Second, its strength cannot be greater than its basic weight.

Example 3 (Cont.) *Consider the weighted graph* $\mathbf{G}_1$ *and assume that* $\sigma \equiv 1$*, i.e., every attack has weight 1.*

$$\mathtt{Str}^{\mathtt{TB}}_{\mathbf{G}_1}(A) = 0.5 \quad \mathtt{Str}^{\mathtt{TB}}_{\mathbf{G}_1}(B_1) = 0.3 \quad \mathtt{Str}^{\mathtt{TB}}_{\mathbf{G}_1}(B_2) = 0.3$$
$$\mathtt{Str}^{\mathtt{TB}}_{\mathbf{G}_1}(B_3) = 0.4 \quad \mathtt{Str}^{\mathtt{TB}}_{\mathbf{G}_1}(B_4) = 0.5$$

1.2.1.3 Ranking Semantics

The third family of semantics is the so-called ranking semantics which have been proposed by Amgoud and Ben-Naim [2013]. Unlike the two other families that compute numerical/qualitative strengths, these focus rather on ranking arguments w.r.t. their strengths from the strongest to the weakest. These semantics are useful in applications like decision making where a comparison of arguments is crucial.

Definition 14 (Ranking)**.** *A ranking on a set* X *is a binary relation* $\preceq$ *on* X *such that:* $\preceq$ *is total (i.e.,* $\forall A, B \in \mathtt{Arg}(\mathcal{L})$*,*$A \preceq B$ *or* $B \preceq A$*) and transitive (i.e.,* $\forall A, B, C \in \mathtt{Arg}(\mathcal{L})$*, if* $A \preceq B$ *and* $B \preceq C$*, then* $A \preceq C$*). Intuitively,* $A \preceq B$ *means that* B *is at least as acceptable as* A*. We will note* $A \prec B$ *when* $B \npreceq A$ *which means that* B *is strictly more acceptable than A and* $A \simeq B$ *when* $A \preceq B$ *and* $B \preceq A$ *which means that* A *and* B *are equally acceptable.*

Definition 15 (Ranking semantics)**.** *A ranking semantics is a function* $\mathbf{S}$ *transforming any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ *into a ranking* $\preceq^{\mathbf{S}}$ *on* $\mathcal{A}$*.*

Obviously every gradual semantics gives birth to a ranking one. However, the converse is not true as we can see into Burden-based semantics. Before defining this semantics let us first recall some useful notions, namely lexicographical order.

Definition 16 (Lexicographical order)**.** *A lexicographical order between two vectors of real numbers* $V = \langle V_1, \cdots, V_n \rangle$ *and* $V' = \langle V'_1, \cdots, V'_n \rangle$ *is defined as* $V' \preceq_{lex} V$ *iff*

$\forall j \in \{1, \cdots, n\}$, $V'_j \leq V_j$. $V \simeq_{lex} V'$ means that $V \preceq_{lex} V'$ and $V' \preceq_{lex} V$; and $V' \prec_{lex} V$ means that $V' \preceq_{lex} V$ and $V \not\preceq_{lex} V'$.

Example 4 (Cont.) *Recall that under h-Categoriser semantics, the argument of the graph $\mathbf{G}_2$ set the following values:*

$$\mathtt{Str}^{\mathbf{S_h}}_{\mathbf{G}_2}(A) = 1 \quad \mathtt{Str}^{\mathbf{S_h}}_{\mathbf{G}_2}(B) = 0.403 \quad \mathtt{Str}^{\mathbf{S_h}}_{\mathbf{G}_2}(C) = 0.481$$

$$\mathtt{Str}^{\mathbf{S_h}}_{\mathbf{G}_2}(D) = 0.675 \quad \mathtt{Str}^{\mathbf{S_h}}_{\mathbf{G}_2}(E) = 0.618$$

Thus, the arguments are ranked as follows:

$$B \prec^{\mathbf{S_h}}_{lex} C \prec^{\mathbf{S_h}}_{lex} E \prec^{\mathbf{S_h}}_{lex} D \prec^{\mathbf{S_h}}_{lex} A$$

Example 3 (Cont.) *Recall that under Weighted h-Categoriser, the arguments of the graph $\mathbf{G}_1$ set the following values:*

$$\mathtt{Str}^{\mathbf{S_{wh}}}_{\mathbf{G}_1}(A) = 0.28 \quad \mathtt{Str}^{\mathbf{S_{wh}}}_{\mathbf{G}_1}(B_1) = 0.3 \quad \mathtt{Str}^{\mathbf{S_{wh}}}_{\mathbf{G}_1}(B_2) = 0.3$$

$$\mathtt{Str}^{\mathbf{S_{wh}}}_{\mathbf{G}_1}(B_3) = 0.4 \quad \mathtt{Str}^{\mathbf{S_{wh}}}_{\mathbf{G}_1}(B_4) = 0.5$$

Hence:

$$A \prec^{\mathbf{S_{wh}}}_{lex} B_1 \simeq^{\mathbf{S_{wh}}}_{lex} B_2 \prec^{\mathbf{S_{wh}}}_{lex} B_3 \prec^{\mathbf{S_{wh}}}_{lex} B_4$$

In the literature, there exist "pure" ranking semantics, i.e., semantics that are not based on gradual semantics. An example is Burden-based semantics defined by Amgoud and Ben-Naim [2013] for flat graphs. To define this semantics, let us first introduce the notion of burden number.

Definition 17 (Burden number)**.** *Let $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w} \equiv 1, \mathcal{R}, \sigma \equiv 1 \rangle$ be a flat argumentation framework, $i \in \{0, 1, \cdots\}$, and $A \in \mathcal{A}$. We denote by $\mathtt{Bur}_i(A)$ the burden number of A in the i^{th} step, i.e.:*

$$\mathtt{Bur}_i(A) = \begin{cases} 1 & \textit{if } i = 0; \\ 1 + \sum\limits_{B \in \mathtt{Att(A)}} \frac{1}{\mathtt{Bur}_{i-1}(B)} & \textit{otherwise.} \end{cases}$$

By convention, if $\mathtt{Att}(A) = \emptyset$, then $\sum\limits_{B \in \mathtt{Att(A)}} \frac{1}{\mathtt{Bur}_{i-1}(B)} = 0$.

To illustrate this definition, let us consider the graph $\mathbf{G}_2$ from Example 4.

Example 4 (Cont.) The burden numbers of each argument are summarised in the table below. Note that these numbers will not change beyond step 7 (approximating the result to 10^-2).

Step i	A	B	C	D	E
0	1	1	1	1	1
1	1	3	3	2	2
2	1	2.33	1.83	1.33	1.5
3	1	2.55	2.18	1.55	1.67
4	1	2.46	2.04	1.46	1.60
5	1	2.49	2.09	1.49	1.63
6	1	2.48	2.07	1.48	1.62
7	1	2.48	2.08	1.48	1.62
8	1	2.48	2.08	1.48	1.62

Table 1.1: Burden numbers on $\mathbf{G}_2$

The Burden-based semantics (Bbs) compares lexicographically two arguments on the basis of their burden numbers.

Definition 18 (Burden-based semantics)**.** Bbs *transforms any flat argumentation framework* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w} \equiv 1, \mathcal{R}, \sigma \equiv 1 \rangle$ *into the ranking* $\texttt{Bbs}(\mathbf{AF})$ *on* $\mathcal{A}$ *such that* $\forall A, B \in \mathcal{A}$, $\langle A, B \rangle \in \texttt{Bbs}(\mathbf{AF})$ *iff one of the two following cases holds:*

- $\forall i \in \{0, 1, \cdots\}$, $\texttt{Bur}_i(A) = \texttt{Bur}_i(B)$;
- $\exists i \in \{0, 1, \cdots\}$, $\texttt{Bur}_i(A) < \texttt{Bur}_i(B)$ *and* $\forall j \in \{0, 1, \cdots, i-1\}$, $\texttt{Bur}_j(A) = \texttt{Bur}_j(B)$.

Note that the lexicographical order used in Bbs is reversed with respect to definition 16, i.e. Bbs prefers an argument when the value is smaller.

Example 4 (Cont.) *From the previous table we obtain that:*

$$B \prec^{\texttt{Bbs}}_{lex} C \prec^{\texttt{Bbs}}_{lex} E \prec^{\texttt{Bbs}}_{lex} D \prec^{\texttt{Bbs}}_{lex} A$$

The three families of existing semantics in the literature have been briefly recalled here. In the rest of the thesis, the focus will be put on the family of gradual semantics.

1.2.2 Principle-based Approach for Semantics

A great number of semantics have been proposed in the literature. Consequently, their comparison became crucial for clarifying their similarities and differences, and also for understanding their foundations. For that purpose, several works have been devoted to the development of principles that semantics may follow. The first work has been done by Baroni and Giacomin [2007], where a list of principles has been proposed for extension-based semantics. The list has further been slightly extended in van der Torre and Vesic

[2017] and has been used for comparing all the existing extension semantics.
In Amgoud and Ben-Naim [2016], other principles have been proposed for gradual semantics, and more generally for any semantics that assigns numerical or qualitative strength to arguments. The initial work focused on flat argumentation graphs, then extended in Amgoud *et al.* [2017] to deal with semi-weighted graphs, and in Amgoud and Doder [2019] for handling weighted argumentation graphs. In what follows, we recall the latter as it is more general than the two previous ones.

The first principle ensures that the strength of an argument does not depend on its identity.

Principle 1 (Anonymity)**.** *A semantics* $\mathbf{S}$ *satisfies Anonymity iff, for any two weighted argumentation graphs* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ *and* $\mathbf{AF}' = \langle \mathcal{A}', \mathbf{w}', \mathcal{R}', \sigma' \rangle$*, for any isomorphism* $\mathbf{f}$ *from* $\mathbf{AF}$ *to* $\mathbf{AF}'$*, the following property holds:* $\forall A \in \mathcal{A}$, $\mathtt{Str}^{\mathbf{S}}_{\mathbf{AF}}(A) = \mathtt{Str}^{\mathbf{S}}_{\mathbf{AF}'}(\mathbf{f}(A))$.

The second principle, called Independence, states that the strength of an argument A should be independent of any argument that is not connected to A (i.e., there is no path from that argument to A, ignoring the direction of the edges).

Principle 2 (Independence)**.** *A semantics* $\mathbf{S}$ *satisfies Independence iff, for any two weighted argumentation graphs* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ *and* $\mathbf{AF}' = \langle \mathcal{A}', \mathbf{w}', \mathcal{R}', \sigma' \rangle$ *such that* $\mathcal{A} \cap \mathcal{A}' = \emptyset$*, the following property holds:* $\forall A \in \mathcal{A}$, $\mathtt{Str}^{\mathbf{S}}_{\mathbf{AF}}(A) = \mathtt{Str}^{\mathbf{S}}_{\mathbf{AF} \oplus \mathbf{AF}'}(A)$.

The next principle, called Directionality, states that the strength of an argument should not depend on the arguments it itself attacks.

Principle 3 (Directionality)**.** *A semantics* $\mathbf{S}$ *satisfies Directionality iff, for any two weighted argumentation graphs* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$ *such that* $A, B \in \mathcal{A}$ *and* $\mathbf{AF}' = \langle \mathcal{A}', \mathbf{w}', \mathcal{R}', \sigma' \rangle$ *such that* $\mathcal{A} = \mathcal{A}'$, $\mathbf{w} = \mathbf{w}'$, $\mathcal{R}' = \mathcal{R} \cup \{(A, B)\}$ *and* $\forall (x, y) \in \mathcal{R},\ \sigma((x, y)) = \sigma'((x, y))$*, the following holds:* $\forall x \in \mathcal{A}$*, if there is no path from* B *to* x*, then* $\mathtt{Str}^{\mathbf{S}}_{\mathbf{AF}}(x) = \mathtt{Str}^{\mathbf{S}}_{\mathbf{AF}'}(x)$.

The next principle called Equivalence states that the strength of an argument depends only on its initial weight, the strength of its attackers and the weight of its direct attacks.

Principle 4 (Equivalence)**.** *A semantics* $\mathbf{S}$ *satisfies Equivalence iff, for any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, for all* $A, B \in \mathcal{A}$*, if*

- $\mathbf{w}(A) = \mathbf{w}(B)$,
- *there exists a bijective function* $\mathbf{f}$ *from* $\mathtt{Att}(A)$ *to* $\mathtt{Att}(B)$ *s.t.* $\forall x \in \mathtt{Att}(A)$*:*
  - $\mathtt{Str}^{\mathbf{S}}(x) = \mathtt{Str}^{\mathbf{S}}(\mathbf{f}(x))$,

- $\sigma((x, A)) = \sigma((\mathbf{f}(x), B))$,

then $\texttt{Str}^{\mathbf{S}}(A) = \texttt{Str}^{\mathbf{S}}(B)$.

The next principle, called Maximality, states that the strength of an argument is equal to its initial weight if it is not attacked.

Principle 5 (Maximality)**.** *A semantics* $\mathbf{S}$ *satisfies Maximality iff, for any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, for any* $A \in \mathcal{A}$*, if* $\texttt{Att}(A) = \emptyset$*, then* $\texttt{Str}^{\mathbf{S}}(A) = \mathbf{w}(A)$.

The next principle, called Neutrality, says that the strength of an argument does not take into account any attacker whose strength is equal to 0 and any attack whose weight is 0.

Principle 6 (Neutrality)**.** *A semantics* $\mathbf{S}$ *satisfies Neutrality iff, for any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, for all* $A, B \in \mathcal{A}$*, if*

- $\mathbf{w}(A) = \mathbf{w}(B)$,
- $\texttt{Att}(B) = \texttt{Att}(A) \cup \{x\}$ *such that* $x \in \mathcal{A} \setminus \texttt{Att}(A)$ *and (*$\texttt{Str}^{\mathbf{S}}(x) = 0$ *or* $\sigma(x, B) = 0$*),*

then $\texttt{Str}^{\mathbf{S}}(A) = \texttt{Str}^{\mathbf{S}}(B)$.

The next principle, called Weakening, states that an argument loses weight if it has at least one serious attack from a serious attacker.

Principle 7 (Weakening)**.** *A semantics* $\mathbf{S}$ *satisfies Weakening iff, for any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, for any* $A \in \mathcal{A}$*, if*

- $\mathbf{w}(A) > 0$ *and*
- $\exists B \in \texttt{Att}(A)$ *such that* $\mathbf{w}(B) > 0$ *and* $\sigma((B, A)) > 0$,

then $\texttt{Str}^{\mathbf{S}}(A) < \mathbf{w}(A)$.

The following principle, called Resilience, states that an attack cannot kill an argument.This principle is satisfied by most gradual semantics while it is violated by all extension-based semantics.

Principle 8 (Resilience)**.** *A semantics* $\mathbf{S}$ *satisfies resilience iff, for any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, for any* $A \in \mathcal{A}$*, if* $\mathbf{w}(A) > 0$ *then* $\texttt{Str}^{\mathbf{S}}(A) > 0$.

The next principle, called Proportionality, ensures that the evaluation of an argument is sensitive to the basic weight of the argument.

Principle 9 (Proportionality)**.** *A semantics* $\mathbf{S}$ *satisfies Proportionality iff, for any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, for all* $A, B \in \mathcal{A}$*, if*

- $\texttt{Att}(A) = \texttt{Att}(B)$*,*
- $\forall x \in \texttt{Att}(A)$*,* $\sigma((x, A)) = \sigma((x, B))$*,*
- $\mathbf{w}(A) > \mathbf{w}(B)$*,*
- $\texttt{Str}^{\mathbf{S}}(A) > 0$*,*

then $\texttt{Str}^{\mathbf{S}}(A) > \texttt{Str}^{\mathbf{S}}(B)$.

The next principle, called Reinforcement, ensures that the evaluation of an argument is sensitive to the strength of its attackers.

Principle 10 (Reinforcement)**.** *A semantics* $\mathbf{S}$ *satisfies Reinforcement iff, for any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, for all* $A, B \in \mathcal{A}$*, if*

- $\mathbf{w}(A) = \mathbf{w}(B)$*,*
- $\texttt{Str}^{\mathbf{S}}(A) > 0$*,*
- $\texttt{Att}(A) \setminus \texttt{Att}(B) = \{x\}$*,* $\texttt{Att}(B) \setminus \texttt{Att}(A) = \{y\}$*,*
- $\texttt{Str}^{\mathbf{S}}(y) > \texttt{Str}^{\mathbf{S}}(x)$*,*
- $\sigma((x, A)) = \sigma((y, B))$*,*

then $\texttt{Str}^{\mathbf{S}}(A) > \texttt{Str}^{\mathbf{S}}(B)$.

The following principle, called Monotony, ensures that an argument cannot become stronger when its set of attackers gets bigger.

Principle 11 (Monotony)**.** *A semantics* $\mathbf{S}$ *satisfies Monotony iff, for any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, for all* $A, B \in \mathcal{A}$*, if*

- $\mathbf{w}(A) = \mathbf{w}(B)$*,*
- $\texttt{Att}(B) = \texttt{Att}(A) \cup \{x\}$*, with* $\sigma((x, B)) > 0$ *and* $\texttt{Str}^{\mathbf{S}}(x) > 0$*,*
- $\texttt{Str}^{\mathbf{S}}(A) > 0$*,*

then $\texttt{Str}^{\mathbf{S}}(A) > \texttt{Str}^{\mathbf{S}}(B)$.

The next principle, named Attack-Sensitivity, states that the stronger the weight of an attack, the greater its impact on the targeted argument.

Principle 12 (Attack-Sensitivity)**.** *A semantics* $\mathbf{S}$ *satisfies Attack-Sensitivity iff, for any weighted argumentation graph* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, for all* $A, B \in \mathcal{A}$*, if*

- $\mathbf{w}(A) = \mathbf{w}(B)$*,*
- $\texttt{Att}(A) \setminus \texttt{Att}(B) = \{x\}$*,* $\texttt{Att}(B) \setminus \texttt{Att}(A) = \{y\}$*,*
- $\texttt{Str}^{\mathbf{S}}(x) = \texttt{Str}^{\mathbf{S}}(y)$*,*
- $\sigma((y, B)) > \sigma((x, A))$*,*
- $\texttt{Str}^{\mathbf{S}}(A) > 0$*,*

then $\texttt{Str}^{\mathbf{S}}(A) > \texttt{Str}^{\mathbf{S}}(B)$*.*

It has been shown in Amgoud and Doder [2019] that the 12 principles are compatible together, i.e., there exists at least one semantics that satisfies all of them together.

Theorem 1. *The twelve principles are compatible.*

It is worth mentioning that Weighted h-Categoriser (Def. 12) satisfies all the twelve principles and for extension-based semantics, it is possible to study these principles on them, as defined in Amgoud and Ben-Naim [2016]. From the acceptability status of an argument a strength may be assigned.

Definition 19 (Acceptability strength)**.** *Let* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w} \equiv 1, \mathcal{R}, \sigma \equiv 1 \rangle$ *be a flat argumentation graph and* $x \in \{ad, co, gr, pr, st\}$ *be an extension-based semantics among admissible, complete, grounded, preferred and stable. Let* $A \in \mathcal{A}$*:*

- *if* A *is sceptically accepted then* $\texttt{Str}^{x}_{\mathbf{AF}}(A) = 1$*.*
- *if* A *is credulously accepted then* $\texttt{Str}^{x}_{\mathbf{AF}}(A) = 0.5$*.*
- *if* A *is undefined then* $\texttt{Str}^{x}_{\mathbf{AF}}(A) = 0.3$*.*
- *if* A *is rejected then* $\texttt{Str}^{x}_{\mathbf{AF}}(A) = 0$*.*

The table below summarises how the extension-based semantics behave with respect to the twelve principles in case of flat argumentation framework (i.e. $\langle \mathcal{A}, \mathbf{w} \equiv 1, \mathcal{R}, \sigma \equiv 1 \rangle$). Note that Resilience is violated by the four extension-based semantics, Equivalence is satisfied only by Grounded and Proportionality and Attack-Sensitivity are not applicable on flat graphs.

	Grounded	Stable	Preferred	Complete
Anonymity	•	•	•	•
Independence	•	◦	•	•
Directionality	•	◦	•	•
Equivalence	•	◦	◦	◦
Maximality	•	◦	•	•
Neutrality	•	•	◦	◦
Weakening	•	•	•	•
Resilience	◦	◦	◦	◦
Reinforcement	•	•	◦	◦
Monotony	•	•	•	•
Proportionality	–	–	–	–
Attack-Sensitivity	–	–	–	–

The symbol • (resp.◦) means the principle is satisfied (resp. violated) by the semantics whereas – means that the principle may not be applied to the semantics.

Table 1.2: Satisfaction of the principles of extension-based semantics

1.2.3 Evaluation Methods

The idea is to define a gradual semantics by an evaluation method, which is a tuple of aggregation functions. Each of them must satisfy certain properties. Such approach has at least four advantages:

1. Firstly, it makes transparent the different operations performed by a semantics (e.g. , accruing strengths of attackers, adjusting weights, etc.) and formalizes them through aggregation functions.
2. Secondly, it indicates the main parameters to be tuned to define the different semantics.
3. Third, it facilitates the study of combinations of functions that lead to reasonable semantics.
4. Fourth, it has recently been shown in Amgoud and Doder [2018] that the properties of aggregation functions are closely related to the principles recalled in the previous section.

In what follows, we present a simplified version of this general framework, where we consider the unit interval [0,1] for all functions (the following definition comes from Amgoud and Doder [2019]).

Definition 20 (EM). *An* evaluation method *(EM) is a tuple* $\mathrm{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{h} \rangle$ *such that:*

- $\mathbf{h} : [0,1] \times [0,1] \to [0,1]$,
- $\mathbf{g} : \bigcup_{k=0}^{+\infty} [0,1]^k \to [0,+\infty[$, *such that* $\mathbf{g}$ *is symmetric,*
- $\mathbf{f} : [0,1] \times \texttt{Range}(\mathbf{g})$ [3] $\to [0,1]$.

The function $\mathbf{h}$ calculates the strength of an attack by aggregating the weight of the attack (given by σ) with the strength of the attacker (given by $\texttt{Str}$).
The function $\mathbf{g}$ evaluates the combined strength of the attacks of an argument. To do this, $\mathbf{g}$ aggregates the strength of all attacks (obtained by $\mathbf{h}$) received by the argument. Since the ordering of attackers should not be important, the function should respect the condition of symmetry, i.e.

$$\mathbf{g}(x_1, \cdots, x_n) = \mathbf{g}(x_{\rho(1)}, \cdots, x_{\rho(n)}),$$

for any permutation ρ of the set $\{1, \cdots, n\}$.
The function $\mathbf{f}$ returns the strength of an argument by combining its initial weight (given by $\mathbf{w}$) with the value returned by $\mathbf{g}$.
The following table shows some possibilities for functions $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$, including well-known T-norms (Klement *et al.* [2000]) for $\mathbf{h}$ and aggregation functions for $\mathbf{g}$. It can be noted that most of them are already (implicitly) used in literature.

$\mathbf{f}_{comp}(x_1, x_2) = x_1(1 - x_2)$	$\mathbf{g}_{sum}(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i$
$\mathbf{f}_{exp}(x_1, x_2) = x_1 e^{-x_2}$	$\mathbf{g}_{sum,\alpha}(x_1, \cdots, x_n) = (\sum_{i=1}^{n} (x_i)^\alpha)^{\frac{1}{\alpha}}$
$\mathbf{f}_{frac}(x_1, x_2) = \frac{x_1}{1+x_2}$	$\mathbf{g}_{max}(x_1, \cdots, x_n) = \texttt{max}\{x_1, \cdots, x_n\}$
$\mathbf{f}_{min}(x_1, x_2) = \texttt{min}\{x_1, 1 - x_2\}$	$\mathbf{g}_{psum}(x_1, \cdots, x_n) = x_1 \ominus \cdots \ominus x_n$, where $x_1 \ominus x_2 = x_1 + x_2 - x_1 x_2$

$\mathbf{h}_{prod}(x_1, x_2) = x_1 x_2$
$\mathbf{h}_{prod,\alpha}(x_1, x_2) = x_1^\alpha x_2$, $\alpha > 0$
$\mathbf{h}_{min}(x_1, x_2) = \texttt{min}\{x_1, x_2\}$
$\mathbf{h}_{Ham}(x_1, x_2) = \frac{x_1 x_2}{x_1 + x_2 - x_1 x_2}$, where $\mathbf{h}_{Ham}(x_1, x_2) = 0$ if $x_1 = x_2 = 0$

Table 1.3: Examples of functions $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$

Let us define a gradual semantics based on an evaluation method.

Definition 21 (Gradual Semantics based on an $\texttt{EM}$)**.** *A gradual semantics* $\mathbf{S}$ *based on an evaluation method* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{h} \rangle$ *is a function assigning to every* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*, a weighting* $\texttt{Str}^{\mathbf{S}}$ *such that for every* $A \in \mathcal{A}$, $\texttt{Str}^{\mathbf{S}}(A) =$

$$\mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\Bigg(\mathbf{h}\Big(\sigma(B_1, A), \texttt{Str}^{\mathbf{S}}(B_1)\Big), \cdots, \mathbf{h}\Big(\sigma(B_k, A), \texttt{Str}^{\mathbf{S}}(B_k)\Big)\Bigg)\Bigg),$$

[3] $\texttt{Range}(\mathbf{g})$ denotes the co-domain of $\mathbf{g}$

where $\{B_1, \cdots, B_k\} = \mathtt{Att}(A)$.

To illustrate the definition, let us use the Trust-Based semantics (Def. 13) on a simple example.

Example 5. *Consider the graph* $\mathbf{G}_3$ *depicted below and whose weights of arguments and attacks are all equal to 1.*

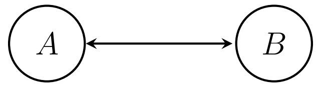


Figure 1.3: Argumentation graph $\mathbf{G}_3$

It is easy to check that $\mathtt{Str}^{TB}_{\mathbf{G}_3}(A) = \mathtt{Str}^{TB}_{\mathbf{G}_3}(B) = \frac{1}{2}$.

It was shown in (da Costa Pereira *et al.* [2011]), that TB leads to the following equation:

$$\alpha(A) = \mathtt{min}\{\mathbf{w}(A), 1 - \max_{(B,A)\in\mathcal{R}} \alpha(B)\}$$

Hence, $\mathtt{Str}^{TB}$ is an instance of α. The above equation can be decomposed into an evaluation method, called TB evaluation method: $\mathbf{M}_{TB} = \langle \mathbf{f}_{min}, \mathbf{g}_{max}, \mathbf{h}_{prod} \rangle$. So, the TB semantics is based on the evaluation method $\mathbf{M}_{TB}$. Note that Trust-Based semantics is defined on semi-weighted graphs, i.e. $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \equiv 1 \rangle$ and we took for example $\mathbf{h}_{prod}$ in $\mathbf{M}_{TB}$ but to be more general, the semantics TB is based on any evaluation method $\mathbf{M}^{\beta}_{TB} = \langle \mathbf{f}_{min}, \mathbf{g}_{max}, \mathbf{h}_{\beta} \rangle$ such as for any $x \in [0, 1]$, $\mathbf{h}_{\beta}(1, x) = x$.

Definition 21 shows that evaluating arguments with a semantics amounts to solving a system of equations (one equation per argument). Indeed, the solutions $v(A) = \mathtt{Str}^{\mathbf{S}}(A)$ of the system of equations

$$v(A) = \mathbf{f}(\mathbf{w}(A), \mathbf{g}(\mathbf{h}(\sigma((B_1, A)), v(B_1)), \cdots, \mathbf{h}(\sigma((B_n, A)), v(B_n)))) \tag{1.1}$$

for each argument $A \in \mathcal{A}$ with $\{B_1, \cdots, B_n\} = \mathtt{Att}(A)$, correspond to a semantics $\mathbf{S}$ based on an evaluation method $\mathbf{M}$ noted $\mathtt{Str}^{\mathbf{S}}$.

The following result shown in (Amgoud and Doder [2019]) ensures that the above system has at least one solution when the functions of the evaluation method are continuous.

Theorem 2. *If* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{h} \rangle$ *is an evaluation method such that* $\mathbf{g}$ *is continuous, and* $\mathbf{f}$ *and* $\mathbf{h}$ *are continuous on the second variable, then there exists a semantics based on* $\mathbf{M}$.

The system of equations 1.1 may thus have one or several solutions. Consider the case of the TB evaluation method $\mathbf{M}_{TB}$.

Example 5 (Cont.) Let $\mathbf{M}_{TB} = \langle \mathbf{f}_{min}, \mathbf{g}_{max}, \mathbf{h}_{prod} \rangle$ and recall that $\mathtt{Str}_G^{TB}(A) = \mathtt{Str}_G^{TB}(B) = \frac{1}{2}$. This is a solution of the system of equations 1.2 below.

$$v(A) = 1 - v(B), \;\; v(B) = 1 - v(A) \tag{1.2}$$

However, this system has infinitely many solutions including $(v(A), v(B)) = (0, 1)$. Then, the semantics $\mathbf{S}'$ defined by:

- $\mathtt{Str}_{\mathbf{G}_3}^{\mathbf{S}'}(A) = 0$,
- $\mathtt{Str}_{\mathbf{G}_3}^{\mathbf{S}'}(B) = 1$,
- $\mathtt{Str}_{\mathbf{G}'_3}^{\mathbf{S}'} \equiv \mathtt{Str}_{\mathbf{G}'_3}^{TB}$ for all $\mathbf{G}'_3 \neq \mathbf{G}_3$,

is also based on $\mathbf{M}_{TB}$. This means that $\mathbf{M}_{TB}$ does not characterize Trust-based semantics. However, a gradual semantics should be based on an evaluation method which characterizes it.

For that purpose, in (Amgoud and Doder [2019]), the authors extend the existing general setting by integrating a characterization condition. They introduce the concept of determinative evaluation methods, i.e., methods that characterize semantics.

Definition 22 (Determinative EM)**.** *An evaluation method* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{h} \rangle$ *is determinative iff there is a unique semantics* $\mathbf{S}$ *which is based on* $\mathbf{M}$. *We denote by* $\mathbf{S}(\mathbf{M})$ *the semantics characterized by a determinative evaluation method* $\mathbf{M}$.

As we have shown in example 5, the TB evaluation method is not determinative for all graphs. However, $\mathbf{M}_{TB}$, like any other evaluation method, is determinative on acyclic graphs, i.e. it produces unique semantics for these graphs.

What we would like, is to have evaluation methods that produce a single result (i.e. determinative) and also using functions with desirable properties. To this point, no constraints are imposed on the functions of an evaluation method, except the symmetry on g. In the following definition, the authors in Amgoud and Doder [2019] have introduced the notion of well-behaved evaluation methods, i.e. using functions satisfying properties that control their behaviour. They consider a subset of the properties of (Cayrol and Lagasquie-Schiex [2005]; Egilmez *et al.* [2013]), thus broadening the framework.

Definition 23 (Well-Behaved EM)**.** *An evaluation method* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{h} \rangle$ *is well-behaved iff the following holds:*

1. (a) $\mathbf{f}$ *is increasing in the first variable, decreasing in the second variable whenever the first variable is not equal to 0,*

   (b) $\mathbf{f}(x, 0) = x$,

   (c) $\mathbf{f}(0, x) = 0$.

2. (a) $\mathbf{g}() = 0$,

   (b) $\mathbf{g}(x) = x$,

   (c) $\mathbf{g}(x_1, \cdots, x_n) = \mathbf{g}(x_1, \cdots, x_n, 0)$,

   (d) $\mathbf{g}(x_1, \cdots, x_n, y) \leq \mathbf{g}(x_1, \cdots, x_n, z)$ *if* $y \leq z$,

   (e) $\mathbf{g}$ *is symmetric.*

3. (a) $\mathbf{h}(0, x) = 0$,

   (b) $\mathbf{h}(1, x) = x$,

   (c) $\mathbf{h}(x, y) > 0$ *whenever* $xy > 0$,

   (d) $\mathbf{h}$ *is non-decreasing in both components.*

Now, let us see how the authors have defined a determinative family of well behaved evaluation methods.

Definition 24 (A determinative family of EM)**.** *Let* $\mathbf{M}^*$ *be the set of all well-behaved evaluation methods* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{h} \rangle$ *such that:*

- $\lim\limits_{x_2 \to x_0} \mathbf{f}(x_1, x_2) = \mathbf{f}(x_1, x_0)$, $\forall x_0 \neq 0$.
- $\lim\limits_{x \to x_0} \mathbf{g}(x_1, \cdots, x_k, x) = \mathbf{g}(x_1, \cdots, x_k, x_0)$, $\forall x_0 \neq 0$.
- $\mathbf{h}$ *is continuous on the second variable.*
- $\lambda \mathbf{f}(x_1, \lambda x_2) < \mathbf{f}(x_1, x_2)$, $\forall \lambda \in [0, 1[$, $x_1 \neq 0$.
- $\mathbf{g}(\mathbf{h}(y_1, \lambda x_1), \cdots, \mathbf{h}(y_k, \lambda x_k)) \geq \lambda \mathbf{g}(\mathbf{h}(y_1, x_1), \cdots, \mathbf{h}(y_k, x_k))$, $\forall \lambda \in [0, 1]$.

Let us have a closer look at some of these constraints. For the first two conditions (from the top) of $\mathbf{M}^*$, they relax the continuity conditions of Theorem 2 by excluding the value 0; they are weakened in order to capture semantics sensitive to the number of attackers (such as the Weighted Card-based (Amgoud *et al.* [2017])), where even a weak attacker can have a significant impact. The last two conditions in $\mathbf{M}^*$ are specific contraction conditions. The fifth condition is satisfied by most combinations of aggregation functions (for $\mathbf{g}$) and T-norms (for $\mathbf{h}$).

Theorem 3. *Any evaluation* $\mathbf{M} \in \mathbf{M}^*$ *is determinative.*

Definition 25 (Family of $\mathbf{S}^*$ semantics)**.** *Let* $\mathbf{S}^*$ *be the set of all semantics which are based on an evaluation method in* $\mathbf{M}^*$*, i.e.,*

$$\mathbf{S}^* = \{\mathbf{S}(\mathbf{M}) \mid \mathbf{M} \in \mathbf{M}^*\}.$$

It has been shown (in Amgoud and Doder [2019]) that the semantics of S* satisfy nine principles defined in section 1.1.

Theorem 4. *For any gradual semantics* $\mathbf{S} \in \mathbf{S}^*$ *,* $\mathbf{S}$ *satisfies Anonymity, Independence, Directionality, Equivalence, Maximality, Neutrality, Weakening, Proportionality, and Resilience.*

Now, if we want the semantics defined by an evaluation method belonging to $\mathbf{M}^*$ to satisfy the twelve principles of section 1.2.2, then we should add two additional constraints, one on g and one on $\mathbf{h}$ as follows:

Definition 26 ($\mathbf{M}_e^*$)**.** *Let* $\mathbf{M}_e^*$ *(e stands for extended) as the set of all evaluation methods* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{h} \rangle$ *such that:*

- $\mathbf{M} \in \mathbf{M}^*$,
- $\mathbf{g}(x_1, \cdots, x_k, y) > \mathbf{g}(x_1, \cdots, x_k, z)$ *whenever* $y > z$,
- $\mathbf{h}(x_1, y) > \mathbf{h}(x_2, y)$ *whenever* $x_1 > x_2$, $y \neq 0$.

This gives us a new family of semantics.

Definition 27 (Family of $\mathbf{M}_e^*$ semantics)**.** *Let* $\mathbf{S}_e^*$ *the set of all semantics which are based on an evaluation method in* $\mathbf{M}_e^*$*, i.e.,*

$$\mathbf{S}_e^* = \{\mathbf{S}(\mathbf{M}) \mid \mathbf{M} \in \mathbf{M}_e^*\}.$$

The semantics of $\mathbf{S}_e^*$ satisfy all the principles seen previously.

Theorem 5. *For any gradual semantics* $\mathbf{S} \in \mathbf{S}^*$*,* $\mathbf{S}$ *satisfies all the twelve principles.*

To illustrate these definitions and results, let us consider again the gradual semantics Weighted h-Categoriser which belongs to $\mathbf{S}^*$ (Amgoud and Doder [2019]). It has been shown that it is based on the evaluation method $\mathbf{M}_{\mathtt{wh}} = \langle \mathbf{f}_{frac}, \mathbf{g}_{sum}, \mathbf{h}_{prod} \rangle$ (see table 1.3 for the definition of the functions).

In the source paper, the authors proposed an algorithm that computes the strength of an argument under any semantics of the set $\mathbf{S}^*$. The idea is that at each step, a value is assigned to each argument. In the initial step, the value of an argument is its basic weight. Then, at each step, the value is recalculated on the basis of the weights of the arguments and attacks as well as the values of the attackers of the argument in the previous step.

Theorem 6. *Let* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{h} \rangle \in \mathbf{M}^*$, $\mathbf{S} = \mathbf{S}(\mathbf{M})$*, and* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$*. For every* $A \in \mathcal{A}$*, we define the sequence* $\{s(A)^{(n)}\}_{n=1}^{+\infty}$ *in the following way:*

- $s(A)^{(1)} = \mathbf{w}(A)$,
- $s(A)^{(n+1)} = \mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\Bigg(\mathbf{h}\Big(\sigma(B_1, A), s(B_1)^{(n)}\Big), \cdots, \mathbf{h}\Big(\sigma(B_k, A), s(B_k)^{(n)}\Big)\Bigg)\Bigg)$, *where* $\{B_1, \cdots, B_k\} = \mathtt{Att}(A)$.

Then, for every $A \in \mathcal{A}$*:*

1. $\{s(A)^{(n)}\}_{n=1}^{+\infty}$ *converges, and*
2. $\lim\limits_{n \to +\infty} s(A)^{(n)} = \mathtt{Str}^{\mathbf{S}}_{\mathbf{AF}}(A)$.

1.3 Similarity

This section introduces the challenges of this thesis. Then it gives an overview of the state of the art on the notion of similarity in artificial intelligence and more specifically in argumentation.

1.3.1 Motivation

Let us consider again the debate on how to reduce the debts of a country (graph $\mathbf{G}_1$ of Example 3). The following five arguments have been exchanged.

A: Increasing taxes, decreasing financial market borrowing and allowing government to finance itself through money creation, reduce the country's debt.

B_1: For a better living standards for all, taxes must not be increased.

B_2: To improve the quality of life, taxes must not be increased.

B_3: For a better healthcare and social justice, taxes must not be increased.

B_4: The purpose of borrowing is to prevent the inflation caused by money creation, therefore decreasing financial market borrowing and allowing government to finance itself through money creation do not imply a reduction of the debt.

Recall also that B_1, B_2, B_3, B_4 attack A. When $\mathbf{w}(A) = 0.7$, $\mathbf{w}(B_1) = 0.3$, $\mathbf{w}(B_2) = 0.3$, $\mathbf{w}(B_3) = 0.4$ and $\mathbf{w}(B_4) = 0.5$, and $\sigma \equiv 1$, we have seen that under Weighted h-Categoriser semantics, $\forall i \in \{1, 2, 3, 4\}$, $\mathtt{Str}(B_i) = \mathbf{w}(B_i)$ while $\mathtt{Str}(A) = 0.28$, loosing thus weight.

We have seen that Weighted h-Categoriser semantics satisfies the Monotony principle, i.e. when an argument receives an additional (non-null) attack the strength of that argument must decrease. However, this should only hold if the new attacker brings new information (compared to the other attackers of the argument). Let us consider four new graphs $\mathbf{G}_{11}$, $\mathbf{G}_{12}$, $\mathbf{G}_{13}$ and $\mathbf{G}_{14}$ where we rename the argument A into A_i when A_i belongs to $\mathbf{G}_{1i}$.

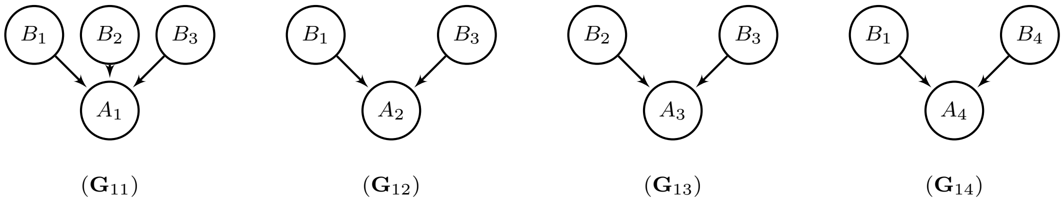


Figure 1.4: Argumentation graphs $\mathbf{G}_{11}$, $\mathbf{G}_{12}$, $\mathbf{G}_{13}$ and $\mathbf{G}_{14}$

According to Monotony, the strength of A_1 should be weaker than the strength of A_2 due to the additional attack from B_2. However the two arguments B_1 and B_2, are identical or *totally similar* since they support the same claim with the same evidence. One of them is thus redundant, and considering both in the evaluation of A_1 is questionable. A reasonable semantics would declare A_1, A_2 and A_3 as equally strong.
Consider now B_1 and B_3. They are *partially similar* since each of them brings a new piece of evidence (eg., entertainment for B_1, and social justice for B_3) in addition to the common one (better healthcare which is part of better living standards). Finally, B_4 is based on a completely different evidence, making it dissimilar to the three others. Hence, one would expect to declare A_2 (resp. A_1, A_3) as stronger than A_4 since the group $\{B_1, B_3\}$ of A_2's attackers is weaker than the group $\{B_1, B_4\}$ of A_4's ones. Indeed, the former contains some redundancy which should be removed, while the latter does not (B_1 and B_4 being different).

To sum up, ignoring (total or partial) similarities would lead to inaccurate evaluations of arguments, and thus to wrong recommendations by argumentation systems. Therefore, an argumentation framework should be equipped with a similarity measure that assesses similarity between arguments, and semantics should be able to take it into account.

1.3.2 Existing Works

1.3.2.1 Similarity Measures in Artificial Intelligence

The notion of similarity measure is widely studied in the domain of computer science (information retrieval, classification, image processing, etc.) and particularly in artificial intelligence. It is worth mentioning that the field of machine learning frequently uses a similarity measure to compare objects between them. For instance, in clustering tasks (i.e. grouping data from an unstructured set), similarity values directly influence the resulting data subgroups (e.g. Irani *et al.* [2016] is a survey presenting different clustering categories with their most frequently employed similarity measure).

We may distinguish two main families of similarity measures (across all domains) that are characterised by the nature of objects to compare. There are those that use symbolic information (e.g. natural or logical language) and those that work with numerical data (e.g. pixel values or coordinates). Note that, symbolic information may be transformed into a numerical form to compute a degree of similarity. For example, for a given object containing a set of features, a binary vector may be produced, with 0 meaning that the object does not have that feature and 1 meaning that it does. If the objects are texts, for instance, common elements and differences may be counted. For a study of similarity measures on binary and numerical data see Lesot *et al.* [2009b]; Choi *et al.* [2010], and for a study of similarity measures applied to texts see Gomaa *et al.* [2013]; Vijaymeena and Kavitha [2016].

Below we recall some measures that have been used in this thesis. They compare arbitrary pairs of non-empty objects (X and Y). Let $a = |X \cap Y|$, $b = |X - Y|$, and $c = |Y - X|$ where $|\cdot|$ denotes the cardinality of a set. Table 1.4 recalls the most prominent measures, namely the Jaccard measure (Jaccard [1901]), Dice measure (Dice [1945]), Sorensen one (Sørensen [1948]), and those proposed in (Anderberg [1973]; Sneath *et al.* [1973]; Ochiai [1957]; Kulczynski [1927]).

Jaccard	Jaccard [1901]	$s_{\texttt{jac}}(X,Y) = \frac{a}{a+b+c}$
Dice	Dice [1945]	$s_{\texttt{dic}}(X,Y) = \frac{2a}{2a+b+c}$
Sorensen	Sørensen [1948]	$s_{\texttt{sor}}(X,Y) = \frac{4a}{4a+b+c}$
Symmetric Anderberg	Anderberg [1973]	$s_{\texttt{and}}(X,Y) = \frac{8a}{8a+b+c}$
Sokal and Sneath 2	Sneath *et al.* [1973]	$s_{\texttt{ss2}}(X,Y) = \frac{a}{a+2(b+c)}$
Ochiai	Ochiai [1957]	$s_{\texttt{och}}(X,Y) = \frac{a}{\sqrt{a+b}\sqrt{a+c}}$
Kulczynski 2	Kulczynski [1927]	$s_{\texttt{ku2}}(X,Y) = \frac{1}{2}\left(\frac{a}{a+b} + \frac{a}{a+c}\right)$

Table 1.4: Similarity measures for sets of objects

Regarding similarity measures on natural language data, it is interesting to briefly present the field of argument mining and then to mention some research related to our purpose. It aims to automatically extracting structured arguments from unstructured textual documents. This is a topical issue, especially because of its potential to process information from the web, particularly from social media. Moreover, thanks to the progress made in machine learning methods, more and more promising applications in the fields of social and economic sciences, policy development and information technology are considered (Lippi and Torroni [2016]). More importantly for our work, it will be possible to automate the generation of argument graphs, which is necessary to automate the whole argumentation process. Note that similarity between pairs of arguments has been studied in the context of argument mining (Misra *et al.* [2016]; Stein [2016]; Konat *et al.* [2016]). The aim is to detect paraphrases, i.e. redundant textual arguments.

1.3.2.2 Similarity in Argumentation

In the argumentation literature, similarity has been studied within an argument (Walton *et al.* [2008]; Walton [2010, 2013]). Walton has discussed different argumentation schemes such as analogical arguments or similarity arguments. These arguments have premises comparing different objects to conclude their claim. Regarding the use of similarity in logical argumentation, in Wooldridge *et al.* [2006] and Amgoud *et al.* [2014]) the authors studied equivalence, i.e., complete similarity, between logical arguments. We will see that our approach generalizes these proposals. Indeed, it assigns the maximum value to each pair of equivalent arguments. Finally, Budàn *et al.* [2015] have defined a general measure evaluating the similarity between pairs of analogical arguments. It is based on an assumed mapping function applied between the features of a pair of arguments. Then in Budan *et al.* [2020], the authors have proposed a definition of similarity measure in the framework of bipolar argumentation (with attacks and supports). Unlike our work where we will use a logical language to represent the information in an argument, they consider abstract arguments extended by a set of descriptors defined in pairs (domain, value); for example, a descriptor on the domain of "activity" may be: $(general\_activity, \{walk, watch\_movie, go\_out\})$. They also offer the possibility of evaluating similarity in different contexts, i.e., with different degrees of importance between the domains of different descriptors.

Besides the fact that they use a bipolar graph of enriched arguments, another difference is that they use extension-based semantics (while we use gradual semantics). Furthermore, similarity is not used for eliminating redundant attackers but rather to validate attack and support relations. This idea may be compared within a weighted argumentation framework using similarity as a relevance weight on relations. In this view, if two arguments connected by a relation are similar then the relation is validated in the opposite case the relation is ignored. On the other hand, it would seem interesting to check definition 13 of Budan *et al.* [2020] (degree of similarity between arguments) with respect to definition 12 (similarity coefficient of a descriptor). Indeed, it is written that the similarity between arguments is between 0 and 1. However, as it is defined, it is possible to obtain results strictly higher than 1. In definition 12, the similarity of a descriptor may be strictly higher than 1 and in definition 13, if we take for example the maximum t-conorme, it is possible to obtain a similarity between arguments strictly higher than 1.

The last work that tackled similarity in argumentation has been done in Amgoud *et al.* [2018]. The authors have considered semi-weighted argumentation frameworks, and have used two families of similarity measures between abstract arguments, an n-ary family between an argument and a set of arguments and, as in our case, a binary family between a pair of arguments. Finally, they have proposed necessary principles for gradual semantics considering similarity, as well as new semantics using these different similarity measures.

Chapter 2

Similarity Measures for Logical Arguments

Contents

THIS chapter starts by introducing some notions of propositional logic and logical arguments. Afterwards, we introduce the notion of similarity measure between pairs of arguments and discuss some properties it should satisfy. Then, we introduce the concept of concise argument, which solves some weaknesses of the original definition. Finally, we propose some similarity measures and discuss their properties.

2.1 Background on Logic

2.1.1 Fundamental Concepts

Throughout this chapter, we consider propositional logic, i.e. a pair $(\mathcal{L}, \vdash)$ where $\mathcal{L}$ is a propositional language built up from a *finite* set $\mathcal{P}$ of variables, the two Boolean constants $\top$ (true) and $\bot$ (false), and the usual connectives ($\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$), and $\vdash$ is the consequence relation of the logic. A literal of $\mathcal{L}$ is either a variable of $\mathcal{P}$ or the negation of a variable of $\mathcal{P}$, the set of all literals is denoted $\mathcal{P}^{\pm}$. Two formulas $\phi, \psi \in \mathcal{L}$ are *logically equivalent*, denoted by $\phi \equiv \psi$, iff $\phi \vdash \psi$ and $\psi \vdash \phi$.

In what follows, we introduce different functions that will be used in the rest of the chapter. We recall below the notion of Negation Normal Form of a propositional formula.

Definition 28 (Negation Normal Form)**.** *Let a formula $\phi \in \mathcal{L}$. ϕ is in negation normal form (NNF) if and only if it does not contain implication or equivalence symbols, and every negation symbol occurs directly in front of an atom.*

Following Lang *et al.* [2003], we slightly abuse words and denote by $\texttt{NNF}(\phi)$ the formula in NNF obtained from ϕ by "pushing down" every occurrence of $\neg$ (using De Morgan's law) and eliminating double negations. For instance, $\texttt{NNF}(\neg((p \rightarrow q) \vee \neg t)) = p \wedge \neg q \wedge t$.

Notations: let $\phi \in \mathcal{L}$.

- **Variables:** we denote by $\texttt{Var}(\phi)$, the function returning all the variables occurring in the formula ϕ. For instance, $\texttt{Var}(\neg p \wedge q) = \{p, q\}$.
- **Literals:** we denote by $\texttt{Lit}(\phi)$ the set of literals occurring in $\texttt{NNF}(\phi)$. For instance, $\texttt{Lit}(\neg((p \rightarrow q) \vee \neg t)) = \{p, \neg q, t\}$.

Lang *et al.* [2003] have defined when a formula depends on a given literal.

Definition 29 (Literals dependence)**.** *Let $\phi \in \mathcal{L}$ and $l \in \mathcal{P}^{\pm}$. ϕ is* independent *from l iff $\exists \psi \in \mathcal{L}$ such that $\phi \equiv \psi$ and $l \notin \texttt{Lit}(\psi)$. Otherwise, ϕ is* dependent *on l. $\texttt{DepLit}(\phi)$ denotes the set of all literals of $\mathcal{P}^{\pm}$ that ϕ is dependent on.*

For instance, $\texttt{DepLit}((\neg p \vee q) \wedge (\neg p \vee \neg q)) = \{\neg p\}$ while $\texttt{DepLit}(\neg p \wedge q) = \{\neg p, q\}$.

Definition 30 (Logical consequences - $\mathtt{CN}$). *Let $\phi \in \mathcal{L}$. The function $\mathtt{CN}(\phi)$ is the set of all logical consequences of ϕ, i.e.*

$$\mathtt{CN}(\phi) = \{\psi \in \mathcal{L} \mid \phi \vdash \psi\}.$$

Note that, this set of logical consequences is infinite even when the set of literals is finite. This can be explained by the repetition of literals in conjunctions. For example: p, $p \wedge p$, $p \wedge p \wedge p$, etc.

It is worth recalling that a model of a formula ϕ is an interpretation (i.e., a total function from $\mathcal{P}$ to $\{0, 1\}$) that makes ϕ true in the usual truth-functional way.

Definition 31 (Models - $\mathtt{Mod}$). *Let $\phi \in \mathcal{L}$. The function $\mathtt{Mod}(\phi)$ denotes the set of all models of the formula ϕ, i.e.,*

$$\mathtt{Mod}(\phi) = \{\omega \in \mathcal{W} \mid \omega \models \phi\},$$

where $\mathcal{W}$ is the set of all interpretations.

Definition 32 (Isomorphic formulas). *Let $\phi, \psi \in \mathcal{L}$. The two formulas are isomorphic if and only if there exists a permutation (i.e. a bijective renaming function) $\pi : \mathcal{P} \to \mathcal{P} \setminus \mathtt{Var}(\phi)$ of the variables of ϕ such that ψ and $\pi(\phi)$[1] become logically equivalent. We say that ϕ and ψ are isomorphic w.r.t. π.*

For instance, the formulas $p \wedge \neg q$ and $t \wedge \neg v$ are isomorphic w.r.t. the renaming function π, where $\pi(t) = p$, $\pi(v) = q$, hence $\pi(t \wedge \neg v) = p \wedge \neg q$.

Definition 33 (Consistence of a set of formulas). *Let $\Phi \subseteq_f \mathcal{L}$. Φ is consistent iff $\Phi \nvdash \bot$, it is inconsistent otherwise.*

For instance, the set of formulas $\{p, \neg q\}$ is consistent while $\{p, \neg p\}$ is inconsistent.

Let us now define when two finite sets Φ and Ψ of formulas are equivalent. A natural definition is when the two sets have the same logical consequences, i.e., $\{\phi \in \mathcal{L} \mid \Phi \vdash \phi\} = \{\psi \in \mathcal{L} \mid \Psi \vdash \psi\}$. Thus, the three sets $\{p, q\}$, $\{p \wedge p, \neg\neg q\}$, and $\{p \wedge q\}$ are pairwise equivalent. This definition is strong since it considers any inconsistent sets as equivalent. For instance, $\{p, \neg p\}$ and $\{q, \neg q\}$ are equivalent even if the *contents* (i.e. meaning of variables and formulas) of the two sets are unrelated (assume that p and q stand respectively for "the sky is flamboyant red" and "this boat is a sailing ship").

[1] $\pi(\phi)$ denotes the formula obtained by replacing in ϕ each variable $v \in \mathtt{Var}(\phi)$ by $\pi(v)$.

Furthermore, it considers the two sets $\{p, p \rightarrow q\}$ and $\{q, q \rightarrow p\}$ as equivalent while their contents are different as well. Indeed, "a square is a rectangle” is different from "a rectangle is a square" where p stand for "it’s a square" and q for "it’s a rectangle". One may be true while the other is not. In what follows, we consider the following definition borrowed from Amgoud *et al.* [2014]. It compares the formulas contained in sets instead of the logical consequences of the sets.

Definition 34 (Equivalent Sets of Formulas)**.** *Two sets of formulas* $\Phi, \Psi \subseteq_f \mathcal{L}$ *are* equivalent*, denoted by* $\Phi \cong \Psi$*, iff* $\forall \phi \in \Phi$*,* $\exists \psi \in \Psi$ *such that* $\phi \equiv \psi$ *and* $\forall \psi' \in \Psi$*,* $\exists \phi' \in \Phi$ *such that* $\phi' \equiv \psi'$*. We write* $\Phi \ncong \Psi$ *otherwise.*

Note that $\{p, p \rightarrow q\} \ncong \{q, q \rightarrow p\}$, $\{p, \neg p\} \ncong \{q, \neg q\}$, and $\{p, q\} \ncong \{p \wedge q\}$ while $\{p, q\} \cong \{p \wedge p, \neg\neg q\}$.

Between two sets of formulas, the following function allows to return formulas in the first set that have an equivalent one in the second set.

Definition 35 ($\mathtt{Co}$)**.** *For* $\Phi, \Psi \subseteq_f \mathcal{L}$*,* $\mathtt{Co}(\Phi, \Psi) = \{\phi \in \Phi \mid \exists \psi \in \Psi$ *such that* $\phi \equiv \psi\}$.

Property 1. *For all* $\Phi, \Psi \subseteq_f \mathcal{L}$*,* $\Phi \cong \Psi$ *iff* $\mathtt{Co}(\Phi, \Psi) = \Phi$ *and* $\mathtt{Co}(\Psi, \Phi) = \Psi$.

2.1.2 Logical Arguments

Let us now, define the notion of argument under propositional logic $(\mathcal{L}, \vdash)$. Following Besnard and Hunter [2001], an argument is a pair set of: propositional formulas representing the support of the argument, and a propositional formulas representing its conclusion.

Definition 36. *[Logical argument] An* argument *built under the logic* $(\mathcal{L}, \vdash)$ *is a pair* $\langle \Phi, \phi \rangle$*, where* $\Phi \subseteq_f \mathcal{L}$ *and* $\phi \in \mathcal{L}$*, such that:*

- Φ *is consistent,* *(Consistency)*
- $\Phi \vdash \phi$*,* *(Validity)*
- $\nexists \Phi' \subset \Phi$ *such that* $\Phi' \vdash \phi$*.* *(Minimality)*

An argument $\langle \Phi, \phi \rangle$ *is* trivial *iff* $\Phi = \emptyset$ *and* $\phi \equiv \top$*.* Φ *is called the support of the argument and* ϕ *its conclusion.*

Example 6. *The following are examples of arguments:* $\langle \{p \wedge q\}, p \rangle$*,* $\langle \{p, q\}, p \wedge q \rangle$*,* $\langle \{p\}, p \rangle$*,* $\langle \{p\}, p \vee q \rangle$*,* $\langle \emptyset, p \vee \neg p \rangle$*.*

Notations:

- We denote by $\texttt{Arg}(\mathcal{L})$ the set of all arguments that can be built in $(\mathcal{L}, \vdash)$ in the sense of Definition 36 above.
- For any $A = \langle \Phi, \phi \rangle \in \texttt{Arg}(\mathcal{L})$, the functions $\texttt{Supp}$ and $\texttt{Conc}$ return respectively the *support* ($\texttt{Supp}(A) = \Phi$) and the *conclusion* ($\texttt{Conc}(A) = \phi$) of A.

Let us now introduce the notion of *sub-argument* of an argument A. It is an argument whose support is a subset of the support of A.

Definition 37 (Sub-argument)**.** *Let $A, B \in \texttt{Arg}(\mathcal{L})$. A is a* sub-argument *of B, denoted by $A \sqsubset B$, iff* $\texttt{Supp}(A) \subseteq \texttt{Supp}(B)$.

Note that an argument may be a sub-argument of itself according to Definition 37.

Example 6 (Cont.) *The argument $\langle \{p\}, p \rangle$ is sub-argument of $\langle \{p, q\}, p \wedge q \rangle$ while is not sub-argument of $\langle \{p \wedge q\}, p \wedge q \rangle$. Note that $\langle \emptyset, p \vee \neg p \rangle$ is sub-argument of $\langle \{p \wedge q\}, p \rangle$, $\langle \{p, q\}, p \wedge q \rangle$, $\langle \{p\}, p \rangle$, $\langle \{p\}, p \vee q \rangle$ and $\langle \emptyset, p \vee \neg p \rangle$.*

Proposition 1. *Let $A, B \in \texttt{Arg}(\mathcal{L})$. If A is trivial then for any B, $A \sqsubset B$.*

We now define the notion of *isomorphic* arguments.

Definition 38 (Isomorphic Arguments)**.** *Two arguments $A, B \in \texttt{Arg}(\mathcal{L})$ are* isomorphic *with respect to a renaming function π iff the two following conditions hold:*

- *there exists a bijective function $f : \texttt{Supp}(A) \to \texttt{Supp}(B)$ such that for any $\phi \in \texttt{Supp}(A)$, ϕ and $f(\phi)$ are isomorphic w.r.t. π,*
- *$\texttt{Conc}(A)$ and $\texttt{Conc}(B)$ are isomorphic w.r.t. π.*

Example 7. *Let π be a renaming function such that $\pi(r) = p$, $\pi(v) = q$. The arguments $\langle \{p \wedge q\}, p \wedge q \rangle$ and $\langle \{r \wedge v\}, r \wedge v \rangle$ are isomorphic w.r.t. π while $\langle \{p \wedge q\}, p \wedge q \rangle$ and $\langle \{p \to q\}, p \to q \rangle$ are not.*

In Amgoud *et al.* [2014], the authors studied when two arguments are *equivalent*. Two arguments are equivalent if their supports (respectively their conclusions) are equivalent.

Definition 39 (Equivalent Arguments)**.** *Two arguments $A, B \in \texttt{Arg}(\mathcal{L})$ are equivalent, denoted by $A \approx B$, iff*

$$(\texttt{Supp}(A) \cong \texttt{Supp}(B)) \textit{ and } (\texttt{Conc}(A) \equiv \texttt{Conc}(B)).$$

Isomorphic arguments are not necessarily equivalent. For instance, $\langle\{p \wedge q\}, p \wedge q\rangle$ and $\langle\{r \wedge v\}, r \wedge v\rangle$ are isomorphic but not equivalent. All trivial arguments are equivalent.

Property 2. *All trivial arguments are pairwise equivalent.*

Next, we present a useful property of the function $\mathtt{Co}$, which states that the number of equivalent formulas in the first set is equal to the number of equivalent formulas in the second set. It holds in the case of arguments but not in general (e.g. for sets with equivalent formulas, the cardinality may vary).

Property 3. *For all* $A, B \in \mathtt{Arg}(\mathcal{L})$, $|\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))| = |\mathtt{Co}(\mathtt{Supp}(B), \mathtt{Supp}(A))|$.

When the supports of arguments are equivalent their number of formulas is equal.

Property 4. *For all* $A, B \in \mathtt{Arg}(\mathcal{L})$, *if* $\mathtt{Supp}(A) \cong \mathtt{Supp}(B)$ *then* $|\mathtt{Supp}(A)| = |\mathtt{Supp}(B)|$.

2.2 Axiomatic Foundations of Similarity Measures

Our goal is to evaluate the extent to which pairs of logical arguments are similar. For this purpose, we define a *similarity measure*, i.e. a function that assigns a value in the unit interval [0, 1] to each pair of arguments. The greater the value, the more similar the arguments are.

Definition 40 (Similarity Measure)**.** *A* similarity measure *is a function*

$$\mathtt{sim} : \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L}) \to [0, 1].$$

This definition is very general in that it accepts any function. In what follows, we restrict the possible candidate functions by proposing a set of *principles* that any reasonable similarity measure should satisfy. Principles are basic and desirable properties of a measure $\mathtt{sim}$. We may distinguish these principles according to their nature.

2.2.1 Principles

Indeed, for any pair of objects applied to a similarity measure, four principles may be defined. In addition to these general principles, we propose six principles specific to similarity measures applied between pairs of logical arguments.

2.2.1.1 General Principles

It is worth mentioning that despite the wide range of similarity measures in the literature (see Lesot *et al.* [2009a]; Choi *et al.* [2010] for surveys of existing measures), there are only two formal properties that have been identified in the literature: Maximality and Symmetry which are presented below.

The first principle, called Maximality, deals with the case of total similarity. Maximality asserts that the similarity between an object and itself is maximal, i.e. here equal to 1.

Principle 1 (Maximality)**.** *A similarity measure* $\texttt{sim}$ *satisfies* Maximality *iff for any* $A \in \texttt{Arg}(\mathcal{L})$*,* $\texttt{sim}(A, A) = 1$*.*

Symmetry states that similarity is a symmetric notion.

Principle 2 (Symmetry)**.** *A similarity measure* $\texttt{sim}$ *satisfies* Symmetry *iff for all* $A, B \in \texttt{Arg}(\mathcal{L})$*,* $\texttt{sim}(A, B) = \texttt{sim}(B, A)$*.*

The following principle is a well-known property in the field of dissimilarity. The Triangle Inequality guarantees that if a pair of objects is very similar to a third object, then the first two objects are also very similar.

Principle 3 (Triangle Inequality)**.** *For all* $A, B, C \in \texttt{Arg}(\mathcal{L})$ *the following holds:*

$$1 + \texttt{sim}(A, C) \geq \texttt{sim}(A, B) + \texttt{sim}(B, C).$$

The next general principle, called Substitution, states that two fully similar objects are equally similar to any third object.

Principle 4 (Substitution)**.** *A similarity measure* $\texttt{sim}$ *satisfies* Substitution *iff for all* $A, B, C \in \texttt{Arg}(\mathcal{L})$*, if* $\texttt{sim}(A, B) = 1$ *then* $\texttt{sim}(A, C) = \texttt{sim}(B, C)$*.*

2.2.1.2 Principles based on Logical Arguments

The following principle states that the similarity between arguments must be independent from the syntax (i.e. the name of the variables).

Principle 5 (Syntax Independence)**.** *A similarity measure* $\texttt{sim}$ *satisfies* Syntax Independence *iff for any renaming function* π*, for all* $A, B, A', B' \in \texttt{Arg}(\mathcal{L})$ *such that:*

- *A and A' are isomorphic w.r.t. π,*
- *B and B' are isomorphic w.r.t. π,*

it holds that $\texttt{sim}(A, B) = \texttt{sim}(A', B')$*.*

The next principle, called Minimality, ensures that similarity depends on the *content* of the arguments. It states that if two arguments do not share any variables, then they are completely different. An example of such arguments are $\langle\{p\}, p \vee q\rangle$ and $\langle\{t\}, t\rangle$. Note that a variable may appear in the conclusion of an argument even though it is not used in the support (q in the case of $\langle\{p\}, p \vee q\rangle$).

Principle 6 (Minimality)**.** *A similarity measure* $\mathtt{sim}$ *satisfies* Minimality *iff for all* $A, B \in \mathtt{Arg}(\mathcal{L})$*, if*

- *A and B are not equivalent,*
- $\bigcup_{\phi_i \in \mathtt{Supp}(A)} \mathtt{Var}(\phi_i) \cap \bigcup_{\phi_j \in \mathtt{Supp}(B)} \mathtt{Var}(\phi_j) = \emptyset$ *and*
- $\mathtt{Var}(\mathtt{Conc}(A)) \cap \mathtt{Var}(\mathtt{Conc}(B)) = \emptyset$*,*

then $\mathtt{sim}(A, B) = 0$.

Note that the first condition excludes the special case of two trivial arguments while keeping the possibility to compare a trivial with a non-trivial.

The following principle, called Non-Zero, considers that when two arguments have common information (i.e. equivalent formulas) in their support, they present some similarity.

Principle 7 (Non-Zero)**.** *A similarity measure* $\mathtt{sim}$ *satisfies* Non-Zero *iff for all* $A, B \in \mathtt{Arg}(\mathcal{L})$*, if* $\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B)) \neq \emptyset$*, then* $\mathtt{sim}(A, B) > 0$.

The next principle, called (Strict) Monotony, states that the similarity between two arguments increases as the supports of said arguments share more formulas. This means the similarity increases with the addition of common logically equivalent formulas or the deletion of distinct formulas.

Principle 8 (Monotony – Strict Monotony)**.** *A similarity measure* $\mathtt{sim}$ *satisfies* Monotony *iff for all* $A, B, C \in \mathtt{Arg}(\mathcal{L})$*, if*

1. $\mathtt{Conc}(A) \equiv \mathtt{Conc}(B)$ *or* $\mathtt{Var}(\mathtt{Conc}(A)) \cap \mathtt{Var}(\mathtt{Conc}(C)) = \emptyset$*,*
2. $\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(C)) \subseteq \mathtt{Co}(\mathtt{Supp}(A)), \mathtt{Supp}(B))$*,*
3. $\mathtt{Supp}(B) \setminus \mathtt{Co}(\mathtt{Supp}(B), \mathtt{Supp}(A)) = \mathtt{Co}(\mathtt{Supp}(B) \setminus \mathtt{Co}(\mathtt{Supp}(B), \mathtt{Supp}(A)), \mathtt{Supp}(C) \setminus \mathtt{Co}(\mathtt{Supp}(C), \mathtt{Supp}(A)))$*,*

then the following hold:

- $\texttt{sim}(A, B) \geq \texttt{sim}(A, C)$ ***(Monotony)***
- *If*
  - *the inclusion in condition 2 is strict, OR*
  - $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C)) \neq \emptyset$ *and* $|\texttt{Supp}(C) \setminus \texttt{Co}(\texttt{Supp}(C), \texttt{Supp}(A))| > |\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A))|$

  then $\texttt{sim}(A, B) > \texttt{sim}(A, C)$. ***(Strict Monotony)***

The (Strict) Monotony fundamentally compares the elements in the support of the arguments. For this reason, the first condition is to avoid the conclusions having an impact on the comparison, by ensuring that the conclusions of A and B are equivalent or that those of A and C are totally different. Then condition 2 indicates an inclusion of the elements in common between A and C in comparison to those between A and B. While condition 3, on the other hand, requires that elements of B distinct from A are also elements of C. In Amgoud and David [2018], the condition 3 was $\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) \subseteq \texttt{Supp}(C) \setminus \texttt{Co}(\texttt{Supp}(C), \texttt{Supp}(A))$, but there is a problem with the set inclusion. It is syntax dependent, i.e., equivalent formulas are not considered in the inclusion. This is why we propose a new condition 3.

Example 8. *Consider the arguments below, it illustrate the problem with the old condition 3.*

- $A = \langle \{p, p \rightarrow q\}, q \rangle$,
- $B = \langle \{p, t\}, p \wedge t \rangle$,
- $C = \langle \{t \wedge t\}, t \rangle$.

Here we can observe that B shares more information with A than C with A.Then B and C have (t, $t \wedge t$) in their supports.But given that they are syntactically different, with the old version of monotony we could not apply it. Therefore we propose the new condition 3 that holds for equivalent formulas.

We may observe that the similarity between the supports of B and A is greater than that between the supports of C and A, because:

- The common formulas between the supports of C and A are included in the common one between B and A, i.e. the empty set (i.e., $\emptyset$) is included in $\{p\}$.
- The different formulas of the support of B with respect to A and that of C with respect to A are equivalent (t and $t \wedge t$). However, they are syntactically different, and according to the old version of Monotony, we may not apply it. Here we propose a new condition 3 that holds for equivalent formulas.

Let us summarise this principle with another example.

Example 9. *Consider the arguments below.*

- $A = \langle \{p, p \to q\}, q \rangle$,
- $B = \langle \{p\}, p \rangle$,
- $C = \langle \{t\}, t \rangle$,
- $D = \langle \emptyset, t \vee \neg t \rangle$.

Monotony ensures that $\texttt{sim}(A, B) \geq \texttt{sim}(A, C)$*,* $\texttt{sim}(D, A) \geq \texttt{sim}(D, B)$ *and* $\texttt{sim}(D, B) \geq \texttt{sim}(D, A)$ *while Strict Monotony states that* $\texttt{sim}(A, B) > \texttt{sim}(A, C)$*. Note that if we extend the definition of Strict Monotony by allowing strict inclusion in condition 3, then we get* $\texttt{sim}(D, B) > \texttt{sim}(D, A) \geq 0$*. Hence,* $\texttt{sim}(D, B) > 0$ *which is counter-intuitive.*

When we compare the similarity between two supports, the question is whether we should use the function $\texttt{Co}$ or $\texttt{CN}$? The advantages of one are the disadvantages of the other, and vice versa. Two examples will illustrate this issue:

- First: $\{p, p \to q\}$ and $\{q, q \to p\}$. Using $\texttt{Co}$, we distinguish the difference of reasoning between the two sets of formulas, but not with $\texttt{CN}$.
- Second: $\{p, q\}$ and $\{p \wedge q\}$. Using $\texttt{Co}$, the two sets are different while with $\texttt{CN}$, they are equivalent.

Since a principle is a desirable property, it is better to choose the more cautious choice. Here using $\texttt{Co}$ we are able to tell the difference in the first example but not the similarity in the second while $\texttt{CN}$ forces the first example to be similar (wrongly) but detects the similarity in the second case. Therefore, $\texttt{Co}$ is more cautious in comparing supports.

The next principle, called Dominance, ensures that the similarity between two logical arguments also depends on the conclusions of the arguments. The more consequences the conclusions have in common, the greater the similarity.

In the case of conclusions, the $\texttt{Co}$ function is not sufficient. It only allows determining if two conclusions are equivalent, but that is not precise enough. Regarding $\texttt{CN}$, between two conclusions, we may not have any loss of information due to an inference between several formulas (problematic of $\texttt{CN}$ between two supports). However, we are going to show that without restriction on $\texttt{CN}$ this Dominance is incompatible with Minimality.

We begin by presenting the function that we will use under the conditions of (Strict) Dominance. Once we define the constraints, we will discuss the different functions considered with their advantages and disadvantages.

In what follows, we define a finite $\texttt{CN}$ using only the dependent literals of a formula. In order to avoid working "up to equivalence" we will use a fixed set of formulas $\mathcal{F}$, which contains one formula of $\mathcal{L}$ per equivalence class (i.e., for every $\phi \in \mathcal{L}$, there exists a unique $\psi \in \mathcal{F}$ such that $\phi \equiv \psi$). Moreover, in order to simplify the presentation and be homogeneous, we assume that formulas from $\mathcal{F}$ are simplified with the minimum of literals and we assume that each $\phi \in \mathcal{F}$ contains only dependent literals.

Definition 41 (Dependent finite CN). *Let $\phi \in \mathcal{L}$, the dependent finite $\texttt{CN}$ is defined by* $\texttt{CN}_{df}(\phi) = \{\psi \in \texttt{CN}(\phi)$ *s.t.* $\psi \in \mathcal{F}$ *and* $\texttt{Lit}(\psi) \subseteq \texttt{DepLit}(\phi)\}$.

Let us illustrate this dependent finite CN.

Example 10. *Let $\phi = p$, $\psi = (p \vee q) \wedge (p \vee \neg q), \lambda = p \vee q, \delta = p \wedge q \in \mathcal{L}$.*
Then we obtain:

- $\texttt{CN}_{df}(\phi) = \{p\}$,
- $\texttt{CN}_{df}(\psi) = \{p\}$,
- $\texttt{CN}_{df}(\lambda) = \{p \vee q\}$,
- $\texttt{CN}_{df}(\delta) = \{p, q, p \vee q, p \wedge q\}$.

Let us present the conditions of the principle using this dependent finite CN.

Principle 9. *[Dominance – Strict Dominance] A similarity measure* $\texttt{sim}$ *satisfies* Dominance *iff for all $A, B, C \in \texttt{Arg}(\mathcal{L})$, if*

1. $\texttt{Supp}(B) \cong \texttt{Supp}(C)$,
2. $\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C)) \subseteq \texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(B))$,
3. $\texttt{CN}_{df}(\texttt{Conc}(B)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A)) \subseteq \texttt{CN}_{df}(\texttt{Conc}(C)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A))$,

then the following hold:

- $\texttt{sim}(A, B) \geq \texttt{sim}(A, C)$. ***(Dominance)***
- *If the inclusion in condition 2 is strict or,* $\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C)) \neq \emptyset$ *and condition 3 is strict, then* $\texttt{sim}(A, B) > \texttt{sim}(A, C)$. ***(Strict Dominance)***

Let us illustrate this last principle.

Example 11. *Consider the three arguments below.*

- $A = \langle \{p \wedge q \wedge t\}, p \rangle$,
- $B = \langle \{p \wedge q \wedge t\}, p \wedge q \rangle$,
- $C = \langle \{p \wedge q \wedge t\}, p \wedge q \wedge t \rangle$.

Dominance ensures that $\mathtt{sim}(A, B) \geq \mathtt{sim}(A, C)$ *and* $\mathtt{sim}(C, B) \geq \mathtt{sim}(C, A)$. *Strict Dominance ensures* $\mathtt{sim}(A, B) > \mathtt{sim}(A, C)$ *and* $\mathtt{sim}(C, B) > \mathtt{sim}(C, A)$.

First, let us see why we defined this dependent finite CN instead of using the classical CN. The reason is that we do not want to give similarity between arguments that have no information in common (i.e. Minimality). To illustrate the problem with classical CN, consider the following example.

Example 12. *Consider the three arguments below.*

- $A = \langle \{p\}, p \rangle$,
- $B = \langle \{q \wedge t\}, q \rangle$,
- $C = \langle \{q \wedge t\}, q \wedge t \rangle$.

If we apply the Strict Dominance with the classical CN, we get that $\mathtt{sim}(A, B) > \mathtt{sim}(A, C)$ which is incompatible with the Minimality principle (which ensures that sim $(A, B) = 0$). The problem comes from the fact that we may deduce a formula from any other formula (e.g., $p \vdash p \vee q$).

Hence, every formula has a deduction in common (e.g. p and q both infer $p \vee q$). Our goal is to give a degree of similarity based only on the content present in the formulas (i.e. the literals). The remaining question is whether to restrict inferences to present literals or dependent literals. As for the choice between Co and CN in the supports, we have chosen the less restrictive one. It is worth noting that using present literals allows a better accuracy. For instance, between the formulas $(p \vee q) \wedge (p \vee \neg q)$ and $p \vee q$, we will detect the common inference $p \vee q$. Whereas with dependent literals this is not possible (because the first formula reduces to p and thus $p \vee q$ is no longer inferable). However, with dependent literals, we ensure that for any equivalent conclusion (i.e. let $\phi, \psi \in \mathcal{L}$, $\mathtt{CN}(\phi) = \mathtt{CN}(\psi)$), they obtain the same degree of similarity. On the other hand, with the use of a CN restricted to literals this is not guaranteed. This is because with different literals we can construct different equivalent formulas (e.g., $(p \vee q) \wedge (p \vee \neg q) \equiv p$). The formulas deduced from a CN restricted to present literals will therefore not be the same, whereas for a CN restricted to dependent literals they will be (thanks to the uniqueness of the dependent literals). This is why we used the dependent finite CN to define (Strict) Dominance. About the finitude of this new $\mathtt{CN}$, its addition is due to the pointlessness of having several equivalent formulas, and it will be useful for the definition of the similarity measure.

Note that, in order not to limit our representation of arguments, we accept that an argument may conclude new literals by disjunction. However, once the argument is defined, we do not want to use the set of inferences on the non-present literals (to reason only on the present information).

Moreover, to check for common information between conclusions, we also thought of using models (i.e., $\mathtt{Mod}$). Unfortunately, between inconsistent formulas the models do not allow to detect common information.

Example 13. *Consider the three arguments below.*

- $A = \langle\{p \wedge \neg t\}, p \wedge \neg t\rangle$,
- $B = \langle\{p \wedge t\}, p \wedge t\rangle$,
- $C = \langle\{p \wedge t\}, t\rangle$.

Strict Dominance ensures $\mathtt{sim}(A, B) > \mathtt{sim}(A, C)$ *because we use* $\mathtt{CN}$ *while using models we have* $\mathtt{Mod}(\mathtt{Conc}(A)) \cap \mathtt{Mod}(\mathtt{Conc}(B)) = \mathtt{Mod}(\mathtt{Conc}(A)) \cap \mathtt{Mod}(\mathtt{Conc}(C)) = \emptyset$ *and so we cannot distinguish any difference between these conclusions.*

Note that compared to the definition in Amgoud and David [2018], we remove the condition in the strict version saying that the supports of A and B must have a common element. This principle focuses on the conclusion.

Example 14. *Let* $A, B, C \in \mathtt{Arg}(\mathcal{L})$ *such that:*

- $A = \langle\{t, p\}, t \wedge (p \vee q)\rangle$,
- $B = \langle\{t, q\}, t \wedge (p \vee q)\rangle$,
- $C = \langle\{t, q\}, t \wedge q\rangle$.

The principle Strict Dominance ensure that $\mathtt{sim}(A, B) > \mathtt{sim}(A, C)$.

Two limitations can be observed in the old version:

1. Constraining the supports to have a common formula is limited by the syntax, as we will see below with the arguments $\{A_1, B_1, C_1\}$ there is no common formula due to their syntax whereas they are semantically equivalent.

2. As explained before, it is accepted that an argument infers a literal not present in the support, therefore without information in common in the support (as we will see below with arguments $\{A_2, B_2, C_2\}$) we can still have similarity in the conclusions.

Therefore if we accept to use the Strict Dominance in the example 14 then we can accept and generalize the principle to be used in the example 15.

Example 15. *Let* $A_1, B_1, C_1, A_2, B_2, C_2 \in \texttt{Arg}(\mathcal{L})$ *such that:*

- $A_1 = \langle \{t, p\}, t \wedge (p \vee q) \rangle$,
- $B_1 = \langle \{t \wedge q\}, t \wedge (p \vee q) \rangle$,
- $C_1 = \langle \{t \wedge q\}, t \wedge q \rangle$.
- $A_2 = \langle \{p\}, p \vee q \rangle$,
- $B_2 = \langle \{q\}, p \vee q \rangle$,
- $C_2 = \langle \{q\}, q \rangle$.

The principle Strict Dominance ensure that $\texttt{sim}(A_1, B_1) > \texttt{sim}(A_1, C_1)$ *and* $\texttt{sim}(A_2, B_2) > \texttt{sim}(A_2, C_2)$.

The next principle, called Independent Distribution, concerns the non-importance of the location of different elements between pairs of arguments. It ensures that when two pairs of arguments have the same common and different elements, no matter how these different elements are distributed, their similarity is equal.

Principle 10 (Independent Distribution)**.** *A similarity measure* sim *satisfies* Independent Distribution *iff for all* $A, B, A', B' \in \texttt{Arg}(\mathcal{L})$*, if*

- $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(B)) = \texttt{Var}(\texttt{Conc}(A')) \cap \texttt{Var}(\texttt{Conc}(B')) = \emptyset$,
- $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) \cong \texttt{Co}(\texttt{Supp}(A'), \texttt{Supp}(B'))$,
- $\texttt{Supp}(A) \cup \texttt{Supp}(B) \cong \texttt{Supp}(A') \cup \texttt{Supp}(B')$,

then $\texttt{sim}(A, B) = \texttt{sim}(A', B')$

Example 16. *Let* $A, B, A', B' \in \texttt{Arg}(\mathcal{L})$*:*

- $A = \langle \{p, p \rightarrow q\}, q \rangle$,
- $B = \langle \{p, p \rightarrow r\}, r \rangle$,
- $A' = \langle \{p, p \rightarrow q, p \rightarrow r\}, q \wedge r \rangle$,
- $B' = \langle \{p\}, p \rangle$.

With the axiom of Independent Distribution we have, $\texttt{sim}(A, B) = \texttt{sim}(A', B')$.

To conclude the axiomatic study, let us see the links between these principles.

2.2.2 Compatibility and Dependency Results

All the principles (1 to 10) are compatible, in that they can be satisfied all together by a similarity measure.

Proposition 2. *All the principles are compatible.*

The principles are *independent*, i.e. none of them follows from the others. A notable exception is Substitution, which follows from a subset of principles.

Proposition 3. *If a similarity measure* $\mathtt{sim}$ *satisfies Symmetry, Maximality, Strict Monotony, Dominance, and Strict Dominance, then* $\mathtt{sim}$ *satisfies Substitution.*

Regarding Non-Zero, one may wonder whether Strict Monotony and Minimality imply it, but they do not. Let two arguments A and B having common formulas and different formulas in their support, and C an argument completely different from A and B.

- According to Non-Zero the similarity between A and B is strictly greater than 0.
- According to Minimality the similarity between A and C is equal to 0.
- We would like to apply Strict Monotony such that $\mathtt{sim}(A, B) > \mathtt{sim}(A, C)$ and knowing that $\mathtt{sim}(A, C) = 0$ (Minimality), we would have the same result as in Non-Zero.
- However, Strict Monotony may not be applied, as the different formulas of B with respect to A are not in C (see condition 3 of Monotony). In other words, Strict Monotony with Minimality implies only a special case of Non-Zero (the one where the differences of B from A are included in C).

Let us consider some consequences of satisfying the proposed principles. We propose a characterization of all cases where the similarity between two arguments is maximal (equal to 1). Let us present the result gradually. First, we show that any measure satisfying Maximality and Monotony declares that equivalent arguments are completely similar.

Theorem 7. *Let* $\mathtt{sim}$ *be a similarity measure that satisfies Maximality and Monotony. For all* $A, B \in \mathtt{Arg}(\mathcal{L})$,

$$\text{if } A \approx B, \text{ then } \mathtt{sim}(A, B) = 1.$$

Then we show that, if in addition to Maximality, a similarity measure satisfies Strict Monotony and Strict Dominance, then two fully similar arguments are necessarily equivalent.

Theorem 8. *Let* $\mathtt{sim}$ *be a similarity measure that satisfies Maximality, Strict Monotony, and Strict Dominance. For all* $A, B \in \mathtt{Arg}(\mathcal{L})$ *the following holds:*

$$\textit{if } \mathtt{sim}(A, B) = 1 \textit{ then } A \approx B.$$

From the two previous results, it follows that any similarity measure that satisfies Maximality, Monotony, Strict Monotony and Strict Dominance assigns the maximum value 1 to pairs of equivalent arguments and *only* to equivalent pairs.

Corollary 1. *Let* $\mathtt{sim}$ *be a similarity measure that satisfies Maximality, Monotony, Strict Monotony, and Strict Dominance. For all* $A, B \in \mathtt{Arg}(\mathcal{L})$ *the following holds:*

$$\mathtt{sim}(A, B) = 1 \textit{ iff } A \approx B.$$

The following result shows that a non-trivial argument is completely different from any trivial arguments. This is ensured when the similarity measure satisfies Minimality, and Substitution.

Proposition 4. *Let* $\mathtt{sim}$ *be a similarity measure which satisfies Minimality, and Substitution. For all* $A, B \in \mathtt{Arg}(\mathcal{L})$*,*

$$\textit{if } A \textit{ is non-trivial and } B \textit{ is trivial}, \textit{ then } \mathtt{sim}(A, B) = 0.$$

The next result shows that a non-trivial sub-argument has always some similarity with its argument. This is the case when the similarity measure satisfies Strict Monotony, or Non-Zero.

Proposition 5. *Let* $\mathtt{sim}$ *be a similarity measure which satisfies Strict Monotony, or Non-Zero. For all* $A, B \in \mathtt{Arg}(\mathcal{L})$*,*

$$\textit{if } B \sqsubset A \textit{ and B is non-trivial}, \textit{ then } \mathtt{sim}(A, B) > 0.$$

Similarity measures satisfying Monotony satisfy some monotony property regarding the sub-argument relationship between arguments.

Proposition 6. *Let* $\mathtt{sim}$ *be a similarity measure which satisfies Monotony. For all* $A, B, C \in \mathtt{Arg}(\mathcal{L})$*, if*

- $\mathtt{Var}(\mathtt{Conc}(A)) \cap \mathtt{Var}(\mathtt{Conc}(C)) = \emptyset$*, and*
- $C \sqsubset B \sqsubset A$*,*

then $\mathtt{sim}(A, B) \geq \mathtt{sim}(A, C)$*.*

Strict Dominance ensures that the more consequences shared by the conclusions of two arguments, the more similar the arguments are.

Proposition 7. *Let* $\texttt{sim}$ *be a similarity measure which satisfies Strict Dominance. For all* $A, B, C \in \texttt{Arg}(\mathcal{L})$*, if*

- A, B, C *are non trivial,*
- $\texttt{Supp}(A) \cong \texttt{Supp}(B) \cong \texttt{Supp}(C)$*,*
- $\texttt{Conc}(A) \vdash \texttt{Conc}(B) \vdash \texttt{Conc}(C)$*,*
- $\texttt{Conc}(C) \nvdash \texttt{Conc}(B), \texttt{Conc}(B) \nvdash \texttt{Conc}(A)$*,*

then $\texttt{sim}(A, B) > \texttt{sim}(A, C)$.

The last result states that the union of supports of two fully similar arguments is consistent. This is particularly the case for similarity measures that satisfy Maximality, Strict Monotony, and Strict Dominance.

Proposition 8. *Let* $\texttt{sim}$ *be a similarity measure which satisfies Maximality, Strict Monotony, and Strict Dominance. For all* $A, B \in \texttt{Arg}(\mathcal{L})$*, if* $\texttt{sim}(A, B) = 1$*, then* $\texttt{Supp}(A) \cup \texttt{Supp}(B)$ *is consistent.*

We have seen a set of principles for similarity measures between logical arguments, followed by a study of these properties. However, it turns out that for the principles related to supports, another vision of similarity is possible. Indeed, it is possible according to definition 36, that an argument has irrelevant information in its support. For instance the argument $\langle \{p \wedge q\}, p \rangle\rangle$ is well-formed, i.e. consistent, minimal in the sense of set inclusion and the support infers the conclusion. However, the knowledge q is useless for inferring p, as it is only syntactically related to p. Such arguments are non-concise and assessment of their similarities with other arguments may lead to inaccurate results.

2.3 Concise Arguments

To determine the degree of similarity between logical arguments, different methods may be designed. Since arguments have a set of logical formulas in their support and a logical formula in their conclusion, intuitively it may be worthwhile to adapt the symbolic data similarity measures to compute a degree of similarity between logical arguments. For instance, the function Co defined in the previous section 2.1 may be used to count the number of formulas in common. However, as seen before, using classical definition of a logical argument (Def. 36) may result in irrelevant information in formulas used for the conclusion. This may affect the accuracy of a similarity measure.

Let us illustrate the problem of assigning a degree of similarity between arguments when one of them has irrelevant information. Assume that they are measured according to the common equivalent formulas included in the arguments. Hence, the two arguments $A = \langle\{p \wedge q\}, p\rangle$ and $B = \langle\{p\}, p\rangle$ are not completely similar, since $p \wedge q \not\equiv p$. Nevertheless, they support the same conclusion and on basis of the same ground (p). This is due to the non-concision of A, which contains the unnecessary information q in its support. In what follows, we refine the arguments by removing such information from their supports. The idea is to weaken the formulas in the supports.

Definition 42 (Refinement)**.** *Let* $A, B \in \mathtt{Arg}(\mathcal{L})$ *such that* $A = \langle\{\phi_1, \cdots, \phi_n\}, \phi\rangle$ *and* $B = \langle\{\phi'_1, \cdots, \phi'_n\}, \phi'\rangle$. *B is a* refinement *of A iff:*

1. $\phi = \phi'$,
2. *There exists a permutation* ρ *of the set* $\{1, \cdots, n\}$ *such that* $\forall k \in \{1, \cdots, n\}$, $\phi_k \vdash \phi'_{\rho(k)}$ *and* $\mathtt{Lit}(\phi'_{\rho(k)}) \subseteq \mathtt{DepLit}(\phi_k)$.

Let Ref *be a function that returns the set of all refinements of a given argument.*

The first condition ensures that a refined argument does not change the conclusion of its original argument. In addition, we set a strong condition with equality and not equivalence in order to simplify the set of refinements of an argument.

The second condition states that each formula of an argument's support is weakened. Furthermore, novel literals are not allowed in the weakening step since such literals would negatively impact similarity between supports of arguments. Finally, literals from which a formula is independent should be removed since they are useless for inferring the conclusion of an argument.

It is worth mentioning that an argument may have several refinements as shown in the following example.

Example 17. *The following pairs are all arguments.*

$A = \langle\{p \wedge q\}, p\rangle$	$B = \langle\{p\}, p\rangle$
$C = \langle\{p \wedge q \wedge r\}, r\rangle$	$D = \langle\{p \wedge q, p \wedge r\}, p \wedge q \wedge r\rangle$
$E = \langle\{p \wedge q, (p \vee q) \rightarrow r\}, r\rangle$	$F = \langle\{p \wedge q\}, p \vee q\rangle$

The following set are subset of refinement of these arguments.

- $\{\langle\{p\}, p\rangle, \langle\{p \wedge p\}, p\rangle\} \subseteq \mathtt{Ref}(A)$
- $\{\langle\{p \wedge r\}, r\rangle, \langle\{q \wedge r\}, r\rangle, \langle\{r\}, r\rangle\} \subseteq \mathtt{Ref}(C)$

- $\{\langle\{p \wedge q, r\}, p \wedge q \wedge r\rangle, \langle\{q, p \wedge r\}, p \wedge q \wedge r\rangle\} \subseteq \texttt{Ref}(D)$
- $\{\langle\{p \vee q, (p \vee q) \rightarrow r\}, r\rangle, \langle\{p, p \rightarrow r\}, r\rangle, \langle\{q, q \rightarrow r\}, r\rangle\} \subseteq \texttt{Ref}(E)$
- $\{\langle\{p\}, p \vee q\rangle, \langle\{q\}, p \vee q\rangle, \langle\{p \vee q\}, p \vee q\rangle\} \subseteq \texttt{Ref}(F)$

The following property shows that there exists a unique possible permutation ρ for each refinement of an argument.

Proposition 9. *For all $A = \langle\{\phi_1, \cdots, \phi_n\}, \phi\rangle, B = \langle\{\phi'_1, \cdots, \phi'_n\}, \phi\rangle \in \texttt{Arg}(\mathcal{L})$ such that $B \in \texttt{Ref}(A)$, there exists a unique permutation ρ of the set $\{1, \cdots, n\}$ such that $\forall k \in \{1, \cdots, n\}$, $\phi_k \vdash \phi'_{\rho(k)}$.*

A trivial argument is the only refinement of itself.

Proposition 10. *For any trivial argument $A \in \texttt{Arg}(\mathcal{L})$, $\texttt{Ref}(A) = \{A\}$.*

A non-trivial argument has a non-empty set of refinements. Moreover, it is a refinement of itself only if the formulas of its support do not contain literals from which they are independent.

Proposition 11. *Let $A \in \texttt{Arg}(\mathcal{L})$ be a non-trivial argument. The following hold:*

- $\texttt{Ref}(A) \neq \emptyset$,
- $A \in \texttt{Ref}(A)$ *iff* $\forall \phi \in \texttt{Supp}(A)$, $\texttt{Lit}(\phi) = \texttt{DepLit}(\phi)$.

We show next that the function Ref is idempotent and that equivalent arguments have the same refinements.

Proposition 12. *Let $A, B \in \texttt{Arg}(\mathcal{L})$. The following hold:*

- *If $B \in \texttt{Ref}(A)$, then $\texttt{Ref}(B) \subseteq \texttt{Ref}(A)$.*
- *If $A \approx B$, then $\texttt{Ref}(A) = \texttt{Ref}(B)$.*

We are now ready to define the notion of concise argument. An argument is concise if it is equivalent to any of its refinements. This means that a concise argument cannot be further refined.

Definition 43 (Conciseness)**.** *An argument $A \in \texttt{Arg}(\mathcal{L})$ is* concise *iff for all $B \in \texttt{Ref}(A)$, $A \approx B$.*

Example 17 (Cont.) The two refinements $\langle\{p \wedge r\}, r\rangle$ and $\langle\{q \wedge r\}, r\rangle$ of the argument C are not concise. Indeed, $\langle\{r\}, r\rangle \in \texttt{Ref}(\langle\{p \wedge r\}, r\rangle)$, $\langle\{r\}, r\rangle \in \texttt{Ref}(\langle\{q \wedge r\}, r\rangle)$ while $\langle\{r\}, r\rangle \not\approx \langle\{p \wedge r\}, r\rangle$, and $\langle\{r\}, r\rangle \not\approx \langle\{q \wedge r\}, r\rangle$.

For any argument from $\texttt{Arg}(\mathcal{L})$, we generate its concise versions. The latter are simply its concise refinements.

Definition 44 (Concise Refinements)**.** *A concise refinement of an argument $A \in \texttt{Arg}(\mathcal{L})$ is any concise argument B such that $B \in \texttt{Ref}(A)$. We denote the set of all concise refinements of A by $\texttt{CR}(A)$.*

Example 17 (Cont.)

- $\langle \{p\}, p\rangle \in \texttt{CR}(A)$
- $\langle \{r\}, r\rangle \in \texttt{CR}(C)$
- $\{\langle \{p \wedge q, r\}, p \wedge q \wedge r\rangle, \langle \{q, p \wedge r\}, p \wedge q \wedge r\rangle\} \subseteq \texttt{CR}(D)$
- $\{\langle \{p \vee q, (p \vee q) \rightarrow r\}, r\rangle, \langle \{p, p \rightarrow r\}, r\rangle, \langle \{q, q \rightarrow r\}, r\rangle\} \subseteq \texttt{CR}(E)$
- $\{\langle \{p\}, p \vee q\rangle, \langle \{q\}, p \vee q\rangle, \langle \{p \vee q\}, p \vee q\rangle\} \subseteq \texttt{CR}(F)$

Next we state some properties of concise refinements.

Proposition 13. *Let $A \in \texttt{Arg}(\mathcal{L})$. The following hold:*

1. *For any $B \in \texttt{CR}(A)$ the following hold: $B \in \texttt{Ref}(B)$ and $\forall C \in \texttt{Ref}(B)$, $C \approx B$.*
2. $\texttt{CR}(A) \neq \emptyset$.
3. *If A is non-trivial, then $\texttt{CR}(A)$ is infinite.*
4. *If $A \approx B$, then $\texttt{CR}(A) = \texttt{CR}(B)$.*
5. $\forall B \in \texttt{Ref}(A)$, $\texttt{CR}(B) \subseteq \texttt{CR}(A)$.

The following result shows that any formula in the support of a concise argument cannot be further weakened without introducing additional literals.

Proposition 14. *Let $A, B \in \texttt{Arg}(\mathcal{L})$ such that $B \in \texttt{CR}(A)$. For any $\phi \in \texttt{Supp}(B)$, if $\exists \psi \in \mathcal{L}$ such that $\phi \vdash \psi$, $\psi \nvdash \phi$, and $\langle (\texttt{Supp}(B) \setminus \{\phi\}) \cup \{\psi\}, \texttt{Conc}(B)\rangle \in \texttt{Arg}(\mathcal{L})$, then $\texttt{Lit}(\psi) \setminus \texttt{Lit}(\phi) \neq \emptyset$.*

2.4 Similarity Measures

In this section, we will present various similarity measure, for logical arguments. We start by those that deal only with concise arguments. In other words, we assume that arguments are concise. We provide two kinds of measures: syntactic measures and mixed ones. The latter are based on syntax for comparing supports and semantics for comparing conclusions of arguments. In a second step we will see how to use these measures on non-concise arguments.

2.4.1 Syntactic Similarity Measures

The measures we propose are based on measures that exist in the litterature, namely those recalled in table 1.4, in Section 1.3.2.1. Indeed, we adapt Jaccard measure (Jaccard [1901]), Dice measure (Dice [1945]), Sorensen one (Sørensen [1948]), and those proposed in (Anderberg [1973]; Sneath *et al.* [1973]; Ochiai [1957]; Kulczynski [1927]). These measures are suitable in the argumentation context since an argument may be seen as a pair of two sets: one set containing the formulas of the support and another one containing the conclusion. In what follows, we use these measures for assessing similarity between supports (respectively conclusions) of pairs of arguments. However, those measures cannot be applied directly to supports of arguments since supports may have different but still equivalent formulas. For instance, the two sets $\{p\}$ and $\{p \wedge p\}$ are equivalent while their intersection is empty. Thus, we extend each measure of Table 1.4 using the function Co as shown in Table 2.1 in case of non-empty sets.

Extended Jaccard	$s_{\mathtt{j}}(\Phi, \Psi) = \frac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert+\lvert\Psi\rvert-\lvert\mathtt{Co}(\Phi,\Psi)\rvert}$
Extended Dice	$s_{\mathtt{d}}(\Phi, \Psi) = \frac{2\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert+\lvert\Psi\rvert}$
Extended Sorensen	$s_{\mathtt{s}}(\Phi, \Psi) = \frac{4\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert+\lvert\Psi\rvert+2\lvert\mathtt{Co}(\Phi,\Psi)\rvert}$
Extended Symmetric Anderberg	$s_{\mathtt{a}}(\Phi, \Psi) = \frac{8\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert+\lvert\Psi\rvert+6\lvert\mathtt{Co}(\Phi,\Psi)\rvert}$
Extended Sokal and Sneath 2	$s_{\mathtt{ss}}(\Phi, \Psi) = \frac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{2(\lvert\Phi\rvert+\lvert\Psi\rvert)-3\lvert\mathtt{Co}(\Phi,\Psi)\rvert}$
Extended Ochiai	$s_{\mathtt{o}}(\Phi, \Psi) = \frac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\sqrt{\lvert\Phi\rvert}\sqrt{\lvert\Psi\rvert}}$
Extended Kulczynski 2	$s_{\mathtt{ku}}(\Phi, \Psi) = \frac{1}{2}\left(\frac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert} + \frac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Psi\rvert}\right)$

Table 2.1: Similarity measures for sets $\Phi, \Psi \subseteq_f \mathcal{L}$

Note that the original definitions compare non-empty sets. In the argumentation context, trivial arguments have an empty support. Thus, the definition of each measure follows the following schema that we illustrate with the Jaccard-based measure. For all $\Phi, \Psi \subseteq_f \mathcal{L}$,

$$s_{\mathtt{j}}(\Phi, \Psi) = \begin{cases} \frac{\lvert\mathtt{Co}(\Phi,\Psi)\rvert}{\lvert\Phi\rvert+\lvert\Psi\rvert-\lvert\mathtt{Co}(\Phi,\Psi)\rvert} & \text{if } \Phi \neq \emptyset, \Psi \neq \emptyset \\ 1 & \text{if } \Phi = \Psi = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Let us illustrate the definition of extended Jaccard measure by the following example.

Example 18. *Consider the following sets of formulas:*

- $\Phi_0 = \{p, q\}$,
- $\Phi_1 = \{r, s, r \wedge s \rightarrow t\}$,

- $\Phi_2 = \{r, s, z, r \wedge s \wedge z \rightarrow u\}$,
- $\Phi_3 = \{\neg\neg r, s\}$, *and*
- $\Phi_4 = \{r, \neg\neg s\}$.

It can be checked that $s_{\mathtt{j}}(\Phi_0, \Phi_1) = 0$, $s_{\mathtt{j}}(\Phi_1, \Phi_2) = 0.4$, $s_{\mathtt{j}}(\Phi_1, \Phi_3) = 0.66$, $s_{\mathtt{j}}(\Phi_2, \Phi_3) = 0.5$, *and* $s_{\mathtt{j}}(\Phi_3, \Phi_4) = 1$.

The measures of Table 2.1 evaluate in the same way pairs of sets containing each one formula. They assign value 1 if the two formulas of the sets are equivalent and 0 otherwise.

Proposition 15. *For any* $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, *for all* $\phi, \psi \in \mathcal{L}$, *the following holds:*

$$s_x(\{\phi\}, \{\psi\}) = \begin{cases} 1 & \text{if } \phi \equiv \psi \\ 0 & \text{otherwise.} \end{cases}$$

We are now ready to introduce our similarity measures between pairs of logical arguments. They are *syntactic* in nature, and are based on a parameter $\sigma \in]0, 1[$ which allows a user to give different importance degrees to supports and conclusions. Indeed, one may declare two arguments as similar as soon as they have quite equivalent supports, or my be more requiring by ensuring that the conclusions also are equivalent. Due to the previous result, the same measure is used for assessing similarity between supports an similarity between conclusions of pairs of logical arguments.

Definition 45 (Extended Measures)**.** *Let* $0 < \sigma < 1$. *We define* $\mathtt{sim}_x^\sigma$, *with* $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, *as a function assigning to any pair* $(A, B) \in \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$ *a value*

$$\mathtt{sim}_x^\sigma(A, B) = \sigma \cdot s_x(\mathtt{Supp}(A), \mathtt{Supp}(B)) \ + (1 - \sigma) \cdot s_x(\{\mathtt{Conc}(A)\}, \{\mathtt{Conc}(B)\}).$$

Note that σ cannot take the value 0 since the corresponding similarity measure would ignore the supports of arguments, and cannot get value 1 since the measure would ignore the conclusions. Both cases are undesirable since an argument is a pair (support, conclusion).

Example 18 (Cont.) Let $\sigma = 0.5$ and $x = s_{\mathtt{j}}$. Consider the following arguments:

- $A_0 = \langle\{p, q\}, p \wedge q\rangle$,
- $A_1 = \langle\{r, s, r \wedge s \rightarrow t\}, t\rangle$,
- $A_2 = \langle\{r, s, z, r \wedge s \wedge z \rightarrow u\}, u\rangle$,

- $A_3 = \langle \{\neg\neg r, s\}, r \wedge s \rangle$, and
- $A_4 = \langle \{r, \neg\neg s\}, r \wedge \neg\neg s \rangle$.

It can be checked that we get the following values: $\mathtt{sim}_{\mathtt{j}}^{0.5}(A_0, A_1) = 0$, $\mathtt{sim}_{\mathtt{j}}^{0.5}(A_1, A_2) = 0.5 \times 0.4 + 0.5 \times 0 = 0.2$, $\mathtt{sim}_{\mathtt{j}}^{0.5}(A_1, A_3) = 0.5 \times 0.66 + 0.5 \times 0 = 0.33$, $\mathtt{sim}_{\mathtt{j}}^{0.5}(A_2, A_3) = 0.5 \times 0.5 + 0.5 \times 0 = 0.25$, and $\mathtt{sim}_{\mathtt{j}}^{0.5}(A_3, A_4) = 1$.

Due to Proposition 15, the definition of the Extended Measures can be simplified as follows:

Proposition 16. *For any $0 < \sigma < 1$, for any $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, for all $(A, B) \in \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$, the following property holds:*

$$\mathtt{sim}_x^\sigma(A, B) = \begin{cases} \sigma \cdot s_x(\mathtt{Supp}(A), \mathtt{Supp}(B)) + (1 - \sigma) \\ \qquad\qquad \textit{if } \mathtt{Conc}(A) \equiv \mathtt{Conc}(B) \\ \sigma \cdot s_x(\mathtt{Supp}(A), \mathtt{Supp}(B)) \ \textit{otherwise.} \end{cases}$$

We show that any measure $\mathtt{sim}_x^\sigma$ assigns values from the unit interval $[0, 1]$ to any pair of arguments. Thus, any $\mathtt{sim}_x^\sigma$ is a similarity measure in the sense of Definition 40.

Proposition 17. *For any $0 < \sigma < 1$, for any $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, for all $A, B \in \mathtt{Arg}(\mathcal{L})$, $\mathtt{sim}_x^\sigma(A, B) \in [0, 1]$. Hence, $\mathtt{sim}_x^\sigma$ is a similarity measure.*

Similarity measures $\mathtt{sim}_x^\sigma$ satisfy a same group of principles, violate all Strict Dominance and the satisfaction of principles Triangle Inequality and Independent Distribution differ.

Theorem 9. *For any $0 < \sigma < 1$:*

- *for any $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, $\mathtt{sim}_x^\sigma$*
  - *violates Strict Dominance, and*
  - *satisfies Maximality, Symmetry, Substitution, Syntax Independence, Minimality, Non-Zero, Monotony, Strict Monotony and Dominance.*
- *for any $x \in \{\mathtt{d}, \mathtt{s}, \mathtt{a}\}$, $\mathtt{sim}_x^\sigma$*
  - *violates Triangle Inequality, and*
  - *satisfies Independent Distribution.*
- *for any $x \in \{\mathtt{o}, \mathtt{ku}\}$, $\mathtt{sim}_x^\sigma$*
  - *violates Triangle Inequality and Independent Distribution.*

- *for any* $x \in \{\mathtt{j}, \mathtt{ss}\}$, $\mathtt{sim}_x^\sigma$
  - *satisfies Triangle Inequality and Independent Distribution.*

As we showed in Theorem 9, some measures $\mathtt{sim}_x^\sigma$ with $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, satisfy and violate the same set of principles. An interesting question is thus: are there any links between these measures? Do they return the same values? For answering these questions, we introduce the notion of *equivalent measures*, borrowed from Lesot *et al.* [2009a] and other like Lerman [1967].

Definition 46 (Equivalent measures)**.** *Two similarity measures* $\mathtt{sim}$ *and* $\mathtt{sim}'$ *are equivalent iff for all* $A, B, C, D \in \mathtt{Arg}(\mathcal{L})$*,*

$$\mathtt{sim}(A, B) \leq \mathtt{sim}(C, D) \iff \mathtt{sim}'(A, B) \leq \mathtt{sim}'(C, D).$$

The following result compares the values assigned by each measure for a given pair of arguments. It shows that for a fixed σ, the three measure $\mathtt{sim}_{\mathtt{ss}}^\sigma$, $\mathtt{sim}_{\mathtt{j}}^\sigma$, $\mathtt{sim}_{\mathtt{d}}^\sigma$ give the lowest degree of similarity.

Theorem 10. *Let* $0 < \sigma < 1$*, for any* $A, B \in \mathtt{Arg}(\mathcal{L})$*:*

- $\mathtt{sim}_{\mathtt{ss}}^\sigma(A, B) \leq \mathtt{sim}_{\mathtt{j}}^\sigma(A, B) \leq \mathtt{sim}_{\mathtt{d}}^\sigma(A, B) \leq \mathtt{sim}_{\mathtt{s}}^\sigma(A, B) \leq \mathtt{sim}_{\mathtt{a}}^\sigma(A, B)$.
- $\mathtt{sim}_{\mathtt{ss}}^\sigma(A, B) \leq \mathtt{sim}_{\mathtt{j}}^\sigma(A, B) \leq \mathtt{sim}_{\mathtt{d}}^\sigma(A, B) \leq \mathtt{sim}_{\mathtt{o}}^\sigma(A, B) \leq \mathtt{sim}_{\mathtt{ku}}^\sigma(A, B)$.

From these results, it follows that there are two sets of equivalent similarity measures.

Corollary 2. *Let* $0 < \sigma < 1$.

- $\mathtt{sim}_{\mathtt{ss}}^\sigma$, $\mathtt{sim}_{\mathtt{j}}^\sigma$, $\mathtt{sim}_{\mathtt{d}}^\sigma$, $\mathtt{sim}_{\mathtt{s}}^\sigma$ *and* $\mathtt{sim}_{\mathtt{a}}^\sigma$ *are pairwise equivalent, and*
- $\mathtt{sim}_{\mathtt{ss}}^\sigma$, $\mathtt{sim}_{\mathtt{j}}^\sigma$, $\mathtt{sim}_{\mathtt{d}}^\sigma$, $\mathtt{sim}_{\mathtt{o}}^\sigma$ *and* $\mathtt{sim}_{\mathtt{ku}}^\sigma$ *are pairwise equivalent.*

Despite the fact of violating Strict Dominance, any measure $\mathtt{sim}_x^\sigma$ assigns the maximal value 1 only to equivalent arguments. This result generalizes the binary similarity measure defined in Amgoud *et al.* [2014], where arguments are either equivalent (value 1) or completely different (value 0).

Theorem 11. *For any* $0 < \sigma < 1$*, for any* $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$*, for all* $A, B \in \mathtt{Arg}(\mathcal{L})$*,*

$$\mathtt{sim}_x^\sigma(A, B) = 1 \text{ iff } A \approx B.$$

Measures $\mathtt{sim}_x^\sigma$ assign the minimal value 0 to pairs of arguments whose conclusions are not equivalent and their supports do not share any equivalent formula.

Theorem 12. *For any* $0 < \sigma < 1$, *for all* $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, *for all* $A, B \in \mathtt{Arg}(\mathcal{L})$,

$$\mathtt{sim}_x^\sigma(A, B) = 0 \text{ iff } \begin{cases} \mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = \emptyset \text{ and} \\ \mathtt{Conc}(A) \not\equiv \mathtt{Conc}(B). \end{cases}$$

As explained before (after Principle 8), assessing similarity semantically over a set of formulas (support) may lead to undesirable results. Therefore, it is safer to restrict to syntactic similarity concerning supports. However, regarding similarity between conclusions, it is interesting to take into account this semantic notion (as explained after the principle 9 - Dominance). In what follows, we introduce measures that combine a syntactic measure (for supports) with a semantic measure (for conclusions).

2.4.2 Mixed Syntactic and Semantic Similarity Measure

We have seen in the previous section, that syntactic measures $\mathtt{sim}_x^\sigma$ violate Strict Dominance. Thus, they do not distinguish between arguments such as $A = \langle\{p \wedge q \wedge t\}, p\rangle$, $B = \langle\{p \wedge q \wedge t\}, p \wedge q\rangle$, and $C = \langle\{p \wedge q \wedge t\}, p \wedge q \wedge t\rangle$. They all return $\mathtt{sim}_x^\sigma(A, B) = \mathtt{sim}_x^\sigma(A, C)$ while A is more similar to B than to C. Indeed, those measures are not able to capture the fact that A's conclusion is closer to B's conclusion than to C's. We therefore propose to use a semantic approach to compare conclusions. The objective of the following measure is to semantically capture the similarities between two formulas. To do so with inferences in common (such as p and $p \wedge q$) and without giving similarities between formulas with no dependent literals in common (such as p and q), we use the function $\mathtt{CN}_{df}$.

Definition 47 (CN-based Jaccard Measure)**.** *The CN-based Jaccard measure is a function* $s_{\mathtt{cnj}}$ *assigning for all* $\phi, \psi \in \mathcal{L}$, *the value:*

$$s_{\mathtt{cnj}}(\phi, \psi) = \frac{|\mathtt{CN}_{df}(\phi) \cap \mathtt{CN}_{df}(\psi)|}{|\mathtt{CN}_{df}(\phi) \cup \mathtt{CN}_{df}(\psi)|}$$

Let us illustrate the measure with an example.

Example 19. *Let* $\phi = p$, $\psi = q, \delta = p \vee q, \gamma = p \wedge q \in \mathcal{L}$.
By applying $s_{\mathtt{cnj}}$ *between these formulas we get:*

- $s_{\mathtt{cnj}}(\phi, \psi) = 0$,
- $s_{\mathtt{cnj}}(\phi, \delta) = 0$,
- $s_{\mathtt{cnj}}(\phi, \gamma) = \frac{1}{4} = 0.25$,
- $s_{\mathtt{cnj}}(\delta, \gamma) = \frac{1}{4} = 0.25$.

We are now ready to introduce the second set of similarity measures. They use any of the previous measures on supports and the novel $s_{\mathtt{cnj}}$ on conclusions.

Definition 48 (Mixed Extended Measure)**.** *Let* $0 < \sigma < 1$. *We define* $\mathtt{sim}^{\sigma}_{\mathtt{x-cnj}}$, *with* $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, *as a function assigning to any pair* $(A, B) \in \mathtt{Arg}(\mathcal{L}) \times \mathtt{Arg}(\mathcal{L})$ *a value*

$$\mathtt{sim}^{\sigma}_{\mathtt{x-cnj}}(A, B) = \sigma . s_{\mathtt{x}}(\mathtt{Supp}(A), \mathtt{Supp}(B)) + (1-\sigma) s_{\mathtt{cnj}}(\mathtt{Conc}(A), \mathtt{Conc}(B)).$$

The objective of these mixed measures is to overcome the lack of precision on the evaluation of the similarity between arguments. In terms of principles, this results in the aim of satisfying Strict Monotony. Having already studied the behavior of the 7 syntactic measures between supports, we will restrict the mixed measures to the $\mathtt{sim}_{\mathtt{j-cnj}}$ measure in the rest of the document.

Notation: we simplify $\mathtt{sim}_{\mathtt{j-cnj}}$ by $\mathtt{sim}_{\mathtt{cnj}}$ called Mixed CN-based Jaccard Measure.

Theorem 13. *For any* $0 < \sigma < 1$, *the similarity measure* $\mathtt{sim}^{\sigma}_{\mathtt{cnj}}$ *satisfies all the principles.*

From Corollary 1 and Theorem 13, we can deduce that $\mathtt{sim}^{\sigma}_{\mathtt{cnj}}$ gives the maximum degree of similarity only if the arguments are equivalent.

Corollary 3. *Let* $A, B \in \mathtt{Arg}(\mathcal{L})$, *for any* $0 < \sigma < 1$, $\mathtt{sim}^{\sigma}_{\mathtt{cnj}}(A, B) = 1$ *iff* $A \approx B$.

The following table summarises the satisfaction of the principles by the different similarity measures for $\sigma \in]0, 1[$.

	$\mathtt{sim}^{\sigma}_{\mathtt{j}}$	$\mathtt{sim}^{\sigma}_{\mathtt{d}}$	$\mathtt{sim}^{\sigma}_{\mathtt{s}}$	$\mathtt{sim}^{\sigma}_{\mathtt{a}}$	$\mathtt{sim}^{\sigma}_{\mathtt{ss}}$	$\mathtt{sim}^{\sigma}_{\mathtt{o}}$	$\mathtt{sim}^{\sigma}_{\mathtt{ku}}$	$\mathtt{sim}^{\sigma}_{\mathtt{cnj}}$
Maximality	•	•	•	•	•	•	•	•
Symmetry	•	•	•	•	•	•	•	•
Triangular Inequality	•	◦	◦	◦	•	◦	◦	•
Substitution	•	•	•	•	•	•	•	•
Syntax Independence	•	•	•	•	•	•	•	•
Minimality	•	•	•	•	•	•	•	•
Non-Zero	•	•	•	•	•	•	•	•
Monotony	•	•	•	•	•	•	•	•
Strict Monotony	•	•	•	•	•	•	•	•
Dominance	•	•	•	•	•	•	•	•
Strict Dominance	◦	◦	◦	◦	◦	◦	◦	•
Independent Distribution	•	•	•	•	•	◦	◦	•

The symbol • means the measure satisfies the principle and ◦ means the measure violates the principle.

Table 2.2: Satisfaction of the principles of similarity measures for concise arguments

Note that in Amgoud and David [2018], the authors proposed a mixed similarity measure (named Model-based Measure) looking like $\mathtt{sim}^{\sigma}_{\mathtt{cnj}}$. It also uses s_j between the supports but for the conclusions it is a different measure. This semantic measure, called Model-based Jaccard, is defined as $s_{\mathtt{cnj}}$, but instead of applying the function $\mathtt{CN}_{df}$ between conclusions, it uses Mod (i.e. models). However, as discussed after the definition of Dominance (Principle 9), the use of models is not desirable because it assigns some degree of similarity even for different formulas, i.e., formulas without any common literal. This leads to the violation of the Minimality principle. Moreover, in the case where the conclusions contain both common and contradictory information (see Example 13), the use of models does not allow the measurement of redundant information. This leads to the violation of the Strict Dominance principle.

In this section we considered concise arguments. In the next section we propose measures that deal properly with non-concise arguments.

2.4.3 Similarity Measures for Non-Concise Arguments

As already said in previous sections, although the similarity measures from Definition 45 return reasonable results in most cases, they might lead to inaccurate assessments if the arguments are not concise. Indeed, as we illustrated in Section 2.3, the measures from Definitions 45 and 48, declare the two arguments $A = \langle\{p \wedge q\}, p\rangle$ and $B = \langle\{p\}, p\rangle$ as not completely similar, while they support the same conclusion with the same ground (p).

In this section, we extend those measures in two ways, leading to two families of similarity measures, using concise refinements of arguments, and we show that they properly resolve the drawbacks of the existing measures. Note that by Proposition 13(3), every non-trivial argument A has infinitely many concise refinements. This is due to the fact that every formula α from a support of a concise refinement can be equivalently rewritten in infinitely many ways using the same set of literals (eg. $\alpha \equiv \alpha \wedge \alpha \equiv \alpha \wedge \alpha \wedge \alpha \equiv \cdots$). In the rest of this thesis, we will consider only one argument from $\texttt{CR}(A)$ per equivalence class.

Definition 49. *Let $A \in \texttt{Arg}(\mathcal{L})$. We define the set*

$$\overline{\texttt{CR}}(A) = \{B \in \texttt{CR}(A) \mid \texttt{Supp}(B) \subset \mathcal{F}\}.$$

In this way, we obtain a finite set of non-equivalent concise refinements.

Proposition 18. *For every $A \in \texttt{Arg}(\mathcal{L})$, the set $\overline{\texttt{CR}}(A)$ is finite.*

We propose now our first family of similarity measures. In the following definition, for $A \in \texttt{Arg}(\mathcal{L})$, $\Sigma \subseteq_f \texttt{Arg}(\mathcal{L})$ and a similarity measure $\texttt{sim}$ from Definition 45 or 48, we denote by $\texttt{Max}(A, \Sigma, \texttt{sim})$ the maximal similarity value between A and an argument from Σ according to $\texttt{sim}$, i.e.,

$$\texttt{Max}(A, \Sigma, \texttt{sim}) = \max_{B \in \Sigma} \texttt{sim}(A, B).$$

The first family of measures compares the sets of concise refinements of the two arguments under study. Indeed, the similarity between A and B is the average of maximal similarities (using any existing measure from Definition 45 or 48) between any concise refinement of A and those of B.

Definition 50 (A-CR Similarity Measures)**.** *Let $A, B \in \texttt{Arg}(\mathcal{L})$, and let $\texttt{sim}$ be a similarity measure from Definition 45 or 48. We define* A-CR *similarity measure*[2] *by*

$$\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}) = \frac{\sum\limits_{A_i \in \overline{\texttt{CR}}(A)} \texttt{Max}(A_i, \overline{\texttt{CR}}(B), \texttt{sim}) + \sum\limits_{B_j \in \overline{\texttt{CR}}(B)} \texttt{Max}(B_j, \overline{\texttt{CR}}(A), \texttt{sim})}{|\overline{\texttt{CR}}(A)| + |\overline{\texttt{CR}}(B)|}.$$

The value of A-CR similarity measure always belongs to the unit interval.

Proposition 19. *Let $A, B \in \texttt{Arg}(\mathcal{L})$, $\texttt{sim}^{\sigma}_{x}$ a similarity measure where $x \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}, \texttt{cnj}\}$ and $0 < \sigma < 1$. Then $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{x}) \in [0, 1]$.*

[2] A in A-CR stands for "average".

Next we show that the new measure properly resolves the problem of non-conciseness of the argument $A = \langle\{p \wedge q\}, p\rangle$ from our running example. We illustrate that by considering Extended Jaccard Measure with the parameter $\sigma = 0.5$.[3]

Example 17 (Cont.) It is easy to check that $\overline{\texttt{CR}}(A) = \{\langle\{p\}, p\rangle\}$ and $\overline{\texttt{CR}}(B) = \{\langle\{p\}, p\rangle\}$. Then $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{0.5}_{\texttt{j}}) = 1$ while $\texttt{sim}^{0.5}_{\texttt{j}}(A, B) = 0.5$.

We define now our second family of similarity measures, which is based on comparison of sets obtained by merging supports of concise refinements of arguments. For an argument $A \in \texttt{Arg}(\mathcal{L})$, we denote that set by

$$\texttt{US}(A) = \bigcup_{A' \in \overline{\texttt{CR}}(A)} \texttt{Supp}(A').$$

Definition 51 (U-CR Similarity Measures)**.** *Let $A, B \in \texttt{Arg}(\mathcal{L})$, $0 < \sigma < 1$, and s_x be a similarity measure from Table 2.1 and s_y be a similarity measure from Table 2.1 merge to the s_{cnj}. We define* U-CR *similarity measure*[4] *by*

$$\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_y, \sigma) = \sigma \cdot s_x(\texttt{US}(A), \texttt{US}(B)) + (1 - \sigma) \cdot s_y(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(B)\}).$$

Next example illustrates that U-CR also properly resolves the problem of non-conciseness of the argument $A = \langle\{p \wedge q\}, p\rangle$ from our running example.

Example 17 (Cont.) Let $\sigma = 0.5$ and $x = y = \texttt{j}$. It is easy to check that $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{j}}, s_{\texttt{j}}, 0.5) = 1$ while $\texttt{sim}^{0.5}_{\texttt{j}}(A, B) = 0.5$.

Let us now consider another more complex example where existing similarity measures provide inaccurate values while the new ones perform well.

Example 20. *Let us consider the following arguments:*

- $A = \langle\{p \wedge q, (p \vee q) \rightarrow t, (p \vee t) \rightarrow r\}, t \wedge r\rangle$
- $B = \langle\{p, p \rightarrow t, p \rightarrow r\}, t \wedge r\rangle$

It is easy to check that $\overline{\texttt{CR}}(A) = \{A_1, A_2, A_3, A_4, A_5\}$ and $\overline{\texttt{CR}}(B) = \{B_1\}$, where:

- $A_1 = \langle\{p, p \rightarrow t, p \rightarrow r\},\ t \wedge r\rangle$
- $A_2 = \langle\{p, p \rightarrow t, t \rightarrow r\},\ t \wedge r\rangle$

[3] In this section, we slightly relax the notation by simply assuming that $p \in \mathcal{F}$. We will make similar assumptions throughout this section.

[4] U in U-CR stands for "union".

- $A_3 = \langle \{q, q \to t, t \to r\}, t \wedge r \rangle$
- $A_4 = \langle \{p \vee q, (p \vee q) \to t, t \to r\},\ t \wedge r \rangle$
- $A_5 = \langle \{p \wedge q, q \to t, p \to r\},\ t \wedge r \rangle$
- $B_1 = \langle \{p, p \to t, p \to r\}, t \wedge r \rangle$

It is worth noticing that the Extended Jaccard Measure could not detect any similarity between the supports of A and B while their concise arguments A_1 and B_1 are identical. Indeed, $s_{\mathtt{j}}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = 0$ and $\mathtt{sim}^{0.5}_{\mathtt{j}}(A, B) = 0.5 \cdot 0 + 0.5 \cdot 1 = 0.5$ while $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{\mathtt{j}}, s_{\mathtt{j}}, 0.5) = 0.5 \cdot \frac{3}{9} + 0.5 \cdot 1 = \frac{2}{3} = 0.666$ and $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{0.5}_{\mathtt{j}}) = 0.5 \cdot \frac{9}{20} + 0.5 \cdot 1 = \frac{29}{40} = 0.725$.

The following proposition characterizes the arguments which are totally similar according to the novel measures. It states that total similarity is obtained exactly in the case when two arguments have equivalent concise refinements.

Proposition 20. *Let $A, B \in \mathtt{Arg}(\mathcal{L})$, $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$. Then $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{y}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_x, s_y, \sigma) = 1$ iff:*

- *$\forall A' \in \overline{\mathtt{CR}}(A)$, $\exists B' \in \overline{\mathtt{CR}}(B)$ such that $\mathtt{Supp}(A') \cong \mathtt{Supp}(B')$, $\mathtt{Conc}(A') \equiv \mathtt{Conc}(B')$ and*
- *$\forall B' \in \overline{\mathtt{CR}}(B)$, $\exists A' \in \overline{\mathtt{CR}}(A)$ such that $\mathtt{Supp}(B') \cong \mathtt{Supp}(A')$, $\mathtt{Conc}(B') \equiv \mathtt{Conc}(A')$.*

The following result shows the behavior of these new measures regarding the principles.

Theorem 14. *Let $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$. The following hold:*

Satisfaction of the Principles

[Syntax Independence] *Let π be a permutation on the set of variables, and A, B, A', $B' \in \mathtt{Arg}(\mathcal{L})$ such that*

- *A' is obtained by replacing each variable p in A with $\pi(p)$,*
- *B' is obtained by replacing each variable p in B with $\pi(p)$.*

Then $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{y}) = \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{y})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_x, s_y, \sigma) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_x, s_y, \sigma)$.

[Maximality] *For every* $A \in \texttt{Arg}(\mathcal{L})$, $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, A, \texttt{sim}^{\sigma}_{y}) = \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, A, s_x, s_y, \sigma) = 1$.

[Symmetry] *For all* $A, B \in \texttt{Arg}(\mathcal{L})$, $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{y}) = \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(B, A, \texttt{sim}^{\sigma}_{y})$ *and* $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_y, \sigma) = \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(B, A, s_x, s_y, \sigma)$.

[Substitution] *For all* $A, B, C \in \texttt{Arg}(\mathcal{L})$,

- *if* $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{y}) = 1$, *then* $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{\sigma}_{y}) = \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(B, C, \texttt{sim}^{\sigma}_{y})$,
- *if* $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_y, \sigma) = 1$, *then* $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_x, s_y, \sigma) = \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(B, C, s_x, s_y, \sigma)$.

[Minimality] *For all* $A, B \in \texttt{Arg}(\mathcal{L})$, *if*

- A *and* B *are not equivalent,*
- $\bigcup_{\phi_i \in \texttt{Supp}(A)} \texttt{Var}(\phi_i) \cap \bigcup_{\phi_j \in \texttt{Supp}(B)} \texttt{Var}(\phi_j) = \emptyset$, *and*
- $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(B)) = \emptyset$,

then $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{\texttt{y}}) = \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{x}}, s_{\texttt{y}}, \sigma) = 0$.

[Triangle Inequality] *For all* $A, B, C \in \texttt{Arg}(\mathcal{L})$, *if* $\texttt{sim}^{\sigma}_{\texttt{y}}(A, B)$ *satisfies Triangle Inequality then*

$$1 + \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_{\texttt{x}}, s_{\texttt{y}}, \sigma) \geq \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{x}}, s_{\texttt{y}}, \sigma) + \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(B, C, s_{\texttt{x}}, s_{\texttt{y}}, \sigma).$$

[(Strict) Dominance] *For all* $A, B, C \in \texttt{Arg}(\mathcal{L})$, *such that*

1. $\texttt{Supp}(B) \cong \texttt{Supp}(C)$,
2. $\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C)) \subseteq \texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(B))$,
3. $\texttt{CN}_{df}(\texttt{Conc}(B)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A)) \subseteq \texttt{CN}_{df}(\texttt{Conc}(C)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A))$,

there exists a $\sigma \in]0, 1[$ *such that:*

- $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{j}}, s_{\texttt{cnj}}, \sigma) > \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_{\texttt{j}}, s_{\texttt{cnj}}, \sigma)$ *and*
- $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{\texttt{cnj}}) > \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{\sigma}_{\texttt{cnj}})$.

Violation of the Principles

[Triangle Inequality] *There exists* $A, B, C \in \texttt{Arg}(\mathcal{L})$, *such that*

$$1 + \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{\sigma}_{\texttt{y}}) < \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{\texttt{y}}) + \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(B, C, \texttt{sim}^{\sigma}_{\texttt{y}}).$$

[Non-Zero] *There exists* $A, B \in \texttt{Arg}(\mathcal{L})$*, such that* $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) \neq \emptyset$ *and* $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{\texttt{y}}) = \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{x}}, s_{\texttt{y}}, \sigma) = 0$.

[(Strict) Monotony] *There exists* $A, B, C \in \texttt{Arg}(\mathcal{L})$*, such that*

1. $\texttt{Conc}(A) \equiv \texttt{Conc}(B)$ *or* $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(C)) = \emptyset$,
2. $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C)) \subseteq \texttt{Co}(\texttt{Supp}(A)), \texttt{Supp}(B))$,
3. $\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) = \texttt{Co}(\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)), \texttt{Supp}(C) \setminus \texttt{Co}(\texttt{Supp}(C), \texttt{Supp}(A)))$,

and

- $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{y}) < \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{\sigma}_{y})$.
- $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_y, \sigma) < \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_x, s_y, \sigma)$.

[(Strict) Dominance] *There exists* $A, B, C \in \texttt{Arg}(\mathcal{L})$*, such that*

1. $\texttt{Supp}(B) \cong \texttt{Supp}(C)$,
2. $\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C)) \subseteq \texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(B))$,
3. $\texttt{CN}_{df}(\texttt{Conc}(B)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A)) \subseteq \texttt{CN}_{df}(\texttt{Conc}(C)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A))$,

and

- $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{x}) < \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{\sigma}_{x})$.
- $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_x, \sigma) < \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_x, s_x, \sigma)$.

[Independent Distribution] *There exists* $A, B, A', B' \in \texttt{Arg}(\mathcal{L})$*, such that*

- $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(B)) = \texttt{Var}(\texttt{Conc}(A')) \cap \texttt{Var}(\texttt{Conc}(B')) = \emptyset$,
- $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) \cong \texttt{Co}(\texttt{Supp}(A'), \texttt{Supp}(B'))$,
- $\texttt{Supp}(A) \cup \texttt{Supp}(B) \cong \texttt{Supp}(A') \cup \texttt{Supp}(B')$,

and

- $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{y}) \neq \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A', B', \texttt{sim}^{\sigma}_{y})$.
- $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_y, \sigma) \neq \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A', B', s_x, s_y, \sigma)$.

We may observe that $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}$ and $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}$ violate Non-Zero, (Strict) Monotony, Dominance and Independent Distribution for any parameterised measure. The violation is due to the definition of the principles themselves. Indeed, they are based on the arguments' support. Regarding the $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}$ measures, they do not handle arguments' support as in the principles. Instead, they compute similarity on the concise refinements of the arguments (i.e. without the irrelevant information). We also see that using the mixed similarity measure, for any arguments under the conditions of Strict Dominance, there exists a σ such that the measure satisfies the principle. On the other hand, we observe that the satisfaction of the principles between the two measures ($\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}$) are identical except for the Triangular Inequality.

Let us display a summary of the satisfactions in the form of a table.

	$\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(x)$	$\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(x)$	$\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(y)$	$\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(y)$
Maximality	•	•	•	•
Symmetry	•	•	•	•
Substitution	•	•	•	•
Syntax Independence	•	•	•	•
Minimality	•	•	•	•
Non-Zero	○	○	○	○
Monotony	○	○	○	○
Strict Monotony	○	○	○	○
Dominance	○	○	○	○
Strict Dominance	○	○	⊛	⊛
Triangular Inequality	○	⊠	○	•
Independent Distribution	○	○	○	○

Let x is a syntactic measure and y is the mixed measure, • (resp. ⊛) means the measure satisfy the principle for any σ (resp. according to σ), ⊠ means the measure satisfies the principle if its parameterised measure satisfies it and ○ means the measure doesn't satisfy the principle.

Table 2.3: Satisfaction of the principles of similarity measure for non-concise arguments

It may be seen that the conciseness of the arguments has an impact on many principles. It would be interesting in a future work to analyse and redefine the principles according to this notion of concise argument.

The following proposition shows that if we apply A-CR or U-CR to any similarity measure $\mathtt{sim}^{\sigma}_{x}$ from Definition 45 (respectively s_x from Table 2.1) or from Definition 48, then the two families of measures will coincide with $\mathtt{sim}^{\sigma}_{x}$ when applied to concise arguments.

Proposition 21. *Let* $A, B \in \mathtt{Arg}(\mathcal{L})$ *be two concise arguments. Then, for every* $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ *and* $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ *with the condition that if*

$y = \texttt{cnj}$ then $x = \texttt{j}$, otherwise $y = x$, it holds

$$\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{y}) = \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_y, \sigma) = \texttt{sim}^{\sigma}_{y}(A, B). \tag{2.1}$$

Remark. *Note that the equations (2.1) might also hold for some A and B that are not concise. For example, let $A = \langle\{p \wedge q, t \wedge s\}, p \wedge t\rangle$ and $B = \langle\{p, t \wedge s\}, p \wedge s\rangle$. Then $\overline{\texttt{CR}}(A) = \{\langle\{p, t\}, p \wedge t\rangle\}$ and $\overline{\texttt{CR}}(B) = \{\langle\{p, s\}, p \wedge s\rangle\}$, so $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{0.5}_{\texttt{j}}) = \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{j}}, s_{\texttt{j}}, 0.5) = \texttt{sim}^{0.5}_{\texttt{j}}(A, B) = 0.25$.*

The following example shows that A-CR and U-CR may return different results. Indeed, it is possible for three arguments A, B and C that A is more similar to B than to C according to one measure, but not according to the other one.

Example 21. *Let $A = \langle\{p, p \to q_1 \wedge q_2\}, q_1 \vee q_2\rangle$, $B = \langle\{p, s\}, p \wedge s\rangle$ and $C = \langle\{p \to q_1\}, p \to q_1\rangle$. We have $\overline{\texttt{CR}}(A) = \{\langle\{p, p \to q_1\}, q_1 \vee q_2\rangle, \langle\{p, p \to q_2\}, q_1 \vee q_2\rangle, \langle\{p, p \to q_1 \vee q_2\}, q_1 \vee q_2\rangle\}$, $\overline{\texttt{CR}}(B) = \{\langle\{p, s\}, p \wedge s\rangle\}$, $\overline{\texttt{CR}}(C) = \{\langle\{p \to q_1\}, p \to q_1\rangle\}$. Consequently:*

- $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{0.5}_{\texttt{j}}) = \frac{1}{6} > \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{0.5}_{\texttt{j}}) = \frac{1}{8}$, *but*
- $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{j}}, s_{\texttt{j}}, 0.5) = \frac{1}{10} < \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_{\texttt{j}}, s_{\texttt{j}}, 0.5) = \frac{1}{8}$.

The next example shows that none of the two novel measures dominates the other. Indeed, some pairs of arguments have greater similarity value according to A-CR, and other pairs have greater similarity value using U-CR.

Example 21 (Cont.) Note that $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{j}}, s_{\texttt{j}}, 0.5) < \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{0.5}_{\texttt{j}})$. Let us consider $A' = \langle\{p \wedge q\}, p \vee q\rangle$, $B' = \langle\{p, q\}, p \wedge q\rangle \in \texttt{Arg}(\mathcal{L})$. $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A', B', s_{\texttt{j}}, s_{\texttt{j}}, 0.5) = 0.5 \cdot \frac{2}{3} + 0.5 \cdot 0 = \frac{1}{3} = 0.333$ and $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A', B', \texttt{sim}^{0.5}_{\texttt{j}}) = 0.5 \cdot \frac{3}{8} + 0.5 \cdot 0 = \frac{3}{16} = 0.1875$, thus $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A', B', s_{\texttt{j}}, s_{\texttt{j}}, 0.5) > \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A', B', \texttt{sim}^{0.5}_{\texttt{j}})$.

To investigate the notion of similarity and more precisely how to measure it, we used arguments instantiated in logic. Once we have obtained the degrees of similarity between arguments, we are left with the question *"how to use it in our evaluation methods and semantics to remove redundancy between the arguments?"*. The study of the application of a similarity measure to an evaluation method does not depend on the nature of the content of the arguments. Therefore, in the next chapter, we will incorporate a similarity measure as an input to an abstract argumentation framework.

Chapter 3

Gradual Semantics dealing with Similarity

Contents

THIS chapter discusses *theoretical foundations* that describe *principles* and *processes* involved in defining semantics that deal with similarity between arguments. Focusing on semi-weighted argumentation frameworks, we extend them with a similarity measure. Then, we extend the approach based on evaluation methods for defining gradual semantics that consider similarity.

3.1 Introduction

Two gradual semantics, extending $h-$Categoriser (Besnard and Hunter [2001]) and using a binary similarity measure, have been proposed in Amgoud *et al.* [2018]. They differ in the way they modify the strengths of attackers on the basis of their similarities. While those semantics seem reasonable, the approach followed for defining them is not systematic as the general rules guiding the definition of a semantics in general, and the way of dealing with similarity in particular, have not been discussed. The authors proposed some properties for bridging that gap, but it turns out that both semantics violate some of them. Moreover, those properties are not sufficient for comparing the two semantics. Hence, the approach lacks *theoretical foundations* that describe *principles* and *processes* involved in the definition of semantics that deal with similarity.

This Chapter proposes such theoretical foundations. Rather than focus narrowly on a particular semantics, we propose a general setting for defining systematically gradual semantics that consider similarities. The contributions are six-fold:

1. Clarify the process of defining semantics using three functions:
   - an *adjustment function* - $\mathbf{n}$: that updates the strengths of attackers on the basis of their similarities,
   - an *aggregation function* - $\mathbf{g}$: that computes the strength of the group of attackers,
   - an *influence function* - $\mathbf{f}$: that evaluates the impact of the group on the argument's initial weight.
2. Identify rules for handling similarities, i.e, key properties of an adjustment function.
3. Propose principles that a gradual semantics dealing with similarity would satisfy.
4. Provide a broad family of semantics that satisfy them.
5. Analyse the existing adjustment functions, show that they violate some of the proposed properties and propose novel adjustment functions that satisfy the desirable properties.
6. Extend the h-categoriser semantics with the new adjustment function, and show that the new semantics are instances of the novel family.

3.2 Similarity-based Gradual Semantics

Let us start by presenting the extension of the semi-weighted argumentation framework with a similarity measure.

Definition 52 (SSWAF)**.** *A semi-weighted argumentation framework extended by a similarity measure, abbreviated to* SSWAF*, is a tuple* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$*, where* $\mathcal{A} \subseteq_f \mathtt{Arg}$*,* $\mathbf{w}$ *is a weighting on* $\mathcal{A}$*,* $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ *and* sim *is a similarity measure (a weighting on* $\mathcal{A} \times \mathcal{A}$*).*

An important question is how to define gradual semantics that take into account the adjustment function. We extend the setting that has been proposed by Amgoud and Doder [2018] for semi-weighted argumentation framework where $\mathtt{sim} \equiv 0$[1] and in Cayrol and Lagasquie-Schiex [2005] for simple flat graphs with $\mathtt{sim} \equiv 0$.

Roughly speaking, a gradual semantics dealing with similarity proceeds in a recursive way. For any argument A, if A is not attacked, then its strength is exactly the weight $\mathbf{w}(A)$. Assume now that A is attacked by $A_1, \cdots, A_k$. The semantics starts by evaluating the strength of every attacker A_i, $i = 1, \cdots, k$. Let $x_1, \cdots, x_k$ be numerical values representing those strengths. For computing the strength of A, the semantics follows a *three steps process*:

1. It *adjusts* the values $x_1, \cdots, x_k$ according to the similarities between A_i, A_j where $i, j = 1, \cdots, k$ and $i \neq j$. The goal is to remove redundancy among the attackers, thus the semantics *weakens* the attackers. Let $x'_1, \cdots, x'_k$ denote the adjusted values, x'_i being the new strength of A_i.

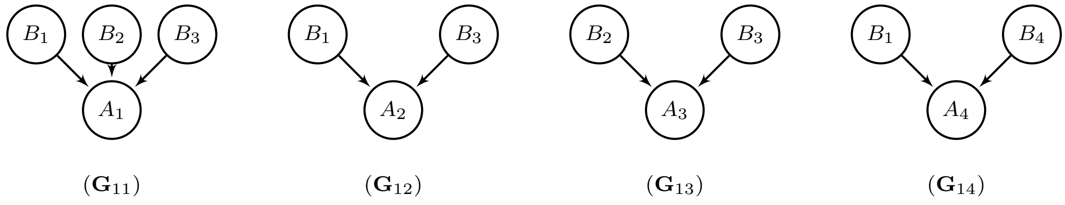
2. It computes the *strength of the group* $\{A_1, \cdots, A_k\}$ by aggregating the values $x'_1, \cdots, x'_k$.

3. It adjusts the initial weight $\mathbf{w}(A)$ on the basis of the strength of the group of attackers.

Let us illustrate this process using the running example. Consider the three graphs $\mathbf{G}_{11}, \mathbf{G}_{12}$, and $\mathbf{G}_{13}$ depicted below.

[1]The notation $\mathtt{sim} \equiv 0$ means all arguments have similarities 0, i.e., they are all completely different.

Example 3 (Cont.)



$\mathbf{w} \equiv 1$, $\mathtt{sim}(B_1, B_2) = 1$, $\mathtt{sim}(B_1, B_3) = \mathtt{sim}(B_2, B_3) = \alpha$ with $0 < \alpha < 1$, $\mathtt{sim}(B_4, B_i) = 0$ for any $i = 1, \cdots, 3$.

Figure 3.1: Running example

According to semantics satisfying the principles described in section 1.2.2 (which do not use similarity), A_1 is strictly weaker than A_2 due to B_2 which further weakens A_1. It is also strictly weaker than A_3 due to B_1 which decreases further the strength of A_1. Assume now that B_1 and B_2 are fully similar, i.e., $\mathtt{sim}(B_1, B_2) = 1$*. Note that B_1 is redundant w.r.t. B_2, and thus considering both* $\mathtt{Str}(B_1)$ *(strength of B_1) and* $\mathtt{Str}(B_2)$ *will lead to an inaccurate evaluation of the argument A. Indeed, A will loose a lot of weight due to redundant information.*

A reasonable gradual semantics would assign strength 1 to each B_i since it is not attacked. For A_1, the semantics would start with the tuple $(1, 1, 1)$, the strengths of B_1, B_2, B_3, and adjusts them. Since $\mathtt{sim}(B_1, B_2) = 1$*, the semantics would for example decide to keep only one of them, say B_1. Hence, it adjusts the score of B_2 from 1 to 0. Regarding B_3, it keeps only its novel part compared to B_1 hence $1 - \alpha$. The adjusted values are thus $(1, 0, 1 - \alpha)$. The semantics computes then the strength of the group $\{B_1, B_2, B_3\}$ using, for instance, the sum aggregation operator and returns the value $2 - \alpha$. Finally, it evaluates the impact of the group on the initial weight of A_1 using for instance the function $f_{frac}(x_1, x_2) = \frac{x_1}{1+x_2}$, hence the strength of $A_1 = \frac{\mathbf{w}(A_1)}{1+2-\alpha} = \frac{1}{3-\alpha}$. Note that if the semantics ignores the similarities, it assigns the score $\frac{1}{4}$ to A_1 and thus A_1 would be much weaker.*

Each step of the process described above can be done in different ways. For instance, a semantics may adjust differently the strengths of B_1, B_2 by weakening both arguments, may aggregate attackers differently, or may use another function than f_{frac}. In what follows, we define a gradual semantics in an abstract way using a tuple of three functions, called *evaluation method*.

Definition 53 (EM)**.** *An* evaluation method *(*EM*) is a tuple* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ *such that:*

- $\mathbf{f} : [0, 1] \times \mathtt{Range}(\mathbf{g})$ [2] $\rightarrow [0, 1]$,

[2] $\mathtt{Range}(\mathbf{g})$ denotes the co-domain of $\mathbf{g}$

- $\mathbf{g} : \bigcup_{k=0}^{+\infty}[0,1]^k \rightarrow [0,+\infty[$,
- $\mathbf{n} : \bigcup_{k=0}^{+\infty} ([0,1] \times \texttt{Arg})^k \rightarrow [0,1]^k$.

Given the set of attackers of a given argument A in an argumentation framework, the function $\mathbf{n}$ adjusts the strength of each attacker based on its similarities with the other attackers of A, $\mathbf{g}$ computes the strength of the group of attackers, and $\mathbf{f}$ evaluates how the latter influences the initial weight of A. Note that the domains of $\mathbf{g}$ and $\mathbf{n}$ are unions because the number of attackers may vary from one argument to another. Note also that $\mathbf{n}$ takes as input two kinds of information: k numerical values and k arguments. Let us illustrate the need of the set of arguments. Consider the two arguments A_3, A_4 in Figure 3.1. Recall that $\texttt{sim}(B_2, B_3) = \alpha > 0$ and $\texttt{sim}(B_1, B_4) = 0$. Since each B_i is not attacked, then its strength is 1. However, the function $\mathbf{n}$ would not alter the values of B_1 and B_4 since the latter are dissimilar, i.e., $\mathbf{n}(1, 1, B_1, B_4) = (1, 1)$ while it modifies those of B_2, B_3, i.e., $\mathbf{n}(1, 1, B_2, B_3) = (x, y)$ as there is some redundancy between the two arguments. This means that the same values (here $(1, 1)$) may be adjusted in different ways according to the arguments they refer to.

We propose below key properties that should be satisfied by each of the three functions $\mathbf{f}, \mathbf{g}, \mathbf{n}$ of an evaluation method. Those properties constrain the range of functions to be considered, and discard those that may exhibit irrational behaviours.

Definition 54. *An evaluation method* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ *is* well-behaved *iff the following conditions hold:*

1. **(a)** $\mathbf{f}$ *is increasing in the first variable, decreasing in the second one if the first variable is not equal to 0,*

   (b) $\mathbf{f}(x, 0) = x$,

   (c) $\mathbf{f}(0, x) = 0$,

2. **(a)** $\mathbf{g}() = 0$,

   (b) $\mathbf{g}(x) = x$,

   (c) $\mathbf{g}(x_1, \cdots, x_k) = \mathbf{g}(x_1, \cdots, x_k, 0)$,

   (d) $\mathbf{g}(x_1, \cdots, x_k, y) \leq \mathbf{g}(x_1, \cdots, x_k, z)$ *if* $y \leq z$,

   (e) $\mathbf{g}$ *is symmetric,*

3. **(a)** $\mathbf{n}() = ()$,

   (b) $\mathbf{n}((x, A)) = (x)$,

   (c) $\mathbf{g}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))) \leq \mathbf{g}(\mathbf{n}((x_1, B_1), \cdots, (x_k, B_k)))$ *if* $\forall i, j \in \{1, \cdots, k\}$, $\texttt{sim}(A_i, A_j) \geq \texttt{sim}(B_i, B_j)$,

(d) *If* $\exists i \in \{1, \cdots, k\}$ *s.t.* $x_i > 0$ *then* $\mathbf{g}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))) > 0$*,*

(e) $\mathbf{g}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))) \leq \mathbf{g}(\mathbf{n}((y_1, A_1), \cdots, (y_k, A_k)))$ *if* $\forall i \in \{1, \cdots, k\}$*,* $x_i \leq y_i$*,*

(f) $\mathbf{n}$ *is symmetric,*

(g) $\mathbf{n}((x_1, A_1), \cdots, (x_{k+1}, A_{k+1})) = (\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)), x_{k+1})$ *if* $\forall i \in \{1, \cdots, k\}$*,* $\mathtt{sim}(A_i, A_{k+1}) = 0$*.*

We say also that $\mathbf{f}, \mathbf{g}, \mathbf{n}$ *are well-behaved.*

Note that the functions $\mathbf{f}, \mathbf{g}, \mathbf{n}$ are defined without referring to any argumentation framework. The idea is to describe their general behaviour. The conditions (2c) and (2d) respectively state that attackers of strength 0 have no impact on their targets, and $\mathbf{g}$ is monotonic in that the greater the individual values, the greater their aggregation. The conditions (3a, $\cdots$, 3g) represent the *core principles for dealing with similarities*. Namely, (3b) states that if a group of attackers contains only one element, then the adjusted value of the latter is equal to the initial one. (3c) states that the greater the similarity between arguments of a set, the *weaker the set*, and (3d) ensures that similarities do not inhibit the attack of a group of arguments. Condition (3e) states that the stronger the individual attackers, the stronger the group. (3g) is an *independence* condition. It states that an argument which is dissimilar to all elements of a group, has no effect on the adjustment of the values of those elements. Furthermore, the argument keeps its initial value.

From the condition (3c), it follows that similarities lead to a decrease in the strength of a group of attackers.

Proposition 22. *If* $\mathbf{g}$ *and* $\mathbf{n}$ *are well-behaved, then for all* $x_1, \cdots, x_k \in [0, 1]$*, for all* $A_1, \cdots, A_k \in \mathtt{Arg}$*, it holds that:*

$$\mathbf{g}(x_1, \cdots, x_k) \geq \mathbf{g}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))).$$

From the condition (3g), it follows that if the arguments of a set are independent (their similarities are all equal to 0), then their initial values remain unchanged by $\mathbf{n}$.

Proposition 23. *Let* $x_1, \cdots, x_k \in [0, 1]$ *and* $A_1, \cdots, A_k \in \mathtt{Arg}$ *such that for all* $i, j \in \{1, \cdots, k\}$*, with* $i \neq j$*,* $\mathtt{sim}(A_i, A_j) = 0$*. If* $\mathbf{n}$ *is well-behaved, then* $\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)) = (x_1, \cdots, x_k)$*.*

Let us now define formally a gradual semantics based on an evaluation method that deals with similarity.

Definition 55 (Gradual Semantics)**.** *A* gradual semantics $\mathbf{S}$ *based on an evaluation method* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ *is a function assigning to every* SSWAF*,* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ *into a weighting* $\mathtt{Str}^{\mathbf{S}} : \mathcal{A} \to [0,1]$ *such that for every* $A \in \mathcal{A}$*,* $\mathtt{Str}^{\mathbf{S}}(A) =$

$$\mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\Bigg(\mathbf{n}\Big((\mathtt{Str}^{\mathbf{S}}(B_1), B_1), \cdots, (\mathtt{Str}^{\mathbf{S}}(B_k), B_k)\Big)\Bigg)\Bigg),$$

where $\{B_1, \cdots, B_k\} = \mathtt{Att}(A)$*.*

The above definition shows that evaluating arguments in an SSWAF amounts to solving a system of equations, one equation per argument. The question of existence of solutions for such systems arises naturally. Note that existence of solutions also means existence of a semantics. The following result shows that if the three functions of an evaluation method are continuous, then a solution exists for every SSWAF.

Theorem 15. *If* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ *is an evaluation method such that* $\mathbf{f}$ *is continuous on the second variable,* $\mathbf{g}$ *is continuous on each variable, and* $\mathbf{n}$ *is continuous on each numerical variable, then there exists a semantics* $\mathbf{S}$ *based on* $\mathbf{M}$*.*

The following result goes further by showing that a system of equations has a single solution for every SSWAF. This is particularly the case when the evaluation method is well-behaved and satisfies some additional constraints. This result shows there is only one semantics that is based on a given evaluation method.

Theorem 16. *Let* $\mathbf{M}^*$ *be the set of all well-behaved evaluation methods* $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ *such that:*

- $\lim\limits_{x_2 \to x_0} \mathbf{f}(x_1, x_2) = \mathbf{f}(x_1, x_0)$, $\forall x_0 \neq 0$.
- $\lim\limits_{x \to x_0} \mathbf{g}(x_1, \cdots, x_k, x) = \mathbf{g}(x_1, \cdots, x_k, x_0)$, $\forall x_0 \neq 0$.
- $\mathbf{n}$ *is continuous on each numerical variable.*
- $\lambda \mathbf{f}(x_1, \lambda x_2) < \mathbf{f}(x_1, x_2)$, $\forall \lambda \in [0, 1[$, $x_1 \neq 0$.
- $\mathbf{g}(\mathbf{n}(\lambda x_1, \cdots, \lambda x_k, B_1, \cdots, B_k)) \geq \lambda \mathbf{g}(\mathbf{n}(x_1, \cdots, x_k, B_1, \cdots, B_k))$, $\forall \lambda \in [0, 1]$.

For any $\mathbf{M} \in \mathbf{M}^*$*, for all gradual semantics* $\mathbf{S}, \mathbf{S}'$*, if* $\mathbf{S}, \mathbf{S}'$ *are based on* $\mathbf{M}$*, then* $\mathbf{S} \equiv \mathbf{S}'$*.*

3.3 Principles of Gradual Semantics dealing with Similarity

So far we have presented a three-step process for defining semantics; at each step a function that obeys to specific conditions is used. We have seen that none of the three (adjustment, aggregation, influence) functions refers to argumentation frameworks, making their impact on argument strength in particular and on the behaviour of gradual semantics in general not clear. This section bridges the gap by proposing *principles* that gradual semantics should satisfy, and relating them to the various conditions of evaluation methods.

Principles are useful properties for understanding underpinnings of semantics. They have recently generated a lot of effort (eg. Bonzon *et al.* [2016]; Amgoud *et al.* [2017]; Amgoud and Ben-Naim [2018]; Mossakowski and Neuhaus [2018]; Baroni *et al.* [2019]). In what follow, we extend those proposed in Amgoud *et al.* [2017], and that are impacted by similarity, namely *Reinforcement*, *Monotony* and *Neutrality*. We also propose a novel one, *Sensitivity to Similarity*.

The Reinforcement principle concerns *strengths of attackers*. It states that the stronger an attacker, the greater its impact on the strength of the argument it is attacking. The original definition does not take into consideration similarities among attackers, and hence may lead to counter-intuitive results in presence of redundancies. Assume for instance that the attacker that is strengthened is redundant with another, in this case it should be ignored by a semantics.

Principle 1 (Reinforcement)**.** *A semantics* $\mathbf{S}$ *satisfies* Reinforcement *iff for any* $\texttt{SSWAF}$, $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \texttt{sim} \rangle$*, for all* $A, B \in \mathcal{A}$*, if*

- $\mathbf{w}(A) = \mathbf{w}(B)$,
- $\texttt{Att}(A) \setminus \texttt{Att}(B) = \{C\}$, $\texttt{Att}(B) \setminus \texttt{Att}(A) = \{D\}$,
- $\forall E \in \texttt{Att}(A) \cap \texttt{Att}(B)$, $\texttt{sim}(C, E) = \texttt{sim}(D, E)$,
- $\texttt{Str}^{\mathbf{S}}(C) \leq \texttt{Str}^{\mathbf{S}}(D)$,

then the following properties hold:

- $\texttt{Str}^{\mathbf{S}}(A) \geq \texttt{Str}^{\mathbf{S}}(B)$. *(Reinforcement)*
- *If* $\texttt{Str}^{\mathbf{S}}(A) > 0$ *and* $\texttt{Str}^{\mathbf{S}}(C) < \texttt{Str}^{\mathbf{S}}(D)$*, then* $\texttt{Str}^{\mathbf{S}}(A) > \texttt{Str}^{\mathbf{S}}(B)$. *(Strict Reinforcement)*

The Monotony principle concerns the *quantity of attackers*. Its original definition states “the more an argument has attackers, the weaker it is”. Hence, an argument A that is attacked by B and C is weaker than if it is only attacked by B. This result is inaccurate when B and C are redundant. A should have the same strength in both cases since one of the attackers should be ignored. The new version of Monotony avoids such inaccurate evaluations and states “the more an argument has dissimilar attackers, the weaker it is”.

Principle 2 (Monotony). *A semantics* $\mathbf{S}$ *satisfies* Monotony *iff for any* $\mathtt{SSWAF}$*,* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$*, for all* $A, B \in \mathcal{A}$*, if*

- $\mathbf{w}(A) = \mathbf{w}(B)$*,*
- $\mathtt{Att}(A) \subseteq \mathtt{Att}(B)$*,*
- *If* $\mathtt{Att}(A) \neq \emptyset$*, then* $\forall C \in \mathtt{Att}(B) \setminus \mathtt{Att}(A)$*,* $\forall D \in \mathtt{Att}(A)$*,* $\mathtt{sim}(C, D) = 0$*,*

then the following properties hold:

- $\mathtt{Str}^{\mathbf{S}}(A) \geq \mathtt{Str}^{\mathbf{S}}(B)$. *(Monotony)*
- *If* $\mathtt{Str}^{\mathbf{S}}(A) > 0$ *and* $\exists C \in \mathtt{Att}(B) \setminus \mathtt{Att}(A)$ *such that* $\mathtt{Str}^{\mathbf{S}}(C) > 0$*,*
  then $\mathtt{Str}^{\mathbf{S}}(A) > \mathtt{Str}^{\mathbf{S}}(B)$. *(Strict Monotony)*

Neutrality states that attackers having strength equal to 0 have no impact on their targets. The new version of the principle ensures that those lifeless attackers are dissimilar to the other attackers.

Principle 3 (Neutrality). *A semantics* $\mathbf{S}$ *satisfies* Neutrality *iff for any* $\mathtt{SSWAF}$*,* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$*, for all* $A, B \in \mathcal{A}$*, if*

- $\mathbf{w}(A) = \mathbf{w}(B)$*,*
- $\mathtt{Att}(B) = \mathtt{Att}(A) \cup \{C\}$ *with* $\mathtt{Str}^{\mathbf{S}}(C) = 0$*,*
- *If* $\mathtt{Att}(A) \neq \emptyset$*, then* $\forall D \in \mathtt{Att}(A)$*,* $\mathtt{sim}(C, D) = 0$*,*

then $\mathtt{Str}^{\mathbf{S}}(A) = \mathtt{Str}^{\mathbf{S}}(B)$*.*

Sensitivity to similarity states that the greater the similarities between attackers of an argument, the stronger the argument. Recall that similarities mean existence of redundancies, and the latter should be removed by semantics.

Principle 4 (Sensitivity to Similarity). *A semantics* $\mathbf{S}$ *is* Sensitive to Similarity *iff for any* $\mathtt{SSWAF}$*,* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$*, for all* $A, B \in \mathcal{A}$ *such that* $\mathbf{w}(A) = \mathbf{w}(B)$*, if there exists a bijective function* $f : \mathtt{Att}(A) \rightarrow \mathtt{Att}(B)$ *such that:*

- $\forall C \in \mathtt{Att}(A)$*,* $\mathtt{Str}^{\mathbf{S}}(C) = \mathtt{Str}^{\mathbf{S}}(f(C))$*,*
- $\forall C, D \in \mathtt{Att}(A)$*,* $\mathtt{sim}(C, D) \geq \mathtt{sim}(f(C), f(D))$*,*

then the following properties hold:

- $\mathtt{Str}^{\mathbf{S}}(A) \geq \mathtt{Str}^{\mathbf{S}}(B)$. *(Sensitivity)*

- *If* $\texttt{Str}^{\mathbf{S}}(A) > 0$ *and* $\exists C, D \in \texttt{Att}(A)$ *such that (*$\texttt{Str}^{\mathbf{S}}(C) > 0$ *or* $\texttt{Str}^{\mathbf{S}}(D) > 0$*) and* $\texttt{sim}(C, D) > \texttt{sim}(f(C), f(D))$*,*
  then, $\texttt{Str}^{\mathbf{S}}(A) > \texttt{Str}^{\mathbf{S}}(B)$. *(Strict Sensitivity)*

Let us show how the above principles relate to the different conditions of evaluation methods. The first result states that any semantics that is based on a well-behaved evaluation method satisfies the non-strict versions of the principles.

Theorem 17. *Let* S *be a gradual semantics based on an evaluation method* M*. If* M *is well-behaved, then* S *satisfies Reinforcement, Monotony, Neutrality and Sensitivity to Similarity.*

In order to guarantee the strict version of Reinforcement, the evaluation method of a semantics should not only be well-behaved but also satisfy the condition below, which is a strict version of the constraint (3e) in Definition 54.

$$\mathbf{g}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))) < \mathbf{g}(\mathbf{n}((y_1, A_1), \cdots, (y_k, A_k)))$$
$$\text{if } \forall i \in \{1, \cdots, k\},\ x_i \leq y_i \text{ and } \exists i \in \{1, \cdots, k\} \text{ s.t. } x_i < y_i. \quad \textbf{(C1)}$$

Theorem 18. *Let* S *be a gradual semantics based on an evaluation method* M*. If* M *is well-behaved and satisfies (**C1**), then* S *satisfies Strict Reinforcement.*

Strict Sensitivity to Similarity is satisfied by a semantics when its evaluation method is well-behaved and enjoys the property (**C2**).

$$\mathbf{g}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))) < \mathbf{g}(\mathbf{n}((x_1, B_1), \cdots, (x_k, B_k)))$$
$$\text{if } \forall i, j \in \{1, \cdots, k\}, \texttt{sim}(A_i, A_j) \geq \texttt{sim}(B_i, B_j) \text{ and}$$
$$\exists i, j \in \{1, \cdots, k\} \text{ s.t. } \texttt{sim}(A_i, A_j) > \texttt{sim}(B_i, B_j) \text{ and}$$
$$(x_i > 0 \text{ or } x_j > 0). \quad \textbf{(C2)}$$

Theorem 19. *Let* S *be a gradual semantics based on an evaluation method* M*. If* M *is well-behaved and satisfies (**C2**), then* S *is Strictly Sensitive to Similarity.*

Strict Monotony is satisfied by a semantics when its evaluation method is well-behaved and enjoys the property (**C3**) below.

$$\mathbf{g}(x_1, \cdots, x_k, y) < \mathbf{g}(x_1, \cdots, x_k, z) \qquad \text{if } y < z \quad \textbf{(C3)}$$

Theorem 20. *Let* S *be a gradual semantics based on an evaluation method* M*. If* M *is well-behaved and satisfies (**C3**), then* S *satisfies Strict Monotony.*

Remark: It is worth mentioning that the conditions (C1), (C2) and (C3) are not part of Def. 54 since they are more demanding than their large versions. In the same way, the large versions of the principles are mandatory while the strict ones are optional and their suitability depends on the application and the type of arguments (deductive, analogical, etc).

3.4 Novel Family of Semantics

We now introduce a broad family of gradual semantics that are able to deal with similarity between arguments. Its members use evaluation methods from the set $\mathbf{M}$ (see Theorem 16). Recall that every evaluation method in this set is well-behaved and satisfies some additional properties, which guarantee that the evaluation method characterizes a single gradual semantics.

Definition 56. *We define by* $\mathbf{S}^*$ *the set of all semantics that are based on an evaluation method from* $\mathbf{M}^*$.

From Theorem 17, it follows that any member of $\mathbf{S}$ satisfies all the large versions of the principles.

Theorem 21. *Any gradual semantics* $\mathbf{S} \in \mathbf{S}^*$ *satisfies Reinforcement, Monotony, Neutrality and Sensitivity to Similarity.*

Obviously, if the evaluation method of a semantics $\mathbf{S} \in \mathbf{S}^*$ satisfies in addition the three constraints (C1), (C2) and (C3), then the semantics would satisfy the Strict versions of Reinforcement, Sensitivity to Similarity and Monotony. In a next section (3.6), we show that the set $\mathbf{S}^*$ is not empty and we discuss some of its instances. But in order to reach the instance of these semantics, we will first discuss different adjustment functions.

3.5 Adjustment Functions

This section presents examples of adjustment functions. Their core idea is that a modified value would represent the *novelty* brought by an attacker to the group of attackers. This amounts at computing approximately the *similarity of the attacker with the group* by aggregating its similarity with every argument of the group. A second central concern when dealing with similarity is how to distribute the redundancy burden among similar arguments. Consider the case of a group of two attackers A, B such that $\mathtt{sim}(A, B) = 1$, the strength of A is equal to 1 and the strength of B is 0.6. The question is: *where should a function* $\mathbf{n}$ *remove redundancy?* There are three possible strategies:

- *Conjunctive*: $\mathbf{n}$ removes the redundancy from the weakest argument B.
- *Disjunctive*: $\mathbf{n}$ removes the redundancy from the strongest argument A.
- *Compensative*: $\mathbf{n}$ distributes the burden to both.

3.5.1 Instances of Adjustment functions

We recall one adjustment function defined in the literature (Amgoud *et al.* [2018]) and we present three new adjustment functions, one per strategy.

In Amgoud *et al.* [2018], the authors proposed two semantics dealing with similarity measures but one of them, i.e. Grouping Weighted h-Categorizer - GHbs, has not an independent adjustment function. That means this gradual semantics mixed the aggregation function (g) with the adjustment function ($\mathbf{n}$). That is why we will not compare this method with ours.

The first adjustment function what has been proposed in Amgoud *et al.* [2018] (without naming it an adjustment function), is based on the average operator (denoted by avg) and follows a compensative strategy.

Definition 57 ($\mathbf{n}_{\mathtt{rs}}$)**.** *Let* sim *a similarity measure,* $A_1, \cdots, A_k \in \mathtt{Arg}$ *and* $x_1, \cdots, x_k \in [0,1]$. $\mathbf{n}_{\mathtt{rs}}((x_1, A_1), \cdots, (x_k, A_k)) =$

$$\left(\underset{x_i \in \{x_1, \cdots, x_k\} \setminus \{x_1\}}{\mathtt{avg}} \left(\frac{\mathtt{avg}(x_1, x_i) \times (2 - \mathtt{sim}(A_1, A_i))}{2} \right), \cdots, \right.$$

$$\left. \underset{x_i \in \{x_1, \cdots, x_k\} \setminus \{x_k\}}{\mathtt{avg}} \left(\frac{\mathtt{avg}(x_k, x_i) \times (2 - \mathtt{sim}(A_k, A_i))}{2} \right) \right).$$

$\mathbf{n}_{\mathtt{rs}}() = ()$ *and* $\mathbf{n}_{\mathtt{rs}}((x_1, A_1)) = (x_1)$ *if* $k = 1$.

Example 3 (Cont.) Using the function $\mathbf{n}_{\mathtt{rs}}$, we get: $\mathbf{n}_{\mathtt{rs}}((1, B_1), (1, B_2), (1, B_3))$ = $(\mathtt{avg}(\frac{1\times1}{2}, \frac{1\times(2-\alpha)}{2}), \mathtt{avg}(\frac{1\times1}{2}, \frac{1\times(2-\alpha)}{2}), \mathtt{avg}(\frac{1\times(2-\alpha)}{2}, \frac{1\times(2-\alpha)}{2})) = (\frac{3-\alpha}{4}, \frac{3-\alpha}{4}, \frac{2-\alpha}{2})$. For $\alpha = 0.5$, we get (0.625, 0.625,0.75). Note that the function weakens both B_1 and B_2, which are identical ($\mathtt{sim}(B_1, B_2) = 1$).

In what follows, we propose novel (family of) functions that compute the degree of similarity of an argument with a set of arguments by aggregating the pairwise similarities using the max operator. They start by rank ordering the initial scores of arguments using a fixed permutation. The new score of an argument is equal to its old value times its novelty with respect to the preceding arguments in the permutation.

Definition 58 (Parameterised Function $\mathbf{n}^{\rho}_{\mathtt{max}}$)**.** *Let* sim *a similarity measure,* $A_1, \cdots, A_k \in \mathtt{Arg}$*,* $x_1, \cdots, x_k \in [0,1]$*, and* ρ *a fixed permutation on the set* $\{1, \cdots, k\}$ *such that if* $x_{\rho(i)} = 0$ *then* $x_{\rho(i+1)} = 0 \; \forall i < k$*, or* $i = k$. $\mathbf{n}^{\rho}_{\mathtt{max}}() = ()$*, otherwise:*
$\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k)) =$

$$\Big(x_{\rho(1)},$$
$$x_{\rho(2)} \cdot (1 - \mathtt{max}(\mathtt{sim}(A_{\rho(1)}, A_{\rho(2)}))),$$

$$\cdots,$$
$$x_{\rho(k)} \cdot (1 - \mathtt{max}(\mathtt{sim}(A_{\rho(1)}, A_{\rho(k)}), \cdots, \mathtt{sim}(A_{\rho(k-1)}, A_{\rho(k)})))\Big).$$

Example 3 (Cont.) Consider the graph $\mathbf{G}_{11}$ in the Figure 3.1. Recall that $\mathtt{sim}(B_1, B_2) = 1$, $\mathtt{sim}(B_1, B_3) = \mathtt{sim}(B_2, B_3) = \alpha$, $0 < \alpha < 1$, and $\mathtt{Att}(A_1) = \{B_1, B_2, B_3\}$. For any reasonable semantics $\mathbf{S}$, $\mathtt{Str}^{\mathbf{S}}(B_1) = \mathtt{Str}^{\mathbf{S}}(B_2) = \mathtt{Str}^{\mathbf{S}}(B_3) = 1$ since they are not attacked. Let $x_i = \mathtt{Str}^{\mathbf{S}}(B_i)$.
We illustrate $\mathbf{n}^{\rho}_{\mathtt{max}}$ using two permutations. $\rho_{\mathtt{min}}$ ranks arguments from the weakest argument with maximal similarity to the strongest with less similar to other attackers. $\rho_{\mathtt{max}}$ ranks arguments from the strongest with minimal similarity to the weakest with more similarity. $\rho_{\mathtt{min}}$ follows thus a conjunctive strategy while $\rho_{\mathtt{max}}$ a disjunctive one. Hence, $\rho_{\mathtt{min}}(x_1, x_2, x_3) = (x_1, x_2, x_3)$ (since B_1, B_2 are the most similar arguments) and $\mathbf{n}^{\rho_{\mathtt{min}}}_{\mathtt{max}}((x_1, B_1), (x_2, B_2), (x_3, B_3)) = (1, 0, 1 - \alpha)$. And, $\rho_{\mathtt{max}}(x_1, x_2, x_3) = (x_3, x_1, x_2)$ (as B_3 is less similar to the others) and $\mathbf{n}^{\rho_{\mathtt{max}}}_{\mathtt{max}}((x_1, B_1), (x_2, B_2), (x_3, B_3)) = (1 - \alpha, 0, 1)$. For $\alpha = 0.5$, we get $\mathbf{n}^{\rho_{\mathtt{min}}}_{\mathtt{max}} = (1, 0, 0.5)$ and $\mathbf{n}^{\rho_{\mathtt{max}}}_{\mathtt{max}} = (0.5, 0, 1)$.

The last adjustment function is based on the gradual semantics Weighted h-Categoriser in the context of weighted argumentation frameworks ($\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \sigma \rangle$). An important question is: *"why a gradual semantics can itself play the role of an adjustment function?"*. The answer lies in the great analogy between the two: both aim at reducing strengths of arguments according to a set of other arguments. Another key question is: *"on which argumentation framework is the semantics applied?"*. Recall that an input of any adjustment function is a tuple of the form $((x_1, A_1), \cdots, (x_k, A_k))$, with $x_i \in [0, 1]$ is given by the gradual semantics that is used and $A_i \in \mathcal{A}$. For every such input, we create a weighted argumentation framework $\langle \mathcal{A}', \mathbf{w}', \mathcal{R}', \sigma' \rangle$ such that:

- $\mathcal{A}' = \{A_1, \cdots, A_k\}$
- For every $A_i \in \mathcal{A}'$, $\mathbf{w}'(A_i) = x_i$
- $\mathcal{R}' = (\mathcal{A}' \times \mathcal{A}') \setminus \{(A_i, A_i) \mid i = 1, \cdots, n\}$
- For every $(A_i, A_j) \in \mathcal{R}'$, $\sigma'((A_i, A_j)) = \mathtt{sim}(A_i, A_j)$

The framework contains thus the set of attackers whose strengths should be readjusted, the initial weight of every argument is its value assigned by the semantics, the attack relation is symmetric and the weight of every attack is the similarity degree between its target and its source. Weighted h-categoriser is applied to this framework and the values assigned to arguments correspond to their readjusted values.

Definition 59 ($\mathbf{n}_{\mathtt{wh}}$)**.** *Let* $\mathtt{sim}$ *a similarity measure,* $A_1, \cdots, A_k \in \mathtt{Arg}$ *and* $x_1, \cdots, x_k \in [0, 1]$*. We define the adjustment function* $\mathbf{n}_{\mathtt{wh}}$ *as follows,* $\mathbf{n}_{\mathtt{wh}}() = ()$*, otherwise:*

$$\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (\mathtt{Deg}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}'}(A_1), \cdots, \mathtt{Deg}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}'}(A_k))$$

where $\mathbf{G}' = \langle \mathcal{A}', \mathbf{w}', \mathcal{R}', \sigma' \rangle$*, such that:*

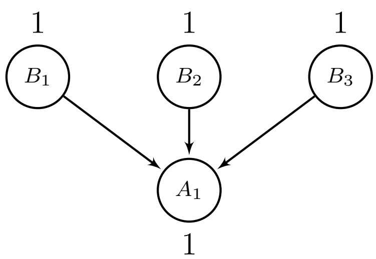
- $\mathcal{A}' = \{A_1, \cdots, A_k\}$,
- $\mathbf{w}'(A_1) = x_1, \cdots, \mathbf{w}'(A_k) = x_k$,
- $\mathcal{R}' = \{(A_1, A_2), \cdots, (A_1, A_k), \cdots, (A_k, A_1), \cdots, (A_k, A_{k-1})\}$,
- *For every* $(A_i, A_j) \in \mathcal{R}'$*,* $\sigma'((A_i, A_j)) = \mathtt{sim}(A_i, A_j)$*,*

Hence, the strength x_i of every attacker A_i will be readjusted to $\mathtt{Deg}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}'}(A_i)$, where

$$\mathtt{Deg}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}'}(A_i) = \frac{x_i}{1 + \sum\limits_{j \in \{1, \cdots, n\} \setminus \{i\}} \mathtt{Deg}^{\mathbf{S}_{\mathtt{wh}}}_{\mathbf{G}'}(A_j) \times \mathtt{sim}(A_j, A_i)}.$$

Let us illustrate the above definition on the graph $\mathbf{G}_{11}$ in the Figure 3.1.

Example 3 (Cont.) We recall the graph $\mathbf{G}_{11}$:

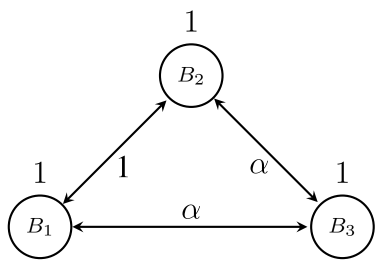


$\mathtt{sim}(B_1, B_2) = 1$

$\mathtt{sim}(B_1, B_3) = \mathtt{sim}(B_2, B_3) = \alpha$

$0 < \alpha < 1$

For any reasonable semantics $\mathbf{S}$, $\mathtt{Str}^{\mathbf{S}}(B_i) = \mathbf{w}(B_i)$ since the arguments are not attacked. Therefore $\mathtt{Str}^{\mathbf{S}}(B_1) = \mathtt{Str}^{\mathbf{S}}(B_2) = \mathtt{Str}^{\mathbf{S}}(B_3) = 1$. According to the Definition 59, to apply $\mathbf{n}_{\mathtt{wh}}$ on the attackers of A_1, let us create the new graph $\mathbf{G}'$ depicted below:



$\mathbf{n}_{\texttt{wh}}$ evaluates the arguments of the above graph using Weighted h-Categoriser. It is easy to check that if $\alpha = 0.5$ then $\texttt{Deg}^{\mathbf{S}_{\texttt{wh}}}_{\mathbf{G}'}(B_1) = 0.537$, $\texttt{Deg}^{\mathbf{S}_{\texttt{wh}}}_{\mathbf{G}'}(B_2) = 0.537$. $\texttt{Deg}^{\mathbf{S}_{\texttt{wh}}}_{\mathbf{G}'}(B_3) = 0.651$. So, $\mathbf{n}_{\texttt{wh}}((1, B_1), (1, B_2), (1, B_3)) = (0.537, 0.537, 0.651)$ meaning that the readjusted value of B_1, B_2 and B_3 are respectively 0.537, 0.537 and 0.651. Therefore, $\mathbf{n}_{\texttt{wh}}$ follows a compensative strategy.

3.5.2 Study of Adjustment Functions

This section investigate the properties of the three adjustment functions. The first result shows that they are indeed adjustment functions since their values are taken from the unit interval [0,1].

Proposition 24. *For all $A_1, \cdots, A_k \in \texttt{Arg}$, for all $x_1, \cdots, x_k \in [0, 1]$,*

- $\mathbf{n}_{\texttt{rs}}((x_1, A_1), \cdots, (x_k, A_k)) \in [0, 1]^k$,
- *for any permutation ρ on the set $\{1, \cdots, k\}$, $\mathbf{n}^{\rho}_{\texttt{max}}((x_1, A_1), \cdots, (x_k, A_k)) \in [0, 1]^k$,*
- $\mathbf{n}_{\texttt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) \in [0, 1]^k$.

Let us now check whether the three functions are well-behaved, i.e., they satisfy the conditions of well-behaved function.

We show that the adjustment function $\mathbf{n}_{\texttt{rs}}$ satisfies almost all the constraints from Def. 54 except (3g). This means that $\mathbf{n}_{\texttt{rs}}$ modifies the values of attackers even when they are all dissimilar. Therefore, $\mathbf{n}_{\texttt{rs}}$ is not well-behaved. However according to a specific g, $\mathbf{n}_{\texttt{rs}}$ satisfies (C1) and (C2).

Proposition 25. *The following properties hold.*

- $\mathbf{n}_{\texttt{rs}}$ *violates the condition (3g) of Def. 54.*
- $\mathbf{n}_{\texttt{rs}}$ *satisfies the conditions (3a), $\cdots$, (3f) of Def. 54.*
- $\mathbf{n}_{\texttt{rs}}$ *is not well-behaved.*
- *If* $\mathbf{g}$ *satisfies (C3), then* $\mathbf{n}_{\texttt{rs}}$ *satisfies (C1) and (C2).*

We show next that the functions $\mathbf{n}^{\rho}_{\texttt{max}}$ satisfy the conditions (3a,$\cdots$,3g) of Definition 54 and those of Theorem 16. However, they violate the conditions (C1) and (C2) because the $\max$ operator considers only the greatest similarity. Hence, increasing small similarity degrees would not impact the result of $\mathbf{n}^{\rho}_{\texttt{max}}$.

Proposition 26. *Let* $\mathbf{f}, \mathbf{g}$ *be well-behaved functions and* $\mathbf{g}$ *satisfies the following property:*

let $\lambda \in [0,1],\ x_1, \cdots, x_k \in [0,1]$, *then* $\mathbf{g}(\lambda x_1, \cdots, \lambda x_k) \geq \lambda \mathbf{g}(x_1, \cdots, x_k)$.

Let $A_1, \cdots, A_k \in \texttt{Arg}$, *the following properties hold:*

- $\mathbf{n}^{\rho}_{\texttt{max}}$ *is well-behaved.*
- $\mathbf{n}^{\rho}_{\texttt{max}}$ *is continuous on numerical variables.*
- $\mathbf{g}(\mathbf{n}^{\rho}_{\texttt{max}}((\lambda x_1, A_1), \cdots, (\lambda x_k, A_k))) \geq \lambda \mathbf{g}(\mathbf{n}^{\rho}_{\texttt{max}}((x_1, A_1), \cdots, (x_k, A_k)))$, $\forall \lambda \in [0,1]$.
- $\mathbf{n}^{\rho}_{\texttt{max}}$ *violates the conditions (C1) and (C2).*

From above, it follows that the functions $\mathbf{n}^{\rho}_{\texttt{max}}$ are used by evaluation methods of the set $\mathbf{M}^*$, and thus by the novel family of semantics.

Proposition 27. *For all functions* $\mathbf{f}, \mathbf{g}$ *that are well-behaved and satisfy the conditions of Theorem 16, it holds that* $\langle \mathbf{f}, \mathbf{g}, \mathbf{n}^{\rho}_{\texttt{max}} \rangle \in \mathbf{M}^*$.

As for $\mathbf{n}_{\texttt{wh}}$, we show that it is also well-behaved and it can be used in an evaluation method of the set $\mathbf{M}^*$. Moreover, this adjustment function satisfies (C1) and (C2) (according to a specific $\mathbf{g}$).

Proposition 28. *Let* $\mathbf{f}, \mathbf{g}$ *be well-behaved functions and* $\mathbf{g}$ *satisfies the following property:*

let $\lambda \in [0,1],\ x_1, \cdots, x_k \in [0,1]$, *then* $\mathbf{g}(\lambda x_1, \cdots, \lambda x_k) \geq \lambda \mathbf{g}(x_1, \cdots, x_k)$.

Let $A_1, \cdots, A_k \in \texttt{Arg}$, *the following properties hold:*

- $\mathbf{n}_{\texttt{wh}}$ *is well-behaved.*
- $\mathbf{n}_{\texttt{wh}}$ *is continuous on numerical variables.*
- $\mathbf{g}(\mathbf{n}_{\texttt{wh}}((\lambda x_1, A_1), \cdots, (\lambda x_k, A_k))) \geq \lambda \mathbf{g}(\mathbf{n}_{\texttt{wh}}((x_1, A_1), \cdots, (x_k, A_k)))$, $\forall \lambda \in [0,1]$.
- *If* $\mathbf{g}$ *satisfies (C3), then* $\mathbf{n}_{\texttt{wh}}$ *satisfies the conditions (C1) and (C2).*
- $\langle \mathbf{f}, \mathbf{g}, \mathbf{n}_{\texttt{wh}} \rangle \in \mathbf{M}^*$.

We naturally observe that it is possible to have different well-behaved adjustment functions with different strategies. We are now going to see in more detail their behaviour with some properties.

The next property states that an adjustment function $\mathbf{n}$ can only reduce the value of an argument.

Property 5. *Let* $\mathbf{n}$ *be an adjustment function. For any* SSWAF, $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$, *for all* $A_1, \cdots, A_k \in \mathcal{A}$, *for all* $x_1, \cdots, x_k \in [0,1]$, *if* $\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$, *then* $\forall i \in \{1, \cdots, n\}$, $x'_i \leq x_i$.

Let us take a look at our functions.

Proposition 29. *Only* $\mathbf{n}^{\rho}_{\mathtt{max}}$ *and* $\mathbf{n}_{\mathtt{wh}}$ *respect the Property 5.*

- $\mathbf{n}_{\mathtt{rs}}$ *violates Property 5,*
- $\mathbf{n}^{\rho}_{\mathtt{max}}$ *satisfies Property 5,*
- $\mathbf{n}_{\mathtt{wh}}$ *satisfies Property 5.*

When an argument having an initial value of 0 and for any similarity with other arguments, this arguments doesn't impact the readjusted values of the other arguments.

Property 6. *Let* $\mathbf{n}$ *be an adjustment function. For any* SSWAF, $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$, *for all* $A_1, \cdots, A_k, B \in \mathcal{A}$, *for all* $x_1, \cdots, x_k, y \in [0,1]$, *if* $y = 0$, *then* $\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k), (y, B)) = (\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)), 0)$.

Below is the result regarding our functions.

Proposition 30. *Only* $\mathbf{n}^{\rho}_{\mathtt{max}}$ *and* $\mathbf{n}_{\mathtt{wh}}$ *respect the Property 6.*

- $\mathbf{n}_{\mathtt{rs}}$ *violates Property 6,*
- $\mathbf{n}^{\rho}_{\mathtt{max}}$ *satisfies Property 6,*
- $\mathbf{n}_{\mathtt{wh}}$ *satisfies Property 6.*

An adjustment function $\mathbf{n}$ cannot readjusts a positive value to 0.

Property 7. *Let* $\mathbf{n}$ *be an adjustment function,* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ *be an* SSWAF, $A_1, \cdots, A_k \in \mathcal{A}$, $x_1, \cdots, x_k \in [0,1]$ *and* $\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$. *For any* $i \in \{1, \cdots, n\}$, *if* $x_i > 0$, *then* $x'_i > 0$.

Proposition 31. *Only* $\mathbf{n}_{\mathtt{rs}}$ *and* $\mathbf{n}_{\mathtt{wh}}$ *respect the Property 7.*

- $\mathbf{n}_{\mathtt{rs}}$ *satisfies Property 7,*
- $\mathbf{n}^{\rho}_{\mathtt{max}}$ *violates Property 7,*
- $\mathbf{n}_{\mathtt{wh}}$ *satisfies Property 7.*

We propose in a second step to analyse the behaviour of our adjustment functions combined with an aggregation function widely used in gradual semantics, i.e. $\mathbf{g}_{sum}$. In this particular case, it may be required that adding an argument that has no strength and no similarity with the other arguments does not affect the value of the group.

Property 8. *Let* $\mathbf{n}$ *an adjustment function,* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ *be an* SSWAF*,* $A_1, \cdots, A_k$*,* $B \in \mathcal{A}$*,* $x_1, \cdots, x_k, y \in [0, 1]$*, if*

- $\forall i \in \{1, \cdots, k\}$, $\mathtt{sim}(A_i, B) = 0$,
- $y = 0$,

Then $\mathbf{g}_{sum}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))) = \mathbf{g}_{sum}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k), (y, B)))$.
This means that adding an attacker dissimilar to all other and whose initial value is 0, cannot increase the sum of readjusted values of the set of attackers.

We know that the adjustment function $\mathbf{n}_{\mathtt{rs}}$ violates condition 3(g) of Definition 54, i.e. an argument that is dissimilar to all other arguments, can have an impact on the readjusted values of the other arguments. Furthermore, when this dissimilar argument has an initial value of 0 and using the $\mathbf{g}_{sum}$ aggregation function, we see that the sum of the adjusted value increases.

Proposition 32. *Only* $\mathbf{n}^{\rho}_{\mathtt{max}}$ *and* $\mathbf{n}_{\mathtt{wh}}$ *respect the Property 8.*

- $\mathbf{n}_{\mathtt{rs}}$ *violates Property 8,*
- $\mathbf{n}^{\rho}_{\mathtt{max}}$ *satisfies Property 8,*
- $\mathbf{n}_{\mathtt{wh}}$ *satisfies Property 8.*

The use of the $\mathbf{g}_{sum}$ aggregation function can also be used to highlight the behaviour of adjustment functions in the case of a set of totally similar arguments. Intuitively, if all the attackers of an argument are totally similar, we would like to have a sum of the adjusted values less than or equal to 1.

Property 9. *Let* $\mathbf{n}$ *be an adjustment function,* $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ *be an* SSWAF*,* $A_1, \cdots, A_k \in \mathcal{A}$*,* $x_1, \cdots, x_k \in [0, 1]$*. If* $\forall i, j \in \{1, \cdots, n\}$, $\mathtt{sim}(A_i, A_j) = 1$ *then* $\mathbf{g}_{sum}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))) \leq 1$.

Proposition 33. *Only* $\mathbf{n}^{\rho}_{\mathtt{max}}$ *respects the Property 9.*

- $\mathbf{n}_{\mathtt{rs}}$ *violates Property 9,*
- $\mathbf{n}^{\rho}_{\mathtt{max}}$ *satisfies Property 9,*

- $\mathbf{n}_{\mathtt{wh}}$ *violates Property 9.*

Table 3.1 summarises the various results presented above. We separate properties 8 and 9 which are defined according to the aggregation function $\mathbf{g}_{\mathtt{sum}}$. Under this assumption (use of $\mathbf{g}_{\mathtt{sum}}$), properties 8 and 9 are important to respect a correct behaviour. The other properties are more general and informative (exception for the well-behaved and $\mathbf{M}^*$ properties).

	$\mathbf{n}_{\mathtt{rs}}$	$\mathbf{n}^{\rho}_{\mathtt{max}}$	$\mathbf{n}_{\mathtt{wh}}$
$\mathbf{n} \in [0, 1]$	•	•	•
$\mathbf{n}$ is well-behaved	○	•	•
$\langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle \in \mathbf{M}^*$	○	•	•
$\mathbf{n}$ satisfies (C1)	•	○	•
$\mathbf{n}$ satisfies (C2)	•	○	•
Property 5	○	•	•
Property 6	○	•	•
Property 7	•	○	•
Property 8	○	•	•
Property 9	○	•	○

Where • means True/Satisfied and ○ means False/Violated.

Table 3.1: Satisfaction of the properties of adjustment functions

Note that, there are 3 parameters that may change the score of $\mathbf{g}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)))$; these are the strength of arguments (x_i), the similarity degrees $(\mathtt{sim}(A_i, A_j))$ and the number of attackers (k). The impact of these 3 parameters is identical according to each adjustment function, i.e.:

- Increasing the strength of an argument cannot decrease the sum.
- Increasing a similarity degree cannot increase the sum.
- Increasing the number of attackers cannot decrease the sum.

However, the variation in the increase or decrease of the sum is not the same:

Variation of	$\mathbf{n}_{\mathtt{rs}}$	$\mathbf{n}^{\rho}_{\mathtt{max}}$	$\mathbf{n}_{\mathtt{wh}}$
Degrees	Constant	Constant	Decreasing
Similarity	Constant	Constant	Decreasing
Number of attackers	Constant	Constant	Decreasing

Table 3.2: Variation of the sum of the adjusted values according to a parameter

To put it clearly, when we increase one of the parameters (i.e. degree, similarity, or number of attackers), we observe a constant variation for $\mathbf{n}_{\mathtt{rs}}$ or $\mathbf{n}^{\rho}_{\mathtt{max}}$ while we observe a decreasing variation for $\mathbf{n}_{\mathtt{wh}}$.

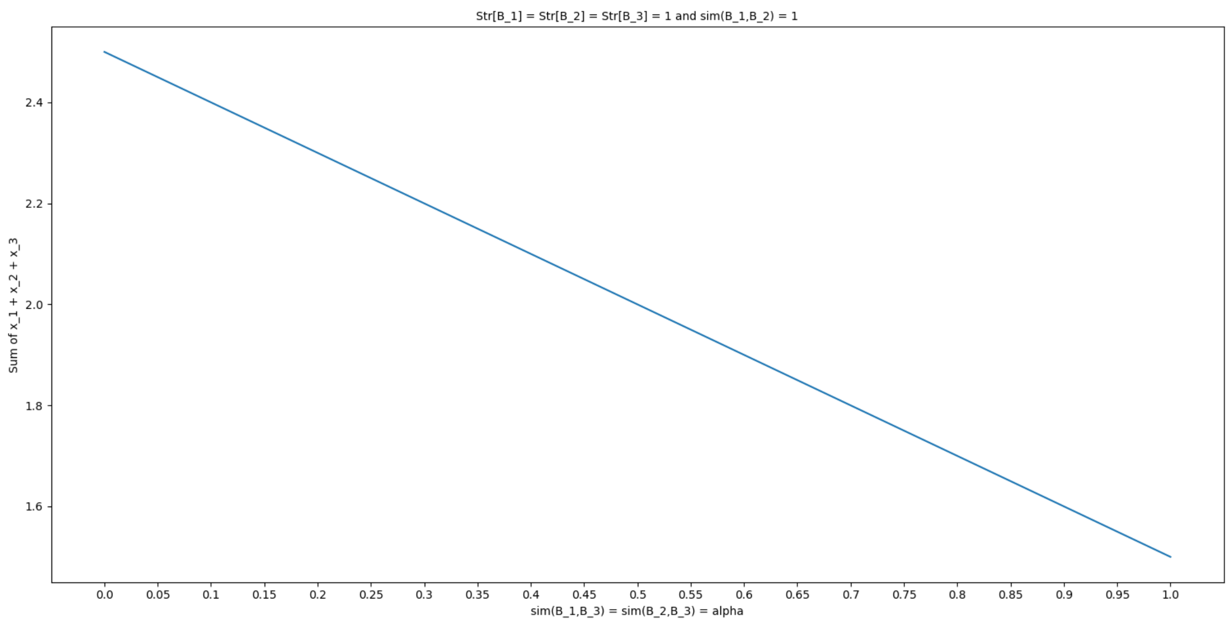
Example 3 (Cont.) Let us illustrate constant and decreasing variation, with $\mathbf{n}_{\mathtt{rs}}$ and $\mathbf{n}_{\mathtt{wh}}$. Recall that we have three arguments $B_1, B_2, B_3 \in \mathtt{Arg}$ such that $\mathtt{sim}(B_1, B_2) = 1$, $\mathtt{sim}(B_1, B_3) = \mathtt{sim}(B_2, B_3) = \alpha$, and $\mathtt{Str}(B_1) = \mathtt{Str}(B_2) = \mathtt{Str}(B_3) = 1$.

If we vary the value of α from 0 to 1 with an increase of 0.1, we obtain these results:

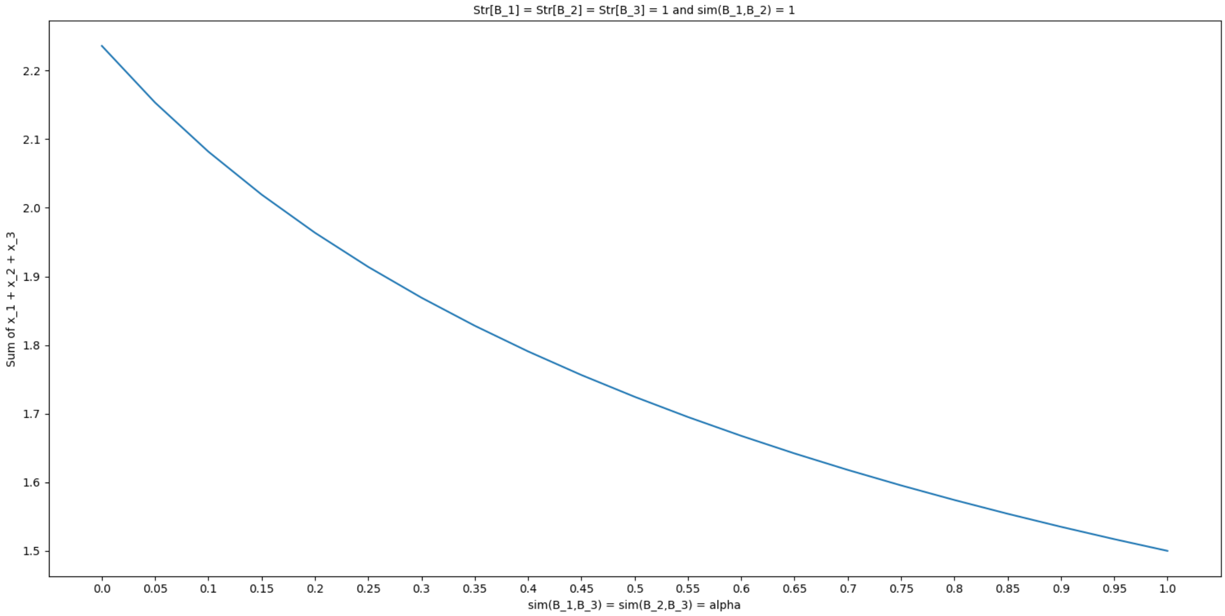
α	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\mathbf{g}_{sum}(\mathbf{n}_{\mathtt{rs}})$	2.5	2.4	2.3	2.2	2.1	2	1.9	1.8	1.7	1.6	1.5
$\mathbf{g}_{sum}(\mathbf{n}_{\mathtt{wh}})$	2.236	2.082	1.963	1.868	1.791	1.725	1.668	1.619	1.574	1.535	1.5

Table 3.3: Variation of the sum of the adjusted values according to similarities

To have a better visualisation of these results, below is their representation in graphical form. We call x_i the readjusted value of $\mathtt{Str}(B_i)$. Let us see the variation of the sum of x_i with respect to the similarities α for $\mathbf{g}_{sum}(\mathbf{n}_{\mathtt{rs}}((\mathtt{Str}(B_1), B_1), (\mathtt{Str}(B_2), B_2), (\mathtt{Str}(B_3), B_3)))$:



Let us see the variation of the sum of x_i with respect to the similarities α for $\mathbf{g}_{sum}(\mathbf{n}_{\mathtt{wh}}((\mathtt{Str}(B_1), B_1), (\mathtt{Str}(B_2), B_2), (\mathtt{Str}(B_3), B_3)))$:



To conclude this section, thanks to the different analyses, we can observe that:

- $\mathbf{n}_{\mathtt{rs}}$ is a compensative adjustment function. It violates the properties 8 and 9 which are desirable. It violates the property 5 which is more informative than desirable. Then it violates property 6 which may be considered as a correct behaviour if we want to consider both arguments with zero strength and those with non-zero value. Moreover, it is not well-behaved due to the violation of condition (3g) of Definition 54. But according to the function $\mathbf{g}$, as for $\mathbf{g}_{\mathtt{sum}}$, this change has no negative effect on the aggregation of adjusted values. Finally, $\mathbf{n}_{\mathtt{rs}}$ has an undesirable behaviour with the aggregation function $\mathbf{g}_{\mathtt{sum}}$ but it has a great sensitivity to the variation of strength, similarity and takes into account arguments with a strength of 0.

- $\mathbf{n}^{\rho}_{\mathtt{max}}$ is a family of conjunctive and disjunctive adjustment functions. They satisfy the property 6 which means that if an argument has a strength of 0 then its similarity has no impact on the other arguments.In addition, they violate conditions (C1) and (C2), which allows a more accurate adjustment. Therefore, the family of functions $\mathbf{n}^{\rho}_{\mathtt{max}}$ is well-behaved, it satisfies all the essential properties but lacks a bit of precision.

- $\mathbf{n}_{\mathtt{wh}}$ is a compensative adjustment function. It violates the property 9 which is a desirable property even if it is defined according to the specific aggregation function $\mathbf{g}_{\mathtt{sum}}$ and it satisfies the debatable property 6. Furthermore, it is interesting to note that the variation of the sum of adjusted values for $\mathbf{n}_{\mathtt{wh}}$ is decreasing, i.e. it makes a non-linear adjustment, while for $\mathbf{n}_{\mathtt{rs}}$ and $\mathbf{n}^{\rho}_{\mathtt{max}}$ they make a linear adjustment.

After studying all these functions, we are able to instantiate our evaluation methods to obtain gradual semantics dealing with similarity.

3.6 Instances of Semantics

In this last section of this chapter, we present instances of the broad family $\mathbf{S}^*$ (Def. 56) that extend h-categoriser (Besnard and Hunter [2001]). They use the well-behaved functions $f_{\mathtt{frac}}$ and $\mathbf{g}_{\mathtt{sum}}$ recalled below and the previously defined adjustment functions $\mathbf{n}_{\mathtt{rs}}$, $\mathbf{n}^{\rho}_{\mathtt{max}}$ and $\mathbf{n}_{\mathtt{wh}}$.

$$\mathbf{f}_{\mathtt{frac}}(x_1, x_2) = \frac{x_1}{1 + x_2} \qquad \mathbf{g}_{\mathtt{sum}}(x_1, \cdots, x_k) = \sum_{i=1}^{k} x_i$$

Definition 60. *Semantics* $\mathbf{S}^{\mathbf{n}}$ *based on the evaluation method* $\langle \mathbf{f}_{\mathtt{frac}}, \mathbf{g}_{\mathtt{sum}}, \mathbf{n} \rangle$ *is a function transforming any* SSWAF, $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$, *into a function* $\mathtt{Str}^{\mathbf{n}}$ *from* $\mathcal{A}$ *to* $[0, 1]$ *s.t. for any* $A \in \mathcal{A}$, $\mathtt{Str}^{\mathbf{n}}(A) =$

$$\frac{\mathbf{w}(A)}{1 + \sum\limits_{i=1}^{k} \left(\mathbf{n}\Big((\mathtt{Str}^{\mathbf{n}}(B_1), B_1), \cdots, (\mathtt{Str}^{\mathbf{n}}(B_k), B_k) \Big) \right)}$$

where $\mathtt{Att}(A) = \{B_1, \cdots, B_k\}$. *If* $\mathtt{Att}(A) = \emptyset$, *then*
$\sum\limits_{i=1}^{k} \left(\mathbf{n}\Big((\mathtt{Str}^{\mathbf{n}}(B_1), B_1), \cdots, (\mathtt{Str}^{\mathbf{n}}(B_k), B_k) \Big) \right) = 0.$

Example 3 (Cont.) Let us consider the adjustment functions $\mathbf{n}^{\rho}_{\mathtt{max}}$, $\mathbf{n}_{\mathtt{rs}}$ and $\mathbf{n}_{\mathtt{wh}}$.

$$\mathtt{Str}^{\mathbf{n}^{\rho_{\mathtt{min}}}_{\mathtt{max}}}(A_1) = \frac{1}{1 + 1 + 0 + 1 - \alpha} = \frac{1}{3 - \alpha}$$

$$\mathtt{Str}^{\mathbf{n}^{\rho_{\mathtt{max}}}_{\mathtt{max}}}(A_1) = \frac{1}{1 + 1 + 1 - \alpha + 0} = \frac{1}{3 - \alpha}$$

$$\mathtt{Str}^{\mathbf{n}_{\mathtt{rs}}}(A_1) = \frac{1}{1 + \frac{3-\alpha}{4} + \frac{3-\alpha}{4} + \frac{2-\alpha}{2}} = \frac{1}{3.5 - \alpha}$$

$$\mathtt{Str}^{\mathbf{n}_{\mathtt{wh}}}(A_1) = \frac{1}{1 + x_1 + x_2 + x_3}$$

Where x_i is the readjusted value of $\texttt{Str}^{\mathbf{n}_{\texttt{wh}}}(B_i)$ and such that:

$$\begin{cases} x_1 = x_2 \\ x_3 = 2x_1^2 + 2x_1 - 1 \\ \alpha = \frac{-x_1^2 - x_1 + 1}{x_1(2x_1^2 + 2x_1 - 1)} \end{cases}$$

Then if $\alpha = 0.5$:

- $\texttt{Str}^{\mathbf{n}^{\rho_{\texttt{min}}}_{\texttt{max}}}(A_1) = \frac{1}{2.5} = 0.4$
- $\texttt{Str}^{\mathbf{n}^{\rho_{\texttt{max}}}_{\texttt{max}}}(A_1) = \frac{1}{2.5} = 0.4$
- $\texttt{Str}^{\mathbf{n}_{\texttt{rs}}}(A_1) = \frac{1}{3} = 0.333$
- $\texttt{Str}^{\mathbf{n}_{\texttt{wh}}}(A_1) = \frac{40}{109} = 0.367$ with $x_1 = x_2 = 0.537$ and $x_3 = 0.651$.

Note that $\mathbf{S}^{\mathbf{n}^{\rho}_{\texttt{max}}}$ covers a range of semantics using different permutations. We show that those semantics are all instances of $\mathbf{S}^*$. They thus satisfy all the (large versions of the) principles. In addition, they satisfy strict monotony, but violate the strict versions of Reinforcement and sensitivity to similarity due to the max operator.

Theorem 22. *For any ρ, it holds that $\mathbf{S}^{\mathbf{n}^{\rho}_{\texttt{max}}} \in \mathbf{S}^*$. Furthermore, $\mathbf{S}^{\mathbf{n}^{\rho}_{\texttt{max}}}$ satisfies Reinforcement, (Strict) Monotony, Neutrality and Sensitivity to Similarity.*

The semantics $\mathbf{S}^{\mathbf{n}_{\texttt{rs}}}$ satisfies all the principles except Neutrality. Note that this semantics extends h-categoriser which satisfies Neutrality in settings where $\texttt{sim} \equiv 0$.

Theorem 23. *The semantics $\mathbf{S}^{\mathbf{n}_{\texttt{rs}}}$ satisfies all the principles except Neutrality. Furthermore, $\mathbf{S}^{\mathbf{n}_{\texttt{rs}}} \notin \mathbf{S}^*$.*

The semantics $\mathbf{S}^{\mathbf{n}_{\texttt{wh}}}$ satisfies all the principles.

Theorem 24. *The semantics $\mathbf{S}^{\mathbf{n}_{\texttt{wh}}}$ satisfies all the principles. Furthermore, $\mathbf{S}^{\mathbf{n}_{\texttt{wh}}} \in \mathbf{S}^*$.*

Note that, when the arguments are all distinct (i.e., similarities are equal to 0), the above semantics assign the same values to all arguments, and coincide with the semi-weighted h-categoriser semantics (denoted by $\texttt{Str}^h$) that assigns for any $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \texttt{sim} \rangle$, to every argument $A \in \mathcal{A}$,

$$\texttt{Str}^h(A) = \frac{\mathbf{w}(A)}{1 + \sum_{B_i \in \texttt{Att}(A)} \texttt{Str}^h(B_i)} \tag{3.1}$$

Theorem 25. *For any* $\mathtt{SSWAF}$, $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ *and any permutation* ρ, *if* $\mathtt{sim} \equiv 0$, *then*

$$\mathtt{Str}^{\mathbf{n}^{\rho}_{\mathtt{max}}} \equiv \mathtt{Str}^{\mathbf{n}_{\mathtt{rs}}} \equiv \mathtt{Str}^{\mathbf{n}_{\mathtt{wh}}} \equiv \mathtt{Str}^{h}.$$

This shows that these semantics extend semi-weighted h-categoriser by considering similarity degrees of attackers.

Chapter 4

Conclusions

Contents

4.1 Contributions

In this thesis, I studied the notion of similarity in argumentation. I tackled two research questions:

- how to measure similarity between two arguments, and
- how to define semantics, that are able to deal with similarity.

Regarding the first question, I focused on logical arguments, and proposed properties that a reasonable similarity measure should satisfy in this context of arguments. Then, I extended in several ways well-known similarity measure, from the literature. The measure are syntax-dependent, thus I used various notions (like concise arguments, refinements of arguments) for solving some tricky issues. This part of the thesis is novel, and there is almost no work in the literature neither on principles for measure, nor on the definition of measure. Two notable exceptions are the work by Amgoud *et al.* [2014] where the authors studied equivalence of logical argument the latter corresponds to full similarity in our setting. The second measure was proposed by Budàn *et al.* [2015] and it is based on a simple comparison of features associated to arguments.

The second part of the thesis is also novel in the literature. Indeed, I proposed the first principles for semantics dealing with similarity. I also proposed a general setting (using three functions) for defining in a systematic way gradual semantics. Each function should obey to certain rules ensuring reasonable behavior in dealing with similarity. I investigated different readjustment function, and proposed a broad family of gradual semantics that encompasses almost all the existing gradual semantics.

4.2 Perspectives

This work may be extended in different directions, and both argumentation settings: logical or abstract. Let us start with the case of logical argumentation. Firstly, a possible research direction consists of getting rid of the syntax-dependency of our similarity measures. For instance, one may want to capture the similarity between the supports for the following arguments $\langle\{p\}, p\rangle$ and $\langle\{p \wedge q\}, p \wedge q\rangle$. Indeed, the formulas p and $p \wedge q$ have a common content, p. Therefore, similarity measures applied to supports should be improved in terms of accuracy. A solution to this problem might be, like concise arguments, to decompose each argument into a minimal structure. A solution might be the use of causal formulas, i.e., the formulas in the support of an argument are clauses.
As far as concise arguments are concerned, we can observe that similarity measures dealing with non-concise arguments do not respect support-based principles (since a refined argument may modify the support). Therefore, new principles that are based on concise arguments will have to be defined.
Finally, it would be interesting to study similarity under more complex logics such as first order logics, or modal logic.

Let us now look at the different perspectives for abstract argumentation dealing with a similarity measure. A first line of research is to study fair adjustment functions, which remove the exact amount of redundancy. Furthermore, the combination of influence and aggregation functions ($\mathbf{f}$ and $\mathbf{g}$) may also be studied further with adjustment functions $\mathbf{n}$. For example, we have seen that $\mathbf{g}_{\mathtt{sum}}$ and $\mathbf{n}_{\mathtt{wh}}$ do not combine so well. To get a more complete evaluation method, we can extend it by adding a function $\mathbf{h}$ which allows to compute the strength of an attack (with the strength of an argument and the weight of a relation). Then, another characterisation including all functions of an evaluation method must be done to ensure the definition of one and only one gradual semantics.
As we have seen in the state of the art, there are different types of semantics (extensions, gradual, ranking). It would also be interesting to study how ranking and extension semantics would deal with similarity.

We plan to apply the new semantics in applications such as analogical reasoning and argument evaluation in debate platforms.
Finally, in this thesis we have studied a binary similarity measure (between pairs of arguments). It is easier (from a human's point of view) to know how similar two arguments are to each other than to know the similarity between a set of arguments. However, it is not possible to deduce the exact degree of similarity between a set of arguments from a binary measure set. Therefore, for greater accuracy and to deal with similarity in its generality, it will be necessary to investigate n-ary similarity measures.

Chapter 5

Appendix

5.1 Proofs of Chapter 2

5.1.1 Proofs of section 2.1: Background on Logic

Proof. [Property 1] Assume that $\texttt{Co}(\Phi, \Psi) = \Phi$ and $\texttt{Co}(\Psi, \Phi) = \Psi$. We can deduce that:

- $\texttt{Co}(\Phi, \Psi) = \Phi$ implies $\forall \phi \in \Phi$, $\exists \psi \in \Psi$ such that $\phi \equiv \psi$,
- $\texttt{Co}(\Psi, \Phi) = \Psi$ implies $\forall \psi \in \Psi$, $\exists \phi \in \Phi$ such that $\psi \equiv \phi$.

Therefore, $\Phi \cong \Psi$ according to the definition 34. The other way follows also trivially from Definition 34. □

Proof. [Property 2] From Definition 36, an argument $\langle \Phi, \phi \rangle$ is *trivial* iff $\Phi = \emptyset$ and $\phi \equiv \top$. From Definition 39, two arguments are equivalent iff their support and conclusion are logically equivalent. Because any empty set are equivalent and $\top \equiv \top$, therefore any pair of trivial argument are equivalent. □

Proof. [Property 3] Let $A, B \in \texttt{Arg}(\mathcal{L})$. We distinguish two cases: i) $\texttt{Supp}(A) = \emptyset$ or $\texttt{Supp}(B) = \emptyset$. By definition, $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) = \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) = \emptyset$. Hence, $|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))| = |\texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A))| = 0$. ii) $\texttt{Supp}(A) \neq \emptyset$ and $\texttt{Supp}(B) \neq \emptyset$. If $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) = \emptyset$, then $\texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) = \emptyset$. Assume now that $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) \neq \emptyset$. Assume that $|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))| < |\texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A))|$. Thus, there exists at least two formulas $\phi, \psi \in \texttt{Co}(\texttt{Supp}(B),$ $\texttt{Supp}(A))$ such that $\phi \equiv \lambda$ and $\psi \equiv \lambda$, with $\lambda \in \texttt{Supp}(A)$. This means that $\phi \equiv \psi$. This contradicts the fact that $\texttt{Supp}(B)$ is minimal for set inclusion. □

Proof. [Property 4] Let $A, B \in \texttt{Arg}(\mathcal{L})$ be such that $\texttt{Supp}(A) \cong \texttt{Supp}(B)$. Property 1 implies $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) = \texttt{Supp}(A)$ and $\texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) = \texttt{Supp}(B)$. Property 3 implies that $|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))| = |\texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A))|$. Hence, $|\texttt{Supp}(A)| = |\texttt{Supp}(B)|$. □

5.1.2 Proofs of section 2.2.2: Compatibility and Dependency Results

Proof. [Proposition 2] Follows from Theorem 13. □

Proof. [Proposition 3] Let $\texttt{sim}$ be a similarity measure which satisfies Maximality, Symmetry, Strict Monotony, Dominance, and Strict Dominance. Let $A, B, C \in \texttt{Arg}(\mathcal{L})$ such that $\texttt{sim}(A, B) = 1$. From Theorem 8, it holds that $\texttt{Supp}(A) \cong \texttt{Supp}(B)$ and $\texttt{Conc}(A) \equiv \texttt{Conc}(B)$. By applying Dominance twice, we get $\texttt{sim}(C, A) \geq \texttt{sim}(C, B)$ and $\texttt{sim}(C, B) \geq \texttt{sim}(C, A)$. Hence, $\texttt{sim}(C, A) = \texttt{sim}(C, B)$. Symmetry implies $\texttt{sim}(C, A) = \texttt{sim}(A, C) = \texttt{sim}(C, B) = \texttt{sim}(B, C)$. □

Proof. [Theorem 7] Let $\texttt{sim}$ be a similarity measure which satisfies Maximality and Monotony. Let $A, B \in \texttt{Arg}(\mathcal{L})$ be such that $A \approx B$. Let us show that $\texttt{sim}(A, B) = 1$. From Definition 39, $\texttt{Supp}(A) \cong \texttt{Supp}(B)$ and $\texttt{Conc}(A) \equiv \texttt{Conc}(B)$. From Monotony, it follows that $\texttt{sim}(A, A) \geq \texttt{sim}(A, B)$ and $\texttt{sim}(A, B) \geq \texttt{sim}(A, A)$. Therefore, $\texttt{sim}(A, A) = \texttt{sim}(A, B)$. From Maximality, $\texttt{sim}(A, A) = 1$, so $\texttt{sim}(A, B) = 1$. □

Proof. [Theorem 8] Let $\texttt{sim}$ be a similarity measure which satisfies Maximality, Strict Monotony and Strict Dominance. Let $A, B \in \texttt{Arg}(\mathcal{L})$ be such that $\texttt{sim}(A, B) = 1$. Let us show that $A \approx B$. There are two cases:

i) A and B are trivial: From Property 2, it holds that $A \approx B$.

ii) A is non-trivial: Assume that $A \not\approx B$. By definition, $\texttt{Supp}(A) \ncong \texttt{Supp}(B)$ or $\texttt{Conc}(A) \not\equiv \texttt{Conc}(B)$.

Consider the case where $\texttt{Supp}(A) \ncong \texttt{Supp}(B)$. Clearly,

- $\texttt{Conc}(A) \equiv \texttt{Conc}(A)$,
- $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) \subset \texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(A)) = \texttt{Supp}(A)$ (this inclusion is strict since $\texttt{Supp}(A) \neq \emptyset$ and $\texttt{Supp}(A) \ncong \texttt{Supp}(B)$),
- $\texttt{Supp}(A) \setminus \texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(A)) = \texttt{Co}(\texttt{Supp}(A) \setminus \texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(A)), \texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)))$,

By applying Strict Monotony, we get $\texttt{sim}(A, A) > \texttt{sim}(A, B)$. From Maximality $\texttt{sim}(A, A)$
$= 1$, so $\texttt{sim}(A, B) < 1$. This shows that $\texttt{Supp}(A) \cong \texttt{Supp}(B)$.

Consider now the case where $\texttt{Supp}(A) \cong \texttt{Supp}(B)$ and $\texttt{Conc}(A) \not\equiv \texttt{Conc}(B)$. The conditions of Strict Dominance are verified, indeed:

- $\texttt{Supp}(A) \cong \texttt{Supp}(B)$,
- $\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(B)) \subset \texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(A)) = \texttt{CN}_{df}(\texttt{Conc}(A))$. The implication is strict since $\texttt{Conc}(A) \not\equiv \texttt{Conc}(B)$.
- $\texttt{CN}_{df}(\texttt{Conc}(A)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A)) \subset \texttt{CN}_{df}(\texttt{Conc}(B)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A))$.
- $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(A)) = \texttt{Supp}(A)$. Since a is non trivial, then $\texttt{Supp}(A) \neq \emptyset$.

Strict Dominance ensures $\texttt{sim}(A, A) > \texttt{sim}(A, B)$ while Maximality ensures $\texttt{sim}(A, A) = 1$, so $\texttt{sim}(A, B) < 1$.

Note that the case where B is non-trivial is similar to the previous case. □

Proof. [Proposition 4] Let sim be a similarity measure which satisfies Minimality, and Substitution. Let $A, B, C \in \texttt{Arg}(\mathcal{L})$ such that A is non-trivial, B is trivial and C is trivial and $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(C)) = \emptyset$. Moreover, from Definition 36, $\texttt{Supp}(A) \neq \emptyset$, $\texttt{Supp}(C) = \emptyset$ and because $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(C)) = \emptyset$, therefore:

- A and C are not equivalent,
- $\bigcup_{\phi_i \in \texttt{Supp}(A)} \texttt{Var}(\phi_i) \cap \bigcup_{\phi_j \in \texttt{Supp}(C)} \texttt{Var}(\phi_j) = \emptyset$ and
- $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(C)) = \emptyset$.

Hence, from Minimality $\texttt{sim}(A, C) = 0$. From Property 2, $B \approx C$, then from Substitution $\texttt{sim}(A, B) = \texttt{sim}(A, C) = 0$. □

Proof. [Proposition 5] Let sim be a similarity measure which satisfies Strict Monotony or Non-Zero. Let $A, B \in \texttt{Arg}(\mathcal{L})$ such that $B \sqsubset A$ and B is non-trivial.
Start by the case of Strict Monotony.
Let $C \in \texttt{Arg}(\mathcal{L})$ such that :

- $\texttt{Supp}(C) = \{\phi \in \mathcal{L} \mid \forall \psi \in \texttt{Supp}(A),\ \phi \not\equiv \psi\}$.
- $\texttt{Var}(\texttt{Conc}(C)) \cap \texttt{Var}(\texttt{Conc}(A)) = \emptyset$.

Therefore the 3 conditions of the principle Monotony are satisfied. Additionally, given that $B \sqsubset A$ and B is non-trivial, then there exists an equivalent formula between the support of A and B. While there is no equivalent formula between the support of A and C. Hence, the inclusion in condition 2 is strict, i.e. we can use the result of Strict Monotony. Finally, given that from Definition 40, $\texttt{sim}(A, C) \geq 0$. Therefore, $\texttt{sim}(A, B) > 0$.

Let see now the case of Non-Zero.
From the Definition 37, and the fact that $B \sqsubset A$, we have $\texttt{Supp}(B) \subseteq \texttt{Supp}(A)$. Given

that B is non-trivial then $\exists \phi \in \texttt{Supp}(B)$ such that $\phi \in \texttt{Supp}(A)$. Consequently, $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) \neq \emptyset$, and using the principle Non-Zero we obtain $\texttt{sim}(A, B) > 0$.

□

Proof. [Proposition 6] Let $\texttt{sim}$ be a similarity measure which satisfies Monotony. Let $A, B, C \in \texttt{Arg}(\mathcal{L})$ be such that:

- $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(C)) = \emptyset$, and
- $C \sqsubset B \sqsubset A$.

It can be checked below that the conditions of Monotony are guaranteed. Indeed,

- $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(C)) = \emptyset$,
- $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C)) = \texttt{Supp}(C) \subseteq \texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) = \texttt{Supp}(B)$,
- $\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) = \texttt{Co}(\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)), \texttt{Supp}(C) \setminus \texttt{Co}(\texttt{Supp}(C), \texttt{Supp}(A))) = \emptyset$.

The second and third conditions are satisfied because $\texttt{Supp}(C) \subseteq \texttt{Supp}(A)$, $\texttt{Supp}(C) \subseteq \texttt{Supp}(B)$ and $\texttt{Supp}(B) \subseteq \texttt{Supp}(A)$.
Therefore, Monotony ensures $\texttt{sim}(A, B) \geq \texttt{sim}(A, C)$. □

Proof. [Proposition 7] Let $\texttt{sim}$ be a similarity measure which satisfies Strict Dominance. Let $A, B, C \in \texttt{Arg}(\mathcal{L})$ be such that:

1. A, B, C are non trivial,
2. $\texttt{Supp}(A) \cong \texttt{Supp}(B) \cong \texttt{Supp}(C)$,
3. $\texttt{Conc}(A) \vdash \texttt{Conc}(B) \vdash \texttt{Conc}(C)$,
4. $\texttt{Conc}(C) \nvdash \texttt{Conc}(B), \texttt{Conc}(B) \nvdash \texttt{Conc}(A)$,

The conditions of Strict Dominance are guaranteed, indeed:

- $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) \neq \emptyset$ (from condition 1),
- $\texttt{Supp}(B) \cong \texttt{Supp}(C)$ (from condition 2),
- $\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C)) \subset \texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(B))$ (from conditions 3, 4),
- $\texttt{CN}_{df}(\texttt{Conc}(B)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A)) = \texttt{CN}_{df}(\texttt{Conc}(C)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A)) = \emptyset$ (from condition 3).

Hence, Strict Dominance leads to $\mathtt{sim}(A,B) > \mathtt{sim}(A,C)$. □

Proof. [Proposition 8] Let $\mathtt{sim}$ be a similarity measure which satisfies Maximality, Strict Monotony, and Strict Dominance. From Theorem 8, it follows that $\mathtt{Supp}(A) \cong \mathtt{Supp}(B)$. Hence, $\mathtt{Supp}(A) \cup \mathtt{Supp}(B) \cong \mathtt{Supp}(A)$. Furthermore, by definition (36) of an argument, $\mathtt{Supp}(A)$ is consistent. So is for $\mathtt{Supp}(A) \cup \mathtt{Supp}(B)$. □

5.1.3 Proofs of section 2.3: Concise Arguments

Proof. [Proposition 9] Let $A = \langle\{\phi_1, \ldots, \phi_n\}, \phi\rangle, B = \langle\{\phi'_1, \ldots, \phi'_n\}, \phi\rangle \in \mathtt{Arg}(\mathcal{L})$ such that $B \in \mathtt{Ref}(A)$. Assume that there exist two different permutations ρ_1, ρ_2 of the set $\{1, \ldots, n\}$ such that $\forall k \in \{1, \ldots, n\}$, $\phi_k \vdash \phi'_{\rho_1(k)}$ and $\phi_k \vdash \phi'_{\rho_2(k)}$. Let $\ell \in \{1, \ldots, n\}$ be a number such that $\rho_1(\ell) \neq \rho_2(\ell)$. Then $\phi_\ell \vdash \phi'_{\rho_1(\ell)}$ and $\phi_\ell \vdash \phi'_{\rho_2(\ell)}$. Let $i \neq \ell$ be the number such that $\rho_1(i) = \rho_2(\ell)$. Then we have both $\phi_i \vdash \phi'_{\rho_1(i)}$ and $\phi_\ell \vdash \phi'_{\rho_1(i)}$. Thus, $\mathtt{Supp}(A) \setminus \{\phi_\ell, \phi_i\} \vdash \psi$ for every $\psi \in \{\phi'_{\rho_1(k)} \mid k \neq i, \ell\}$ and $\phi_\ell \vdash \phi'_{\rho_1(\ell)} \wedge \phi'_{\rho_1(i)}$. Then $\mathtt{Supp}(A) \setminus \{\phi_i\} \vdash \psi$ for every $\psi \in \mathtt{Supp}(B)$. Consequently, since $\mathtt{Supp}(B) \vdash \phi$, we have $\mathtt{Supp}(A) \setminus \{\phi_i\} \vdash \phi$ as well. This contradicts the Minimality condition from Definition 36. □

Proof. [Proposition 10] Let $A \in \mathtt{Arg}(\mathcal{L})$ be a trivial argument. Therefore $\mathtt{Supp}(A) = \emptyset$ and $\mathtt{Conc}(A) \equiv \top$. In that case $A \in \mathtt{Ref}(A)$ obviously holds (we can say that all the elements of $\mathtt{Supp}(A)$ contain only dependent variables). Suppose that there is $B \neq A$ such that $B \in \mathtt{Ref}(A)$. Then $\mathtt{Conc}(B) = \mathtt{Conc}(A)$ by Definition 42, so $\mathtt{Supp}(B) \neq \mathtt{Supp}(A)$. Thus, $|\mathtt{Supp}(B)| \neq |\mathtt{Supp}(A)|$, so there is no bijection between the elements of $\mathtt{Supp}(B)$ and $\mathtt{Supp}(A)$. Consequently, a permutation ρ from the second condition of Definition 42 doesn't exist, so $B \notin \mathtt{Ref}(A)$. □

Proof. [Proposition 11]

- Let $A = \langle\{\phi_1, \ldots, \phi_n\}, \phi\rangle \in \mathtt{Arg}(\mathcal{L})$. For every formula $\phi_i \in \mathtt{Supp}(A)$ there exists an equivalent formula $\phi'_i \equiv \phi_i$ such that it contains only dependent literals, i.e., $\mathtt{Lit}(\phi'_i) = \mathtt{DepLit}(\phi_i)$. Let us consider $B = \langle\{\phi'_1, \ldots, \phi'_n\}, \phi\rangle$. For the permutation $\rho = Id$ (i.e., $\rho(i) = i$ for all $i \in \{1, \ldots, n\}$), we have that $\forall k \in \{1, \ldots, n\}$, $\phi_k \vdash \phi'_{\rho(k)}$ and $\mathtt{Lit}(\phi'_{\rho(k)}) \subseteq \mathtt{DepLit}(\phi_k)$. Moreover, $\mathtt{Conc}(B) = \mathtt{Conc}(A)$, so by Definition 42 B is a refinement of A.

- If $\mathtt{Lit}(\phi) = \mathtt{DepLit}(\phi)$ for all $\phi \in \mathtt{Supp}(A)$, then we can prove that $A \in \mathtt{Ref}(A)$ in the same way as we proved that $B \in \mathtt{Ref}(A)$ in the proof of the first part of the Proposition. Conversely, suppose that $A \in \mathtt{Ref}(A)$. Then the unique (by Proposition 9) permutation ρ from Definition 36 must be identity, i.e., $\rho(i) = i$ for all

$i \in \{1, \ldots, n\}$. From the second condition of Definition 36 it follows that Definition $\texttt{Lit}(\phi_i) \subseteq \texttt{DepLit}(\phi_i)$ for all $i \in \{1, \ldots, n\}$.

□

Proof. [Proposition 12]

- Let $A = \langle\{\phi_1, \ldots, \phi_n\}, \phi\rangle \in \texttt{Arg}(\mathcal{L})$ and let $B = \langle\{\phi'_1, \ldots, \phi'_n\}, \phi\rangle$ be a refinement of A. Then there exists a permutation ρ of the set $\{1, \ldots, n\}$ such that $\forall k \in \{1, \ldots, n\}$, $\phi_k \vdash \phi'_{\rho(k)}$ and $\texttt{Lit}(\phi'_{\rho(k)}) \subseteq \texttt{DepLit}(\phi_k)$. If $B = \langle\{\phi''_1, \ldots, \phi''_n\}, \phi\rangle$ is a refinement of B, then there exists a permutation σ of $\{1, \ldots, n\}$ such that $\forall k \in \{1, \ldots, n\}$, $\phi'_k \vdash \phi''_{\sigma(k)}$ and $\texttt{Lit}(\phi''_{\rho(k)}) \subseteq \texttt{DepLit}(\phi'_k)$. In order to prove that C is a refinement of A, we will consider the permutation $\sigma \circ \rho$. For every $k \in \{1, \ldots, n\}$, we have both $\phi_k \vdash \phi'_{\rho(k)}$ and $\phi'_{\rho(k)} \vdash \phi''_{\sigma((\rho(k))}$, together they imply $\phi_k \vdash \phi''_{\sigma((\rho(k))}$. Moreover, from $\texttt{Lit}(\phi'_{\rho(k)}) \subseteq \texttt{DepLit}(\phi_k)$ and $\texttt{Lit}(\phi''_{\sigma((\rho(k))}) \subseteq \texttt{DepLit}(\phi'_{\rho(k)})$, using $\texttt{DepLit}(\phi'_{\rho(k)}) \subseteq \texttt{Lit}(\phi'_{\rho(k)})$ we obtain $\texttt{Lit}(\phi''_{\sigma((\rho(k))}) \subseteq \texttt{DepLit}(\phi_k)$. Therefore $C \in \texttt{Ref}(A)$.

- Let $A = \langle\{\phi_1, \ldots, \phi_n\}, \phi\rangle \in \texttt{Arg}(\mathcal{L})$ and let $A \approx B$. Then B is of the form $\langle\{\psi_1, \ldots, \psi_n\}, \phi\rangle$ such that $\phi_i \equiv \psi_i$, for each $i \in \{1, \ldots, n\}$. Let $C = \langle\{\phi'_1, \ldots, \phi'_n\}, \phi\rangle \in \texttt{Ref}(A)$. Then there exists a permutation ρ of the set $\{1, \ldots, n\}$ such that $\forall k \in \{1, \ldots, n\}$, $\phi_k \vdash \phi'_{\rho(k)}$ and $\texttt{Lit}(\phi'_{\rho(k)}) \subseteq \texttt{DepLit}(\phi_k)$. Let us consider arbitrary $i \in \{1, \ldots, n\}$. Since $\phi_i \equiv \psi_i$, we have $\phi_i \vdash \phi'_{\rho(i)}$. Moreover, $\texttt{DepLit}(\phi_i) = \texttt{DepLit}(\psi_i)$, so $\texttt{Lit}(\phi'_{\rho(i)}) \subseteq \texttt{DepLit}(\psi_i)$. By Definition 42, $C \in \texttt{Ref}(B)$. Therefore $\texttt{Ref}(A) \subseteq \texttt{Ref}(B)$. In the same way we can prove that $\texttt{Ref}(B) \subseteq \texttt{Ref}(A)$. Consequently, $\texttt{Ref}(A) = \texttt{Ref}(B)$.

□

Proof. [Proposition 13]

1. Let $B \in \texttt{CR}(A)$. Then $B \in \texttt{Ref}(B)$ by Definition 44. Moreover, B is concise, so if $C \in \texttt{Ref}(B)$, then $C \approx B$ by Definition 43.

2. Let $A = \langle\{\phi_1, \ldots, \phi_n\}, \phi\rangle \in \texttt{Arg}(\mathcal{L})$, and let $\texttt{Lit}_A = \bigcup_{i \in \{1,\ldots,n\}} \texttt{Lit}(\phi_i)$. Since $\texttt{Lit}_A$ is finite, there are finitely many valuations from $\texttt{Lit}_A$ to $\{true, false\}$, so by Completeness theorem for propositional logic, there are finitely many classes of equivalence $\equiv$ on $\texttt{Lit}_A$. By Proposition 11, $\texttt{Ref}(A) \neq \emptyset$. Let $B_1 = \langle\{\phi'_1, \ldots, \phi'_n\}, \phi\rangle \in \texttt{Ref}(A)$. If B_1 is concise, then $B_1 \in \texttt{CR}(A)$, so $\texttt{CR}(A) \neq \emptyset$. Otherwise, there exists $B_2 \in \texttt{Ref}(B_1)$ such that $B_2 \not\approx B_1$. By Proposition 12, $B_2 \in \texttt{Ref}(A)$. Therefore, if B_2 is concise, then $B_2 \in \texttt{CR}(A)$, so $\texttt{CR}(A) \neq \emptyset$. Otherwise, there exists

$B_3 \in \mathtt{Ref}(B_2)$ such that $B_3 \not\approx B_2$. This process must be finite, since for each n we have that all formulas of B_n are built form the elements of $\mathtt{Lit}_A$, and there are finitely many classes of equivalence $\equiv$ on $\mathtt{Lit}_A$. This means that there is k such that $B_k \in \mathtt{Ref}(B_{k-1})$ and for every $C \in \mathtt{Ref}(B_k)$, $C \approx B_k$. Therefore B_k is concise and $B_k \in \mathtt{Ref}(A)$ (Proposition 12), so $B_k \in \mathtt{CR}(A)$. Thus $\mathtt{CR}(A) \neq \emptyset$.

3. By Proposition 11, there exists $B \in \mathtt{CR}(A)$. If A is non-trivial, then $\mathtt{Supp}(B) \neq \emptyset$. Let $B = \langle \{\phi_1, \dots, \phi_n\}, \phi \rangle$. For every positive integer n, let us denote by $\phi_1^{(n)}$ the formula $\phi_1 \wedge \cdots \wedge \phi_1$, where the conjunction $\wedge$ is applied n times. Note that the formulas $\phi_1^{(n)}$ and ϕ_1 have the same sets of literals. Furthermore, let $B^{(n)} = \langle \{\phi_1^{(n)}, \phi_2 \dots, \phi_n\}, \phi \rangle$. Then for every n we have $B^{(n)} \approx B$. Then from conciseness of B we can conclude that $B^{(n)}$ is also concise. Moreover, from $B \in \mathtt{Ref}(A)$ we obtain $B^{(n)} \in \mathtt{Ref}(A)$, for every n. Therefore $\{B^{(n)} \mid n = 1, 2, \dots\} \subseteq \mathtt{CR}(A)$, so $\mathtt{CR}(A)$ is infinite.

4. Let $A \approx B$. From Proposition 12 we obtain $\mathtt{Ref}(A) = \mathtt{Ref}(B)$. Consequently, the set of concise arguments form $\mathtt{Ref}(A)$ coincide with the set of concise arguments form $\mathtt{Ref}(B)$, i.e., $\mathtt{CR}(A) = \mathtt{CR}(B)$.

5. Let $B \in \mathtt{Ref}(A)$. From Proposition 12 we obtain $\mathtt{Ref}(B) \subseteq \mathtt{Ref}(A)$. Consequently, each concise argument from $\mathtt{Ref}(B)$ is also a concise argument from $\mathtt{Ref}(A)$, i.e., $\mathtt{CR}(B) \subseteq \mathtt{CR}(A)$.

□

Proof. [Proposition 14] Let $B \in \mathtt{CR}(A)$. Let $\phi \in \mathtt{Supp}(B)$ and let $\psi \in \mathcal{L}$ be such that $\phi \vdash \psi$ and $\psi \not\equiv \phi$. If $C = \langle (\mathtt{Supp}(B) \setminus \{\phi\}) \cup \{\psi\}, \mathtt{Conc}(B) \rangle \in \mathtt{Arg}(\mathcal{L})$, and if $\mathtt{Lit}(\psi) \setminus \mathtt{Lit}(\phi) = \emptyset$, then $\mathtt{Lit}(\psi) \subseteq \mathtt{Lit}(\phi)$. Note that $\mathtt{Lit}(\phi) = \mathtt{DepLit}(\phi)$, since $B \in \mathtt{CR}(A)$. Then C would be a refinement of B such that $C \not\approx B$, which contradicts the assumption that B is concise.

□

5.1.4 Proofs of section 2.4.1: Syntactic Similarity Measures

Proof. [Proposition 15] Follows from the definition of the measures (table 2.1). The size of each of the compared sets is 1. □

Proof. [Proposition 16] Follows from Proposition 15. □

Proof. [Proposition 17] Let $A, B \in \mathtt{Arg}(\mathcal{L})$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, and $0 < \sigma < 1$. $s_x(\mathtt{Supp}(A), \mathtt{Supp}(B)) \in [0, 1]$ and $s_x(\mathtt{Conc}(A), \mathtt{Conc}(B)) \in [0, 1]$. Hence, $\mathtt{sim}_x^\sigma(A, B) \in [0, 1]$. □

Proof. [Theorem 9] For all the principles except Triangle Inequality and Independent Distribution, we prove the result for Extended Jaccard Measure. The same reasoning holds for the others. Let $0 < \sigma < 1$.

Maximality: Let $A \in \texttt{Arg}(\mathcal{L})$. There are two cases:

i) A is trivial, hence $\texttt{Supp}(A) = \emptyset$. By definition of Extended Jaccard Measure, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(A)) = 1$.

ii) A is non-trivial. Hence, $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(A)) = \texttt{Supp}(A)$. Thus, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(A)) = 1$. Furthermore, from Proposition 15, $s_{\texttt{j}}(\texttt{Conc}(A), \texttt{Conc}(A)) = 1$. Hence, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, A) = 1$.

Symmetry: Let $A, B \in \texttt{Arg}(\mathcal{L})$. We show that $s^{\sigma}_{\texttt{j}}(A, B) = s^{\sigma}_{\texttt{j}}(B, A)$. There are three cases:

i) A and B are both trivial. Then, $\texttt{Supp}(A) = \texttt{Supp}(B) = \emptyset$ and $\texttt{Conc}(A) \equiv \texttt{Conc}(B)$. Hence, by definition of Extended Measure, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) = 1$ and from Proposition 15, $s_{\texttt{j}}(\texttt{Conc}(A), \texttt{Conc}(B)) = 1$. Hence, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = \texttt{sim}^{\sigma}_{\texttt{j}}(B, A) = 1$.

ii) A is trivial and B is non-trivial. Then, $\texttt{Supp}(A) = \emptyset$ and $\texttt{Conc}(A) \not\equiv \texttt{Conc}(B)$. By definition, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) = s_{\texttt{j}}(\texttt{Supp}(B), \texttt{Supp}(A)) = 0$ and from Proposition 15, $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(B)\}) = 0$. So, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = \texttt{sim}^{\sigma}_{\texttt{j}}(B, A) = 0$.

iii) Both A and B are not trivial, i.e., $\texttt{Supp}(A) \neq \emptyset$ and $\texttt{Supp}(B) \neq \emptyset$. From Property 3, $|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))| = |\texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A))|$. So, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) = s_{\texttt{j}}(\texttt{Supp}(B), \texttt{Supp}(A))$. From Proposition 15, $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(B)\}) = s_{\texttt{j}}(\{\texttt{Conc}(B)\}, \{\texttt{Conc}(A)\})$. Thus, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = \texttt{sim}^{\sigma}_{\texttt{j}}(B, A)$.

Substitution: Let $A, B, C \in \texttt{Arg}(\mathcal{L})$ such that $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = 1$. From Theorem 11, it holds that $a \approx b$. Hence, $\texttt{Supp}(A) \cong \texttt{Supp}(B)$ and $\texttt{Conc}(A) \equiv \texttt{Conc}(B)$. If $\texttt{Conc}(A) \equiv \texttt{Conc}(C)$, then $\texttt{Conc}(B) \equiv \texttt{Conc}(C)$. So, $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(C)\}) = s_{\texttt{j}}(\{\texttt{Conc}(B)\}, \{\texttt{Conc}(C)\})$. It is thus sufficient to check the equality $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(C)) = s_{\texttt{j}}(\texttt{Supp}(B), \texttt{Supp}(C))$. From Property 1, $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) = \texttt{Supp}(A)$ and $\texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) = \texttt{Supp}(B)$. From Property 3, $|\texttt{Supp}(A)| = |\texttt{Supp}(B)|$. Furthermore, $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C)) = \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(C))$. Hence, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(C)) = s_{\texttt{j}}(\texttt{Supp}(B), \texttt{Supp}(C))$. Consequently, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, C) = \texttt{sim}^{\sigma}_{\texttt{j}}(B, C)$.

Syntax Independence: The similarity measure is based on two functions, Co (common logical formulas) and $|\cdot|$ (cardinality of a set of formula), which are not looking for some specific name of information. In other terms, these functions will return the same result applying any renaming function.

Minimality: Let $A, B \in \texttt{Arg}(\mathcal{L})$ such that:

1. A and B are not equivalent,
2. $\bigcup_{\phi_i \in \texttt{Supp}(A)} \texttt{Var}(\phi_i) \cap \bigcup_{\phi_j \in \texttt{Supp}(B)} \texttt{Var}(\phi_j) = \emptyset$ and
3. $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(B)) = \emptyset$.

From Property 2, the condition 1 implies that both argument cannot be trivial, then $|\texttt{Supp}(A)| > 0$ or $|\texttt{Supp}(B)| > 0$. The condition 2 implies that $\nexists \phi \in \texttt{Supp}(A)$ such that $\exists \psi \in \texttt{Supp}(B)$ such that $\phi \equiv \psi$, i.e. $|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))| = 0$. Therefore, combining the condition 1 and 2 we obtain that $s_j(\texttt{Supp}(A), \texttt{Supp}(B)) = 0$.
Then condition 1 implies also $\texttt{Conc}(A) \not\equiv \top$ or $\texttt{Conc}(B) \not\equiv \top$. The condition 3 implies that if $\texttt{Conc}(A)$ and $\texttt{Conc}(B)$ are not both equal to $\top$, then $\texttt{Conc}(A) \not\equiv \texttt{Conc}(B)$. Therefore, combining the condition 1 and 3 we obtain that $|\texttt{Co}(\texttt{Conc}(A), \texttt{Conc}(B))| = 0$, i.e. $s_j(\texttt{Conc}(A), \texttt{Conc}(B)) = 0$.
That's why for any $\sigma \in [0, 1]$, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = 0$.

Non-Zero: Let $A, B \in \texttt{Arg}(\mathcal{L})$, such that $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B)) \neq \emptyset$.
Then, from this condition we know that $\frac{|\texttt{Co}(\texttt{Supp}(A),\texttt{Supp}(B))|}{|\texttt{Supp}(A)|+|\texttt{Supp}(B)|-|\texttt{Co}(\texttt{Supp}(A),\texttt{Supp}(B))|} > 0$, i.e. $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) > 0$. From Definition 45, an Extended Similarity Measure cannot ignore the support ($\sigma > 0$), therefore $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) > 0$.

Monotony - Strict Monotony: Let $0 < \sigma < 1$ and $A, B, C \in \texttt{Arg}(\mathcal{L})$ be such that:

1. $\texttt{Conc}(A) \equiv \texttt{Conc}(B)$ or $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(C)) = \emptyset$,
2. $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C)) \subseteq \texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))$,
3. $\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) = \texttt{Co}(\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)), \texttt{Supp}(C) \setminus \texttt{Co}(\texttt{Supp}(C), \texttt{Supp}(A)))$

There are two cases:

- C is trivial, i.e., $\texttt{Supp}(C) = \emptyset$. Condition 3) implies $\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) = \emptyset$, hence $\texttt{Supp}(B) = \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A))$ and $\texttt{Supp}(A) \cong \texttt{Supp}(B)$. Consequently, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) = 1$. Since $\sigma > 0$, then $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) > 0$.
  - Assume that A is trivial. Since $\texttt{Supp}(B) \cong \texttt{Supp}(A)$, then B is also trivial. From Theorem 11, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = \texttt{sim}^{\sigma}_{\texttt{j}}(A, C) = 1$.

– Assume that A is not trivial. Then, $\texttt{Conc}(A) \not\equiv \texttt{Conc}(C)$ and $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(C)\}) = 0$. Furthermore, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(C)) = 0$. Hence, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, C) = 0$. So, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) > \texttt{sim}^{\sigma}_{\texttt{j}}(A, C)$.

- C is not trivial, i.e., $\texttt{Supp}(C) \neq \emptyset$ and $\texttt{Conc}(C) \not\equiv \top$.

  – Assume that A is trivial. So, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(C)) = 0$ and $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(C)\}) = 0$ leading to $\texttt{sim}^{\sigma}_{\texttt{j}}(A, C) = 0$. If B is trivial, then $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = 1$ (from Theorem 11). If B is not trivial, then $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) = 0$ and $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(B)\}) = 0$ leading to $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = 0$.

  – Assume that A is not trivial. Assume that B is trivial, then $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) = 0$ and $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(B)\}) = 0$ leading to $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = 0$. Condition 2) implies that $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C) = \emptyset$. So, $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(C)) = 0$. Note that $\texttt{Conc}(A) \not\equiv \texttt{Conc}(B)$ (since A is not trivial), then $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(C)) = \emptyset$. Thus, $\texttt{Conc}(A) \not\equiv \texttt{Conc}(C)$ and so $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(C)\}) = 0$ leading to $\texttt{sim}^{\sigma}_{\texttt{j}}(A, C) = 0$. Thus, $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) = \texttt{sim}^{\sigma}_{\texttt{j}}(A, C)$.

    Assume now that B is not trivial (i.e., the 3 arguments are not trivial). From condition 2) it holds that $|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C))| \leq |\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))|$ and from condition 3) $\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)) = \texttt{Co}(\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)), \texttt{Supp}(C) \setminus \texttt{Co}(\texttt{Supp}(C), \texttt{Supp}(A)))$.
    Since $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) =$

$$\frac{|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))|}{|\texttt{Supp}(A)| + |\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A)))|},$$

    we get $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) \geq s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(C))$. Since $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(B)\}) = 1$ or $s_{\texttt{j}}(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(C)\}) = 0$, then $s^{\sigma}_{\texttt{j}}(A, B) \geq s^{\sigma}_{\texttt{j}}(A, C)$.

If the condition 2 is strict then $|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))| > |\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C))|$ and thus $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) > \texttt{sim}^{\sigma}_{\texttt{j}}(A, C)$.
If $\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C)) \neq \emptyset$ and $|\texttt{Supp}(C) \setminus \texttt{Co}(\texttt{Supp}(C), \texttt{Supp}(A))| > |\texttt{Supp}(B) \setminus \texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(A))|$ then $s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(B)) > s_{\texttt{j}}(\texttt{Supp}(A), \texttt{Supp}(C))$ therefore $\texttt{sim}^{\sigma}_{\texttt{j}}(A, B) > \texttt{sim}^{\sigma}_{\texttt{j}}(A, C)$.

Dominance: Let $A, B, C \in \texttt{Arg}(\mathcal{L})$ such that:

1. $\texttt{Supp}(B) \cong \texttt{Supp}(C)$,

2. $\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C)) \subseteq \texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(B))$,

3. $\mathtt{CN}_{df}(\mathtt{Conc}(B)) \setminus \mathtt{CN}_{df}(\mathtt{Conc}(A)) \subseteq \mathtt{CN}_{df}(\mathtt{Conc}(C)) \setminus \mathtt{CN}_{df}(\mathtt{Conc}(A))$.

Condition 1 implies that $\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = \mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(C))$. From Property 4, $|\mathtt{Supp}(B)| = |\mathtt{Supp}(C)|$. Hence, $s_{\mathtt{j}}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = s_{\mathtt{j}}(\mathtt{Supp}(A), \mathtt{Supp}(C))$. Assume now that $\mathtt{Conc}(A) \equiv \mathtt{Conc}(B)$. Thus, $s_{\mathtt{j}}(\mathtt{Conc}(A), \mathtt{Conc}(B)) = 1$. Now we have two cases:

$\mathtt{Conc}(A) \equiv \mathtt{Conc}(C)$: in this case $s_{\mathtt{j}}(\mathtt{Conc}(A), \mathtt{Conc}(C)) = 1$ and therefore $\mathtt{sim}(A, B) = \mathtt{sim}(A, C)$.

$\mathtt{Conc}(A) \not\equiv \mathtt{Conc}(C)$: in this case from Proposition 15, $s_{\mathtt{j}}(\mathtt{Conc}(A), \mathtt{Conc}(C)) = 0$ and therefore $\mathtt{sim}(A, B) > \mathtt{sim}(A, C)$.

Assume now that $\mathtt{Conc}(A) \not\equiv \mathtt{Conc}(B)$ then from condition 2 we know that $\mathtt{Conc}(A) \not\equiv \mathtt{Conc}(C)$ as well. Thus from Proposition 15, $s_{\mathtt{j}}(\mathtt{Conc}(A), \mathtt{Conc}(B)) = s_{\mathtt{j}}(\mathtt{Conc}(A), \mathtt{Conc}(C)) = 0$ and therefore $\mathtt{sim}(A, B) = \mathtt{sim}(A, C)$.

Example 11 shows that $\mathtt{sim}_{\mathtt{j}}^{\sigma}$ violate Strict Dominance:
For any $0 < \sigma < 1$, $\mathtt{sim}_{\mathtt{j}}^{\sigma}(A, B) = \mathtt{sim}_{\mathtt{j}}^{\sigma}(A, C)$ while Strict Dominance ensure that $\mathtt{sim}_{\mathtt{j}}^{\sigma}(A, B) > \mathtt{sim}_{\mathtt{j}}^{\sigma}(A, C)$.

Triangle Inequality:
Start by the measures (Jaccard and Sokal and Sneath 2) satisfying the Triangle Inequality. In the paper Bren and Batagelj [2006], the proof was done considering these measures as a dissimilarity d. The Triangle Inequality for a dissimilarity is defined as $d(A, C) \leq d(A, B) + d(B, C)$. Defining similarity $\mathtt{sim}$ as the dual of the dissimilarity: $\mathtt{sim} = 1 - d$, we obtain the Triangle Inequality for similarity as the form: $1 + \mathtt{sim}(A, C) \geq \mathtt{sim}(A, B) + \mathtt{sim}(B, C)$.
Because $\mathtt{sim} = 1 - d \iff \mathtt{sim} - 1 = -d \iff 1 - \mathtt{sim} = d$ then $d(A, C) \leq d(A, B) + d(B, C) \iff 1 - \mathtt{sim}(A, C) \leq 1 - \mathtt{sim}(A, B) + 1 - \mathtt{sim}(B, C) \iff -\mathtt{sim}(A, C) \leq 1 - \mathtt{sim}(A, B) - \mathtt{sim}(B, C) \iff \mathtt{sim}(A, B) + \mathtt{sim}(B, C) \leq \mathtt{sim}(A, C) + 1$.
However the proofs are done for $1 + \mathtt{sim}(A, C) \geq \mathtt{sim}(A, B) + \mathtt{sim}(B, C)$ and the Extended Measures are defined as $\mathtt{sim}_{x}^{\alpha}(A, C) = \alpha \cdot s_x(\mathtt{Supp}(A), \mathtt{Supp}(C)) + (1 - \alpha) \cdot s_x(\mathtt{Conc}(A), \mathtt{Conc}(C))$.
From Bren and Batagelj [2006], for $x \in \{j, ss2\}$, we know that:

$$1 + s_x(\mathtt{Supp}(A), \mathtt{Supp}(C)) \geq s_x(\mathtt{Supp}(A), \mathtt{Supp}(B)) + s_x(\mathtt{Supp}(B), \mathtt{Supp}(C))$$

$$\iff \alpha + \alpha \cdot s_x(\mathtt{Supp}(A), \mathtt{Supp}(C)) \geq \alpha \cdot (s_x(\mathtt{Supp}(A), \mathtt{Supp}(B)) + s_x(\mathtt{Supp}(B), \mathtt{Supp}(C)))$$

and

$$1 + s_x(\texttt{Conc}(A), \texttt{Conc}(C)) \geq s_x(\texttt{Conc}(A), \texttt{Conc}(B)) + s_x(\texttt{Conc}(B), \texttt{Conc}(C))$$

$$\iff (1-\alpha) + (1-\alpha) \cdot s_x(\texttt{Conc}(A), \texttt{Conc}(C)) \geq (1-\alpha) \cdot (s_x(\texttt{Conc}(A), \texttt{Conc}(B)) +$$

$$s_x(\texttt{Conc}(B), \texttt{Conc}(C)))$$

therefore

$$1 + \alpha \cdot s_x(\texttt{Supp}(A), \texttt{Supp}(C)) + (1-\alpha) \cdot s_x(\texttt{Conc}(A), \texttt{Conc}(C)) \geq$$

$$\alpha \cdot (s_x(\texttt{Supp}(A), \texttt{Supp}(B)) + s_x(\texttt{Supp}(B), \texttt{Supp}(C))) + (1-\alpha) \cdot (s_x(\texttt{Conc}(A), \texttt{Conc}(B)) +$$

$$s_x(\texttt{Conc}(B), \texttt{Conc}(C))).$$

About those that don't respect the Triangle Inequality (Dice, Sorensen, Symmetric Anderberg, Ochiai, Kulczynski 2), in Bren and Batagelj [2006], the proof was done for the measures: Dice, Ochiai and Kulczynski 2. For Sorensen and Symmetric Anderberg we give a counter example.

Let $A, B, C \in \texttt{Arg}(\mathcal{L})$ such that:
$A = \langle \{u, v, x, y, (u \wedge v \wedge x \wedge y) \rightarrow t\}, t \rangle$,
$B = \langle \{w, x, y, z, (w \wedge x \wedge y \wedge z) \rightarrow t\}, t \rangle$,
$C = \langle \{w, y, z, (w \wedge y \wedge z) \rightarrow t\}, t \rangle$.
Then $|\texttt{Supp}(A)| = |\texttt{Supp}(B)| = 5$, $|\texttt{Supp}(C)| = 4$ and $|\texttt{Conc}(A)| = |\texttt{Conc}(B)| = |\texttt{Conc}(C)| = 1$, $|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(B))| = 2$, $|\texttt{Co}(\texttt{Supp}(A), \texttt{Supp}(C))| = 1$, $|\texttt{Co}(\texttt{Supp}(B), \texttt{Supp}(C))| = 3$ and $|\texttt{Co}(\texttt{Conc}(A), \texttt{Conc}(B))| = |\texttt{Co}(\texttt{Conc}(A), \texttt{Conc}(C))| = |\texttt{Co}(\texttt{Conc}(B), \texttt{Conc}(C))| = 1$.

For any $0 < \sigma < 1$:

$1 + \texttt{sim}_s^\sigma(A, C) = 1 + \frac{4}{11} \cdot \sigma + 1 \cdot (1-\sigma) = 2 - \frac{7}{11}\sigma$, $\texttt{sim}_s^\sigma(A, B) + \texttt{sim}_d^\sigma(B, C) = \frac{8}{14} \cdot \sigma + 1 \cdot (1-\sigma) + \frac{12}{15} \cdot \sigma + 1 \cdot (1-\sigma) = 2 - \frac{22}{35}\sigma$ then $2 - \frac{7}{11}\sigma < 2 - \frac{22}{35}\sigma$ because $\frac{7}{11} \approx 0.6364 > 0.6286 \approx \frac{22}{35}$. Therefore, $1 + \texttt{sim}_s^\sigma(A, C) < \texttt{sim}_s^\sigma(A, B) + \texttt{sim}_s^\sigma(B, C)$.

$1 + \texttt{sim}_a^\sigma(A, C) = 1 + \frac{8}{15} \cdot \sigma + 1 \cdot (1-\sigma) = 2 - \frac{7}{15} \cdot \sigma$, $\texttt{sim}_a^\sigma(A, B) + \texttt{sim}_a^\sigma(B, C) = \frac{16}{22} \cdot \sigma + 1 \cdot (1-\sigma) + \frac{24}{27} \cdot \sigma + 1 \cdot (1-\sigma) = 2 - \frac{38}{99} \cdot \sigma$ then $2 - \frac{7}{15} \cdot \sigma < 2 - \frac{38}{99} \cdot \sigma$ because $\frac{7}{15} \approx 0.4667 > 0.3838 \approx \frac{38}{99}$. Therefore, $1 + \texttt{sim}_a^\sigma(A, C) < \texttt{sim}_a^\sigma(A, B) + \texttt{sim}_a^\sigma(B, C)$.

Independent Distribution: Let $A, B, A', B' \in \mathtt{Arg}(\mathcal{L})$ such that:

1. $\mathtt{Var}(\mathtt{Conc}(A)) \cap \mathtt{Var}(\mathtt{Conc}(B)) = \mathtt{Var}(\mathtt{Conc}(A')) \cap \mathtt{Var}(\mathtt{Conc}(B')) = \emptyset$,
2. $\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B)) \cong \mathtt{Co}(\mathtt{Supp}(A'), \mathtt{Supp}(B'))$,
3. $\mathtt{Supp}(A) \cup \mathtt{Supp}(B) \cong \mathtt{Supp}(A') \cup \mathtt{Supp}(B')$.

The Extended Jaccard, Dice, Sorensen, Symmetric Anderberg and Sokal and Sneath 2 have the same form:

$$\frac{\alpha|\mathtt{Co}(\Phi,\Psi)|}{\beta(|\Phi|+|\Psi|)+\gamma|\mathtt{Co}(\Phi,\Psi)|}$$

such that Extended Jaccard: ($\alpha = 1, \beta = 1, \gamma = -1$), Dice: ($\alpha = 2, \beta = 1, \gamma = 0$), Sorensen: ($\alpha = 4, \beta = 1, \gamma = 2$), Symmetric Anderberg: ($\alpha = 8, \beta = 1, \gamma = 6$) and Sokal and Sneath 2: ($\alpha = 1, \beta = 2, \gamma = -3$).
To start, for all the syntactic similarity measures, the first condition ensure that the similarity on the conclusion is 0.
The second condition ensure that the numerator between $\mathtt{sim}(A, B)$ and $\mathtt{sim}(A', B')$ is equal. And the third condition say that $\mathtt{Supp}(A) \cup \mathtt{Supp}(B) \cong \mathtt{Supp}(A') \cup \mathtt{Supp}(B')$, i.e. $|\mathtt{Supp}(A)|+|\mathtt{Supp}(B)|-|\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))| = |\mathtt{Supp}(A')|+|\mathtt{Supp}(B')|-|\mathtt{Co}(\mathtt{Supp}(A'), \mathtt{Supp}(B'))|$. Given that $|\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))| = |\mathtt{Co}(\mathtt{Supp}(A'), \mathtt{Supp}(B'))|$ from the condition 2 then the denominator is also equal.

For the counter example of Ochiai and Kulczynski 2, let us take the example 16, with any $0 < \sigma < 1$:
$\mathtt{sim}^{\sigma}_{o}(A, B) = \frac{1}{\sqrt{2}\sqrt{2}} \cdot \sigma + 0 \cdot (1-\sigma) = \frac{1}{2} \cdot \sigma.$
$\mathtt{sim}^{\sigma}_{o}(A', B') = \frac{1}{\sqrt{3}\sqrt{1}} \cdot \sigma + 0 \cdot (1-\sigma) = \frac{1}{\sqrt{3}} \cdot \sigma.$
$\mathtt{sim}^{\sigma}_{ku}(A, B) = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{2}\right) \cdot \sigma + 0 \cdot (1-\sigma) = \frac{1}{2} \cdot \sigma.$
$\mathtt{sim}^{\sigma}_{ku}(A', B') = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{1}\right) \cdot \sigma + 0 \cdot (1-\sigma) = \frac{2}{3} \cdot \sigma.$ □

Proof. [Theorem 10] The theorem talk about similarity measure on arguments which use the same similarity on set of formulas on the support and the conclusion. Then showing the ranking on the similarity measure on set of formulas will show also the ranking between the similarity measure on arguments. The following ranking: $\mathtt{sim}_{\mathtt{ss}} \leq \mathtt{sim}_{\mathtt{j}} \leq \mathtt{sim}_{\mathtt{d}} \leq \mathtt{sim}_{\mathtt{s}} \leq \mathtt{sim}_{\mathtt{a}}$ was proved in Bouchon-Meunier *et al.* [2009].
To prove the following ranking: $\mathtt{sim}_{\mathtt{ss}} \leq \mathtt{sim}_{\mathtt{j}} \leq \mathtt{sim}_{\mathtt{d}} \leq \mathtt{sim}_{\mathtt{o}} \leq \mathtt{sim}_{\mathtt{ku}}$, we have only to show the part between the three last similarity (d,o,ku).
For the proofs we will use the equivalent form of the measures (defined in table 1.4) using the notation with a, b, c such that $a \geq 0$, $b \geq 0$ and $c \geq 0$.
$\mathtt{sim}_{\mathtt{d}} \leq \mathtt{sim}_{\mathtt{o}}$:

$$
\begin{aligned}
&\frac{2a}{2a+b+c} \leq \frac{a}{\sqrt{a+b}\sqrt{a+c}} \\
\Leftrightarrow\ &\frac{2a}{2a+b+c} \leq \frac{2a}{2(\sqrt{a+b}\sqrt{a+c})} \\
\Leftrightarrow\ &2a+b+c \geq 2(\sqrt{a+b}\sqrt{a+c}) \\
\Leftrightarrow\ &(2a+b+c)^2 \geq (2(\sqrt{a+b}\sqrt{a+c}))^2 \\
\Leftrightarrow\ &4a^2+4ab+4ac+b^2+2bc+c^2 \geq 4a^2+4ab+4ac+4bc \\
\Leftrightarrow\ &b^2+c^2 \geq 2bc \\
\Leftrightarrow\ &b^2+c^2-2bc \geq 0 \\
\Leftrightarrow\ &(b-c)^2 \geq 0
\end{aligned}
$$

$\texttt{sim}_{\circ} \leq \texttt{sim}_{\texttt{ku}}$:

$$
\begin{aligned}
&\frac{a}{\sqrt{a+b}\sqrt{a+c}} \leq \frac{1}{2}\left(\frac{a}{a+b}+\frac{a}{a+c}\right) \\
\Leftrightarrow\ &\frac{a}{\sqrt{a+b}\sqrt{a+c}} \leq \frac{a(a+c)+a(a+b)}{2(a+b)(a+c)} \\
\Leftrightarrow\ &\frac{a}{\sqrt{a+b}\sqrt{a+c}} \leq \frac{a+\frac{1}{2}b+\frac{1}{2}c}{a+b+c+\frac{bc}{a}} \\
\Leftrightarrow\ &\frac{a(a+b+c+\frac{bc}{a})}{(\sqrt{a+b}\sqrt{a+c})(a+b+c+\frac{bc}{a})} \leq \frac{(a+\frac{1}{2}b+\frac{1}{2}c)(\sqrt{a+b}\sqrt{a+c})}{(a+b+c+\frac{bc}{a})(\sqrt{a+b}\sqrt{a+c})} \\
\Leftrightarrow\ &[a(a+b+c+\frac{bc}{a})]^2 \leq [(a+\frac{1}{2}b+\frac{1}{2}c)(\sqrt{a+b}\sqrt{a+c})]^2 \\
\Leftrightarrow\ &a^4+2a^3b+2a^3c+a^2b^2+4a^2bc+a^2c^2+2ab^2c+2abc^2+b^2c^2 \leq \\
&a^4+2a^3b+2a^3c+\frac{5a^2b^2}{4}+\frac{7a^2bc}{2}+\frac{5a^2c^2}{4}+\frac{ab^3}{4}+\frac{7ab^2c}{4}+\frac{7abc^2}{4}+ \\
&\frac{ac^3}{4}+\frac{b^3c}{4}+\frac{b^2c^2}{2}+\frac{bc^3}{4} \\
\Leftrightarrow\ &a^2\left(\frac{bc}{2}\right)+a\left(\frac{b^2c+bc^2}{4}\right)+\frac{b^2c^2}{2} \leq a^2\left(\frac{b^2+c^2}{4}\right)+a\left(\frac{b^3+c^3}{4}\right)+\frac{b^3c+bc^3}{4}
\end{aligned}
$$

We will investigate now the 3 cases:

1.

$$\begin{aligned}
& a^2\left(\frac{bc}{2}\right) \leq a^2\left(\frac{b^2+c^2}{4}\right) \\
\Leftrightarrow\ & \frac{bc}{2} \leq \frac{b^2+c^2}{4} \\
\Leftrightarrow\ & 0 \leq b^2+c^2-2bc \\
\Leftrightarrow\ & 0 \leq (b-c)^2
\end{aligned}$$

2.

$$\begin{aligned}
& a\left(\frac{b^2c+bc^2}{4}\right) \leq a\left(\frac{b^3+c^3}{4}\right) \\
\Leftrightarrow\ & b^2c+bc^2 \leq b^3+c^3 \\
\Leftrightarrow\ & 0 \leq b^3-b^2c+c^3-bc^2 \\
\Leftrightarrow\ & 0 \leq b^2(b-c)+c^2(c-b)
\end{aligned}$$

Let's study the three different cases:

- if $b=c$ then $0=0$
- if $b<c$ then $b-c<c-b$, $b^2<c^2$, $0<c-b$ therefore $0<b^2(b-c)+c^2(c-b)$
- if $c<b$ then with the same reasoning than in the second case: $0<b^2(b-c)+c^2(c-b)$

3.

$$\begin{aligned}
& \frac{b^2c^2}{2} \leq \frac{b^3c+bc^3}{4} \\
\Leftrightarrow\ & 2b^2c^2 \leq b^3c+bc^3 \\
\Leftrightarrow\ & 0 \leq b^3c+bc^3-2b^2c^2 \\
\Leftrightarrow\ & 0 \leq bc(b-c)^2
\end{aligned}$$

Moreover, we show that $\texttt{sim}_\text{s}$, $\texttt{sim}_\text{a}$ and $\texttt{sim}_\text{o}$, $\texttt{sim}_\text{ku}$ can obtain contradictory results. Given that $\texttt{sim}_\text{s} \leq \texttt{sim}_\text{a}$ and $\texttt{sim}_\text{o} \leq \texttt{sim}_\text{ku}$ we will only show that there exists sets such that $\texttt{sim}_\text{ku} < \texttt{sim}_\text{s}$ and $\texttt{sim}_\text{o} > \texttt{sim}_\text{a}$.
Let's $a=1,\ b=1,\ c=2$ then $\texttt{sim}_\text{ku} = \frac{5}{12} = 0.417$ and $\texttt{sim}_\text{s} = \frac{4}{7} = 0.571$ then $\texttt{sim}_\text{ku} < \texttt{sim}_\text{s}$.

Let's $a = 1,\ b = 1,\ c = 200$ then $\mathtt{sim_o} = \frac{1}{\sqrt{402}} = 0.05$ and $\mathtt{sim_a} = \frac{8}{209} = 0.038$ then $\mathtt{sim_o} > \mathtt{sim_a}$.

□

Proof. [Theorem 11] We show the result for Extended Jaccard-based Measures. The same reasoning holds for the other measures. Let $A, B \in \mathtt{Arg}(\mathcal{L})$ and $\sigma \in]0, 1[$. Assume that $A \approx B$, then

i) $\mathtt{Supp}(A) \cong \mathtt{Supp}(B)$ and

ii) $\mathtt{Conc}(A) \equiv \mathtt{Conc}(B)$.

From i) and Property 1, $\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = \mathtt{Supp}(A)$. From Property 4, $|\mathtt{Supp}(A)| = |\mathtt{Supp}(B)|$. Thus, $s_{\mathtt{j}}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = 1$.
From ii) and Proposition 15, $s_{\mathtt{j}}(\mathtt{Conc}(A), \mathtt{Conc}(B)) = 1$. So, $\mathtt{sim}_{\mathtt{j}}^{\sigma}(A, B) = 1$.

Assume that $\mathtt{sim}_{\mathtt{j}}^{\sigma}(A, B) = 1$.
Since $\sigma \in]0, 1[$, then $s_{\mathtt{j}}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = 1$ and $s_{\mathtt{j}}(\mathtt{Conc}(A), \mathtt{Conc}(B)) = 1$. From Proposition 15, it holds that $\mathtt{Conc}(A) \equiv \mathtt{Conc}(B)$. Recall that $s_{\mathtt{j}}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = \frac{|\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))|}{|\mathtt{Supp}(A)| + |\mathtt{Supp}(B)| - |\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))|} = 1$.
Furthermore, $|\mathtt{Supp}(A)| + |\mathtt{Supp}(B)| - |\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))| = |\mathtt{Supp}(A) \setminus \mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))| + |\mathtt{Supp}(B) \setminus \mathtt{Co}(\mathtt{Supp}(B), \mathtt{Supp}(A))| + |\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))|$. Thus, $|\mathtt{Supp}(A) \setminus \mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))| + |\mathtt{Supp}(B) \setminus \mathtt{Co}(\mathtt{Supp}(B), \mathtt{Supp}(A))| + |\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))| = |\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))|$, i.e. $|\mathtt{Supp}(A) \setminus \mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))| + |\mathtt{Supp}(B) \setminus \mathtt{Co}(\mathtt{Supp}(B), \mathtt{Supp}(A))| = 0$. So, $|\mathtt{Supp}(A) \setminus \mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))| = 0$ and $|\mathtt{Supp}(B) \setminus \mathtt{Co}(\mathtt{Supp}(B), \mathtt{Supp}(A))| = 0$. Then $\mathtt{Supp}(A) = \mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B))$ and $\mathtt{Supp}(B) = \mathtt{Co}(\mathtt{Supp}(B), \mathtt{Supp}(A))$. Thus, $\mathtt{Supp}(A) \cong \mathtt{Supp}(B)$, and so $A \approx B$. □

Proof. [Theorem 12] Let $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, $\sigma \in (0, 1)$, and $A, B \in \mathtt{Arg}(\mathcal{L})$. Assume that $\mathtt{sim}_x^{\sigma}(A, B) = 0$. Since $\sigma > 0$, then

i) $s_x(\mathtt{Supp}(A), \mathtt{Supp}(B)) = 0$ and

ii) $s_x(\{\mathtt{Conc}(A)\}, \{\mathtt{Conc}(B)\}) = 0$.

From Proposition 15, $\mathtt{Conc}(A) \not\equiv \mathtt{Conc}(B)$. This means also that either A or B is not trivial. There are thus two cases regarding ii):
Case 1. $\mathtt{Supp}(A) = \emptyset$ or $\mathtt{Supp}(B) = \emptyset$. Hence, $\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = \emptyset$.
Case 2. $\mathtt{Supp}(A) \neq \emptyset$ and $\mathtt{Supp}(B) \neq \emptyset$. Thus, $\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = \emptyset$.

Assume now that i) $\mathtt{Co}(\mathtt{Supp}(A), \mathtt{Supp}(B)) = \emptyset$ and ii) $\mathtt{Conc}(A) \not\equiv \mathtt{Conc}(B)$. From Proposition 15, $s_x(\{\mathtt{Conc}(A)\}, \{\mathtt{Conc}(B)\}) = 0$. Regarding i) there are two cases:

Case 1. $\texttt{Supp}(A) \neq \emptyset$ and $\texttt{Supp}(B) \neq \emptyset$; and
Case 2. $\texttt{Supp}(A) = \emptyset$ or $\texttt{Supp}(B) \neq \emptyset$ (But not both since $\texttt{Conc}(A) \not\equiv \texttt{Conc}(B)$).
In both cases, $s_x(\texttt{Supp}(A), \texttt{Supp}(B)) = 0$. Hence, $\texttt{sim}_x^\sigma(A, B) = 0$. □

5.1.5 Proofs of section 2.4.2: Mixed Syntactic and Semantic Similarity Measure

Theorem 13. According to the Theorem 9, for any $0 < \sigma < 1$, $\texttt{sim}_j^\sigma$, violates Strict Dominance and satisfies all the remaining principles. The same reasoning as in the proof of Theorem 9 holds with $\texttt{sim}_{cnj}^\sigma$.

Except for (Strict) Dominance, only Minimality, (Strict) Monotony and Independent Distribution have a condition on the conclusions. In fact, there are only two different constraints and they are treated the same way by $s_\texttt{j}$ and $s_\texttt{cnj}$. Let $A, B \in \texttt{Arg}(\mathcal{L})$:

- if $\texttt{Conc}(A) \equiv \texttt{Conc}(B)$ then $s_\texttt{j}(A, B) = s_\texttt{cnj}(A, B) = 1$.
- if $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(C)) = \emptyset$ then $s_\texttt{j}(A, B) = s_\texttt{cnj}(A, B) = 0$.

Let us see the satisfaction of the remaining principle, (Strict) Dominance.

[(Strict) Dominance]: Let $A, B, C \in \texttt{Arg}(\mathcal{L})$, such that

1. $\texttt{Supp}(B) \cong \texttt{Supp}(C)$,
2. $\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C)) \subseteq \texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(B))$,
3. $\texttt{CN}_{df}(\texttt{Conc}(B)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A)) \subseteq \texttt{CN}_{df}(\texttt{Conc}(C)) \setminus \texttt{CN}_{df}(\texttt{Conc}(A))$,

And either

- the inclusion in condition 2 is strict or,
- $\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C)) \neq \emptyset$ and condition 3 is strict.

Let us recall the definition 47:

$$s_\texttt{cnj}(\phi, \psi) = \frac{|\texttt{CN}_{df}(\phi) \cap \texttt{CN}_{df}(\psi)|}{|\texttt{CN}_{df}(\phi) \cup \texttt{CN}_{df}(\psi)|}$$

Assume now the first case where the inclusion in condition 2 is strict.
From condition 2, $|\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C))| < |\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(B))|$. From condition 2 and 3, we know that $|\texttt{CN}_{df}(\texttt{Conc}(B))| \leq |\texttt{CN}_{df}(\texttt{Conc}(C))|$. Given that $|\texttt{CN}_{df}(\texttt{Conc}(A)) \cup \texttt{CN}_{df}(\texttt{Conc}(C))| = |\texttt{CN}_{df}(\texttt{Conc}(A))| + |\texttt{CN}_{df}(\texttt{Conc}(C))| - |\texttt{CN}_{df}(\texttt{Conc}(A)) \cap \texttt{CN}_{df}(\texttt{Conc}(C))|$ and $|\texttt{CN}_{df}(\texttt{Conc}(A)) \cup \texttt{CN}_{df}(\texttt{Conc}(B))| = |\texttt{CN}_{df}(\texttt{Conc}(A))| + |\texttt{CN}_{df}(\texttt{Conc}$

$(B))| - |\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cap \mathtt{CN}_{df}(\mathtt{Conc}(B))|$ therefore $|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cup \mathtt{CN}_{df}(\mathtt{Conc}(C))| \geq |\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cup \mathtt{CN}_{df}(\mathtt{Conc}(B))|$. In this first case we have:

$$\frac{|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cap \mathtt{CN}_{df}(\mathtt{Conc}(C))|}{|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cup \mathtt{CN}_{df}(\mathtt{Conc}(C))|} < \frac{|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cap \mathtt{CN}_{df}(\mathtt{Conc}(B))|}{|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cup \mathtt{CN}_{df}(\mathtt{Conc}(B))|}.$$

Assume now the second case where $\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cap \mathtt{CN}_{df}(\mathtt{Conc}(C)) \neq \emptyset$ and condition 3 is strict.
Here $|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cap \mathtt{CN}_{df}(\mathtt{Conc}(C))| \leq |\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cap \mathtt{CN}_{df}(\mathtt{Conc}(B))|$. Using the same reasoning as before but with the new condition we obtain this time: $|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cup \mathtt{CN}_{df}(\mathtt{Conc}(C))| > |\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cup \mathtt{CN}_{df}(\mathtt{Conc}(B))|$. Therefore

$$\frac{|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cap \mathtt{CN}_{df}(\mathtt{Conc}(C))|}{|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cup \mathtt{CN}_{df}(\mathtt{Conc}(C))|} < \frac{|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cap \mathtt{CN}_{df}(\mathtt{Conc}(B))|}{|\mathtt{CN}_{df}(\mathtt{Conc}(A)) \cup \mathtt{CN}_{df}(\mathtt{Conc}(B))|}.$$

□

5.1.6 Proofs of section 2.4.3: Similarity Measures for Non-Concise Arguments

Proof. [Proposition 18] Let $A = \langle\{\phi_1, \dots, \phi_n\}, \phi\rangle \in \mathtt{Arg}(\mathcal{L})$, and let $\mathtt{Lit}_A = \bigcup_{i\in\{1,\dots,n\}} \mathtt{Lit}(\phi_i)$. Similarly as in the proof of Proposition 13(2), we obtain that there are finitely many classes of equivalence $\equiv$ on $\mathtt{Lit}_A$. Let us denote by $\mathcal{F}(\mathtt{Lit}_A)$ the set of all formulas from $\mathcal{F}$ whose set of literals is a subset of $\mathtt{Lit}_A$. Since $\mathcal{F}$ contains only one formula per equivalence class, we obtain that the set $\mathcal{F}(\mathtt{Lit}_A)$ is finite. Since for each $B = \langle\{\phi'_1, \dots, \phi'_n\}, \phi'\rangle \in \overline{\mathtt{CR}}(A)$ we have:

- $\{\phi'_1, \dots, \phi'_n\} \subseteq \mathcal{F}(\mathtt{Lit}_A)$, and
- $\phi' = \phi$,

the set $\overline{\mathtt{CR}}(A)$ is finite.

□

Proof. [Proposition 19] Obviously $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{x}) \geq 0$. On the other hand, for any pair of arguments $C, D \in \mathtt{Arg}(\mathcal{L})$, any $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ and $0 < \sigma < 1$, we have $\mathtt{sim}^{\sigma}_{x}(C, D) \leq 1$. Therefore, for any finite set of arguments Σ, $\mathtt{Max}(C, \Sigma, \mathtt{sim}) = \max_{C'\in\Sigma} \mathtt{sim}^{\sigma}_{x}(C, C') \leq 1$. Then for the considered $A, B \in \mathtt{Arg}(\mathcal{L})$, we have $\sum\limits_{A_i\in\overline{\mathtt{CR}}(A)} \mathtt{Max}(A_i, \overline{\mathtt{CR}}(B), \mathtt{sim}^{\sigma}_{x}) \leq |\overline{\mathtt{CR}}(A)|$ and $\sum\limits_{B_j\in\overline{\mathtt{CR}}(B)} \mathtt{Max}(B_j, \overline{\mathtt{CR}}(A), \mathtt{sim}^{\sigma}_{x}) \leq |\overline{\mathtt{CR}}(B)|$. Consequently, $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{x}) \leq 1$. Then $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{x}) \in [0, 1]$.

□

Proof. [Proposition 20] Here we will use the result from Theorem 11, which states that, for all σ such that $0 < \sigma < 1$ and $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$, $\mathtt{sim}_x^\sigma(A, B) = 1$ iff both $\mathtt{Supp}(A) \cong \mathtt{Supp}(B)$ and $\mathtt{Conc}(A) \equiv \mathtt{Conc}(B)$.

First we show the implication from right to left. Let $A, B \in \mathtt{Arg}(\mathcal{L})$ such that

(a) $\forall A' \in \overline{\mathtt{CR}}(A), \exists B' \in \overline{\mathtt{CR}}(B)$ such that $\mathtt{Supp}(A') \cong \mathtt{Supp}(B'), \mathtt{Conc}(A') \equiv \mathtt{Conc}(B')$ and

(b) $\forall B' \in \overline{\mathtt{CR}}(B), \exists A' \in \overline{\mathtt{CR}}(A)$ such that $\mathtt{Supp}(B') \cong \mathtt{Supp}(A'), \mathtt{Conc}(B') \equiv \mathtt{Conc}(A')$.

Let $0 < \sigma < 1$ and $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$. If $A_i \in \overline{\mathtt{CR}}(A)$ then, by (a), there exists $B_i \in \overline{\mathtt{CR}}(B)$ such that $\mathtt{Supp}(A_i) \cong \mathtt{Supp}(B_i)$ and $\mathtt{Conc}(A_i) \equiv \mathtt{Conc}(B_i)$. Consequently, $\mathtt{sim}_x^\sigma(A_i, B_i) = 1$, so $\mathtt{Max}(A_i, \overline{\mathtt{CR}}(B), \mathtt{sim}_x^\sigma) = 1$. Since this holds for every $A_i \in \overline{\mathtt{CR}}(A)$, we have $\sum\limits_{A_i \in \overline{\mathtt{CR}}(A)} \mathtt{Max}(A_i, \overline{\mathtt{CR}}(B), \mathtt{sim}_x^\sigma) = |\overline{\mathtt{CR}}(A)|$. In the same way we can use (b) to conclude $\sum\limits_{B_j \in \overline{\mathtt{CR}}(B)} \mathtt{Max}(B_j, \overline{\mathtt{CR}}(A), \mathtt{sim}_x^\sigma) = |\overline{\mathtt{CR}}(B)|$. Consequently, $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{A}}(A, B, \mathtt{sim}_x^\sigma) = 1$.

From (a) we can obtain that for every $A' \in \overline{\mathtt{CR}}(A)$ and $\phi \in \mathtt{Supp}(A')$ there exists $B' \in \overline{\mathtt{CR}}(B)$ and $\psi \in \mathtt{Supp}(B')$ such that $\psi \equiv \phi$. From (b) we have that for every $B' \in \overline{\mathtt{CR}}(B)$ and $\psi \in \mathtt{Supp}(B')$ there exists $A' \in \overline{\mathtt{CR}}(A)$ and $\phi \in \mathtt{Supp}(A')$ such that $\psi \equiv \phi$. Together we have $\bigcup\limits_{A' \in \overline{\mathtt{CR}}(A)} \mathtt{Supp}(A') \cong \bigcup\limits_{B' \in \overline{\mathtt{CR}}(B)} \mathtt{Supp}(B')$, i.e., $\mathtt{US}(A) \cong \mathtt{US}(B)$. Therefore $s_x(\mathtt{US}(A), \mathtt{US}(B)) = 1$. Moreover, for every $A' \in \overline{\mathtt{CR}}(A)$ $\exists B' \in \overline{\mathtt{CR}}(B)$ such that $\mathtt{Conc}(A') \equiv \mathtt{Conc}(B')$. From $\mathtt{Conc}(A) = \mathtt{Conc}(A')$ and $\mathtt{Conc}(B) = \mathtt{Conc}(B')$ we obtain $\mathtt{Conc}(A) \equiv \mathtt{Conc}(B)$, so $s_x(\{\mathtt{Conc}(A)\}, \{\mathtt{Conc}(B)\}) = 1$. Consequently, $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A, B, s_x, s_x, \sigma) = \sigma \cdot s_x(\mathtt{US}(A), \mathtt{US}(B)) + (1-\sigma) \cdot s_x(\{\mathtt{Conc}(A)\}, \{\mathtt{Conc}(B)\}) = \sigma + (1 - \sigma) = 1$.

Now let us suppose that $A, B \in \mathtt{Arg}(\mathcal{L})$, $0 < \sigma < 1$ and $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ are such that $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{A}}(A, B, \mathtt{sim}_x^\sigma) = \mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A, B, s_x, s_x, \sigma) = 1$. From $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{A}}(A, B, \mathtt{sim}_x^\sigma) = 1$ we have both $\sum\limits_{A_i \in \overline{\mathtt{CR}}(A)} \mathtt{Max}(A_i, \overline{\mathtt{CR}}(B), \mathtt{sim}_x^\sigma) = |\overline{\mathtt{CR}}(A)|$ and $\sum\limits_{B_j \in \overline{\mathtt{CR}}(B)} \mathtt{Max}(B_j, \overline{\mathtt{CR}}(A), \mathtt{sim}_x^\sigma) = |\overline{\mathtt{CR}}(B)|$. Then for every $A_i \in \overline{\mathtt{CR}}(A)$ we have $\mathtt{Max}(A_i, \overline{\mathtt{CR}}(B), \mathtt{sim}_x^\sigma) = 1$, i.e., there exists $B_i \in \overline{\mathtt{CR}}(B)$ such that $\mathtt{sim}_x^\sigma(A_i, B_i) = 1$. Then $\mathtt{Supp}(A_i) \cong \mathtt{Supp}(B_i)$ and $\mathtt{Conc}(A_i) \equiv \mathtt{Conc}(B_i)$. Thus, (a) holds. In the same way we can obtain (b).

From Corollary 3, we know that for any σ such that $0 < \sigma < 1$, $\mathtt{sim}_{\mathtt{cnj}}^\sigma(A, B) = 1$ iff both $\mathtt{Supp}(A) \cong \mathtt{Supp}(B)$ and $\mathtt{Conc}(A) \equiv \mathtt{Conc}(B)$. Then, using the same reasoning as before (note that for $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}$ we use $s_x = s_{\mathtt{j}}$ and $s_y = s_{\mathtt{cnj}}$) the result holds for $\mathtt{sim}_{\mathtt{cnj}}^\sigma$.

□

Proof. [Theorem 14]

Satisfaction of the Principles

[Syntax Independence]

For a permutation on the set of variables π, and $A \in \mathtt{Arg}(\mathcal{L})$, let A^π denotes the argument obtained by replacing each variable p in A with $\pi(p)$. It is shown in Theorems 9 and 13, that for every pair of arguments $A, B \in \mathtt{Arg}(\mathcal{L})$, every σ such that $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$, it holds

$$\mathtt{sim}_y^\sigma(A, B) = \mathtt{sim}_y^\sigma(A^\pi, B^\pi).$$

Also, from the fact that $\mathtt{CR}(A^\pi) = \{B^\pi \mid B \in \mathtt{CR}(A)\}$, we have

- $|\overline{\mathtt{CR}}(A)| = |\overline{\mathtt{CR}}(A^\pi)|$
- for every $A_i \in \overline{\mathtt{CR}}(A)$, $\mathtt{Max}(A_i, \overline{\mathtt{CR}}(B), \mathtt{sim}_y^\sigma) = \mathtt{Max}(A_i^\pi, \overline{\mathtt{CR}}(B^\pi), \mathtt{sim}_y^\sigma)$, and for every $B_i \in \overline{\mathtt{CR}}(B)$, $\mathtt{Max}(B_i, \overline{\mathtt{CR}}(A), \mathtt{sim}_y^\sigma) = \mathtt{Max}(B_i^\pi, \overline{\mathtt{CR}}(A^\pi), \mathtt{sim}_y^\sigma)$.

Thus, $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{A}}(A, B, \mathtt{sim}_y^\sigma) = \mathtt{sim}_{\mathtt{CR}}^{\mathtt{A}}(A^\pi, B^\pi, \mathtt{sim}_y^\sigma)$. Let us now switch to the second measure. $\mathtt{US}(A^\pi) = \bigcup_{C \in \overline{\mathtt{CR}}(A^\pi)} \mathtt{Supp}(C) = \bigcup_{D \in \overline{\mathtt{CR}}(A)} \mathtt{Supp}(D^\pi) = \{D^\pi \mid D \in \mathtt{US}(A)$. Similarly, $\mathtt{US}(B^\pi) = \{F^\pi \mid F \in \mathtt{US}(B)$. It was shown in Theorem 9 and 13, that for any similarity measure s_y, and two sets of formulas $\Phi, \Psi \subseteq_f \mathcal{L}$, $s_y(\Phi, \Psi) = s_y(\{E^\pi \mid E \in \Phi\}, \{E^\pi \mid E \in \Psi\})$. Now we obtain $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A, B, s_x, s_y, \sigma) = \mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A^\pi, B^\pi, s_x, s_y, \sigma)$ directly form the definition of U-CR.

[Maximality] Let $A \in \mathtt{Arg}(\mathcal{L})$, σ such that $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$. From Theorems 9 and 13, we know that each $\mathtt{sim}_y^\sigma$ satisfies Maximality, i.e. $\forall y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$, $\mathtt{sim}_y^\sigma(A, A) = 1$. From Proposition 13(4) and the fact that $A \approx A$, $\mathtt{CR}(A) = \mathtt{CR}(A)$ and so $\overline{\mathtt{CR}}(A) = \overline{\mathtt{CR}}(A)$. Therefore, $\sum_{A_i \in \overline{\mathtt{CR}}(A)} \mathtt{Max}(A_i, \overline{\mathtt{CR}}(A), \mathtt{sim}_y^\sigma) = |\overline{\mathtt{CR}}(A)|$, consequently $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{A}}(A, A, \mathtt{sim}_y^\sigma) = 1$. As already proved in Proposition 20, given that $\overline{\mathtt{CR}}(A) = \overline{\mathtt{CR}}(A)$, we have $\mathtt{US}(A) = \mathtt{US}(A)$ and $\mathtt{Conc}(A) = \mathtt{Conc}(A)$, hence $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A, A, s_x, s_y, \sigma) = 1$.

[Symmetry] Let $A, B \in \mathtt{Arg}(\mathcal{L})$, $\sigma \in]0, 1[$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$. From Definition 51:

$$\mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A, B, s_x, s_y, \sigma) = \sigma \cdot s_x(\mathtt{US}(A), \mathtt{US}(B)) + (1-\sigma) \cdot s_y(\{\mathtt{Conc}(A)\}, \{\mathtt{Conc}(B)\}).$$

From Theorems 9 and 13, we know that each $\mathtt{sim}_y^\sigma$ satisfies Symmetry, therefore $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A, B, s_x, s_y, \sigma) = \mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(B, A, s_x, s_y, \sigma)$.

From Definition 50 and because the addition is commutative:

$$\frac{\sum\limits_{A_i \in \overline{\mathtt{CR}}(A)} \mathtt{Max}(A_i, \overline{\mathtt{CR}}(B), \mathtt{sim}) + \sum\limits_{B_j \in \overline{\mathtt{CR}}(B)} \mathtt{Max}(B_j, \overline{\mathtt{CR}}(A), \mathtt{sim})}{|\overline{\mathtt{CR}}(A)| + |\overline{\mathtt{CR}}(B)|}$$

$$=$$

$$\frac{\sum\limits_{B_i \in \overline{\mathtt{CR}}(B)} \mathtt{Max}(B_i, \overline{\mathtt{CR}}(A), \mathtt{sim}) + \sum\limits_{A_j \in \overline{\mathtt{CR}}(A)} \mathtt{Max}(A_j, \overline{\mathtt{CR}}(B), \mathtt{sim})}{|\overline{\mathtt{CR}}(B)| + |\overline{\mathtt{CR}}(A)|}$$

$$\Leftrightarrow$$

$$\mathtt{sim}_{\mathtt{CR}}^{\mathtt{A}}(A, B, \mathtt{sim}) = \mathtt{sim}_{\mathtt{CR}}^{\mathtt{A}}(B, A, \mathtt{sim})$$

[Substitution] Let $A, B \in \mathtt{Arg}(\mathcal{L})$, $\sigma \in]0,1[$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$. From Proposition 20, $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{A}}(A, B, \mathtt{sim}) = \mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A, B, s_x, s_y, \sigma) = 1$ iff

- $\forall A' \in \overline{\mathtt{CR}}(A)$, $\exists B' \in \overline{\mathtt{CR}}(B)$ such that $\mathtt{Supp}(A') \cong \mathtt{Supp}(B')$, $\mathtt{Conc}(A') \equiv \mathtt{Conc}(B')$ and
- $\forall B' \in \overline{\mathtt{CR}}(B)$, $\exists A' \in \overline{\mathtt{CR}}(A)$ such that $\mathtt{Supp}(B') \cong \mathtt{Supp}(A')$, $\mathtt{Conc}(B') \equiv \mathtt{Conc}(A')$.

Given that each concise refinement argument of A has an equivalent argument in $\overline{\mathtt{CR}}(B)$ and vice versa, and because from Theorems 9 and 13, the principle Substitution is satisfied by each $\mathtt{sim}_y^\sigma$, for any $C \in \mathtt{Arg}(\mathcal{L})$:

- $\forall A_i \in \overline{\mathtt{CR}}(A)$, $\exists B_j \in \overline{\mathtt{CR}}(B)$ s.t. $\max_{C \in \overline{\mathtt{CR}}(C)} \mathtt{sim}_y^\sigma(A, C) = \max_{C \in \overline{\mathtt{CR}}(C)} \mathtt{sim}_y^\sigma(B, C)$, i.e. $\mathtt{Max}(A, \overline{\mathtt{CR}}(C), \mathtt{sim}_y^\sigma) = \mathtt{Max}(B, \overline{\mathtt{CR}}(C), \mathtt{sim}_y^\sigma)$.
- $\forall B_i \in \overline{\mathtt{CR}}(B)$, $\exists A_j \in \overline{\mathtt{CR}}(A)$ s.t. $\max_{C \in \overline{\mathtt{CR}}(C)} \mathtt{sim}_y^\sigma(A, C) = \max_{C \in \overline{\mathtt{CR}}(C)} \mathtt{sim}_y^\sigma(B, C)$, i.e. $\mathtt{Max}(A, \overline{\mathtt{CR}}(C), \mathtt{sim}_y^\sigma) = \mathtt{Max}(B, \overline{\mathtt{CR}}(C), \mathtt{sim}_y^\sigma)$.

This means:

$$\sum_{A_i \in \overline{\mathtt{CR}}(A)} \mathtt{Max}(A_i, \overline{\mathtt{CR}}(C), \mathtt{sim}_y^\sigma) + \sum_{C_j \in \overline{\mathtt{CR}}(C)} \mathtt{Max}(C_j, \overline{\mathtt{CR}}(A), \mathtt{sim}_y^\sigma)$$
$$= \sum_{B_i \in \overline{\mathtt{CR}}(B)} \mathtt{Max}(B_i, \overline{\mathtt{CR}}(C), \mathtt{sim}_y^\sigma) + \sum_{C_j \in \overline{\mathtt{CR}}(C)} \mathtt{Max}(C_j, \overline{\mathtt{CR}}(B), \mathtt{sim}_y^\sigma).$$

For the second similarity measure $\mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}$, from Proposition 20 we can deduce that

$\bigcup_{A' \in \overline{\texttt{CR}}(A)} \texttt{Supp}(A') \cong \bigcup_{B' \in \overline{\texttt{CR}}(B)} \texttt{Supp}(B')$, i.e. $\texttt{US}(A) \cong \texttt{US}(B)$ and $\texttt{Conc}(A) \equiv \texttt{Conc}(B)$. Moreover, from Theorems 9 and 13, each s_x and s_y satisfy the principle Substitution, therefore:

$$\sigma \cdot s_x(\texttt{US}(A), \texttt{US}(C)) + (1-\sigma) \cdot s_y(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(C)\})$$

$$= \sigma \cdot s_x(\texttt{US}(B), \texttt{US}(C)) + (1-\sigma) \cdot s_y(\{\texttt{Conc}(B)\}, \{\texttt{Conc}(C)\}).$$

[Minimality] Let $\sigma \in]0,1[$, $x \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}\}$ and $y \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}, \texttt{cnj}\}$ with the condition that if $y = \texttt{cnj}$ then $x = \texttt{j}$, otherwise $y = x$. Let $A, B \in \texttt{Arg}(\mathcal{L})$, such that

- A and B are not equivalent,
- $\bigcup_{\phi_i \in \texttt{Supp}(A)} \texttt{Var}(\phi_i) \cap \bigcup_{\phi_j \in \texttt{Supp}(B)} \texttt{Var}(\phi_j) = \emptyset$ and
- $\texttt{Var}(\texttt{Conc}(A)) \cap \texttt{Var}(\texttt{Conc}(B)) = \emptyset$.

Let $\overline{\texttt{CR}}(A) = \{A_1, \cdots, A_n\}$, $\overline{\texttt{CR}}(B) = \{B_1, \cdots, B_m\}$.

From the Definition 42, refinement arguments do not add new literals according to the original argument, and so do not add new variables in the support and in the conclusion. Therefore, for two arguments having no common variable in their support and conclusion, they cannot have any common variable in their concise arguments.

For every $A_i \in \overline{\texttt{CR}}(A)$, $B_j \in \overline{\texttt{CR}}(B)$, $|\texttt{Co}(\texttt{Supp}(A_i), \texttt{Supp}(B_j))| = 0$, $|\texttt{Co}(\texttt{Conc}(A_i), \texttt{Conc}(B_j))| = 0$ and so $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{x}) = 0$, and $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{x}}, s_{\texttt{x}}, \sigma) = 0$.

Moreover, in the absence of common literals in the conclusions, the two formulas cannot have a common inference using dependent finite CN, i.e., $|\texttt{CN}_{df}(\texttt{Conc}(A_i)) \cap \texttt{CN}_{df}(\texttt{Conc}(B_j))| = 0$. Consequently, $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{\texttt{cnj}}) = 0$, and $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_{\texttt{j}}, s_{\texttt{cnj}}, \sigma) = 0$.

[Triangle Inequality] Let $\sigma \in]0,1[$, $x \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}\}$ and $y \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}, \texttt{cnj}\}$ with the condition that if $y = \texttt{cnj}$ then $x = \texttt{j}$, otherwise $y = x$. Let $A, B \in \texttt{Arg}(\mathcal{L})$. From definition 51:

$$\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_y, \sigma) = \sigma \cdot s_x(\texttt{US}(A), \texttt{US}(B)) + (1-\sigma) \cdot s_y(\{\texttt{Conc}(A)\}, \{\texttt{Conc}(B)\}).$$

In other words, $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_y, \sigma) = \texttt{sim}^{\sigma}_{y}(A', B')$ such that $A' = \langle \texttt{US}(A), \texttt{Conc}(A) \rangle$ and $B' = \langle \texttt{US}(B), \texttt{Conc}(B) \rangle$. Cases where $A', B' \notin \texttt{Arg}(\mathcal{L})$ because the support is not minimal, are not important to the rest of the proof, we will simply calculate the similarity between set of formulas, regardless of whether they satisfy the constraint of an argument.

From Theorem 9 and Theorem 13, we show that only the similarity measures $\mathtt{sim}_j^\sigma$, $\mathtt{sim}_{ss}^\sigma$ and $\mathtt{sim}_{cnj}^\sigma$ satisfy Triangle Inequality. Therefore when $x \in \{\mathtt{j}, \mathtt{ss}\}$, $y \in \{\mathtt{j}, \mathtt{ss}, \mathtt{cnj}\}$, and for any $C \in \mathtt{Arg}(\mathcal{L})$ such that $C' = \langle \mathtt{US}(C), \mathtt{Conc}(C)\rangle$:

$$1 + \mathtt{sim}_y^\sigma(A', C') \geq \mathtt{sim}_y^\sigma(A', B') + \mathtt{sim}_y^\sigma(B', C').$$

This means:

$$1 + \mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A, C, s_x, s_y, \sigma) \geq \mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(A, B, s_x, s_y, \sigma) + \mathtt{sim}_{\mathtt{CR}}^{\mathtt{U}}(B, C, s_x, s_y, \sigma).$$

[Strict Dominance] Let $\mathtt{sim}_{\mathtt{cnj}}^\sigma$ and $A, B, C \in \mathtt{Arg}(\mathcal{L})$ such that from the principle Strict Dominance, for any $\sigma \in]0, 1[$, $\mathtt{sim}_{\mathtt{cnj}}^\sigma(A, B) > \mathtt{sim}_{\mathtt{cnj}}^\sigma(A, C)$ i.e.

$$\sigma s_j(\mathtt{Supp}(A), \mathtt{Supp}(B)) + (1-\sigma) s_{\mathtt{cnj}}(\mathtt{Conc}(A), \mathtt{Conc}(B))$$

$$> \sigma s_j(\mathtt{Supp}(A), \mathtt{Supp}(C)) + (1-\sigma) s_{\mathtt{cnj}}(\mathtt{Conc}(A), \mathtt{Conc}(C))$$

$$\Leftrightarrow \sigma x_1 + (1-\sigma) y_1 > \sigma x_2 + (1-\sigma) y_2$$

Like it was proved in Theorem 13, given that $\mathtt{Supp}(B) \cong \mathtt{Supp}(C)$ we obtain $x_1 = x_2$ and with the conditions of Strict Dominance $y_1 > y_2$, hence:

$$(1-\sigma) y_1 > (1-\sigma) y_2$$

Now, we know that using concise arguments can change their support and therefore it is possible that $x_1 \neq x_2$, but the conclusions don't change, i.e. $y_1 > y_2$.

We will show that for any $x_1, x_2, y_1, y_2 \in [0, 1]$ such that $y_1 - y_2 = \alpha > 0$, $\exists \sigma \in]0, 1[$ such that $\sigma x_1 + (1-\sigma) y_1 > \sigma x_2 + (1-\sigma) y_2$ is true.

$$\sigma x_1 + (1-\sigma) y_1 > \sigma x_2 + (1-\sigma) y_2$$

$$\Leftrightarrow (1-\sigma)\alpha > \sigma(x_2 - x_1)$$

In the case where $x_2 \leq x_1$, it is obviously true.

When $x_2 > x_1$, let us take the most critical case when $x_2 = 1$ and $x_1 = 0$, we have then $(1-\sigma)\alpha > \sigma \Leftrightarrow \alpha > \sigma(1+\alpha) \Leftrightarrow \frac{\alpha}{1+\alpha} > \sigma$. Given that $\frac{\alpha}{1+\alpha} \in]0, 1[$ like σ, and because the number of element in this interval is infinite, then for any α there exists a σ such that $\frac{\alpha}{1+\alpha} > \sigma$.

For the measure $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}$ we can apply directly the definition from $\sigma x_1 + (1-\sigma)y_1 > \sigma x_2 + (1-\sigma)y_2$ and the reasoning holds.
For the measure $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}$ we can rewrite its definition as:

$$\frac{\sum\limits_{A_i \in \overline{\mathtt{CR}}(A)} \mathtt{Max}(A_i, \overline{\mathtt{CR}}(B), \mathtt{sim}) + \sum\limits_{B_j \in \overline{\mathtt{CR}}(B)} \mathtt{Max}(B_j, \overline{\mathtt{CR}}(A), \mathtt{sim})}{|\overline{\mathtt{CR}}(A)| + |\overline{\mathtt{CR}}(B)|}$$

$$\Leftrightarrow \frac{(\sigma x_1 + (1-\sigma)y) + \cdots + (\sigma x_n + (1-\sigma)y)}{n+m}$$
$$+ \frac{(\sigma x_{n+1} + (1-\sigma)y) + \cdots + (\sigma x_{n+m} + (1-\sigma)y)}{n+m}$$

$$\Leftrightarrow \sigma x' + (1-\sigma)y, \text{ where } x' = \frac{\sum\limits_{i \in \{1,\cdots,n+m\}} x_i}{n+m}$$

Therefore using the same reasoning as before, $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}$ using $\mathtt{sim}^{\sigma}_{\mathtt{cnj}}$ satisfies Strict Dominance according to the good σ.

Violation of the Principles

[Non-Zero] Let $A, B \in \mathtt{Arg}(\mathcal{L})$ such that:

- $A = \langle \{p \wedge q\}, p \rangle$,
- $B = \langle \{p \wedge q\}, q \rangle$.

$\overline{\mathtt{CR}}(A) = \{A_1\}$, $\overline{\mathtt{CR}}(B) = \{B_1\}$, where:

- $A_1 = \langle \{p\}, p \rangle$,
- $B_1 = \langle \{q\}, q \rangle$.

Let σ s.t. $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$. From the axiom Non-Zero, $\mathtt{sim}^{\sigma}_{y}(A, B) > 0$ while $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{y}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_x, s_y, \sigma) = 0$.

[Monotony] Let σ s.t. $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$. Let $A, B, C \in \mathtt{Arg}(\mathcal{L})$ such that:

- $A = \langle \{p, t, (p \wedge t) \rightarrow r\}, r \rangle$,
- $B = \langle \{p \wedge q, t\}, p \wedge t \rangle$,

- $C = \langle \{t\}, t \rangle$.

$\overline{\mathtt{CR}}(A) = \{A_1\}$, $\overline{\mathtt{CR}}(B) = \{B_1\}$, $\overline{\mathtt{CR}}(C) = \{C_1\}$, where:

- $A_1 = \langle \{p, t, (p \wedge t) \rightarrow r\}, r \rangle$,
- $B_1 = \langle \{p, t\}, p \wedge t \rangle$,
- $C_1 = \langle \{t\}, t \rangle$.

From Proposition 21, because A and C are concise arguments:

$$\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{y}) = \mathtt{sim}^{\sigma}_{y}(A, C),$$
$$\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_x, s_y, \sigma) = \mathtt{sim}^{\sigma}_{y}(A, C).$$

Then, let us calculate for each parameterised measure the similarity obtained in this example.

- Extended Jaccard:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{j}) = \sigma \cdot \frac{2}{3}$
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_j, \sigma) = \sigma \cdot \frac{2}{3}$
  * $\mathtt{sim}^{\sigma}_{j}(A, C) = \sigma \cdot \frac{1}{3}$

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{j}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{j})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_j, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_j, s_j, \sigma)$.

- Extended Dice:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{d}) = \sigma \cdot \frac{4}{5}$
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_d, s_d, \sigma) = \sigma \cdot \frac{4}{5}$
  * $\mathtt{sim}^{\sigma}_{d}(A_1, C_1) = \sigma \cdot \frac{2}{4}$

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{d}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{d})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_d, s_d, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_d, s_d, \sigma)$.

- Extended Sorensen:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{s}) = \sigma \cdot \frac{8}{9}$
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_s, s_s, \sigma) = \sigma \cdot \frac{8}{9}$
  * $\mathtt{sim}^{\sigma}_{s}(A_1, C_1) = \sigma \cdot \frac{4}{6}$

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{s}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, S\mathtt{sim}^{\sigma}_{s})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_s, s_s, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_s, s_s, \sigma)$.

- Extended Symmetric Anderberg:

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{a}) = \sigma \cdot \frac{16}{17}$
* $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_a, s_a, \sigma) = \sigma \cdot \frac{16}{17}$
* $\mathtt{sim}^{\sigma}_{a}(A_1, C_1) = \sigma \cdot \frac{8}{10}$

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{a}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{a})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_a, s_a, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_a, s_a, \sigma)$.

– Extended Sokal and Sneath 2:

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ss}) = \sigma \cdot \frac{2}{4}$
* $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ss}, s_{ss}, \sigma) = \sigma \cdot \frac{2}{4}$
* $\mathtt{sim}^{\sigma}_{ss}(A_1, C_1) = \sigma \cdot \frac{1}{5}$

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ss}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{ss})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ss}, s_{ss}, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_{ss}, s_{ss}, \sigma)$.

– Extended Ochiai:

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{o}) = \sigma \cdot \frac{2}{\sqrt{3}\cdot\sqrt{2}} + 0 \approx \sigma \cdot \frac{2}{2.449} \approx \sigma \cdot 0.816$
* $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_o, s_o, \sigma) \approx \sigma \cdot 0.816$
* $\mathtt{sim}^{\sigma}_{o}(A_1, C_1) = \sigma \cdot \frac{1}{\sqrt{3}} + 0 \approx \sigma \cdot \frac{1}{1.732} \approx \sigma \cdot 0.577$

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{o}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{o})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_o, s_o, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_o, s_o, \sigma)$.

– Extended Kulczynski 2:

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ku}) = \sigma \cdot \frac{5}{6}$
* $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ku}, s_{ku}, \sigma) = \sigma \cdot \frac{5}{6}$
* $\mathtt{sim}^{\sigma}_{ku}(A_1, C_1) = \sigma \cdot \frac{4}{6}$

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ku}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{ku})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ku}, s_{ku}, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_{ku}, s_{ku}, \sigma)$.

– Mixed CN-based Jaccard Measure:

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{\mathtt{cnj}}) = \sigma \cdot \frac{2}{3}$
* $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_{\mathtt{cnj}}, \sigma) = \sigma \cdot \frac{2}{3}$
* $\mathtt{sim}^{\sigma}_{\mathtt{cnj}}(A_1, C_1) = \sigma \cdot \frac{1}{3}$

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{\mathtt{cnj}}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{\mathtt{cnj}})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_{\mathtt{cnj}}, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_j, s_{\mathtt{cnj}}, \sigma)$.

From the axiom of Monotony, $\mathtt{sim}(A, C) \geq \mathtt{sim}(A, B)$ while for each measure with any $0 < \sigma < 1$, $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{y}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{y})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_x, s_y, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_x, s_y, \sigma)$.

[Strict Monotony] Strict Monotony is violated because, according to the example above in Monotony, the strict version is applicable but also violated.

[Dominance] Let σ s.t. $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$. Let $A, B, C \in \mathtt{Arg}(\mathcal{L})$ such that:

- $A = \langle\{p, t\}, (p \vee q) \wedge t\rangle$,
- $B = \langle\{p \wedge q\}, p \vee q\rangle$,
- $C = \langle\{p \wedge q\}, p\rangle$.

$\overline{\mathtt{CR}}(A) = \{A_1\}$, $\overline{\mathtt{CR}}(B) = \{B_1, B_2, B_3\}$, $\overline{\mathtt{CR}}(C) = \{C_1\}$, where:

- $A_1 = \langle\{p, t\}, (p \vee q) \wedge t\rangle$,
- $B_1 = \langle\{p\}, p \vee q\rangle$, $B_2 = \langle\{q\}, p \vee q\rangle$, $B_3 = \langle\{p \vee q\}, p \vee q\rangle$,
- $C_1 = \langle\{p\}, p\rangle$.

From definition 50, using the definition 45, we obtain in our example:

$$\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{y}) = \frac{\mathtt{sim}^{\sigma}_{y}(A_1, B_1) + \mathtt{sim}^{\sigma}_{y}(A_1, B_1) + \mathtt{sim}^{\sigma}_{y}(A_1, B_2) + \mathtt{sim}^{\sigma}_{y}(A_1, B_3)}{4}.$$

From definition 51:

$$\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_x, s_y, \sigma) = \sigma \cdot s_x(\mathtt{US}(A), \mathtt{US}(B)) + (1-\sigma) \cdot s_y(\{\mathtt{Conc}(A)\}, \{\mathtt{Conc}(B)\}).$$

Let compute for every $0 < \sigma < 1$ and $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$:

- Extended Jaccard:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{j}) = \frac{\sigma \cdot \frac{1}{2} + \sigma \cdot \frac{1}{2} + 0 + 0}{4} = \sigma \cdot \frac{1}{4}$,
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_j, \sigma) = \sigma \cdot \frac{1}{4}$,
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{j}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_j, s_j, \sigma) = \frac{1}{2}$.

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{j}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{j})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_j, s_j, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_j, \sigma)$.
- Extended Dice:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{d}) = \frac{\sigma \cdot \frac{2}{3} + \sigma \cdot \frac{2}{3} + 0 + 0}{4} = \sigma \cdot \frac{1}{3}$,
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_d, s_d, \sigma) = \sigma \cdot \frac{2}{5}$,

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{d}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_d, s_d, \sigma) = \frac{2}{3}$.

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{d}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{d})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_d, s_d, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_d, s_d, \sigma)$.

- Extended Sorensen:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{s}) = \frac{\sigma \cdot \frac{4}{5} + \sigma \cdot \frac{4}{5} + 0 + 0}{4} = \sigma \cdot \frac{2}{5}$,
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_s, s_s, \sigma) = \sigma \cdot \frac{4}{9}$,
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{s}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_s, s_s, \sigma) = \frac{4}{5}$.

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{s}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, S\mathtt{sim}^{\sigma}_{s})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_s, s_s, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_s, s_s, \sigma)$.

- Extended Symmetric Anderberg:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{a}) = \frac{\sigma \cdot \frac{8}{9} + \sigma \cdot \frac{8}{9} + 0 + 0}{4} = \sigma \cdot \frac{4}{9}$,
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_a, s_a, \sigma) = \sigma \cdot \frac{8}{11}$,
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{a}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_a, s_a, \sigma) = \frac{8}{9}$.

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{a}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{a})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_a, s_a, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_a, s_a, \sigma)$.

- Extended Sokal and Sneath 2:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ss}) = \frac{\sigma \cdot \frac{1}{3} + \sigma \cdot \frac{1}{3} + 0 + 0}{4} = \sigma \cdot \frac{1}{6}$,
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ss}, s_{ss}, \sigma) = \sigma \cdot \frac{1}{7}$,
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{ss}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_{ss}, s_{ss}, \sigma) = \frac{1}{3}$.

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{ss}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ss})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_{ss}, s_{ss}, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ss}, s_{ss}, \sigma)$.

- Extended Ochiai:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{o}) = \frac{\sigma \cdot \frac{1}{\sqrt{2}} + \sigma \cdot \frac{1}{\sqrt{2}} + 0 + 0}{4} = \sigma \cdot \frac{1}{2\sqrt{2}} \approx \sigma \cdot 0.354$,
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_o, s_o, \sigma) = \sigma \cdot \frac{1}{\sqrt{2} \cdot \sqrt{3}} + 0 \approx \sigma \cdot 0.408$,
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{o}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_o, s_o, \sigma) = \sigma \cdot \frac{1}{\sqrt{2}} \approx \sigma \cdot 0.707$.

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{o}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{o})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_o, s_o, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_o, s_o, \sigma)$.

- Extended Kulczynski 2:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ku}) = \frac{\sigma \cdot \frac{3}{4} + \sigma \cdot \frac{3}{4} + 0 + 0}{4} = \sigma \cdot \frac{3}{8}$,
  * $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ku}, s_{ku}, \sigma) = \sigma \cdot \frac{5}{12}$,
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{ku}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_{ku}, s_{ku}, \sigma) = \frac{3}{4}$.

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{ku}) > \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ku})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, C, s_{ku}, s_{ku}, \sigma) > \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ku}, s_{ku}, \sigma)$.

– Mixed CN-based Jaccard Measure:
  Let $A, B, C \in \texttt{Arg}(\mathcal{L})$ such that:
  * $A = \langle\{p, p \to t\}, t\rangle$,
  * $B = \langle\{p \land q\}, p \land q\rangle$,
  * $C = \langle\{p \land q\}, p\rangle$.

  $\overline{\texttt{CR}}(A) = \{A_1\}$, $\overline{\texttt{CR}}(B) = \{B_1\}$, $\overline{\texttt{CR}}(C) = \{C_1\}$, where:
  * $A_1 = \langle\{p, p \to t\}, t\rangle$,
  * $B_1 = \langle\{p \land q\}, p \land q\rangle$,
  * $C_1 = \langle\{p\}, p\rangle$.

  We obtain:
  * $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{cnj}) = \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_j, s_{cnj}, \sigma) = 0$,
  * $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{\sigma}_{cnj}) = \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_j, s_{cnj}, \sigma) = \sigma \cdot \frac{1}{2}$.

  Then for every $0 < \sigma < 1$: $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{\sigma}_{j}) > \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{j})$ and $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_j, s_{cnj}, \sigma) > \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_j, s_{cnj}, \sigma)$.

From the principle Dominance, $\texttt{sim}(A, B) \geq \texttt{sim}(A, C)$ while for every $0 < \sigma < 1$, $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{y}) > \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{\sigma}_{y})$ and $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_x, s_y, \sigma) > \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_y, \sigma)$.

[Strict Dominance] For the syntactic measures $x \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}\}$, the example used above for Dominance, satisfies the conditions of Strict Monotony, i.e. $\texttt{sim}(A, B) > \texttt{sim}(A, C)$ while for every $0 < \sigma < 1$, $\texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, B, \texttt{sim}^{\sigma}_{x}) > \texttt{sim}^{\texttt{A}}_{\texttt{CR}}(A, C, \texttt{sim}^{\sigma}_{x})$ and $\texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, C, s_x, s_x, \sigma) > \texttt{sim}^{\texttt{U}}_{\texttt{CR}}(A, B, s_x, s_x, \sigma)$.

[Independent Distribution] Let σ s.t. $0 < \sigma < 1$, $x \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}\}$ and $y \in \{\texttt{j}, \texttt{d}, \texttt{s}, \texttt{a}, \texttt{ss}, \texttt{o}, \texttt{ku}, \texttt{cnj}\}$ with the condition that if $y = \texttt{cnj}$ then $x = \texttt{j}$, otherwise $y = x$. Let $A, B, A', B' \in \texttt{Arg}(\mathcal{L})$ such that:

– $A = \langle\{p, q, (p \land q) \to r\}, r\rangle$,
– $B = \langle\{p, q \land s, (p \land q) \to t\}, t\rangle$,
– $A' = \langle\{p, q, (p \land q) \to r\}, r\rangle$,
– $B' = \langle\{p, q \land s, (p \land q) \to t\}, t \land s\rangle$.

$\overline{\texttt{CR}}(A) = \{A_1\}$, $\overline{\texttt{CR}}(B) = \{B_1\}$, $\overline{\texttt{CR}}(A') = \{A'_1\}$, $\overline{\texttt{CR}}(B') = \{B'_1\}$, where:

- $A_1 = \langle \{p, q, (p \wedge q) \rightarrow r\}, r \rangle$,
- $B_1 = \langle \{p, q, (p \wedge q) \rightarrow t\}, t \rangle$,
- $A'_1 = \langle \{p, q, (p \wedge q) \rightarrow r\}, r \rangle$,
- $B'_1 = \langle \{p, q \wedge s, (p \wedge q) \rightarrow t\}, t \wedge s \rangle$.

From Proposition 21, because A', B' are concise arguments:

$$\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{y}) = \mathtt{sim}^{\sigma}_{y}(A'_1, B'_1),$$
$$\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_x, s_y, \sigma) = \mathtt{sim}^{\sigma}_{y}(A'_1, B'_1).$$

Let compute the similarity degree for each similarity measure.

- Extended Jaccard:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{j}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_j, \sigma) = \sigma \cdot \frac{1}{2}$
  * $\mathtt{sim}^{\sigma}_{j}(A'_1, B'_1) = \sigma \cdot \frac{1}{5}$

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{j}) \neq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{j})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_j, \sigma) \neq \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_j, s_j, \sigma)$.
- Extended Dice:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{d}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_d, s_d, \sigma) = \sigma \cdot \frac{2}{3}$
  * $\mathtt{sim}^{\sigma}_{d}(A'_1, B'_1) = \sigma \cdot \frac{1}{3}$

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{d}) \neq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{d})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_d, s_d, \sigma) \neq \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_d, s_d, \sigma)$.
- Extended Sorensen:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{s}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_s, s_s, \sigma) = \sigma \cdot \frac{4}{5}$
  * $\mathtt{sim}^{\sigma}_{s}(A'_1, B'_1) = \sigma \cdot \frac{1}{2}$

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{s}) \neq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', S\mathtt{sim}^{\sigma}_{s})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_s, s_s, \sigma) \neq \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_s, s_s, \sigma)$.
- Extended Symmetric Anderberg:
  * $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{a}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_a, s_a, \sigma) = \sigma \cdot \frac{8}{9}$
  * $\mathtt{sim}^{\sigma}_{a}(A'_1, B'_1) = \sigma \cdot \frac{2}{3}$

  Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{a}) \neq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{a})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_a, s_a, \sigma) \neq \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_a, s_a, \sigma)$.
- Extended Sokal and Sneath 2:

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ss}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ss}, s_{ss}, \sigma) = \sigma \cdot \frac{1}{3}$
* $\mathtt{sim}^{\sigma}_{ss}(A'_1, B'_1) = \sigma \cdot \frac{1}{9}$

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ss}) \neq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{ss})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ss}, s_{ss}, \sigma) \neq \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_{ss}, s_{ss}, \sigma)$.

– Extended Ochiai:

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{o}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_o, s_o, \sigma) = \sigma \cdot \frac{2}{3}$
* $\mathtt{sim}^{\sigma}_{o}(A'_1, B'_1) = \sigma \cdot \frac{1}{3}$

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{o}) \neq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{o})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_o, s_o, \sigma) \neq \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_o, s_o, \sigma)$.

– Extended Kulczynski 2:

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ku}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ku}, s_{ku}, \sigma) = \sigma \cdot \frac{2}{3}$
* $\mathtt{sim}^{\sigma}_{ku}(A'_1, B'_1) = \sigma \cdot \frac{1}{3}$

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{ku}) \neq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{ku})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_{ku}, s_{ku}, \sigma) \neq \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_{ku}, s_{ku}, \sigma)$.

– Mixed CN-based Jaccard Measure:

* $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{cnj}) = \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_{cnj}, \sigma) = \sigma \cdot \frac{1}{2}$
* $\mathtt{sim}^{\sigma}_{cnj}(A'_1, B'_1) = \sigma \cdot \frac{1}{5}$

Then for every $0 < \sigma < 1$: $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{cnj}) \neq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{cnj})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_j, s_{cnj}, \sigma) \neq \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_j, s_{cnj}, \sigma)$.

From the axiom Independent Distribution, $\mathtt{sim}(A, B) = \mathtt{sim}(A', B')$ while $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{x}) \neq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A', B', \mathtt{sim}^{\sigma}_{x})$ and $\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_x, \sigma) \neq \mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A', B', s_x, \sigma)$.

[Triangle Inequality] Let $A, B, C \in \mathtt{Arg}(\mathcal{L})$ such that:

– $A = \langle\{p \wedge q, (p \wedge q) \rightarrow t\}, p \wedge t\rangle$,

– $B = \langle\{p \wedge q, q \wedge (p \wedge q) \rightarrow t\}, t\rangle$,

– $C = \langle\{p, q \wedge (p \wedge q) \rightarrow t\}, t\rangle$.

$\overline{\mathtt{CR}}(A) = \{A_1\}$, $\overline{\mathtt{CR}}(B) = \{B_1, B_2\}$, $\overline{\mathtt{CR}}(C) = \{C_1\}$, where:

– $A_1 = \langle\{p \wedge q, (p \wedge q) \rightarrow t\}, p \wedge t\rangle$,

– $B_1 = \langle\{p \wedge q, (p \wedge q) \rightarrow t\}, t\rangle$, $B_2 = \langle\{p, q \wedge (p \wedge q) \rightarrow t\}, t\rangle$,

– $C_1 = \langle\{p, q \wedge (p \wedge q) \rightarrow t\}, t\rangle$.

Let compute for every $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$.

- $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{x}) = 0$,
- $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{x}) = \frac{\sigma+\sigma+0}{3} = \sigma \cdot \frac{2}{3}$,
- $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(B, C, \mathtt{sim}^{\sigma}_{x}) = \frac{(1-\sigma)+1+1}{3} = 1 - \sigma \cdot \frac{1}{3}$.

Then, $1 + \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{x}) < \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{x}) + \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(B, C, \mathtt{sim}^{\sigma}_{j})$, because $1 < 1 + \frac{1}{3} \cdot \sigma$.

Let see for Mixed CN-based Jaccard Measure:

- $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{cnj}) = (1-\sigma) \cdot \frac{1}{3}$,
- $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{cnj}) = \frac{(\sigma+\sigma+0)+3(1-\sigma)\cdot\frac{1}{3}}{3} = \sigma \cdot \frac{2}{3} + (1-\sigma) \cdot \frac{1}{3}$,
- $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(B, C, \mathtt{sim}^{\sigma}_{cnj}) = \frac{(1-\sigma)+1+1}{3} = 1 - \sigma \cdot \frac{1}{3}$.

Then $1 + \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{cnj}) < \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{cnj}) + \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(B, C, \mathtt{sim}^{\sigma}_{cnj})$, because $1 + (1-\sigma) \cdot \frac{1}{3} < 1 + \frac{1}{3} \cdot \sigma + (1-\sigma) \cdot \frac{1}{3}$.

Let $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$, from the axiom Triangle Inequality, $1 + \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{y}) \geq \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{y}) + \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(B, C, \mathtt{sim}^{\sigma}_{y})$ while $1 + \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, C, \mathtt{sim}^{\sigma}_{y}) < \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{y}) + \mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(B, C, \mathtt{sim}^{\sigma}_{y})$.

□

Proof. [Proposition 21] If $A, B \in \mathtt{Arg}(\mathcal{L})$ are concise arguments, then by Proposition 13(1) and Definition 49 we have that $\overline{\mathtt{CR}}(A) = \{A'\}$ such that $\mathtt{Supp}(A') \cong \mathtt{Supp}(A)$ and $\mathtt{Conc}(A') = \mathtt{Conc}(A)$, and $\overline{\mathtt{CR}}(B) = \{B'\}$ such that $\mathtt{Supp}(B') \cong \mathtt{Supp}(B)$ and $\mathtt{Conc}(B') = \mathtt{Conc}(B)$. Let σ such that $0 < \sigma < 1$, $x \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}\}$ and $y \in \{\mathtt{j}, \mathtt{d}, \mathtt{s}, \mathtt{a}, \mathtt{ss}, \mathtt{o}, \mathtt{ku}, \mathtt{cnj}\}$ with the condition that if $y = \mathtt{cnj}$ then $x = \mathtt{j}$, otherwise $y = x$. Then we obtain $\mathtt{sim}^{\mathtt{A}}_{\mathtt{CR}}(A, B, \mathtt{sim}^{\sigma}_{y}) = \mathtt{sim}^{\sigma}_{y}(A', B')$ and
$\mathtt{sim}^{\mathtt{U}}_{\mathtt{CR}}(A, B, s_x, s_y, \sigma) = \mathtt{sim}^{\sigma}_{y}(A', B')$. Finally, from $A \approx A'$ and $B \approx B'$ we obtain $\mathtt{sim}^{\sigma}_{y}(A', B') = \mathtt{sim}^{\sigma}_{y}(A, B)$. □

5.2 Proofs of Chapter 3

5.2.1 Proofs of section 3.2: Similarity-based Gradual Semantics

Proof. **[Proposition 22]**
Let $\mathrm{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ be a well-behaved evaluation method, $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ be a SSWAF, $x_1, \cdots, x_k \in [0,1]$, and $\mathcal{X} = \{A_1, \cdots, A_k\} \subseteq \mathcal{A}$. Let $\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$.

- **Case 1.** $\mathcal{X} = \emptyset$. Since $\mathrm{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ is well-behaved, then from condition 3a (Def. 54), it follows that $\mathbf{n}() = ()$. Hence, $\mathbf{g}() = \mathbf{g}(\mathbf{n}()) = 0$ (from condition 2a (Def. 54)).

- **Case 2.** $k = 1$. Since $\mathrm{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ is well-behaved, then from condition 3b (Def. 54), $\mathbf{n}((x_1, A_1)) = (x_1)$. Hence, $\mathbf{g}(x_1) = \mathbf{g}(\mathbf{n}((x_1, A_1))) = x_1$ (from condition 2b (Def. 54)).

- **Case 3.** $k > 1$. We distinguish two cases:

  - **Case 3.1.** for all $i, j \in \{1, \cdots, k\}$, with $i \neq j$, $\mathtt{sim}(A_i, A_j) = 0$. From Proposition 23, $\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)) = (x_1, \cdots, x_k)$. Hence, $\mathbf{g}(x_1, \cdots, x_k) = \mathbf{g}(\mathbf{n}((x_1, B_1), \cdots, (x_k, B_k)))$.
  - **Case 3.2.** There exist some similarities between elements of $\mathcal{X}$. Let $B_1, \cdots, B_k$ be arguments such that for all $i, j \in \{1, \cdots, k\}$, with $i \neq j$, $\mathtt{sim}(B_i, B_j) = 0$. From Proposition 23, $\mathbf{n}((x_1, B_1), \cdots, (x_k, B_k)) = (x_1, \cdots, x_k)$. Furthermore, for all $i, j \in \{1, \cdots, k\}$, $\mathtt{sim}(A_i, A_j) \geq \mathtt{sim}(B_i, B_j)$. Since $\mathrm{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ is well-behaved, then from condition 3c (Def. 54), $\mathbf{g}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))) \leq \mathbf{g}(\mathbf{n}((x_1, B_1), \cdots, (x_k, B_k)))$. So, $\mathbf{g}(\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k))) \leq \mathbf{g}(x_1, \cdots, x_k)$.

□

Proof. **[Proposition 23]**
Let $\mathrm{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ be a well-behaved evaluation method, $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ be a SSWAF, $x_1, \cdots, x_k \in [0,1]$, and $\mathcal{X} = \{A_1, \cdots, A_k\} \subseteq \mathcal{A}$. Let $\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$. Assume that for all $i, j \in \{1, \cdots, k\}$, with $i \neq j$, $\mathtt{sim}(A_i, A_j) = 0$. Since $\mathrm{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$ is a well-behaved evaluation method, then for every $i \in \{1, \cdots, k\}$, $x_i = x'_i$ (from condition 3g (Def. 54)). □

Proof. **[Theorem 15]**
This result follows from Brouwer's fixed point theorem. Indeed, if a SSWAF, $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$

has k arguments, then each semantics $\mathbf{S}$ corresponds to a solution of the system of k equations, $\mathtt{Str}^{\mathbf{S}}(A) =$

$$\mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\Bigg(\mathbf{n}\Big((\mathtt{Str}^{\mathbf{S}}(B_1), B_1), \cdots, (\mathtt{Str}^{\mathbf{S}}(B_k), B_k)\Big)\Bigg)\Bigg)$$

where $\{B_1, \cdots, B_k\} = \mathtt{Att}(A)$, for each $A \in \mathcal{A}$. Now the result follows from the fact that the composition of functions $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{n}$ as presented in Definition 55 is continuous function, so the k-ary function $F : [0,1]^k \to [0,1]^k$, whose components are given by the equations of Definition 55 (one per argument), is continuous function on the compact set $[0,1]^k$. By Brouwer's fixed point theorem, F has a fixed point, and that point is a k-tuple which is a solution of the considered system of equations. □

Proof. **[Theorem 16]**
Let $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle \in \mathbf{M}^*$ be an evaluation method. We need to prove that there is one and only one semantics $\mathbf{S}$ such that $\mathbf{S}$ is based on $\mathbf{M}$ for any SSWAF, $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$. The proof contains two steps:

- First, we define one semantics, denoted by $\mathbf{S}'(\mathbf{M})$. For that, for each graph SSWAF, $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$, we need to define a weighting on the arguments. Let us fix a graph $\mathbf{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ and assume a fixed enumeration of the arguments $\mathcal{A} = \{A_1, \cdots, A_n\}$. We define a sequence $\{\mathbf{u}^{(i)}\}_{i=1}^{+\infty}$ of vectors from $[0,1]^n$ in the following way:

  - $\mathbf{u}^{(1)} = (\mathbf{w}(A_1), \cdots, \mathbf{w}(A_n))$.
  - In order to define $\mathbf{u}^{(i)}$ for each $i \geq 2$, we first define the mapping $\mathcal{Q} : [0,1]^n \to [0,1]^n$: $\mathcal{Q}(\mathbf{v}) = [\mathcal{Q}_1(\mathbf{v}), \cdots, \mathcal{Q}_n(\mathbf{v})]$, where for every $\mathbf{v} = (v_1, \cdots, v_n)$, $k \in \{1, \cdots, n\}$,

  $$\mathcal{Q}_k(\mathbf{v}) = \mathbf{f}(\mathbf{w}(A_k), \mathbf{g}(\mathbf{n}(v_{\ell_k(1)}, \cdots, v_{\ell_k(n_k)}, b_1^k, \cdots, b_{nk}^k))), \tag{5.1}$$

  where $\{B_1^k, \cdots, b_{nk}^k\} = \mathtt{Att}_{\mathbf{AF}}(A_k)$ and, for every $j \leq n_k$, $\ell_k(j) = m$ iff $b_j^k = A_m$. Then, for each $i \geq 2$, we define

  $$\mathbf{u}^{(i)} = (u_1^{(i)}, \cdots, u_n^{(i)}) = \mathcal{Q}(\mathbf{u}^{(i-1)}) \tag{5.2}$$

  We will prove that the sequence $\{\mathbf{u}^{(i)}\}_{i=1}^{+\infty}$ converges. Then we define the semantics

S' by assigning to the graph the weighting $\mathtt{Str}^{\mathbf{S}'}$ such that

$$(\mathtt{Str}^{\mathbf{S}'}(A_1), \cdots, \mathtt{Str}^{\mathbf{S}'}(A_n)) = \lim_{i \to +\infty} \mathbf{u}^{(i)}.$$

(We define a weighting on arguments, in the previously described way, for every graph.)

- Then we prove that S' is based on $\mathbf{M}$. Finally, using the same sequence $\{\mathbf{u}^{(i)}\}_{i=1}^{+\infty}$, we show that every semantics based on $\mathbf{M}$ coincides with S'.

Let $\mathrm{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ and let $\mathcal{A} = \{A_1, \cdots, A_n\}$. Let $\{\mathbf{u}^{(i)}\}_{i=1}^{+\infty}$ be the sequence of vectors constructed above. Let $\mathtt{Att}_{\mathbf{AF}}(A_k) = \{b_1^k, \cdots, b_{n_k}^k\} \subseteq \mathcal{A}$, for every $j \leq n_k$, let $\ell_k(j) = m$ iff $b_j^k = A_m$. If $\mathbf{u} = (u_1, \cdots, u_n)$ and $\mathbf{v} = (v_1, \cdots, v_n)$ are two vectors, we write $\mathbf{u} \leq \mathbf{v}$ if and only if for every $k \in \{1, \cdots, n\}$ it holds that $u_k \leq v_k$. Suppose that $\mathbf{u} \leq \mathbf{v}$. We will use the fact that $\mathbf{M}$ is a well-behaved evaluation method. For arbitrary $k \in \{1, \cdots, n\}$, from condition 3e (Def. 54)

$$\mathbf{g}\Big(\mathbf{n}\Big((u_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (u_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Big) \leq$$
$$\mathbf{g}\Big(\mathbf{n}\Big((v_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (v_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Big).$$

Finally, by condition 1a (Def. 54) we have

$$\mathbf{f}\Big(\mathbf{w}(A_k), \mathbf{g}\Big(\mathbf{n}\Big((v_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (v_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Big)\Big) \leq$$
$$\mathbf{f}\Big(\mathbf{w}(A_k), \mathbf{g}\Big(\mathbf{n}\Big((u_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (u_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Big)\Big),$$

i.e.,

$$\mathcal{Q}_k(v) \leq \mathcal{Q}_k(u).$$

Thus, we proved

$$\mathbf{u} \leq \mathbf{v} \implies \mathcal{Q}(\mathbf{v}) \leq \mathcal{Q}(\mathbf{u}). \tag{5.3}$$

Consequently, we obtain

$$\mathbf{u} \leq \mathbf{v} \implies \mathcal{Q}(\mathcal{Q}(\mathbf{u})) \leq \mathcal{Q}(\mathcal{Q}(\mathbf{v})). \tag{5.4}$$

by applying $\mathcal{Q}$ to (5.3). Now we can show that for every $i \in \mathbb{N}$

$$\mathbf{u}^{(2)} \leq \mathbf{u}^{(4)} \leq \cdots \leq \mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+1)} \leq \cdots \leq \mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}. \tag{5.5}$$

From condition 1a (Def. 54) and equation (5.1) we obtain that

$$(\forall i \in \mathbb{N})\ \mathbf{u}^{(i)} \leq (\mathbf{w}(A_1), \cdots, \mathbf{w}(A_n)) = \mathbf{u}^{(1)}.$$

Specially, $\mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}$. From (5.4) we obtain

$$\cdots \leq \mathbf{u}^{(2i+1)} \leq \cdots \leq \mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}. \tag{5.6}$$

From $\mathbf{u}^{(3)} \leq \mathbf{u}^{(1)}$ we conclude $\mathbf{u}^{(2)} \leq \mathbf{u}^{(4)}$ by (5.3). From (5.4) we obtain

$$\mathbf{u}^{(2)} \leq \mathbf{u}^{(4)} \leq \cdots \leq \mathbf{u}^{(2i)} \leq \cdots \tag{5.7}$$

From $\mathbf{u}^{(2)} \leq \mathbf{u}^{(1)}$, by (5.4) we obtain

$$(\forall i \in \mathbb{N})\ \mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+1)}. \tag{5.8}$$

Now (5.5) follows from (5.6), (5.7) and (5.8).

Note that from

$$\mathbf{u}^{(2i+2)} \leq \mathbf{u}^{(2i+3)} \leq \mathbf{u}^{(2i+1)}, \tag{5.9}$$

we can obtain that for every $i \in \mathbb{N}$, there exists $0 < \pi \leq 1$ such that

$$\pi u^{(2i-1)} \leq u^{(2i)}.$$

Now we define $\pi_i = \sup\{\pi \mid \pi\mathbf{u}^{(2i-1)} \leq \mathbf{u}^{(2i)}\}$. Obviously, for every $i \in \mathbb{N}$,

$$\pi_i \mathbf{u}^{(2i-1)} \leq \mathbf{u}^{(2i)}.$$

Also, if $\pi\mathbf{u}^{(2i-1)} \leq \mathbf{u}^{(2i)}$, from

$$\mathbf{u}^{(2i)} \leq \mathbf{u}^{(2i+2)} \leq \mathbf{u}^{(2i+1)} \leq \mathbf{u}^{(2i-1)},$$

we obtain $\pi\mathbf{u}^{(2i+1)} \leq \mathbf{u}^{(2i+2)}$. Consequently,

$$(\forall i \in \mathbb{N})\ \pi_i \leq \pi_{i+1}.$$

Thus, the sequence $\{\pi_i\}_{i=1}^{+\infty}$ is non-decreasing. Since it is also bounded by 1, we obtain that it converges. Let us denote $\pi = \lim_{i\to+\infty} \pi_i$.

Next we prove that $\pi = 1$.

Let $\mathbf{v} = (v_1, \cdots, v_n) \in [0,1]^n$ be a vector and let $\lambda \leq 1$ be a positive number. From the last condition of Theorem 16, for every $k \in \{1, \cdots, n\}$ we obtain

$$\mathbf{g}\Bigg(\mathbf{n}\Big((\lambda v_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (\lambda v_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Bigg) \geq \lambda \mathbf{g}\Bigg(\mathbf{n}\Big((v_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (v_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Bigg).$$

From condition 1a (Def. 54) we derive

$$\mathbf{f}\Bigg(\mathbf{w}(A_k), \mathbf{g}\Bigg(\mathbf{n}\Big((\lambda v_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (\lambda v_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Bigg)\Bigg) \leq \mathbf{f}\Bigg(\mathbf{w}(A_k), \lambda\mathbf{g}\Bigg(\mathbf{n}\Big((v_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (v_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Bigg)\Bigg). \tag{5.10}$$

Let $g^* = \sup_{\overline{x}\in\bigcup_{n=0}^{+\infty}[0,1]^n} \mathbf{g}(\overline{x})$. Note that, by condition 1a (Def. 54), for every r from the unit interval of reals there exists the function

$$\varphi_r : \ ]f_r^*, f_r^{**}] \to [0, +\infty[$$

such that $f_r^* = \lim_{y\to g^*_-} \mathbf{f}(r, y)$, $f_r^{**} = \lim_{y\to 0_+} \mathbf{f}(r, y)$, and

$$\mathbf{f}(r, \varphi_r(y)) = y. \tag{5.11}$$

Note that φ_r is the inverse function of the function obtained by $\mathbf{f}$ by fixing the first variable to be r. Since $\mathbf{f}$ is decreasing on the second variable, φ_r is decreasing as well. Also, from continuity of $\mathbf{f}$ we obtain that φ_r is continuous.

It is easy to check that for every $k \in \{1, \cdots, n\}$ and every $\mathbf{x} = (x_1, \cdots, x_n) \in [0,1]^n$

$$\mathbf{f}(\mathbf{w}(A_k), \varphi_{\mathbf{w}(A_k)}(\mathcal{Q}_k(\mathbf{x}))) = \mathbf{f}\Bigg(\mathbf{w}(A_k), \mathbf{g}\Bigg(\mathbf{n}\Big((x_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (x_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Bigg)\Bigg). \tag{5.12}$$

If we denote $\lambda\mathbf{v} = (\lambda v_1, \cdots, \lambda v_n)$, note that

$$\mathcal{Q}_k(\lambda\mathbf{v}) = \mathbf{f}\Bigg(\mathbf{w}(A_k), \mathbf{g}\Bigg(\mathbf{n}\Big((\lambda v_{\ell_k(1)}, A_{\ell_k(1)}), \cdots, (\lambda v_{\ell_k(n_k)}, A_{\ell_k(n_k)})\Big)\Bigg)\Bigg). \tag{5.13}$$

Using (5.12) and (5.13), we can rewrite (5.10) as

$$\mathcal{Q}_k(\lambda\mathbf{v}) \leq \mathbf{f}(\mathbf{w}(A_k), \lambda\varphi_{\mathbf{w}(A_k)}(\mathcal{Q}_k(\mathbf{v}))). \tag{5.14}$$

Since $\pi_i\mathbf{u}^{(2i-1)} \leq \mathbf{u}^{(2i)}$ for every $i \in \mathbb{N}$, from (5.3) we obtain $\mathcal{Q}(\mathbf{u}^{(2i)}) \leq \mathcal{Q}(\pi_i\mathbf{u}^{(2i-1)})$. Thus,

$$\mathcal{Q}_k(\mathbf{u}^{(2i)}) \leq \mathcal{Q}_k(\pi_i\mathbf{u}^{(2i-1)}), \tag{5.15}$$

for every $k \in \{1, \cdots, n\}$. On the other hand, from (5.14) we obtain

$$\mathcal{Q}_k(\pi_i\mathbf{u}^{(2i-1)}) \leq \mathbf{f}(\mathbf{w}(A_k), \pi_i\varphi_{\mathbf{w}(A_k)}(\mathcal{Q}_k(\mathbf{u}^{(2i-1)}))). \tag{5.16}$$

From (5.15) and (5.16) we obtain

$$\mathcal{Q}_k(\mathbf{u}^{(2i)}) \leq \mathbf{f}(\mathbf{w}(A_k), \pi_i\varphi_{\mathbf{w}(A_k)}(\mathcal{Q}_k(\mathbf{u}^{(2i-1)}))), \tag{5.17}$$

i.e.,

$$u_k^{(2i+1)} \leq \mathbf{f}(\mathbf{w}(A_k), \pi_i\varphi_{\mathbf{w}(A_k)}(u_k^{(2i)})). \tag{5.18}$$

From (5.18) we obtain

$$\frac{u_k^{(2i+2)}}{\mathbf{f}(\mathbf{w}(A_k), \pi_i\varphi_{\mathbf{w}(A_k)}(u_k^{(2i)}))} u_k^{(2i+1)} \leq u_k^{(2i+2)}, \tag{5.19}$$

for every $k \in \{1, \cdots, n\}$. Since $\pi_i = \sup\{\pi \mid \pi\mathbf{u}^{(2i-1)} \leq \mathbf{u}^{(2i)}\}$, From (5.19) we obtain that for every $i \in \mathbb{N}$, there exists $k_i \in \{1, \cdots, n\}$ such that

$$\frac{u_{k_i}^{(2i+2)}}{\mathbf{f}(\mathbf{w}(A_{k_i}), \pi_i\varphi_{\mathbf{w}(A_{k_i})}(u_{k_i}^{(2i)}))} \leq \pi_{i+1}, \tag{5.20}$$

Note that the sequence $\{\mathbf{u}^{(2i)}\}_{i=1}^{+\infty}$ converges, since it is non-decreasing (by (5.7)) and bounded by $\mathbf{u}^{(1)}$. Let us denote the limit of the sequence by $\mathbf{u}^*$, i.e.,

$$\lim_{i\to+\infty} \mathbf{u}^{(2i)} = \mathbf{u}^*.$$

Note also that the sequence $\{k_i\}_{i=1}^{+\infty}$ is an infinite sequence, and that all the elements of the sequence belong to the finite set $\{1, \cdots, n\}$. Thus, there exists $l \in \{1, \cdots, n\}$ such that l appears infinitely many times in the sequence. Let's apply limit to the inequality

(5.20) using the subsequence obtained by taking only those i for which $k_i = l$. We obtain

$$\frac{u_l^*}{\mathbf{f}(\mathbf{w}(A_l), \pi\varphi_{\mathbf{w}(A_l)}(u_l^*))} \leq \pi, \tag{5.21}$$

i.e.,

$$u_l^* \leq \pi\mathbf{f}(\mathbf{w}(A_l), \pi\varphi_{\mathbf{w}(A_l)}(u_l^*)). \tag{5.22}$$

We know that $\pi \leq 1$. If $\pi < 1$, from the fifth condition of Theorem 16,we obtain

$$\pi\mathbf{f}(\mathbf{w}(A_l), \pi\varphi_{\mathbf{w}(A_l)}(u_l^*)) < \mathbf{f}(\mathbf{w}(A_l), \varphi_{\mathbf{w}(A_l)}(u_l^*)). \tag{5.23}$$

By (5.11), we have

$$\mathbf{f}(\mathbf{w}(A_l), \varphi_{\mathbf{w}(A_l)}(u_l^*)) = u_l^*. \tag{5.24}$$

From (5.22), (5.23) and (5.24), we obtain $u_l^* < u_l^*$; a contradiction. Thus, $\pi = 1$.

Note that the sequence $\{\mathbf{u}^{(2i+1)}\}_{i=1}^{+\infty}$ converges, since it is non-increasing (by (5.6)) and bounded (for example, by $\mathbf{u}^{(2)}$). Let us denote the limit of the sequence by $\mathbf{u}_*$, i.e.,

$$\lim_{i\to+\infty} \mathbf{u}^{(2i+1)} = \mathbf{u}_*.$$

From (5.5) we obtain

$$\mathbf{u}^* \leq \mathbf{u}_*.$$

On the other hand,

$$\pi_i\mathbf{u}^{(2i-1)} \leq \mathbf{u}^{(2i)}$$

for every $i \in \mathbb{N}$. Letting $i \to +\infty$ we obtain

$$\mathbf{u}_* \leq \mathbf{u}^*.$$

Consequently, $\mathbf{u}_* = \mathbf{u}^*$, so the sequence $\{\mathbf{u}^{(i)}\}_{i=1}^{+\infty}$ converges.

Note that the argumentation graph AF is not arbitrary chosen. We define the semantics $\mathrm{S}'(\mathbf{M})$ by let

$$(\mathtt{Str}_{AF}^{\mathbf{S}'(\mathbf{M})}(A_1), \cdots, \mathtt{Str}_{AF}^{\mathbf{S}'(\mathbf{M})}(A_n)) := \lim_{i\to+\infty} \mathbf{u}^{(i)},$$

for every $\mathrm{AF} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim}\rangle$.

If we let $i \to +\infty$ in (5.2), using the first four condition of Theorem 16, we obtain for every $k \in \{1, \cdots, n\}$

$$\mathtt{Str}_{AF}^{\mathbf{S}'(\mathbf{M})}(A_k) = \mathbf{f}\Bigg(\mathbf{w}(A_k), \mathbf{g}\Bigg(\mathbf{n}\Big((\mathtt{Str}_{AF}^{\mathbf{S}}(b_1^k), b_1^k), \cdots, (\mathtt{Str}_{AF}^{\mathbf{S}}(b_{n_k}^k), \cdots, b_{n_k}^k)\Big)\Bigg)\Bigg),$$

where $\{b_1^k, \cdots, b_{nk}^k\} = \mathtt{Att}_{AF}(A_k)$. Thus, the semantics $\mathbf{S}'(\mathbf{M})$ is based on the evaluation method $\mathbf{M}$.
Suppose now that there is another semantics $\mathbf{S}^*$ such that $\mathbf{S}^*$ is also based on the method $\mathbf{M}$. Then, for the vector

$$\mathbf{v} = (\mathtt{Str}_{AF}^{\mathbf{S}^*}(A_1), \cdots, \mathtt{Str}_{AF}^{\mathbf{S}^*}(A_n))$$

we have $\mathcal{Q}(\mathbf{v}) = \mathbf{v}$. Let us define the constant sequence $\mathbf{v}^{(i)} = \mathbf{v}$, for every $i \in \mathbb{N}$. Note that, by condition 1a (Def. 54), $\mathbf{v} \leq \mathbf{u}^{(1)}$.
Since both $\mathcal{Q}(\mathbf{v}^{(i)}) = \mathbf{v}^{(i+1)}$ and $\mathcal{Q}(\mathbf{u}^{(i)}) = \mathbf{u}^{(i+1)}$, applying $\mathcal{Q}$ to $\mathbf{v}^{(1)} \leq \mathbf{u}^{(1)}$ and using (5.3) and (5.4) we obtain

$$(\forall i \in \mathbb{N})\ \mathbf{v}^{(2i+1)} \leq \mathbf{u}^{(2i+1)}$$

and

$$(\forall i \in \mathbb{N})\ \mathbf{v}^{(2i)} \geq \mathbf{u}^{(2i)}.$$

By letting $i \to +\infty$ we obtain $\lim_{i\to+\infty} \mathbf{v}^{(i)} = \lim_{i\to+\infty} \mathbf{u}^{(i)}$, i.e.,

$$\mathtt{Str}_{AF}^{\mathbf{S}^*} \equiv \mathtt{Str}_{AF}^{\mathbf{S}'}.$$

□

5.2.2 Proofs of section 3.3: Principles of Gradual Semantics dealing with Similarity

Proof. **[Theorem 17]**
Let $\mathbf{S}$ be a gradual semantics based on a well-behaved evaluation method $\mathbf{M} = \langle \mathbf{f}, \mathbf{g}, \mathbf{n} \rangle$. $\mathbf{S}$ satisfies:

- **Reinforcement**:
  Let $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ be an arbitrary but fixed SSWAF, and $A, B \in \mathcal{A}$ such that:
  - $\mathbf{w}(A) = \mathbf{w}(B)$,
  - $\mathtt{Att}(A) \setminus \mathtt{Att}(B) = \{x\}$, $\mathtt{Att}(B) \setminus \mathtt{Att}(A) = \{y\}$,
  - $\forall z \in \mathtt{Att}(A) \cap \mathtt{Att}(B)$, $\mathtt{sim}(x, z) = \mathtt{sim}(y, z)$,
  - $\mathtt{Str}^{\mathbf{S}}(x) \leq \mathtt{Str}^{\mathbf{S}}(y)$.

From Definition 55, we have the following equations:

$$\texttt{Str}^{\mathbf{S}}(A) = \mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\bigg(\mathbf{n}\Big((\texttt{Str}^{\mathbf{S}}(B_1), B_1), \cdots, (\texttt{Str}^{\mathbf{S}}(B_k), B_k), (\texttt{Str}^{\mathbf{S}}(x), x)\Big)\bigg)\Bigg),$$

where $\{B_1, \cdots, B_k, x\} = \texttt{Att}(A)$.

$$\texttt{Str}^{\mathbf{S}}(B) = \mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\bigg(\mathbf{n}\Big((\texttt{Str}^{\mathbf{S}}(B_1), B_1), \cdots, (\texttt{Str}^{\mathbf{S}}(B_k), B_k), (\texttt{Str}^{\mathbf{S}}(y), y)\Big)\bigg)\Bigg),$$

where $\{B_1, \cdots, B_k, y\} = \texttt{Att}(B)$.

Case 1: $\texttt{Att}(A) \cap \texttt{Att}(B) = \emptyset$. Since $\mathbf{M}$ is well behaved, then from conditions 3b and 2b (both Def. 54) we have: $\mathbf{g}(\mathbf{n}((\texttt{Str}^{\mathbf{S}}(x), x))) = \mathbf{g}(\texttt{Str}^{\mathbf{S}}(x)) = \texttt{Str}^{\mathbf{S}}(x)$ and $\mathbf{g}(\mathbf{n}((\texttt{Str}^{\mathbf{S}}(y), y))) = \mathbf{g}(\texttt{Str}^{\mathbf{S}}(y)) = \texttt{Str}^{\mathbf{S}}(y)$. If $\mathbf{w}(A) = \mathbf{w}(B) = 0$, then from condition 1c (Def. 54) of well behaved $\mathbf{f}$, $\texttt{Str}^{\mathbf{S}}(A) = \texttt{Str}^{\mathbf{S}}(B) = 0$. If $\mathbf{w}(A) > 0$, then since $\texttt{Str}^{\mathbf{S}}(x) \leq \texttt{Str}^{\mathbf{S}}(y)$ and $\mathbf{w}(A) = \mathbf{w}(B) > 0$ and $\mathbf{f}$ is decreasing on the second variable (1a Def. 54), then $\texttt{Str}^{\mathbf{S}}(A) \geq \texttt{Str}^{\mathbf{S}}(B)$.

Case 2: $\texttt{Att}(A) \cap \texttt{Att}(B) \neq \emptyset$.
Since $\texttt{Str}^{\mathbf{S}}(x) \leq \texttt{Str}^{\mathbf{S}}(y)$ then from condition 3e (Def. 54) of a well behaved $\mathbf{M}$, $\mathbf{g}\bigg(\mathbf{n}\Big((\texttt{Str}^{\mathbf{S}}(B_1), B_1), \cdots, (\texttt{Str}^{\mathbf{S}}(B_k), B_k), (\texttt{Str}^{\mathbf{S}}(x), x)\Big)\bigg) \leq$
$\mathbf{g}\bigg(\mathbf{n}\Big((\texttt{Str}^{\mathbf{S}}(B_1), B_1), \cdots, (\texttt{Str}^{\mathbf{S}}(B_k), B_k), (\texttt{Str}^{\mathbf{S}}(y), y)\Big)\bigg)$.
If $\mathbf{w}(A) = 0$, then from condition 1c (Def. 54) of well behaved $\mathbf{f}$, $\texttt{Str}^{\mathbf{S}}(A) = \texttt{Str}^{\mathbf{S}}(B) = 0$.
If $\mathbf{w}(A) > 0$, since $\mathbf{f}$ is decreasing on the second variable and $\mathbf{w}(A) = \mathbf{w}(B) > 0$, then $\texttt{Str}^{\mathbf{S}}(A) \geq \texttt{Str}^{\mathbf{S}}(B)$.

- **Sensitivity to Similarity**:
Let $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \texttt{sim} \rangle$ be a $\texttt{SSWAF}$, and $A, B \in \mathcal{A}$ such that $\mathbf{w}(A) = \mathbf{w}(B)$ and there exists a bijective function $f : \texttt{Att}(A) \rightarrow \texttt{Att}(B)$ such that:

  - $\forall x \in \texttt{Att}(A)$, $\texttt{Str}^{\mathbf{S}}(x) = \texttt{Str}^{\mathbf{S}}(f(x))$,
  - $\forall x, y \in \texttt{Att}(A)$, $\texttt{sim}(x, y) \geq \texttt{sim}(f(x), f(y))$.

  Let $\texttt{Att}(A) = \{A_1, \cdots, A_k\}$, $\texttt{Att}(B) = \{B_1, \cdots, B_k\}$ and $\forall i \in \{1, \cdots, k\}$, $\texttt{Str}^{\mathbf{S}}(A_i) = \texttt{Str}^{\mathbf{S}}(B_i) = x_i$.

Hence $\mathtt{Str}^{\mathbf{S}}(A) = \mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\bigg(\mathbf{n}\Big((x_1, A_1), \cdots, (x_k, A_k)\Big)\bigg)\Bigg)$ and

$$\mathtt{Str}^{\mathbf{S}}(B) = \mathbf{f}\Bigg(\mathbf{w}(B), \mathbf{g}\bigg(\mathbf{n}\Big((x_1, B_1), \cdots, (x_k, B_k)\Big)\bigg)\Bigg).$$

From condition 3c (Def. 54), $\mathbf{g}\bigg(\mathbf{n}\Big((x_1, A_1), \cdots, (x_k, A_k)\Big)\bigg) \leq$

$$\mathbf{g}\bigg(\mathbf{n}\Big((x_1, B_1), \cdots, (x_k, B_k)\Big)\bigg).$$

Since $\mathbf{w}(A) = \mathbf{w}(B)$ and $\mathbf{f}$ is decreasing on the second variable (condition 1a from Definition 54), then $\mathtt{Str}^{\mathbf{S}}(A) \geq \mathtt{Str}^{\mathbf{S}}(B)$.

- **Monotony**:
  Let $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ be a SSWAF, and $A, B \in \mathcal{A}$, such that
  - $\mathbf{w}(A) = \mathbf{w}(B)$,
  - $\mathtt{Att}(A) \subset \mathtt{Att}(B)$,
  - If $\mathtt{Att}(A) \neq \emptyset$, then $\forall x \in \mathtt{Att}(B) \setminus \mathtt{Att}(A)$, $\forall y \in \mathtt{Att}(A)$, $\mathtt{sim}(x, y) = 0$.

Case 1. Assume that $\mathtt{Att}(A) = \emptyset$.
Hence, from Def. 54:

- $\mathbf{n}() = ()$ from 3a,
- $\mathbf{g}() = 0$ from 2a,
- $\mathbf{f}(x, 0) = x$ from 1b, i.e., $\mathtt{Str}^{\mathbf{S}}(A) = \mathbf{w}(A)$.

Let $\mathtt{Att}(B) = \{A_1, \cdots, A_k\}$.

Case 1.1. $k = 1$. Hence $\mathtt{Str}^{\mathbf{S}}(B) = \mathbf{f}\Bigg(\mathbf{w}(B), \mathbf{g}\bigg(\mathbf{n}\Big((x_1, A_1)\Big)\bigg)\Bigg)$.
From condition 3b (Def. 54), $\mathbf{n}((x_1, A_1)) = (x_1)$.
From condition 2b (Def. 54), $\mathbf{g}(x_1) = x_1$.
Hence, $\mathtt{Str}^{\mathbf{S}}(B) = \mathbf{f}(\mathbf{w}(B), x_1)$.
If $x_1 = 0$ then from 1b (Def. 54), $\mathtt{Str}^{\mathbf{S}}(B) = \mathbf{w}(B) = \mathtt{Str}^{\mathbf{S}}(A)$.
If $x_1 > 0$ then from 1a (Def. 54), $\mathbf{f}$ is decreasing on the second variable, thus $\mathbf{f}(\mathbf{w}(B), x_1) \leq \mathbf{f}(\mathbf{w}(B), 0)$, i.e. $\mathbf{f}(\mathbf{w}(B), x_1) \leq \mathbf{w}(B) = \mathtt{Str}^{\mathbf{S}}(A)$.

Case 1.2. $k > 1$. Let $\texttt{Str}^{\mathbf{S}}(B) = \mathbf{f}\Bigg(\mathbf{w}(B), \mathbf{g}\Big(x'_1, \cdots, x'_k\Big)\Bigg)$.

From conditions 2a, 2b and 2c (Def. 54), $\mathbf{g}() = \mathbf{g}(0) = \mathbf{g}(0, \cdots, 0) = 0$. From condition 2d (Def. 54) and because $\forall i \in \{1, \cdots, k\}$, $x'_i \geq 0$, $\mathbf{g}(0, \cdots, 0) \leq \mathbf{g}(x'_1, \cdots, x'_k)$. Since $\mathbf{w}(A) = \mathbf{w}(B)$, $\mathbf{f}$ is decreasing on the second variable then $\mathbf{f}(\mathbf{w}(A), \mathbf{g}(0, \cdots, 0)) \geq \mathbf{f}(\mathbf{w}(B), \mathbf{g}(x'_1, \cdots, x'_k))$, i.e. $\texttt{Str}^{\mathbf{S}}(A) \geq \texttt{Str}^{\mathbf{S}}(B)$.

Case 2. Assume that $\texttt{Att}(A) \neq \emptyset$.

Let $\texttt{Att}(A) = \{A_1, \cdots, A_k\}$ and $\texttt{Att}(B) = \{A_1, \cdots, A_k, B_1, \cdots, B_m\}$ such that $\forall i \in \{1, \cdots, k\}$, $\forall j \in \{1, \cdots, m\}$, $\texttt{sim}(A_i, B_j) = 0$.

$\texttt{Str}^{\mathbf{S}}(A) = \mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\Bigg(\mathbf{n}\Big((\texttt{Str}^{\mathbf{S}}(A_1), A_1), \cdots, (\texttt{Str}^{\mathbf{S}}(A_k), A_k)\Big)\Bigg)\Bigg) =$

$\mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\Big(x'_1, \cdots, x'_k\Big)\Bigg)$.

$\texttt{Str}^{\mathbf{S}}(B) = \mathbf{f}\Bigg(\mathbf{w}(B), \mathbf{g}\Bigg(\mathbf{n}\Big((\texttt{Str}^{\mathbf{S}}(A_1), A_1), \cdots, (\texttt{Str}^{\mathbf{S}}(A_k), A_k), (\texttt{Str}^{\mathbf{S}}(B_1),$

$B_1), \cdots, (\texttt{Str}^{\mathbf{S}}(B_m), B_m)\Big)\Bigg)\Bigg) = \mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\Big(x''_1, \cdots, x''_k, y'_1, \cdots, y'_m\Big)\Bigg)$.

Let us show first that $\forall i \in \{1, \cdots, k\}$, $x'_i = x''_i$.

From condition 3g (Def. 54), $\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k), (y_1, B_1)) = (\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k)), y_1)$ since $\forall i \in \{1, \cdots, k\}$, $\texttt{sim}(A_i, B_1) = 0$. We repeat the same operation, and get $\mathbf{n}((x_1, A_1), \cdots, (x_k, A_k), (y_1, B_1), \cdots, (y_m, B_m))$ $= (\mathbf{n}((x_1, A_1), \cdots, (x_k, A_K)), y_1, \cdots, y_m) = (x'_1, \cdots, x'_k, y_1, \cdots, y_m)$.

From condition 2c (Def. 54), $\mathbf{g}(x'_1, \cdots, x'_k, 0, \cdots, 0) = \mathbf{g}(x'_1, \cdots, x'_k)$.

From condition 2d (Def. 54), $\mathbf{g}(x'_1, \cdots, x'_k, 0, \cdots, 0) \leq \mathbf{g}(x'_1, \cdots, x'_k, y'_1, \cdots, y'_m)$, i.e. $\mathbf{g}(x'_1, \cdots, x'_k) \leq \mathbf{g}(x'_1, \cdots, x'_k, y'_1, \cdots, y'_m)$.

Since $\mathbf{w}(A) = \mathbf{w}(B)$ and $\mathbf{f}$ is decreasing on the second variable, then $\mathbf{f}(\mathbf{w}(A), \mathbf{g}(x'_1, \cdots, x'_k)) \geq \mathbf{f}(\mathbf{w}(A), \mathbf{g}(x'_1, \cdots, x'_k, y'_1, \cdots, y'_m))$. Therefore $\texttt{Str}^{\mathbf{S}}(A) \geq \texttt{Str}^{\mathbf{S}}(B)$.

- **Neutrality**:
  Let $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \texttt{sim} \rangle$ be a SSWAF, and $A, B \in \mathcal{A}$, such that

  - $\mathbf{w}(A) = \mathbf{w}(B)$,
  - $\texttt{Att}(B) = \texttt{Att}(A) \cup \{x\}$ with $\texttt{Str}^{\mathbf{S}}(x) = 0$,
  - If $\texttt{Att}(A) \neq \emptyset$, then $\forall y \in \texttt{Att}(A)$, $\texttt{sim}(x, y) = 0$.

Case 1. $\texttt{Att}(A) = \emptyset$:

$\texttt{Str}^{\mathbf{S}}(A) = \mathbf{w}(A)$ since $\mathbf{n}() = ()$, $\mathbf{g}() = 0$ and $\mathbf{f}(\mathbf{w}(A), 0) = \mathbf{w}(A)$ (from 3a, 2a, 1b of Def. 54). $\texttt{Str}^{\mathbf{S}}(B) = \mathbf{w}(B)$ since from 3b (Def. 54), $\mathbf{n}((0, x)) = 0$, from 2c (Def. 54) $\mathbf{g}(0) = \mathbf{g}()$, from 2a (Def. 54) $\mathbf{g}() = 0$ and from 1b (Def. 54) $\mathbf{f}(\mathbf{w}(B), 0) = \mathbf{w}(B) = \texttt{Str}^{\mathbf{S}}(B)$. Therefore $\texttt{Str}^{\mathbf{S}}(A) = \texttt{Str}^{\mathbf{S}}(B)$.

Case 2. $\texttt{Att}(A) \neq \emptyset$:

Let $\texttt{Att}(A) = \{B_1, \cdots, B_i\}$ and $\texttt{Att}(B) = \{B_1, \cdots, B_i, x\}$. $\texttt{Str}^{\mathbf{S}}(A) = \mathbf{f}\Bigg(\mathbf{w}(A), \mathbf{g}\bigg(\mathbf{n}\Big((x_1, B_1), \cdots, (x_i, B_i)\Big)\bigg)\Bigg) = \mathbf{f}(\mathbf{w}(A), \mathbf{g}(x'_1, \cdots, x'_i))$.

$\texttt{Str}^{\mathbf{S}}(B) = \mathbf{f}\Bigg(\mathbf{w}(B), \mathbf{g}\bigg(\mathbf{n}\Big((x_1, B_1), \cdots, (x_i, B_i), (0, x)\Big)\bigg)\Bigg)$.

From condition 3g (Def. 54) and given that $\forall y \in \texttt{Att}(A)$, $\texttt{sim}(x, y) = 0$, $\mathbf{g}\bigg(\mathbf{n}\Big((x_1, B_1), \cdots, (x_i, B_i), (0, x)\Big)\bigg) = \mathbf{g}((x'_1, \cdots, x'_i), 0)$. From 2c (Def. 54), $\mathbf{g}((x'_1, \cdots, x'_i), 0) = \mathbf{g}(x'_1, \cdots, x'_i)$. Since $\mathbf{w}(A) = \mathbf{w}(B)$, then $\texttt{Str}^{\mathbf{S}}(A) = \texttt{Str}^{\mathbf{S}}(B)$.

□

Proof. [**Theorem 18**]
Same reasoning as proof of Theorem 17 (Reinforcement):

Case 1. $\texttt{Att}(A) \cap \texttt{Att}(B) = \emptyset$.

$\texttt{Str}^{\mathbf{S}}(A) = \mathbf{f}(\mathbf{g}(\mathbf{n}((\texttt{Str}^{\mathbf{S}}(x), x))))$, $\texttt{Str}^{\mathbf{S}}(B) = \mathbf{f}(\mathbf{g}(\mathbf{n}((\texttt{Str}^{\mathbf{S}}(y), y))))$, $\texttt{Str}^{\mathbf{S}}(x) < \texttt{Str}^{\mathbf{S}}(y)$ then $\texttt{Str}^{\mathbf{S}}(A) > \texttt{Str}^{\mathbf{S}}(B)$.

Case 2. $\texttt{Att}(A) \cap \texttt{Att}(B) \neq \emptyset$.

Use the condition C1 instead of condition 3e (Def. 54) and we obtain $\texttt{Str}^{\mathbf{S}}(A) > \texttt{Str}^{\mathbf{S}}(B)$.

□

Proof. [**Theorem 19**]
Same reasoning as proof of Theorem 17 (Sensitivity to Similarities), but using the condition C2 instead of condition 3c (Def. 54). □

Proof. [**Theorem 20**]
Same reasoning as proof of Theorem 17 (Monotony), but using the condition C3 instead of condition 2d (Def. 54). □

5.2.3 Proof of section 3.4: Novel Family of Semantics

Proof. [**Theorem 21**]

Follows from proof of Theorem 17. □

5.2.4 Proofs of section 3.5: Adjustment Functions

Proof. [**Proposition 24**]

Let see for $\mathbf{n}_{\mathtt{wh}}$.

It has been shown in Amgoud and Doder [2019] that the semantics $\mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}$ always returns a degree between 0 and 1. Since the adjustment function $\mathbf{n}_{\mathtt{wh}}$ is based on $\mathtt{Str}^{\mathbf{S}_{\mathtt{wh}}}$, it therefore also returns readjusted values between 0 and 1.

For $\mathbf{n}_{\mathtt{rs}}$.

From its definition, we know that each readjusted value is computed as follows: $\underset{x_i \in \{x_1,\cdots,x_k\}\setminus\{x_1\}}{\mathtt{avg}} \left(\frac{\mathtt{avg}(x_1,x_i)\times(2-\mathtt{sim}(A_1,A_i))}{2} \right)$. Given that $\mathtt{avg}(x_1, x_i) \in [0,1]$ and, $2 - \mathtt{sim}(A_1, A_i)) \in [1,2]$ then $\frac{\mathtt{avg}(x_1,x_i)\times(2-\mathtt{sim}(A_1,A_i))}{2} \in [0,1]$ (and averaging between values belonging to $[0,1]$ keeps this interval).

For $\mathbf{n}^{\rho}_{\mathtt{max}}$.

Let a permutation ρ of $\{1,\cdots,k\}$, $x_1,\cdots,x_k \in [0,1]$.We denote the returned vector of the parameterised function $\mathbf{n}^{\rho}_{\mathtt{max}}((x_1,B_1),\cdots,(x_k,B_k)) = (x'_{\rho(1)},\cdots,x'_{\rho(k)})$. From Definition 58, $x'_{\rho(1)} = x_{\rho(1)}$ and the permutation doesn't change the value therefore $x'_{\rho(1)} \in [0,1]$.

Moreover from Definition 1, we know that $\mathtt{sim} : \mathcal{A} \times \mathcal{A} \to [0,1]$. Additionally with Definition 58, we can compute the domain $\forall i \in \{2,\cdots,k\}$, of $x'_{\rho(i)}$. We have the equation $x'_{\rho(i)} = x_{\rho(i)} \cdot (1 - \mathtt{max}(\mathtt{sim}(B_{\rho(1)}, B_{\rho(i)}),\cdots,\mathtt{sim}(B_{\rho(i-1)}, B_{\rho(i)})))$.

We can rewrite the equation in a more simplifying way: $x'_i = x_i \cdot (1-s)$ where $x_i, s \in [0,1]$. Given that $(1-s) \in [0,1]$, the product of two values belonging to $[0,1]$ stay in $[0,1]$, that's why $x'_i \in [0,1]$. Do the same for any x'_i, $\forall i \in \{2,\cdots,k\}$, $x'_{\rho(i)} \in [0,1]$. □

Proof. [**Proposition 25**]

Let $\mathbf{f}$, $\mathbf{g}$ be well-behaved functions.

- 3a) $\mathbf{n}_{\mathtt{rs}}() = ()$, from Definition 57,
- 3b) $\mathbf{n}_{\mathtt{rs}}((x_1, A_1)) = (x_1)$, from Definition 57,
- 3c) Let $x_1,\cdots,x_k \in [0,1]$, $A_1,\cdots,A_k,B_1,\cdots,B_k \in \mathtt{Arg}$ such that $\forall i,j \in \{1,\cdots,k\}$ $i \neq j$, $\mathtt{sim}(A_i,A_j) \geq \mathtt{sim}(B_i,B_j)$. For any $i \in \{1,\cdots,k\}$,

$\underset{x_i\in\{x_1,\cdots,x_k\}\setminus\{x_j\}}{\texttt{avg}}\left(\frac{\texttt{avg}(x_j,x_i)\times(2-\texttt{sim}(A_j,A_i))}{2}\right) \leq \underset{x_i\in\{x_1,\cdots,x_k\}\setminus\{x_j\}}{\texttt{avg}}\left(\frac{\texttt{avg}(x_j,x_i)\times(2-\texttt{sim}(B_j,B_i))}{2}\right)$.
Therefore $\mathbf{g}(\mathbf{n_{rs}}((x_1, A_1), \cdots, (x_k, A_k))) \leq \mathbf{g}(\mathbf{n_{rs}}((x_1, B_1), \cdots, (x_k, B_k)))$.

- 3d) From Definition 57, if $\mathbf{n_{rs}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$, then each $x'_i = \underset{x_i\in\{x_1,\cdots,x_k\}\setminus\{x_j\}}{\texttt{avg}}\left(\frac{\texttt{avg}(x_j,x_i)\times(2-\texttt{sim}(A_j,A_i))}{2}\right)$, i.e. given that $\texttt{sim}(A_j, A_i) \leq 1$ if $\exists i \in \{1, \cdots, k\}$ such that $x_i > 0$ then $\forall i \in \{1, \cdots, k\}$, $x'_i > 0$.

- 3e) Let $A_1, \cdots, A_k \in \texttt{Arg}$, $x_1, \cdots, x_k, y_1, \cdots, y_k \in [0, 1]$ such that $\forall i \in \{1, \cdots, k\}$, $x_i \leq y_i$.
  From Definition 57, $\forall i \in \{1, \cdots, k\}$, $\underset{x_i\in\{x_1,\cdots,x_k\}\setminus\{x_j\}}{\texttt{avg}}\left(\frac{\texttt{avg}(x_j,x_i)\times(2-\texttt{sim}(A_j,A_i))}{2}\right) \leq \underset{y_i\in\{y_1,\cdots,y_k\}\setminus\{y_j\}}{\texttt{avg}}\left(\frac{\texttt{avg}(y_j,y_i)\times(2-\texttt{sim}(A_j,A_i))}{2}\right)$

- 3f) $\mathbf{n_{rs}}$ is symmetric because the average, the product and the similarity are symmetric.

- 3g) $\mathbf{n_{rs}}$ violate this condition: Let $x_1 = 0.4, x_2 = 0.8$ and $A_1, A_2 \in \texttt{Arg}$ such that $\texttt{sim}(A_1, A_2) = 0$. However, $\mathbf{n_{rs}}((x_1, A_1)) = (0.4)$, $\mathbf{n_{rs}}((x_1, A_1), (x_2, A_2)) = (0.6, 0.6)$ and so $(0.4, 0.8) \neq (0.6, 0.6)$.

Let g such that it satisfies the condition (C3), i.e. $\mathbf{g}(x_1, \cdots, x_k, y) < \mathbf{g}(x_1, \cdots, x_k, z)$ if $y < z$.

- C1: Let $A_1, \cdots, A_k \in \texttt{Arg}$, $x_1, \cdots, x_k, y_1, \cdots, y_k \in [0, 1]$ such that $\forall i = 1, \cdots, k$, $x_i \leq y_i$ and $\exists i = 1, \cdots, k$ s.t. $x_i < y_i$. Let $\mathbf{n_{rs}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$ and $\mathbf{n_{rs}}((y_1, A_1), \cdots, (y_k, A_k)) = (y'_1, \cdots, y'_k)$.
  Since only the strength increases for every x_i compared to y_i and because for any $\texttt{sim}(A_j, A_i)$, $\underset{x_i\in\{x_1,\cdots,x_k\}\setminus\{x_j\}}{\texttt{avg}}\left(\frac{\texttt{avg}(x_j,x_i)\times(2-\texttt{sim}(A_j,A_i))}{2}\right)$ is strictly increasing on x_i and x_j, therefore $\mathbf{g}(\mathbf{n_{rs}}((x_1, A_1), \cdots, (x_k, A_k))) < \mathbf{g}(\mathbf{n_{rs}}((y_1, A_1), \cdots, (y_k, A_k)))$.

- C2: Let $x_1, \cdots, x_k \in [0, 1]$, $A_1, \cdots, A_k, B_1, \cdots, B_k \in \texttt{Arg}$ such that $\forall i, j \in \{1, \cdots, k\}$, $\texttt{sim}(A_i, A_j) \geq \texttt{sim}(B_i, B_j)$ and $\exists i, j \in \{1, \cdots, k\}$ s.t. $\texttt{sim}(A_i, A_j) > \texttt{sim}(B_i, B_j)$ and ($x_i > 0$ or $x_j > 0$). Let $\mathbf{n_{rs}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$ and $\mathbf{n_{rs}}((x_1, B_1), \cdots, (x_k, B_k)) = (y'_1, \cdots, y'_k)$.
  Hence $\forall i \in \{1, \cdots, k\}$, $x'_i \leq y'_i$. Moreover, since $\exists i, j \in \{1, \cdots, k\}$ s.t. $\texttt{sim}(A_i, A_j) > \texttt{sim}(B_i, B_j)$ and ($x_i > 0$ or $x_j > 0$), then $\exists i \in \{1, \cdots, k\}$ s.t. $x'_i < y'_i$. Therefore $\mathbf{g}(\mathbf{n_{rs}}((x_1, A_1), \cdots, (x_k, A_k))) < \mathbf{g}(\mathbf{n_{rs}}((x_1, B_1), \cdots, (x_k, B_k)))$.

□

Proof. [**Proposition 26**] Let $\mathbf{f}$, $\mathbf{g}$ be well-behaved functions.

- $\mathbf{n}^{\rho}_{\mathtt{max}}$ is well-behaved.

  - 3a) $\mathbf{n}^{\rho}_{\mathtt{max}}() = ()$, from Definition 58,
  - 3b)$\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1)) = (x_1)$, from Definition 58 $x'_{\rho(1)} = x_{\rho(1)}$.
  - 3c) $\mathbf{g}(\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k))) \leq \mathbf{g}(\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, B_1), \cdots, (x_k, B_k)))$ if $\forall i$, $j \in \{1, \cdots, k\}$, $\mathtt{sim}(A_i, A_j) \geq \mathtt{sim}(B_i, B_j)$. For any permutation ρ which respect the condition in Definition 58, let $\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$ and $\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, B_1), \cdots, (x_k, B_k)) = (y'_1, \cdots, y'_k)$. Given that the first element of the tuple doesn't change, $x'_1 = y'_1$. For the other value only the similarity score change between the computation of $x'_{\rho(i)}$ and $y'_{\rho(i)}$ such that $\forall i, j \in \{1, \cdots, k\}$ $i \neq j$, $\mathtt{sim}(A_i, A_j) \geq \mathtt{sim}(B_i, B_j)$. Therefore $\forall i \in \{2, \cdots, k\}$, $x'_i \leq y'_i$. Moreover from condition 2d (Def. 54), $\mathbf{g}$ is monotonic, then $\mathbf{g}(\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k))) \leq \mathbf{g}(\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, B_1), \cdots, (x_k, B_k)))$.

  - 3d) Let $x_1, \cdots, x_k \in [0, 1]$ and ρ a permutation of $\{x_1, \cdots, x_k\}$ such that if $x_{\rho(i)} = 0$ then $x_{\rho(i+1)} = 0$ $\forall i < k$, or $i = k$. Then if there exists a $x_i > 0$, from the above condition $x_{\rho(1)} > 0$. Therefore from conditions 2b, 2c and 2d (Def. 54), $\mathbf{g}(\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k))) \geq x_{\rho(1)} > 0$.
  - 3e) Let $x_1, \cdots, x_k, y_1, \cdots, y_k \in [0, 1]$ such that $\forall i \in \{1, \cdots, k\}$, $x_i \leq y_i$. From Definition 58, $x'_{\rho(1)} = x_{\rho(1)}$ and $\forall i \in \{2, \cdots, k\}$, $x'_{\rho(i)} = x_{\rho(i)} \cdot (1 - \mathtt{max}(\mathtt{sim}(B_{\rho(1)}, B_{\rho(i)}), \cdots, \mathtt{sim}(B_{\rho(i-1)}, B_{\rho(i)})))$ and same for any $y'_{\rho(i)}$.

    Case 1. $i = 1$, $x_{\rho(1)} \leq y_{\rho(1)}$ then $x'_{\rho(1)} \leq y'_{\rho(1)}$.

    Case 2. $i > 1, \forall i \in \{2, \cdots, k\},\ x_i \cdot (1 - s_i) \leq y_i \cdot (1 - s_i)$, because $x_i \leq y_i$.

    Therefore $\forall i \in \{1, \cdots, k\}$, $x'_{\rho(i)} \leq y'_{\rho(i)}$. Add the condition 2d (Def. 54), $\mathbf{g}$ is monotonic, thus $\mathbf{g}(\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k))) \leq \mathbf{g}(\mathbf{n}^{\rho}_{\mathtt{max}}((y_1, A_1), \cdots, (y_k, A_k)))$.
  - 3f) $\mathbf{n}^{\rho}_{\mathtt{max}}$ is symmetric thanks to the fixed permutation ρ.
  - 3g) Let $x_{k+1} \in [0, 1]$, $A_{k+1} \in \mathtt{Arg}$, $\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_{\rho(1)}, \cdots, x'_{\rho(k)})$. If $\forall i \in \{1, \cdots, k\}$, $\mathtt{sim}(A_i, A_{k+1}) = 0$, then either A_{k+1} is not the maximal similarity with A_i or it is with 0.

Case 1. $\mathtt{sim}(A_i, A_{k+1})$ is not the maximal score of similarity for A_i, then A_{k+1} doesn't affect the strength of $x'_{\rho(i)}$.

Case 2. $\mathtt{sim}(A_i, A_{k+1}) = 0$ is the maximal score of similarity: then the strength of $x'_{\rho(i)} = x_{\rho(i)} \cdot 1 = x_{\rho(i)}$, i.e. with or without A_{k+1} the strength of $x'_{\rho(i)}$ doesn't change.

Do this reasoning for each other arguments, we obtain that A_{k+1} doesn't affect the strength of any $x'_{\rho(i)}$. More, having 0 similarity with any other attackers implies that the maximal similarity of A_{k+1} is 0, then $x_{k+1} = x'_{k+1}$. Therefore $\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_{k+1}, A_{k+1})) = (\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k)), x_{k+1})$.

- $\mathbf{n}^{\rho}_{\mathtt{max}}$ is continuous on each numerical variables.
From Definition 58, each variables are compute by polynomial function ($x'_{\rho(i)} = x_{\rho(i)} \cdot (1 - \mathtt{max}(\mathtt{sim}(B_{\rho(1)}, B_{\rho(i)}), \cdots, \mathtt{sim}(B_{\rho(i-1)}, B_{\rho(i)}))))$ which are continuous function.

- Let $\lambda \in [0, 1]$, and $\mathbf{g}$ well-behaved such that $\mathbf{g}(\lambda x'_1, \cdots, \lambda x'_k) \geq \lambda \mathbf{g}(x'_1, \cdots, x'_k)$. Let $\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, B_1), \cdots, (x_k, B_k)) = (x'_1, \cdots, x'_k)$, then let us show that $\mathbf{n}^{\rho}_{\mathtt{max}}((\lambda x_1, B_1), \cdots, (\lambda x_k, B_k)) = (\lambda x'_1, \cdots, \lambda x'_k)$.

$$\begin{aligned}
&\mathbf{n}^{\rho}_{\mathtt{max}}((\lambda x_1, B_1), \cdots, (\lambda x_k, B_k)) = \\
&\quad \Big(\lambda x_{\rho(1)}, \\
&\quad \lambda x_{\rho(2)} \cdot (1 - \mathtt{max}(\mathtt{sim}(B_{\rho(1)}, B_{\rho(2)}))), \\
&\quad \cdots, \\
&\quad \lambda x_{\rho(k)} \cdot (1 - \mathtt{max}(\mathtt{sim}(B_{\rho(1)}, B_{\rho(k)}), \cdots, \mathtt{sim}(B_{\rho(k-1)}, B_{\rho(k)})))\Big) \\
&= (\lambda x'_{\rho(1)}, \cdots, \lambda x'_{\rho(k)}).
\end{aligned}$$

- Let $\mathbf{g}_{\mathtt{sum}}$ which is well-behaved and $\rho_{\mathtt{max}}$ decreasing permutation according to the strength.

C1. Let $x_1 = 1, x_2 = 0.5, y_1 = 1, y_2 = 0.8, A_1, A_2 \in \mathtt{Arg}$ such that $\mathtt{sim}(A_1, A_2) = 1$. $\mathbf{g}_{\mathtt{sum}}(\mathbf{n}^{\rho_{\mathtt{max}}}_{\mathtt{max}}((x_1, A_1), (x_2, A_2))) = (1, 0)$ as $\mathbf{g}_{\mathtt{sum}}(\mathbf{n}^{\rho_{\mathtt{max}}}_{\mathtt{max}}((y_1, A_1), (y_2, A_2))) = (1, 0)$, because $0.5 \cdot (1 - 1) = 0.8 \cdot (1 - 1) = 0$.

C2. Let $x_1 = x_2 = 1, x_3 = 0.5, A_1, A_2, A_3, B_1, B_2, B_3 \in \mathtt{Arg}$ such that $\mathtt{sim}(A_1, A_2) = 0.8$, $\mathtt{sim}(A_1, A_3) = 0.8$, $\mathtt{sim}(A_2, A_3) = 0.5$ and $\mathtt{sim}(B_1, B_2) = 0.8$, $\mathtt{sim}(B_1, B_3) = 0.8$, $\mathtt{sim}(B_2, B_3) = 0.3$. $\mathbf{g}_{\mathtt{sum}}(\mathbf{n}^{\rho_{\mathtt{max}}}_{\mathtt{max}}((x_1, A_1), (x_2, A_2), (x_3, A_3))) = (1, 0.2, 0.1)$ as $\mathbf{g}_{\mathtt{sum}}(\mathbf{n}^{\rho_{\mathtt{max}}}_{\mathtt{max}}((x_1, B_1), (x_2, B_2), (x_3, B_3))) = (1, 0.2, 0.1)$, because $0.5 \cdot (1 - \mathtt{max}(0.8, 0.5)) = 0.5 \cdot (1 - \mathtt{max}(0.8, 0.3)) = 0.1$.

□

Proof. [**Proposition 27**] This result follows the Proposition 26. □

Proof. [**Proposition 28**] Let $\mathbf{f}$, $\mathbf{g}$ be well-behaved functions.

- $\mathbf{n}_{\texttt{wh}}$ is well-behaved.
  - 3a) $\mathbf{n}_{\texttt{wh}}() = ()$, from Definition 59,
  - 3b) $\mathbf{n}_{\texttt{wh}}((x_1, A_1)) = (x_1)$, from Definition 59 $\texttt{Str}^{\mathbf{S}_{\texttt{wh}}}(A_1) = \frac{x_1}{1}$.
  - 3c) $\mathbf{g}(\mathbf{n}_{\texttt{wh}}((x_1, A_1), \cdots, (x_k, A_k))) \leq \mathbf{g}(\mathbf{n}_{\texttt{wh}}((x_1, B_1), \cdots, (x_k, B_k)))$ if $\forall i, j \in \{1, \cdots, k\}$, $\texttt{sim}(A_i, A_j) \geq \texttt{sim}(B_i, B_j)$. From Amgoud and Doder [2019], we know that the Weighted h-Categoriser semantics satisfies the principle Attack-Sensitivity (Principle 12) which means for $\mathbf{n}_{\texttt{wh}}$ a Sensitivity to Similarity. Then let $\mathbf{n}_{\texttt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$ and $\mathbf{n}_{\texttt{wh}}((x_1, B_1), \cdots, (x_k, B_k)) = (y'_1, \cdots, y'_k)$, hence $\forall i \in \{1, \cdots, k\}$, $x'_i \leq y'_i$. Given that $\mathbf{g}$ is well behaved therefore $\mathbf{g}(x'_1, \cdots, x'_k) \leq \mathbf{g}(y'_1, \cdots, y'_k)$.
  - 3d) Let $\exists i \in \{1, \cdots, k\}$ s.t. $x_i > 0$. From Amgoud and Doder [2019], we know that the Weighted h-Categoriser semantics satisfies the principle Resilience (Principle 8) which means in this case that $\mathbf{n}_{\texttt{wh}}$ will return a value $x'_i > 0$. Then from Definition 54, with conditions (b), (c) and (d), we have $\mathbf{g}(\mathbf{n}_{\texttt{wh}}((x_1, A_1), \cdots, (x_k, A_k))) \geq x'_i > 0$.
  - 3e) Let $A_1, \cdots, A_k \in \texttt{Arg}$ and $x_1, \cdots, x_k, y_1, \cdots, y_k \in [0, 1]$ such that $\forall i \in \{1, \cdots, k\}$, $x_i \leq y_i$. Let $\mathbf{n}_{\texttt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$ and $\mathbf{n}_{\texttt{wh}}((y_1, A_1), \cdots, (y_k, A_k)) = (y'_1, \cdots, y'_k)$. Since the similarities are the same because they are the same arguments, this means in this case that $\forall i \in \{1, \cdots, k\}$, $\frac{x_i}{1+\sum\limits_{j \in \{1,\cdots,k\} \setminus \{i\}} x'_j \times \texttt{sim}(A_j, A_i)} \leq \frac{y_i}{1+\sum\limits_{j \in \{1,\cdots,k\} \setminus \{i\}} y'_j \times \texttt{sim}(A_j, A_i)}$,
    i.e. $x'_i \leq y'_i$.
  - 3f) $\mathbf{n}_{\texttt{wh}}$ is symmetrical because for any couple (x_i, A_i), its new value x'_i is calculated thanks to a system of equations and therefore does not depend on their order.
  - 3g) Let $x_1, \cdots, x_{k+1} \in [0, 1]$ and $A_1, \cdots, A_{k+1} \in \texttt{Arg}$ such that $\forall i \in \{1, \cdots, k\}$, $\texttt{sim}(A_i, A_{k+1}) = 0$. Let $\mathbf{n}_{\texttt{wh}}((x_1, A_1), \cdots, (x_{k+1}, A_{k+1})) = (x'_1, \cdots, x'_{k+1})$ and $\mathbf{n}_{\texttt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (y'_1, \cdots, y'_k)$. From Definition 59:
    For any $i \in \{1, \cdots, k\}$:

$$y'_i = \frac{x_i}{1 + \sum\limits_{j \in \{1,\cdots,k\} \setminus \{i\}} y'_j \times \texttt{sim}(A_j, A_i)}$$

and

$$x'_i = \frac{x_i}{1 + \sum\limits_{j \in \{1,\cdots,k\}\setminus\{i\}} x'_j \times \mathtt{sim}(A_j, A_i) + x'_{k+1} \times 0}$$

that means $\forall i \in \{1, \cdots, k\}$, $x'_i = y'_i$.

For $i = k + 1$:

$$x'_{k+1} = \frac{x_{k+1}}{1 + \sum\limits_{j \in \{1,\cdots,k\}} x'_j \times 0} = \frac{x_{k+1}}{1} = x_{k+1}.$$

- $\mathbf{n}_{\mathtt{wh}}$ is continuous on each numerical variables.
  From Definition 59, all variables are computed by using continuous operators.

- Let $\lambda \in [0, 1]$, and $\mathbf{g}$ well-behaved such that $\mathbf{g}(\lambda x'_1, \cdots, \lambda x'_k) \geq \lambda \mathbf{g}(x'_1, \cdots, x'_k)$. Let $\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$, then let us show that $\mathbf{n}_{\mathtt{wh}}((\lambda x_1, A_1), \cdots, (\lambda x_k, A_k)) \geq (\lambda x'_1, \cdots, \lambda x'_k)$.
  For any $i \in \{1, \cdots, k\}$:

$$x'_i = \frac{x_i}{1 + \sum\limits_{j \in \{1,\cdots,k\}\setminus\{i\}} x'_j \times \mathtt{sim}(A_j, A_i)}$$

$$y'_i = \frac{\lambda x_i}{1 + \sum\limits_{j \in \{1,\cdots,k\}\setminus\{i\}} y'_j \times \mathtt{sim}(A_j, A_i)}$$

  If $\lambda \in \{0, 1\}$ then $\lambda x'_i = y'_i$. If $\lambda \in]0, 1[$ then $\lambda x'_i \leq y'_i$, because: if we reduce the score of the y'_i by multiplying in the equations λx_i then it reduces the y'_i but if the y'_i reduces then it reduces the y'_j and therefore it increases y'_i. This is why applying λ in the equations decreases less than applying it after solving the system of equations.

- Let g such that it satisfies the condition (C3), i.e. $\mathbf{g}(x_1, \cdots, x_k, y) < \mathbf{g}(x_1, \cdots, x_k, z)$ if $y < z$.

  C1. Let $A_1, \cdots, A_k \in \mathtt{Arg}$ and $x_1, \cdots, x_k, y_1, \cdots, y_k \in [0, 1]$. Let $\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$ and $\mathbf{n}_{\mathtt{wh}}((y_1, A_1), \cdots, (y_k, A_k)) = (y'_1, \cdots, y'_k)$. With the same reasoning as in the proof for the condition 3e) of the Definition 54, if we add the knowledge that $\exists i \in \{1, \cdots, k\}$ such that $x_i < y_i$ then $\frac{x_i}{1+\sum_{j \in \{1,\cdots,k\}\setminus\{i\}} x'_j \times \mathtt{sim}(A_j, A_i)} < \frac{y_i}{1+\sum_{j \in \{1,\cdots,k\}\setminus\{i\}} y'_j \times \mathtt{sim}(A_j, A_i)}$. Therefore $\forall j \in \{1, \cdots, k\}$, $x'_j \leq y'_j$ and $x'_i < y'_i$. Given that $\mathbf{g}$ satisfies the condition (C3), hence $\mathbf{g}(\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k))) < \mathbf{g}(\mathbf{n}_{\mathtt{wh}}((y_1, A_1), \cdots, (y_k, A_k)))$.

  C2. Let $x_1, \cdots, x_k \in [0, 1]$, $A_1, \cdots, A_k, B_1, \cdots, B_k \in \mathtt{Arg}$ such that $\forall i, j \in \{1, \cdots, k\}$, $\mathtt{sim}(A_i, A_j) \geq \mathtt{sim}(B_i, B_j)$ and $\exists i, j \in \{1, \cdots, k\}$ s.t. $\mathtt{sim}(A_i, A_j)$

$> \mathtt{sim}(B_i, B_j)$ and ($x_i > 0$ or $x_j > 0$). Let $\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$ and $\mathbf{n}_{\mathtt{wh}}((x_1, B_1), \cdots, (x_k, B_k)) = (y'_1, \cdots, y'_k)$. With the same reasoning as in the proof for the condition 3c) of the Definition 54, adding the knowledge that $\exists i, j \in \{1, \cdots, k\}$ s.t. $\mathtt{sim}(A_i, A_j) > \mathtt{sim}(B_i, B_j)$ and ($x_i > 0$) then $\frac{x_i}{1+\sum\limits_{j\in\{1,\cdots,k\}\setminus\{i\}} x'_j\times\mathtt{sim}(A_j,A_i)} < \frac{x_i}{1+\sum\limits_{j\in\{1,\cdots,k\}\setminus\{i\}} y'_j\times\mathtt{sim}(B_j,B_i)}$. Therefore $\forall j \in \{1, \cdots, k\}$, $x'_j \leq y'_j$ and $x'_i < y'_i$. Given that g satisfies the condition (C3), hence $\mathbf{g}(\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k))) < \mathbf{g}(\mathbf{n}_{\mathtt{wh}}((x_1, B_1), \cdots, (x_k, B_k)))$.

- Let $\lambda \in [0, 1]$, $x_1, \cdots, x_k \in [0, 1]$ and $\mathbf{f}, \mathbf{g}$ are well behaved such that $\mathbf{g}(\lambda x_1, \cdots, \lambda x_k) \geq \lambda\mathbf{g}(x_1, \cdots, x_k)$. From the previous proofs it follows from Theorem 16 that $\langle \mathbf{f}, \mathbf{g}, \mathbf{n}_{\mathtt{wh}}\rangle \in \mathbf{M}^*$.

□

Proof. [**Proposition 29**] For $\mathbf{n}_{\mathtt{rs}}$:
Let $A_1, A_2 \in \mathtt{Arg}$ such that $\mathtt{sim}(A_1, A_2) = 1$ and $x_1 = 0.4$, $x_2 = 0.6$. $\mathbf{n}_{\mathtt{rs}}((x_1, A_1), (x_2, A_2)) = (x'_1, x'_2) = (0.5, 0.5)$. Then $x'_1 > x_1$.

For $\mathbf{n}^{\rho}_{\mathtt{max}}$:
Let $A_1, \cdots, A_k \in \mathtt{Arg}$, $x_1, \cdots, x_k$ and $\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$. From Definition 58, $\forall i \in \{1, \cdots, k\}$, either $x'_i = x_i$ or $x'_i = x_i \cdot X$ such that $X \in [0, 1]$ then $x'_i \leq x_i$.

For $\mathbf{n}_{\mathtt{wh}}$:
Let $\mathbf{G} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim}\rangle$ be a SSWAF, $A_1, \cdots, A_n \in \mathcal{A}$ and $x_1, \cdots, x_n \in [0, 1]$ such that $\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (\mathtt{Str}(A_1), \cdots, \mathtt{Str}(A_k))$. For any $i \in \{1, \cdots, n\}$, from Definition 59, $\mathtt{Str}(A_i) = \frac{x_i}{1+X}$ such that $X \in [0, +\infty[$ therefore $\mathtt{Str}(A_i) \leq x_i$. □

Proof. [**Proposition 30**] For $\mathbf{n}_{\mathtt{rs}}$:
Let $A_1, A_2 \in \mathtt{Arg}$ such that $\mathtt{sim}(A_1, A_2) = 1$ and $x_1 = 0, x_2 = 0.2$. $\mathbf{n}_{\mathtt{rs}}((x_1, A_1), (x_2, A_2)) = (x'_1, x'_2) = (0.1, 0.1)$. Then $x'_2 \neq x_2$.

For $\mathbf{n}^{\rho}_{\mathtt{max}}$:
Let $\mathtt{sim}$ a similarity measure, $A_1, \cdots, A_k \in \mathtt{Arg}$ and $x_1, \cdots, x_k \in [0, 1]$. From Definition 58, we know that ρ is a fixed permutation on the set $\{1, \cdots, k\}$ such that if $x_{\rho(i)} = 0$ then $x_{\rho(i+1)} = 0$ $\forall i < k$, or $i = k$. Additionally, each adjusted value is computed as follows: $x_{\rho(k)} \cdot (1 - \mathtt{max}(\mathtt{sim}(A_{\rho(1)}, A_{\rho(k)}), \cdots, \mathtt{sim}(A_{\rho(k-1)}, A_{\rho(k)})))$. Therefore, $\nexists i \in \{2, \cdots, k\}$, such that if $x_{\rho(i)} \neq 0$ then $x_{\rho(i-1)} = 0$. Consequently, initially strictly positive values are

not affected by values of 0 (no matter how similar they are) and 0 values remain at 0 anyway.

For $\mathbf{n}_{\mathtt{wh}}$:
Let $\mathbf{G} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ be a SSWAF, $A_1, \cdots, A_k, B_1 \in \mathcal{A}$ and $x_1, \cdots, x_k, y \in [0, 1]$ such that $y = 0$. From Definition 59 we have $\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (\mathtt{Deg_1}(A_1), \cdots, \mathtt{Deg_1}(A_k)) = \mathtt{Deg_1}^{\mathbf{S}_{\mathtt{wh}}}_{G'}$, where

$$\mathtt{Deg_1}^{\mathbf{S}_{\mathtt{wh}}}_{G'} = \begin{cases} \mathtt{Deg_1}(A_1) = \frac{x_1}{1+\mathtt{Deg_1}(A_2)\times\mathtt{sim}(A_1,A_2)+\cdots+\mathtt{Deg_1}(A_k)\times\mathtt{sim}(A_1,A_k)} \\ \cdots \\ \mathtt{Deg_1}(A_k) = \frac{x_k}{1+\mathtt{Deg_1}(A_1)\times\mathtt{sim}(A_k,A_1)+\cdots+\mathtt{Deg_1}(A_{n-1})\times\mathtt{sim}(A_n,A_{n-1})} \end{cases}$$

and $\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k), (y, B_1)) = (\mathtt{Deg_2}(A_1), \cdots, \mathtt{Deg_2}(A_k), \mathtt{Deg_2}(B_1)) = \mathtt{Deg_2}^{\mathbf{S}_{\mathtt{wh}}}_{G'}$, where

$$\mathtt{Deg_2}^{\mathbf{S}_{\mathtt{wh}}}_{G'} = \begin{cases} \mathtt{Deg_2}(A_1) = \frac{x_1}{1+\mathtt{Deg_2}(A_2)\times\mathtt{sim}(A_1,A_2)+\cdots+\mathtt{Deg_2}(A_k)\times\mathtt{sim}(A_1,A_k)+\mathtt{Deg_2}(B_1)\times\mathtt{sim}(A_1,B_1)} \\ \cdots \\ \mathtt{Deg_2}(A_k) = \frac{x_k}{1+\mathtt{Deg_2}(A_1)\times\mathtt{sim}(A_k,A_1)+\cdots+\mathtt{Deg_2}(A_{n-1})\times\mathtt{sim}(A_n,A_{n-1})+\mathtt{Deg_2}(B_1)\times\mathtt{sim}(A_k,B_1)} \\ \mathtt{Deg_2}(B_1) = \frac{y_1}{1+\mathtt{Deg_2}(A_1)\times\mathtt{sim}(B_1,A_1)+\cdots+\mathtt{Deg_2}(A_n)\times\mathtt{sim}(B_1,A_n)} \end{cases}$$

Given that $y = 0$, $\mathtt{Deg_2}(B_1) = 0$, so for every $i \in \{1, \cdots, k\}$, $\mathtt{Deg_1}(A_i) = \mathtt{Deg_2}(A_i)$. □

Proof. [**Proposition 31**] For $\mathbf{n}_{\mathtt{rs}}$:
Let $A_1, \cdots, A_k \in \mathtt{Arg}$, $x_1, \cdots, x_k$ and $\mathbf{n}_{\mathtt{rs}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$. From Definition 57: $\forall j \in \{1, \cdots, k\}$, $x'_j = \underset{x_i \in \{x_1,\cdots,x_k\}\setminus\{x_j\}}{\mathtt{avg}} \left(\frac{\mathtt{avg}(x_j,x_i)\times(2-\mathtt{sim}(A_j,A_i))}{2} \right)$.
Then $\forall x_j \in]0, 1]$, $\forall x_i \in [0, 1]$, $\mathtt{avg}(x_j, x_i) > 0$ then $\frac{\mathtt{avg}(x_j,x_i)\times(2-\mathtt{sim}(A_j,A_i))}{2} > 0$ (because $\mathtt{sim}(A_j, A_i) \in [0, 1]$) and so $\underset{x_i \in \{x_1,\cdots,x_k\}\setminus\{x_j\}}{\mathtt{avg}} \left(\frac{\mathtt{avg}(x_j,x_i)\times(2-\mathtt{sim}(A_j,A_i))}{2} \right) > 0$, i.e. $x'_j > 0$.

For $\mathbf{n}^{\rho}_{\mathtt{max}}$:
Let $A_1, A_2 \in \mathtt{Arg}$ such that $\mathtt{sim}(A_1, A_2) = 1$ and $x_1 = 1, x_2 = 1$. $\mathbf{n}^{\rho_{\mathtt{max}}}_{\mathtt{max}}((x_1, A_1), (x_2, A_2)) = (x'_1, x'_2) = (1, 0)$. Then $x'_2 = 0$ while $x_2 > 0$.

For $\mathbf{n}_{\mathtt{wh}}$:
Let $\mathbf{G} = \langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$ be a SSWAF, $A_1, \cdots, A_k \in \mathcal{A}$, $x_1, \cdots, x_k \in [0, 1]$ and $\mathbf{n}_{\mathtt{wh}}((x_1, A_1), \cdots, (x_k, A_k)) = (\mathtt{Str}(A_1), \cdots, \mathtt{Str}(A_k))$. For any $i \in \{1, \cdots, k\}$, from Definition 59, $\mathtt{Str}(A_i) = \frac{x_i}{1+X}$ such that $X \in [0, +\infty[$ therefore if $x_i > 0$, then $\mathtt{Str}(A_i) > 0$. □

Proof. [**Proposition 32**] For $\mathbf{n}_{\mathtt{rs}}$:
Let $A_1, A_2, A_3 \in \mathtt{Arg}$ such that $\mathtt{sim}(A_1, A_2) = 0.5$, $\mathtt{sim}(A_1, A_3) = 0$, $\mathtt{sim}(A_2, A_3) = 0$

and $x_1 = 1$, $x_2 = 0.8$, $x_3 = 0$ then $\mathbf{n}_{\mathtt{rs}}((x_1, A_1), (x_2, A_2)) = (0.675, 0.675)$ and $\mathbf{n}_{\mathtt{rs}}((x_1, A_1), (x_2, A_2), (x_3, A_3)) = (0.5875, 0.5375, 0.45)$. Therefore, we have that $0.675 + 0.675 = 1.35 < 1.575 = 0.5875 + 0.5375 + 0.45$, i.e.
$\mathbf{g}_{\mathtt{sum}}(\mathbf{n}_{\mathtt{rs}}((x_1, A_1), (x_2, A_2))) < \mathbf{g}_{\mathtt{sum}}(\mathbf{n}_{\mathtt{rs}}((x_1, A_1), (x_2, A_2), (x_3, A_3)))$.

For $\mathbf{n}^{\rho}_{\mathtt{max}}$:
Follows from Proposition 30.

For $\mathbf{n}_{\mathtt{wh}}$:
Follows from Proposition 30.

□

Proof. [**Proposition 33**] For $\mathbf{n}_{\mathtt{rs}}$:
Follows from Example 3, when $\alpha = 1$, $\mathbf{g}_{\mathtt{sum}}(\mathbf{n}_{\mathtt{rs}}((x_1, B_1), (x_2, B_2), (x_3, B_3))) = 1.5$.

For $\mathbf{n}^{\rho}_{\mathtt{max}}$:
Let $x_1, \cdots, x_k \in [0, 1]$ and $A_1, \cdots, A_k \in \mathtt{Arg}$ such that $\forall i, j \in \{1, \cdots, n\}$, $\mathtt{sim}(A_i, A_j) = 1$. Let $\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k)) = (x'_1, \cdots, x'_k)$. From Definition 58, we know that $\exists i \in \{1, \cdots, k\}$ such that $x'_i = x_i$ and $\forall j \in \{1, \cdots, k\} \setminus \{i\}$, $x'_j = 0$. Hence,
$\mathbf{g}_{sum}(\mathbf{n}^{\rho}_{\mathtt{max}}((x_1, A_1), \cdots, (x_k, A_k))) \leq 1$.

For $\mathbf{n}_{\mathtt{wh}}$:
Follows from Example 3, when $\alpha = 1$, $\mathbf{g}_{\mathtt{sum}}(\mathbf{n}_{\mathtt{wh}}((x_1, B_1), (x_2, B_2), (x_3, B_3))) = 1.5$.

□

5.2.5 Proofs of section 3.6: Instances of Semantics

Proof. [**Theorem 22**] $\mathbf{S}^{\mathbf{n}^{\rho}_{\mathtt{max}}} \in \mathbf{S}^*$ follows the result of the Proposition 26. The functions $f_{\mathtt{frac}}$ and $\mathbf{g}_{\mathtt{sum}}$ satisfy the constraints 1,2,4 of the Theorem 16 (already proved in Amgoud and Doder [2019]) and combining with $\mathbf{n}^{\rho}_{\mathtt{max}}$ they satisfy the constraints 3 and 4 (proof of Proposition 26).

□

Proof. [**Theorem 23**] The semantics $\mathbf{S}^{\mathbf{n}_{\mathtt{rs}}}$ satisfies all the principles except Neutrality.

Reinforcement. For any SSWAF, $\langle \mathcal{A}, \mathrm{w}, \mathcal{R}, \mathtt{sim} \rangle$, for all $A, B \in \mathcal{A}$, such that

- $\mathbf{w}(A) = \mathbf{w}(B)$,
- $\mathtt{Att}(A) \setminus \mathtt{Att}(B) = \{x\}$, $\mathtt{Att}(B) \setminus \mathtt{Att}(A) = \{y\}$,

- $\forall z \in \mathtt{Att}(A) \cap \mathtt{Att}(B)$, $\mathtt{sim}(x, z) = \mathtt{sim}(y, z)$,
- $\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(x) \leq \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(y)$.

Let $\mathtt{Att}(A) \cap \mathtt{Att}(B) = \{z_1, \cdots, z_k\}$.
Using the conditions of Reinforcement we have: $\forall j \in \{1, \cdots, k\}$,
$\underset{z_i \in \{z_1,\cdots,z_k\}\setminus\{z_j\}}{\mathtt{avg}} \left(\frac{\mathtt{avg}(\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_j), \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(x)) \times (2 - \mathtt{sim}(z_j, x))}{2} \right) \leq$
$\underset{z_i \in \{z_1,\cdots,z_k\}\setminus\{z_j\}}{\mathtt{avg}} \left(\frac{\mathtt{avg}(\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_j), \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(y)) \times (2 - \mathtt{sim}(z_j, y))}{2} \right)$ when $\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(x) \leq \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(y)$.
Then $\mathbf{n_{rs}}\Big((\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_1), z_1), \cdots, (\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_k), z_k), (\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(x), x) \Big) \leq$
$\mathbf{n_{rs}}\Big(\mathtt{Str}^{(\mathbf{S^{n_{rs}}}}(z_1), z_1), \cdots, (\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_k), z_k), (\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(y), y) \Big)$. That implies
$\frac{\mathbf{w}(A)}{1 + \sum_{i=1}^{k} \left(\mathbf{n_{rs}}\left((\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_1), z_1), \cdots, (\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_k), z_k), (\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(x), x) \right) \right)} \geq$
$\frac{\mathbf{w}(B)}{1 + \sum_{i=1}^{k} \left(\mathbf{n_{rs}}\left((\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_1), z_1), \cdots, (\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_k), z_k), (\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(y), y) \right) \right)}$, i.e. $\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(A) \geq \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(B)$.
Same reasoning in the case of $\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(A) > 0$ and $\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(x) < \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(y)$ and we obtain $\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(A) > \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(B)$.

Sensitivity to similarity. For any SSWAF, $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$, for all $A, B \in \mathcal{A}$ such that $\mathbf{w}(A) = \mathbf{w}(B)$, there exists a bijective function $f : \mathtt{Att}(A) \to \mathtt{Att}(B)$ such that:

- $\forall x \in \mathtt{Att}(A)$, $\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(x) = \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(f(x))$,
- $\forall x, y \in \mathtt{Att}(A)$, $\mathtt{sim}(x, y) \geq \mathtt{sim}(f(x), f(y))$,

Same reasoning as Reinforcement and Strict Reinforcement, but instead of increase the strength, we increase the score of similarity. In this case the more similar the attackers, the stronger the attacked argument. Because the similarity score is subtracted.

Monotony. For any SSWAF, $\langle \mathcal{A}, \mathbf{w}, \mathcal{R}, \mathtt{sim} \rangle$, for all $A, B \in \mathcal{A}$, such that

- $\mathbf{w}(A) = \mathbf{w}(B)$,
- $\mathtt{Att}(A) \subset \mathtt{Att}(B)$,
- If $\mathtt{Att}(A) \neq \emptyset$, then $\forall x \in \mathtt{Att}(B) \setminus \mathtt{Att}(A)$, $\forall y \in \mathtt{Att}(A)$, $\mathtt{sim}(x, y) = 0$.

Case 1. $\mathtt{Att}(A) = \emptyset$:
Let $\mathtt{Att}(B) = \{z_1, \cdots, z_k\}$.

Case 1.1. $\nexists z_i \in \texttt{Att}(B)$ such that $\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_i) > 0$ then
$\mathbf{g_{sum}}\Bigg(\mathbf{n_{rs}}\Big((\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_1), z_1), \cdots, (\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_k), z_k)\Big)\Bigg) = \mathbf{g_{sum}}\Bigg(\mathbf{n_{rs}}\Big(\Big)\Bigg)$
$= 0$. Then $\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(A) = \texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(B)$.

Case 1.2. $\exists z_i \in \texttt{Att}(B)$ such that $\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_i) > 0$ then
$\mathbf{g_{sum}}\Bigg(\mathbf{n_{rs}}\Big((\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_1), z_1), \cdots, (\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_k), z_k)\Big)\Bigg) > \mathbf{g_{sum}}\Bigg(\mathbf{n_{rs}}\Big(\Big)\Bigg)$
because from the proposition 3 the functions satisfy the conditions 2a, 3a, 3d (Def. 54). Therefore if $\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(A) > 0$, then $\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(A) > \texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(B)$.

Case 2. $\texttt{Att}(A) \neq \emptyset$:
Let $\texttt{Att}(A) = \{z_1, \cdots, z_k\}$.
Let $\texttt{Att}(B) = \{z_1, \cdots, z_k, z_{k+1}, \cdots, z_{k+m}\}$ such that $\forall i \in \{k+1, \cdots, k+m\}, \forall j \in \{1, \cdots, k\}$, $\texttt{sim}(z_i, z_j) = 0$.
Let $\mathbf{n_{rs}}((\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_1), z_1), \cdots, (\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_k), z_k)) = (z'_1, \cdots, z'_k)$.
Start by develop the sum of output value of the function $\mathbf{n_{rs}}$ on the attackers of A.
Denote by

$$x_{ij} = \frac{\texttt{avg}(\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_j), \texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_i)) \times (2 - \texttt{sim}(z_j, z_i))}{2}.$$

Then we can rewrite

$$z'_j = \underset{\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_i) \in \{\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_1), \cdots, \texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_k)\} \setminus \{\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_j)\}}{\texttt{avg}}$$

$$\left(\frac{\texttt{avg}(\texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_j), \texttt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(z_i)) \times (2 - \texttt{sim}(z_j, z_i))}{2}\right)$$

$$= \frac{x_{1j} + \cdots + x_{j-1j} + x_{j+1j} + \cdots + x_{kj}}{k-1}.$$

And thus, $\sum_{i=1}^{k} z'_i =$

$$\frac{2x_{12} + \cdots + 2x_{1k} + 2x_{23} + \cdots + 2x_{2k} + \cdots + 2x_{k-1k}}{k-1}$$

$$= \frac{X}{k-1}.$$

Do the same for $\mathbf{n_{rs}}$ on the attackers of b.

$$z'_j = \frac{x_{1j} + \cdots + x_{j-1j} + x_{j+1j} + \cdots + x_{k+mj}}{k-1+m}.$$

Case 2.1. $\forall i \in \{k+1, \cdots, k+m\},\ \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_i) = 0.$

Let us take the minimal case where the attackers in b which are not in a have zero strength. $\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_i) > 0.$

Then adding the fact that each attacker z_i such that $i \in \{k+1, \cdots, k+m\}$ have zero similarity with any other attackers, we can instantiate the score of the m last value:

$$\forall j \in \{1, \cdots k\},\ z'_j = \frac{x_{1j} + \cdots + x_{j-1j} + x_{j+1j} + \cdots + x_{kj}}{k-1+m} + \frac{\frac{m \cdot \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_j)}{2}}{k-1+m}.$$

$$\forall j \in \{k+1, \cdots, k+m\},\ z'_j = \frac{\frac{\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_1)}{2} + \cdots + \frac{\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_k)}{2}}{k-1+m}.$$

We obtain the following sum: $\sum_{i=1}^{k+m} z'_i =$

$$\frac{2x_{12} + \cdots + 2x_{1k} + 2x_{23} + \cdots + 2x_{2k} + \cdots + 2x_{k-1k}}{k-1+m} + \frac{2m \cdot (\frac{\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_1)}{2} + \cdots + \frac{\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_k)}{2})}{k-1+m}$$

$$= \frac{X + m(\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_1) + \cdots + \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_k))}{k-1+m}.$$

To summarise we want to check that

$$\frac{X}{k-1} \leq \frac{X + m(\mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_1) + \cdots + \mathtt{Str}^{\mathbf{S^{n_{rs}}}}(z_k))}{k-1+m}.$$

The value of X depends on the strength of each attackers z_i and on there similarities.

For a fix set of strength, the lower value of X is when each argument are fully similar and the biggest value of X is when they didn't exist any similarities.

Case 2.1.1. $\mathtt{sim} \equiv 0$, i.e. the maximal score of X.

$\mathbf{n_{rs}}((x_1, A_1), \cdots, (x_k, A_k)) =$

$$\left(\underset{x_i \in \{x_1, \cdots, x_k\} \setminus \{x_1\}}{\mathtt{avg}} \left(\frac{\mathtt{avg}(x_1, x_i) \times (2)}{2} \right), \ \cdots, \ \underset{x_i \in \{x_1, \cdots, x_k\} \setminus \{x_k\}}{\mathtt{avg}} \left(\frac{\mathtt{avg}(x_k, x_i) \times (2)}{2} \right) \right).$$

$$= \left(\frac{(k-1) \cdot x_1 + x_2 + \cdots + x_k}{2k-2}, \cdots , \right.$$

$$\left. \frac{x_1 + x_2 + \cdots + (k-1) \cdot x_k}{2k-2} \right).$$

The sum of each element in this case is equal to:

$$\frac{(2k-2) \cdot x_1 + \cdots , (2k-2) \cdot x_k}{2k-2} = \sum_{i=1}^{k} x_i.$$

In other words, $\mathbf{g}_{\mathtt{sum}}(x_1, \cdots , x_k) = \mathbf{g}_{\mathtt{sum}}(\mathbf{n}_{\mathrm{rs}}((x_1, A_1), \cdots , (x_k, A_k)))$, i.e. $\sum_{i=1}^{k} z'_i = \sum_{i=1}^{k} \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_i)$.
Then in this case $X = (k-1) \sum_{i=1}^{k} \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_i)$.
Thus in the maximal case we have an equality:

$$\frac{(k-1) \sum_{i=1}^{k} \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_i)}{k-1}$$

$$= \frac{(k-1) \sum_{i=1}^{k} \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_i) + m(\mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_1) + \cdots + \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_k))}{k-1+m}$$

$$= \sum_{i=1}^{k} \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_i).$$

Therefore

$$\sum_{i=1}^{k} \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_i) = \sum_{i=1}^{k+m} \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_i)$$

$$\iff \frac{X}{k-1} = \frac{X + m(\mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_1) + \cdots + \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_k))}{k-1+m}$$

$$\iff \frac{(k-1)(\mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_1) + \cdots + \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_k))}{k-1} =$$

$$\frac{(k-1)(\mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_1) + \cdots + \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_k))}{k-1+m} +$$

$$\iff \frac{m(\mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_1) + \cdots + \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(z_k))}{k-1+m}$$

$$\Longrightarrow \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(A) = \mathtt{Str}^{\mathbf{S}^{\mathbf{n}_{\mathrm{rs}}}}(B).$$

Case 2.1.2. $\exists i, j \in \{1, \cdots , k\}$, $i \neq j$, s.t. $\mathtt{sim}(z_i, z_j) > 0$.

We know that

$$X = \frac{2x_{12} + \cdots + 2x_{1k} + 2x_{23} + \cdots + 2x_{2k} + \cdots + 2x_{k-1k}}{k-1+m}$$

and $\forall i, j \in \{1, \cdots, k\}, i \neq j$,

$$2x_{ij} = \texttt{avg}(\texttt{Str}^{\mathbf{S^{nrs}}}(z_j), \texttt{Str}^{\mathbf{S^{nrs}}}(z_i)) \times (2 - \texttt{sim}(z_j, z_i)).$$

That's why, increasing the similarity ($\texttt{sim}(z_j, z_i)$) will decrease $2xij$, hence X. From the case 2.1.1 we can rewrite when $\texttt{sim} \equiv 0$,

$$\frac{X}{k-1} = \frac{X + m(\texttt{Str}^{\mathbf{S^{nrs}}}(z_1) + \cdots + \texttt{Str}^{\mathbf{S^{nrs}}}(z_k))}{k-1+m}.$$

By

$$\iff \frac{(k-1)\alpha}{k-1} = \frac{(k-1)\alpha + m\alpha}{k-1+m}.$$

And when the similarities increase, as we have seen before α will decrease to β, i.e. $\alpha < \beta$ and we obtain:

$$\iff \frac{(k-1)\beta}{k-1} < \frac{(k-1)\beta + m\alpha}{k-1+m}$$

$$\implies \texttt{Str}^{\mathbf{S^{nrs}}}(A) > \texttt{Str}^{\mathbf{S^{nrs}}}(B).$$

Case 2.2. $\exists i \in \{k+1, \cdots, k+m\}$ such that $\texttt{Str}^{\mathbf{S^{nrs}}}(z_i) > 0$ and $\texttt{Str}^{\mathbf{S^{nrs}}}(A) > 0$. (Strict Monotony)

Given that the method $\texttt{Str}^{\mathbf{S^{nrs}}}$ satisfy the Strict Reinforcement principle, increasing the strength of an attacker of b which is not in a will strictly decrease the score of b. We denote by $\delta \in]0, +\infty[$, the strength of attackers which is in b and not in a.

$$\frac{(k-1)\beta}{k-1} < \frac{(k-1)\beta + m\alpha}{k-1+m} < \frac{(k-1)\beta + m(\alpha+\delta)}{k-1+m}$$

$$\implies \texttt{Str}^{\mathbf{S^{nrs}}}(A) > \texttt{Str}^{\mathbf{S^{nrs}}}(B).$$

Neutrality. Let $\mathbf{G} = \langle \mathcal{A}, \mathrm{w}, \mathcal{R}, \texttt{sim} \rangle$ be a $\texttt{SSWAF}$, $x_1, x_2, x_3 \in \mathcal{A}$ such that $\texttt{sim}(x_1, x_2) = 0.5$, $\texttt{sim}(x_1, x_3) = 0$, $\texttt{sim}(x_2, x_3) = 0$ and $\texttt{Str}^{\mathbf{S^{nrs}}}(x_1) = 1$, $\texttt{Str}^{\mathbf{S^{nrs}}}(x_2) = 1$, $\texttt{Str}^{\mathbf{S^{nrs}}}(x_3) = 0$.

$\mathbf{n_{rs}}((\mathtt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(x_1), x_1), (\mathtt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(x_2), x_2)) = (0.75, 0.75)$.

$\mathbf{n_{rs}}((\mathtt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(x_1), x_1), (\mathtt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(x_2), x_2), (\mathtt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(x_3), x_3)) = (0.625, 0.625, 0.5)$.

Let $A, B \in \mathcal{A}$ such that $\mathbf{w}(A) = \mathbf{w}(B)$ and $\mathtt{Att}(A) = \{x_1, x_2\}$, $\mathtt{Att}(B) = \{x_1, x_2, x_3\}$.

Then $\mathtt{Str}^{\mathbf{n_{rs}}}(A) =$

$$\frac{\mathbf{w}(A)}{1 + \sum_{i=1}^{2} \Big(\mathbf{n_{rs}}\Big((\mathtt{Str}^{\mathbf{n_{rs}}}(x_1), x_1), (\mathtt{Str}^{\mathbf{n_{rs}}}(x_2), x_2)\Big)\Big)}$$

$$= \frac{\mathbf{w}(A)}{1 + 1.5}.$$

$\mathtt{Str}^{\mathbf{n_{rs}}}(B) =$

$$\frac{\mathbf{w}(B)}{1 + \sum_{i=1}^{3} \Big(\mathbf{n_{rs}}\Big((\mathtt{Str}^{\mathbf{n_{rs}}}(x_1), x_1), (\mathtt{Str}^{\mathbf{n_{rs}}}(x_2), x_2), (\mathtt{Str}^{\mathbf{n_{rs}}}(x_3), x_3)\Big)\Big)}$$

$$= \frac{\mathbf{w}(B)}{1 + 1.75}.$$

Therefore when $\mathbf{w}(A) > 0$, $\mathtt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(A) > \mathtt{Str}^{\mathbf{S}^{\mathbf{n_{rs}}}}(B)$.

□

Proof. [**Theorem 24**] From Proposition 28 and Amgoud and Doder [2019], $\langle \mathbf{f_{frac}}, \mathbf{g_{sum}}, \mathbf{n_{wh}} \rangle \in \mathbf{M}^*$ then $\mathbf{S}^{\mathbf{n_{wh}}} \in \mathbf{S}^*$. From Theorem 21, $\mathbf{S}^{\mathbf{n_{wh}}}$ satisfies Reinforcement, Monotony, Neutrality and Sensitivity to Similarity. Moreover from Proposition 28 we know that $\langle \mathbf{f_{frac}}, \mathbf{g_{sum}}, \mathbf{n_{wh}} \rangle$ satisfies the conditions (C1), (C2), (C3) then from Theorems 18, 19 and 20, it satisfies also the strict versions, i.e. all the principles of the section 3.3. □

Proof. [**Theorem 25**] Let us show that:

$$\mathtt{Str}^{\mathbf{n}_{\max^\rho}} \equiv \mathtt{Str}^{\mathbf{n_{rs}}} \equiv \mathtt{Str}^{\mathbf{n_{wh}}} \equiv \mathtt{Str}^{h}.$$

The difference between these functions come from the function of adjustment, that's why if $\sum_{i=1}^{k}\left(\mathbf{n}\Big((\mathtt{Str}^{\mathbf{n}}(B_1), B_1), \cdots, (\mathtt{Str}^{\mathbf{n}}(B_k), B_k)\Big)\right) = \sum_{B_i \in \mathtt{Att}(A)} \mathtt{Str}^h(B_i)$ the theorem is verified.

For $\mathtt{Str}^{\mathbf{n}_{\mathtt{max}^\rho}}$, if $\mathtt{sim} \equiv 0$ then $\forall i \in \{1, \cdots, k\}$, $x'_{\rho(i)} = x_{\rho(i)}$, i.e. $\mathtt{Str}^{\mathbf{n}_{\mathtt{max}^\rho}} \equiv \mathtt{Str}^h$.

For $\mathtt{Str}^{\mathbf{n}_{\mathtt{rs}}}$, we know from the proof of the Theorem 23 (Monotony) that in the case of $\mathtt{sim} \equiv 0$, the sum of the output of $\mathbf{n}_{\mathtt{rs}}$ is equal to the sum of the numerical input.

For $\mathtt{Str}^{\mathbf{n}_{\mathtt{wh}}}$, if $\mathtt{sim} \equiv 0$ then $\forall i \in \{1, \cdots, k\}$, $x'_i = \frac{x_i}{1+0} = x_i$, i.e. $\mathtt{Str}^{\mathbf{n}_{\mathtt{wh}}} \equiv \mathtt{Str}^h$.

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