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HYPERBOLICITY AND CERTAIN STATISTICAL PROPERTIES OF CHAOTIC BILLIARD SYSTEMS

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HYPERBOLICITY AND CERTAIN STATISTICAL PROPERTIES OF CHAOTIC BILLIARD SYSTEMS

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by

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ABSTRACT

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OF CHAOTIC BILLIARD SYSTEMS

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In this thesis, we address some questions about certain chaotic dynamical systems. In particular, the objects of our studies are chaotic billiards. A billiard is a dynamical system that describes the motions of point particles in a table where the particles collide elastically with the boundary and with each other.

Among the dynamical systems, billiards have a very important position. They are models for many problems in acoustics, optics, classical and quantum mechanics, etc.. Despite of the rather simple description, billiards of different shapes of tables exhibit a wide range of dynamical properties from being complete integrable to chaotic. A very important and also very interesting type of billiards is chaotic (or hyperbolic) billiards. In a hyperbolic billiard system, two nearby trajectories in the phase space can be separated exponentially fast in future.

In the first two Chapters, we prove the Central Limit Theorem and the Almost Sure Invariance Principle for a class of billiard systems with flat points. They are two among the important statistical properties for chaotic systems. In the last chapter, we introduce a random perturbation to a wide class of billiards and prove that even if the original system is completely integrable, the perturbed system can be chaotic even under arbitrarily small random perturbation.

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CHAPTER 1

INTRODUCTION

The mathematical area of dynamical systems studies models in which things evolve over time and in a state space. The motivations of such models come from many different fields, including mathematics, physics, chemistry, engineering, economics, finance. Methods used for analysing these systems include but not limited to analysis, geometry, probability and measure theory, numerical computation and simulation.

Perhaps a dynamical system that is familiar to everyone is the weather. There are many different variables considered when models are built to predict the temperature or other conditions of the atmosphere in the future for any given location. However, long-term prediction of the weather is nearly impossible. This is partly due to of the huge number of variables involved that cannot be all included in a mathematical model. Even within a fixed model, there are many small errors in measurements and computations. These small differences in initial conditions at the beginning could lead to very different outcomes in the future. This is an example of a chaotic deterministic dynamical system. See [64] for a more detailed treatment.

In this thesis, we address some problems about certain chaotic dynamical systems. In particular, the objects of our studies are dynamical billiards. In the first 2 chapters, we prove the Central Limit Theorem and the Almost Sure Invariance Principle for billiards with zero curvature points on the boundary. In the last chapter, we introduce a random perturbation to a wide class of billiards and prove that the perturbed systems are chaotic, even if the random perturbation is arbitrarily small.

Before continuing, we need to introduce some notation to describe chaos in dynamical systems. In mathematics, a discrete-time dynamical system consists of phase space M and a function $F : M \to M$. We will assume that M is a complete separable metric space. If we start from the initial state x, then at time 1 the new state is $x_1 = F(x_0)$, and similarly the state at time n is given by $x_n = F^n(x)$. A mathematical structure will be equipped to the set M depending on each problem. In our setting, the set M and the map F are in the category of probability and measure spaces, but usually also has other structures. There are many texts on introduction to measure theory such as [66], [28]. We also include in this thesis a short appendix on basic measure theory to fix notation.

Consider a σ-algebra $\mathcal{B}$ of subsets of M that contains the empty set $\emptyset$ and is closed under complementation and countable unions, then the pair $(M, \mathcal{B})$ is a measurable space. If there is a function $\mu : \mathcal{B} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and μ satisfies the countable additivity condition then the triple $(M, \mathcal{B}, \mu)$ is called a *measure space.* Furthermore, if $\mu(M) = 1$ then $(M, \mathcal{B}, \mu)$ is a *probability space*; an element B in $\mathcal{B}$ could be viewed as an event and $\mu(B)$ is the likelihood of that event. Given two measurable spaces $(M, \mathcal{B})$ and $(G, \mathcal{G})$, a map $f : (M, \mathcal{B}) \to (G, \mathcal{G})$ is called *measurable* if $f^{-1}(E) \in \mathcal{B}$, where $f^{-1}(E) = \{x \in M : f(x) \in E\}$, for any $E \in \mathcal{G}$,

We consider a probability space $(M, \mathcal{B}, \mu)$ and a measurable map $F : (M, \mathcal{B}) \to (M, \mathcal{B})$. Assume further that μ is F-invariant, that is if $\mu(F^{-1}(B)) = \mu(B)$ for any $B \in \mathcal{B}$. Then $(M, \mathcal{B}, \mu, F)$ is called a *measure-preserving dynamical system.*

For measure-preserving dynamical system, there are several properties to characterise different levels of chaos, including hyperbolicity, ergodicity, weakly mixing, mixing, multiple mixing, K-mixing, Bernoulli which represent the increasing degree of chaos [[17] chapter 7]. However, we give in this thesis an example where the system is ergodic but not hyperbolic. A great presentation on introduction to ergodic theory is [65]. The dynamical system $(M, \mathcal{B}, \mu, F)$ is called *ergodic* if whenever $F^{-1}B = B$ we must have that $\mu(B)$ is either 0 or 1. In an ergodic system, μ-almost every point moving in M will eventually visit every set of positive measure. A stronger level of chaos is called *mixing.* The map F is called *mixing* with respect to μ if for all measurable sets $A, B \in \mathcal{B}$,

$$\lim_{n\to\infty} |\mu(F^{-n}(A) \cap B) - \mu(F^{-n}(A))\mu(B)| = 0. \tag{1.1}$$

The mixing property says that for the event $x \in B$ at present will become asymptotically independent on the event $F^n(x) \in A$ in the future at time n.

We denote by $\mu(f)$ the integral $\int_M f(x)\mu(dx)$ for any $f : M \to \mathbf{R}$ any measurable function (or also known as *observable*) that is integrable. Let $L^2_\mu(M)$ be the space of square integrable functions. The *correlation function* of any two square integrable functions f and g on M are given by:

$$
\begin{aligned}
C_{f,g}(n) &= \int_M f(x)\cdot(g\circ F^n)(x)\mu(dx) - \int_M f(x)\mu(dx)\int_M g(x)\mu(dx) \\
&= \mu(f\cdot(g\circ F^n)) - \mu(f)\mu(g).
\end{aligned} \tag{1.2}
$$

It is very important to know rate of decay of the correlation for any $f, g \in L^2_\mu(M)$ as it characterises the mixing speed. The mixing condition (1.1) is in fact equivalent to the convergence to 0 of the correlation as $n \to \infty$ for every $f, g \in L^2_\mu(M)$.[[17] page 302.]

Let $f : M \to \mathbf{R}$ be any observable. The sequence $X_n = f \circ F^n$ is a stationary stochastic process defined on (M, μ) [[17] Lemma 7.1] . The statistical properties of this process, such as the decay rate of correlations, Central Limit Theorem, Invariance Principles and other limit theorems, display the similarity between the dynamics given by (M, μ, F, f) and sequences of independent identically distributed random variables. See [17], Chapter 7 for these definitions and a nice introduction to statistical properties. A great article on this topic is [69].

The Birkhoff partial sum S_n of the process (X_n) is defined by:

$$
S_n = X_0 + X_1 + \cdots + X_{n-1}. \tag{1.3}
$$

If the system is *ergodic* and the observables are *integrable*, then the process $(X_n)_{n\geq 0}$ satisfies the Birkhoff's Ergodic Theorem:

$$
\lim_{n\to\infty} \frac{S_n}{n} = \mu(f). \tag{1.4}
$$

We say that the process $(X_n)_{n\geq 0}$ satisfies the Central Limit Theorem if we have:

$$
\lim_{n\to\infty} \mu\left\{\frac{S_n - n\mu(f)}{\sqrt{n}}\right\} = \frac{1}{\sqrt{2\pi}\sigma_f}\int_{-\infty}^{z} e^{-\frac{z^2}{2\sigma_f^2}}\,dz. \tag{1.5}
$$

for $-\infty < z < \infty$. The constant $\sigma_f \geq 0$ is given by the Green-Kubo equation:

$$\sigma_f^2 = C_{f,f}(0) + 2\sum_{n=1}^{\infty} C_{f,f}(n) \tag{1.6}$$

The next step in the study of the statistical properties of the sequence $X_n = f \circ F^n$ is the invariance principle. We say that the process $(X_n)_{n\geq 0}$ satisfies the Almost Sure Invariance Principle if there exists a standard Brownian motion $W(\cdot)$ on M with respect to the measure μ so that for some $\lambda > 0$ we have:

$$\left| \frac{S_n - n\mu(f)}{\sigma_f \sqrt{N}} - W(\frac{n}{N}) \right| = \mathcal{O}(N^{-\lambda}) \tag{1.7}$$

for μ-almost every $x \in M$, integers $N \geq 1$ and $0 \leq n \leq N$.

From a different viewpoint, a typical characteristic of a chaotic dynamical system is its sensitivity to initial conditions. A chaotic system in this sense is also called a *hyperbolic system.* To study hyperbolicity, we need to view the phase space M as a compact Riemannian manifold and let $F : M \to M$ be a diffeomorphism on an open dense subset of full measure in M, and also μ-preseving. The mathematical tool to measure the senitivity to initial conditions of the system $F : M \to M$ is the Lyapunov exponents. Their definition is given by the Oseledets's multiplicative ergodic theorem:

Theorem 1.1 (Oseledets) *[17] Let M be a compact Riemannian manifold and F : $M \to M$ a C^2 diffeomorphism on an open dense subset of full measure, preserving a Borel probability measure μ on M. Suppose that*

$$\int_M \log^+ \|D_xF\|\mu(dx) < \infty \text{ and } \int_M \log^+ \|D_xF^{-1}\|\mu(dx) < \infty, \tag{1.8}$$

where $\log^+ = \max\{\log, 0\}$. *Then there exists an* F*-invariant set* $H \subset M$ *of full measure, on which all iterations of* F *are defined on* H*, such that and for each* $x \in H$ *there is a* DF*-invariant decomposition of the tangent space:*

$$T_xM = E_1(x) \oplus \cdots \oplus E_m(x) \tag{1.9}$$

for some m *depends on* x*, such that for each non-zero vector* $v \in E_i(x)$ *the following limit exists:*

$$\lim_{n\to\infty} \frac{1}{n} \log \|D_xF^nv\| = \lambda_i(x) \tag{1.10}$$

where $\lambda_1(x) > \cdots > \lambda_m(x)$.

Remark *The Lyapunov exponents are invariant under the map* F*. If the map* F *is ergodic with respect to* μ *then the Lyapunonv exponents are constant* μ*-almost everywhere.*

By (1.10), we can see that if $\lambda_i > 0$ then any non-zero tangent vector $v \in E_i(x)$ will grow with rate approximately $\lambda_i(x)$ in the future. A point x is said to be *hyperbolic* if $\lambda_i(x)$ exists and $\neq 0$ for all i. The system $F : M \to M$ is said to be *hyperbolic* if μ-almost every point in M is hyperbolic.

Among the dynamical systems, billiards have a very important position. They are models for many problems in acoustics, optics, classical and quantum mechanics, etc.. Billiards also appear in the study of Riemann surfaces. Inside a billiard, one or more point particles move and collide with the boundary and with each other. The collision is elastic: the pre-collisional angle of incidence equals to the post-collisional angle of reflection. Despite of the rather simple description, billiards of different shapes exhibit a wide range of dynamical properties from being complete integrable to chaotic. Birkhoff showed that elliptic billiards are integrable. The collision space M for the billiard map in an elliptic billiard is foliated by 1-dimensional invariant manifolds. He also conjectured that the only strictly convex integrable billiards are elliptic billiard. A nice discussion on Birkhoff's theorem and Birkhoff's conjecture is in [63] chapter 5.

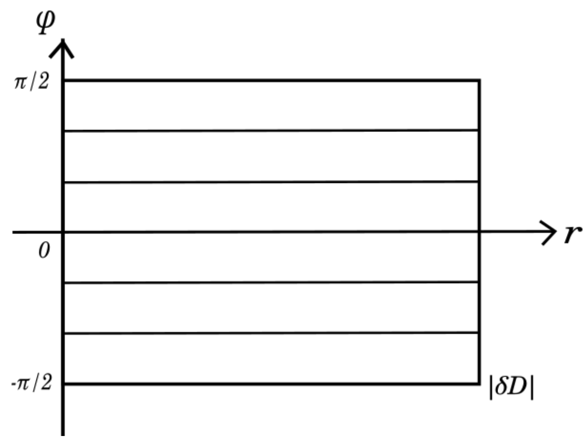


Figure 1. Phase space of a circular billiard

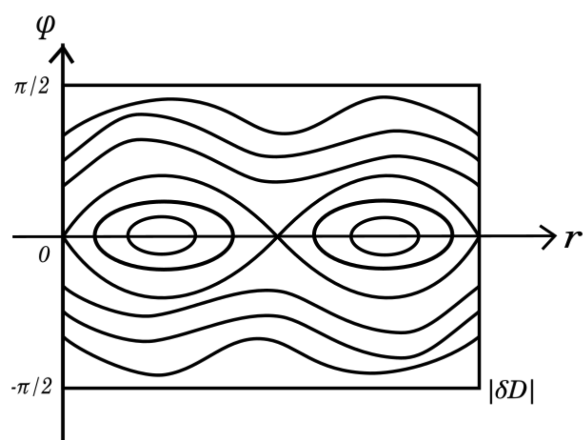


Figure 2. Phase space of an elliptic billiard

The conjecture is still an open problem and attracts lots of interest. At the other end of the spectrum, billiards with chaotic properties have been also studied for a long time such as the Boltzmann hard balls models, Lorentz gas. But only until 1970 the mathematical theory of chaotic billiards was introduced by Sinai in [60]. In his seminal paper, he constructed the dispersing billiards in which a wavefront of parallel trajectories will disperse after colliding with the dispersing obstacles. Dispersing billiards have strong chaotic properties: ergodic, mixing, Bernoulli, hyperbolic and exponential decay of correlations [60] and thus satisfies limit theorems [[17], Lemma 7.60].

With exponential decay of correlations, the Central Limit Theorem is known to be true [8]. Since then, the central limit theorem and other limit theorems have been proved for various billiards, including ones with slow mixing rate. A common assumption in these examples is that the observables are Hölder continuous and the diffusion constant is given as an infinite series by the Green-Kubo formula in equation (1.6).

By using the martingale approximation technique on induced systems, we proved the Central Limit Theorem and the Almost Sure Invariance Principle billiards with flat points. For the Central Limit theorem, the observables are assumed to be only piecewise Hölder continuous functions and moreover, we are able to represent the diffusion constants in an explicit and simple formula. With the Almost Sure Invariance Principle, the observables are integrable but could be unbounded. However, they provide good approximation for most regular observables.

Since the discover of dispersing billiards, many billiard models with focusing arcs have been studied. Bunimovich discovered the elegant defocusing mechanism [7] in billiards with focusing arcs and the hyperbolicity has been proved for many models. After that, Wojtkowski, Markarian, Donnay and Bunimovich developed methods to design hyperbolic billiards with focusing boundary components [67, 47, 26, 6]. The idea was to used the invariant cones or quadratic forms. Interesting billiards, for instance the Lemon billiards [11, 10], were proved to be hyperbolic using this invariant cones technique. However, there are still many classes of billiards whose hyperbolicity or ergodicity is still not confirmed for example the Moon billiards [21]. In this methods, we have to construct a cone in the tangent space at each point on either the collision space M or a subset of M. Then we need to show that the cones will be at least eventually strictly invariant under iterations of the derivative map DF or induced map if the cones are on a subset of M. In any case we have to keep track of the dynamics of the cones along each trajectory. This is a problem for many billiards as the task of choosing the right moment for the cones to shrink is rather challenging.

Instead of keeping track of each individual and deterministic trajectory, we add a small randomness into the systems, so that the image of points in M are determined not just by the billiard map. In this new setting, the evolution of the system is governed by a Markov transition function $P(x, B)$, for each $x \in M$ and $B \in \mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra of M. For each point $x \in M$, it jumps to $F(x)$ and then perturbed to a nearby point according to the distribution $P(x, \cdot)$. By iterating this process, we obtain a Markov chain with values in M. If there is no perturbation, any realisation of this Markov chain is a real orbit of a point in the phase space

M. By evaluating the derivative map DF along this Markov chain, we obtain a stationary sequence of invertible matrices. It is shown in [44] that the Lyapunov exponents $\lambda_1 \geq \lambda_2$ exist for this process of matrices (there are at most two distinct Lyapunov exponents for billiards since $\dim M = 2$). Moreover, in [45] and later in [1], a necessary condition for $\lambda_1 = \lambda_2$ is presented. If $\lambda_1 = \lambda_2$, a special measurability condition must be satisfied [1, 45].

There has been several works on random billiards. The perturbation to the system in these works also described by a Markov transition function on the phase space. The randomness introduced into the systems may be due to external force as in [15], the microscopic surface structures as in [31], [32], [30],[20], [49], change in table configuration as in [61], [24]. Also the hyperbolicity of the random billiards are not addressed in many cases. In our case, the random billiards have the same invariant measure as the original ones, and this invariant measure is in fact the only one. Hyperbolicity is also established for many random billiards. Two interesting examples are circular billiards and non-circular elliptic billiards. Their random versions are all ergodic, but the random circular billiards still have zero Lyapunov exponent at all point, while the random elliptic billiards have positive Lyapunov exponent for any magnitude of the noise.

The thesis is organised as follows:

Chapter 2 is the paper [54]. This is a joint work with HongKun Zhang. We proved the central limit theorem for billiards with flat points.

Chapter 3 is the paper [12]. This is a joint work with Jianyu Chen. We proved the invariance principles for ergodic systems with slow α-mixing inducing base.

Chapter 4 is a joint work with Jinxin Xue and HongKun Zhang. We introduced a perturbation to several classes of billiards and study the ergodicity and hyperbolicity of the perturbed systems.

CHAPTER 2

CENTRAL LIMIT THEOREM FOR BILLIARDS WITH FLAT POINTS

2.1 Introduction to the main result

Billiards are natural models to many different physical problems, especially in classical and statistical mechanics. They have a wide range of properties depending on the shape of the tables. Sinai introduced in 1970 the so-called Sinai (or dispersing) billiards where the boundary of the table is smooth and concave with positive curvature. These billiards are strongly chaotic: they are ergodic, mixing and have exponential decay of correlations. The central limit theorem is known to be true for these systems, see [8]. Since then, the central limit theorem and other limit theorems have been proved for various billiards, including ones with slow mixing rate. In many

This chapter is a slightly modified version of [54].

cases, the observables considered in those examples are Hölder continuous and the diffusion constant is given as an infinite series by the Green-Kubo formula.

In their paper [18], Chernov and Zhang introduced a family of dispersing billiard models. They were able to prove that the correlations for the collision map decay as $\mathcal{O}(1/n^a)$ for any constant $a \in (1, \infty)$, by introducing an induced system together with a first return time function. Instead of using the traditional methods, we constructed a filtration generated by the first return time function. Then we are able to construct a stationary martingale difference sequence to approximate the process adapted to this filtration. With this new tool, we are going to the central limit theorem for this billiard family for a class of piecewise Hölder continuous functions. One achievement of our results is that we are able to represent the diffusion constants in an explicit and simple formula, comparing to the infinite series using the Green Kubo formula. Before we proceed to the main result, let us briefly recall some basic notions; more detailed exposition can be found in, for example, [17].

The billiard table $\mathcal{D}$ considered in [18] is bounded by the curves $y = |x|^\beta + 1$, $y = -(|x|^\beta + 1)$ and some strictly inward convex curves with nowhere vanishing curvature and no cusps. A point mass moves inside the table and bounces off its boundary $\partial\mathcal{D}$ elastically.

Let $\mathcal{M}$ be the collision space of the billiard dynamics on $\mathcal{D}$. We parameterize $\partial\mathcal{D}$ by arclength in the clockwise direction and thus each collision is determined by its position r on $\partial\mathcal{D}$ and its angle of reflection $-\pi/2 \leq \varphi \leq \pi/2$ (that formed with the inward normal vector). They are natural coordinates $\mathcal{M}$ and we can write $\mathcal{M} = [0, |\partial\mathcal{D}|] \times [-\pi/2, \pi/2]$, where $|\partial\mathcal{D}|$ is the length of $\partial\mathcal{D}$. The collision map

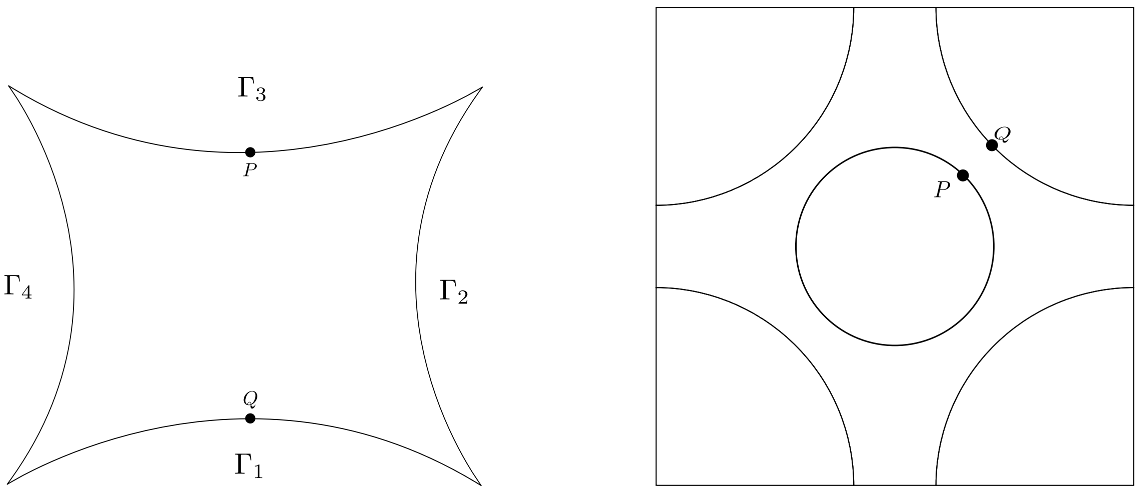


Figure 3. In either table, P and Q are the only flat points with zero curvature.

$\mathcal{F} : \mathcal{M} \to \mathcal{M}$ preserves a smooth probability measure μ on $\mathcal{M}$ defined by:

$$d\mu = \frac{1}{2|\partial \mathcal{D}|} \cos(\varphi) dr d\varphi. \tag{2.1}$$

Let $f, g \in L^2(\mathcal{M}, \mu)$ be two piecewise Hölder continuous with singularities coincide with those of $\mathcal{F}^k$ for some k. The correlations of f and g are defined by:

$$C_n(f, g, \mathcal{F}, \mu) = \int_{\mathcal{M}} (f \circ \mathcal{F}^n) \cdot g d\mu - \int_{\mathcal{M}} f d\mu \int_{\mathcal{M}} g d\mu. \tag{2.2}$$

Chernov and Zhang proved in [18] that these correlations decay polynomially, that is:

$$|C_n(f, g, \mathcal{F}, \mu)| \leq C \frac{(\ln n)^{a+1}}{n^a}, \tag{2.3}$$

where $a = \frac{\beta+2}{\beta-2}$ and C is some fixed constant.

For systems with slow rates of decay of correlations like this, it is typical to study the dynamics on a subset of the phase space such that the induced system has exponential decay of correlations, then extend the results to the original space.

Let $M \subset \mathcal{M}$ be a subset of $\mathcal{M}$ obtained by removing the collisions that happen in an arbitrarily small neighbourhood of the flat points. The first return time function $R : M \to \mathbf{N}$ is defined almost everywhere by:

$$R(z) = \inf\{n \geq 1 : \mathcal{F}^n(z) \in M\}. \tag{2.4}$$

Let $M_n = \{R = n\} \subset M$ be the n-th level set of R, for each $n \geq 1$. Moreover, for $n, m \geq 1$, we denote

$$p_{n,m} := \frac{\nu(F^{-1}M_m \cap M_n)}{\nu(M_n)} \tag{2.5}$$

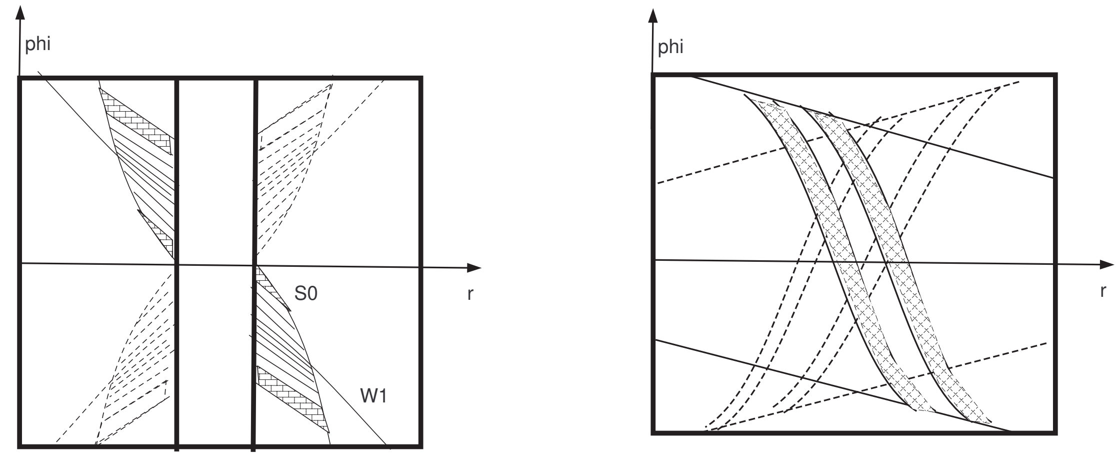
The quantities $p_{n,m}$ can be thought of as the transition probability of going from cell M_n to cell M_m in one iteration. It is important to note that everything in M_n with $n \geq 3$ must go to M_1 if the neighbourhood is sufficiently small. From M_2, although it cannot go to cells of higher indices, it is possible, however, to go back to itself because of the presence of period-four-orbit-like trajectories. There is a positive probability to go from M_1 to any cells.

Now consider the induced collision map $F : M \to M$ given by: $F(z) = \mathcal{F}^{R(z)}(z)$. The function F is discontinuous on the lines separating the cells M_n's. Moreover, F preserves the conditional measure ν on M, where for each $B \subset M$, $\nu(B) := \frac{\mu(B)}{\mu(M)}$. The map $F : M \to M$ is strongly hyperbolic and has exponential decay of correlations.

Since the set M is partitioned by the cells M_n's, we also have a partition for $\mathcal{M}$:

$$\mathcal{M} = \cup_{n=1}^{\infty} \cup_{k=0}^{n-1} \mathcal{F}^k M_n.$$

An element $z \in \mathcal{M}$ can be represented by the pair (y, i) where $\Pi(z) = y$ is the projection onto the base M and $z = \mathcal{F}^i(y)$ with $0 \leq i \leq R(y) - 1$. Let $\mathcal{F}_0^{\mathcal{M}}$ be the



(a) (b)

σ-algebra generated by this partition of $\mathcal{M}$. We now state the main theorem of this paper:

Theorem 2.1 *Let $\mathcal{D}$ be the billiard table with flat points. Let $f : \mathcal{M} \to \mathbf{R}$ be a bounded $\mathcal{F}_0^{\mathcal{M}}$-measurable function and $\mu(f) = 0$. Then we have:*

$$\lim_{n\to\infty} \mu \left\{ \frac{\mathcal{S}_n f}{\sqrt{n}} \leq t \right\} = \frac{1}{\sqrt{2\pi}\sigma_f} \int_{-\infty}^{t} e^{-\frac{s^2}{2\sigma_f^2}} ds. \tag{2.6}$$

for all $-\infty < t < \infty$. Here:

$$\mathcal{S}_n f = f + f \circ \mathcal{F} + \cdots + f \circ \mathcal{F}^{n-1}.$$

Moreover, $\sigma_f^2 = \frac{\sigma_{\tilde{f}}^2}{\nu(R)}$, where $\sigma_{\tilde{f}}^2$ is given in Theorem 2.3.

Remark *Since $R \in L^{2+\delta}$ with $\delta > 0$ (see Lemma 2.2 below), the bounded condition on f can actually be replaced by $f \in L^{2+2/\delta}(\mathcal{M}, \mu)$, see [48].*

2.2 Induced function

In order to prove Theorem 2.1, we will first prove that the induced function of f also satisfies a central limit theorem. The induced function of f is given by:

$$\tilde{f} := f + f \circ \mathcal{F} + \cdots + f \circ \mathcal{F}^{R-1}.$$

Lemma 2.2 *We have that* $R \in L^{2+\delta}(M, \nu)$ *for any* $0 < \delta < a - 1$.

Proof. The verification of this lemma is straightforward, since:

$$\nu(R > n) \leq C' \cdot n^{-a-1} \tag{2.7}$$

for every $n \geq 1$ and some uniform constant C' (see [18]). We recall that $a = \frac{\beta+2}{\beta-2} > 1$.

□ ◇

Suppose that $f(z) : \mathcal{M} \to \mathbf{R}$ is $\mathcal{F}_0^{\mathcal{M}}$-measurable. Then one can check that $\tilde{f}$ is constant on each cell M_n and furthermore $\tilde{f} \in L^2(M, \nu)$ since $f \in L^\infty(\mathcal{M}, \mu)$.

Theorem 2.3 (CLT for the induced function) *Let* $f : \mathcal{M} \to \mathbf{R}$ *be defined as in Theorem 2.1 and* $\tilde{f}$ *its induced function on* M. *Then we have*

$$\lim_{n\to\infty} \nu \left\{ \frac{S_n\tilde{f} - n\nu(\tilde{f})}{\sqrt{n}} \leq t \right\} = \frac{1}{\sqrt{2\pi}\sigma_{\tilde{f}}} \int_{-\infty}^{t} e^{-\frac{s^2}{2\sigma_{\tilde{f}}^2}} ds. \tag{2.8}$$

for all $-\infty < t < \infty$, *where*

$$S_n\tilde{f} = \tilde{f} + \tilde{f} \circ F + \cdots + \tilde{f} \circ F^{n-1}.$$

and

$$\sigma_{\tilde{f}}^2 = \mathrm{Var}(\tilde{f}) - 2(\mathbf{E}(\tilde{f}|M_1))^2\nu(M_1) + 2\frac{p_{2,2}}{p_{1,2}}\mathbf{E}(\tilde{f}|M_2)\nu(M_2)\big(\mathbf{E}(\tilde{f}|M_1)(p_{1,1}-1) + \mathbf{E}(\tilde{f}|M_2)p_{1,2} - \mathbf{E}(\tilde{f}\circ F|M_1)\big).$$

An important special case of Theorem 2.3 is when $\tilde{f}$ is the return time function:

Corollary 2.4 *Let f be defined by:*

$$f(z) = \begin{cases} 1 & \text{if } z \in \mathcal{M} \setminus M \\ 1 - \nu(R) & \text{if } z \in M, \end{cases} \tag{2.9}$$

then $\tilde{f} = R - \nu(R)$. Thus the (centralised) return time function $R - \nu(R)$ also satisfies the central limit theorem, that is:

$$\lim_{n\to\infty} \nu \left\{ \frac{S_n R - n\nu(R)}{\sqrt{n}} \le t \right\} = \frac{1}{\sqrt{2\pi}\sigma_R} \int_{-\infty}^{t} e^{-\frac{s^2}{2\sigma_R^2}} ds. \tag{2.10}$$

for all $-\infty < t < \infty$, with

$$\sigma_R^2 = \mathrm{Var}(R) - 2(1-\nu(R))^2\nu(M_1) + \frac{p_{2,2}}{p_{1,2}}(4-2\nu(R))\nu(M_2)\Big(p_{1,2} - p_{1,3} + \nu(R)(1+p_{1,3}) - \mathbf{E}(R \circ F | M_1)\Big).$$

Assuming Theorem 2.3, we now show that the Theorem 2.1 is true. This standard result is proved in several references, for example, [2] and [17]. For completeness, we give a proof here. But before we go to the proof of this lemma, we need some basic results.

Lemma 2.5 *For each $n \ge 1$, let $n_x(n)$ be the number of times the point mass comes back to M during the first n iterations. Then for ν-a.e. $x \in M$ we have:*

$$\lim_{n\to\infty} \frac{n}{n_x(n)} = \nu(R).$$

Proof. We first note that, for ν-almost every x, $n_x(n) \to \infty$ as $n \to \infty$. The set of x such that the sequence $\{n_x(n)\}$ is bounded has measure 0: it is the countable union of all preimages of the set $\{R = \infty\}$.

The induced map F is ergodic and $R \in L^1(M,\nu)$, therefore we have, by Birkhoff ergodic theorem:

$$\lim_{n\to\infty} \frac{S_n R}{n} = \nu(R)$$

for almost every $x \in M$. For such an $x \in M$, since $S_{n_x(n)}R \le n < S_{n_x(n)+1}R$, we have that:

$$\frac{S_{n_x(n)}R}{n_x(n)} \le \frac{n}{n_x(n)} \le \frac{S_{n_x(n)+1}R}{n_x(n)+1} \cdot \frac{n_x(n)+1}{n_x(n)}.$$

Therefore we have for almost every $x \in M$ that

$$\lim_{n\to\infty} \frac{n}{n_x(n)} = \nu(R). \qquad \square$$

$\diamond$

Corollary 2.6 *We have:*

$$\lim_{n\to\infty} \nu\left(\frac{n_x(n) - n/\nu(R)}{\sqrt{n}} \le t\right) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{t} e^{-\frac{s^2}{2\sigma^2}}\,ds.$$

for all $t \in (-\infty,\infty)$ *and* $\sigma^2 = \sigma_R^2/(\nu(R))^3$.

Lemma 2.7 *Theorem 2.3 implies Theorem 2.1.*

Proof.

In this proof, we will assume for simplicity that the function f is bounded. See [33], Appendix A for a similar but longer proof of the more general case. Without loss of generality, assume that $\mu(f) = 0$ and therefore we also have $\nu(\tilde{f}) = 0$. Let $m = m(n) = \lfloor n/\nu(R) \rfloor$. Corollary 2.6 implies that for any $\epsilon > 0$, there exists $A_\varepsilon > 0$ such that

$$\nu(|n_x - m| \ge A\sqrt{n}) \le \varepsilon.$$

First we prove that with respect to μ on $\mathcal{M}$ we have:

$$\frac{\mathcal{S}_n f \circ \Pi}{\sqrt{n}} \implies N(0, \sigma_f^2).$$

We have:

$$\frac{\mathcal{S}_n f}{\sqrt{n}} = \frac{S_{n_1}\tilde{f}}{\sqrt{n}} + \frac{S_{n_2}\tilde{f} - S_{n_1}\tilde{f}}{\sqrt{n}} + \frac{\mathcal{S}_n f - S_{n_2}\tilde{f}}{\sqrt{n}}.$$

The first term converges to $N(0, \sigma_f^2)$ with respect to ν by our assumption and $\sigma_f^2 = \sigma_{\tilde{f}}^2/\nu(R)$. The second and third terms converge to 0 in probability, by Birkhoff ergodic theorem and the fact that f is a bounded function. Thus we have shown that on (M, ν):

$$\frac{\mathcal{S}_n f}{\sqrt{n}} \implies N(0, \sigma_f^2). \tag{2.11}$$

We define a new probability measure ξ on M by $d\xi = R/\nu(R)d\nu$. Since $\xi << \nu$, the central limit theorem (2.11) also holds with respect to ξ. We have:

$$\int_{\mathcal{M}} exp\left(it\frac{\mathcal{S}_n f \circ \Pi}{\sqrt{n}}\right) d\mu = \int_M R \exp\left(it\frac{\mathcal{S}_n f}{\sqrt{n}}\right) d\mu = \int_M \frac{R}{\nu(R)} \exp\left(it\frac{\mathcal{S}_n f}{\sqrt{n}}\right) d\nu. \tag{2.12}$$

This shows that on $(\mathcal{M}, \mu)$:

$$\frac{\mathcal{S}_n f \circ \Pi}{\sqrt{n}} \implies N(0, \sigma_f^2).$$

To complete the prove of this lemma, we will show that $\frac{\mathcal{S}_n f}{\sqrt{n}} - \frac{\mathcal{S}_n f \circ \Pi}{\sqrt{n}} \longrightarrow 0$ in probability.

$$\begin{aligned}\mathcal{S}_n f(y,i) - \mathcal{S}_n f(y,0) &= \sum_{k=0}^{n-1} f \circ \mathcal{F}^k(y,i) - \sum_{k=0}^{n-1} f \circ \mathcal{F}^k(y,0) \\ &= -\sum_{k=0}^{i-1} f \circ \mathcal{F}^k(y,0) + \sum_{k=n}^{n+i-1} f \circ \mathcal{F}^k(y,0) \\ &= -\sum_{k=0}^{i-1} f(y,k) + \sum_{k=0}^{i-1} f \circ \mathcal{F}^n(y,k).\end{aligned}$$

Since $|\mathcal{S}_n f(y,i) - \mathcal{S}_n f(y,0)| \leq 2 \|f\|_\infty R$, we have that

$$\frac{\mathcal{S}_n f}{\sqrt{n}} - \frac{\mathcal{S}_n f \circ \Pi}{\sqrt{n}} \longrightarrow 0 \text{ in probability.}$$

Thus we have shown that Theorem 2.3 implies Theorem 2.1. $\square$ $\diamond$

2.3 Central limit theorem for the induced function

We devote this section to prove a central limit theorem on the induced system (M, F, ν) of which Theorem 2.3 is a special case:

Theorem 2.8 *Let $X : M \to \mathbf{R}$ be an $\mathcal{F}_0$-measurable function such that $X \in L^2(M,\nu)$ and $\mathbf{E}(X) = 0$. Then*

$$\frac{S_n X}{\sqrt{n}} \Rightarrow N(0, \sigma_X^2), \tag{2.13}$$

where the variance σ_X^2 is given by formula (2.31).

There is a filtration of σ-algebras on M:

$$\mathcal{F}_n = \sigma(R \circ F^k : -n \leq k \leq n) \tag{2.14}$$

for $n \geq 0$ and $\mathcal{F}_n = \{\emptyset, M\}$ for $n < 0$. Let $X_n = X \circ F^n$ for $n \geq 0$. Because F preserves the probability measure ν, the sequence $\{X_n\}_{n\geq 0}$ is a stationary stochastic process adapted to the filtration $\{\mathcal{F}_n\}$. By replacing X by $X - \mathbf{E}(X)$, we can assume that $\mathbf{E}(X) = 0$.

Our method in proving that $\frac{S_n X}{\sqrt{n}}$ converges to a normal distribution as $n \to \infty$ is to approximate the Birkhoff sum by a series of martingale differences for which a central limit theorem is already proved, see [36]:

Lemma 2.9 *Let $\{Z_j : j \geq 1\}$ be a stationary ergodic sequence of martingale differences such that $\mathbf{E}(Z_1^2) = \sigma^2 < \infty$. Then we have*

$$\frac{S_n Z}{\sqrt{n}} \implies N(0, \sigma^2).$$

The convergence here is in distribution.

Our approximation is as follows. Fix any large integer $k \geq 1$. Then for any $n \geq 1$ we have a decomposition:

$$X_n = \mathbf{E}(X_n|\mathcal{F}_{n-k}) + h_k \circ F^{n-1} + u_n^k - v_n^k, \tag{2.15}$$

where $h_k = \sum_{i=1}^k \big(\mathbf{E}(X_i|\mathcal{F}_1) - \mathbf{E}(X_i|\mathcal{F}_0)$, $v_{n-1}^k = u_n^k$ and

$$u_n^k = \sum_{i=0}^{k-2} \left(\mathbf{E}(X_{n+i}|\mathcal{F}_{n-1}) - \mathbf{E}(X_{n+i}|\mathcal{F}_{n-k+i})\right). \tag{2.16}$$

Therefore:

$$X_0 + \cdots + X_{n-1} = \sum_{i=0}^{n-2} h_k \circ F^i + \mathbf{E}(X_0 + \cdots + X_{k-1}|\mathcal{F}_0) - v_{n-1}^k + \sum_{i=k}^{n-1} \mathbf{E}(X_i|\mathcal{F}_{i-k}). \tag{2.17}$$

Note that $\mathbf{E}(X_i|\mathcal{F}_{i-k}) = \mathbf{E}(X_k|\mathcal{F}_0) \circ F^{i-k}$ for $i \geq k$, and $v_n^k = v_k^k \circ F^{n-k}$. We have:

$$\begin{aligned}
&\limsup_{n\to\infty} n^{-1}\mathbf{E}\big(X_0 + \cdots + X_{n-1} - \sum_{i=0}^{n-2} h_k \circ F^i\big)^2 \\
&= \limsup_{n\to\infty} n^{-1}\mathbf{E}\Big(\mathbf{E}(X_0 + \cdots + X_{k-1}|\mathcal{F}_0) - v_{n-1}^k + \sum_{i=k}^{n-1} \mathbf{E}(X_i|\mathcal{F}_{i-k})\Big)^2 \\
&\leq 3\limsup_{n\to\infty} n^{-1}\mathbf{E}\Big(\sum_{i=k}^{n-1} \mathbf{E}(X_i|\mathcal{F}_{i-k})\Big)^2 \\
&= 3\limsup_{n\to\infty} \left(\frac{n-k}{n}\mathbf{E}\big(\mathbf{E}(X_k|\mathcal{F}_0)\big)^2 + \frac{2}{n}\sum_{i=1}^{n-k-1}(n-k-i)\mathbf{E}\Big(\mathbf{E}(X_k|\mathcal{F}_0)\cdot\mathbf{E}(X_k|\mathcal{F}_0)\circ F^i\Big)\right).
\end{aligned}$$

Since $X : M \to \mathbf{R}$ is an $\mathcal{F}_0$-measurable function, we can compute the quantities $\mathbf{E}(X_k|\mathcal{F}_0)$ rather explicitly.

Lemma 2.10 *Let* $\mathbf{E}(X_k|\mathcal{F}_0) = \sum_{n=1}^{\infty} a_n^{(k)} \chi_{M_n}$, *where* $a_n^{(k)} = \mathbf{E}(X_k|M_n)$ *for* $n \geq 1$ *and* $k \geq 0$. *We have a recurrence relation:*

$$a_i^{(k+1)} = \sum_{m=1}^{\infty} a_m^{(k)} p_{i,m} \text{ for } i = 1,2, \text{ and } a_n^{(k+1)} = a_1^{(k)} \text{ for } k \geq 0 \text{ and } n \geq 3. \tag{2.18}$$

Moreover,

$$\lim_{k\to\infty} a_i^{(k)} = 0, \text{ for } i = 1,2. \tag{2.19}$$

Proof. Suppose that $\mathbf{E}(X_k|\mathcal{F}_0) = \sum_{n=1}^{\infty} a_n^{(k)} \chi_{M_n}$. Then

$$\mathbf{E}(X_{k+1}|\mathcal{F}_1) = \mathbf{E}(X_k|\mathcal{F}_0) \circ F = \sum_{n=1}^{\infty} a_n^{(k)} \chi_{F^{-1}M_n},$$

and thus:

$$\mathbf{E}(X_{k+1}|\mathcal{F}_0) = \sum_{n=1}^{\infty} \mathbf{E}(\mathbf{E}(X_{k+1}|\mathcal{F}_1)|M_n)\chi_{M_n} = \sum_{n=1}^{\infty} \big(\sum_{m=1}^{\infty} a_m^{(k)} p_{n,m}\big)\chi_{M_n},$$

where we define $p_{n,m}$ as in (2.5). for $n, m \geq 1$. It is straightforward that $\sum_{m=1}^{\infty} p_{nm} = 1$ for any $n \geq 1$ since the cells M_n's are disjoint and the map F is invertible. Suppose that $x \in M_n$; that means the point mass will enter the neighbourhood of the flat points and come out after $n-1$ collisions with the boundary. For $n \geq 3$, by shrinking the neighbourhood if necessary, once the point mass come out it will not come back to the neighbourhood after at least 2 collisions with the good part of the boundary of the table. That is to say $F^{-1}M_1 \cap M_n = M_n$, hence $p_{n,1} = 1$, for $n \geq 3$. In essence, we have a three-state Markov chain. Therefore we have that:

$$\mathbf{E}(X_{k+1}|\mathcal{F}_0) = \Big(\sum_{m=1}^{\infty} a_m^{(k)} p_{1,m}\Big)\chi_{M_1} + \Big(\sum_{m=1}^{\infty} a_m^{(k)} p_{2,m}\Big)\chi_{M_2} + a_1^{(k)} \sum_{n=3}^{\infty} \chi_{M_n}. \tag{2.20}$$

Let $z_k = (a_1^{(k)}, a_2^{(k)}, a_1^{(k-1)})^t$, and

$$(A_{ij}) = \begin{pmatrix} p_{1,1} & p_{1,2} & 1 - p_{1,1} - p_{1,2} \\ p_{2,1} & 1 - p_{2,1} & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.21}$$

The recurrence can then be written in matrix form as:

$$z_{k+1} = Az_k \quad \text{for } k \geq 1; \qquad z_1 = (a_1^{(1)}, a_2^{(1)}, a_1^{(0)})^t. \tag{2.22}$$

We note that the first row of A is strictly positive, thus A is an irreducible, aperiodic stochastic matrix and the unique stationary probability vector is $\pi = (\nu(M_1), \nu(M_2), \nu(M_{n\geq 3}))$:

$$\nu(M_1)A_{12} = \nu(M_2)A_{21}$$

$$\nu(M_1)A_{13} = \nu(M_{n\geq 3})$$

It follows that $\lim_{k\to\infty} a_i^{(k)} = \pi\cdot z_1$ for $i = 1, 2$. Furthermore, $\pi\cdot z_1 = \mathbf{E}(\mathbf{E}(X_1|\mathcal{F}_0)) = 0$. Thus we have:

$$\lim_{k\to\infty} a_i^{(k)} = 0, \text{ for } i = 1, 2. \qquad \square$$

$\diamond$

Lemma 2.11

$$\lim_{k\to\infty} \mathbf{E}(\mathbf{E}(X_k|\mathcal{F}_0))^2 = 0.$$

Proof. We recall that

$$\mathbf{E}(X_k|\mathcal{F}_0) = \sum_{n\geq 1} a_n^{(k)} \chi_{M_n} = a_1^{(k)}\chi_{M_1} + a_2^{(k)}\chi_{M_2} + a_1^{(k-1)}(1 - \chi_{M_1} - \chi_{M_2}).$$

Therefore:

$$\mathbf{E}\left(\mathbf{E}(X_k|\mathcal{F}_0)\right)^2 = (a_1^{(k)})^2\nu(M_1) + (a_2^{(k)})^2\nu(M_2) + (a_1^{(k-1)})^2(1 - \nu(M_1) - \nu(M_2)).$$

Thus we have:

$$\lim_{k\to\infty} \mathbf{E}\left(\mathbf{E}(X_k|\mathcal{F}_0)\right)^2 = 0. \qquad \square$$

$\diamond$

Lemma 2.12

$$\lim_{k\to\infty} \sum_{i=1}^{\infty} \mathbf{E}\big(\mathbf{E}(X_k|\mathcal{F}_0) \cdot \mathbf{E}(X_k|\mathcal{F}_0) \circ F^i\big) = 0.$$

Proof. As before we have

$$\mathbf{E}(X_k|\mathcal{F}_0)\cdot\mathbf{E}(X_{k+i}|\mathcal{F}_0) = a_1^{(k)}a_1^{(k+i)}\chi_{M_1} + a_2^{(k)}a_2^{(k+i)}\chi_{M_2} + a_1^{(k-1)}a_1^{(k+i-1)}(1 - \chi_{M_1} - \chi_{M_2}).$$

Taking the expectation we have:

$$\begin{aligned}\mathbf{E}(\mathbf{E}(X_k|\mathcal{F}_0)\cdot\mathbf{E}(X_{k+i}|\mathcal{F}_0)) &= a_1^{(k)}a_1^{(k+i)}\nu(M_1)+a_2^{(k)}a_2^{(k+i)}\nu(M_2)+\\ &+a_1^{(k-1)}a_1^{(k+i-1)}(1-\nu(M_1)-\nu(M_2))\\ &= a_1^{(k)}\nu(M_1)(a_1^{(k+i)}-a_1^{(k+i-1)})+a_2^{(k)}\nu(M_2)(a_2^{(k+i)}-a_1^{(k+i-1)}).\end{aligned}$$

To deal with the last term, we have for $n\geq 2$ that:

$$a_2^{(n)}-a_1^{(n-1)}=(a_2^{(n-1)}-a_1^{(n-1)})A_{22} \tag{2.23}$$

$$a_2^{(n-1)}-a_1^{(n-1)}=\frac{a_1^{(n)}-a_1^{(n-1)}}{A_{12}}+\frac{(a_1^{(n-1)}-a_1^{(n-2)})A_{13}}{A_{12}}. \tag{2.24}$$

Thus the series $\sum_{i=1}^{\infty}\mathbf{E}(\mathbf{E}(X_k|\mathcal{F}_0)\cdot\mathbf{E}(X_k|\mathcal{F}_0)\circ F^i)$ is in fact a telescoping series and noting that $a_i^{(k)}\to 0$ as $k\to\infty$ for $i=1,2$, it must be the case that:

$$\lim_{k\to\infty}\sum_{i=1}^{\infty}\mathbf{E}(\mathbf{E}(X_k|\mathcal{F}_0)\cdot\mathbf{E}(X_k|\mathcal{F}_0)\circ F^i)=0. \qquad \square$$

$\diamond$

Thus for any positive sequence $\varepsilon_k\to 0$ as $k\to\infty$, there exists a sequence $n_k\to\infty$ as $k\to\infty$ such that

$$\limsup_{n\to\infty} n^{-1}\mathbf{E}\big(X_0+\cdots+X_{n-1}-\sum_{i=0}^{n-2}h_{n_k}\circ F^i\big)^2<\varepsilon_k.$$

The sequence $\{h_{n_k}\circ F^i\}_{i\geq 0}$ is a stationary sequence of martingale differences adapted to the filtration $\{\mathcal{F}_i\}$. The CLT holds for this sequence:

$$n^{-1/2}\sum_{i=0}^{n-1}h_{n_k}\circ F^i \implies N(0,\sigma_k^2) \tag{2.25}$$

where $\sigma_k^2=\mathbf{E}(h_{n_k}^2)$.

Next, we show that the sequence $\{\sigma_k\}$ converges to some limit as $k \to \infty$.

$$\begin{aligned}(\sigma_i - \sigma_j)^2 &\leq \mathbf{E}\Big(h_{n_i} - h_{n_j}\Big)^2 \\ &= n^{-1}\mathbf{E}\Big(\sum_{m=0}^{n-1}(h_{n_i} - h_{n_j}) \circ F^m\Big)^2 \\ &\leq 2(\varepsilon_i + \varepsilon_j).\end{aligned}$$

Therefore $\{\sigma_k\}$ is a Cauchy sequence and hence $\sigma_k \to \sigma_X$ as $k \to \infty$ for some constant σ_X and

$$\lim_{n\to\infty} \mathbf{E}\left(\frac{S_n X}{\sqrt{n}}\right)^2 = \sigma_X^2.$$

Finally, the variance σ_X^2 can be computed directly as below:

For any $n \geq 1$:

$$\begin{aligned}\mathrm{Cov}(X, X \circ F^n) &= \mathbf{E}\left(\mathbf{E}(X \circ F^n | \mathcal{F}_0) \cdot X\right) \\ &= \mathbf{E}\left(\left(a_1^{(n)}\chi_{M_1} + a_2^{(n)}\chi_{M_2} + a_1^{(n-1)}\sum_{m\geq 3}\chi_{M_m}\right) \cdot \sum_{m=1}^{\infty} a_m^{(0)}\chi_{M_m}\right) \\ &= a_1^{(n)}a_1^{(0)}\nu(M_1) + a_2^{(n)}a_2^{(0)}\nu(M_2) + a_1^{(n-1)}\sum_{m=3}^{\infty} a_m^{(0)}\nu(M_m) \\ &= (a_1^{(n)} - a_1^{(n-1)})a_1^{(0)}\nu(M_1) + (a_2^{(n)} - a_1^{(n-1)})a_2^{(0)}\nu(M_2).\end{aligned}$$

In particular, for $n = 1$:

$$\mathrm{Cov}(X, X \circ F) = (a_1^{(1)} - a_1^{(0)})a_1^{(0)}\nu(M_1) + (a_2^{(1)} - a_1^{(0)})a_2^{(0)}\nu(M_2) \tag{2.26}$$

$$= (a_1^{(1)} - a_1^{(0)})a_1^{(0)}\nu(M_1) + (a_2^{(0)} - a_1^{(0)})A_{22}a_2^{(0)}\nu(M_2). \tag{2.27}$$

For $n \geq 2$, we have:

$$a_2^{(n)} - a_1^{(n-1)} = (a_2^{(n-1)} - a_1^{(n-1)})A_{22} \tag{2.28}$$

$$a_2^{(n-1)} - a_1^{(n-1)} = \frac{a_1^{(n)} - a_1^{(n-1)}}{A_{12}} + \frac{(a_1^{(n-1)} - a_1^{(n-2)})A_{13}}{A_{12}}. \tag{2.29}$$

Therefore:

$$\begin{aligned}
&\operatorname{Cov}(X, X \circ F^n) = (a_1^{(n)} - a_1^{(n-1)})a_1^{(0)}\nu(M_1) \\
&+ \left(a_1^{(n)} - a_1^{(n-1)} + (a_1^{(n-1)} - a_1^{(n-2)})A_{13}\right) \frac{A_{22}}{A_{12}} a_2^{(0)}\nu(M_2). \\
&= (a_1^{(n)} - a_1^{(n-1)})(a_1^{(0)}\nu(M_1) + W) + (a_1^{(n-1)} - a_1^{(n-2)})A_{13}W,
\end{aligned}$$

where:

$$W = \frac{A_{22}}{A_{12}} a_2^{(0)}\nu(M_2). \tag{2.30}$$

We can then compute the variance of $\frac{S_n X}{\sqrt{n}}$ as follows:

$$\begin{aligned}
\operatorname{Var}\left(\frac{S_n X}{\sqrt{n}}\right) &= \operatorname{Var}(X) + \frac{2}{n}\sum_{k=1}^{n-1}(n-k)\operatorname{Cov}(X, X \circ F^k) \\
&= \operatorname{Var}(X) + 2\sum_{k=1}^{n-1}\operatorname{Cov}(X, X \circ F^k) - \frac{2}{n}\sum_{k=1}^{n-1} k\operatorname{Cov}(X, X \circ F^k).
\end{aligned}$$

The second term is:

$$\sum_{k=1}^{n-1}\operatorname{Cov}(X, X \circ F^k) = \operatorname{Cov}(X, X \circ F) + (a_1^{(n-1)} - a_1^{(1)})(a_1^{(0)}\nu(M_1) + W) + (a_1^{(n-2)} - a_1^{(0)})A_{13}W.$$

Taking limit as $n \to \infty$, the third term converges to 0 by Kronecker's lemma or by direct verificaton. Thus we have:

$$\sigma_X^2 = \lim_{n\to\infty}\operatorname{Var}\left(\frac{S_n X}{\sqrt{n}}\right) = \operatorname{Var}(X) - 2(a_1^{(0)})^2\nu(M_1) + 2W\left(a_1^{(0)}A_{11} + a_2^{(0)}A_{12} - a_1^{(0)} - a_1^{(1)}\right). \tag{2.31}$$

Thus we have shown that $\frac{S_n X}{\sqrt{n}} \implies N(0, \sigma_X^2)$ in distribution and completed the proof of Theorem 2.8.

Remark *Our method also works for functions X that are $\mathcal{F}_m$-measurable for any $m \geq 0$. The martingale approximation is virtually the same, and the estimations of the errors are easily reduced to estimation of the case X is $\mathcal{F}_0$-measurable since we are dealing with stationary stochastic sequences. Thus the central limit theorem actually holds for a much larger class of observables than those considered in Theorem 2.1. However, a drawback is that a formula for the diffusion constant would be more complicated.*

CHAPTER 3

INVARIANCE PRINCIPLES FOR ERGODIC SYSTEMS WITH SLOWLY α-MIXING INDUCING BASE

3.1 Introduction

As a functional generalization of the central limit theorems, the almost sure invariance principle (ASIP) asserts the the partial sum of a random process can be well approximated by a Brownian motion with an almost sure error. There has been a great deal of work on the invariance principles in probability theory, such as [55, 3, 29, 59, 68, 23], etc., as well as in the context of dynamical systems, for instance, [9, 13, 69, 70, 56, 37, 62, 50, 14, 51, 34, 2, 25, 38], etc.. Three major approaches are exploited in the proof of invariance principles: (1) the martingale approximation method (e.g. [55, 14]); (2) the inducing and Young towers (e.g. [50, 51]); (3) the

This chapter is a slightly modified version of [12].

spectral method for transfer operators (e.g. [56, 34]).

In the paper, we study the almost sure invariance principle (ASIP) for a class of ergodic dynamical systems with a slowly α-mixing inducing base. Our setting is rather abstract, and does not have any smooth structures. Also, we assume very low regularity for the observable that generates the stationary process, that is, the observable is only integrable but could be unbounded. In this situation, we are able to prove the ASIP for stationary processes that are generated by any adapted observables. Although adapted observables might be a quite narrowed class of functions, they can provide good approximations for most regular observales.

This paper is organized as follows. In Section 3.2, we shall introduce Assumption (**H**1) on the inducing base and Assumption (**H**2) for the first return time, and state our main theorem. In Section 3.3, we deliver the proof of the ASIP in four subsections. In Section 3.4, we apply our main result to intermittent maps and billiards with flat points.

3.2 Statement of Results

Let $\mathcal{T}$ be an ergodic measure-preserving transformation on a standard probability space $(\mathcal{M}, \mathcal{B}, \mu)$. We choose a subset $M \subset \mathcal{M}$ of positive μ-measure, and denote the first return time to M by

$$R(x) = \inf\{n \geq 1 : \mathcal{T}^n(x) \in M\}, \quad \text{for any } x \in M.$$

Consider the induced base transformation $T : (M, \mathcal{B}_M, \nu) \circlearrowleft$, where

- $T(x) = \mathcal{T}^{R(x)}(x)$ for any $x \in M$;
- $\mathcal{B}_M := \{B \cap M : \ B \in \mathcal{B}\}$;
- ν is the conditional measure of μ on M, i.e., $\nu(\cdot) = \mu(\cdot|\ M)$.

By Poincaré recurrence and the ergodicity of $\mathcal{T}$, we have

$$M = \bigcup_{n=1}^{\infty} \{R = n\} \pmod{\nu}, \quad \text{and} \quad \mathcal{M} = \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{n-1} \mathcal{T}^k \{R = n\} \pmod{\mu}.$$

Remark *The induced map T must be ergodic, since the original map $\mathcal{T}$ is ergodic. However, $\mathcal{T}$ may not be mixing, even if T is mixing.*

We now impose the following assumptions.

(**H**1) T admits a generating partition ξ, i.e., $\mathcal{F}_0^\infty = \mathcal{B}_M \pmod{\nu}$, where $\mathcal{F}_s^t := \sigma\left(T^{-s}\xi \vee \cdots \vee T^{-t}\xi\right)$. for any $0 \leq s \leq t \leq \infty$. Moreover, the family $\mathfrak{F} := \{\mathcal{F}_s^t\}_{0 \leq s \leq t \leq \infty}$ is α-mixing with polynomial rate $\mathcal{O}(n^{-\beta})$ for some $\beta > 2$, that is,

$$\alpha_{\mathfrak{F}}(n) = \sup_{t \geq 0} \sup_{A \in \mathcal{F}_0^t} \sup_{B \in \mathcal{F}_{t+n}^\infty} |\nu(A \cap B) - \nu(A)\nu(B)| = \mathcal{O}(n^{-\beta}). \tag{3.1}$$

(**H**2) $R \in L^p(M, \nu)$ for some $p > 2$ satisfying $\frac{1}{\beta} + \frac{1}{p} < \frac{1}{2}$, or equivalently,

$$\nu\{R > k\} = \mathcal{O}(k^{-p}). \tag{3.2}$$

Refining ξ if necessary, one may assume that $\{R = n\} \in \mathcal{F}_0^0$ for each $n \geq 1$. We then naturally lift the partition ξ to the partition $\widetilde{\xi}$ on $\mathcal{M}$, to be precise,

$$\widetilde{\xi} := \left\{A \subset \mathcal{T}^k\{R = n\} : \ \mathcal{T}^{-k}A \in \xi, \ n \geq 1, \ 0 \leq k \leq n-1\right\}.$$

It is clear that $\widetilde{\xi}$ is a generating partition for $\mathcal{T}$. We denote $\widetilde{\mathcal{F}}_s^t := \sigma\left(\mathcal{T}^{-s}\widetilde{\xi} \vee \cdots \vee \mathcal{T}^{-t}\widetilde{\xi}\right)$ for any $0 \le s \le t \le \infty$.

A measurable function $f : \mathcal{M} \to \mathbb{R}$ (or $f : M \to \mathbb{R}$) is said to be an *adapted* function if f is $\widetilde{\mathcal{F}}_s^t$-measurable (or $\mathcal{F}_s^t$-measurable) for some $0 \le s \le t < \infty$. In particular, the first return time R is adapted.

Our main result is the following.

Theorem 3.1 *Let $q > 2$ be such that $\frac{1}{\beta} + \frac{1}{p} + \frac{1}{q} < \frac{1}{2}$. Suppose that $f \in L^q(\mathcal{M}, \mu)$ with $\mathbb{E}_\mu(f) = 0$, and f is an adapted function on $\mathcal{M}$. Then the stationary process $\mathbf{X}_f := \{f \circ \mathcal{T}^n\}_{n \ge 0}$ satisfies an almost sure invariance principle (ASIP) as follows: for any $\lambda \in \left(\max\left\{\frac{1}{4}, \frac{1}{\beta} + \frac{1}{p} + \frac{1}{q}\right\}, \frac{1}{2}\right)$, enlarging to a richer probability space $(\mathcal{M}', \mu')$ if necessary, there exists a standard Brownian motion $W(\cdot)$ such that*

$$\left|\sum_{k=0}^{n-1} f \circ \mathcal{T}^k - W(n\sigma^2)\right| = \mathcal{O}(n^\lambda), \quad \mu' - a.s. \tag{3.3}$$

where $\sigma = \sigma(f)$ is defined by (3.18) *in Section 3.3.4.*

It is obvious from (3.3) that $\sigma = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}_\mu\left(\sum_{k=0}^{n-1} f \circ \mathcal{T}^k\right)^2$. We shall provide an alternative formula in (3.18) for σ from the induced system.

Remark *We could easily extend Theorem 3.1 in the invertible case, with the only modification on the families $\mathcal{F}_s^t$ and $\widetilde{\mathcal{F}}_s^t$ to be two sided, i.e., $-\infty \le s \le t \le \infty$.*

3.3 Proof of Theorem 3.1

3.3.1 The induced function $\widehat{f}$

For any measurable function $f : \mathcal{M} \to \mathbb{R}$, we define the induced function on M by

$$\widehat{f}(x) := \sum_{k=0}^{R(x)-1} f \circ \mathcal{T}^k(x), \quad x \in M.$$

Lemma 3.2 *Let $f : \mathcal{M} \to \mathbb{R}$ be a function that satisfies Theorem 3.1. Then*

(1) $\mathbb{E}_\nu(\widehat{f}) = 0$;

(2) $\widehat{f} \in L^r(M, \nu)$ for any $r \in \left(2, \frac{pq}{p+q}\right)$;

(3) For each $n \geq 0$, the function $\widehat{f} \circ T^n$ is adapted on M.

Proof. (1) By Kac formula, i.e., $\int_M \widehat{f} d\mu = \int f d\mu$, and the fact that $\nu(\cdot) = \mu(\cdot|M)$, we have that $\mathbb{E}_\nu(\widehat{f}) = 0$ if $\mathbb{E}_\mu(f) = 0$.

(2) Note that $\widehat{f} = \sum_{k=0}^\infty f \circ \mathcal{T}^k 1_{\{R>k\}}$, then by Minkowski's inequality, Hölder inequality and $\mathcal{T}$-invariance of μ, we have

$$\begin{aligned}
\|\widehat{f}\|_{L^r(\nu)} &\leq \sum_{k=0}^\infty \||f| \circ \mathcal{T}^k 1_{\{R>k\}}\|_{L^r(\nu)} \\
&= \mu(M)^{-\frac{1}{r}} \sum_{k=0}^\infty \left(\int |f|^r \circ \mathcal{T}^k 1_{\{R>k\}} d\mu \right)^{\frac{1}{r}} \\
&\leq \mu(M)^{-\frac{1}{r}} \sum_{k=0}^\infty \|f \circ \mathcal{T}^k\|_{L^q(\mu)} \left(\mu\{R > k\}\right)^{1/r-1/q} \\
&= \mu(M)^{-\frac{1}{q}} \|f\|_{L^q(\mu)} \sum_{k=0}^\infty \left(\nu\{R > k\}\right)^{1/r-1/q}.
\end{aligned}$$

The last summation is finite due to Condition (3.2), i.e.,

$$\sum_{k=0}^{\infty} (\mu\{R > k\})^{1/r-1/q} = 1 + \mathcal{O}\left(\sum_{k=1}^{\infty} \left(k^{-p}\right)^{1/r-1/q}\right) < \infty,$$

since $p(1/r - 1/q) > 1$. Therefore, $\|\widehat{f}\|_{L^r(\nu)} < \infty$ and thus $\widehat{f} \in L^r(M, \nu)$.

(3) Since f is adapted, there are $0 \leq s \leq t < \infty$ such that f is $\widetilde{\mathcal{F}}_s^t$-measurable. It is easy to see that $\widehat{f}$ is $\mathcal{F}_s^t$-measurable. Moreover, we have that $\widehat{f} \circ T^n$ is $\mathcal{F}_{s+n}^{t+n}$-measurable for each $n \geq 0$, since $T^{-n}\mathcal{F}_s^t = \mathcal{F}_{s+n}^{t+n}$. $\diamond$

We shall first study the induced process $\mathbf{X}_{\widehat{f}} := \{\widehat{f} \circ T^n\}_{n \geq 1}$ on (M, ν).

3.3.2 ASIP for the induced process $\mathbf{X}_{\widehat{f}}$

In this subsection, we establish an ASIP for the induced process $\mathbf{X}_{\widehat{f}} = \{\widehat{f} \circ T^n\}_{n\geq 1}$. We first recall the following special case of an ASIP result by Shao and Lu [59].

Definition 3.3 *Given a random process* $\mathbf{X} = \{X_n\}_{n\geq 0}$ *on* (M, ν)*, we denote*

$$\mathcal{G}_m^n(\mathbf{X}) := \sigma\left\{X_m, X_{m+1}, \ldots, X_n\right\}$$

for any $0 \leq m \leq n \leq \infty$*. The* α*-mixing coefficient of the process is defined by*

$$\alpha_{\mathbf{X}}(n) = \sup_{k \geq 0} \sup_{A \in \mathcal{G}_0^k(\mathbf{X})} \sup_{B \in \mathcal{G}_{k+n}^{\infty}(\mathbf{X})} \left|\nu(A \cap B) - \nu(A)\nu(B)\right|.$$

Proposition 3.4 *Let* $\delta \in (0, 2]$ *and* $r \in (2 + \delta, \infty]$*. If* $\mathbf{X} = \{X_n\}_{n\geq 0}$ *is a zero-mean random process such that*

(i) $\sup_{n\geq 0} \|X_n\|_{L^r} < \infty$*;*

(ii) $\sum_{n=1}^{\infty} \alpha_{\mathbf{X}}(n)^{\frac{1}{2+\delta} - \frac{1}{r}} < \infty$*;*

(iii) $\liminf_{n\to\infty} \frac{a_n}{n} > 0$, *where* $a_n := \mathbb{E}_\nu(\sum_{k=0}^{n-1} X_k)^2$,

then for any $\epsilon > 0$, *enlarging to a richer probability space* (M', ν') *if necessary, there exists a standard Brownian motion* $W(\cdot)$ *such that*

$$\left|\sum_{k=0}^{n-1} X_k - W(a_n)\right| = \mathcal{O}\left(a_n^{\frac{1}{2+\delta}+\epsilon}\right), \quad \nu' - a.s.$$

We now directly apply Proposition 3.4 to adapted stationary processes on (M, ν).

Lemma 3.5 *Let* $r > 2$ *be such that* $\frac{1}{\beta} + \frac{1}{r} < \frac{1}{2}$. *Suppose that* $g \in L^r(M, \nu)$ *with* $\mathbb{E}_\nu(g) = 0$, *and* g *is an adapted function on* M. *Then the stationary process* $\mathbf{X}_g = \{g \circ T^n\}_{n\geq 0}$ *satisfies an ASIP as follows: for any* $\lambda \in \left(\max\left\{\frac{1}{4}, \frac{1}{\beta} + \frac{1}{r}\right\}, \frac{1}{2}\right)$, *enlarging to a richer probability space* (M', ν') *if necessary, there exists a standard Brownian motion* $W(\cdot)$ *such that*

$$\left|\sum_{k=0}^{n-1} g \circ T^k - W\left(n\sigma_g^2\right)\right| = \mathcal{O}\left(n^\lambda\right), \quad \nu' - a.s. \tag{3.4}$$

where σ_g^2 *is given by*

$$\sigma_g^2 := \sum_{n=-\infty}^{\infty} \mathbb{E}_\nu(g \cdot g \circ T^n) = \sum_{n=-\infty}^{\infty} \int g \cdot g \circ T^n \, d\nu. \tag{3.5}$$

Proof. In the degenerate case when $\sigma_g = 0$, it is well known that g is a coboundary, i.e., there exists a measurable function $h : M \to \mathbb{R}$ such that $g = h - h \circ T$ (see e.g. [40], Theorem 18.2.2), and thus (3.8) is automatic.

We now consider the non-degenerate case when $\sigma_g > 0$, and check conditions in Proposition 3.4 for the stationary process $\mathbf{X}_g := \{g \circ T^n\}_{n\geq 0}$ as follows.

As $\lambda \in \left(\max\left\{\frac{1}{4}, \frac{1}{\beta} + \frac{1}{r}\right\}, \frac{1}{2}\right)$, we pick a sufficiently small $\delta \in \left(0, \frac{1}{\lambda} - 2\right)$ such that $\frac{1}{r} < \frac{1}{2+\delta} - \frac{1}{\beta}$. By T-invariance of ν, we have $\mathbb{E}_\nu(g \circ T^n) = \mathbb{E}_\nu(g) = 0$ for any

$n \geq 0$, that is, the process is of zero mean. Also, $\|g \circ T^n\|_{L^r(\nu)} = \|g\|_{L^r(\nu)}$, and thus Condition (i) in Proposition 3.4 holds.

For Condition (ii), we recall that $\mathcal{G}_m^n(\mathbf{X}_g)$ is the σ-algebra generated by $g \circ T^m, \ldots, g \circ T^n$, where $0 \leq m \leq n \leq \infty$. Since g is an adapted function, there are some $0 \leq s \leq t < \infty$ such that g is $\mathcal{F}_s^t$-measurable. Therefore, $g \circ T^n$ is $\mathcal{F}_{s+n}^{t+n}$-measurable, and hence $\mathcal{G}_m^n(\mathbf{X}_g) \subset \mathcal{F}_{s+m}^{t+n}$. Hence by (3.1),

$$\alpha_{\mathbf{X}_g}(n) \leq \alpha_{\mathfrak{F}}(n+s-t) = \mathcal{O}\left((n+s-t)^{-\beta}\right) = \mathcal{O}\left(n^{-\beta}\right),$$

as $n \to \infty$, which immediately implies Condition (ii) since $\beta\left(\frac{1}{2+\delta} - \frac{1}{r}\right) > 1$.

By the covariance inequality in Lemma 7.2.1 in [55], we have

$$\begin{aligned} |\mathbb{E}_\nu(g \cdot g \circ T^n)| &\leq 10\alpha_{\mathbf{X}_g}(n)^{1-\frac{2}{r}}\|g\|_{L^r(\nu)}\|g \circ T^n\|_{L^r(\nu)} \\ &\leq 10\|g\|^2_{L^r(\nu)}\mathcal{O}\left(n^{-\beta(1-\frac{2}{r})}\right) =: \mathcal{O}\left(n^{-\beta_1}\right), \end{aligned}$$

where we set $\beta_1 := \beta(1-\frac{2}{r}) > 2$. Hence the series in (3.5) absolutely converges. We now check Condition (iii).

$$\begin{aligned} a_n = \mathbb{E}_\nu\left(\sum_{k=0}^{n-1} g \circ T^k\right)^2 &= n\mathbb{E}_\nu(g)^2 + 2\sum_{k=1}^{n-1}(n-k)\mathbb{E}_\nu(g \cdot g \circ T^k) \\ &= n\sigma_g^2 - n\sum_{|k|\geq n}\mathbb{E}_\nu(g \cdot g \circ T^k) - 2\sum_{k=1}^{n-1}k\mathbb{E}_\nu(g \cdot g \circ T^k) \\ &= n\sigma_g^2 + \mathcal{O}(n\sum_{|k|\geq n}k^{-\beta_1}) + \mathcal{O}\left(\sum_{k=1}^{n-1}k^{1-\beta_1}\right) \\ &= n\sigma_g^2 + \mathcal{O}(1), \end{aligned}$$

Therefore, $\lim_{n\to\infty}\frac{a_n}{n} = \sigma_g^2 > 0$.

By Proposition 3.4, for any $\epsilon \in (0, \lambda - \frac{1}{2+\delta})$, enlarging to a richer probability space (M', ν') if necessary, there exists a standard Brownian motion $W(\cdot)$ such that

$$\left| \sum_{k=0}^{n-1} g \circ T^k - W(a_n) \right| = \mathcal{O}\left(n^{\frac{1}{2+\delta}+\epsilon}\right) = \mathcal{O}(n^{\lambda}), \quad \nu' - a.s. \tag{3.6}$$

We recall the following property of standard Brownian motions: for any $s \geq 0$ and $t > 0$, the increment $W(s+t) - W(s)$ has the same distribution as $Z(t)$, where $Z(t)$ is normally distributed with mean 0 and variance t. Also, it is well known that $\mathbb{E}\,|Z(t)|^{2\ell} = t^{\ell}(2\ell-1)!!$ for any $\ell \in \mathbb{N}$, where the double factorial is defined by $(2\ell-1)!! = \prod_{k=1}^{\ell}(2k-1)$. In particular, $\mathbb{E}\,|Z(t)|^4 = 3t^2$. See e.g. [27] for details.

Now we compare $W(a_n)$ and $W\left(n\sigma_g^2\right)$ as follows. Since $a_n = n\sigma_g^2 + \mathcal{O}(1)$, by Markov's inequality,

$$\begin{aligned} \sum_{n=1}^{\infty} \nu'\left\{\left|W(a_n) - W\left(n\sigma_g^2\right)\right| \geq n^{\lambda}\right\} &\leq \sum_{n=1}^{\infty} \frac{\mathbb{E}_{\nu'}\left|Z(|a_n - n\sigma_g^2|)\right|^4}{n^{4\lambda}} \\ &= \sum_{n=1}^{\infty} n^{-4\lambda} \cdot 3\left|a_n - n\sigma_g^2\right|^2 \\ &= \mathcal{O}\left(\sum_{n=1}^{\infty} n^{-4\lambda}\right) < \infty, \end{aligned}$$

as $\lambda > \frac{1}{4}$. Then by Borel-Cantelli Lemma,

$$\left|W(a_n) - W\left(n\sigma_g^2\right)\right| = \mathcal{O}(n^{\lambda}), \quad \nu' - a.s. \tag{3.7}$$

Therefore, (3.4) immediately follows from (3.6) and (3.7). ◇

Applying Lemma 3.5 to the induced processes, we obtain

Lemma 3.6 *The induced process $\mathbf{X}_{\widehat{f}} = \{\widehat{f} \circ T^n\}_{n\geq 0}$ satisfies an ASIP as follows: for any $\lambda \in \left(\max\left\{\frac{1}{4}, \frac{1}{\beta} + \frac{1}{p} + \frac{1}{q}\right\}, \frac{1}{2}\right)$, enlarging to a richer probability space (M', ν')*

if necessary, there exists a standard Brownian motion $W(\cdot)$ such that

$$\left|\sum_{k=0}^{n-1} \widehat{f} \circ T^k - W\left(n\sigma_{\widehat{f}}^2\right)\right| = \mathcal{O}(n^\lambda), \quad \nu' - a.s. \tag{3.8}$$

where $\sigma_{\widehat{f}}^2$ is given by (3.5).

Proof. Recall that $\lambda \in \left(\max\left\{\frac{1}{4}, \frac{1}{\beta} + \frac{1}{p} + \frac{1}{q}\right\}, \frac{1}{2}\right)$. Pick a sufficiently small $\delta \in \left(0, \frac{1}{\lambda} - 2\right)$, and choose some $r > 2$ such that

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{r} < \lambda - \frac{1}{\beta}. \tag{3.9}$$

By Lemma 3.2, $\widehat{f} \in L^r(M, r)$ and $\mathbb{E}_\nu(\widehat{f}) = 0$, and $\widehat{f}$ is an adapted function on M. Then (3.8) holds by Lemma 3.5. ◇

3.3.3 *Comparison between $\mathbf{X}_f$ and $\mathbf{X}_{\widehat{f}}$*

We now regard ν as a probability measure on $\mathcal{M}$, although it is not $\mathcal{T}$-invariant. Note that ν-a.s. $x \in \mathcal{M}$ belongs to the induced space M. In this subsection, we shall show that the induced process $\mathbf{X}_{\widehat{f}} = \{\widehat{f} \circ T^n\}_{n\geq 1}$ on (M, ν) is comparable to the original process $\mathbf{X}_f = \{f \circ \mathcal{T}^n\}_{n\geq 1}$ on $(\mathcal{M}, \nu)$.

For any point $x \in M$, or equivalently, for ν-a.s. $x \in \mathcal{M}$, we define the following time functions: for any $n \geq 1$, there is a unique integer $\widehat{n} = \widehat{n}(x, n)$ such that

$$\widehat{n} = \widehat{n}(x, n) := \max\left\{m \geq 1 : \sum_{k=0}^{m-1} R \circ T^k(x) \leq n\right\}. \tag{3.10}$$

We set $\widehat{n} = 0$ if the above set is empty. Also, we let

$$\widetilde{n} = \widetilde{n}(x, n) := n - \sum_{k=0}^{\widehat{n}-1} R \circ T^k(x). \tag{3.11}$$

Lemma 3.7 *For any $\epsilon > 0$, we have*

$$|\widehat{n} - n\mu(M)| = \mathcal{O}\left(n^{\frac{1}{2}+\varepsilon}\right), \quad \nu - a.s. \tag{3.12}$$

Proof. We first apply Lemma 3.5 to the stationary process

$$\mathbf{X}_R := \{R \circ T^m - \mathbb{E}_\nu(R)\}_{m\geq 0} = \{(R - \mathbb{E}_\nu(R)) \circ T^m\}_{m\geq 0}$$

on the probability space (M, ν). Indeed, $R - \mathbb{E}_\nu(R) \in L^p(\nu)$ and it is of zero mean. Furthermore, R is $\mathcal{F}_0^0$-measurable, and so is $R - \mathbb{E}_\nu(R)$. Hence by Lemma 3.5, enlarging to a richer probability space (M', ν') if necessary, there exists a standard Brownian motion $W_1(\cdot)$ such that

$$\left|\sum_{k=0}^{m-1} R \circ T^k - m\mathbb{E}_\nu(R) - W_1\left(m\ \sigma^2_{R-\mathbb{E}_\nu(R)}\right)\right| = \mathcal{O}\left(m^{\frac{1}{2}}\right), \quad \nu' - a.s. \tag{3.13}$$

By Kac formula, we have $\mathbb{E}_\nu(R) = \frac{1}{\mu(M)}$. It is well known (or use Borel-Cantelli Lemma) that for any $\varepsilon > 0$, $W_1\left(m\ \sigma^2_{R-\mathbb{E}_\nu(R)}\right) = \mathcal{O}\left(m^{\frac{1}{2}+\varepsilon}\right)$, ν'-a.s.. Hence (3.13) implies that

$$\sum_{k=0}^{m-1} R \circ T^k = \frac{m}{\mu(M)} + \mathcal{O}\left(m^{\frac{1}{2}+\varepsilon}\right), \quad \nu - a.s. \tag{3.14}$$

By the definitions in (3.10) and (3.11), we have

$$|\widetilde{n}| = \left|n - \sum_{k=0}^{\widehat{n}-1} R \circ T^k\right| \leq R \circ T^{\widehat{n}} = \mathcal{O}\left(\widehat{n}^{\frac{1}{p}+\varepsilon}\right) = \mathcal{O}\left(\widehat{n}^{\frac{1}{2}}\right), \ \nu - a.s. \tag{3.15}$$

where we use that $R \in L^p(\nu)$ and $p > 2$. Hence by (3.14) and (3.15),

$$n = \frac{\widehat{n}}{\mu(M)} + \mathcal{O}\left(\widehat{n}^{\frac{1}{2}+\varepsilon}\right),$$

for ν-a.s. $x \in M$. In particular, it follows that $\widehat{n} \to \infty$ a.s. if and only if $n \to \infty$, and $\widehat{n} = \mathcal{O}(n)$. Therefore,

$$n = \frac{\widehat{n}}{\mu(M)} + \mathcal{O}\left(n^{\frac{1}{2}+\varepsilon}\right), \quad \nu - a.s.$$

from which (3.12) holds. ◇

To compare the partial sums of $\mathbf{X}_f$ and $\mathbf{X}_{\widehat{f}}$, we consider

$$\Delta_n(x) := \sum_{k=0}^{n-1} f \circ \mathcal{T}^k(x) - \sum_{j=0}^{\widehat{n}-1} \widehat{f} \circ T^j(x) = \sum_{k=0}^{\widetilde{n}-1} f \circ \mathcal{T}^k(T^{\widehat{n}}(x)). \tag{3.16}$$

for ν-a.s. $x \in M$.

Set $h = |f|$, and let $\widehat{h}$ be its induced function on M. Let λ be given by Theorem 3.1. We choose r as in (3.9) and pick a sufficiently small $\varepsilon > 0$ such that $\frac{1}{r}+\varepsilon < \lambda$. Since $h = |f| \in L^q(\mathcal{M}, \mu)$, by the same argument in the proof of Lemma 3.2 (2), $\widehat{h} \in L^r(M, \nu)$. By Lemma 3.7 and the expression in (3.16), we get

$$|\Delta_n| \le \widehat{h} \circ T^{\widehat{n}} = \mathcal{O}\left(\widehat{n}^{\frac{1}{r}+\varepsilon}\right) = \mathcal{O}\left(n^{\lambda}\right), \quad \nu - a.s. \tag{3.17}$$

3.3.4 ASIP for the original process

We set

$$\sigma = \sigma(f) := \sigma_{\widehat{f}} \sqrt{\mu(M)}. \tag{3.18}$$

where $\sigma_{\widehat{f}}$ is given by (3.5) (in which we let $g = \widehat{f}$).

Lemma 3.8 *For any $\varepsilon > 0$ and any standard Brownian motion $W(\cdot)$ on $(\mathcal{M}, \mu)$,*

$$\left| W\left(n\sigma^2\right) - W\left(\widehat{n}\sigma_{\widehat{f}}^2\right) \right| = \mathcal{O}(n^{\frac{1}{4}+\varepsilon}), \quad a.s., \tag{3.19}$$

Proof. Pick a positive integer $\ell > 1/\varepsilon$. By the basic property of standard Brownian motions, as well as Markov's inequality, Lemma 3.7 and (3.18),

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu\left\{\left|W\left(n\sigma^2\right) - W\left(\widehat{n}\sigma_{\widehat{f}}^2\right)\right| \geq n^{\frac{1}{4}+\varepsilon}\right\} &\leq \sum_{n=1}^{\infty} \frac{\mathbb{E}_\mu \left|Z\left(\left|n\sigma^2 - \widehat{n}\sigma_{\widehat{f}}^2\right|\right)\right|^{2\ell}}{n^{2\ell(\frac{1}{4}+\varepsilon)}} \\
&= \sum_{n=1}^{\infty} n^{-2\ell(\frac{1}{4}+\varepsilon)} \cdot (2\ell-1)!! \left|n\sigma^2 - \widehat{n}\sigma_{\widehat{f}}^2\right|^{\ell} \\
&= \mathcal{O}\left(\sum_{n=1}^{\infty} n^{-\ell\varepsilon}\right) < \infty.
\end{aligned}$$

Here again $Z(t)$ denotes the normal distribution with mean 0 and variance t. Then (3.19) follows from the Borel-Cantelli Lemma. $\diamond$

Let λ be given by Theorem 3.1. Again we regard ν as a probability measure on $\mathcal{M}$, and we show that the original process $\{f \circ \mathcal{T}^k\}_{n\geq 0}$ satisfies an ASIP with rate $\mathcal{O}(n^\lambda)$ with respect to the measure ν.

Note that the almost sure bound for $|\Delta_n|$ in (3.17) also holds with respect to ν since ν is absolutely continuous with respect to μ. Then by Lemmas 3.6, 3.7 and 3.8, enlarging $(\mathcal{M}, \nu)$ to a richer probability space $(\mathcal{M}', \nu')$ if necessary, there is a standard Brownian motion $W(\cdot)$ such that

$$\begin{aligned}
&\left|\sum_{k=0}^{n-1} f \circ \mathcal{T}^k - W\left(n\sigma^2\right)\right| \\
\leq\ & \left|\sum_{k=0}^{n-1} f \circ \mathcal{T}^k - \sum_{j=0}^{\widehat{n}-1} \widehat{f} \circ T^j\right| + \left|\sum_{j=0}^{\widehat{n}-1} \widehat{f} \circ T^j - W\left(\widehat{n}\sigma_{\widehat{f}}^2\right)\right| + \left|W\left(\widehat{n}\sigma_{\widehat{f}}^2\right) - W\left(n\sigma^2\right)\right| \\
=\ & \mathcal{O}(n^\lambda) + \mathcal{O}(\widehat{n}^\lambda) + \mathcal{O}(n^{\frac{1}{4}+\varepsilon}) = \mathcal{O}(n^\lambda), \quad \nu' - a.s.
\end{aligned}$$

Finally, we need to show the ASIP for the original process $\{f \circ \mathcal{T}^k\}_{n\geq 0}$ with respect to the original measure μ, as the Brownian motion $W(\cdot)$ is not defined in a richer space of $(\mathcal{M}, \mu)$. Nevertheless, this issue is recently solved by Korepanov[42] and Gouëzel[35]. Here we quote and state Cororally 1.3 in [35] for the our ergodic system $\mathcal{T} : (\mathcal{M}, \mu) \to (\mathcal{M}, \mu)$ with respect to the two measures ν and μ.

Proposition 3.9 *If the ASIP holds for the process $\{f \circ \mathcal{T}^k\}_{n\geq 0}$ with rate $\mathcal{O}(n^\lambda)$ with respect to ν, and $f \circ \mathcal{T}^n = \mathcal{O}(n^\lambda)$ a.s., with respect to both μ and ν, then the ASIP holds for $\{f \circ \mathcal{T}^k\}_{n\geq 0}$ with the same rate $\mathcal{O}(n^\lambda)$ with respect to μ.*

Applying this proposition, we finish the proof of Theorem 3.1 by confirming $f \circ \mathcal{T}^n = \mathcal{O}(n^\lambda)$. This is due to the fact that $f \in L^q$ and that $\lambda > \frac{1}{q}$.

3.4 Applications

3.4.1 Intermittent maps

A classical example of one-dimensional intermittent maps is provided by the Manneville-Pomeau map $\mathcal{T}_\alpha : [0, 1] \to [0, 1]$ defined by

$$\mathcal{T}_\alpha(x) = x + x^{1+\alpha} \pmod 1,$$

for any $\alpha \in (0, 1)$. It was shown in [46, 70, 58, 39] that bounded Lipschitz observables has the correlation decay in rate $\mathcal{O}\left(n^{1-\frac{1}{\alpha}}\right)$, and satisfies the central limit theorem for $\alpha \in (0, 1/2)$. In [56], Pollicott and Sharp proved the weak invariance principle for $\alpha \in (0, 1/3)$.

We consider the case when $\alpha \in (0, \frac{1}{2})$. We obtain the induced map T_α on $M = [c, 1]$, where $c \in (0, 1)$ is such that $\mathcal{T}_\alpha(c) = 0$. It is well known that the first return time $R \in L^{1/\alpha}$, and the natural partition $\xi := \{[R = n]\}_{n \geq 1}$ is α-mixing with exponential rate. An observable f is adapted if there are $0 \leq s \leq t < \infty$ such that f is constant on each element of $T_\alpha^{-s} \vee \cdots \vee T_\alpha^{-t}$. By Theorem 3.1, the ASIP holds for any L^q adapted function with $q > \frac{\alpha}{1-2\alpha}$.

Remark *Of course, here we do not improve results in [56], since we only deal with adapted functions. Nevertheless, we do include some important functions, such as the first return time R itself, and thus our theorem provides an advanced result on the return time distribution.*

3.4.2 Billiards with flat points

For the basics of chaotic billiards, we refer the reader to [16].

Chernov and Zhang [18] introduced a family of semi-dispersing billiards, for which the decay of correlations for the collision map $\mathcal{T}$ is of order $\mathcal{O}(n^{-a})$ for any $a \in (1, \infty)$. By carefully choosing an inducing domain M, they obtained a generating partition ξ of M given by the first return time $R \in L^{1+a}$. Also, the two-sided σ-filtration exhibits α-mixing with exponential rate. By Remark , our main theorem implies that the ASIP holds for any L^q adapted function with $q > 2\frac{a+1}{a-1}$.

CHAPTER 4

LYAPUNOV EXPONENTS OF RANDOM BILLIARD SYSTEMS

4.1 Introduction

One of the important problems in the area of smooth dynamics is to show that a certain system is hyperbolic. A hyperbolic system is very sensitive to initial conditions. A main tool in the studies of these systems is the Lyapunov exponents; they gives us information about the stability of the dynamics if there is a small change in the initial conditions. Let M be the collision space of the billiard and $F : M \to M$ the billiard map on M. The map F is a diffeomorphism on an open dense subset of M. We also have that F preserves a natural probability measure μ on M. By Oseledet's theorem [[17] Theorem 3.1], under some integrability conditions and boundedness of the curvature of the boundary, the Lyapunov exponents $\lambda_1(x) \geq \lambda_2(x)$ exist for μ-almost every point in M.

A point x is called *hyperbolic* if its Lyapunov exponents are nonzero and the map F is called hyperbolic if μ-almost every point in M is hyperbolic.

For the billiards we are considering in this chapter, the two Lyapunov exponents are of opposite sign [[17] Lemma 3.9], so a point is hyperbolic if its largest exponent $\lambda_1(x)$ is positive and thus there is a direction with strong expansion (and strong contraction in a different direction) at this point. The standard method to prove positivity of Lyapunov exponent is to establish the existence of a strictly invariant cone field on the tangent space level [26]. However, it is difficult to verify this property for many billiard systems, such as the moon billiards in [21].

One approach to the problem is to add random perturbations to a billiard and study the desired properties on the corresponding stochastic version. Even if this method might not actually solve the deterministic problem, but it still gives us insight on how the system behave under small random perturbation. There are several works in literature on stochastic perturbation to billiards: [15], [31], [32], [30],[20], [49], [61], [24]. However, the invariant measure of the perturbed systems in these works is not the natural measure of the deterministic billiard map as in our situation. Also the Lyapunov exponent is not proved to be positive in those works. Our work perhaps is closest to [49] by Markarian et al. and [4] by Blumenthal, Xue and Young. In [4], the authors considered a random perturbation to a dynamical system such that the perturbed system also has the same invariant measure with the unperturbed one. However, the system considered there is the Chirikov standard map, the phase space is a torus and the perturbation is also a bit different.

We will show in this chapter that by adding a small noise, which refers to a distribution, to a system at each iteration, it is possible to obtain the positivity of the largest Lyapunov exponent if there is some source of hyperbolicity at the beginning.

Even with some systems that have zero Lyapunov exponents in a full measure set, the presence of the noise makes the Lyapunov exponent to be positive no matter how small the noise is. On the other hand, we show that the circular billiards cannot have positive Lyapunov exponent even with large perturbations. This is because the noise is added independently of the points, so practically we do not perturb the derivative, whereas the original circular billiard is linear with zero Lyapunov exponent. On the other hand, in the case of non-circular elliptic billiards, we have positive Lyapunov exponents although the systems are also completely integrable just as for circles. The difference here is that in an elliptic billiard, there is a hyperbolic periodic point and it serves as a source of hyperbolicity. We conjecture that circular billiards are the only smooth and convex billiards that are not hyperbolic after the perturbation added.

In the first section of the chapter, we will collect some important background on Markov processes. The perturbation on the billiard map defines a Markov transition function on the collision space. Because of this, in the perturbed system, each trajectory is a realisation path of a Markov process. In the following section we will define the Lyapunov exponent for a stationary sequence of matrices along a stochastic process. We conclude the section with a necessary condition for the Lyapunov exponents to be zero. In the last part of the chapter, we give a detail description of the perturbation to several classical billiards. We show for these billiards that the perturbed systems are ergodic. We also establish the hyperbolicity for many billiards, and show that the random circular billiards are ergodic but not hyperbolic.

4.2 Preliminaries on on Markov processes

In this section, we gather some background on Markov processes. For convenience, basic facts in measure theory can be found in the appendix. Some useful texts on the measure theory and Markov processes are [65], [66], [52], [28], [41].

Let M be a complete separable metric space and $\mathcal{B}$ its Borel σ-algebra.

Definition 4.1 *A Markov transition function on M is a function $P : M \times \mathcal{B} \to [0,1]$ such that:*

1. *for each $x \in M$, the map $P(x,\cdot) : B \mapsto P(x,B)$ is a probability measure on $\mathcal{B}$;*
2. *for each $B \in \mathcal{B}$, the map $P(\cdot,B) : x \mapsto P(x,B)$ is a measurable function on $(M,\mathcal{B})$.*

Definition 4.2 *Let P be a transition function on M. The Ruelle transfer operator $\mathcal{L}$ associated to P is a map $\mathcal{L} : \mathcal{P}(M) \to \mathcal{P}(M)$ on the set $\mathcal{P}(M)$ of probability measures on M, and is defined by:*

$$\mathcal{L}\mu(B) = \int_M P(x,B)\mu(dx) \tag{4.1}$$

for any probability measure $\mu \in \mathcal{P}(M)$ and $B \in \mathcal{B}$.

For any $x \in M$, let δ_x be the Dirac probability measure at x. That is: for every $B \in \mathcal{B}$ we have:

$$\delta_x(B) = \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \notin B. \end{cases} \tag{4.2}$$

Then we can apply the transfer operator $\mathcal{L}$ to δ_x to have:

$$\begin{aligned} \mathcal{L}\delta_x(B) &= \int_M P(y,B)\delta_x(dy) \\ &= P(x,B) \end{aligned} \tag{4.3}$$

for every $B \in \mathcal{B}$. Thus the one-step image $\mathcal{L}\delta_x$ of the Dirac measure δ_x via the transfer operator $\mathcal{L}$ is equal to the probability measure $P(x,\cdot)$.

We can thus define the n^{th} power P^n, $n \geq 0$ and $n \neq 1$, of the transition function P by setting:

$$P^n(x,B) := \mathcal{L}^n\delta_x(B) \tag{4.4}$$

for any $x \in M$ and $B \in \mathcal{B}$. By a straightforward induction we have the following proposition:

Proposition 4.3 *We have the following recursion relation:*

$$P^n(x,B) = \int_M P^{n-1}(y,B)P(x,dy) \tag{4.5}$$

for any $x \in M$ and $B \in \mathcal{B}$.

Definition 4.4 *Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space. A stochastic process defined on $(\Omega, \mathcal{A}, \mathbf{P})$ with values in M is a sequence $(X_n)_{n\geq 0}$ of random variables:*

$$X_n : (\Omega, \mathcal{A}) \to (M, \mathcal{B}).$$

We are going to construct a stochastic process $(X_n)_{n\geq 0}$ on M with the property that at any time $n \geq 0$, if $X_n = x$ then the distribution of X_{n+k} is given by $\mathcal{L}^k\delta_x$ for any $k \geq 1$. We have the following definition:

Definition 4.5 *The stochastic process $(X_n)_{n\geq 0}$ is called a time-homogeneous Markov process with transition probability P and initial distribution μ_0 if for every finite sequence of integers $0 = t_0 < t_1 < \cdots < t_n$ and measurable functions $f_0, \ldots, f_n$:*

$$\begin{aligned} \mathbf{E}_{\mathbf{P}}(\prod_{i=0}^{n} f_i(X_{t_i})) = \int_M \mu_0(dx_0) f_0(x_0) \int_M P^{t_1}(x_0, dx_1) f_1(x_1) \ldots \\ \int_M P^{t_n - t_{n-1}}(x_{n-1}, dx_n) f_n(x_n). \end{aligned} \tag{4.6}$$

Note that the left-hand side of equation (4.6) is the expectation of the product $\prod_{i=0}^{n} f_i(X_{t_i})$ with respect to the probability measure $\mathbf{P}$. Suppose that $B_0, B_1, \ldots, B_n$ is a sequence of measurable sets in $\mathcal{B}$. If we choose, for $0 \leq i \leq n$, $f_i = 1_{B_i}$ the indicator function on B_i, then:

$$\begin{aligned} \mathbf{P}(X_0 \in B_0, X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) = \int_{B_0} \mu_0(dx_0) \int_{B_1} P^{t_1}(x_0, dx_1) \ldots \\ \int_{B_n} P^{t_n - t_{n-1}}(x_{n-1}, dx_n). \end{aligned} \tag{4.7}$$

Proposition 4.6 *Let $(X_n)_{n\geq 0}$ be a Markov process with transition function P. Then:*

$$\mathbf{P}(X_k \in B | X_0 = x) = P^k(x, B).$$

for any integer $k \geq 1$ and $B \in \mathcal{B}$.

Proof. The probability $\mathbf{P}(X_k \in B | X_0 = x)$ is the probability of the event $X_k \in B$ in future, given that at present $X_0 = x$. Thus it can be viewed as the probability of the event $X_k \in B$ when the Markov process $(X_n)_{n\geq 0}$ is equipped with the initial distribution the Dirac measure δ_x at x. The proposition is then a direct application of equation (4.7):

$$\mathbf{P}(X_k \in B | X_0 = x) = \mathbf{P}(X_0 \in M, X_k \in B) = \int_M \delta_x(dx_0) \int_B P^k(x_0, dx_1)$$
$$= P^k(x, B). \tag{4.8}$$

◇

Given a transition function $P : M \times \mathcal{B} \to [0, 1]$ and probability measure μ_0 on M, we now construct a concrete Markov process with transition function P and initial probability measure μ_0. We can achieve this using Kolmogorov's Extension Theorem, see Theorem 12.8 in [41].

Let $\Omega = M^{\mathbf{N}}$ be the Cartesian product space of copies of M indexed by the set of non-negative integers $\mathbf{N}$. Each element $\omega \in \Omega$ is a sequence $\omega = (x_0, x_1, \dots)$, where $x_0, x_1, \dots$ are in M. For each $n \geq 0$, let X_n be the coordinate map $X_n : \Omega \to M$ defined by:

$$X_n(\omega) = X_n(x_0, x_1, \dots) = x_n.$$

A cylinder A is a subset of Ω the form:

$$A = \{\omega \in \Omega : X_0(\omega) \in B_0, X_1(\omega) \in B_1, \dots, X_n(\omega) \in B_n\} \tag{4.9}$$

for some $n \geq 0$ and finite sequence of sets $B_0, B_1, \dots, B_n$ in $\mathcal{B}$. Let $\mathcal{A} = \bigotimes_{\mathbf{N}} \mathcal{B}$ be the σ-algebra generated by these cylinders of Ω. We equip Ω with this σ-algebra. Then the coordinate maps $X_n : \Omega \to M$ are in fact $\mathcal{A}$-measurable functions and thus form a stochastic process $(X_n)_{n \geq 0}$ defined on Ω with values in M.

Given any initial probability measure μ_0 on M, we define a probability measure

μ_{n+1} on M^{n+1} for each $n \geq 0$ by setting:

$$\begin{aligned}\mu_{n+1}(B_0, \cdots, B_n) = \int_{B_0} \mu_0(dx_0) \int_{B_1} P(x_0, dx_1) \cdots \\ \int_{B_{n-1}} P(x_{n-2}, dx_{n-1}) P(x_{n-1}, B_n).\end{aligned} \tag{4.10}$$

for any sequence of sets $B_0, B_1, \ldots, B_n$ in $\mathcal{B}$.

By Kolmogorov's Extension Theorem, there exists a unique probability measure $\mathbf{P}_{\mu_0}$ on $(\Omega, \mathcal{A})$ whose restriction to M^n is equal to μ_n. A statement of the theorem is Theorem A.3.1 in [28].

With the probability measure $\mathbf{P}_{\mu_0}$ equipped on the measurable space $(\Omega, \mathcal{A})$, the process $(X_n)_{n\geq 0}$ becomes a Markov process defined on the probability space $(\Omega, \mathcal{A}, \mathbf{P}_\mu)$ with values in M. The transition function of $(X_n)_{n\geq 0}$ is given by P and the initial distribution is μ_0.

We define a shift map: $\theta : \Omega \to \Omega$ such that

$$\theta(x_0, x_1, \ldots) = (x_1, x_2, \ldots). \tag{4.11}$$

Note that we have $X_{n+1} = X_n \circ \theta = X_0 \circ \theta^n$ for any $n \geq 0$.

Definition 4.7 *Let P be a transition function on M. A probability measure μ on M is called invariant with respect to P if $\mathcal{L}\mu = \mu$. That means:*

$$\mu(B) = \int_M P(x, B)\mu(dx) \tag{4.12}$$

for any $B \in \mathcal{B}$.

Lemma 4.8 *Let $\mu \in \mathcal{P}(M)$ be an invariant measure with respect to the transition function P. Then the shift map $\theta : \Omega \to \Omega$ preserves the measure $\mathbf{P}_\mu$. Equivalently, $\mathbf{P}_\mu(\theta^{-1}(A)) = \mathbf{P}_\mu(A)$ for any set $A \in \mathcal{A}$.*

Proof. We can see that this is true by consider the case where A is a cylinder first, then use the fact that the σ-algebra $\mathcal{A}$ is generated by these cylinders. $\diamond$

We can thus think of $(\Omega, \mathcal{A}, \mathbf{P}_\mu, \theta)$ as a measure-preserving dynamical system. The system is called *ergodic* if for any $A \in \mathcal{A}$ such that $\theta^{-1}(A) = A$ then $\mathbf{P}_\mu(A)$ is either 0 or 1. The system is *mixing* if for any $A, B \in \mathcal{A}$ we have:

$$\mathbf{P}_\mu(\theta^{-n} A \cap B) = \mathbf{P}_\mu(A)\mathbf{P}_\mu(B). \tag{4.13}$$

Definition 4.9 *Suppose that μ is an invariant measure with respect to the transition function P.*

We say that μ is ergodic with respect to P if $(\Omega, \mathcal{A}, \mathbf{P}_\mu, \theta)$ is an ergodic measure-presrving dynamical system.

We say that μ is mixing with respect to P if $(\Omega, \mathcal{A}, \mathbf{P}_\mu, \theta)$ is mixing.

4.3 Lyapunov exponents of stationary sequences of matrices

4.3.1 Existence of the Lyapunov exponents

In this section, we will collect some background information on the Lyapunov exponents of a stationary sequence of matrices. Most of the materials in this section can be found in [44] and [45].

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and $\theta : (\Omega, \mathcal{A}) \to (\Omega, \mathcal{A})$ a measurable map that preserves the probability measure $\mathbf{P}$. We denote by $\mathrm{GL}(2, \mathbf{R})$ the group of 2×2 real invertible matrices. When viewed as a measurable space, the group $\mathrm{GL}(2, \mathbf{R})$

is equipped with its Borel σ-algebra. Let $A : \Omega \to \mathrm{GL}(2, \mathbf{R})$ be a measurable map. Let $A_n = A \circ \theta^n$. Then the sequence $(A_n)_{n\geq 0}$ is a stochastic process defined on the underlying probability space $(\Omega, \mathcal{A}, \mathbf{P})$ with values in $\mathrm{GL}(2, \mathbf{R})$. In fact, $(A_n)_{n\geq 0}$ is stationary stochastic process because of the invariance of the measure $\mathbf{P}$ under the map θ.

We construct another sequence $(A^{(n)})_{n\geq 1}$ of matrices by:

$$A^{(n)}(\omega) := A_{n-1}(\omega) \cdot A_{n-2}(\omega) \cdots A(\omega) \tag{4.14}$$

for any $n \geq 1$.

Definition 4.10 *Let $(\Omega, \mathcal{A}, \mathbf{P}, \theta, A)$ be as above. The Lyapunov exponent at ω of the sequence $(A_n(\omega))_{n\geq 0}$ is defined to be:*

$$\lambda(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \left\| A^{(n)}(\omega) \right\| \tag{4.15}$$

if the limit exists. Here the norm $\|\cdot\|$ is the operator norm of a linear map $\mathbf{R}^2 \to \mathbf{R}^2$.

By the subadditivity of the sequence $(\|A^{(n)}(x)\|)_{n\geq 0}$ and stationarity of the sequence $(A_n)_n \geq 0$, we have the following lemma:

Lemma 4.11 *For $\mathbf{P}$-almost every ω, the Lyapunov exponent at ω of the sequence $(A_n)_{n\geq 0}$ exists in $\mathbf{R} \cup \{-\infty\}$.*

Lemma 4.12 ([44] Proposition 1.1) *Let $(\Omega, \mathcal{A}, \mathbf{P}, \theta, A)$ be as defined above. Suppose that:*

$$\int_\Omega \log^+ \|A(\omega)\| \, \mathbf{P}(d\omega) < \infty, \tag{4.16}$$

where $\log^+ = \max(\log, 0)$. *Then the following two limits*

$$\lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \left\| A^{(n)}(\omega) \right\| \mathbf{P}(d\omega) \tag{4.17}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_\Omega \log |\det A^{(n)}(\omega)| \mathbf{P}(d\omega) \tag{4.18}$$

exist in the extended real line $\mathbf{R} \cup \{-\infty\}$.

Definition 4.13 *Under the conditions of Lemma 4.12, let* λ_1 *and* λ_2 *be real numbers such that:*

$$\lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \left\| A^{(n)}(\omega) \right\| \mathbf{P}(d\omega) = \lambda_1 \tag{4.19}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \int_\Omega \log |\det A^{(n)}(\omega)| \mathbf{P}(d\omega) = \lambda_1 + \lambda_2. \tag{4.20}$$

We set λ_1 *or* λ_2 *to be* $-\infty$ *if the first or second limit is* $-\infty$. *We call the numbers* λ_1 *and* λ_2 *the Lyapunov exponents of the stationary process* $(A_n)_{n\geq 0}$.

We recall that any real square matrix A can always be decomposed as $A = U\Sigma V^T$ where U and V^T are orthogonal matrices and $\Sigma = \begin{pmatrix} \sigma_1(A) & 0 \\ 0 & \sigma_2(A) \end{pmatrix}$ is a diagonal matrix with $\sigma_1(A) \geq \sigma_2(A) \geq 0$. This is called the Singular Value Decomposition of the matrix A. This decomposition tells us that geometrically a linear transformation is a composite of a rotation, a scaling and another rotation. The number $\sigma_1(A)$ is the larger scaling factor among $\sigma_1(A)$ and $\sigma_2(A)$. The columns of V and U tell us the directions in $\mathbf{R}^2$ in which we will see the largest and the smallest scaling, and where those directions move to after the transformation. The following lemma allows us to write the Lyapunov exponents in terms of the scaling factors.

Lemma 4.14 *Under the same conditions as in Lemma 4.12, we have that:*

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log \sigma_i(A^{(n)}(\omega))\mathbf{P}(d\omega) \tag{4.21}$$

for $i = 1, 2$. It is then clear that $\lambda_1 \geq \lambda_2$.

In fact, since we have that

$$\det A^{(n)}(\omega) := \det A_{n-1}(\omega) \cdot \det A_{n-2}(\omega) \cdots \det A(\omega)$$

and that the map $\theta : \Omega \to \Omega$ preserves the measure $\mathbf{P}$, which implies that the sequence $(A_n)_{n\geq 0}$ is stationary, we could drop a limit sign to have:

$$\lambda_1 + \lambda_2 = \lim_{n\to\infty} \frac{1}{n} \int_\Omega \log |\det A^{(n)}(\omega)|\mathbf{P}(d\omega) \tag{4.22}$$

$$= \int_\Omega \log |\det(A(\omega))|\mathbf{P}(d\omega) \tag{4.23}$$

The Lyapunov exponent $\lambda(\omega)$ at ω in Definition 4.10 of the sequence $(A_n(\omega))_{n\geq 0}$ can be viewed as the logarithm of the rate of expansion (or contraction) of vectors along the path starting from ω. If the underlying dynamical system is ergodic, they are constant $\mathbf{P}$-almost everywhere:

Theorem 4.15 ([44] Theorem 2.6) *Let $(\Omega, \mathcal{A}, \mathbf{P}, \theta)$ be an ergodic system, $A : \Omega \to GL(2, \mathbf{R})$ a measurable map such that:*

$$\int_\Omega \log^+ \|A(x)\| \, \mathbf{P}(d\omega) < \infty. \tag{4.24}$$

Let $\lambda_1 \geq \lambda_2$ be the Lyapunov exponents of the stationary process $(A_n)_{n\geq 0}$. Then we have:

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \log \left\| A^{(n)}(\omega) \right\| \qquad \mathbf{P} - a.s. \tag{4.25}$$

and

$$\lambda_1 + \lambda_2 = \lim_{n\to\infty} \frac{1}{n} \log |\det A^{(n)}(\omega)| \qquad \mathbf{P} - a.s. \tag{4.26}$$

We have a straightforward corollary of the Theorem 4.15 in the case the function A takes values in the set of matrices with determinant equals 1. Let $\mathrm{SL}(2, \mathbf{R})$ denote the set of 2×2 real matrices with determinant 1.

Corollary 4.16 *Let $(\Omega, \mathcal{A}, \mathbf{P}, \theta)$ be an ergodic system and $A : \Omega \to SL(2, \mathbf{R})$ a measurable map such that*

$$\int_\Omega \log \max(\|A(\omega)\|, \left\|A(\omega)^{-1}\right\|)\mathbf{P}(d\omega) < \infty.$$

Then both λ_1 and λ_2 are finite and moreover, $\lambda_1 + \lambda_2 = 0$.

We are interested in the necessary conditions to have $\lambda_1 = \lambda_2$. Later we will use these criteria to show that $\lambda_1 \neq \lambda_2$ for certain systems by way of contradiction. The idea behind these necessary conditions is that if the two Lyapunov exponents are equal then a very special condition on measurability must be satisfied. Avila and Viana in [1] discussed this phenomenon in a more general setting.

Let $\mathbb{P}^1$ be the projective space of dimension 1. Elements in $\mathbb{P}^1$ are equivalence classes of the vectors in $\mathbf{R}^2$ where two nonzero vectors v and w are said to be equivalent if they are parallel. For any nonzero vector $v \in \mathbf{R}^2$ we denote by $[v]$ its equivalent class. For any $\omega \in \Omega$, we have $A(\omega)$ is a matrix in $\mathrm{GL}(2, \mathbf{R})$. The matrix

$A(\omega)$ is a linear transformation on the vector space $\mathbf{R}^2$ and hence induces a map on $\mathbb{P}^1$:

$$A(\omega)([v]) = [A(\omega)(v)] \quad \text{for any } [v] \in \mathbb{P}^1. \tag{4.27}$$

Let $\hat{\Omega} = \Omega \times \mathbb{P}^1$ and define a map $\hat{\theta} : \Omega \times \mathbb{P}^1 \to \Omega \times \mathbb{P}^1$ by:

$$\hat{\theta}(\omega, \hat{v}) = (\theta(\omega), A(\omega)(\hat{v})).$$

Let $\pi_1 : \hat{\Omega} \to \Omega$ be the projection map onto the first component. Any probability measure ξ on $\hat{\Omega}$ such that $\pi_{1*}\xi = \mathbf{P}$ can be *disintegrated* into a family $\{\xi_\omega : \omega \in \Omega\}$ of probability measures on $\mathbb{P}^1$ such that the function $\omega \mapsto \xi_\omega$ is $\mathcal{A}$-measurable. This family is essentially unique and each ξ_ω is supported on the fibre $p_1^{-1}(\{\omega\}) \cong \mathbb{P}^1$. We only consider measures ξ that projects to $\mathbf{P}$.

We have the following theorem of Ledrappier:

Theorem 4.17 ([45] Theorem 1) *Let $(\Omega, \mathcal{A}, \mathbf{P}, \theta)$ be a measure-preserving dynamical system, not necessarily ergodic. Let $A : \Omega \to GL(2, \mathbf{R})$ be a measurable function such that*

$$\int_\Omega \log \max(\|A(\omega)\|, \|A(\omega)^{-1}\|)\mathbf{P}(d\omega) < \infty.$$

Let $\mathcal{A}_0 \subset \mathcal{A}$ be a sub σ-algebra such that both θ and A are $\mathcal{A}_0$-measurable and that $\mathcal{A}$ can be generated by all the iterates $\theta^n(\mathcal{A}_0)$ of $\mathcal{A}_0$, $n \in \mathbf{Z}$.

Suppose that $\lambda_1 = \lambda_2$. Then any disintegration of a $\hat{\theta}$-invariant measure ξ is $\mathcal{A}_0$-measurable (modulo null sets).

4.3.2 Lyapunov exponents of a stationary sequence of matrices along a Markov process

Let M be a complete separable metric space and $\mathcal{B}$ its Borel σ-algebra. Let P be a transition function on M and μ an invariant measure for the corresponding transfer operator $\mathcal{L}$. Consider a Markov process $(X_n)_{n\geq 0}$ with transition function P and take values in M with initial probability measure μ. Let $(\Omega, \mathcal{A}, \mathbf{P}_\mu)$ be the canonical underlying probability space for $(X_n)_{n\geq 0}$ constructed as in section 4.2. Recall that an element $\omega \in \Omega$ is of the form $\omega = (x_0, x_1, \dots)$ and the random variables X_n's are coordinate maps:

$$X_n(\omega) = x_n \text{ for } n \geq 0.$$

The shift map θ on Ω is:

$$\theta(x_0, x_1, \dots) = (x_1, x_2, \dots).$$

Since μ is invariant for $\mathcal{L}$, the shift map θ preserves the measure $\mathbf{P}_\mu$ and thus $(X_n)_{n\geq 0}$ is a stationary Markov process.

Let $A : M \to \mathrm{SL}(2, \mathbf{R})$ be a measurable map satisfying the condition:

$$(\mathbf{H1}) \qquad \int_M \log \max(\|A(x)\|, \|A(x)^{-1}\|)\mu(dx) < \infty, \tag{4.28}$$

With a slight abuse of notation, we define a function $A : \Omega \to \mathrm{SL}(2, \mathbf{R})$ by setting:

$$A(\omega) := A(x_0)$$

for any $\omega = (x_0, x_1, \dots) \in \Omega$.

As defined in definition 4.13, let $\lambda_1 \geq \lambda_2$ be the Lyapunov exponents for the process $(A_n = A \circ \theta^n)_{n\geq 0}$. By corollary 4.16 we know that $\lambda_1 + \lambda_2 = 0$.

Lemma 4.18 ([45] Corollary 2) *If $\lambda_1 = 0$ then there exists a measurable family $\{\xi_x : x \in M\}$ of probability measures on $\mathbb{P}^1$ such that for μ-almost every $x \in M$:*

$$\xi_y = (A(x))_* \xi_x \tag{4.29}$$

for $P(x, .)$-almost every y.

4.4 Ergodicity and hyperbolicity of randomly perturbed billiards

In this section, we consider a random perturbation to certain dynamical billiards and prove the ergodicity and hyperbolicity of the perturbed systems. Let us first recall some basic information about billiards. There are several introductory references to billiards including, but not limited to: [17], [43], [63].

Consider a billiard table $\mathcal{D}$ such that the interior $\mathcal{D}_0$ is a compact and connected open domain in $\mathbf{R}^2$, and that the boundary $\partial\mathcal{D}$ satisfies the assumption (**HB**). We call such model a *classical* billiard.

Assumption (**HB**): the boundary $\partial\mathcal{D}$ consists of finitely many piecewise C^3 simple closed curves:

$$\partial\mathcal{D} = \Gamma_1 \cup \Gamma_2 \cdots \cup \Gamma_n, \qquad , n \geq 1. \tag{4.30}$$

Each curve Γ_i is given by a piecewise C^3 map $\gamma_i : [a_i, b_i] \to \mathbf{R}^2$, which is injective on $[a_i, b_i)$ and satisfies $\gamma_i(a_i) = \gamma_i(b_i)$. Assume further that the intervals (a_i, b_i), $i = 1, \ldots, n$, are disjoint.

Fix an orientation on each component Γ_i so that the interior of the table lies on the left-hand side of Γ_i. We parametrised the Γ_i's by their arclengths.

A point particle is moving inside $\mathcal{D}$ and colliding with the boundary $\partial\mathcal{D}$. Let $q(t) \in \mathcal{D}$ be the position and $v(t) \in \mathbf{R}^2$ the velocity of the particle at time $t \in \mathbf{R}$. Between two collisions with the boundary, $q \in \mathcal{D}_0$, the particle moves in the interior with constant velocity. At a collision with the smmooth part of the boundary, $q \in \partial\mathcal{D}$, let v^- and v^+ denote the pre-collisional and post-colllisional velocity vectors, respectively, and let n be the unit normal vector to the boundary at q pointing inward the table. Then we have:

$$v^+ = v^- - 2(v^-, n)n. \tag{4.31}$$

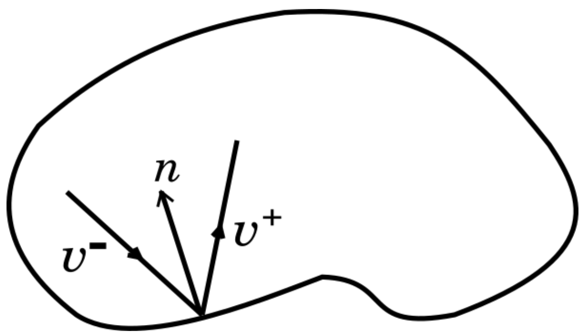


Figure 4. Example of a collision in a billiard table

Let $\mathcal{M}$ be the *collision space* of the billiard map on $\mathcal{D}$. Every point $x \in \mathcal{M}$ is a pair of its position q and post-collisional velocity vector v. The boundary $\partial\mathcal{D}$ is parametrised by the arc-length parameter r in the chosen direction. For each point $x \in \mathcal{M}$, the angle of reflection φ is the directional angle from v to the inward normal vector n. Note that $-\pi/2 \leq \varphi \leq \pi/2$. Thus we have a coordinate system r, φ on $\mathcal{M}$.

For each $i = 1, 2, \ldots, n$, let $\mathcal{M}_i$ be the collision space for collisions that happen on Γ_i, then:

$$\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_n. \tag{4.32}$$

For each Γ_i, since it is parametrised by arclength, we assume that it has length $|\Gamma_i| = b_i - a_i$. Let $\mathcal{R}_i = [a_i, b_i] \times [-\pi/2, \pi/2]$. Then $\mathcal{M}_i$ is a cylinder obtained by identify the two edges $\{r = a_i\}$ and $\{r = b_i\}$ of $\mathcal{R}_i$ with each other.

Let $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ be the billiard map. It sends a point $(r, \varphi) \in \mathcal{M}$ to $(r_1, \varphi_1) \in \mathcal{M}$ at the next collision. Let $|\partial\mathcal{D}|$ be the length of $\partial\mathcal{D}$. By Lemma 2.35 in [17], the collision map $\mathcal{F} : \mathcal{M} \to \mathcal{M}$ preserves a probability measure μ on $\mathcal{M}$ defined by:

$$d\mu = \frac{1}{2|\partial\mathcal{D}|}\cos(\varphi)drd\varphi. \tag{4.33}$$

In this section, however, we will use the coordinate system given by r and s, where $s = \sin(\varphi)$. For each $i = 1, 2, \ldots, n$, let $R_i = [a_i, b_i] \times [-1, 1]$ and M_i the cylinder obtained by identify two edges $\{r = a_i\}$ and $\{r = b_i\}$ of the rectangle R_i with each other. The collision space in this setting is $M = M_1 \cup \cdots \cup M_n$ and the billiard map is now denoted by $F : M \to M$.

Let S_1 be the set of $x \in M$ such that the corresponding trajectory on the billiard table will hit the corners, or tangential to a dispersing wall. The billiard map F is a C^2 diffeomorphism from $M \setminus S_1$ onto its image and S_1 is considered as the *singularity* of F.

Let $x = (r, \sin(\varphi))$ be any point in M and $x_1 = F(x) = (r_1, \sin(\varphi_1))$ be the next collision, where $(r_1, \varphi_1) = \mathcal{F}(r, \varphi)$. We denote by K and K_1 the curvatures of the boundary at the collision points for x and x_1, respectively, and by τ the distance of between those 2 collision points in the table. The differential of the billiard map, in the coordinates r and $s = \sin(\varphi)$, is given by the formula:

$$DF(x) = \begin{pmatrix} 1 & 0 \\ 0 & \cos(\varphi_1) \end{pmatrix} \frac{-1}{\cos(\varphi_1)} \begin{pmatrix} -\tau K + \cos(\varphi) & \tau \\ \tau K K_1 - K\cos(\varphi_1) - K_1 \cos(\varphi) & -\tau K_1 + \cos(\varphi_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\cos(\varphi)} \end{pmatrix} \tag{4.34}$$

Note that $\det(DF(x)) = 1$ and thus the billiard map preserves the multiple of the Lebesgue measure $dm = \frac{1}{2|\partial\mathcal{D}|} dr ds$ on M. We are interested in the Lyapunov exponents of billiard map F. By Oseledets's theorem, we know that the Lyapunov exponents exist at m-almost every point $x \in M$.

Theorem 4.19 ([17] Theorem 3.1) *Let M be a 2-dimensional compact Riemannian manifold and $F : M \to M$ a C^2 diffeomorphism preserving a Borel probability measure m on M. Suppose that*

$$\int_M \log^+ \|DF(x)\| m(dx) < \infty \text{ and } \int_M \log^+ \|DF^{-1}(x)\| m(dx) < \infty, \tag{4.35}$$

where $\log^+ = \max\{\log, 0\}$. Then there exists an F-invariant set $H \subset M$ of full measure, on which all iterations of F are defined on H such that for each $x \in H$ there is a DF-invariant decomposition of the tangent space:

$$T_x M = E_1(x) \oplus \cdots \oplus E_k(x) \tag{4.36}$$

for some $k = k(x)$, such that for each non-zero vector $v \in E_i(x)$ the following limit exists:

$$\lim_{n\to\infty} \frac{1}{n} \log \|DF^n(x)v\| = \lambda_i(x) \tag{4.37}$$

where $\lambda_1(x) > \cdots > \lambda_k(x)$.

The Lyapunov exponents tell us how nearby trajectories will be separated from each other as the system evolves in time. In our case, $k(x)$ is either 1 or 2. Let $\lambda_1(x) \geq \lambda_2(x)$ be its Lyapunov exponents then by Lemma 3.9 in [17]: $\lambda_1(x)+\lambda_2(x) = 0$. A point $x \in M$ is hyperbolic if $\lambda_1(x) > 0$: nearby trajectories are separated exponentially fast in the future. There are many billiards in which the Lyapunov exponents are zero. For example, the Lyapunov exponents for any circular billiard are 0 at all points: the trajectories are separated at most linearly. Other similar examples are elliptic billiards. In each of these billiard models, the systems are completely integrable and their collision spaces are foliated by invariant curves.

In what follows, we are going to add some noise each time there is a collision. In the deterministic setting, a point x is mapped to $F(x)$. With the noise added, now the image of x could be in some open neighbourhood of $F(x)$. In this way, a point can escape a region with slow or no expansion even if it needs many iterations depending on the added noise is.

Fix an $\epsilon > 0$. We denote by $B_\varepsilon(x)$ the ball of radius ε and centred at a point $x \in \mathbf{R}^2$. Consider a probability measure ν_ε on $\mathbf{R}^2$ such that $d\nu_\varepsilon = \rho d\mathbf{m}$ for some measurable density function ρ, here $\mathbf{m}$ is the Lebesgue measure on $\mathbf{R}^2$. Suppose that the support of ρ contains $B_\varepsilon(0,0)$.

Recall that each cylinder M_i is obtained by taking the rectangle R_i and identifying the two vertical edges. The system is randomly perturbed as follows: take a point $x = (r,s) \in M_i$, then perturb this point to another point in M_i with a distribution law given by:

$$\begin{aligned}\eta_\varepsilon^x(B) &= \nu_\varepsilon\left\{u \in \mathbf{R}^2 : u + x \in B \mod \mathbf{Z}\partial R_i\right\} \\ &= C\int_B \sum_{v\in\mathbf{Z}\partial R_i} \rho(u - x - v)\mathbf{m}(du)\end{aligned} \tag{4.38}$$

for any measurable set $B \subset M_i$. The set $\mathbf{Z}\partial R_i$ consists of vectors v such that $\frac{1}{n}v \in \partial R_i$ for some $n \in \mathbf{Z}$; the constant C is the normalising constant.

Example 1 *Let $\rho = \frac{1}{\pi\varepsilon^2}$ on $B_\varepsilon(0,0)$ and $= 0$ elsewhere. A point x goes to $F(x)$ and then jump randomly to a point in a disc of radius ε centred at $F(x)$ following the distribution $\eta_\varepsilon^{F(x)}$. If any part of the disc lies above or below M_i then this part will cut and translated back to M_i by a constant in $\mathbf{Z}\partial R_i$.*

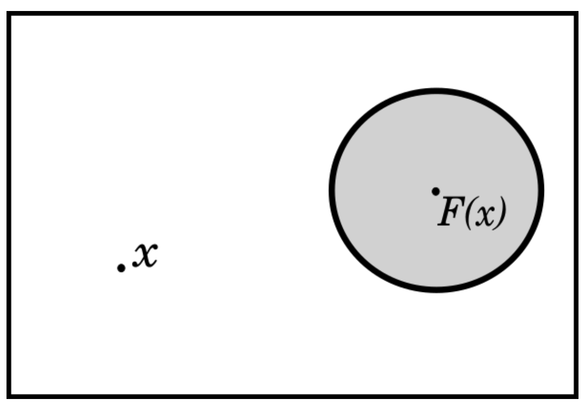


Figure 5. Random perturbation when $F(x)$ is far from the boundary

Definition 4.20 *Given a vector $u \in \mathbf{R}^2$, we define a function $F_u : M \to M$ by:*

$$F_u(x) = F(x) + u. \tag{4.39}$$

The function F_u is called the perturbed billiard map with u.

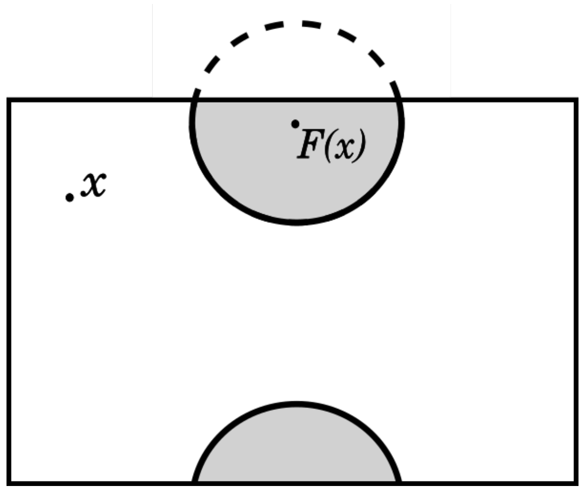


Figure 6. Random perturbation when $F(x)$ is near the boundary

For any sequence of vectors $\underline{u} = (u_0, u_1, \dots)$, the compositions of perturbed map with noise given by $\underline{u}$ is defined by:

$$F_{\underline{u}}^n = F_{u_{n-1}} \circ F_{u_{n-2}} \circ \cdot \circ F_{u_0} \tag{4.40}$$

for any $n \geq 1$.

Let $\Upsilon = (\mathbf{R}^2)^{\mathbf{N}}$ be the space of all sequences of vectors in $\mathbf{R}^2$, equipped with the probability measure $\nu_\varepsilon^{\mathbf{N}}$. This is our sample space for the noise. Let U_n be the coordinate mapping:

$$U_n(\underline{u}) = U_n(u_0, u_1, \dots) = u_n \tag{4.41}$$

for any $\underline{u} = (u_0, u_1, \dots) \in \Upsilon$ and $n \geq 0$.

It is easy to see that the process $\underline{U} = (U_n)_{n \geq 0}$ is a sequence of independent and identically distributed random variables taking values in $\mathbf{R}^2$ with distribution given by the probability measure ν_ε.

We consider a sequence of random variables $(X_n)_{n\geq 0}$ defined on Υ with values in M such that:

- X_0 is a random variable with values in M and some distribution μ_0,
- for each $n \geq 1$, X_n is defined by the recurrence relation:

$$X_n(\underline{u}) = F(X_{n-1}(\underline{u})) + U_{n-1}(\underline{u}). \tag{4.42}$$

Remark *If $\varepsilon = 0$ and there is no perturbation, then $\nu_\varepsilon = \delta_{(0,0)}$ and $\nu_\varepsilon^{\mathbf{N}}$ is the Dirac measure at the sequence $\underline{0}$ of zero vectors. In this case, the recurrence relation (4.42) is the deterministic billiard map and we obtain a trajectory in the phase space given by the unperturbed billiard map, starting from some point $X_0(\underline{0}) \in M$.*

Let $\underline{u} = (u_0, u_1, \dots) \in \Upsilon$ be a realisation of the process $\underline{U}$. Let $x_n = X_n(\underline{u})$. We have:

- $X_0(\underline{u}) = x_0$ is some point in M
- for each $n \geq 1$ we have:

$$\begin{aligned} x_n &= F(x_{n-1}) + u_{n-1} = F_{u_{n-1}}(x_{n-1}) \\ &= F^n_{\underline{u}}(x_0) \end{aligned} \tag{4.43}$$

Lemma 4.21 *Let $P : M \times \mathcal{B} \to [0,1]$ be a function defined by:*

$$P(x, B) = \eta_\varepsilon^{F(x)}(B) \tag{4.44}$$

for m-almost every $x \in M$ and $B \in \mathcal{B}$. Then P is a Markov transition function on M.

Proof. It is clear from the definition of the perturbation in (4.38). ◇

Lemma 4.22 *The process* (X_n) *defined by equation (4.42) is a Markov process with Markov kernel given by the function* P *with initial distribution* μ_0*.*

Proof. It is straightforward from the definition of the process $(X_n)_{n\geq 0}$ in (4.42). ◇

Example 2 *Suppose that* $\mu_0 = \delta_x$ *for some* $x \in M$*. The distribution of* X_n *tells us all possible images of* x *under* n *iterations of the randomly perturbed billiard map.*

Lemma 4.23 *For any vector* $u \in \mathbf{R}^2$*, the map* F_u *preserves the measure* m*.*

Proof. The map F_u is the composite of the billiard map F with the translation by u. The measure m is F-invariant and also translation-invariant, therefore it is F_u-invariant. ◇

Lemma 4.24 *The natural measure* m *on* M *is an invariant measure with respect to the Markov transition function* P *defined in Lemma 4.21.*

Proof. For every $B \in \mathcal{B}$ we have:

$$\begin{aligned}\mathcal{L}m(B) &= \int_M P(x,B)m(dx)\\ &= \int_M \nu_\varepsilon(u : F(x)+u \in B)m(dx)\\ &= \int_M \int_{\mathbf{R}^2} 1_{\{F_u(x)\in B\}}(u,x)\nu_\varepsilon(du)m(dx)\\ &= \int_{\mathbf{R}^2} \int_M 1_{\{F_u(x)\in B\}}(u,x)m(dx)\nu_\varepsilon(du)\\ &= \int_{\mathbf{R}^2} m(B)\nu_\varepsilon(du)\\ &= m(B).\end{aligned}$$

Therefore we have $\mathcal{L}m = m$. ◇

Theorem 4.25 *Let $\mathcal{D}$ be a classical billiard such that the table's boundary satisfies Assumption* (**HB**)*. Consider a random perturbation to the system as described in (4.38). Then the resulting random billiard is ergodic.*

Proof. By Lemma 4.22, we know that the process (X_n) defined by the composition of the perturbed billiard map is a Markov chain with Markov kernel:

$$P(x,B) = \eta_\varepsilon^{F(x)}(B)$$

for $x \in M$ and $B \in \mathcal{B}$. We need to prove that the measure m on M is ergodic for this Markov process.

By Lemma 4.24, the measure m is an invariant probability measure for this Markov process. We will now show that this is in fact the unique invariant measure.

Since the transition probability has a density function, we observe that if $B \in \mathcal{B}$ such that $m(B) = 0$ then $P(x, B) = 0$ as functions of x. Thus the support of every invariant measure for $\mathcal{L}$ is also has positive measure with respect to the measure m. Thus there can be at most countably many of invariant measures for $\mathcal{L}$.

Recall that such that for any $x \in M$, the density function of the transition probability $P(x, dy)$ is positive on $B_\varepsilon(F(x))$. This condition means that if we start from x then in next step of the process we are allowed to go to anywhere in a ball of radius ε centred at the point $F(x)$. Under this condition, there can be almost countably many ergodic invariant measures with respect to P.

Two nearby points are in the same ergodic components due to the perturbation. Because two distinct ergodic measures are either coincide or mutually singular, the Lebesgue measure must be the only ergodic measure. In fact, this implies that it is the only invariant measure with respect to P.

$\diamond$

Theorem 4.26 *Let $\mathcal{D}$ be a classical billiard such that the table's boundary satisfies Assumption* (**HB**). *Consider a random perturbation to the system given by the transition function P as defined in Lemma 4.21. Assume that the derivative $DF : M \to SL(2, \mathbf{R})$ satisfies one of the two hypotheses* (**H2**) *and* (**H3**)*:*

(**H2**)*: there exist non-empty open sets U and V in M such that DF has distinct eigenvalues on U and only complex eigenvalues on V.*

(**H3**)*: there exist non-empty open sets U and V in M such that DF has distinct eigenvalues on U and V but the eigenvectors on U are different from those on V.*

Let $(X_n)_{n\geq 0}$ be the Markov process with transition P and initial distribution m. Let $\lambda_1 \geq \lambda_2$ be the Lyapunov exponents associated to the Markov process $(X_n)_{n\geq 0}$ and the derivative map DF. Then $\lambda_1 > 0$.

Proof. Suppose to the contrary that $\lambda_1 = 0$. The billiard map satisfies the condition (**H1**) in Lemma 4.18 as shown in Lemma 3.6 of [17]. Therefore there exists a measurable family $\xi : x \mapsto \xi_x$ of probability measures on $\mathbb{P}^1$ indexed by M such that for m-almost every $x \in M$:

$$\xi_y = (DF(x))_* \xi_x \tag{4.45}$$

for $P(x, .)$-almost every y.

For any $x \in M$ the support of $P(x, .)$ contains a ball $B_\varepsilon(F(x))$ of radius $\varepsilon > 0$ and centred at $F(x)$. Consider a partition of M by squares of size $\varepsilon/2$. Since F is an invertible map, ξ is constant on the union of any 4 adjacent squares and hence m-almost everywhere on M. So there exists a subset $S \subset M$ with $m(S) = 1$ such that ξ is constant at every point in S. From now on, we will also use ξ to denote the measure $\xi(x)$ of any $x \in S$.

Let $x \in S$, we have that $\xi = (DF(x))_* \xi$. By iterating the matrix $DF(x)$ we have

$$\xi = (DF(x))^n_* \xi$$

for every $n \geq 1$.

Suppose that $DF(x_1)$ has distinct real eigenvalues α_1 and α_2 for some $x_1 \in S$. As $\det DF(x_1) = 1$, we can assume that $|\alpha_1| > 1 > |\alpha_2|$. Let $v_i \in \mathbf{R}^2$ be a unit eigenvector for α_i, $i = 1, 2$. Let $v \in \mathbf{R}^2$ be any nonzero vector such that $v \neq v_2$ and consider the sequence of unit vectors:

$$u_n = \frac{DF(x_1)^n(v)}{\|DF(x_1)^n(v)\|}.$$

As $n \to \infty$, the angle between u_n and v_1 converges to 0 or π. Therefore the probability measure ξ must be concentrated only on the direction of v_1 and v_2. In other words, we must have that $\xi = c_1\delta_{[v_1]} + c_2\delta_{[v_2]}$ for some constants c_1 and c_2.

Let $x_2 \in S$ such that $DF(x_2)$ has complex eigenvalues. By a change of coordinates, $DF(x_2)$ becomes a rotation matrix. We could assume that $DF(x_2)$'s rotational angle is an irrational multiple of 2π as the rotational angle varies continuously wherever the billiard map is C^2. If the rotational angle is an irrational multiple of 2π then the probability measure ξ must be the Lebesgue measure on $\mathbb{P}^1$. This is a contradiction to the fact that ξ is discrete.

Let $x_3 \in S$ such that $DF(x_3)$ has distinct real eigenvalues and eigenvectors w_1 and w_2, such that $\{[w_1], [w_2]\} \cap \{[v_1], [v_2]\} = \emptyset$. Then $\xi = d_1\delta_{[w_1]} + d_2\delta_{[w_2]}$ for some constants d_1 and d_2. But this contradicts the fact that $\xi = c_1\delta_{[v_1]} + c_2\delta_{[v_2]}$.

$\diamond$

4.4.1 *Random non-circular elliptic billiards*

Theorem 4.27 *Let $\mathcal{D}$ be any non-circular elliptic billiard table. Consider any random perturbation to the billiard map on $\mathcal{D}$ as defined in (4.38). The resulting random billiard is ergodic and hyperbolic.*

Proof. Let $F : M \to M$ be the billiard map with coordinates r and $s = \sin(\phi)$ as usual. The derivative map DF in the coordinates r and $s = \sin(\varphi)$ is given by

$$DF(x) = \begin{pmatrix} 1 & 0 \\ 0 & \cos(\varphi_1) \end{pmatrix} \frac{-1}{\cos(\varphi_1)} \begin{pmatrix} -\tau K + \cos(\varphi) & \tau \\ \tau K K_1 - K\cos(\varphi_1) - K_1 \cos(\varphi) & -\tau K_1 + \cos(\varphi_1) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\cos(\varphi)} \end{pmatrix}$$

We have $\det(DF(x)) = 1$ and

$$\mathrm{trace}(DF(x)) = \frac{\tau K - \cos(\varphi)}{\cos(\varphi_1)} + \frac{\tau K_1 - \cos(\varphi_1)}{\cos(\varphi)}. \tag{4.46}$$

At the point $x_1 \in M$ corresponding to $u = a, v = 0$, $\varphi = \varphi_1 = 0$, we have $\tau = 2a$, $K = K_1 = \frac{a}{b^2}$ and therefore $\mathrm{trace}(DF(x_1)) = \frac{4a^2}{b^2} - 2 > 2$.

At the point $x_2 \in M$ corresponding to $u = 0, v = b$, $\varphi = \varphi_1 = 0$, we have $\tau = 2b$, $K = K_1 = \frac{b}{a^2}$ and therefore $\mathrm{trace}(DF(x_2)) = \frac{4b^2}{a^2} - 2$. It's clear that $|\mathrm{trace}(DF(x_2))| < 2$.

Since the derivative is a smooth function, $\mathrm{trace}(DF(x)) > 2$ for any x sufficiently close to x_1 and similarly $|\mathrm{trace}(DF(x))| < 2$ for x sufficiently close to x_2. Among those x such that $DF(x)$ has complex eigenvalues, there is a subset of them with positive measure such that at those points the derivative corresponds to irrational rotations. Because of this mixture of real and complex eigenvalues, there cannot be any probability measure on $\mathbb{P}^1$ that is invariant under $DF(x)$ for m-almost every x.

◇

4.4.2 Random circular billiards

Theorem 4.28 *Let $\mathcal{D}$ be any circular billiard table of radius $R > 0$. Consider any random perturbation to the billiard map on $\mathcal{D}$ as defined in (4.38). The resulting random billiard is ergodic, but the Lyapunov exponents are always 0.*

Proof. The derivative of the billiard map in this case is:

$$DF(x) = \begin{pmatrix} 1 & \frac{-2R}{\sqrt{1-s^2}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{-2R}{\cos(\varphi)} \\ 0 & 1 \end{pmatrix} \tag{4.47}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \cos(\varphi) \end{pmatrix} \begin{pmatrix} 1 & -2R \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\cos(\varphi)} \end{pmatrix}. \tag{4.48}$$

Let $(X_n)_{n\geq 0}$ be the Markov process with transition P given in Lemma 4.21 and initial distribution m. By Theorem 4.25, the dynamical system $(\Omega, \mathcal{A}, \mathbf{P}_m, \theta)$ associated to $(X_n)_{n\geq 0}$ is ergodic, and thus the largest Lyapunov exponent in this case is:

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \log \left\| \begin{pmatrix} 1 & -2R\left(\frac{1}{\cos(\varphi_0)} + \cdots + \frac{1}{\cos(\varphi_{n-1})}\right) \\ 0 & 1 \end{pmatrix} \right\| \tag{4.49}$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \left(2R\left(\frac{1}{\cos(\varphi_0)} + \cdots + \frac{1}{\cos(\varphi_{n-1})}\right)\right). \tag{4.50}$$

for $\mathbf{P}_m$-almost every sequence $\omega = (x_0, x_1, \dots)$. In the above equality, we used the max norm for the matrices.

Recall that any point $x \in M$ has two coordinates r and $s = \sin(\varphi)$. Let $g : M \to [1, \infty)$ be the function defined by:

$$g(r, s) = \frac{1}{\cos(\varphi)},$$

where $s = \sin(\varphi)$. We have that:

$$\int_M g(x) m(dx) = \frac{1}{2|\partial\mathcal{D}|} \int_M \frac{1}{\cos(\varphi)} ds dr \tag{4.51}$$

$$= \frac{1}{2|\partial\mathcal{D}|} \int_{-\pi/2}^{\pi/2} d\varphi \int_{\partial\mathcal{D}} dr \tag{4.52}$$

$$= \frac{\pi}{2}. \tag{4.53}$$

Therefore, by the Birkhoff's Ergodic Theorem, we have:

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{\cos(\varphi_0)} + \frac{1}{\cos(\varphi_1)} + \cdots + \frac{1}{\cos(\varphi_{n-1})}\right) = \frac{\pi}{2}. \tag{4.54}$$

for $\mathbf{P}_m$-almost every sequence $\omega = (x_0, x_1, \ldots)$.

Let $\omega = (x_0, x_1, \ldots)$ be a sequence such that both Eq. (4.49) and (4.54) hold for ω. Then there exists an integer $N > 0$ large enough such that for every $n \geq N$ we have:

$$\frac{\pi}{3} \leq \frac{1}{n}\left(\frac{1}{\cos(\varphi_0)} + \frac{1}{\cos(\varphi_1)} + \cdots + \frac{1}{\cos(\varphi_{n-1})}\right) \leq \pi.$$

Therefore:

$$\lambda_1 \leq \lim_{n\to\infty} \frac{1}{n}(\log(2R) + \log(n\pi)) = 0. \tag{4.55}$$

This implies that for $0 \leq \lambda_1 \leq 0$, therefore $\lambda_1 = 0$.

◇

4.5 Mixing property of randomly perturbed smooth and convex billiards

In this section, we consider smooth and convex billiards with random perturbation. Markarian et al. proved in [49] the exponential convergence to the unique invariant measure. Although the random perturbation considered there is different from the perturbation defined in (4.38), the same proof still works in our case.

Lemma 4.29 ([49], Theorem 1) *Let $\mathcal{D}$ be a smooth and convex billiard table whose boundary is C^3. Consider any random perturbation to the billiard map on $\mathcal{D}$ as defined in (4.38). There exist $\gamma_\varepsilon > 0$ such that, for any probability measure μ on $\mathcal{B}$ the Borel σ-algebra on M and $n \in \mathbf{N}$, we have:*

$$\|\mathcal{L}^n \mu - m\| \leq e^{-\gamma_\varepsilon n}, \tag{4.56}$$

where for any probability measures μ and ν on $\mathcal{B}$:

$$\|\mu - \nu\| = \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)|.$$

Theorem 4.30 *Let $\mathcal{D}$ be a smooth (C^3) and convex billiard. Consider a random perturbation to the system given by the transition function P as defined in Lemma 4.21.*

Let $(X_n)_{n\geq 0}$ be the Markov process with transition P and initial distribution m. Then the process $(X_n)_{n\geq 0}$ is exponential mixing.

Proof. Let $B_0 \in \mathcal{B}$ be a measurable set on M with $m(B_0) \neq 0$, and consider the probability measure μ on $\mathcal{B}$ such that:

$$\mu(A_0) = \frac{m(A_0 \cap B_0)}{m(B_0)} \tag{4.57}$$

for any $A_0 \in \mathcal{B}$. Then for $n \geq 0$:

$$\begin{aligned} \mathbf{P}_m(X_n \in A_0, X_0 \in B_0) &= \mathbf{P}_m(X_n \in A_0 | X_0 \in B_0)\mathbf{P}_m(X_0 \in B_0) \\ &= (\mathcal{L}^n\mu)(A_0)m(B_0) \end{aligned} \tag{4.58}$$

By Lemma 4.29, $\lim_{n\to\infty} \|\mathcal{L}^n\mu - m\| = 0$ with exponential rate, we also have that:

$$\lim_{n\to\infty} \mathbf{P}_m(X_n \in A_0, X_0 \in B_0) = m(A_0)m(B_0) \tag{4.59}$$

with exponential rate.

To prove that the Markov process $(X_n)_{n\geq 0}$ is mixing, it suffices to verify the mixing property for the elementary cylinders. Consider two elementary cylinders

$$A = \{\omega = (x_0, x_1, \dots) : x_i \in A_i, i = 0, \dots, k\}$$

and

$$B = \{\omega = (x_0, x_1, \dots) : x_i \in B_i, i = 0, \dots, l\}$$

for some integers $k, l \geq 0$ and sequences $A_1, A_2, \dots, A_k$ and $B_0, B_1, \dots, B_k$ in $\mathcal{B}$.

For $n > l$:

$$\begin{aligned}
\mathbf{P}_m(\theta^{-n} A \cap B) &= \mathbf{P}_m(X_{n+k} \in A_k, \dots, X_n \in A_0, X_l \in B_l, \dots, X_0 \in B_0) \\
&= \mathbf{P}_m(X_{n+k} \in A_k, \dots, X_n \in A_k | X_l \in B_l, \dots, X_0 \in B_0)\mathbf{P}_m(X_l \in B_l, \dots, X_0 \in B_0) \\
&= \mathbf{P}_m(X_{n+k} \in A_k | X_{n+k-1} \in A_{k-1}) \cdots \mathbf{P}_m(X_{n+1} \in A_1 | X_n \in A_0) \times \\
&\quad \times \mathbf{P}_m(X_n \in A_0 | X_l \in B_l)\mathbf{P}_m(B) \\
&= \mathbf{P}_m(X_k \in A_k | X_{k-1} \in A_{k-1}) \cdots \mathbf{P}_m(X_1 \in A_1 | X_0 \in A_0) \times \\
&\quad \times \mathbf{P}_m(X_{n-l} \in A_0 | X_0 \in B_l)\mathbf{P}_m(B).
\end{aligned} \tag{4.60}$$

By (4.59), we have that

$$\lim_{n\to\infty} \mathbf{P}_m(X_{n-l} \in A_0 | X_0 \in B_l) = m(A_0). \tag{4.61}$$

Therefore

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbf{P}_m(X_k \in A_k | X_{k-1} \in A_{k-1}) \cdots \mathbf{P}_m(X_1 \in A_1 | X_0 \in A_0) \mathbf{P}_m(X_{n-l} \in A_0 | X_0 \in B_l) \\
&= \mathbf{P}_m(X_k \in A_k | X_{k-1} \in A_{k-1}) \cdots \mathbf{P}_m(X_1 \in A_1 | X_0 \in A_0) m(A_0) \\
&= \mathbf{P}_m(X_k \in A_k, \ldots, X_0 \in A_0) \\
&= \mathbf{P}_m(A).
\end{aligned} \tag{4.62}
$$

Substitute this into (4.60), we have that:

$$\lim_{n\to\infty} \mathbf{P}_m(\theta^{-n} A \cap B) = \mathbf{P}_m(A)\mathbf{P}_m(B) \tag{4.63}$$

and the rate of convergence is exponential.

Thus we have proved that the Markov process $(X_n)_{n\geq 0}$ is exponential mixing. $\diamond$

APPENDIX A

Measure theory

Let E be any set. A σ-algebra $\mathcal{E}$ on E is a collection of subsets of E such that it contains the empty set $\emptyset$ and satisfies the following two conditions:

1. If $A \in \mathcal{E}$ then its complement $A^c \in \mathcal{E}$

2. If $(A_n)_{n\geq 0}$ is a sequence in $\mathcal{E}$ then $\bigcup_{n\geq 0} A_n \in \mathcal{E}$.

We call the pair $(E, \mathcal{E})$ a *measurable space* and elements in $\mathcal{E}$ *measurable sets.*

A *measure* μ on the pair $\mathcal{E}$ is a function $\mu : \mathcal{E} \to [0, \infty]$ such that $\mu(\emptyset) = 0$ and if $(A_n)_{n\geq 0}$ is a sequence of disjoint elements in $\mathcal{E}$ then

$$\mu(\bigcup_{n\geq 0} A_n) = \sum_{n\geq 0} \mu(A_n).$$

We call the triple $(E, \mathcal{E}, \mu)$ a *measure space.* When $\mu(E) = 1$, it is also called a *probability measure* and $(E, \mathcal{E}, \mu)$ is called a probability space. Depending on the context, we could understand the number $\mu(A)$, for a measurable set A, as a sort of size of A, or as the probability of event A happening if μ is a probability measure.

Let $(E, \mathcal{E})$ and $(G, \mathcal{G})$ be 2 measurable spaces. A function $f : (E, \mathcal{E}) \to (G, \mathcal{G})$ between the 2 spaces is *measurable* if for any $B \in \mathcal{G}$, its inverse image $f^{-1}(B) \in \mathcal{E}$. Let μ be a measure on $\mathcal{E}$ then the *pushforward measure* of μ is a probability measure $f_*\mu = \mu \circ f^{-1}$ on $\mathcal{G}$ and defined by:

$$f_*\mu(B) = \mu(f^{-1}(B)).$$

Let $(\Omega, \mathcal{A}, \mathbf{P})$ be a probability space and $(E, \mathcal{E})$ a measurable space. A measurable function $X : (\Omega, \mathcal{A}) \to (E, \mathcal{E})$ is called a *random variable* with values in E. The pushfoward measure $\mu_X = \mathbf{P} \circ X^{-1}$ is called the *distribution* of X in E.

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