INSTABILITIES AND TRANSITION TO CHAOS IN PLANE WAKES

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This is a report on the initial investigation and progress of the project on instabilities and transition to chaos in plane wake behind a flat plate parallel to the upstream uniform flow. The shear flow system to be studied is an important flow in practice and, in particular, is of interest in aerodynamics. An improved understanding of the instabilities and transition processes would open significant possibilities to the design engineers to delay transition and control or promote turbulence.

Our investigation applies, presently, finite difference methods and stability analysis of disturbances about a fully nonlinear and nonparallel base flow and, later, a Lyapunov exponents analysis to determine the most critical disturbances and the physical mechanisms that drive the transition to chaos. The results will be used to detect the type of instability that can cause transition to chaos in a wake.

By generalizing both the concept of convective instability and the definition of Lyapunov exponents, we plan to investigate transition to chaos for the present open flow system.

Our investigation addresses many issues such as the forms of the transitional large scale flow structures, two and three–dimensional and nonparallel effects (for both base flow and disturbances) on transition to chaos, times/space locations at which important chaotic processes occur, subcriticality/supercriticality of chaos and gradual/fast transitional aspects.

1. Introduction

The present study concerns the problem of the transition to chaos in plane wake behind a flat plate of length l parallel to the upstream uniform flow with velocity u_∞. We assume that the fluid is homogeneous and incompressible. In order to detect the origin of the transition to chaos, we need to investigate instabilities, due to the most dangerous disturbances, which can occur beyond the critical value R_c of the Reynolds number $R = u_\infty l/\nu$ below which no instabilities are possible. Here ν is the kinematic viscosity. A review of the literature on the subject indicate that very few studies have been done on the calculation of R_c for the present problems, and these studies have been approximate and incomplete at best. For example, Taneda (1963) studies the stability of a two–dimensional laminar wake both experimentally and theoretically. He restricted his theoretical studies on disturbances with large or small wave numbers and for parallel flow approximation. He found that $R_c = 3.2$ for wake behind a cylinder (diameter = l) which is about 3. times that predicted by his experimental studies. The source of this discrepancy seems to lie in his theoretical assumptions of the parallel mean flow and of particular disturbances with large or small wavelength. For wake behind a flat plate with length l, he finds theoretically that $R_c = 3.8$, but he does not report any experimental values for R_c in this case.

Furthermore, most of the theoretical and computational instability investigations that have been done in the past for plane wakes (Meiburg and Lasheras, 1988; Mattingly and Criminale, 1972; Sato and Kuriki, 1961; Papageorgiou and Smith, 1988, 1989) are for inviscid case where R is very large, and, hence, they do not provide satisfactory information on the subject of the present study. We, therefore, decided to start from investigation of primary instability of the base flow due to both two and three–dimensional disturbances, which take place beyond R_c, determination of R_c for more general base flow and disturbances and then apply a Lyapunov exponent analysis to search for possible transition to chaos. If this search does not lead to any indication of transition to chaotic motion, we will then investigate secondary instability and beyond until we can detect indication of transition to chaotic motion. In this report, we present the results and analysis of some initial and preliminary investigation of primary instability in plane wakes in Section 3. We present the relatively new concept of the so–called short–time Lyapunov exponent analysis in Section 2. Some conclusions are given in Section 4. The complete flow charts for for the Fortran programs, which have been developed to compute the primary instability, based on finite difference methods, are provided at the end of the report.

2. Short–time Lyapunov exponent analysis

Our main investigation deals with the so–called short–time Lyapunov exponent analysis (Vastano and Moser, 1991) which is a new approach to study chaotic motion and its stability properties. Our study is based on the linear stability system for small disturbances acting on the base flow. We plan to compute a set of N Lyapunov exponents (LES) by the method of approach similar to that described in detail by Vastano and Moser (1991). It is described briefly here. Following Benettin et al. (1980), an initially orthogonal set of N perturbations will be evolved by integrating the equations for perturbations using finite difference methods. Alternatively, a possible well established accurate spectral/finite difference code can be used. Before integrating these equations, we will transform the original coordinate system (x,y,z) into a moving one designated by (ξ,η,ζ), where

$$(\xi,\eta,\zeta) = (x,y,z) - \underset{\sim}{u}_b\, t$$

Here $\underset{\sim}{u}_b$ is the base flow velocity. This transformation will enable us to deal adequately with perturbations in the present open flow system. The base flow velocity $\underset{\sim}{u}_b$ known as a set of data from previous computation will be assumed constant within two consecutive adjacent grid points in independent variables of the base flow velocity. Each perturbation will then be decomposed, say at time t, onto a set of basic vectors. As time proceeds, the components will grow or decay exponentially according to Lyapunov exponents λ_i . Before the set of perturbations become numerically indistinguishable, say at time $t + \Delta t$, a Gram–Schmidt reorthogonalization will be performed. The first vector in the set (the numbering here is arbitrary), whose growth rate is maximum among all the other vectors, will be renormalized and thereafter the normalization factor is the growth undergone by vectors since the last renormalization. The logarithm of the factor divided by Δt will be the short–time contribution to the long–time average exponent λ_1 . The normalization procedure will be based on L_2–norm in (ξ, η, ζ) space. The second vector in the set will then be made orthogonal to the first, which has the effect of subtracting out the growth of the first vector. The fastest growing component of the remainder will then be considered. Since the short–time contribution $\lambda_1(t)$ will then be determined by normalizing the first vector, the normalization factor for the orthogonalized second vector will be the short–time contribution $\lambda_2(t)$. Proceeding iteratively, the normalization of the n–th vector $(1 \leq n <= N)$ will give the short–time contribution $\lambda_n(t)$, and thus all N *LES* will be computed.

The analysis of the short–time contributions to the long–time average exponents described above can provide important information about the physical mechanism driving the transition to chaos, and this anlaysis does not suffer from the computational difficulties associated with t he traditional LES computation (Vastano and Moser, 1990). The vectors associated with positive λ_i characterize the space of growing perturbations (Greene and Kim, 1987) and, in partiuclar, the first vector is the perturbation to which the flow system is most sensitive in the long run. Such perturbation is of importance and will allow us to determine the instability mechanism leading to chaos in the present system.

Once we obtain the results based on the Lyapunov exponent analysis described above, we will determine important times corresponding to large positive and negative peaks in $\lambda_1(t)$, will examine what mechanisms are responsible for the exponential growth of perturbations, will compute the wave lengths of LE perturbations, will investigate the structure of the most dangerous perturbation(s) at different values of (ξ, η, ζ), will compute velocity contours of base flows and perturbations, will compute contours of spanwise/transverse/streamwise vorticity of the base flow(s) as well as those of the perturbations. We will estimate the dimension of the chaotic attractor using the Kaplan–Yorke type conjecture (Fredrickson et al., 1983) which is based on computed $\lambda_i (i = 1, ..., N)$. We will check to see if fractal dimension of the chaotic motion increases with R and see if the estimated dimension support low–dimensional chaos or not. We will compute perturbation kinetic energy as a function of (ξ, η, ζ) to see where it is generated, where it dissipates and where it is maximum. We will determine the integrated form of the perturbation kinetic energy to describe generally the time evolution of the perturbation. We would like to determine if the transition to chaos is gradual or not. This can be done by checking the attractor's dimension to see if it increases continuously as R increases. We would like to examine the spatial dependence of perturbations corresponding to λ_i which can indicate if the perturbations are spatially localized or not. Such results may be indication of which flow features are important. We will determine the critical R_t of R at the onset of transition to chaos. Since the flow may well be sensitive with respect to disturbances for R close to R_t, we will use an iterative procedure and approach R_t from either above ($R > R_t$) or below ($R < R_t$).

3. Primary Instability

As we discussed before, we find it appropriate to start our study by investigating the instability of the base flow for $R > R_c$, where R_c is the critical value of R below which no instability is possible, with respect to infinitesimal disturbances.

First, we determine the base flow. We assume that the base flow is steady, two–dimensional, subjected to the boundary layer approximation, and the pressure gradient for the base flow is negligible. The non–dimensional form of the system of the equations and the boundary conditions for the base flow can be written as (Schlitching, 1979; Berger, 1971):

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = \frac{1}{R}\frac{\partial^2 \bar{u}}{\partial y^2}, \tag{1a}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{1b}$$

$$\frac{\partial \bar{u}}{\partial y} = \bar{v} = 0 \text{ at } y = 0 \tag{1c}$$

$$\bar{u} = 1 \text{ as } y \rightarrow \infty \tag{1d}$$

$$\underset{\sim}{\bar{u}} = \underset{\sim}{\bar{u}}_0 \quad \text{at } x = x_0 \tag{1e}$$

Here $\underset{\sim}{\bar{u}} = (\bar{u}, \bar{v})$ is the velocity vector, R is the Reynolds number $(R = u_\infty l/\nu)$, x–axis is along the flat plate, while y–axis is in transverse direction. Also, $\bar{u}_0$ is the initial base flow profile at x_0 station. Note that no parallel flow assumption is made.

We solve the system (1) for both far wake and near wake regions using an implicit finite difference model for (1a) and an explicit finite difference model for (1b) (Anderson et al., 1989; Hoffmann, 1989; White, 1991). We apply certain conditions on the mesh intervals Δx and Δy to insure that the scheme for (1b) is numerically stable. The number of grid points along y–direction increases as x increases since the wake region widens with increasing x. Since the wake region is symmetric with respect to x–axis, only the upper–half region is considered to determine the base flow solution. To determine accurate base flow solution, we need to know good initial values $\underset{\sim}{\bar{u}}_0$ at x_0 , values of $\underset{\sim}{\bar{u}}$ at the upper edge of the wake and some good estimation of the shape of the wake. These initial and boundary conditions were obtained by using the available analytical estimation of $\underset{\sim}{\bar{u}}$ for both far and near wake regions. For far wake case, we use the known similarity solution for $\bar{u}$ (Schlitching, 1979).

$$\bar{u} = 1 - (0.664/\sqrt{\pi x})\exp(-y^2R/4x). \tag{2a}$$

Using (1b) and (2), we find the expression for $\bar{v}$

$$\bar{v} = -(0.332/\sqrt{\pi x})(y/x)\exp(-y^2R/4x). \tag{2b}$$

For near wake case, we use the well known Blasius analytical solns for $\bar{u}$ and $\bar{v}$ (Berger, 1971).

Preliminary investigation considers computations for far wake case at $R = 30$ and $3 \le x \le 6$ and for near wake case at $R = 4$ and $0 \le x \le 1$. Plots of finite difference solutions for $\bar{u} = (u, v)$ versus y for different x–stations are given in the accompanying figures 1–4 for far and near wake cases. Symmetry and anti–symmetry of $\bar{u}$ and $\bar{v}$ profiles, respectively, with respect to $y = 0$ axis are apparent from these figures. Computational domain along x–axis are expected to be reasonable for both near wake (Berger, 1971) and far wake (Schlitching, 1979) cases. All the preliminary computations reported in this report were made at the University of Illinois, vmd main frame system (IBM 3081–GX2).

Preliminary investigation of the instability of the base flow, for both near and far wake cases, considers two–dimensional disturbances. We consider vorticity–stream function formulation for total quantities (i.e. sum of the base flow and disturbance quantities). The governing equations for total vorticity ω and total stream function ψ can be written in the following form:

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{1}{R}\nabla^2 \omega, \tag{3a}$$

$$\nabla^2 \psi = -\omega, \tag{3b}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

These equations must be solved subject to the following conditions:

$$\frac{\partial \psi}{\partial x} = 0, \quad \frac{\partial \psi}{\partial y} = 1 \quad \text{as } y \rightarrow \pm\, \infty, \tag{3c}$$

$$\psi = \psi_0 \text{ at } x = x_0, \tag{3d}$$

$$\psi = \psi_1 \text{ at } x = x_1, \tag{3e}$$

We used a forward time centered space difference (FTCS method (White, 1991)) explicit for (3a). Initial values for ω and ψ are used from computed base flow velocity components. A Gauss Siedel procedure is used to convert velocity component values at grid points into ψ and ω values. We have chosen sufficiently small time step Δt to ensure numerically stable scheme. New computed ω is then used in the right–hand side of (3b). This equation is solved for ψ using ADI method (White, 1991). Upstream and downstream boundary conditions (3d,e) are determined by an iterative procedure. Initial values of this iteration are based on the results at the previous time step. For the first time step $t = \Delta t$, initial values of the iterative procedure are based on the already computed base flow. Iteration continues until procedure converges. For $t = \Delta t$, iterative procedure converges after about 70 iterations for far wake case. For $t = n\,\Delta t$ $(n > 1)$, iterative procedure converges after about 2–3 iterations for far wake cases. We determined results for n up to about 6. We computed the volume averaged integral E for perturbation kinetic energy and found that E increases with n for both test cases (far wake case, $R = 30$; near wake case, $R = 4$). These results indicate that for far wake we have instability and that R_c, should be smaller than the value of $R = 30$ used in the computation. For near wake case, we found stability which indicates that R_c should be larger than the value of $R = 4$ used in the computation. Flow charts of the computer Fortran programs for the primary instability are provided for both far and near wake cases.

4. Some Conclusions

The results of the preliminary and initial investigations of the present research studies indicate that it is, indeed, possible to carry out a careful and successful investigation of realistic nonparallel wake flows and their instabilities using finite difference methods and stability analysis.

The amount of data storage, computer time and memory requirement during the preliminary computation of two–dimensional time dependent nonparallel wake flow indicated that our next three dimensional computations of instabilities and transition to chaos may well require supercomputing resources. Also, access to possible well tested computational codes can substantially save us time and can enable us to spend more of our time and efforts on the physical aspects and the improvements of the formulations and analysis.

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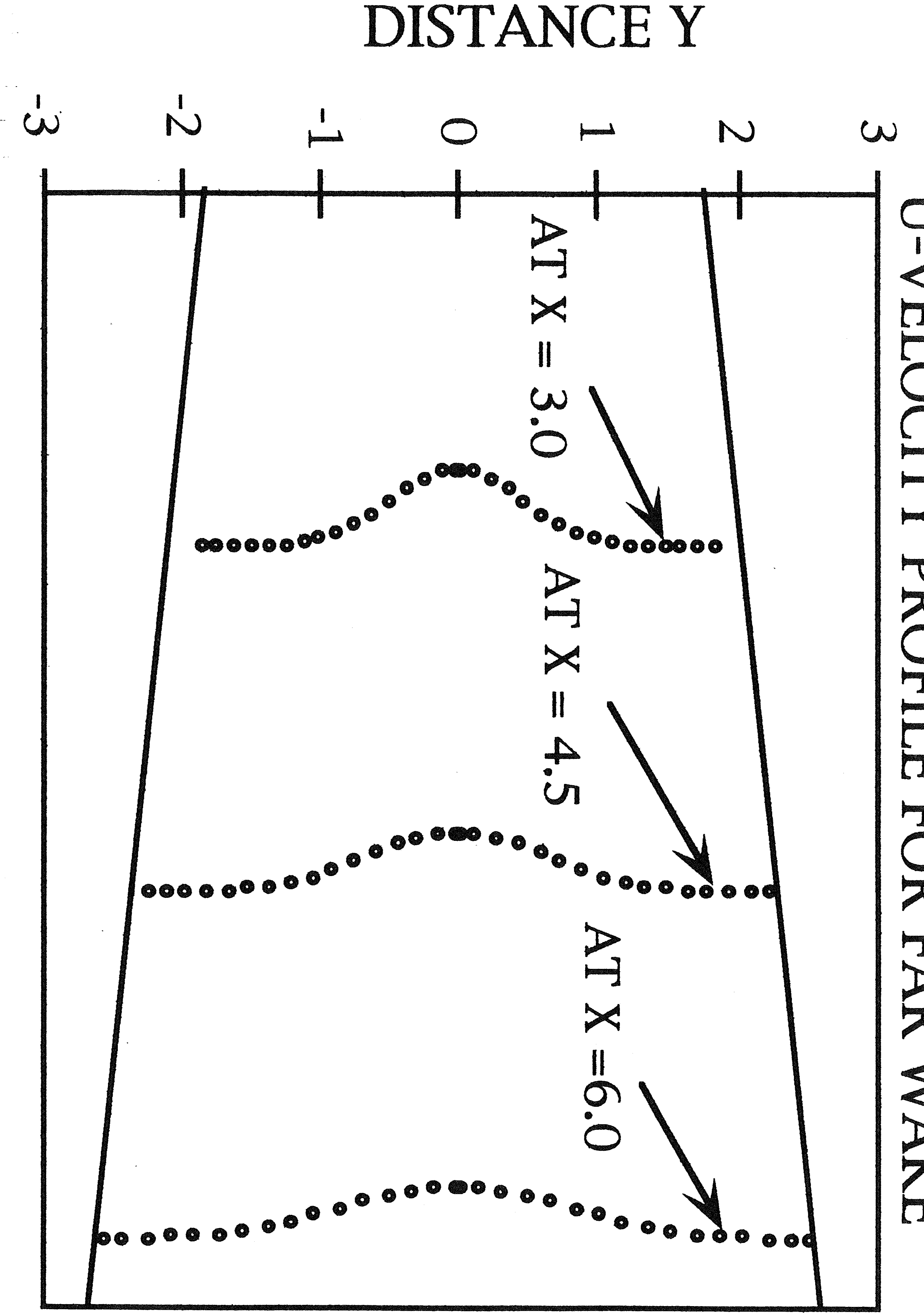

U-VELOCITY PROFILE FOR FAR WAKE
DISTANCE Y
3
2
1
0
-1
-2
-3
AT X = 3.0
AT X = 4.5
AT X =6.0

Fig. 1

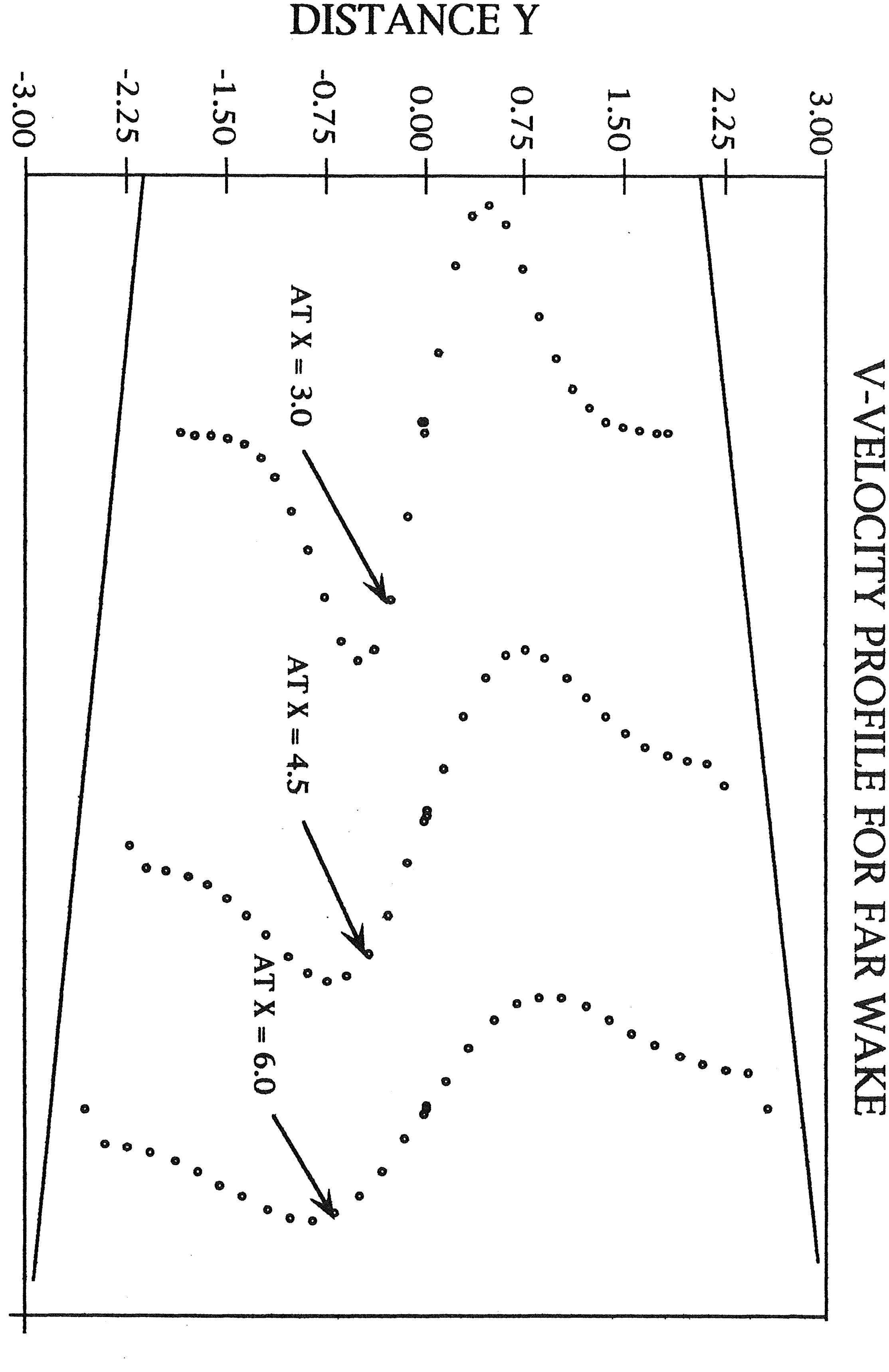

V-VELOCITY PROFILE FOR FAR WAKE
DISTANCE Y
-3.00
-2.25
-1.50
-0.75
0.00
0.75
1.50
2.25
3.00
AT X = 3.0
AT X = 4.5
AT X = 6.0

Fig. 2

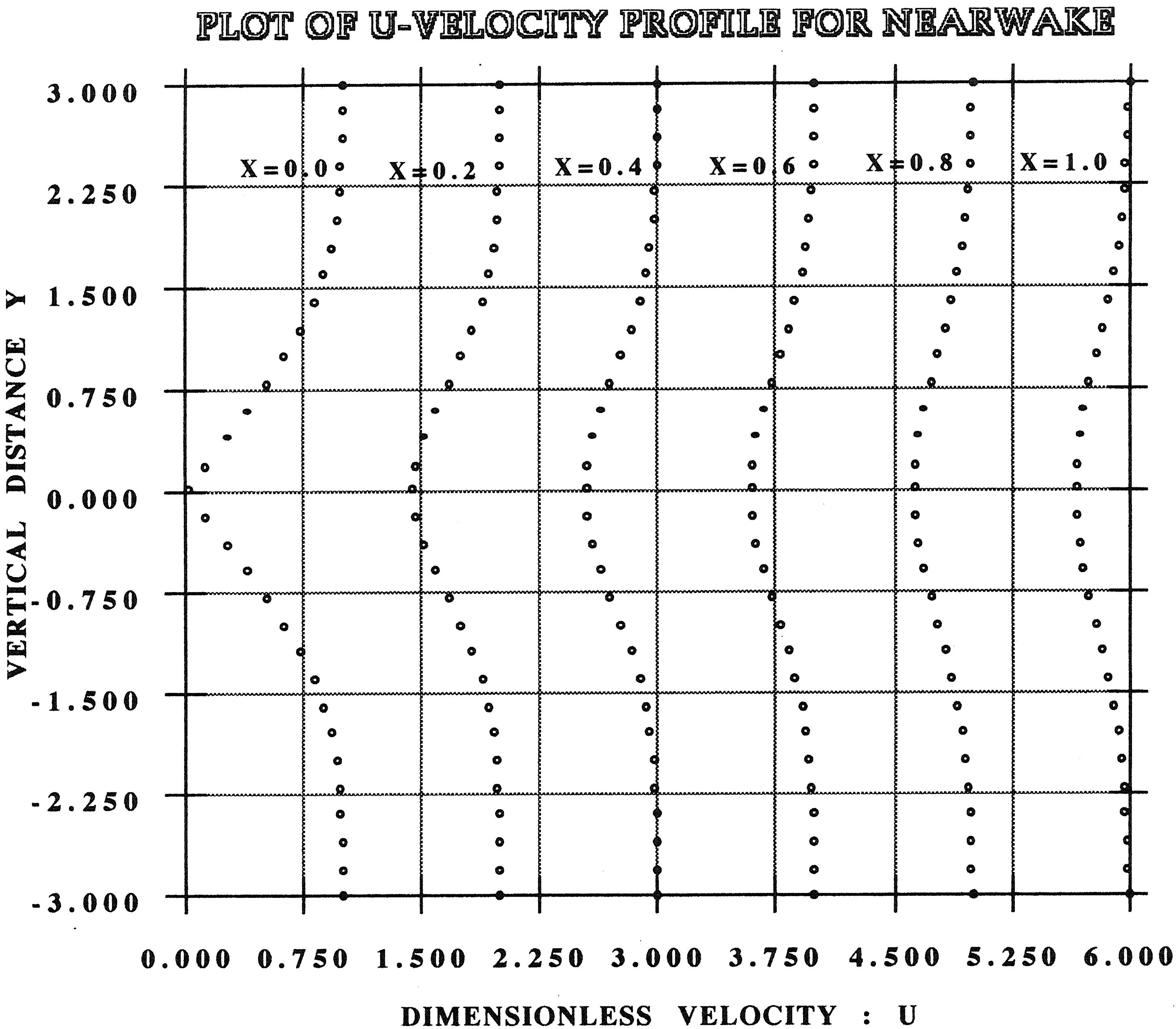
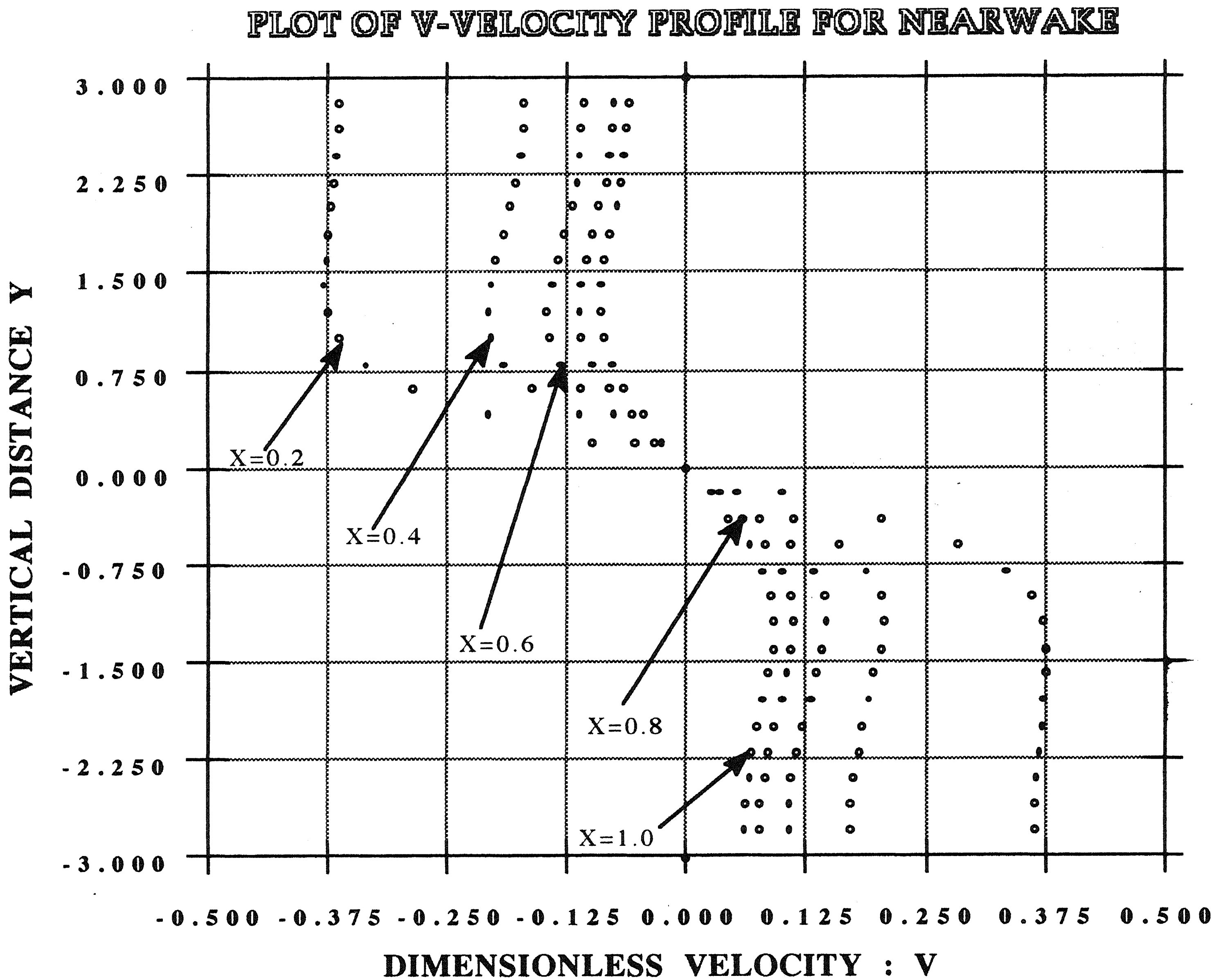
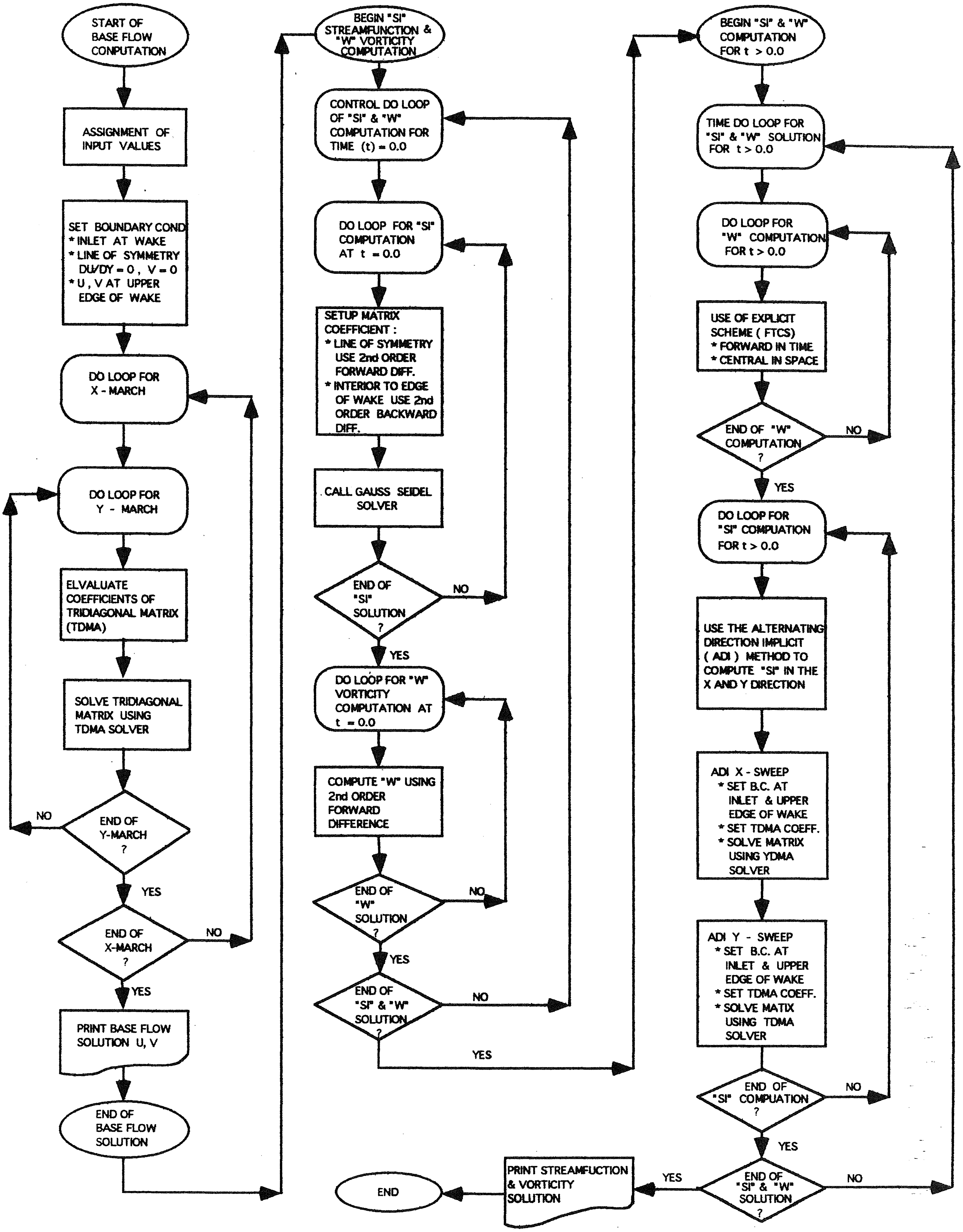

PLOT OF U-VELOCITY PROFILE FOR NEARWAKE
X=0.0
X=0.2
X=0.4
X=0.6
X=0.8
X=1.0
3.000
2.250
1.500
0.750
0.000
-0.750
-1.500
-2.250
-3.000
VERTICAL DISTANCE Y
0.000
0.750
1.500
2.250
3.000
3.750
4.500
5.250
6.000
DIMENSIONLESS VELOCITY : U

Fig. 3

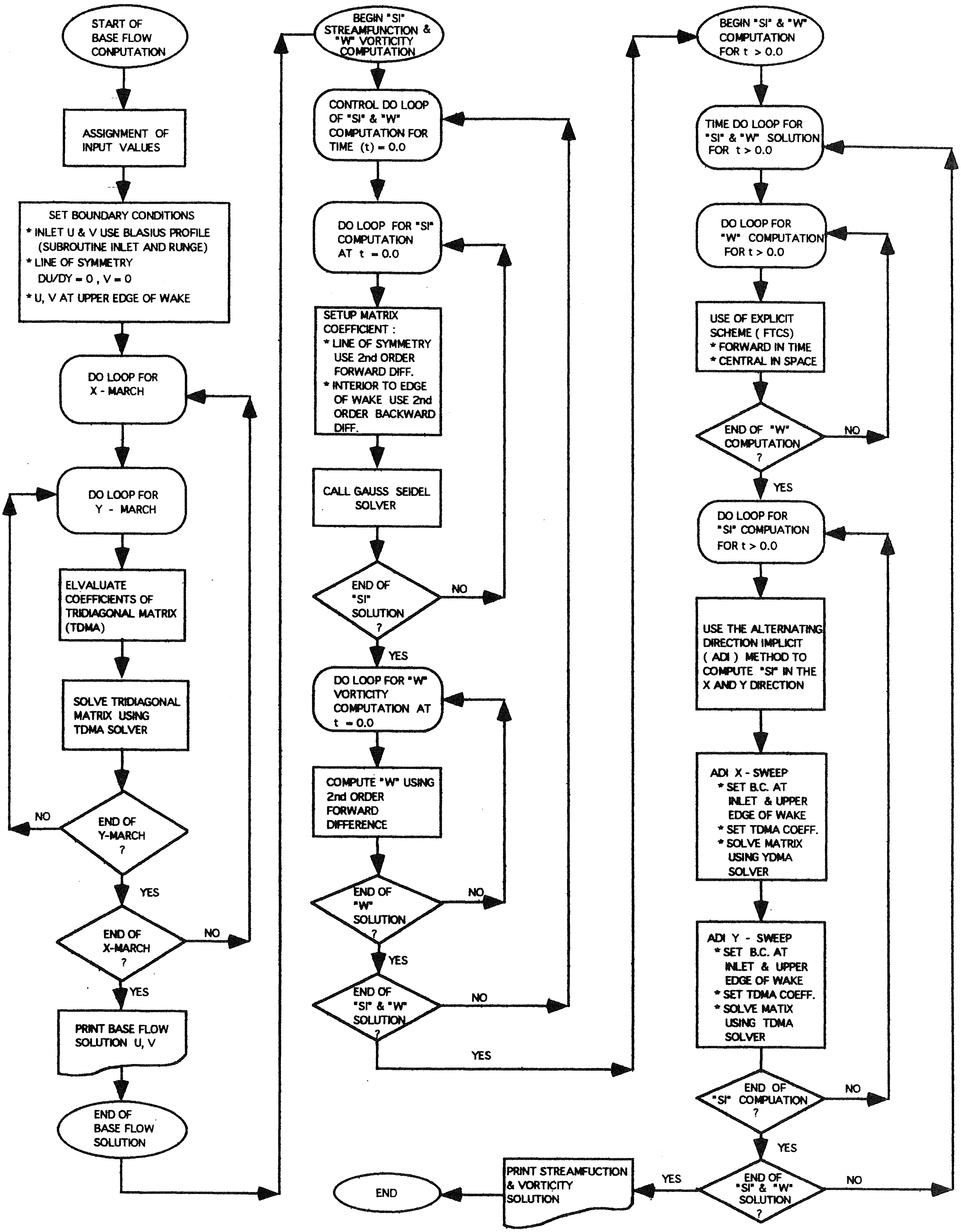


PLOT OF V-VELOCITY PROFILE FOR NEARWAKE
3.000
2.250
1.500
0.750
0.000
-0.750
-1.500
-2.250
-3.000
VERTICAL DISTANCE Y
X=0.2
X=0.4
X=0.6
X=0.8
X=1.0
-0.500 -0.375 -0.250 -0.125 0.000 0.125 0.250 0.375 0.500
DIMENSIONLESS VELOCITY : V

Fig. 4



GENERAL FLOWCHART FOR PROGRAM STABILITY **FARWAKE**



GENERAL FLOWCHART FOR PROGRAM STABILITY **NEARWAKE**