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Dominant Graphs for Rectilinear Network Design with Barriers

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Abstract

Given a set of points on a Cartesian plane and the coordinate axes, the rectilinear network design problem is to find a network, with sides parallel and perpendicular to the axes, which minimizes the fixed and the variable costs of interactions between a specified set of pairs of points. We show that, even in the presence of barriers, an optimal solution to the problem is contained in a grid graph defined by the set of given points and the barriers. This converts the spatial problem to a combinatorial problem. Finally, we show connections between the rectilinear network design problem and a number of well-known problems.

Key Words: Networks and graphs, layout, location, barriers

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1. Introduction

Many network design problems can be formalized as the following problem: Given a set of points and a set of flows described by a pair of points, find a network which permits every flow such that the sum of fixed cost of the network and variable cost of flows is minimized. In many situations the network to be designed is constrained to a subnetwork of a given graph, whereas in other cases it may not be the case. Problems with no prespecified graph may still have some other restrictions on the kinds of edges which can be included in the network. For example, in plant layout, the network to be designed may represent the material handling structure, and may be restricted to take only right angle turns. In the design of piping and electrical wiring in buildings, or connections in printed circuit boards, similar restrictions are required. We will call the network design problem which is constrained such that only horizontal and vertical arcs are allowed in the network as the *rectilinear network design problem.* We consider a general case of the network design problem in which there are regions on the plane, called barriers, through which no flow is allowed.

Magnanti and Wong (1986) describe a discrete choice network design problem in which the network to be designed has to be a subnetwork of a given graph. For our problem no prespecified graph is given of which the solution has to be a subnetwork. Since the rectilinear network design problem is posed on a Cartesian plane; there are potentially an infinite number of horizontal and vertical arcs which can be added to the network. In this paper, we will show that only a small number of these line segments are required to find an optimal solution. This permits us to pose the problem on a graph formed by these line segments instead of the plane, thereby reducing the complexity of the problem. The consequent discretization of the problem has several benefits, including reduction in the search space and ability to model as an integer program. The importance of discretization of continuous problems through the establishment of a finite dominating set has been well documented in location theory literature. We refer the reader to the paper by Hooker et al. (1991) for a comprehensive survey of finite dominating set results in location theory.

As we show later, the network design problem can be viewed as a generalization of several well-known problems. In particular we show that the problem reduces to the rectilinear Steiner tree problem (Hannan, 1966) and the shortest r-flow network problem (Chhajed et al., 1992). A variant of the problem is also related to the 1-median problem (Hakimi, 1966), the p-median problem with barriers (Larson and Sadiq, 1983), the rectilinear multifacility location problem (Francis and White, 1974) and the integrated station location and network design problem (Chhajed et al., 1992). Finite dominating set or dominating graph solutions are known to exist for each of these problems without barriers and for the p-median problem with barriers.

Our proof is different from the proof for the Steiner tree problem in Hannan (1966) but follows the proof technique in Chhajed (1991). An interesting element of our proof is the use of some results from the rectilinear location theory.

In the next section we will formally describe the problem and embed a network called the grid graph on the Cartesian plane. In section 3, we consider the special case where all barriers are isothetic polygons. These results are extended to arbitrarily shaped barriers in section 4. We show connection of the rectilinear network design problem to other well-known problems in section 5. Finally, we present conclusions in section 6.

2. Problem Statement and Grid Graph Construction

Let S be a set of n points on the two dimensional Cartesian plane. Along with the set S we have a *flow set* F whose elements are unordered pairs of points in S. The n points in S may represent locations of demand/supply points or input/output stations in a manufacturing plant and F may be the set of those pairs of points which have non-zero interaction. Let f_{ij} be the interaction cost per unit distance for flow $[i,j] \in F$ and C be the fixed cost per unit length of the network. Further, let $B = \cup B_i$ be the set of barriers. Each B_i represents an open and bounded region. Note that the barriers can be non-convex. The feasible region for the problem is $FR = \Re^2 \setminus B$. If x_i (y_i) is the x-coordinate (y-coordinate) of some point $i \in S$, then d(i,j) is defined as the length of a shortest path consisting of only horizontal and vertical line segments. Note that, in the presence of barriers d(i,j) may be greater than or equal to $|x_i - x_j| + |y_i - y_j|$, the rectilinear distance between points i and j.

Let N = (V(N), E(N)) represent a network with node set V(N) and arc set E(N), which consists of only horizontal and vertical line segments. Such a network is called a *rectilinear network.* Let d(i,j;N) be the length of a shortest path between nodes i and j on the network N. If there is no path between nodes i and j on the network N then d(i,j;N) is defined to be an arbitrary high number. An arc of E(N) adjacent to nodes i and j will be represented by (i,j). Let L(k,r) represent the length of an arc (k,r). The rectilinear network design problem can be written as:

$$\text{(P)} \qquad \underset{N \subset FR}{Min}\ Z(N) = \sum_{[i,j]\in F} f_{ij} d(i,j;N) + C \sum_{(k,r)\in E(N)} L(k,r) \qquad (1)$$

subject to:

There is a path in N between i and j for all $[i,j] \in F$. (2)

In problem (P), the network to be designed can have only horizontal or vertical arcs. While there are infinitely many solutions to this problem we now describe a dominating graph that is guaranteed to contain an optimal solution.

A *grid graph* for a set of points S, in the absence of barriers, is constructed as follows: Draw a horizontal and a vertical line through each point of S. The intersection of each horizontal line and vertical line defines a grid point. Take the intersection of these lines with the smallest rectangle containing S with sides parallel to the axes. The union of the grid points, the given set S, the rectangle, and the collection of horizontal and vertical lines contained in this rectangle is the grid graph, $\Pi(S)$ (Figure 1).

In the presence of barriers, the grid graph is drawn by adding extra nodes and arcs as follows: The smallest circumscribing rectangle must now contain the set S and the barriers. In addition to the horizontal and vertical lines through each point in S, draw all the possible supporting vertical and horizontal lines for every barrier. The point at which these lines support a barrier are called *barrier vertices* (BV). The lines through points in S and the supporting lines are continued until they intersect with the smallest circumscribing rectangle or a barrier. We shall denote the resulting grid graph as $\Pi_B(S)$. The extra grid lines generated due to the presence of barriers are shown by heavy lines in Figure 2.

The creation of a grid graph divides the feasible part of the Cartesian plane into *cells*. Formally, a cell is a minimal closed region bounded by either grid lines or barriers such that there are no grid lines in its interior. There are two types of cells :

i) Rectangular cells - These cells are rectangular in shape and may or may not share a boundary with a barrier. If a rectangular cell shares a boundary with a barrier then the intersection of the barrier boundary and the cell lies on a grid line. If the barriers are isothetic polygons, i.e., polygons with each side parallel to one of the co-ordinate axis, then all cells formed by the grid graph will be rectangular.

ii) Irregular cells - These cells are not rectangular. For such cells the intersection with a barrier boundary does not lie on a grid line. The boundary of an irregular cell is composed of sections formed by grid lines and those formed by the boundary of barriers. Irregular cells in which the grid line portion of the boundary is contiguous are called *deadend cells* ; while cells where the grid line portion of the boundary is non-contiguous are called *alley cells.* Figure 3 shows several cases of deadend and alley cells.

Associated with each cell are points called *cell vertices.* These are points generated by the intersection of either two grid lines or by a grid line and a barrier boundary.

3. Network Design with Isothetic Barriers

In this section we will consider the case where all barriers are isothetic polygons. This implies that the cells formed by the grid graph are rectangular.

Suppose we are given a feasible solution N to (P) embedded on a plane with objective function value Z(N). We assume that there is a node at every point where there is a change in the direction while following any path on this embedded network. If this assumption is not satisfied, we can introduce additional nodes to modify the network so that it satisfies the assumption without affecting the solution value (see Figure 4 for an example of such a network). Henceforth, we will assume that very arc in E(N) is either horizontal or vertical. We will now show that we can construct another feasible solution with objective function value no larger than Z(N) such that it is a subgraph of $\Pi_B(S)$.

Let P(i,j) be the set of arcs of N used by a shortest path between nodes i and j in N. If multiple paths exist we simply designate any one of them as the shortest path and use it to define P(i,j). Without loss of generality, we assume that every arc of N will be in some P(i,j), $[i,j] \in F$. If this is not true for an arc (a,b) then we can delete it to obtain a better feasible solution. Further, define T(i,j) to be the set of paths which contain the arc (i,j) , i.e., $T(i,j)=\{[i',j'] \in F \mid (i,j) \in P(i',j')\}$. For a feasible network N, we now define quantities u_{ij} , v_{ij} and w_{ij} as follows:

$$u_{ij} = \begin{cases} C + \sum_{[i',j'] \in T(i,j)} f_{i'j'} & \text{if } i,j \in S, j > i, (i,j) \in E(N); \\ 0 & \text{otherwise.} \end{cases} \qquad (3)$$

$$v_{ij} = \begin{cases} C + \sum_{[i',j'] \in T(i,j)} f_{i'j'} & \text{if } i \in M, j \in S, (i,j) \in E(N); \\ 0 & \text{otherwise.} \end{cases} \qquad (4)$$

$$w_{ij} = \begin{cases} C + \sum_{[i',j'] \in T(i,j)} f_{i'j'} & \text{if } i,j \in M, j > i, (i,j) \in E(N); \\ 0 & \text{otherwise.} \end{cases} \qquad (5)$$

Notice that $u = \{u_{ij}\}$, $v = \{v_{ij}\}$ and $w = \{w_{ij}\}$ are identically defined but over different node sets. These quantities will be used to establish a relationship with the rectilinear multifacility location problem. First, we have the following lemmas.

Lemma 1: If N is a feasible solution to problem (P) and u, v, and w are as defined in (3) and (4), respectively, then,

$$Z(N) = \sum_{i\in M}\sum_{j\in M} w_{ij}L(i,j) + \sum_{i\in M}\sum_{j\in S} v_{ij}L(i,j) + \sum_{i\in S}\sum_{j\in S} u_{ij}L(i,j).$$

Proof: Consider the right had side of the above expression,

$$\sum_{i\in M}\sum_{j\in M} w_{ij}L(i,j) + \sum_{i\in M}\sum_{j\in S} v_{ij}L(i,j) + \sum_{i\in S}\sum_{j\in S} u_{ij}L(i,j)$$

Substituting for u_{ij}, v_{ij} and w_{ij} using (3), (4), and (5):

$$= \sum_{i,j\in M, j>i, (i,j)\in E(N)} \left\{C + \sum_{[i',j']\in T(i,j)} f_{i'j'}\right\} L(i,j)$$

$$+ \sum_{i\in M, j\in S, (i,j)\in E(N)} \left\{C + \sum_{[i',j']\in T(i,j)} f_{i'j'}\right\} L(i,j)$$

$$+ \sum_{i,j\in S, j>i, (i,j)\in E(N)} \left\{C + \sum_{[i',j']\in T(i,j)} f_{i'j'}\right\} L(i,j).$$

$$= C\sum_{(i,j)\in E(N)} L(i,j) + \sum_{(i,j)\in E(N)} L(i,j) \sum_{[i',j']\in T(i,j)} f_{i'j'}$$

$$= C\sum_{(i,j)\in E(N)} L(i,j) + \sum_{[i',j']\in F} f_{i'j'} d(i',j';N) = Z(N). \text{ «»}$$

With every node i∈ M, we associate a set of nodes n(i) and a region r(i). These are defined as follows (see Figure 5):

(i) if i coincides with a node of the grid graph then n(i) and r(i) are both set to this node of the grid graph;

(ii) if i is on an arc of the grid graph then r(i) is the arc and n(i) includes the two end-nodes of the arc; and

(iii) if i is in the interior of a cell then r(i) is the cell while the four corner nodes of the cell constitute n(i).

Note that whenever w_{ij} is strictly positive, there is a horizontal or a vertical arc between nodes i and j. Also, between any two points i' ∈ r(i) and j' ∈ r(j), there exists a path of rectilinear length that does not cross a barrier. The same is also true when $v_{ij} > 0$. When $u_{ij} > 0$, both r(i) and r(j) are singleton sets consisting of the end nodes i and j, respectively, of arc (i,j) and the above observation is trivially true.

Now we consider the problem in which the locations of nodes in M are allowed to vary anywhere on the plane (including the barriers). Let the variable z_i denote the location of new facility i. We designate the nodes in M as *new facilities*, and the nodes in $R' = \cup_{i \in M} n(i) \cup S$ as *existing facilities*, and consider the following problem:

$$Q(N): \min_{z_i \in \Re^2} \sum_{i \in M} \sum_{j \in M} w_{ij} d(z_i, z_j) + \sum_{i \in M} \sum_{j \in S} v_{ij} d(z_i, j) + \theta \sum_{i \in M} \sum_{j \in n(i)} d(z_i, j),$$

where v and w are defined in (3) and (4), and θ is a large number.

Q(N) is equivalent to the *rectilinear multifacility location problem* (Francis and White, 1974) in which, given a set of existing facilities (here R'), the locations of m new facilities (here nodes in M) are to be determined to minimize the weighted sum of distance between pairs of new facilities and between new and existing facilities. The first term in Q(N) accounts for the interactions between pairs of new facilities while the second and the third terms are the interactions between pairs of new and existing facilities. We have the following lemma from Francis and White (1974):

Lemma 2: An optimal solution to the rectilinear multifacility location problem, with existing facilities σ, is contained in the grid graph $\Pi(\sigma)$. «»

Notice that in the above lemma, there are no barriers on the plane. We are now ready for our next result.

Lemma 3: Given a feasible solution N to the network design problem and the corresponding problem Q(N), it is never optimal to locate a new facility inside a barrier.
Proof: For any new facility $i \in M$, consider the term

$$\theta \sum_{j \in n(i)} d(z_i, j). \tag{6}$$

The third term in Q(N) consists of these terms summed over all $i \in M$. For any $z_i \in r(i)$, (6) remains constant and is strictly greater than this when $z_i \notin r(i)$ (see Figure 5). Note that r(i) is defined as a property of the network N and does not change with z_i. Since θ is an arbitrary large number, it is not optimal to locate i outside r(i). Finally, there are no common points between the interior of a barrier and r(i) and thus the lemma follows. «»

Recall that $\Pi(R)$ is the grid graph defined by the smallest rectangle circumscribing R, and horizontal and vertical lines through every point of R. While constructing $\Pi(R)$, any barriers present on the plane are ignored. Further, let $\pi(R)$ be the graph defined by removing from $\Pi(R)$ all the nodes and the associated arcs that are inside the barriers. By lemmas 2 and 3, it follows that the new facilities will be located on the nodes of the graph $\pi(R')$.

Lemma 4: Given a feasible solution to the network design problem, there exists an alternate feasible solution of equal or lesser objective function value which is entirely on the graph $\pi(R')$.

Proof: Suppose we have a feasible solution N to (P) with objective function value Z(N). We form the multifacility location problem Q(N). With the current location of new facility i as z'_i for all $i \in M$ (as defined by N), the objective function value of Q(N) is,

$$\sum_{i\in M}\sum_{j\in M} w_{ij}d(z'_i,z'_j)+\sum_{i\in M}\sum_{j\in S} v_{ij}d(z'_i,j)+\theta\sum_{i\in M}\sum_{j\in n(i)} d(z'_i,j)$$
$$=\sum_{i\in M}\sum_{j\in M} w_{ij}d(z'_i,z'_j)+\sum_{i\in M}\sum_{j\in S} v_{ij}d(z'_i,j)+K_1,$$

where $K_1=\theta\sum_{i\in M}\sum_{j\in n(i)} d(z'_i,j)$.

Let us now consider an optimal solution $\{z_i^*: i\in M\}$ to Q(N). From the proof of lemma 3, the new facilities will be located within r(i) and with these locations, the term K_1 is a constant. Therefore, optimizing Q(N) minimizes the first two terms in Q(N).

Given an optimal solution to Q(N), we now construct an alternate network N* as follows. For every pair of new facilities with $w_{ij} > 0$, construct a rectilinear length path between i and j that is contained in $\pi(R')$. Such a path must exist as argued in the discussion following lemma 1. Similarly, we connect pairs of new and existing facilities and pairs of existing facilities. Because the existing facilities do not move, the arcs between pairs of existing facilities will be the same as in N. Although in network N* every arc may not be either horizontal or vertical (they may be L shaped), the length of arc (i,j) will be d(i,j) for every arc.

Now consider the solution value of the network N*.

$$Z(N^*)=\sum_{i\in M}\sum_{j\in M} w_{ij}L(z^*_i,z^*_j)+\sum_{i\in M}\sum_{j\in S} v_{ij}L(z^*_i,j)+\sum_{i\in S}\sum_{j\in S} u_{ij}L(i,j)$$
$$=\sum_{i\in M}\sum_{j\in M} w_{ij}d(z^*_i,z^*_j)+\sum_{i\in M}\sum_{j\in S} v_{ij}d(z^*_i,j)+\sum_{i\in S}\sum_{j\in S} u_{ij}d(i,j) \quad (7)$$

$\le Z(N)$, as the last term in (7) is constant and the first two terms are minimized in Q(N). Therefore, the theorem follows. «»

The node set R' is a subset of $\{S\cup BV\}$. This implies $\pi(R') \subseteq \pi(S\cup BV)$. From this and lemma 4, the solution to (P) must lie on $\pi(S\cup BV)$. There is a close relationship between $\pi(S\cup BV)$ and $\Pi_B(S)$. In constructing $\Pi_B(S)$, the horizontal and vertical lines were terminated on reaching a barrier whereas, in $\pi(S \cup BV)$, the lines are continued past the barriers. Thus, $\Pi_B(S)$ can be obtained from $\pi(S \cup BV)$ by the removal of these *continuation line segments*. We now show that the continuation line segment can be ignored while solving (P).

Theorem 1: Given a set of points S and a set of isothetic barriers, there exists an optimal solution to (P) that is contained in the grid graph $\Pi_B(S)$.

Proof: From lemma 4 and the discussion following it, we have an optimal solution N* that is contained in $\pi(S \cup BV)$. We show that there exists an alternate optimal solution that does not involve the continuation line segments. Let A be the path in $\pi(S \cup BV)$ that is defined by one such continuation line segment such that $N^* \cap A \neq \emptyset$. Note that the path A will be either horizontal or vertical. By definition there are no points of S in A and therefore nodes in $A \cap N^*$ can be moved.

Similar to our approach earlier, we form a multifacility location problem with the nodes in $N^* \cap A$ as new facilities and all the nodes of N*, adjacent to the nodes in $N^* \cap A$, as existing facilities. The grid graph formed by these existing facilities will not contain any point of A. Thus, the new facilities can be moved from A without increasing the objective function value. Consequently, we can remove the continuation line segments from $\pi(S \cup BV)$. This completes the proof of the theorem.«»

4. Dealing with Arbitrary Barriers

We now consider the problem where barriers may not be isothetic. As discussed in section 2, the feasible region of the Cartesian plane is divided into rectangular cells and irregular cells by the imposition of the grid graph. In the absence of irregular cells we showed in the previous section that the feasible region FR can be replaced by the grid graph. We now proceed to show that after a suitable augmentation of the grid graph, the interiors of cells can be discarded even when irregular cells are present.

We first focus our attention on alley cells. It is possible that the only feasible solution to the overall problem requires traversal through the alley cell. Since the grid network does not contain any path that can cross the alley cell, it is now necessary to augment the grid graph. The augmentation of the grid graph is performed iteratively as follows. From every vertex of the grid cell a ray pointing towards the interior of the cell and oriented parallel to one of the co-ordinate axes is constructed. This ray is terminated when it intersects the boundary of the alley cell. The lines so constructed are then appended to the grid graph. This augmentation results in the creation of new cells in the interior of the alley cell. At most one of these new cells can be an alley cell. The procedure is repeated until there are no alley cells. Figure 6 shows an example of this construction. As a result of the augmentation, the alley cell is divided into deadend cells and rectangular cells. We denote the augmented grid graph by $\Pi_B^a(S)$.

After eliminating all alley cells, we consolidate the deadend cells. This is done as follows: For a deadend cell D_i, let J represent the set of cells that share a common boundary with D_i. If $D_j \in J$ is a deadend cell and $D_i \cup D_j$ is also a deadend cell combine D_i and D_j. Combining two cells involves the removal of the common grid line boundary between the cells. This procedure is terminated when all remaining cells are either rectangular or in combination with D_i result in an alley cell. Note that after consolidation the grid line portion of each deadend cell is either a straight line or L-shaped. Figure 7 shows cells before and after consolidation. Consolidation ensures that a shortest rectilinear path between any two points on the grid line portion of the cell boundary will lie on the cell boundary. After consolidation, let b_k be the grid line portion of the deadend cell D_k and $I_k = D_k \setminus b_k$ be the cell without the grid line portion.

Lemma 6: Given a feasible solution to (P), there exists an alternate feasible solution of equal or smaller length that does not intersect I_k.

<u>Proof :</u> Let N be feasible solution such that $Q' = N \cap I_k \neq \emptyset$. Let Q be the closure of Q'. The graph Q consists of r disconnected components $\{Q_1,\ldots,Q_r\}$ where $Q_i \cap Q_j = \emptyset$, $i \neq j$.

Consider any component Q_j and let J be the set of nodes of Q_j that are also on b_k. Note that $|J| \geq 2$, otherwise Q_j can be deleted from N, since there is no demand point inside the cell. Index the nodes in J as $a_1, a_2, \ldots, a_{|J|}$ so that a_i and a_{i+1} are consecutive nodes on b_k, for $i = 1, \ldots, |J|-1$. Define $P(Q_i)$ as the union of the shortest paths on b_k, between a_i and a_{i+1}, for $i = 1, \ldots, |J|-1$. Note that the length of the shortest path on b_k between any pair of nodes in J is no longer than the length of the shortest path between these nodes on Q_j. Consequently, replacing Q_j by $P(Q_j)$ in N will result in a solution with flow cost no greater than the flow cost of N. Also, the total length of $P(Q_j)$ is no larger than the sum of the length of edges in Q_j.

Application of this to all Q_j, $j = 1, \ldots, |J|$ will result in a solution that satisfies the theorem. «»

We have shown that there exists a solution to (P) that does not intersect the interior of a deadend cell created by the graph $\Pi_B^a(S)$. The remaining cells are all rectangular and thus by theorem 1 we can state the following result.

Theorem 2 : Given problem (P) with arbitrary barriers, there exists an optimal solution contained in the graph $\Pi_B^a(S)$.

Although theorem 2 combinatorializes the network design problem, the augmented grid graph may be large. To ameliorate the problem, the concept of a rectilinear hull is used to reduce the size of the grid graph. A *rectilinear hull*, RH(Ω), of a point set Ω is a smallest connected set containing Ω such that for any pair of points a, b in RH(Ω), there is a rectilinear path of length d(a,b) between them contained in RH(Ω). This is defined as a cr-rectilinear hull in Ottmann et al., 1983. The rectilinear hull of n points can be constructed in O(nlogn) time. The notion of a rectilinear hull can be extended to problems with barriers. From this definition, it is sufficient to consider the portion of the grid graph contained in RH(Ω) to obtain an optimal solution to the rectilinear network design problem. This generalizes the result of Provan (1988) for the rectilinear Steiner tree.

5. Discussion of Related Problems

As mentioned in the introduction the network design problem can be shown to be a generalization of several well-known problems. After discussing two such problems, we present an extension of the network design problem which shows connection to other problems.

<u>Rectilinear Steiner Tree Problem:</u> Given a set of points S on the plane, find a shortest connected tree spanning S. Note that the Steiner tree may contain some points which are not in S. The rectilinear Steiner tree problem has applications to wire layout for printed circuit boards and utilities connections in buildings (Hannan, 1966).

To model this problem as (P), choose an $i^\circ \in S$ and let the flow set $F = \{[i^\circ,j] : \forall j \in S \backslash i^\circ\}$. Set all $f_{i^\circ j} = 0$ and C = 1. Any solution, N, to problem (P) with these data will give a network which is connected and spans all points of S. The objective function accounts for only the length of the network. It is easy to see that N will be a rectilinear Steiner tree.

From theorem 2, not only can we obtain Hannan's result that an optimal rectilinear tree is contained in the grid graph, but extend it to the case of rectilinear Steiner trees in the presence of arbitrary barriers.

<u>Shortest r-Flow Network:</u> Given a set F, find a rectilinear network N, of smallest length

such that $d(i,j;N)=d(i,j)\ \forall\ [i,j] \in F$. In this problem, the routing cost is minimized by forcing each flow to follow a shortest path. With this requirement, the design now calls for minimization of the fixed cost of the network. This problem is defined in Chhajed et al . (1992) and arises in the context of material handling systems design.

Set $f_{ij} = \theta$ (a very high number) $\forall\ [i,j] \in F$ and C=1. This will force the network to have a shortest possible path for every flow (a path of rectilinear length) and minimize the length of the network.

As before, theorem 2 generalizes the grid graph result of Chhajed (1989) to the shortest r-flow network problem in the presence of barriers.

While the previous two problems were direct specializations of the network design problem, we now consider a generalization of the network design problem: In addition to the points in S in the rectilinear network design, we are given an additional set of points $S°$ whose locations are to be determined. Now the flow set consists of pairs of points in $S \cup S°$. All the results presented in sections 3 and 4 can be extended to this problem if the set of points $S°$ are added to the set M defined in section 3. With this generalization, we can now model several well-known problems as rectilinear network design problems.

1-median problem : The 1-median problem is to locate a facility on a rectilinear plane such that the weighted distance of the facility from a given set of existing facilities is minimized. To model as a rectilinear network design problem let

S = the set of existing facilities

$S° = \{h\}$, where h is the facility to be located

$F = \{[h,i] : i \in S\}$; $f_{hj} = w_j$, where w_j is the weight associated with $j \in S$;

$C = 0$.

It is possible to obtain the grid graph optimality result for 1-median problem with barriers (Larson and Sadiq, 1983) from theorem 2.

While we cannot formulate the p-median problem as a rectilinear network design problem, the grid graph optimality in the presence of barriers for the p-median problem follows from the result of the 1-median problem (Larson and Sadiq, 1983).

Multifacility location problem: The multifacility location problem is to find the optimal locations for a given number of new facilities in relation to a given set of existing facilities. In this problem a new facility may interact with the existing facilities as well as with other new facilities. Let w_{ij} represent the interaction between new facility i and existing facility j and v_{ij} represent the interaction between new facilities i and j. The objective is to minimize the weighted distance between all facilities. To formulate as a rectilinear network design problem let,

S = set of existing facilities; $S°$ = set of new facilities;

$F = S° \times S° \cup S \times S°$;

$$f_{ij} = \begin{cases} w_{ij} & \text{if } i \in S°, j \in S \\ v_{ij} & \text{if } i \in S°, j \in S° \end{cases}$$

$C = 0$.

The well known intersection point optimality result for this problem (Francis and White, 1974) may be derived from theorem 1. This should not be surprising as the multifacility location problem without barriers was used to derive theorem 1. However, using theorem 2, we can now extend this result to the case of multifacility location with barriers.

Integrated Station Location and Flow Network Design Problem: Many researchers have begun to investigate the problem of layout design beyond the basic block design (Chhajed et al., 1992; Montreuil, 1987; O'Brien and Abdul Barr, 1980). One problem that arises in this context is to locate exactly one input/output station in each department and design a flow network (material handling network) connecting them. Such a problem also arises in the automated guided vehicle system design (Usher et al., 1988).

This problem is a special case of problem (P) when rectilinear travel is assumed and all departments are rectangular shaped. We have $S°$={one input/output station for each department}, $F=\{[i,j] : i,j \in S°$ and interact$\}$, f_{ij} = annualized flow between departments i and j and C = fixed cost per unit length of the material handling network. To restrict the location of each input/output station to be within its department, define t(i) as the four corner vertices on the contour of department i. Set $f_{ij} = \theta$ for $i \in S°$ and $j \in t(i)$, and $S = \cup_i t(i)$.

Chhajed et al. (1991) have shown the existence of a dominating graph for the case when each department is an isothetic simple polygon following an approach similar to ours. Although barriers are not considered in the above cited literature, they may exist in a plant layout, e.g., preexisting machines or fixtures, restriction by certain departments (clean room) on material flow through them.

6. Conclusion

In this paper we have considered a rectilinear network design problem on a plane in the presence of barriers. We have established that an optimal solution to the problem lies on a specially constructed grid graph. The consequent discretization of the problem has several benefits, including reduction in the search space and ability to model as an integer program. We further show that the size of the grid graph can be reduced by considering a rectilinear hull of the demand points and barriers.

The rectilinear network design problem is shown to contain several well-known problems. Therefore the sufficiency of the grid graph for an optimal solution follows for all these problems.

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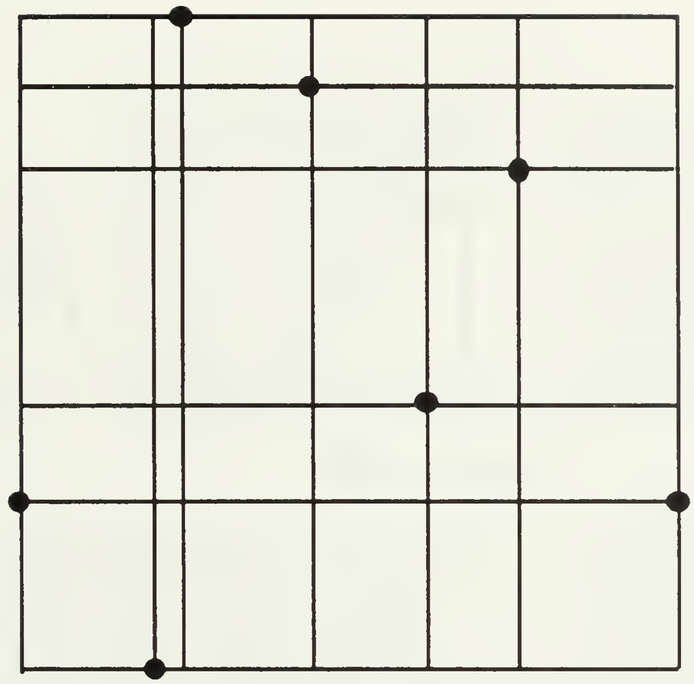


Figure 1. Set S (dark points) and Grid Graph Π(S)

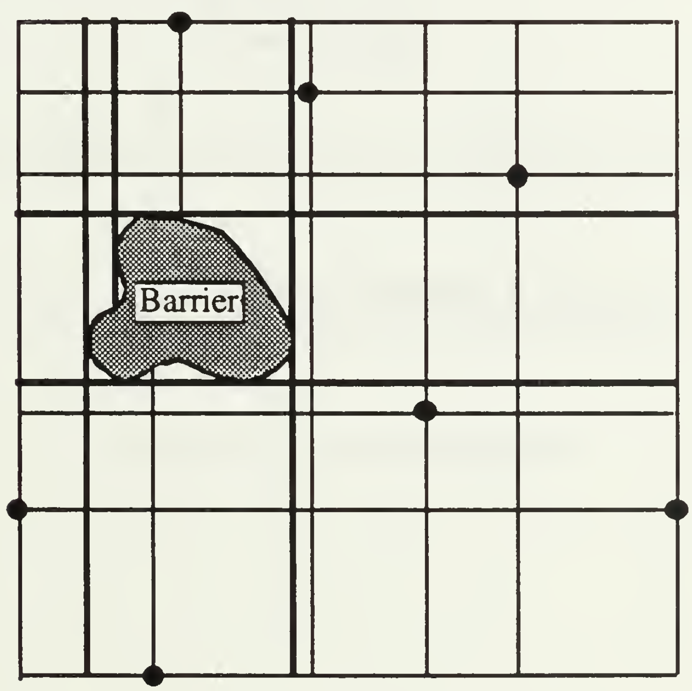
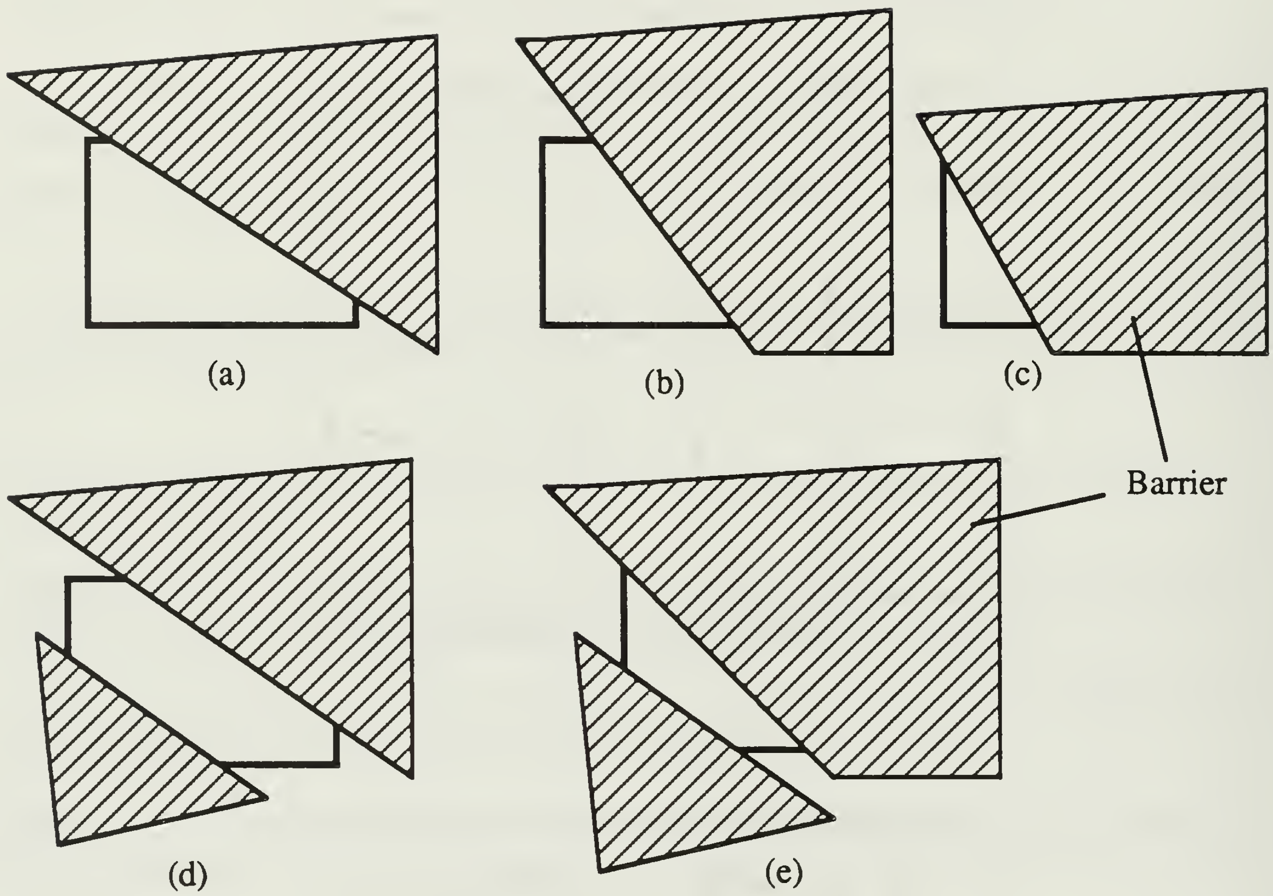
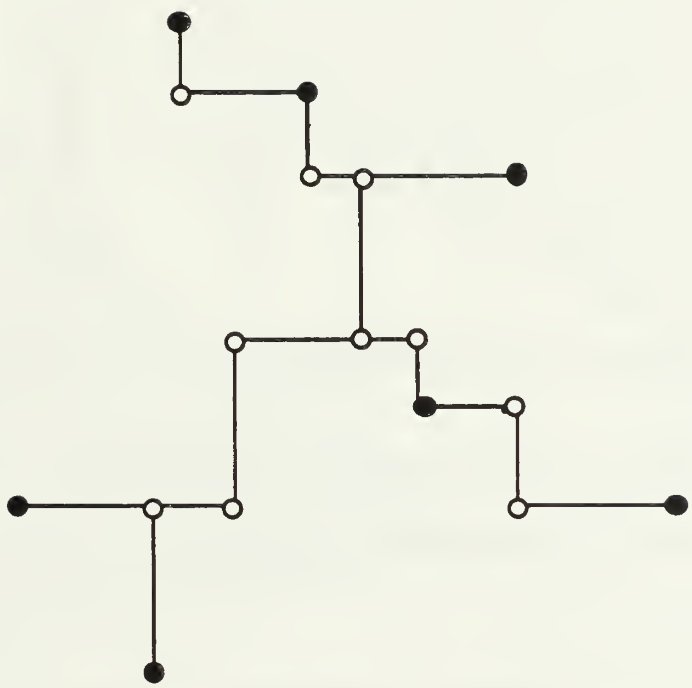


Figure 2. Grid Graph in the Presence of a Barrier



(a), (b), (c) are deadend cells and (d) and (e) are alley cells.

Figure 3. Deadend and Alley Cells



Set S (dark points) and Set M (hollow points)

Figure 4. A Feasible Network N

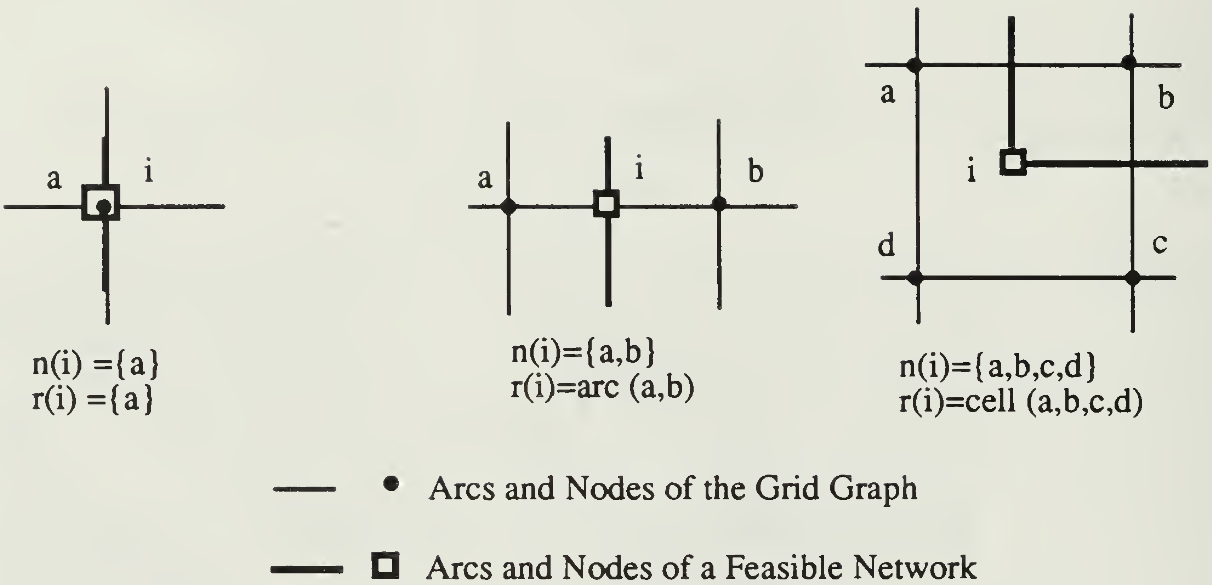
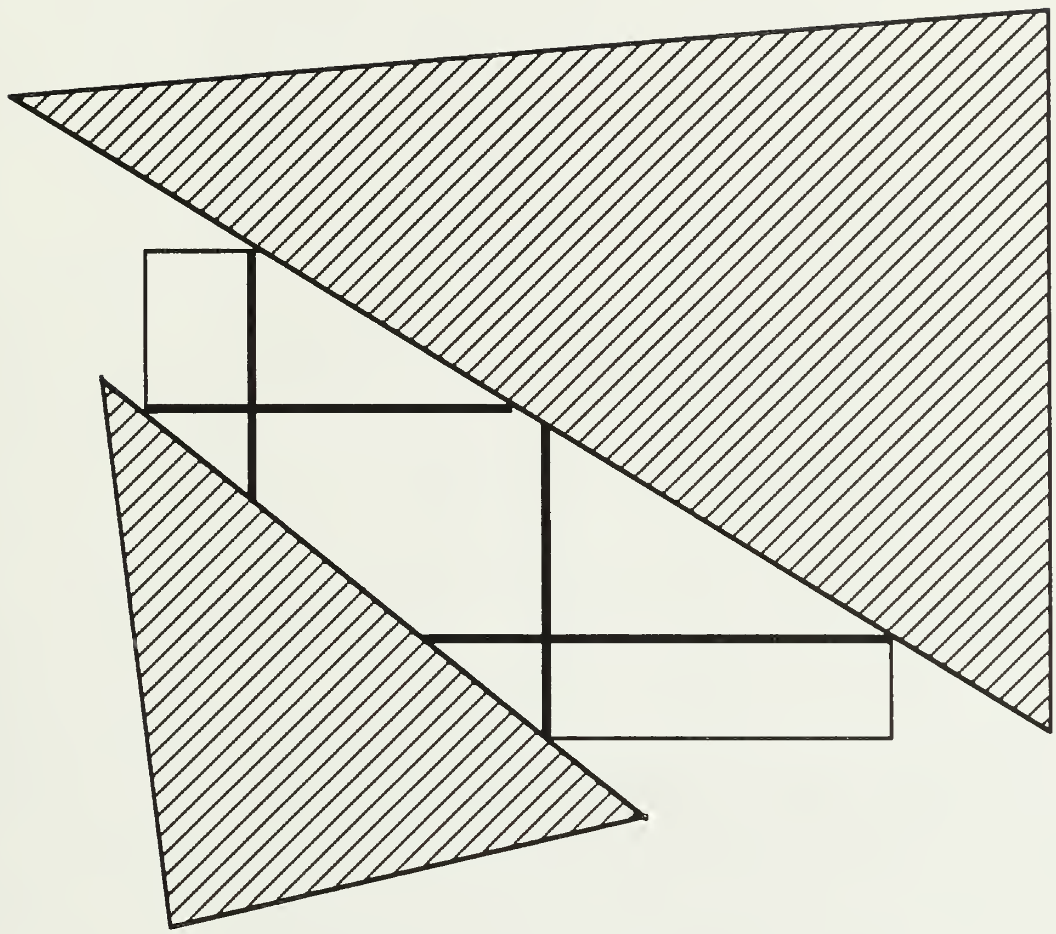
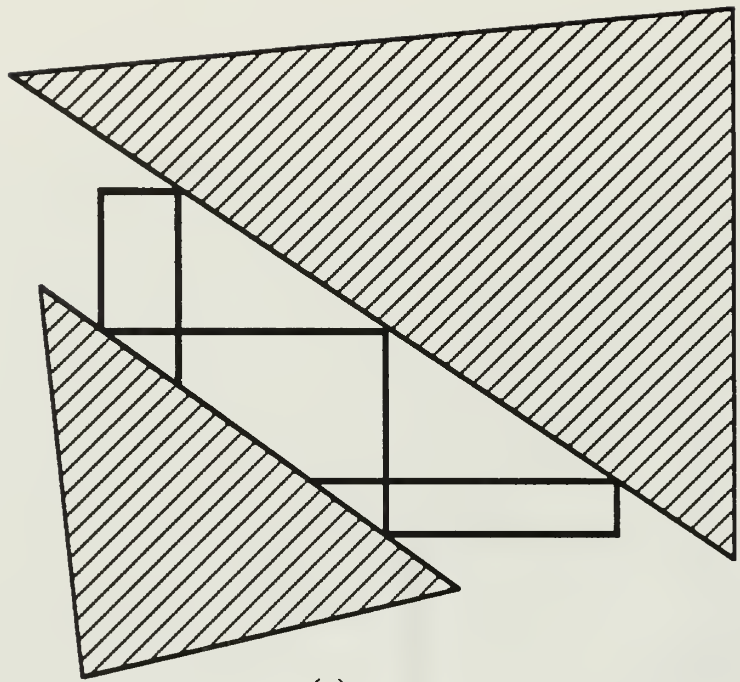


Figure 5. Definitions of Sets n(i) and r(i)

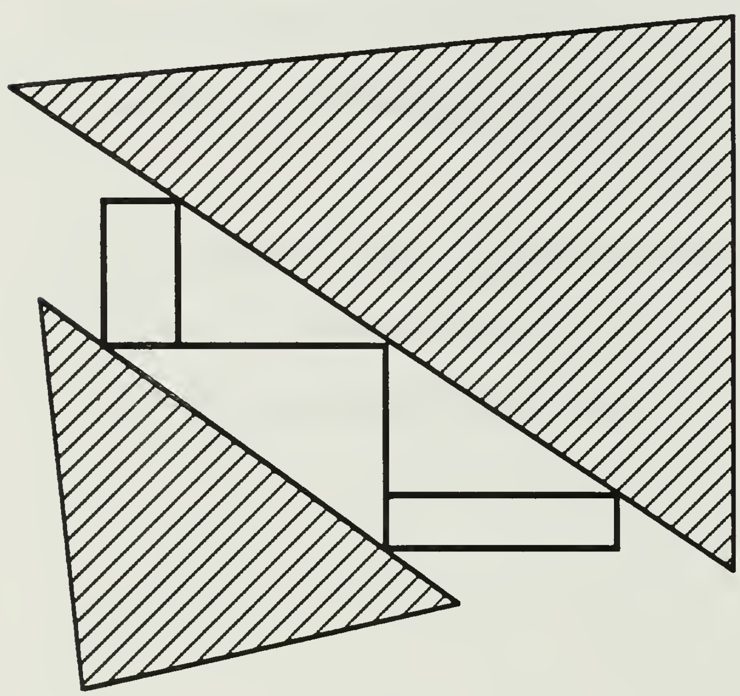


The extra grid lines generated are indicated in bold.

Figure 6. Augmented Network Due to Alley Cell



(a)



(b)

Figure 7. Deadend Cells (a) Before and (b) After Consolidation.