The Hecke algebra and structure constants of the ring of symmetric polynomials

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Abstract

We give half a dozen bases of the Hecke algebra of the symmetric group, and relate them to the basis of Geck-Rouquier, and to the basis of Jones, using matrices of change of bases of the ring of symmetric polynomials.

Key words. Hecke algebra, Center, Symmetric functions.

1 Introduction

To determine the characters of the symmetric group $\mathfrak{S}_n$, Frobenius defined a linear morphism from $\mathbb{C}[\mathfrak{S}_n]$, the group algebra of $\mathfrak{S}_n$, to the space $\mathfrak{Sym}$ of symmetric polynomials of degree n. This allowed him to identify the center of $\mathbb{C}[\mathfrak{S}_n]$ and $\mathfrak{Sym}$. Several natural bases of $\mathfrak{Sym}$ correspond to specific bases of the center of $\mathbb{C}[\mathfrak{S}_n]$. This explains why transition matrices of symmetric functions occur as transition matrices of different bases of the center.

The identification between $\mathfrak{Sym}$ and the center of the Hecke algebra of the symmetric group is not so clear. We still have a basis of central idempotents, and a basis, due to Geck and Rouquier, which extends conjugacy classes. Jones [10] (see also [8]) gave another natural basis. It happens that the transition matrix of Jones' basis to the one of Geck and Rouquier is, up to powers of a parameter Q, equal to the transition matrix between power sums and monomial functions (Th. 12).

This is this phenomenon that we want to explain in this text, and to generalize by describing other bases in Theorems 7, 10, 17. In our opinion, the most satisfactory explanation of the appearance of the transition matrices of the usual symmetric functions is due to the connection with the theory of non-commutative symmetric functions given in Theorem 9.

Recall that the Hecke algebra $\mathcal{H}_n$ of the symmetric group $\mathfrak{S}_n$, with coefficients in a commutative ring containing q, q^{-1}, is the algebra generated by elements $T_1, \ldots, T_{n-1}$ satisfying the braid relations

$$\begin{cases} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\ T_i T_j = T_j T_i \ , \quad (|j-i| > 1) \end{cases} ,$$

together with the Hecke relations:

$$(T_i - q)(T_i + 1/q) = 0$$

or, equivalently, with $Q = q - 1/q$,

$$T_i^2 = Q T_i + 1 \, . \tag{1}$$

Let us write $[k]$ for the q-integer $(q^k - q^{-k})/(q - q^{-1})$, and $[i.j \ldots k]$ for the product $[i]\,[j] \cdots [k]$.

To define the bases that we denote $\{\mathcal{N}_l(e_{\lambda^\natural}\}$, $\{\mathcal{N}_\lambda(1)\}$, $\{\mathcal{N}_\lambda(T^2_{\omega_\lambda}\}$, we need the ring of coefficients of $\mathcal{H}_n$ to contain Q^{-1} as well as the rational numbers. For the three other bases, we furthermore need that the q-integers $[1], \ldots, [n]$ be invertible.

On $\mathcal{H}_n$, one has a natural scalar product, with respect to which the basis $\{T_w : w \in \mathfrak{S}_n\}$ is orthonormal:

$$(T_w, T_v) = \delta_{w,v} \, .$$

It is such that $(T_w T_i, T_v) = (T_w, T_v T_i)$, $(T_i T_w, T_v) = (T_w, T_i T_v)$.

Define $\mathfrak{Comp}(n)$ to be the set of *compositions* of n, i.e. the set of vectors with positive integral components whose sum is n. For any composition I, one defines a Young subgroup $\mathfrak{S}_I$, and its corresponding Hecke algebra $\mathcal{H}(\mathfrak{S}_I)$. Elements of $\mathfrak{S}_I$ are denoted $w^1 \times w^2 \times \cdots$. There is natural morphism, called *normalization*, from $\mathcal{H}(\mathfrak{S}_I)$ to $\mathcal{H}_n$:

$$\mathcal{H}(\mathfrak{S}_I) \ni h \to \mathcal{N}_I(h) := \sum_{w \in \mathfrak{S}_n / \mathfrak{S}_I} T_w \, h \, T_{w^{-1}} \, . \tag{2}$$

One will find in [10] many properties of more general norms. We shall need only the following lemma, which shows that the normalization morphism allows to construct central elements.

Lemma 1 *If h is central in $\mathcal{H}(\mathfrak{S}_I)$, then $\mathcal{N}_I(h)$ is central in $\mathcal{H}_n$.*

Given I, one has also a projection $\mathfrak{p}^I : \mathcal{H}_n \to \mathcal{H}(\mathfrak{S}_I)$, which, for a permutation w, consists in considering it as a word cut into factors of respective lengths $i_1, i_2, \ldots, i_r$, then renormalizing the values inside each factor.

We shall use different bases of symmetric functions, and the corresponding matrices of change of basis $H2M, P2M, S2M, \ldots$ (see last section, and [18]).

2 Yang-Baxter basis

Given an integral vector $v \in \mathbb{Z}^n$ (with components all different), one defines a linear basis $\{\mathcal{Y}_w^v : w \in \mathfrak{S}_n\}$ of $\mathcal{H}_n$ recursively as follows:

$$\mathcal{Y}_{ws_i}^v = \mathcal{Y}_w^v \left(T_i - \frac{q^k}{[k]} \right), \; \ell(ws_i) > \ell(w), \; k = v_{w_{i+1}} - v_{w_i} \,,$$

starting with $\mathcal{Y}_1 = 1$.

To any sequence of parameters $[z_1, \ldots, z_n]$ all different, one can in fact associate a Yang-Baxter basis. Here we have taken as "spectral parameters" powers of q. We shall use only the cases $v = [1, 2, \ldots, n]$ and $v = [n, \ldots, 1]$.

An important property of Yang-Baxter bases is a duality property that is given in [16] for general parameters. Taking into account that we do not take the same scalar product, this duality formulates as follows.

Theorem 2 *Given $v \in \mathbb{Z}^n$, let $u = [v_n, \ldots, v_1]$. Define $\widehat{\mathcal{Y}}_w^v := T_\omega \mathcal{Y}_{\omega w}^u$, $w \in \mathfrak{S}_n$. Then $\{\widehat{\mathcal{Y}}_w^v\}$ is the basis adjoint to $\{\mathcal{Y}_w^v\}$, i.e.*

$$\left(\mathcal{Y}_w^v \,,\, \widehat{\mathcal{Y}}_{w'}^v \right) = \delta_{w,w'} \,.$$

The following two special cases are of interest, giving the two 1-dimensional idempotents of $\mathcal{H}_n$, up to a normalizing factor (cf. [16]).

Lemma 3 *Let ω be the maximal permutation of $\mathfrak{S}_n$. Then*

$$\mathcal{Y}_\omega^{[1,\ldots,n]} = \sum_{w \in \mathfrak{S}_n} (-q)^{\ell(\omega)-\ell(w)} \, T_w \,, \tag{3}$$

$$\mathcal{Y}_\omega^{[n,\ldots,1]} = \sum_{w \in \mathfrak{S}_n} q^{\ell(w)-\ell(\omega)} \, T_w \,. \tag{4}$$

3 The center of the Hecke algebra

It is clear that conjugacy classes $\mathcal{C}_\lambda : \lambda \in \mathfrak{Part}(n)$ (considered as sums in the group algebra) are a linear basis of the center of the group algebra of $\mathfrak{S}_n$.

Geck and Rouquier [13] have shown that this basis extends canonically to a basis Γ_λ of the center of $\mathcal{H}_n$.

According to Francis [6], the elements Γ_λ are characterized by the property that

- Γ_λ specializes to $\mathcal{C}_\lambda$ for $Q = 0$.
- The difference $\Gamma_\lambda - \mathcal{C}_\lambda$ involves no permutation which is of minimal length in its conjugacy class (for the symmetric group).

Let us write

$$\zeta_n = T_{n-1} \cdots T_2 T_1 \,,$$

and define accordingly, by direct product, for any composition $I = [i_1, \ldots, i_r]$ of n, an element ζ_I. Explicitely, let $K = [i_1, i_1 + i_2, \ldots, i_1 + i_2 + \cdots + i_r]$. Then ζ_I is the element T_{w_I} indexed by the following permutation in $\mathfrak{S}_I$:

$$w_I = [k_1, 1, \ldots, k_1 - 1, k_2, k + 1 + 1, \ldots, k_2 - 1, \ldots, k_n, k_{n-1} + 1, \ldots, k_m - 1] \,.$$

Reordering the factors, one sees that $\{\zeta_I : I \in \mathfrak{Comp}(n)\}$ is the set of subwords of $T_{n-1} \cdots T_2 T_1$.

For example, for $I = [3, 2, 4]$, then

$$\zeta_I = \zeta_3 \times \zeta_2 \times \zeta_4 = T_{312\,54\,9678} = (T_2 T_1)\,(T_4)\,(T_8 T_7 T_6) = (T_8 T_7 T_6)(T_4)(T_2 T_1) \,.$$

There is one such $w_\lambda : \lambda \in \mathfrak{Part}(n)$ in each conjugacy class for the symmetric group, and it is of minimal length in its conjugacy class. Therefore, any central element g decomposes as the sum

$$g = \sum_{\lambda \in \mathfrak{Part}(n)} (g, \zeta_\lambda)\, \Gamma_\lambda \,.$$

We shall mostly use the following property of the elements ζ_J, that is easy to check by induction.

Lemma 4 *Given a composition J, given any permutation w, then*

$$(T_w, \,, \, \zeta_J T_w) = Q^{|J| - r}$$

if w is of maximal length in its coset $w\mathfrak{S}_J$, and otherwise $(T_w \,, \, \zeta_J T_w) = 0$.

This can be formulated in more striking terms. Define a *recoil* of a permutation w to be any integer i such that $i+1$ is left of i in w. Write $T_i \in \zeta_J$ if T_i appears in a reduced decomposition of ζ_J. Then w is of maximal length in its coset $w\mathfrak{S}_J$ iff w has the recoil i for all i such that $T_i \in \zeta_J$.

An important remark follows from the lemma: the matrix representing the left multiplication by ζ_J in $\mathcal{H}_n$ has the same diagonal as the matrix representing the same multiplication in the 0-Hecke algebra (with generators satisfying $T_i^2 = QT_i$). In other words, one can use the 0-Hecke algebra to compute traces of elements ζ_J.

4 Jucys-Murphy elements

Let the JM elements, first used by Bernstein, be $\xi_1 = 1$, $\xi_2 = T_1T_1$, $\xi_3 = T_2T_1T_1T_2$,

$$\xi_i = T_{i-1}\cdots T_1\, T_1 \cdots T_{i-1}\,. \tag{5}$$

These elements generate a subcommutative algebra (in fact maximal) of $\mathcal{H}_n$. The center of $\mathcal{H}_n$ coincides with the symmetric functions in $\xi_1,\ldots,\xi_n$ (this was a conjecture of Dipper-James [4], just settled by Francis and Graham [7]). However, these symmetric functions satisfy many relations which are not easy to control, and do not directly furnish the answer to our problem of identifying the symmetric polynomials of degree n with the center of $\mathcal{H}_n$.

We shall write exponentially a monomial

$$\xi_1^{u_1}\cdots\xi_n^{u_n} = \xi^u\,.$$

More generally, given any composition $I = [i_1,\ldots,i_r]$, given any sequence of integral vectors $\mathbb{u} = [u^1, u^2, \ldots, u^r]$, with $u^j \in \mathbb{N}^{i_j}$, let us write

$$\xi^{\mathbb{u}}$$

for the direct product n $\mathcal{H}(\mathfrak{S}_I)$ of monomials in JM elements. One can also view $\mathbb{u}$ as a single composition decomposed into factors of respective lengths $i_1, i_2, \ldots$ (we say that $\mathbb{u}$ is *compatible* with I).

We shall only consider standard monomials $\xi^{\mathbb{u}}$, i.e. such that all components are boolean vectors (0-1 vectors).

Lemma 5 *Given a sequence $\mathbb{u}$ of boolean vectors, there exists a permutation $\sigma(\mathbb{u})$ such that*

$$\xi^{\mathbb{u}} = T_{\sigma(\mathbb{u})}\, T_{\sigma(\mathbb{u})^{-1}}\,.$$

Proof. The general statement is obtained by direct product from the case of a single vector $u \in \{0,1\}^r$. Write

$$\sqrt{\xi^u} := (1)^{u_1}(T_1)^{u_2}\cdots(T_{r-1}\cdots T_1)^{u_r}\,.$$

This product is reduced, and therefore there exists a unique permutation $\sigma(u)$ such that $\sqrt{\xi^u} = T_{\sigma(u)}$. Moreover, using the commutation relations $T_i\,(T_k\cdots T_1) = (T_k\cdots T_1)\,T_{i+1}$, $i<k$, one sees that

$$\xi^u = \sqrt{\xi^u}(\sqrt{\xi^u})^{\omega}\,,$$

denoting by z^{ω} the reverse of a word z. QED

For example, for $u=[0,1,0,1,0]$,

$$\sqrt{\xi^u} = T_1\cdot T_3T_2T_1 = T_{4213} \quad , \quad \xi^u = T_1T_3T_2T_1\cdot T_1T_2T_3T_1 = T_{4213}T_{3241}.$$

Notice that for any composition I and compatible $\mathbb{u}$, then all products $T_wT_{\sigma(\mathbb{u})}$, for $w\in\mathfrak{S}_n/\mathfrak{S}_I$, are reduced, and the permutations $w\sigma(\mathbb{u})$ are exactly those permutations w' such that $\mathfrak{p}^I(w') = \sigma(\mathbb{u})$.

For example, with $I=[4,3]$ and $\mathbb{u} = [0101\,,\,001]$, then the set

$$\{w\sigma(\mathbb{u}):\ w\in\mathfrak{S}_n/\mathfrak{S}_I\} = \{[4213\,756],\ldots,[7546\,312]\}$$

consists of all permutations which project by $\mathfrak{p}^{43}$ onto $[4213\,756]$.

Theorem 6 *Given two compositions I,J of n, given $\mathbb{u}$ compatible with I, then*

$$(\mathcal{N}_I(\xi^{\mathbb{u}})\,,\,\zeta_J)\,Q^{\ell(J)-n}$$

is equal to the number of permutations in $\{w\sigma(\mathbb{u}) : w\in\mathfrak{S}_n/\mathfrak{S}_I\}$ having recoil i, for all i such that $T_i\in\zeta_J$.

Proof. Extending the notation $\sqrt{\xi^u}$ to the case of $\mathbb{u}$, one rewrites

$$\begin{aligned}\left(\sum_{w\in\mathfrak{S}_n/\mathfrak{S}_I} T_w\,\xi^{\mathbb{u}}\,T_{w^{-1}},\zeta_J\right) &= \sum\left(T_w\sqrt{\xi^{\mathbb{u}}}\,,\,\zeta_J\,T_w\sqrt{\xi^{\mathbb{u}}}\right)\\ &= \sum\left(T_{w\sigma(\mathbb{u})}\,,\,\zeta_J\,T_{w\sigma(\mathbb{u})}\right)\end{aligned}$$

and one concludes with the help of Lemma 4. QED

For example, for $I = [3,2]$, $\mathfrak{w} = [001, 01]$, one has $\sqrt{\xi^{\mathfrak{w}}} = T_2T_1 \cdot T_4$, $\sigma(\mathfrak{w}) = [3,1,2,5,4]$. Taking $\zeta_J = T_1T_3$, one finds that there are two permutations in the set $\{w\,[3,1,2,5,4]\}$ having recoils 1 and 3. They are [42351] and [52431]. In consequence,

$$(\mathcal{N}_{32}(T_2T_1T_1T_2\,T_4T_4)\,,\,T_1T_3) = 2\,Q^2\,.$$

Taking the exponents $\mathfrak{w}$ with all components equal to 0, i.e. inducing from the identity, one gets a basis for the center:

Theorem 7 *The set $\{\mathcal{N}_\lambda(1)\}_{\lambda\in\mathfrak{Part}(n)}$ is a basis of the center of $\mathcal{H}_n$, and the matrix expressing it in the basis $\{\Gamma_\lambda\}$ is*

$$E2M \cdot D\,,$$

D being the diagonal matrix with entries $Q^{n-\ell(\lambda)}$, $\lambda \in \mathfrak{Part}(n)$.

Proof. According to the preceding theorem, the entry $[\lambda,\mu]$ of the matrix is equal, up to powers of Q to the number of permutations $w \in \mathfrak{S}_n/\mathfrak{S}_\lambda\}$ having a recoil i for all i such that $T_i \in \zeta_\mu$. But this the number of 0-1 matrices having row sums λ and column sums μ, which is also the coefficient of the monomial function m_μ in the expansion of $e_\lambda = e_{\lambda_1}e_{\lambda_2}\cdots$. Furthermore, the matrix having a non zero determinant, the set $\{\mathcal{N}_\lambda(1)\}$ is a basis. QED

For example, for $n = 4$, the matrix is

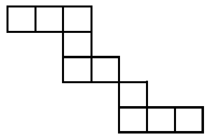
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & Q & 4 \\ 0 & 0 & Q^2 & 2Q & 6 \\ 0 & Q^2 & 2Q^2 & 5Q & 12 \\ Q^3 & 4Q^2 & 6Q^2 & 12Q & 24 \end{bmatrix}.$$

Its third line is explained by the fact that, among the six permutations in $\mathfrak{S}_4/\mathfrak{S}_{22}$, there are two with recoil 1: [2314] and [2413], and one recoils 1, 3: [2413].

5 The Solomon module

We shall see in this section that the square roots $\sqrt{\xi^{\mathfrak{w}}}$ provide a connection with non-commutative symmetric functions.

A *ribbon* θ is a skew diagram, not necessarily connected, (cf. [18]) which contains no 2×2 block of boxes. One can write a ribbon as a sequence of compositions, recording the number of boxes in each row, and passing to a new composition for every connected component. For example,

$$[[3,1,2],[1,3]] =$$  .

A permutation w is *compatible* with θ if, writing $w_1, w_2, \ldots$ in the successive boxes of θ, the result is a *skew standard tableau*, i.e. increases in rows (from left to right), and columns (from bootm to top). A *hook* is a connected ribbon of the type $[1, \ldots, 1, k]$.

Given a ribbon θ, the *ribbon function* $R[\theta]$ is the sum, in the Hecke algebra, of all permutations compatible with θ. If θ is not connected, then $R[\theta]$ is equal[1] to a sum of $R[J]$, J composition of n.

The *Solomon* module[2] is the linear span of $R[J]$, $J \in \mathfrak{Comp}(n)$.

We shall need another basis, generated by shuffle (for a general theory, cf. [9] and [22], and also the different papers intitled `NCSFx` on the page of J-Y Thibon.

Given a composition J, let $L[J]$ be the sum of all T_w: w is of maximal length in its coset $\mathfrak{S}_J\, w$. For example,

$$L[3,2] = T_{w_{3,2,1,5,4}} + T_{w_{3,2,5,1,4}} + T_{w_{3,2,5,4,1}} + T_{w_{3,5,2,1,4}} + T_{w_{3,5,2,4,1}} \\ + T_{w_{3,5,4,2,1}} + T_{w_{5,3,2,1,4}} + T_{w_{5,3,2,4,1}} + T_{w_{5,3,4,2,1}} + T_{w_{5,4,3,2,1}} \,.$$

Notice that, for an y $w \in \mathfrak{S}_n$, then

$$\big(T_w \,,\, L[J]\big) = 1 \text{ or } 0 \;, \tag{6}$$

according to whether w has a recoil in i for all i such that $T_i \in \zeta_J$ or not.

Let $E_n(z)$ denote the product

$$(1 + zT_1)(1 + zT_2T_1) \cdots (1 + zT_{n-1} \cdots T_1)\,,$$

[1]This results from the iteration of the identity

$$R[[\clubsuit, a], [b, \diamondsuit]] = R[[\clubsuit, a, b, \diamondsuit] + R[[\clubsuit, a+b, \diamondsuit]\,,$$

where $\clubsuit, \diamondsuit$ are arbitray sequences, and a, b are arbitrary positive integers, cf. [9].

[2]As a subspace of $\mathbb{C}[\mathfrak{S}_n]$, it is a sub-algebra.

and, accordingly, define for a composition $J = [j_1, \ldots, j_r]$ the direct product

$$E_J(z_1, \ldots, z_r) = E_{j_1}(z_1) \times \cdots \times E_{j_r}(z_r)\,.$$

Similarly, let $E_n^\xi(z)$ be the product

$$E_n^\xi(z) := (1 + z\xi_2)(1 + z\xi_3) \cdots (1 + z\xi_n)\,,$$

and $E_J^\xi(z_1, \ldots, z_r)$ be the direct product

$$E_J^\xi(z_1, \ldots, z_r) = E_{j_1}^\xi(z_1) \times \cdots \times E_{j_r}^\xi(z_r)\,.$$

Proposition 8 *For any composition J of n, then*

$$\sum_{w \in \mathfrak{S}_n/\mathfrak{S}_J} T_w\, E_J(z_1, \ldots, z_r)$$

belongs to the Solomon module, being equal to

$$\sum_{I \le J - 1^r} z^I\, R\big[[1^{i_1}, j_1 - i_1], \ldots, [1^{i_r}, j_r - i_r]\big]\,.$$

Proof. Every monomial appearing in the expansion of $E_J(z_1, \ldots, z_r)$ is reduced, and equal to some $\sigma(\mathfrak{w})$, with $\mathfrak{w}$ compatible with J. Conversely, one gets in this way all $\mathfrak{w}$ compatible with J (the first component of each composition inside $\mathfrak{w}$ has to be ignored, because $\xi_1 = 1$).

In fact, $E_n(z)$ is equal to $\sum_{i=0^n-1} z^i R[1^i, n-i]$, and, by direct product,

$$E_J(z_1, \ldots, z_r) = \sum_{I \le J - 1^r} z^I\, R\big[[1^{i_1}, j_1 - i_1], \ldots, [1^{i_r}, j_r - i_r]\big] \bigcap \mathcal{H}(\mathfrak{S}_J)\,,$$

each permutation belonging to $\mathfrak{S}_J$ and being compatible with a direct product of hooks.

We have already noted that for any $w \in \mathfrak{S}_n/\mathfrak{S}_J$, the product $T_w T_{\sigma(\mathfrak{w})}$ is reduced, and therefore the condition for the ribbons to belong to the subalgebra $\mathcal{H}(\mathfrak{S}_J)$ has to be lifted, multiplication by all T_w permuting the values in such a way as to preserve the ribbon shape. QED

For example, for $J = [3, 2]$, then

$$\begin{aligned} E_{32}(z_1, z_2) = {} & z^{00} R[[3],[2]] + z^{10} R[[1,2],[2]] + z^{01} R[[3],[11]] + z^{20} R[[1,1,1],[2]] \\ & + z^{11} R[[1,2],[1,1]] + z^{21} R[[1,1,1],[1,1]] \bigcap \mathcal{H}_3 \times \mathcal{H}_2\,. \end{aligned}$$

The terms

$$T_w\left((\sqrt{\xi_2}+\sqrt{\xi_3})\times\sqrt{\xi_2}\right) = T_w\left((T_1+T_2T_1)\,T_4\right),$$

$w\in\mathfrak{S}_5/\mathfrak{S}_{32}$, are exactly all the T_v such that v be compatible with $[[1,2],[1,1]]$, i.e. v can be written as a ribbon tableau

$$\begin{array}{ccc} \boxed{v_1} & & \\ \boxed{v_2} & \boxed{v_3} & \\ & & \boxed{v_4} \\ & & \boxed{v_5} \end{array} \; : \; v_1>v_2,\; v_2<v_3,\; v_4>v_5\,.$$

Theorem 9 *For any pair of compositions $J\in\mathbb{N}^r$, K of n, then*

$$\left(\sum\nolimits_{w\in\mathfrak{S}_n/\mathfrak{S}_J} T_w E_J(z_1,\ldots,z_r)\,,\, L[K]\right) \\ = \left(\mathcal{N}_J(E^{\xi}_J(z_1,\ldots,z_r)\,,\,\zeta_K\right) Q^{\ell(K)-n}\,, \quad (7)$$

and this polynomial in z is equal to the coefficient of m_K in the expansion of the product of symmetric functions

$$\left(S_{1^{j_1}}+z_1S_{2,1^{j_1-2}}+\cdots+z_1^{j_1-1}S_{j_1}\right)\cdots\left(S_{1^{j_r}}+z_rS_{2,1^{j_r-2}}+\cdots+z_r^{j_r-1}S_{j_r}\right).$$

Proof. Theorem 6 can be rewritten

$$\left(\mathcal{N}_J(\xi^{\mathfrak{w}}\,,\,\zeta_K\right)Q^{\ell(K)-n} = \left(\sum\nolimits_{w\in\mathfrak{S}_n/\mathfrak{S}_J} T_{w\sigma(\mathfrak{w})}\,,\, L[K]\right),$$

because $L[K]$ is precisely the sum of all T_v such that v has a recoil in i for all $i:\,T_i\in\zeta_k$. Taking the generating function of all monomials $\xi^{\mathfrak{w}}$, one gets

$$\left(\mathcal{N}_J(E^{\xi}_J(z_1,\ldots,z_r)))\,,\,\zeta_K\right)Q^{\ell(K)-n} = \left(\sum\nolimits_{w\in\mathfrak{S}_n/\mathfrak{S}_J} T_wE_J(z_1,\ldots,z_r)\,,\, L[K]\right).$$

On the other hand, $\sum T_wE_J(z_1,\ldots,z_r)$ belongs to the Solomon module. Scalar products in this space can be evaluated at the commutative level (cf. [17], [9]) and this gives the last statement. QED

For example, for $J=[3,2]$, the scalar products

$$\left(((1+z_1\xi_2)(1+z_1\xi_3))\times\left(1+z_2\xi_2\right),\, L[K]\right) \\ = \left((1+z_1T_1T_1)(1+z_1T_2T_1T_1T_2)(1+z_2T_4T_4)\,,\, L[K]\right)$$

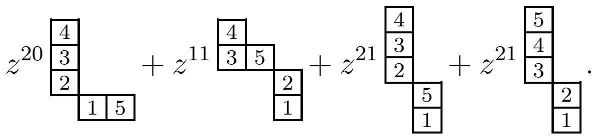
and the scalar products

$$\Big(\mathcal{N}_{32}\big(E^{\xi}_{32}(z_1,z_2)\big)\,,\,\zeta_K\Big)Q^{\ell(K)-5}$$

are the coefficients of the monomial functions in the expansion of

$$\begin{aligned}&(S_{111}+z_1S_{21}+z_1^2S_3)\,(S_{11}+z_2S_2)\\ &\quad = z^{21}m_5+(z^{20}+z^{11}+2z^{21})m_{41}+(z^{10}+2z^{11}+z^{20}+3z^{21})m_{32}+\cdots\end{aligned}$$

To explain the coefficient, say, of m_{41}, we exhibit the contributing permutations:



6 Basis of maximal products of JM elements

Given a composition I, let us take $\mathfrak{w} = \mathfrak{w}(I) := [1^{i_1},\ldots,1^{i_r}]$. Then $\xi^{\mathfrak{w}}$ is the product of all JM elements in each component of $\mathcal{H}(\mathfrak{S}_I)$, and is equal to $T^2_{\omega_I}$, where ω_I is the permutation of maximal length in $\mathfrak{S}_I$. Moreover, the permutations $w\sigma(\mathfrak{w})$ are maximal coset representatives for the cosets $\mathfrak{S}_n/\mathfrak{S}_I$ (i.e. are decreasing in each block).

Taking the term of maximal degree in z in Th. 9, one gets:

Theorem 10 *The set of elements $\{\mathcal{N}_\lambda\big(T^2_{\omega_\lambda}\big),\ \lambda\in\mathfrak{Part}(n)\}$, is a basis of the center of $\mathcal{H}_n$.*

The matrix expressing this basis in terms of the basis Γ_λ is equal to

$$H2M\cdot D\,,$$

D being the diagonal matrix used in Th.7

For example, for $n=5$, writing the partitions in the order $5, 41, 32, 311, 221, 2111, 11111$, the matrix of change of basis is equal to

$$\begin{bmatrix} Q^4 & Q^3 & Q^3 & Q^2 & Q^2 & Q & 1\\ Q^4 & 2Q^3 & 2Q^3 & 3Q^2 & 3Q^2 & 4Q & 5\\ Q^4 & 2Q^3 & 3Q^3 & 4Q^2 & 5Q^2 & 7Q & 10\\ Q^4 & 3Q^3 & 4Q^3 & 7Q^2 & 8Q^2 & 13Q & 20\\ Q^4 & 3Q^3 & 5Q^3 & 8Q^2 & 11Q^2 & 18Q & 30\\ Q^4 & 4Q^3 & 7Q^3 & 13Q^2 & 18Q^2 & 33Q & 60\\ Q^4 & 5Q^3 & 10Q^3 & 20Q^2 & 30Q^2 & 60Q & 120 \end{bmatrix}.$$

The third line, for example, is obtained by expressing $\mathcal{N}_{32}\,(T_1T_1T_2T_1T_1T_2T_4T_4) = \mathcal{N}_{32}\,(T^2_{32154})$ in the basis Γ_λ.

7 Basis of Jones

The first description of a basis of the center over $\mathbb{Q}[q, q^{-1}]$ was obtained by Jones [10]. We shall rather follow the more recent paper of Francis and Jones [8], giving an explicit description of this basis in terms of the basis of Geck and Rouquier.

We now use the original JM elements [11, 19] $x_1, \ldots, x_n$ defined by $x_1 = 0$, $x_2 = T_1$, $x_3 = T_2 + T_2T_1T_2$,

$$x_j = \sum_{i<j} T_{(i,j)}\,,$$

sum over transpositions (i, j).

It is immediate that

$$\xi_j = 1 + Qx_j\,,$$

so that statements about the ξ_j's can be translated in terms of the x_j's and conversely.

In $\mathcal{H}_n$, let e_i, $i = 0 \ldots n-1$ denote the elementary symmetric function of degree i in $x_1, \ldots, x_n$. More generally, given a composition $K = [k_1, k_2, \ldots]$, let $e_{k_1} \times e_{k_2} \times \cdots$ denote a direct product of elementary symmetric functions.

We have already induced central elements, starting from $1 = e_0 \times \cdots \times e_0$. There is another case of special interest, taking the maximum (i.e non-vanishing) elementary symmetric functions. The following lemma (due to Jones, [10, Lemma 3.8]) first describes the case of a composition with one part.

Lemma 11 *The maximal product $x_2 \cdots x_n$ of JM elements is equal to Γ_n, and to*

$$\mathcal{N}_{1^{n-1}}(\zeta_n) := \sum_{w \in \mathfrak{S}_{n-1}} T_w\,\zeta_n\,T_{w^{-1}}\,.$$

Proof. Notice that

$$Q^{n-1}\,x_2 \cdots x_n = \xi_2 \cdots \xi_n = (-1)^{n-1} E^{\xi}_{[n]}(-1)\,.$$

In the case $J = [n]$, Th. 9 states that, for any $K \in \mathfrak{Comp}(n)$, then

$$\big((1-)^{n-1}E_{[n]}(-1)\,,\, L[K]\big) = \big((1-)^{n-1}E^{\xi}_{[n]}(-1)\,,\, \zeta_K\big)$$

is the coefficient of m_K in

$$(-1)^{n-1}\left(S_{1^n} - S_{2,1^{n-2}} + \cdots + (-1)^{n-1}S_n\right) = m_n = p_n\,.$$

Therefore, only for $K = [n]$ does the scalar product $(x_2 \cdots x_n\,,\, \zeta_K)$ be different from 0, and $x_2 \cdots x_n = \Gamma_n$.

The element $X = \mathcal{N}_{1^{n-1}}(\zeta_n)$ (considering $\mathfrak{S}_{n-1}$ as the Young subgroup $\mathfrak{S}_{n-1} \times \mathfrak{S}_1$) is central in $\mathcal{H}_n$. Each scalar product

$$\big(T_w\zeta_nT_{w^{-1}}\,,\, \zeta_J\big) = \big(T_w\zeta_n\,,\, \zeta_JT_w\big)$$

is null whenever $J \neq [n]$, because $T_w\zeta_n$ is equal to some T_v, with $\ell(v) = \ell(w) + n - 1$, and no term of that length appears in the expansion of ζ_JT_w. Since

$$T_w\,\zeta_n = \zeta_nT_{1\times w}\,,$$

where $1 \times w = [1, w_1 + 1, \ldots, w_{n-1} + 1]$, the only permutation w which gives a non-zero scalar product is the identity. QED

By direct product, defining for any composition I the composition

$$I^{\natural} := [i_1 - 1, \ldots, i_r - 1]$$

then $e_{I^\natural}$ is the product of all non zero JM elements in each component of $\mathcal{H}(\mathfrak{S}_I)$, as well as the direct product $\Gamma_{i_1} \times \cdots \times \Gamma_{i_r} \times \cdots$.

Specializing Th. 9 in $z_1 = -1 = z_2 = \cdots$, and using Lemma 11, one gets a basis of the center of $\mathcal{H}_n$ [8] (Francis and Jones use another formulation, in term of the order[3] of the intersections of conjugacy classes with Young subgroups).

Theorem 12 *The set of elements $\big\{\mathcal{N}_\lambda(e_{\lambda^\natural})\big\}_{\lambda\in\mathfrak{Part}(n)}$, is a basis of the center of $\mathcal{H}_n$.*

The transition matrix to the basis $\{\Gamma_\lambda\}$ is equal to the product

$$D^{-1} \cdot P2M \cdot D\,,$$

D being the diagonal matrix used in Th.7.

[3]The matrix with entry $[\mu, \lambda]$, $\mu, \lambda \in \mathfrak{Part}(n)$, equal to

$$n!\big|\mathcal{C}_\mu \cap \mathfrak{S}_\lambda\big|\,\big|\mathcal{C}_\mu\big|^{-1}\big|\mathfrak{S}_\lambda\big|^{-1}$$

is in fact equal to $P2M$.

For example, for $n = 5$, writing the partitions in the order $5, 41, 32, 311, 221, 2111, 11111$, the matrix of Francis-Jones is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ Q & 1 & 0 & 0 & 0 & 0 & 0 \\ Q & 0 & 1 & 0 & 0 & 0 & 0 \\ Q^2 & 2Q & Q & 2 & 0 & 0 & 0 \\ Q^2 & Q & 2Q & 0 & 2 & 0 & 0 \\ Q^3 & 3Q^2 & 4Q^2 & 6Q & 6Q & 6 & 0 \\ Q^4 & 5Q^3 & 10Q^3 & 20Q^2 & 30Q^2 & 60Q & 120 \end{bmatrix}$$

One could have used other elementary symmetric functions in the JM elements. In the case where $I = [n]$, Lemma 11 generalizes to

$$e_k(x_2, \ldots, x_n) = \sum \Gamma_{J:\,\ell(J)=n-k} \,,$$

for any $k : 0 \le k < n$, and one can induce bases of the center from direct products of such elements.

8 Characters

Central elements can also be determined by their characters. In this section, we shall record properties of characters for the Hecke algebra, which will not be needed later, but which illustrate the usefulness of both symmetric functions and Yang-Baxter elements and complete the relevant chapter of [12].

In the case of the symmetric group $\mathfrak{S}_n$, characters are constant on a conjugacy class, and consequently, it is sufficient to determine their values on a single element in each conjugacy class. One way to compute characters is to use the *Frobenius morphism* $\phi : \mathbb{C}[\mathfrak{S}_n] \to \mathfrak{Sym}$, which sends a permutation w to the product of power sums corresponding to its cycle type decomposition:

$$\phi(w) = p^{cycle(w)} \,.$$

The characters of w appear then as coefficients in the expansion of $\phi(w)$ in the basis of Schur functions. In fact, the Frobenius morphism is the easiest way to relate the center of the algebra of the symmetric group to the ring of symmetric polynomials.

In the case of the Hecke algebra, one also call "Table of characters" the table of characters of the elements ζ_λ (which are canonical elements of conjugacy classes), but a general element of $\mathfrak{S}_n$ will not have its characters in this table.

The table of characters of $\mathcal{H}_n$ has been determined by Carter [2], Desarménien [3], Ram [20]. Let us formulate their findings by transforming the Frobenius morphism.

For an alphabet A, write $\mathcal{S}^k$ for the "modified" complete function

$$\mathcal{S}^k(A) := S^k(-QA)\,Q^{-1},\ k \geq 1\,,$$

and write exponentially $\mathcal{S}^{ij\cdots}$ the product $\mathcal{S}^i\mathcal{S}^j\cdots$ of such functions.

Define $\Phi : \mathcal{H}_n \to \mathfrak{Sym}$ to be the morphism

$$\Phi(T_w) = \mathcal{S}^{cycle(w)}\ .$$

Then, according to the cited authors, one has

Theorem 13 *Given a composition J of n, the characters of ζ_J are the coefficients of the expansion of $(-1)^{\ell(J)}\mathcal{S}^J$ in the basis of Schur functions.*

We shall now show that one can easily transform the table of characters into a matrix independent of q.

Instead of ζ_J, let us take, up to some normalizing factor, the Yang-Baxter element corresponding to the same permutation.

Define

$$\Upsilon_n := (T_{n-1}-q)\left(T_{n-2}-\frac{q^2}{[2]}\right)\cdots\left(T_1-\frac{q^{n-1}}{[n-1]}\right)\left(\frac{-1}{[n]}\right)$$

and, by direct product, Υ_J, for any composition J of n.

For example,

$$\Upsilon_{43} = (T_3-q)\left(T_2-\frac{q^2}{[2]}\right)\left(T_1-\frac{q^3}{[3]}\right)\left(\frac{-1}{[4]}\right)(T_6-q)\left(T_5-\frac{q^2}{[2]}\right)\left(\frac{-1}{[3]}\right)\ .$$

The characters of such elements are simpler than those of ζ_J.

Theorem 14 *Given any composition J of n, then*

$$\Phi(\Upsilon_J) = h^J\,.$$

The table of characters of all the $\Upsilon_\lambda : \lambda \in \mathfrak{Part}(n)$ is the Kostka matrix $(E2S)^{tr}$.

Proof. The first statement comes, by direct product, from the case of a composition with a single part. Let us show that one can compute $\Phi(\Upsilon_n)$ by induction on n, filtering the product $(T_{n-1}-\alpha)\cdots(T_1-\beta)$ according to the maximum right factor of type $T_j\cdots T_2T_1$. For example,

$$\begin{aligned}(T_3-q)\left(T_2-\frac{q^2}{[2]}\right)\left(T_1-\frac{q^3}{[3]}\right)\left(\frac{-1}{[4]}\right)\\ = T_3T_2T_1\frac{-1}{[4]}+(-q)T_2T_1\frac{-1}{[4]}+(T_3-q)\,\frac{-1}{[2]}q^2T_1\frac{-1}{[4]}\\ +(T_3-q)\left(T_2-\frac{q^2}{[2]}\right)\frac{-1}{[3]}q^3\frac{-1}{[4]}\,.\end{aligned}$$

Taking the image under Φ of the right-hand side for a general n, one gets $\mathcal{S}^n\frac{1}{[n]}+\mathcal{S}^{n-1}\,qS^1\frac{1}{[n]}+\cdots+\mathcal{S}^1\,q^{n-1}S^{n-1}\frac{1}{[n]}$.

One notices then with pleasure that the first property enunciated in the theorem reduces to the identity $S^n(-QA+qA)=S^n(-(q-1/q)A+qA)=S^n(A/q)$ (see [15] for more identities using the λ-ring structure of the ring of symmetric polynomials).

The expansion of any Υ_J produces only subwords of $T_{n-1}\cdots T_1$, which have the same characters as some ζ_λ. Therefore, the characters of Υ_J are the coefficients of the expansion of $\Phi(\Upsilon_J)$ in the basis of Schur functions, and the matrix of characters of the Υ_λ is the transpose[4] of $E2S$ QED

Thus the table of characters[5] is independent of q. Moreover, one could in fact have directly computed the traces of the matrices representing Υ_J, and therefore obtained the theory of characters of $\mathcal{H}_n$ without using the results of Carter, Désarménien, Ram.

As an example, for $n=4$, the table of characters of $\zeta_4,\zeta_{31},\zeta_{22},\zeta_{211},\zeta_{1111}$,

[4]We have here followed the usual disposition for the table of characters, and this introduces a transposition.

[5]Using Th.14, one can easily write determinants of elements of the Hecke algebra which are sent under Φ onto Schur functions. The table of characters now becomes the identity matrix. For example, the determinant $\begin{vmatrix}\Upsilon_3 & \Upsilon_4\\ -1 & (T_4-q)(-1/[2])\end{vmatrix}$, expanded by rows or by columns, is sent onto the Schur functions s_{32}.

written by columns, is

$$\mathfrak{Car}_4 = \begin{bmatrix} q^3 & q^2 & q^2 & q & 1 \\ -q & qQ & q^2-2 & 2q-1/q & 3 \\ 0 & -1 & [4]/[2] & Q & 2 \\ 1/q & -Q/q & 1/q^2-2 & q-2/q & 3 \\ -1/q^3 & 1/q^2 & 1/q^2 & -1/q & 1 \end{bmatrix}$$

and the table of characters of $\Upsilon_4, \Upsilon_{31}, \Upsilon_{22}, \Upsilon_{211}, \Upsilon_{1111}$ is

$$\mathfrak{Car}_4^{\Upsilon} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

The first column of the last matrix tells us that Υ_4 has zero trace in every irreducible representation, except only one.

The matrix $\mathfrak{Car}_n$ specializes, for $q = 1$, to the table of characters $P2M^{tr}$ for the symmetric group (i.e. to the transpose of the matrix from the power sums to the Schur basis).

The relation between the two matrices has been studied by Carter [2], Ueno-Shibukawa [23]. The following property, that we shall not prove, gives another factorization than theirs.

Lemma 15 *For any n, let D_3, D be the diagonal matrices with respective diagonals*[6] $\left[Q^{\ell(\lambda)-n}\prod_{i\in\lambda}[i]z_\lambda^{-1},\ \lambda\in\mathfrak{Part}(n)\right]$, $\left[Q^{n-\ell(\lambda)},\ \lambda\in\mathfrak{Part}(n)\right]$. *Then*

$$\mathfrak{Car}_n = (P2S)^{tr}\cdot D_3\cdot P2M\cdot D\,.$$

[6]For a partition written exponentially: $\lambda = 1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\cdots$, then $z_\lambda = 1^{\alpha_1}\alpha_1!\,2^{\alpha_2}\alpha_2!\,3^{\alpha_3}\alpha_3!\cdots$.

For example,

$$\mathfrak{Car}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 & 3 \\ 0 & -1 & 2 & 0 & 2 \\ 1 & 0 & -1 & -1 & 3 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{[4]}{4}Q^{-3} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \frac{[3]}{3}Q^{-2} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{[2.2]}{8}Q^{-2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{[2]}{4}Q^{-1} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \frac{1}{24} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 \\ 1 & 4 & 6 & 12 & 24 \end{bmatrix} \cdot \begin{bmatrix} Q^3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & Q^2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & Q^2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & Q & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

We have already met the matrix $P2M$ in the preceding section.

9 Bases, by computing characters

Let us now evaluate characters for different bases of central elements.

Recall that Young [25, 21] defined orthogonal idempotents, indexed by standard tableaux, for the group algebra of the symmetric group. These can be generalized to idempotents $\mho_t$ for $\mathcal{H}_n$, the sum

$$\mho_\lambda := \sum_t \mho_t$$

over all tableaux of shape λ being a central idempotent. Let us mention that one can view idempotents as polynomials to compute them efficiently[14].

The idempotents $\mho_t$ belong to the algebra generated by the JM elements, and they are left and right eigenvectors for them [24] :

$$\mho_t\, \xi_i = \xi_i\, \mho_t = \mho_t\, q^{2c(i,t)}\,,$$

where $c(i,t)$ is the content of i in the tableau t.

We shall not require these properties, but compute characters by evaluating traces or scalar products. Indeed, the value of the character of index λ over an element $g \in \mathcal{H}_n$ is equal to the scalar product

$$(\mho_\lambda\,,\, g)\,.$$

We first take the basis of Jones, i.e. the family $\{\mathcal{N}_\lambda(e^{\lambda^\natural}) : \lambda \in \mathfrak{Part}(n)\}$.

Theorem 16 *Given n, let D_1 and D_2 be the diagonal matrices with respective entries*

$$\prod_i \frac{1}{[\lambda_i]} \qquad , \qquad q^{\sum contents} n! \prod_{h=hook} \frac{[h]}{h} \; ,$$

product over all hooks h of the diagram of λ, for all $\lambda \in \mathfrak{Part}(n)$.

Then the matrix of characters of the elements $\mathcal{N}_\lambda\big(e^{\lambda^\natural}\big)$ is equal to

$$D_1 \cdot P2S \cdot D_2 \; .$$

For example, for $n = 4$, the matrix of characters is

$$\begin{bmatrix} q^6[2.3] & -3q^2[2] & 0 & 3q^{-2}[2] & -q^{-6}[2.3] \\ q^6[2.4] & 0 & -2[2.2] & 0 & q^{-6}[2.4] \\ q^6[3.4]/[2] & -3q^2[4]/[2] & 4[3] & -3q^{-2}[4]/[2] & q^{-6}[3.4]/[2] \\ q^6[3.4] & 3q^2[4] & 0 & -3q^{-2}[4] & -q^{-6}[3.4] \\ q^6[2.3.4] & 9q^2[2.4] & 4[2.2.3] & 9[2.4]q^{-2} & q^{-6}[2.3.4] \end{bmatrix}$$

$$= \begin{bmatrix} 1/[4] & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1/[3] & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1/[2.2] & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1/[2] & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 2 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 3 & 2 & 3 & 1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} q^6[2.3.4] & \cdot & \cdot & \cdot & \cdot \\ \cdot & 3q^2[2.4] & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2[2.2.3] & \cdot & \cdot \\ \cdot & \cdot & \cdot & 3q^{-2}[2.4] & \cdot \\ \cdot & \cdot & \cdot & \cdot & q^{-6}[2.3.4] \end{bmatrix}$$

We are now in position to produce other bases of the center by computing characters.

Given a composition I define

$$\Box_I := \sum_{w \in \mathfrak{S}_I} q^{\ell(w)} T_w \quad \& \quad \nabla_I := \sum_{w \in \mathfrak{S}_I} (-q)^{-\ell(w)} T_w.$$

According to (3), (4), both elements are proportional to some Yang-Baxter elements. Indeed,

$$\Box_I = q^{\ell(\omega_I)} \, \mathcal{Y}_{\omega_I}^{[n,\ldots,1]}$$

and

$$\nabla_I = (-q)^{-\ell(\omega_I)}\, \mathcal{Y}_{\omega_I}^{[1,\ldots,n]}\,,$$

where ω_I is the permutation of maximal length in $\mathfrak{S}_I$.

Both elements are central in $\mathcal{H}(\mathfrak{S}_I)$ and generate a 1-dimensional representation of $\mathcal{H}(\mathfrak{S}_I)$:

$$\square_I T_w = \square_I\, q^{\ell(w)} \quad \& \quad \nabla_I T_w = \nabla_I (-q)^{-\ell(w)} \quad \text{for } w \in \mathfrak{S}_I\ .$$

Theorem 17 *Both sets* $\{\mathcal{N}_\lambda(\square_\lambda)\}_{\lambda\in\mathfrak{Part}(n)}$ *and* $\{\mathcal{N}_\lambda(\nabla_\lambda)\}_{\lambda\in\mathfrak{Part}(n)}$ *are a basis of the center of* $\mathcal{H}_n$*.*

Their characters are respectively

$$H2S\cdot D_2 \qquad \& \qquad E2S\cdot D_2\ ,$$

where D_2 *is the diagonal matrix defined in Th. 16.*

Proof. The fact that we have two bases will result from the fact that their matrices of characters be invertible.

For each λ, the element $g_\lambda := \sum_{w\in\mathfrak{S}_n} T_w \square_\lambda T_{w^{-1}}$ is equal to

$$\sum_{w\in\mathfrak{S}_n/\mathfrak{S}_\lambda,\, u\in\mathfrak{S}_\lambda} T_w\, T_u\, \square_\lambda\, T_{u^{-1}}\, T_{w^{-1}} = \left(\sum_u q^{2\ell(u)}\right) \mathcal{N}_\lambda(\square_\lambda)\,.$$

Therefore, one can use g_λ instead of $\mathcal{N}_\lambda(\square_\lambda)$. However, the characters of g_λ are the same as the characters of $\sum_{w\in\mathfrak{S}_n} \square_\lambda T_{w^{-1}} T_w$. Since $\sum_w T_{w^{-1}} T_w$ is a central element, we are finally reduce to compute the characters of $\square_\lambda$.

Given a representation of $\mathfrak{S}_n$, let us decompose it into irreducible subrepresentations of $\mathfrak{S}_\lambda$. In the present case, given a partition μ, and the irreducible representation of $\mathfrak{S}_n$ with basis indexed by the standard tableaux of shape μ, we take any subspace with basis all the tableaux differing from each other by a permutation in $\mathfrak{S}_\lambda$. Then $\square_\lambda$ vanishes on this space, except if there is a tableau t containing the subwords $1,\ldots,\lambda_1$, $\lambda_1+1,\ldots,\lambda_1+\lambda_2,\ldots$, i.e., which, considered as a permutation, is of minimal length in its coset $\mathfrak{S}_\lambda\, t$. In that case, knowing that the trace of $\square_k$ is equal to $q^{k(k-1)/2}\,[1]\,[2]\cdots[k]$, then the trace of $\square_\lambda$ is equal to

$$\prod_i q^{i(i-1)/2} \prod_{j=1}^{\lambda_i} [j]\ .$$

The dependency on μ is therefore only the number of tableaux which are of minimal length in their cosets $\mathfrak{S}_\lambda\, t$. This is one of the descriptions of the Kostka matrix $H2S$.

As for the set $\{\mathcal{N}_\lambda(\nabla_\lambda)\}$, one obtains it from the first one by the involution $q \to -1/q$ and the exchange of the complete functions with the elementary ones (this corresponds to conjugating the partitions indexing irreducible representations). QED

For example, for $n = 4$, the matrix of characters of $\mathcal{N}_4(\square_4), \mathcal{N}_{31}(\square_{31}), \mathcal{N}_{22}(\square_{22}), \mathcal{N}_{211}(\square_{211}), \mathcal{N}_{1111}(\square_{1111})$ (read by rows) is

$$\begin{bmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&1&1&0&0\\ 1&2&1&1&0\\ 1&3&2&3&1 \end{bmatrix} \begin{bmatrix} q^6[2.3.4] & \cdot & \cdot & \cdot & \cdot \\ \cdot & 3q^2[2.4] & \cdot & \cdot & \cdot \\ \cdot & \cdot & 2[2.2.3] & \cdot & \cdot \\ \cdot & \cdot & \cdot & 3q^{-2}[2.4] & \cdot \\ \cdot & \cdot & \cdot & \cdot & q^{-6}[2.3.4] \end{bmatrix},$$

the matrix of characters of $\square_4, \square_{31}, \square_{22}, \square_{211}, \square_{1111}$ is

$$\begin{bmatrix} q^6[2.3.4] & \cdot & \cdot & \cdot & \cdot \\ \cdot & q^3[2.3] & \cdot & \cdot & \cdot \\ \cdot & \cdot & q^3[2.2] & \cdot & \cdot \\ \cdot & \cdot & \cdot & q[2] & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&1&1&0&0\\ 1&2&1&1&0\\ 1&3&2&3&1 \end{bmatrix}.$$

the matrix of characters of $\nabla_4, \nabla_{31}, \nabla_{22}, \nabla_{211}, \nabla_{1111}$ is

$$\begin{bmatrix} q^{-6}[2.3.4] & \cdot & \cdot & \cdot & \cdot \\ \cdot & q^{-3}[2.3] & \cdot & \cdot & \cdot \\ \cdot & \cdot & q^{-3}[2.2] & \cdot & \cdot \\ \cdot & \cdot & \cdot & q^{-1}[2] & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} 0&0&0&0&1\\ 0&0&0&1&1\\ 0&0&1&1&1\\ 0&1&1&2&1\\ 1&3&2&3&1 \end{bmatrix}$$

The relations between the different bases that we have written may be summarized into the following diagram:

$$\begin{tikzcd}
\mathcal{N}_\lambda(\nabla_\lambda) \arrow[r, "E2P\cdot D_1^{-1}D^{-1}\cdot P2H"] \arrow[d, "E2S\cdot D_2"'] & \mathcal{N}_\lambda(T^2_{\omega_\lambda}) \arrow[r, "H2E"] \arrow[d] & \mathcal{N}_\lambda(1) \arrow[d, "E2P\cdot D"] & \\
\mho_\lambda \arrow[r, "D_2^{-1}\cdot S2H"'] & \mathcal{N}_\lambda(\square_\lambda) \arrow[r, "H2P\cdot D_1^{-1}"'] & \mathcal{N}_\lambda(e_{\lambda^\natural}) \arrow[r, "D^{-1}\cdot P2M\cdot D"'] & \Gamma_\lambda
\end{tikzcd}$$

Ignoring the diagonal matrices, we have exhibited analogs of the bases e, h, p, m, s of the ring of symmetric polynomials. Notice that Γ_λ should be identified with the basis of monomial symmetric functions, and not with the basis of power sums.

In fact, we have produced two analogs of the bases h^λ and e^λ. The matrix expressing the elements $\mathcal{N}_\lambda(T^2_{\omega_\lambda})$ in terms of the elements $\mathcal{N}_\lambda(\Box_\lambda)$ is

$$H2P \cdot D \cdot D_1 \cdot P2H$$

and becomes the identity when erasing D and D_1 (but there is no specialization of q which sends both D and D_1 to the identity).

Moreover, it is remarkable that the matrix of change of basis between $\mathcal{N}_\lambda\left(T^2_{\omega_\lambda}\right)$ and $\mathcal{N}_\lambda(1)$, which is $H2E$, be self-inverse and independent of q. Let us check directly this fact by showing how to relate normalizing $T^2_{\omega_\lambda}$ and normalizing 1.

Given a composition J of n, consider the lattice of compositions I finer than J (i.e. obtained by replacing the parts of J by compositions of them).

Let us write $\mathfrak{Cos}(I)$ for the sum, in $\mathcal{H}_n$, of the minimum coset representative of $\mathfrak{S}_n/\mathfrak{S}_I$. It is easy to realize that

$$\sum_{I\,finer\,J} (1-)^{n-\ell(I)} \mathfrak{Cos}(I)$$

is the sum of maximal coset representative of $\mathfrak{S}_n/\mathfrak{S}_J$.

Therefore

$$\begin{aligned}\sum_I \sum\nolimits_{w\in\mathfrak{S}_n/\mathfrak{S}_I} T_w\, T_{w^{-1}} &= \sum\nolimits_{w\in\mathfrak{S}_n/\mathfrak{S}_J} T_w T_{\omega_J}\, T_{\omega_J} T_{w^{-1}} \\ &= \mathcal{N}_J\left(T^2_{\omega_J}\right) .\end{aligned}$$

On the other hand, the number of compositions I finer than a composition J, which reorder into a partition μ is, indeed, the coefficient of e^λ in the expansion of the product h^J. QED

10 Note: Symmetric Functions

In this text, we have followed the conventions of Macdonald [18], and not of [15], with the small proviso that we write exponentially the multiplicative bases of the ring of symmetric polynomials: e^λ, h^λ, p^λ are the products of elementary functions, complete functions, power sums, respectively. One has also the monomial functions m_λ and the Schur functions S_λ.

The matrix expressing the basis a_λ (or a^λ) into the basis b_λ (or b^λ) is denoted $A2B$, with capital letters. These matrices, for a given n, are obtained from the command `Sf2TableMat(n,a,b)` in the maple library ACE [1].

The entries of such matrices may be written as scalar products, and have many combinatorial interpretations. Realizing a Schur function, or a skew Schur function as a sum of tableaux of a given (skew)-shape, one gets in particular that the coefficient of m_J in the expansion of a product h^I is equal to the number of non negative integral matrices with row sums I, column sums J. Expanding e^I will similarly involve the 0-1 matrices with row sums I and column sums J (see [18, I. 6] for more details). Indeed, a product of monomials $x^u x^v \cdots x^w$ may be represented by the matrix whose successive rows are $u, v, \ldots, w$.

Consequently, a monomial x^J appearing in a product p^I corresponds to a matrix with row sums I an column sums J, having only one non-zero entry in each row. This is another possible description of the constants in Th. 12.

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