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N° 5567

May, 12 2005

Thème NUM

Rapport de recherche

ISRN INRIA/RR--5567--FR+ENG

N 0249-6399

Integration by parts formula for locally smooth laws and applications to sensitivity computations

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Thème NUM — Systèmes numériques
Projet MATHFI

Rapport de recherche n° 5567 — May, 12 2005 — 54 pages

Abstract: We consider random variables of the form $F = f(V_1, ..., V_n)$ where f is a smooth function and $V_i, i \in \mathbb{N}$ are random variables with absolutely continuous law $p_i(y)$. We assume that $p_i, i = 1, ..., n$ are piecewise differentiable and we develop a differential calculus of Malliavin type based on $\partial \ln p_i$. This allows us to establish an integration by parts formula $E(\partial_i \phi(F)G) = E(\phi(F)H_i(F, G))$ where $H_i(F, G)$ is a random variable constructed using the differential operators acting on F and G. We use this formula in order to give numerical algorithms for sensitivity computations in a model driven by a Lévy process.

Key-words: Lévy process, Malliavin calculus, border terms, Malliavin weights, Monte-Carlo algorithm, sensitivity computations.

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Formule d'intégration par parties pour des lois de probabilité localement régulières et application au calcul des sensibilités

Résumé : On considère des variables aléatoires de la forme $F = f(V_1, ..., V_n)$ où f est une fonction régulière et $V_i, i \in \mathbb{N}$ sont des variables aléatoires de lois $p_i(y)$ absolument continues. On suppose que $p_i, i = 1, ..., n$ sont différentiables par morceaux et on développe un calcul différentiel du type Malliavin basé sur $\partial \ln p_i$. Ceci nous permet d'établir une formule d'intégration par parties $E(\partial_i \phi(F)G) = E(\phi(F)H_i(F, G))$ où $H_i(F, G)$ est une variable aléatoire dépendant des opérateurs différentiels pris en F et G. On utilise cette formule d'intégration par parties afin de donner des algorithmes numériques dans le calcul des sensibilités pour des modèles à sauts.

Mots-clés : Processus de Lévy, calcul de Malliavin, termes de bord, poids de Malliavin, algorithme de Monte-Carlo, calcul des sensibilités.

1 Introduction

In the last years, following the pioneering papers [12], [11] a lot of work concerning the numerical applications of the stochastic variational calculus (Malliavin calculus) has been done. This mainly concerns applications in mathematical finance: computations of conditional expectations (which appear in the American option pricing, for example) and of sensitivities (the so called Greeks). The models at hand are usually log-normal type diffusions and then one may use the standard Malliavin calculus. But nowadays people are more and more interested in jump type diffusions (see [7] for example) and then one has to use the stochastic variational calculus corresponding to Poisson point processes. Such a calculus has already been developped in [4] and [14] concerning the noise coming from the amplitudes of the jumps and in [6] and [9], [18], [17], [21] and [20] concerning the jump moments. Recently N. Bouleau (see [5]) settled the so called error calculus based on the Dirichlet forms language, and showed that both the approaches in [4] and in [6] fit in this frame. Another point of view, based on chaos decomposition may be found in [10], [3], [16], [22], [8] and [15].

The aim of our paper is to give a concrete application of the Malliavin calculus approach for sensitivity computations (Greeks) for pure jump diffusion models. We give three examples: in the first one we use the Malliavin calculus with respect to the jump amplitudes and in the second one we differentiate with respect to the jump moments. In the third one we differentiate with respect to both of them.

The basic tool is an integration by parts formula which is analogues to the one in the standard Malliavin calculus on the Wiener space. We give here an abstract approach which, in particular, permits to treat in an unified way the derivatives with respect to the times and the amplitudes of the jumps of Lévy processes. More precisely we consider functionals of a finite number of random variables $V_i, i = 1, ..., n$. The only assumption is that for each $i = 1, ..., n$ the conditional law of V_i (with respect to $V_j, j \neq i$) is absolutely continuous with respect to the Lebesgue measure and the conditional density $p_i = p_i(\omega, y)$ is piecewise differentiable. Using integration by parts one may settle the duality relation which represents the starting point in Malliavin calculus. But some border terms will appear corresponding to the points in which p_i is not continuous: for example, if V_i has a uniform conditional law on $[0,1]$ the density is $p_i(\omega, y) = 1_{[0,1]}(y)$ and integration by parts produces border terms in 0 and in 1. There is a simple idea which permits to cancel the border terms: we introduce in our calculus some weights π_i which are null in the points of singularity of p_i - in the previous example we may take $\pi_i(y) = y^\alpha(1-y)^\alpha$ with some $\alpha \in (0,1)$. Then we obtain a standard duality relation between the Malliavin derivative and the Skorohod integral and the machinery settled in the Malliavin calculus produces an integration by parts formula. But there is a difficulty hidden in this proceedings: the differential operators involve the weights π_i and their derivatives. In the previous example $\pi_i'(\omega, y) = \alpha(y^{\alpha-1}(1-y)^\alpha - y^\alpha(1-y)^{\alpha-1})$. These derivatives blow up in the neighborhood of the singularity points and this produces some non trivial integrability problems. So one has to realize an equilibrium between the speed of convergence to zero and the speed with which the derivatives of the weights blow up in the singularity points. This leads to a non degeneracy condition which involves the weights and their derivatives.

The integration by parts formula is established in Section 2. Since numerical algorithms involve only functions of a finite number of variables we do not develop here an infinite dimensional Malliavin calculus but restrict ourselves to simple functionals. In Section 3 we use the integration by parts formula in order to compute the Delta (derivative with respect to the initial condition) for European options based on an asset which follows a pure jump diffusion equation and in Section 4 we give numerical results.

2 Malliavin calculus for simple functionals

2.1 The frame

We consider a probability space $(\Omega, \mathcal{F}, P)$, a sub $\sigma-$algebra $\mathcal{G} \subseteq \mathcal{F}$ and a sequence of random variables $V_i, i \in \mathbb{N}$. We denote $\mathcal{G}_i = \mathcal{G} \vee \sigma(V_j, j \neq i)$. Our aim is to settle an integration by parts formula for functionals of V_i, $i \in \mathbb{N}$, which is analogues to the one in the Malliavin calculus. The σ-algebra $\mathcal{G}$ appears in order to describe all the randomness which is not involved in the differential calculus.

We will work on some set $A \in \mathcal{G}$ which will be fixed through this section. We denote $L_{(\infty)}(A)$ the space of the random variables such that $\mathrm{E}\left(|F|^p \, \mathbf{1}_A\right) < \infty$

for all $p \in \mathbb{N}$, and $\mathrm{L}_{(p+)}(A)$ will be the space of the random variables F for which there exists some $\delta > 0$ such that $\mathrm{E}\left(|F|^{p+\delta}\ \mathbf{1}_A\right) < \infty$. We assume that

Hypothesis 2.1 $\qquad V_i \in \mathrm{L}_{(\infty)}(A),\ i \in \mathbb{N}.$

For each $i \in \mathbb{N}$ we consider some $k_i \in \mathbb{N}$ and some $\mathcal{G}_i$-measurable random variables

$$a_i(\omega) = t_i^0(\omega) < t_i^1(\omega) < \ldots < t_i^{k_i}(\omega) < t_i^{k_i+1}(\omega) = b_i(\omega)\,.$$

We denote $D_i(\omega) = \bigcup_{j=0}^{k_i} \left(t_i^j(\omega), t_i^{j+1}(\omega)\right)$. Notice that we may take $a_i = -\infty$ and $b_i = \infty$.

We will work with functions defined on $(a_i(\omega), b_i(\omega))$ which are smooth except for the points $t_i^j, j = 1, \ldots, k_i$. We define $\mathcal{C}_k(D_i)$ to be the set of the measurable functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that, for every ω, $y \to f(\omega, y)$ is k times differentiable on $D_i(\omega)$ and for each $j = 1, \ldots, k_i$, the left hand side and the right hand side limits $f(\omega, t_i^j(\omega)-), f(\omega, t_i^j(\omega)+)$ exist and are finite (for $j = 0$ and $j = k_i + 1$ we assume that the right hand side, respectively the left hand side limit exists and is finite). We denote

$$\Gamma_i(f) = \sum_{j=1}^{k_i} \left(f(\omega, t_i^j(\omega)-) - f(\omega, t_i^j(\omega)+)\right) + f(\omega, b_i(\omega)-) - f(\omega, a_i(\omega)+)\,. \quad (1)$$

For $f, g \in \mathcal{C}_1(D_i)$, the integration by parts formula gives

$$\int_{(a_i,b_i)} f\, g'(\omega, y)\, dy = \Gamma_i(f\, g) - \int_{(a_i,b_i)} f'\, g(\omega, y)\, dy\,, \quad (2)$$

so Γ_i represents the contribution of the border terms - or, put it otherwise, of the singularities of f or g.

Let $n, k \in \mathbb{N}$. We denote by $\mathcal{C}_{n,k}$ the class of the $\mathcal{G} \times \mathcal{B}(\mathbb{R}^n)$ measurable functions $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ such that $I_i(f) \in \mathcal{C}_k(D_i)$, $i = 1, \ldots, n$, where

$$I_i(f)(\omega, y) := f(\omega, V_1, \ldots, V_{i-1}, y, V_{i+1}, \ldots, V_n)\,.$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, n\}^k$, we denote
$\partial^k_\alpha f = \dfrac{\partial^k}{\partial x_{\alpha_1} \dots \partial x_{\alpha_k}} f.$

Moreover we denote by $\mathcal{C}_{n,k}(A)$ the space of the functions $f \in \mathcal{C}_{n,k}$ such that for every $0 \le p \le k$ and every $\alpha = (\alpha_1, \dots, \alpha_p) \in \{1, \dots, n\}^p$, $\partial^p_\alpha f(V_1, \dots, V_n) \in \mathrm{L}_{(\infty)}(A)$.

The points $t^j_i, j = 1, \dots, k_i$ represent singularity points for the functions at hand (notice that f may be discontinuous in t^j_i) and our main propose is to settle a calculus adapted to such functions.

Our basic hypothesis is the following.

Hypothesis 2.2 *For every $i \in \mathbb{N}$ the conditional law of V_i with respect to $\mathcal{G}_i$ is absolutely continuous on (a_i, b_i) with respect to the Lebesgue measure. This means that there exists a $\mathcal{G}_i \times \mathcal{B}(\mathbb{R})$-measurable function $p_i = p_i(\omega, x)$ such that*

$$\mathrm{E}(\Theta\, f(V_i)\, \mathbf{1}_{(a_i,b_i)}(V_i)) = \mathrm{E}\left(\Theta \int_{\mathbb{R}} f(x)\, p_i(\omega, x)\, \mathbf{1}_{(a_i,b_i)}(x)\, dx\right),$$

for every positive, $\mathcal{G}_i$-measurable random variable Θ and every positive, measurable function $f : \mathbb{R} \to \mathbb{R}$.
We assume that $p_i \in \mathcal{C}_1(D_i)$ and $\partial_y \ln p_i(\omega, y) \in \mathrm{L}_{(\infty)}(A)$.

In concrete problems, we consider random variables V_i with conditional densities p_i and then we take t^j_i, $i = 0, \dots, k_{i+1}$ to be the points of singularities of p_i. This means that we choose D_i in such a way that p_i satisfies hypothesis 2.2 on D_i. This is the significance of D_i (in the case where p_i is smooth on the whole $\mathbb{R}$, we may choose $D_i = \mathbb{R}$).

For each $i \in \mathbb{N}$ we consider a $\mathcal{G}_i \times \mathcal{B}(\mathbb{R})$-measurable and positive function $\pi_i : \Omega \times \mathbb{R} \to \mathbb{R}_+$ such that $\pi_i(\omega, y) = 0$ for $y \notin (a_i, b_i)$ and $\pi_i \in \mathcal{C}_1(D_i)$. We assume

Hypothesis 2.3 $\pi_i \in \mathrm{L}_{(\infty)}(A)$ *and* $\pi'_i \in \mathrm{L}_{(1+)}(A)$.

These will be the weights used in our calculus. In the standard Malliavin calculus they appear as re-normalization constants. On the other hand, p_i may have discontinuities in $t^j_i, j = 1, \dots, k_i$ and this will produce some border terms in the integration by parts formula - see (2). We may choose π_i in order to cancel these border terms (as well as the border terms in a_i and b_i).

2.2 Differential operators

In this section we introduce the differential operators which represent the analogues of the Malliavin derivative and of the Skorohod integral.

Simple functionals. A random variable F is called a simple functional if there exists some n and some $\mathcal{G}\times\mathcal{B}(\mathbb{R}^n)$-measurable function $f:\Omega\times\mathbb{R}^n\to\mathbb{R}$ such that

$$F = f(\omega, V_1, ..., V_n)\,.$$

We denote by $\mathcal{S}_{(n,k)}$ the space of the simple functionals such that $f\in\mathcal{C}_{n,k}$ and $\mathcal{S}_{(n,k)}(A)$ will be the space of the simple functionals such that $f\in\mathcal{C}_{n,k}(A)$. We will use the notation $\partial_{V_i}F := \dfrac{\partial f}{\partial x_i}(\omega, V_1,\ldots,V_n)$, $i=1,\ldots,n$.

Simple processes. A simple process of length n is a sequence of random variables $U=(U_i)_{i\leq n}$ such that

$$U_i(\omega) = u_i\left(\omega, V_1(\omega),\ldots,V_n(\omega)\right)\,,$$

where $u_i:\Omega\times\mathbb{R}^n\to\mathbb{R}, i\in\mathbb{N}$ are $\mathcal{G}\times\mathcal{B}(\mathbb{R}^n)$-measurable functions. We denote by $\mathcal{P}_{(n,k)}$ (respectively $\mathcal{P}_{(n,k)}(A)$) the space of the simple processes of length n such that $u_i\in\mathcal{C}_{n,k}$, $i=1,\ldots,n$ (respectively $u_i\in\mathcal{C}_{n,k}(A)$, $i=1,\ldots,n$). Notice that if $U\in\mathcal{P}_{(n,k)}$ then $U_i\in\mathcal{S}_{(n,k)}$ and if $U\in\mathcal{P}_{(n,k)}(A)$ then $U_i\in\mathcal{S}_{(n,k)}(A)$.

On the space of the simple processes we consider the scalar product

$$\langle U,V\rangle_\pi := \sum_{i=1}^n \pi_i(\omega,V_i)\,U_i(\omega)\,V_i(\omega)\,.$$

We define now the differential operators which appear in Malliavin's calculus.

□ **The Malliavin derivative** $D:\mathcal{S}_{(n,1)}\to\mathcal{P}_{(n,0)}$: if $F=f(\omega,V_1,\ldots,V_n)$ then

$$D_iF := \frac{\partial f}{\partial x_i}\left(\omega, V_1(\omega),\ldots,V_n(\omega)\right)\,\mathbf{1}_{D_i(\omega)}(V_i)\,,$$
$$DF = (D_iF)_{i\leq n}\in\mathcal{P}_{(n,0)}\,.$$

□ **The Malliavin covarience matrix.** Given $F=(F^1,\ldots,F^d)$, $F^i=f^i(\omega,V_1,\ldots,V_n)\in\mathcal{S}_{(n,1)}$ the Malliavin covariance matrix is

$$\sigma_F^{ij}=\left\langle DF^i,DF^j\right\rangle_\pi=\sum_{p=1}^n\pi_p(\omega,V_p)\,\partial_pf^i\,\partial_pf^j(\omega,V_1,\ldots,V_n)\,.$$

This is a symmetric positive definite matrix.

□ **The Skorohod integral.** We define $\delta:\mathcal{P}_{(n,1)}\to\mathcal{S}_{(n,0)}$: for $U=(U_i)_{1\leq i\leq n}$ such that $U_i(\omega)=u_i(\omega,V_1,\ldots,V_n)$, we define

$$\begin{aligned}\delta_i(U)&:=-\left(\frac{\partial}{\partial x_i}(\pi_i\,u_i)+(\pi_i\,u_i)\,\partial\ln p_i\right)(\omega,V_1,\ldots,V_n)\,,\\ \delta(U)&:=\sum_{i=1}^n\delta_i(U)\,.\end{aligned}$$

□ **The border term operator**. For $F=f(\omega,V_1,\ldots,V_n)\in\mathcal{S}_{(n,0)}$ and $U=(u_i(\omega,V_1,\ldots,V_n))_{i=1,\ldots,n}\in\mathcal{P}_{(n,0)}$ we define

$$\begin{aligned}[F,U]_\pi&=\sum_{i=1}^n\Gamma_i\left(I_i(f\times u_i)\times\pi_i\times p_i\right)\\ &=\sum_{i=1}^n\sum_{j=1}^{k_i}\big((f\times u_i)(\omega,V_1,\ldots,V_{j-1},t_i^j-,V_{j+1},\ldots,V_n)(\pi_ip_i)(\omega,t_i^j-)\\ &-(f\times u_i)(\omega,V_1,\ldots,V_{j-1},t_i^j+,V_{j+1},\ldots,V_n)(\pi_ip_i)(\omega,t_i^j+)\big)\\ &+\sum_{i=1}^n(f\times u_i)(\omega,V_1,\ldots,V_{j-1},b_i-,V_{j+1},\ldots,V_n)(\pi_ip_i)(\omega,b_i-)\\ &-\sum_{i=1}^n(f\times u_i)(\omega,V_1,\ldots,V_{j-1},a_i+,V_{j+1},\ldots,V_n)(\pi_ip_i)(\omega,a_i+).\end{aligned}$$

Remark 2.1 *If we choose* π_i *such that*

$$\begin{aligned}&\pi_i(\omega,t_i^j+)=\pi_i(\omega,t_i^j-)=0,i=1,\ldots,n,j=1,\ldots,k_i\\ &\pi_i(\omega,a_i+)=\pi_i(\omega,b_i-)=0,i=1,\ldots,n,\end{aligned}\tag{3}$$

then $[F,U]_\pi = 0$ *for every* $F \in \mathcal{S}_{(n,1)}$ *and* $U \in \mathcal{P}_{(n,1)}$*. So there will be no border terms in the duality formula and in the integration by parts formula. This is - one possible - reason of being of the weights. The other one concerns re-normalization.*

In our frame the duality between δ and D is given by the following Proposition.

Proposition 2.1 *Let* $F \in \mathcal{S}_{(n,1)}$ *and* $U \in \mathcal{P}_{(n,1)}$*. Suppose that for every* $i = 1,\ldots,n$

$$\mathrm{E}(|F\,\delta_i(U)|\,\mathbf{1}_A) + \mathrm{E}(\pi_i(\omega,V_i)\,|D_iF \times U_i|\,\mathbf{1}_A) < \infty. \tag{4}$$

Then $\mathrm{E}(|[F,U]_\pi|\,\mathbf{1}_A) < \infty$ *and*

$$\mathrm{E}(\langle DF,U\rangle_\pi\ \mathbf{1}_A) = \mathrm{E}(F\,\delta(U)\,\mathbf{1}_A) + \mathrm{E}([F,U]_\pi\,\mathbf{1}_A)\,. \tag{5}$$

If (3) holds true then

$$\mathrm{E}(\langle DF,U\rangle_\pi\ \mathbf{1}_A) = \mathrm{E}(F\,\delta(U)\,\mathbf{1}_A)\,.$$

Proof. Since $\pi_i = 0$ on $(a_i,b_i)^c$, we have

$$\begin{aligned}
&\mathrm{E}(\langle DF,U\rangle_\pi\ \mathbf{1}_A)\\
=&\mathrm{E}(\sum_{i=1}^n \mathrm{E}\left(\pi_i(\omega,V_i)\,D_iF \times U_i \mid \mathcal{G}_i\right)\mathbf{1}_A)\\
=&\mathrm{E}\left(\mathbf{1}_A\ \sum_{i=1}^n \int_{a_i}^{b_i} (\pi_i\,u_i\,\partial_i f)(\omega,V_1,\ldots,V_{i-1},y,V_{i+1},\ldots,V_n)\,p_i(\omega,y)\,dy\right).
\end{aligned}$$

Using integration by parts (see (2)) we obtain

$$\begin{aligned}
&\int_{a_i}^{b_i} \partial_i f \times (\pi_i\, u_i) \times p_i \\
&= \sum_{j=0}^{k_i} \int_{(t_i^j, t_{i+1}^j)} \partial_i f \times (\pi_i\, u_i) \times p_i \\
&= \Gamma_i\left(I_i(f \times u_i)\, \pi_i\, p_i\right) - \sum_{j=0}^{k_i} \int_{(t_i^j, t_{i+1}^j)} f \times \left(\partial_i(\pi_i\, u_i) \times p_i + (\pi_i\, u_i) \times \partial p_i\right)\right) \\
&= \Gamma_i(I_i(f \times u_i)\pi_i\, p_i) - \int_{a_i}^{b_i} f \times \left(\partial_i(\pi_i\, u_i) + \pi_i\, u_i \partial \ln p_i\right) \times p_i\,.
\end{aligned}$$

By (4) we have

$$\int_{\mathbb{R}} (|u_i\, \partial_i f|\ \pi_i\, p_i)(\omega, V_1, \ldots, V_{i-1}, y, V_{i+1}, \ldots, V_n)\, dy < \infty\,,$$

$$\int_{\mathbb{R}} (|f(\partial_i(\pi_i\, u_i) + \pi_i\, u_i \partial \ln p_i)| \times p_i)(\omega, V_1, \ldots, V_{i-1}, y, V_{i+1}, \ldots, V_n) dy < \infty\,,$$

for almost every $\omega \in A$. So the above integrals make sense. Since $\Gamma_i(I_i(f \times u_i)\, \pi_i\, p_i)$ is the sum of these two integrals we also obtain $\mathrm{E}\,(|\Gamma_i(I_i(f \times u_i)\, \pi_i\, p_i)|\, \mathbf{1}_A) < \infty$ so that $\mathrm{E}\,(|[F, U]_\pi|\ \mathbf{1}_A) < \infty$.

Using the definition of p_i we come back to expectations and we obtain

$$\int_{a_i}^{b_i} (\pi_i\, u_i\, \partial_i f)(\omega, V_1, \ldots, V_{i-1}, y, V_{i+1}, \ldots, V_n)\, p_i(\omega, y)\, dy$$
$$= \mathrm{E}(F\, \delta_i(U) \mid \mathcal{G}_i) + \Gamma_i(I_i(f \times u_i)\, \pi_i\, p_i)\,.$$

One sums over i and the proof is complete. ■

Corollary 2.1 *Let $Q \in \mathcal{S}_{(n,1)}(A)$ which satisfies*

$$\mathrm{E}\left(\mathbf{1}_A\ (|\pi_i\, Q| + |\partial_{V_i}(\pi_i\, Q)|)^{1+\eta}\right) < \infty\,,\ i = 1, \ldots, n\,, \tag{6}$$

for some $\eta > 0$. Then, for every $F \in \mathcal{S}_{(n,1)}(A)$, $U \in \mathcal{P}_{(n,1)}(A)$, one has

$$\mathrm{E}\,(Q\, \langle DF, U\rangle_\pi\, \mathbf{1}_A) = \mathrm{E}\,(F\, \delta(Q\, U)\, \mathbf{1}_A) + \mathrm{E}\,([F, Q\, U]_\pi\ \mathbf{1}_A)\,. \tag{7}$$

Proof. We just have to check that F and $\widetilde{U} = Q\,U$ satisfy (4). We have

$$|\delta_i(Q\,U)| \leq |\partial_{V_i}(\pi_i\,Q)|\ |U_i| + |\pi_i\,Q|\ (|\partial_{V_i}U_i| + |U_i|\ |\partial \ln p_i|)\ .$$

Since $U \in \mathcal{P}_{(n,1)}(A)$, one has U_i, $\partial_{V_i}U_i \in \mathrm{L}_{(\infty)}A$, and by hypothesis 2.2, $\partial \ln p_i \in \mathrm{L}_{(\infty)}A$. So, using (6), $\delta_i(Q\,U) \in \mathrm{L}_{(1+)}A$ and since $F \in \mathrm{L}_{(\infty)}A$, we obtain $\mathrm{E}\left(F\,\delta_i(Q\,U)|\right) < \infty$.
We have D_iF, $U_i \in \mathrm{L}_{(\infty)}A$ and $\pi_i\,Q \in \mathrm{L}_{(1+)}A$, so $\mathrm{E}\left(\pi_i\,|D_iF \times (Q\,U_i)|\right) < \infty$.
■

The Ornstein Uhlenbeck operator: We introduce now $L := \delta(D) : \mathcal{S}_{(n,2)} \to \mathcal{S}_{(n,0)}$:

$$\begin{aligned} LF &:= -\sum_{i=1}^{n} \left(\partial_i(\pi_i\,\partial_i f) + \pi_i\,\partial_i f\,\partial \ln p_i\right)(\omega, V_1, \ldots, V_n) \\ &= -\sum_{i=1}^{n} \left((\pi_i' + \pi_i\,\partial \ln p_i)\,\partial_i f + \pi_i\,\partial_i^2 f\right)(\omega, V_1, \ldots, V_n)\,. \end{aligned}$$

As an immediate consequence of the duality relation (5) we obtain:

Lemma 2.1 *Let $F, G \in \mathcal{S}_{(n,2)}$ and $A \in \mathcal{G}$. Suppose that for every $i = 1, \ldots, n$*

$$\mathrm{E}\left[(|F\,L_iG| + |G\,L_iF| + \pi_i\ |D_iF \times D_iG|)\ \mathbf{1}_A\right] < \infty\,.$$

Then $\mathrm{E}\left(|[F, DG]_\pi|\ \mathbf{1}_A\right) < \infty$, $\mathrm{E}\left(|[G, DF]_\pi|\ \mathbf{1}_A\right) < \infty$ *and*

$$\begin{aligned} \mathrm{E}(F\,LG\,\mathbf{1}_A) + \mathrm{E}([F, DG]_\pi\,\mathbf{1}_A) &= \mathrm{E}(< DF, DG >_\pi\ \mathbf{1}_A) \\ &= \mathrm{E}(G\,LF\,\mathbf{1}_A) + \mathrm{E}([G, DF]_\pi\,\mathbf{1}_A)\,. \end{aligned}$$

We denote by $\mathcal{C}_p^k(\mathbb{R}^d)$ the space of the functions $\phi : \mathbb{R}^d \to \mathbb{R}$ which are k times differentiable and such that ϕ and its derivatives of order less or equal to k have polynomial growth. The standard differential calculus gives the following chain rules.

Lemma 2.2 *i) Let $\phi \in \mathcal{C}_p^1(\mathbb{R}^d)$ and $F = (F^1, \ldots, F^d)$, $F^i \in \mathcal{S}_{(n,1)}(A)$. Then $\phi(F) \in \mathcal{S}_{(n,1)}(A)$ and*

$$D\phi(F) = \sum_{k=1}^{d} \partial_k \phi(F)\, DF^k \,. \tag{8}$$

ii) If $\phi \in \mathcal{C}_p^2(\mathbb{R}^d)$ and $F^i \in \mathcal{S}_{(n,2)}(A)$ then $\phi(F) \in \mathcal{S}_{(n,2)}(A)$ and

$$L\phi(F) = \sum_{k=1}^{d} \partial_k \phi(F)\, LF^k - \sum_{k,p=1}^{d} \partial_{k,p}^2 \phi(F)\, \left\langle DF^k, DF^p \right\rangle_\pi \,.$$

iii) Let $F \in \mathcal{S}_{(n,1)}(A)$ and $U \in \mathcal{P}_{(n,1)}(A)$. Then $FU \in \mathcal{P}_{(n,1)}(A)$ and

$$\delta(F\,U) = F\,\delta(U) - \langle DF, U \rangle_\pi \,.$$

In particular if $F \in \mathcal{S}_{(n,1)}(A)$ and $G \in \mathcal{S}_{(n,2)}(A)$ then $F\,DG \in \mathcal{P}_{(n,1)}(A)$ and

$$\delta(F\,DG) = F\,LG - \langle DF, DG \rangle_\pi \,. \tag{9}$$

Remark 2.2 *Let us define $\mathrm{L}^2_{\pi,n}(A)$ to be the closure of $\mathcal{P}_{(n,0)}$ with respect to the norm associated to the scalar product $\langle U, V \rangle = \mathrm{E}(\langle U, V \rangle_\pi)$. If $[F, U]_\pi$ is not null, then the operator $D : \mathcal{S}_{(n,1)} \subset \mathrm{L}^2(\Omega) \to \mathcal{P}_{(n,0)} \subset \mathrm{L}^2_{\pi,n}(A)$ is not closable. Suppose for example that V_1 is exponentially distributed and V_i, $i = 2, \ldots, n$ are arbitrary and independent of V_1. We take $\pi_1 = 1$ and $\pi_i = 0$, $i = 2, \ldots, n$. So we make our calculus with respect to V_1 only. In this case $a_1 = 0, b_1 = \infty$ and there are no points t_i^j. Take now $F_n = f_n(V_1)$ with $f_n(x) = 1 - nx$ for $0 < x < 1/n$ and $f_n(x) = 0$ for $x \geq 1/n$. Take also $u_1(x) = 1 - x$ for $0 < x < 1$ and $u_1(x) = 0$ for $x \geq 1$ and write the duality formula $\mathrm{E}(\langle DF_n, U \rangle_\pi) = \mathrm{E}(F_n\,\delta(U)) + \mathrm{E}([F_n, U]_\pi)$. Since $[F_n, U]_\pi = 1$ and $F_n \to 0$ in $\mathrm{L}^2(\Omega)$, we obtain $\lim_{n\to\infty} \mathrm{E}(\langle DF_n, U \rangle_\pi) = 1$, and so $DF_n \nrightarrow 0$ in $\mathrm{L}^2_{\pi,n}(A)$. This proves that D is not closable.*
But if $[F, U]_\pi = 0$ for every F, U (this happens for example if we choose π_i to satisfy (3)), then the duality formula (5) guarantees that D and δ are closable. But we stay here in the level of the simple functionals and we do not discuss the extension to the infinite dimensional frame.

Remark 2.3 *The above differential operators and the duality formula (5) represent an abstract version of the operators introduced in Malliavin calculus and of the duality formula used there. In order to see it we consider the simple example of the Euler scheme for a diffusion process, corresponding to the time grid* $0 = s_0 < s_1 < \ldots < s_n = s$. *This is a simple functional depending on the increments of the Brownian motion* B, *that is* $V_i = B(s_i) - B(s_{i-1})$, $i = 1, \ldots, n$. *The variables on which the calculus is based are independent Gaussian variables. It follows that* $p_i(\omega, y) = (2\,\pi\,(s_i - s_{i-1}))^{-1/2}\,\exp\left(-y^2/2\,(s_i - s_{i-1})\right)$. *Since* p_i *is smooth on the whole* $\mathbb{R}$ *and has null limit at infinity, there will be no border terms coming on, so we take* $a_i = -\infty, b_i = \infty$ *and* $k_i = 0$. *If* $F = f(V_1, \ldots, V_n)$, *then* $D_i F = \partial_i f(V_1, \ldots, V_n) = \overline{D}_s F\,\mathbf{1}_{[s_{i-1}, s_i)}(s)$ *where* $\overline{D}_s F$ *is the standard Malliavin derivative. We take* $\pi_i = s_i - s_{i-1}$ *so that*

$$\langle DF, DG\rangle_\pi = \sum_{i=1}^{n} \pi_i\, D_i F\, D_i G = \int_0^s \overline{D}_u F\, \overline{D}_u G\, du\,.$$

We notice that here the weights are used in order to obtain the Lebesgue measure. Moreover, we have $\partial_y \ln p_i(y) = -y/(s_i - s_{i-1})$ *and so*

$$\delta_i(U) = -\sum_{i=1}^{n} \left(\partial_i u_i(V_1, \ldots, V_n)\,(s_i - s_{i-1}) - u_i(V_1, \ldots, V_n) V_i\right)\,.$$

So we find out the standard Malliavin calculus.

Remark 2.4 *If* $[F, G]_\pi = 0$, *the calculus presented here fits in the frame introduced by Bouleau in [5]: in the notation there, the bilinear form* $(F, G) \to \langle DF, DG\rangle_\pi$ *leads to an error structure. A variety of examples and applications of these structures are discussed. That frame mainly focus on the error calculus but examples of applications to sensitivity computations are given as well and an integration by parts formula is derived. This works well in the particular case of a one dimensional functional. Moreover, the differential calculus is based on a single noise* V_i *as in the Corollary 2.2 below (so the weights* π_i *do not come in in the non degeneracy condition). In a more general frame, the non degeracy condition involves the weights* π_i, $i \in \mathbb{N}$ *and a more detailed analysis has to be done (see the following section).*

2.3 The integration by parts formula

We consider $F=(F^1,\dots,F^d)\in\mathcal{S}^d_{(n,1)}(A)$ and we define

$$\Theta_F(A):=\left\{G=\sigma_F\times Q : Q\in\mathcal{S}^d_{(n,1)}(A), Q_i \text{ satisfy } (6)\right\}.$$

We think to $G\in\Theta_F(A)$ as to a random direction in which F is non degenerated (in Malliavin's sense).

The basic integration by parts formula is the following.

Theorem 2.1 *Let $F=(F^1,\dots,F^d)\in\mathcal{S}^d_{(n,2)}(A)$ and $G\in\Theta_F(A)$, $G=\sigma_F\times Q$. Then $\delta\left(\sum_{i=1}^d Q^i\,DF^i\right)$, $[\phi(F),\sum_{i=1}^d Q^i\,DF^i]_\pi\in \mathrm{L}_{(1+)}(A)$ and for every $\phi\in\mathcal{C}^1_p(\mathbb{R}^d)$ one has*

$$\mathrm{E}\left(\langle\nabla\phi(F),G\rangle\,\mathbf{1}_A\right)=\mathrm{E}\left(\phi(F)\,\delta\left(\sum_{i=1}^d Q^i\,DF^i\right)\mathbf{1}_A\right)+\mathrm{E}\left([\phi(F),\sum_{i=1}^d Q^i\,DF^i]_\pi\,\mathbf{1}_A\right). \quad (10)$$

Proof. Using (8),

$$\left\langle D\phi(F),DF^i\right\rangle_\pi=\sum_{j=1}^d\partial_j\phi(F)\left\langle DF^j,DF^i\right\rangle_\pi=\sum_{j=1}^d\partial_j\phi(F)\,\sigma_F^{ij}.$$

Since $G=\sigma_F\times Q$, we obtain

$$\langle\nabla\phi(F),G\rangle=\sum_{j=1}^d\partial_j\phi(F)\,G^j=\sum_{j=1}^d\partial_j\phi(F)\sum_{i=1}^d Q^i\,\sigma_F^{ij}=\sum_{i=1}^d Q^i\sum_{j=1}^d\partial_j\phi(F)\,\sigma_F^{ij}$$
$$=\sum_{i=1}^d Q^i\left\langle D\phi(F),DF^i\right\rangle_\pi.$$

Notice that $\phi(F)\in\mathcal{S}_{(n,1)}(A)$ and $DF^i\in\mathcal{P}_{(n,1)}(A)$. Since Q_i satisfy (6), one may use the duality formula (7) and we obtain (10). ■

We give now a non degeneracy condition on σ_F which guarantees that all the directions are non degenerated for F.

We assume that $\det \sigma_F \neq 0$ on A and we denote by $\gamma_F = \sigma_F^{-1}$. We also assume that $\pi_l\,(\det \gamma_F)^2$, $\pi'_l \det \gamma_F$, $\pi_l\,\pi'_l\,(\det \gamma_F)^2 \in \mathrm{L}_{(1+)}(A)$, for every $l = 1, \ldots, n$. This means that there exists $\eta > 0$ such that

$$\mathrm{E}\left[\mathbf{1}_A\,\left(|\pi_l|\,(\det \gamma_F)^2 + |\pi'_l|\,\left(\det \gamma_F + |\pi_l|\,(\det \gamma_F)^2\right)\right)^{1+\eta}\right] < \infty\,. \qquad (11)$$

Lemma 2.3 *Assume that (11) holds true and $F \in \mathcal{S}^d_{(n,2)}(A)$. Then $\mathcal{S}^d_{(n,1)}(A) \subseteq \Theta_F(A)$.*

Proof. Let $G \in S^d_{(n,1)}(A)$. Then $G = \sigma_F \times Q$ with $Q = \gamma_F \times G$. We write $\gamma_F^{ij} = \widehat{\sigma}_F^{ij} \times \det \gamma_F$, where $\widehat{\sigma}_F^{ij}$ is the algebric complement. It follows that $Q^i = \det \gamma_F \times S^i$ with $S^i = \sum_{j=1}^d G^j\,\widehat{\sigma}_F^{ij}$.

Let us check that (6) holds true for Q^i, $i = 1, \ldots, d$. Since $\pi_l \in L_{(\infty)}(A)$ and $D_l F^i \in L_{(\infty)}(A)$ one has $\widehat{\sigma}_F^{ij}$, $\det \sigma_F \in L_{(\infty)}(A)$. And since $G^j \in L_{(\infty)}(A)$, then we have $S^i \in L_{(\infty)}(A)$. Moreover, by (11), $\pi_l\,\det \gamma_F \in L_{(1+)}(A)$, so $\pi_l\,Q^i = (\pi_l\,\det \gamma_F)\,S^i \in L_{(1+)}(A)$.

We now check that $D_l(\pi_l\,Q^i) \in L_{(1+)}(A)$. We write

$$D_l \sigma_F^{ij} = \pi'_l\,D_l F^i\,D_l F^j + \sum_{k=1}^n \pi_k\,D_l(D_k F^i\,D_k F^j)\,.$$

Since $F \in S^d_{(n,2)}(A)$, we have $D_l F^i\,D_l F^j$, $D_l(D_k F^i\,D_k F^j) \in L_{(\infty)}(A)$ and consequently $D_l \sigma_F^{ij} = \theta_1 + \theta_2\,\pi'_l$ with θ_1, $\theta_2 \in L_{(\infty)}(A)$. Then $D_l(\det \sigma_F) = \mu + \nu\,\pi'_l$ and $D_l S^i = \mu_i + \nu_i\,\pi'_l$ with μ, ν, μ_i, $\nu_i \in L_{(\infty)}(A)$.

Using (11), we obtain

$$\begin{aligned} D_l(\pi_l\,Q^i) &= \pi'_l\,\det \gamma_F\,S^i - \pi_l\,(\det \gamma_F)^2\,D_l(\det \sigma_F)\,S^i + \pi_l\,\det \gamma_F\,D_l S^i \\ &= \pi'_l\,\det \gamma_F\,S^i - \pi_l\,(\det \gamma_F)^2\,(\mu + \nu\,\pi'_l)\,S^i + \pi_l\,\det \gamma_F\,(\mu_i + \nu_i\,\pi'_l) \\ &\in L_{(1+)}(A)\,, \end{aligned}$$

and the proof is complete. ■

As a consequence we obtain

Theorem 2.2 *Let $F=(F^1,\ldots,F^d)\in\mathcal{S}^d_{(n,2)}(A)$ and $G\in\mathcal{S}_{(n,1)}(A)$. Suppose that (11) holds true. Then*
$\delta\left(G\sum_{j=1}^{d}\gamma_F^{ji}\,DF^j\right),\ \left[\phi(F),G\sum_{j=1}^{d}\gamma_F^{ji}\,DF^j\right]_\pi\in \mathrm{L}_{(1+)}(A)$ *and for every* $\phi\in\mathcal{C}^1_p(\mathbb{R}^d)$ *one has*

$$\mathrm{E}(\partial_i\phi(F)\,G\,\mathbf{1}_A)=\mathrm{E}\left[\phi(F)\,\delta\left(G\sum_{j=1}^{d}\gamma_F^{ji}\,DF^j\right)\mathbf{1}_A\right] + \mathrm{E}\left(\left[\phi(F),G\sum_{j=1}^{d}\gamma_F^{ji}\,DF^j\right]_\pi\mathbf{1}_A\right),$$

for every $i=1,\ldots,d$.

Suppose that $\pi_l, l=1,..,n$ satisfy (3). Then

$$\mathrm{E}(\partial_i\phi(F)\,G\,\mathbf{1}_A)=\mathrm{E}(\phi(F)\,H_i(F,G)\,\mathbf{1}_A)\,.$$

with

$$H_i(F,G)=\delta\left(G\sum_{j=1}^{d}\gamma_F^{ji}\,DF^j\right)=\sum_{j=1}^{d}\left(G\,\gamma_F^{ji}\,LF^j-\left\langle D(G\,\gamma_F^{ji}),DF^j\right\rangle_\pi\right)\,.$$

Proof. We take $\widetilde{G}=(0,\ldots,0,G,0,\ldots,0)$ with G on the place i, so that $\partial_i\phi(F)\,G=\left\langle\nabla\phi(F),\widetilde{G}\right\rangle$. In view of Lemma 2.3, $\widetilde{G}\in\Theta_F(A)$ and $\widetilde{G}=\sigma_F\times Q$, with $Q^j=G\,\gamma_F^{ji}$. One employes Theorem 2.1 and concludes. In order to obtain the second equality in the expression of $H_i(F,G)$ one employes (9). ■

There is one particular situation in which the non degeneracy condition (11) does not involve the weights: if F is one dimensional and if the integration by parts formula is based on a single random variable V_i. Then we have the following corollary.

Corollary 2.2 *Let $F=f(V_1,\ldots,V_n)\in\mathcal{S}_{(n,2)}(A)$ and $G\in\mathcal{S}_{(n,1)}(A)$. Suppose that that there exists some $l\in\{1,...,n\}$ such that*

$$\mathrm{E}\left[\mathbf{1}_A\,(D_lF)^{-6\,(1+\eta)}\right]<\infty\,, \tag{12}$$

for some $\eta > 0$*. Consider the weights* $\pi_i = 0$ *for* $i \neq l$ *and* π_l *an arbitrary function which verifies* $\pi_l \in L_{(\infty)}(A)$ *and* $\pi_l' \in L_{(1+)}(A)$*. Then* $\delta(G\,\gamma_F\,DF)$, $[\phi(F), G\,\gamma_F\,DF]_\pi \in \mathrm{L}_{(1+)}(A)$ *and for every* $\phi \in \mathcal{C}_p^1(\mathbb{R})$ *one has*

$$\mathrm{E}(\phi'(F)\,G\,\mathbf{1}_A) = \mathrm{E}\left(\phi(F)\,\delta(G\,\gamma_F\,DF)\,\mathbf{1}_A\right) + \mathrm{E}\left([\phi(F), G\,\gamma_F\,DF]_\pi\,\mathbf{1}_A\right)\,. \quad (13)$$

Proof. Notice that $\sigma_F = \pi_l(V_l)\ |D_l F|^2$. We comme back to the proof of Theorem 2.1 and we write $G = Q\,\sigma_F$ with

$$\begin{aligned} Q &= \tfrac{G}{\pi_l(V_l)\,|D_l F|^2} && \text{if } \pi_l(V_l)\ |D_l F|^2 \neq 0\,, \\ &= 0 && \text{if } \pi_l(V_l)\ |D_l F|^2 = 0\,. \end{aligned}$$

Then $\pi_l(V_l)\,Q = g(V_1, \ldots, V_n)/|D_l F|^2$ and, as a consequence of the hypothesis (12) one has $\pi_l(V_l)\,Q$, $\partial_{V_i}(\pi(V_l)\,Q) \in L_{(1+)}(A)$, $i = 1, \ldots, n$. So we may use the duality relation and conclude. ■

3 Pure jump diffusions

In this section we will use the integration by parts formula presented in the previous section for a pure jump diffusion $(S_t)_{t \geq 0}$. We will use the notations from [13]. We consider a Poisson point measure $N(dt, da)$ on $\mathbb{R}$, with positive and finite intensity measure $\mu(da) \times dt$, that is $\mathrm{E}(N([0,t] \times A)) = \mu(A)t$. We denote by J_t the counting process, that is $J_t := N([0,t] \times \mathbb{R})$, and we denote by T_i, $i \in \mathbb{N}$, the jump times of J_t. We represent the above Poisson point measure by means of a sequence Δ_i, $i \in \mathbb{N}$, of independent random variables of law $\nu(da) = \mu(\mathbb{R})^{-1} \times \mu(da)$. This means that
$N([0,t] \times A) = card\{T_i \leq t : \Delta_i \in A\}$.

We look at S_t solution of the equation

$$\begin{aligned} S_t &= x + \sum_{i=1}^{J_t} c(T_i, \Delta_i, S_{T_i^-}) + \int_0^t g(r, S_r)\,dr\,, && (14) \\ &= x + \int_0^t \int_{\mathbb{R}} c(s, a, S_{s^-})\,dN(s,a) + \int_0^t g(r, S_r)\,dr\,, \qquad 0 \leq t \leq T\,. \end{aligned}$$

We will work under the following hypothesis:

Hypothesis 3.1 *The functions* $(t,x) \to c(t,a,x)$ *and* $x \to g(t,x)$ *are twice differentiable and have bounded derivatives of first and second order. Moreover, we assume that they have linear growth with respect to* x*, uniformly with respect to* t *and* a*, that is* $|c(t,a,x)| + |g(t,x)| \leq K(1+|x|)$.

On each set $\{J_t = n\}$, S_t is a simple functional of $\Delta_1, \ldots, \Delta_n$ and $T_1, \ldots, T_n$. In the first subsection, we present the deterministic calculus which permits to compute the Malliavin derivatives and in the following two subsections we give the integration by parts formula with respect to the amplitude of the jumps respectively to the jump times, separately. Finally in the forth sub-section we briefly present the mixed calculus, with respect to both of them.

We stay here in the one dimensional case (i. e. $S_t \in \mathbb{R}$) because the multi-dimensional case is more involved from a technical point of view. Our purpose is to illustrate the way in which the integration by parts formula works for Poisson Point measures and to emphasize the specific difficulties. The heavy techniques related to the multi-dimensional case would hidden these specific points, but the machinery works in this case as well.

3.1 The deterministic equation

We fix some deterministic $0 = u_0 < u_1 < \ldots < u_n < T$ and we denote by $u = (u_1, \ldots, u_n)$. We also fix $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. To these fixed numbers we associate the deterministic equation

$$s_t = x + \sum_{i=1}^{J_t(u)} c(u_i, a_i, s_{u_i^-}) + \int_0^t g(r, s_r)\, dr\,, \quad 0 \leq t \leq T \tag{15}$$

where $J_t(u) = k$ if $u_k \leq t < u_{k+1}$. We denote by $s_t(u,a)$ or simply by s_t the solution of this equation. This is the deterministic counterpart of our stochastic equation. On the set $\{J_t = n\}$, the solution S_t of (14) is represented as $S_t = s_t(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n)$.

In order to solve this equation, we introduce the flow $\Phi = \Phi_u(t,x)$, $0 \leq u \leq t$, $x \in \mathbb{R}$ solution of the ordinary integral equation

$$\Phi_u(t,x) = x + \int_u^t g(r, \Phi_u(r,x))\, dr, \quad t \geq u\,. \tag{16}$$

The solution s of the equation (15) is given by

$$\begin{aligned} s_0 &= x\,, \\ s_t &= \Phi_{u_i}(t, s_{u_i}) \text{ for } u_i \le t < u_{i+1}\,, \\ s_{u_{i+1}} &= s_{u_{i+1}^-} + c(u_{i+1}, a_{i+1}, s_{u_{i+1}^-}) \\ &= \Phi_{u_i}(u_{i+1}, s_{u_i}) + c(u_{i+1}, a_{i+1}, \Phi_{u_i}(u_{i+1}, s_{u_i}))\,. \end{aligned} \tag{17}$$

Our aim is to compute the derivatives of s with respect to u_j, a_j, $j = 1, \ldots, n$.

We introduce first some notations. We denote by

$$e_{u,t}(x) := \exp\left(\int_u^t \partial_x g(r, \Phi_u(r,x))\, dr\right)\,.$$

Since $\Phi_{u_i}(r, s_{u_i}) = s_r$ for $u_i \le r < u_{i+1}$, we have

$$e_{u_i,t}(s_{u_i}) = \exp\left(\int_{u_i}^t \partial_x g(r, s_r)\, dr\right)\,, \text{ for } u_i \le t < u_{i+1}\,.$$

Since

$$\partial_x \Phi_u(t,x) = 1 + \int_u^t \partial_x g(r, \Phi_u(r,x))\, \partial_x \Phi_u(r,x)\, dr\,,$$

it follows that

$$\partial_x \Phi_u(t,x) = e_{u,t}(x)\,,$$

and since

$$\partial_u \Phi_u(t,x) = -g(u,x) + \int_u^t \partial_x g(r, \Phi_u(r,x))\, \partial_u \Phi_u(r,x)\, dr\,,$$

we have

$$\partial_u \Phi_u(t,x) = -g(u,x)\, e_{u,t}(x)\,.$$

Finally we denote by

$$q(t,\alpha,x) := (\partial_t c + g\, \partial_x c)(t,\alpha,x) + g(t,x) - g(t, x + c(t,\alpha,x))\,.$$

Lemma 3.1 *Suppose that hypothesis 3.1 holds true. Then $s_t(u,a)$ is twice differentiable with respect to $u_j, j=1,\ldots,n$ and with respect to $a_j, j=1,\ldots,n$ and we have the following explicit expressions of the derivatives.*

***A. Derivatives with respect to** u_j. For $t<u_j$, $\partial_{u_j}s_t(u,a)=0$. Moreover*

$$\begin{aligned}\partial_{u_j}s_{u_j-}&=g(u_j,s_{u_j-})\,,\\ \partial_{u_j}s_{u_j}&=(\partial_t c+g\,(1+\partial_x c))(u_j,a_j,s_{u_j-})\,.\end{aligned}$$

For $u_j<t<u_{j+1}$

$$\begin{aligned}\partial_{u_j}s_t&=q(u_j,a_j,s_{u_j-})\,e_{u_j,t}(s_{u_j})\,,\\ \partial_{u_j}s_{u_{j+1}-}&=q(u_j,a_j,s_{u_j-})\,e_{u_j,u_{j+1}}(s_{u_j})\\ \partial_{u_j}s_{u_{j+1}}&=q(u_j,a_j,s_{u_j-})\,(1+\partial_x c(u_{j+1},a_{j+1},s_{u_{j+1}-}))\,e_{u_j,u_{j+1}}(s_{u_j})\,.\end{aligned} \tag{18}$$

Finally, for $p\geq j+1$ and $u_p\leq t<u_{p+1}$ we have the recurrence relations

$$\begin{aligned}\partial_{u_j}s_t&=e_{u_p,t}(s_{u_p})\,\partial_{u_j}s_{u_p}\,,\\ \partial_{u_j}s_{u_{p+1}}&=(1+\partial_x c(u_{p+1},a_{p+1},s_{u_{p+1}-}))\,e_{u_p,u_{p+1}}(s_{u_p})\,\partial_{u_j}s_{u_p}\,.\end{aligned} \tag{19}$$

Denote $T(f):=\partial_t f+g\partial_x f$. The second order derivatives are given by

$$\begin{aligned}\partial^2_{u_j}s_{u_j-}&=T(g)(u_j,a_j,s_{u_j-})\,,\\ \partial^2_{u_j}s_{u_j}&=T(\partial_t c+g\,(1+\partial_x c))(u_j,a_j,s_{u_j-})\,.\end{aligned}$$

Denote

$$\begin{aligned}\rho_j(t)&=\partial_{u_j}e_{u_j,t}(s_{u_j})\\ &=e_{u_j,t}(s_{u_j})\left(-\partial_x g(u_j,s_{u_j})+q(u_j,a_j,s_{u_j-})\int_{u_j}^t\partial_x^2 g(r,s_r)\,e_{u_j,r}(s_{u_j})\,dr\right)\,.\end{aligned}$$

Then, for $u_j<t<u_{j+1}$

$$\partial^2_{u_j}s_t(u,a)=T(q)(u_j,a_j,s_{u_j-}(u,a))\,e_{u_j,t}(s_{u_j})+q(u_j,a_j,s_{u_j-}(u,a))\,\rho_j(t)\,.$$

and

$$\begin{aligned}\partial^2_{u_j} s_{u_{j+1}} &= T(q)(u_j, a_j, s_{u_j-})\,(1+\partial_x c)(u_{j+1}, a_{j+1}, s_{u_{j+1}-})\, e_{u_j,u_{j+1}}(s_{u_j}) \\ &\quad + q^2(u_j, a_j, s_{u_j-})\,\partial_x^2 c(u_{j+1}, a_{j+1}, s_{u_{j+1}-})\, e^2_{u_j,u_{j+1}}(s_{u_j}) \\ &\quad + q(u_j, a_j, s_{u_j-})\,(1+\partial_x c)(u_{j+1}, a_{j+1}, s_{u_{j+1}-})\,\rho_j(u_j)\,.\end{aligned}$$

For $p \geq j+1$ *we denote by*

$$\rho_{j,p}(t) = \partial_{u_j} e_{u_p,t}(s_{u_p}) = e_{u_p,t}(s_{u_p})\,\partial_{u_j} s_{u_p} \int_{u_p}^t \partial_x^2 g(r, s_r)\, e_{u_p,r}(s_{u_p})\, dr\,.$$

Then, for $p \geq j$ *and* $u_p \leq t < u_{p+1}$ *we have the recurrence relations*

$$\begin{aligned}&\partial^2_{u_j} s_t = e_{u_p,t}(s_{u_p})\,\partial^2_{u_j} s_{u_p} + \rho_{j,p}(t,u,a)\,\partial_{u_j} s_{u_p}\,, \\ &\partial^2_{u_j} s_{u_{p+1}} \\ &= \partial_x^2 c(u_{p+1}, a_{p+1}, s_{u_{p+1}-})\,(e_{u_p,u_{p+1}}(s_{u_p})\,\partial_{u_j} s_{u_p})^2 \\ &+ (1+\partial_x c)(u_{p+1}, a_{p+1}, s_{u_{p+1}-})\,(\rho_{j,p}(u_{p+1})\,\partial_{u_j} s_{u_p} + e_{u_p,u_{p+1}}(s_{u_p})\,\partial^2_{u_j} s_{u_p})\,.\end{aligned}$$

B. Derivatives with respect to a_j. *For* $t < u_j, \partial_{a_j} s_{u_j}(u,a) = 0$ *and for* $t \geq u_j$, $\partial_{a_j} s_t(u,a)$ *satisfies the equation*

$$\partial_{a_j} s_t = \partial_a c(u_j, a_j, s_{u_j-}) + \sum_{i=j+1}^{J_t(u)} \partial_x c(u_i, a_i, s_{u_i-})\,\partial_{a_j} s_{u_i-} + \int_{u_j}^t \partial_x g(r, s_r)\,\partial_{a_j} s_r\, dr\,. \quad (20)$$

The second order derivatives satisfy

$$\begin{aligned}\partial^2_{a_j} s_t &= \partial_a^2 c(u_j, a_j, s_{u_j-}) + \sum_{i=j+1}^{J_t(u)} \partial_x^2 c(u_i, a_i, s_{u_i-})\,(\partial_{a_j} s_{u_i-})^2 \qquad (21) \\ &\quad + \int_{u_j}^t \partial_x^2 g(r, s_r)\,(\partial_{a_j} s_r)^2 dr \\ &\quad + \sum_{i=j+1}^{J_t(u)} \partial_x c(u_i, a_i, s_{u_i-})\,\partial^2_{a_j} s_{u_i-} + \int_{u_j}^t \partial_x g(r, s_r)\,\partial^2_{a_j} s_r\, dr\,.\end{aligned}$$

Proof. It is clear that for $t < u_j$, s_t does not depend on u_j and so $\partial_{u_j} s_t = 0$. We compute now

$$\partial_{u_j} s_{u_j-} = \partial_{u_j} \Phi_{u_{j-1}}(u_j, s_{u_{j-1}}) = g\left(u_j, \Phi_{u_{j-1}}(u_j, s_{u_{j-1}})\right) = g(u_j, s_{u_j-})\,.$$

Then

$$\begin{aligned}\partial_{u_j} s_{u_j} &= \partial_{u_j}(s_{u_j-} + c(u_j, a_j, s_{u_j-})) \\ &= \partial_t c(u_j, a_j, s_{u_j-}) + (1 + \partial_x c(u_j, a_j, s_{u_j-}))\, \partial_{u_j} s_{u_j-} \\ &= \partial_t c(u_j, a_j, s_{u_j-}) + (1 + \partial_x c(u_j, a_j, s_{u_j-}))\, g(u_j, s_{u_j-})\,.\end{aligned}$$

For $u_j < t < u_{j+1}$ we have

$$\begin{aligned}&\partial_{u_j} s_t \\ =&\partial_{u_j} \Phi_{u_j}(t, s_{u_j}) = e_{u_j,t}(s_{u_j})\,(-g(u_j, s_{u_j}) + \partial_{u_j} s_{u_j}) \\ =&e_{u_j,t}(s_{u_j})\,\left(-g(u_j, s_{u_j}) + \partial_t c(u_j, a_j, s_{u_j-}) + (1 + \partial_x c(u_j, a_j, s_{u_j-}))\, g(u_j, s_{u_j-})\right) \\ =&e_{u_j,t}(s_{u_j})\, q(u_j, a_j, s_{u_j-})\,.\end{aligned}$$

And the same computation gives $\partial_{u_j} s_{u_{j+1}-} = e_{u_j,u_{j+1}}(s_{u_j})\, q(u_j, a_j, s_{u_j-})$. Finally

$$\begin{aligned}\partial_{u_j} s_{u_{j+1}} &= (1 + \partial_x c(u_{j+1}, a_{j+1}, s_{u_{j+1}-}))\, \partial_{u_j} s_{u_{j+1}-} \\ &= (1 + \partial_x c(u_{j+1}, a_{j+1}, s_{u_{j+1}-}))\, e_{u_j,u_{j+1}}(s_{u_j})\, q(u_j, a_j, s_{u_j-})\,.\end{aligned}$$

We assume now that $u_p \leq t < u_{p+1}$, $p \geq j+1$ and we write

$$\partial_{u_j} s_t = \partial_{u_j} \Phi_{u_p}(t, s_{u_p}) = e_{u_p,t}(s_{u_p})\, \partial_{u_j} s_{u_p}\,,$$

and the same computation gives $\partial_{u_j} s_{u_{p+1}-} = e_{u_p,u_{p+1}}(s_{u_p})\, \partial_{u_j} s_{u_p}$. Finally we have

$$\begin{aligned}\partial_{u_j} s_{u_p} &= \partial_{u_j}(s_{u_p-} + c(u_p, a_p, s_{u_p-})) = (1 + \partial_x c(u_p, a_p, s_{u_p-}))\, \partial_{u_j} s_{u_p-} \\ &= (1 + \partial_x c(u_p, a_p, s_{u_p-}))\, e_{u_{p-1},u_p}(s_{u_{p-1}})\, \partial_{u_j} s_{u_{p-1}}\,.\end{aligned}$$

And the proof is complete for the first order derivatives. The relations concerning the second order derivatives are obtained by direct computations.

B. Using the recurrence relations (17) one verifies that for every $t \in [0,T]$, $a_j \to s_t(u,a)$ is continuously differentiable and then one may differentiate in the equation (15) (this was not possible in the case of the derivatives with respect to u_j because these derivatives are not continuous). ■
As an immediate consequence of the above lemma we obtain:

Corollary 3.1 *Suppose that hypothesis 3.1 holds true and suppose that the starting point x satisfies $|x| \leq K$ for some K. Then for each $n \in \mathbb{N}$ and $T > 0$ there exists a constant $C_n(K,T)$ such that for every $0 < u_1 < \ldots < u_n < T$, $a \in \mathbb{R}^n$ and $0 \leq t \leq T$,*

$$\max_{j=1,\ldots,n} \left(|s_t| + \left|\partial_{u_j} s_t\right| + \left|\partial^2_{u_j} s_t\right| + \left|\partial_{a_j} s_t\right| + \left|\partial^2_{a_j} s_t\right| \right)(u,a) \leq C_n(K,T)\,. \tag{22}$$

Finally, we give a corollary which is useful in order to control the non degeneracy.

Corollary 3.2 *Assume that hypothesis 3.1 holds true and there exists a constant $\eta > 0$ such that for every $(t,a,x) \in [0,T] \times \mathbb{R} \times \mathbb{R}$ one has*

$$\begin{aligned} |1 + \partial_x c(t,a,x)| &\geq \eta\,, \\ |q(t,a,x)| &\geq \eta\,. \end{aligned} \tag{23}$$

Let $n \in \mathbb{N}$ be fixed. Then there exists a constant $\varepsilon_n > 0$ such that for every $j = 1,\ldots,n$ and every $(u,a) \in [0,T]^n \times \mathbb{R}^n$

$$\inf_{t > u_j} \left|\partial_{u_j} s_t(u,a)\right| \geq \varepsilon_n\,. \tag{24}$$

Proof. Since $\partial_x g$ is bounded, there exists a constant $C > 0$ such that $e_{s,t}(x) \geq e^{-CT}$ for
$0 \leq s < t \leq T$. Then one employes (18) and (19). ■

3.2 Integration by parts with respect to the amplitudes of the jumps

In this section we will use the integration by parts formula for S_t which will be regarded as a simple functional of $\Delta_i, i \in \mathbb{N}$. So, with the notation from

Section 2, we have $V_i = \Delta_i$ and $\mathcal{G} = \sigma\{T_i : i \in \mathbb{N}\}$. We recall that $J_t = n$ on $\{T_n \leq t < T_{n+1}\}$. Then, on $\{J_t = n\}$,

$$S_t = s_t(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n)\,,$$

where s_t is defined in the previous section (see (15)).

We assume that the hypothesis 3.1 and 2.1 (that is $\mathrm{E}\,(|\Delta_i|^p) < \infty$ for all $p \in \mathbb{N}$) hold true. Moreover we consider some $q_0 < q_1 < \ldots < q_{k+1}$ and we denote $I = \bigcup_{i=0}^{k}(q_i, q_{i+1})$. We assume:

Hypothesis 3.2 *The law of Δ_i is absolutely continuous on I with respect to the Lebesgue measure and has the density $p = e^{\rho}$, that is*

$$\mathrm{E}\,(f(\Delta_i)\,\mathbf{1}_I(\Delta_i)) = \int_I f(y)\, e^{\rho(y)}\, dy\,,$$

for every measurable and positive function f.
The function ρ is assumed to be continuously differentiable and bounded on I.

So hypothesis 2.2 holds true.

Since ρ is not differentiable on the whole $\mathbb{R}$, we work with the following weight. We take $\alpha \in (0, 1/2)$ and $\beta > \alpha$ and we define

$$\begin{aligned}\pi(y) &= (q_{i+1} - y)^{\alpha}\,(y - q_i)^{\alpha} \text{ for } y \in (q_i, q_{i+1})\,, \quad i = 0, \ldots, k\,,\\ &= 0 \text{ for } y \in (q_0, q_{k+1})^c\,,\end{aligned}$$

We make the following convention: if $b = q_{k+1} = +\infty$ or $a = q_0 = -\infty$, we define

$$\pi(y) = (y - q_k)^{\alpha}\,|y|^{-\beta}\,, \text{ for } y > q_k \text{ and } \pi(y) = (q_1 - y)^{\alpha}\,|y|^{-\beta}\,, \text{ for } y < q_1\,,$$

Then elementary computations show that π satisfies hypothesis 2.3.

Let $A := \{J_t = n\}$. In view of (22),
$(a_1, \ldots, a_n) \to s_t(T_1(\omega), \ldots, T_n(\omega), a_1, \ldots, a_n)$ is twice continuously differentiable and has bounded derivatives. It follows that $S_t \in S_{(n,2)}(A)$.

The differential operators which come on in the integration by parts formula are

$$D_i S_t = \partial_{a_i} s_t(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n)\,,$$

$$LS_t = -\sum_{i=1}^{n} \Big(\pi(\Delta_i)\, \partial^2_{a_i} s_t(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n) \\ +(\pi' + \pi\, \frac{\rho'}{\rho})(\Delta_i)\, \partial_{a_i} s_t(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n) \Big)\,,$$

$$\sigma_{S_t} = \sum_{i=1}^{n} \pi(\Delta_i)\ |D_i S_t|^2 = \sum_{i=1}^{n} \pi(\Delta_i)\ |\partial_{a_i} s_t(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n)|^2\,,$$

$$\gamma_{S_t} = \frac{1}{\sigma_{S_t}} = \frac{1}{\sum_{i=1}^{n} \pi(\Delta_i)\ |\partial_{a_i} s_t(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n)|^2}\,.$$

And all these quantities may be computed using (20) and (21).

The result which is used in sensitivity computations is the following.

Proposition 3.1 *Suppose that hypothesis 3.1 and 3.2 hold true and moreover, suppose that there exists a positive constant* η *such that for every* t, a, x

$$\begin{aligned} i)\, |\partial_a c(t,a,x)| \geq \eta\,, \\ ii)\, |1 + \partial_x c(t,a,x)| \geq \eta\,. \end{aligned} \tag{25}$$

Take $\alpha \in (0, 1/2)$ *and* $\beta > \alpha$*. Then, for every differentiable function* $\phi : \mathbb{R} \to \mathbb{R}$ *which has linear growth, and for every* $n \geq 1$*,*

$$\mathrm{E}(\phi'(S_t)\, \partial_x S_t\, \mathbf{1}_{\{J_t = n\}}) = \mathrm{E}(\phi(S_t)\, H_n\, \mathbf{1}_{\{J_t = n\}})\,,$$

with, on $\{J_t = n\}$

$$\begin{aligned} H_n &:= H_n(S_t, \partial_x S_t) \\ &= \partial_x S_t\, \gamma_{S_t}\, LS_t - \gamma_{S_t}\ < DS_t, D(\partial_x S_t) >_\pi - \partial_x S_t\ < DS_t, D\gamma_{S_t} >_\pi\,. \end{aligned}$$

Proof. Let $n \in \mathbb{N}^*$ be fixed. We already know that $S_t \in \mathcal{S}_{(n,2)}(A)$ with $A = \{J_t = n\}$.
Moreover, $\partial_x S_t = \partial_x s_t(T_1, \dots, T_n, \Delta_1, \dots, \Delta_n)$ and $\partial_x s_t$ is computed by the recurrence relations:

$$\partial_x s_0 = 1 \,,$$
$$\partial_x s_t = (1 + \partial_x c(u_i, a_i, s_{u_i-}))\, \partial_x s_{u_i-} + \int_{u_i}^t \partial_x g(r, s_r)\, \partial_x s_r \, dr \,, \quad u_i \le t < u_{i+1} \,.$$

Then it is easy to check that $\partial_x s_t$ and its derivatives with respect to a_i, $i = 1, \dots, n$ are bounded on the set $\{J_t = n\}$, and consequently $\partial_x S_t \in \mathcal{S}_{(n,1)}(A)$.

• Suppose that $n = 1$. We will use Corollary 2.2, so we check that the non degeneracy condition (12) holds true. One has

$$\partial_{a_1} s_t = \partial_a c(u_1, a_1, s_{u_1-}) + \int_{u_1}^t \partial_x g(r, s_r)\, \partial_{a_1} s_r \, dr \,,$$

so that, using (25),

$$|\partial_{a_1} s_t| = |\partial_a c(u_1, a_1, s_{u_1-})| \, \exp\left(\int_{u_1}^t \partial_x g(r, s_r)\right) \ge c$$

for some positive constant c. And (12) follows. We use then (13) and we notice that by our choice of π the border terms are cancelled.

• Suppose now that $n \ge 2$. In this case, we will use Theorem 2.2 and so we look at the non degeneracy condition (11). Since π is bounded, this amounts to find $\delta > 0$ such that for $i = 1, \dots, n$

$$\mathrm{E}\left[\mathbf{1}_{\{J_t = n\}} \, \left((1 + |\pi'(\Delta_i)|)\, \gamma_{S_t}^2\right)^{1+\delta}\right] < \infty \,. \tag{26}$$

We recall that $\pi(y) = \sum_{i=0}^{k} (q_{i+1} - y)^\alpha \, (y - q_i)^\alpha \, \mathbf{1}_{(q_i, q_{i+1})}(y)$, so

$$\begin{aligned} \pi'(y) &= \alpha\, (q_{i+1} - y)^{\alpha-1}\, (y - q_i)^{\alpha-1}\, (q_i - 2\,y + q_{i+1}) \quad && \text{if } y \in (q_i, q_{i+1}) \\ &= 0 && \text{if } y \in (q_0, q_{k+1})^c \,. \end{aligned}$$

We choose $\delta > 0$ such that $2\,\alpha\,(1+\delta) < 1$ and $(1-\alpha)\,(1+\delta) < 1$ (which is possible because $\alpha \in (0, 1/2)$). In particular, since ρ is bounded on I and Δ_i have finite moments of any order, this gives

$$\mathrm{E}\left(\pi(\Delta_i)^{-2\,(1+\delta)}\right) < \infty \text{ and } \mathrm{E}\left(|\pi'(\Delta_i)|^{1+\delta}\right) < \infty\,.$$

The proof of (26) is different for $i = n$ and $i = 1, \ldots, n-1$.
Take first $i < n$. One has $|\partial_{a_n} s_t| = |\partial_a c(u_n, a_n, s_{u_n -})| \,\exp\left(\int_{u_n}^t \partial_x g(r, s_r)\right) \geq c$, so $\sigma_{S_t} \geq c^2\,\pi^2(\Delta_n)$. Since Δ_i and Δ_n are independent,

$$\begin{aligned}\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\left((1+|\pi'(\Delta_i)|)\,\gamma_{S_t}^2\right)^{1+\delta}\right] &\leq c^{-2}\,\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\left((1+|\pi'(\Delta_i)|)\,\pi^{-2}(\Delta_n)\right)^{1+\delta}\right]\\ &= c^{-2}\,\mathrm{E}\left(\pi^{-2\,(1+\delta)}(\Delta_n)\right)\,\mathrm{E}\left[(1+|\pi'(\Delta_i)|)^{1+\delta}\right]\\ &< \infty\,.\end{aligned}$$

Take now $i = n$ and write $\sigma_{S_t} \geq \pi^2(\Delta_{n-1})\;|D_{n-1}S_t|^2$. A simple computation shows that

$$\partial_{a_{n-1}} s_t = \partial_a c(u_{n-1}, a_{n-1}, s_{u_{n-1}^-})\,(1+\partial_x c(u_n, a_n, s_{u_n^-}))\,\exp\left(\int_{u_{n-1}}^t \partial_x g(r, s_r)\,dr\right)\,.$$

Using (25), we obtain $\partial_{a_{n-1}} s_t \geq c > 0$ so that $\sigma_{S_t} \geq c^2\,\pi^2(\Delta_{n-1})$. Consequently

$$\begin{aligned}&\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\left((1+|\pi'(\Delta_n)|)\,\gamma_{S_t}^2\right)^{1+\delta}\right]\\ \leq& c^{-2}\,\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\left((1+|\pi'(\Delta_i)|)\,\pi^{-2}(\Delta_n)\right)^{1+\delta}\right]\\ =& c^{-2}\,\mathrm{E}\left(\pi^{-2\,(1+\delta)}(\Delta_{n-1})\right)\,\mathrm{E}\left[(1+|\pi'(\Delta_n)|)^{1+\delta}\right] < \infty\,.\end{aligned}$$

And the proof is complete. ■

Remark 3.1 *Suppose now that ρ is differentiable on the whole $\mathbb{R}$. Then we take no weight, $\pi = 1$ and hypothesis (25,i) gives $\sigma_{S_t} \geq c$, on $\{J_t = n\}$ for all $n \in \mathbb{N}^*$. So the above integrability problems disappear. In particular, hypothesis (25,ii) is no more necessary. This case is discussed in* [1].

3.3 Integration by parts with respect to the jump times

In this section we differentiate with respect to the jump times T_i, $i \in \mathbb{N}$. It is well known (see [2]) that conditionally to $\{J_t = n\}$, the law of the vector $(T_1, \ldots, T_n)$ is absolutely continuous with respect to the Lebesgue measure and has the following density

$$p(\omega, t_1, \ldots, t_n) = \frac{n!}{t^n} \mathbf{1}_{\{0<t_1<\ldots<t_n<t\}}(t_1, \ldots, t_n)\, \mathbf{1}_{\{J_t(\omega)=n\}} .$$

In particular, for a given $i = 1, \ldots, n$, conditionally to $\{J_t = n\}$ and to $\{T_j,\ j \neq i\}$, T_i is uniformly distributed on $[T_{i-1}(\omega), T_{i+1}(\omega)]$. So it has the density (with the convention $T_0 = 0, T_{n+1} = t$)

$$p_i(\omega, u) = \frac{1}{T_{i+1}(\omega) - T_{i-1}(\omega)} \mathbf{1}_{[T_{i-1}(\omega), T_{i+1}(\omega)]}(u)\, du, \quad i = 1, \ldots, n .$$

Since p_i is not differentiable with respect to u, we have to use the following weights:

$$\pi_i(\omega, u) = (T_{i+1}(\omega) - u)^\alpha\, (u - T_{i-1}(\omega))^\alpha\, \mathbf{1}_{[T_{i-1}(\omega), T_{i+1}(\omega)]}(u)\,, \quad i = 1, \ldots, n$$

with $\alpha \in (0, 1/2)$.

In order to fit in the notation from the first section we take $V_i = T_i$, $k_i = 2$, $t_i^1 = T_{i-1}$ and $t_i^2 = T_{i+1}$. We have $\mathcal{G} = \sigma\,(\Delta_i, i \in \mathbb{N}) \vee \sigma(J_t)$. We fix n and we work on the set $A = \{J_t = n\}$. Then hypothesis 2.1, 2.2 and 2.3 hold true and
$S_t = s_t(T_1, \ldots, T_n, \Delta_1(\omega), \ldots, \Delta_n(\omega))$. So S_t is a simple functional and the function which represents S_t is twice differentiable and has continuous derivatives on the whole $\mathbb{R}^n$. The differential operators are

$$\begin{aligned} D_i S_t &= \partial_{u_i} s_t(T_1, \ldots, T_n, \Delta_1(\omega), \ldots, \Delta_n(\omega))\,, \\ \sigma_{S_t} &= \sum_{i=1}^n \pi_i(\omega, T_i)\ \left|\partial_{u_i} s_t(T_1, \ldots, T_n, \Delta_1(\omega), \ldots, \Delta_n(\omega))\right|^2\,, \\ L_i S_t &= -\left(\pi_i'\, \partial_{u_i} s_t + \pi_i\, \partial_{u_i}^2 s_t)(T_1, \ldots, T_n, \Delta_1(\omega), \ldots, \Delta_n(\omega)\right)\,. \end{aligned}$$

And all these quantities may be computed using Lemma 3.1.

Proposition 3.2 *Suppose that hypothesis 3.1 holds true. Suppose moreover that (23) is satisfied, that is*

$$|q(t,a,x)| \geq \eta > 0\,,$$
$$|(1+\partial_x c)(t,a,x)| \geq \eta > 0\,,$$

for some $\eta > 0$*. Take* $\alpha \in (0, \frac{1}{2})$*. Then for every* $n \geq 4$ *and every continuously differentiable function* ϕ *which has linear growth we have*

$$\mathrm{E}(\phi'(S_t)\,\partial_x S_t\,\mathbf{1}_{\{J_t=n\}}) = \mathrm{E}(\phi(S_t)\,H_n\,\mathbf{1}_{\{J_t=n\}})\,,$$

with, on $\{J_t = n\}$

$$\begin{aligned} H_n &:= H_n(S_t, \partial_x S_t) \\ &= \partial_x S_t\,\gamma_{S_t}\,LS_t - \gamma_{S_t} < DS_t, D(\partial_x S_t) >_\pi - \partial_x S_t < DS_t, D\gamma_{S_t} >_\pi \;. \end{aligned}$$

Proof. From (22) we know that $s_t(u,a)$ and its derivatives up to order two with respect to u_i, $i = 1, \ldots, n$ are bounded on $[0,T]^n$. It follows that $S_t \in \mathcal{S}_{(n,2)}(A)$.

Since π_i are bounded, the non degeneracy condition (11) amounts to

$$\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\gamma_{S_t}^{2(1+\eta)}\right] < \infty \text{ and } \mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\gamma_{S_t}^{2(1+\eta)}\,|\pi_i'(T_i)|^{1+\eta}\right] < \infty\,,$$

for some $\eta > 0$.
Let us prove that $\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\gamma_{S_t}^{2(1+\eta)}\,|\pi_i'(T_i)|^{1+\eta}\right] < \infty$. We denote $\delta_i = T_i - T_{i-1}$, $i = 1, \ldots, n$ so that $\pi_i = \delta_i^\alpha\,\delta_{i+1}^\alpha$. We use (24) in order to obtain

$$\sigma_{S_t} = \sum_{i=1}^n \delta_{i+1}^\alpha\,\delta_i^\alpha\;|\partial_{u_i} s_t(T_1, \ldots, T_n, \Delta_1, \ldots, \Delta_n)|^2 \geqslant \varepsilon^2 \sum_{i=1}^n \delta_{i+1}^\alpha\,\delta_i^\alpha\,.$$

Since $\pi_i'(T_i) = \alpha(-\delta_{i+1}^{\alpha-1}\,\delta_i^\alpha + \delta_{i+1}^\alpha\,\delta_i^{\alpha-1})$, we have to check that, for every $i = 1, \ldots, n$

$$\mathrm{E}\left[\left(\delta_i^{\alpha-1}\delta_{i+1}^\alpha + \delta_i^\alpha\delta_{i+1}^{\alpha-1}\right)^{1+\eta}\left(\sum_{j=1}^n \delta_{j+1}^\alpha\,\delta_j^\alpha\right)^{-2(1+\eta)}\right] < \infty\,.$$

Take $i = 1$ and write

$$\mathrm{E}\left[\left(\delta_1^{\alpha-1}\,\delta_2^{\alpha}\right)^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)}\right] \le \mathrm{E}\left[\left(\delta_1^{\alpha-1}\,\delta_2^{\alpha}\right)^{1+\eta}\left(\delta_2^{\alpha}\,\delta_3^{\alpha}\right)^{-2(1+\eta)}\right]$$
$$= \mathrm{E}\left(\delta_1^{(\alpha-1)(1+\eta)}\right)\mathrm{E}\left(\delta_2^{-\alpha(1+\eta)}\right)\mathrm{E}\left(\delta_3^{-2\alpha(1+\eta)}\right).$$

Since δ_i is exponentially distributed of parameter $\mu(\mathbb{R})$, a necessary and sufficient condition in order to have $\mathrm{E}(\delta_i^{-p}) < \infty$ is $p < 1$. Then, we choose η small enough such that $(1-\alpha)(1+\eta) < 1$ and $\alpha(1+\eta) < 2\alpha(1+\eta) < 1$ (which is possible because $0 < \alpha < 1/2$), and we have $\mathrm{E}\left(\delta_1^{(\alpha-1)(1+\eta)}\right) < \infty$, $\mathrm{E}\left(\delta_2^{-\alpha(1+\eta)}\right) < \infty$ and $\mathrm{E}\left(\delta_3^{-2\alpha(1+\eta)}\right) < \infty$. So

$$\mathrm{E}\left[\left(\delta_1^{\alpha-1}\,\delta_2^{\alpha}\right)^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)}\right] < \infty\,.$$

We write now

$$\mathrm{E}\left[\left(\delta_1^{\alpha}\,\delta_2^{\alpha-1}\right)^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)}\right] \le \mathrm{E}\left[\left(\delta_1^{\alpha}\,\delta_2^{\alpha-1}\right)^{1+\eta}\left(\delta_3^{\alpha}\,\delta_4^{\alpha}\right)^{-2(1+\eta)}\right]$$
$$= \mathrm{E}\left(\delta_2^{(\alpha-1)(1+\eta)}\right)\mathrm{E}\left(\delta_1^{\alpha(1+\eta)}\right)\mathrm{E}\left(\delta_3^{-2\alpha(1+\eta)}\right), \mathrm{E}\left(\delta_4^{-2\alpha(1+\eta)}\right).$$

Recalling that δ_i has finite moments of any order, by the choice of η, we obtain

$$\mathrm{E}\left[\left(\delta_1^{\alpha}\,\delta_2^{\alpha-1}\right)^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)}\right] < \infty\,.$$

Since $n \ge 4$, the same argument works for $i = 2, \ldots, n$, and leads to $\mathrm{E}\left[\mathbf{1}_{\{J_t=n\}}\,\gamma_{S_t}^{2\,(1+\eta)}\right] < \infty$.
■

Remark 3.2 *Suppose that* $n = 2$. *Then*

$$(\delta_1^{\alpha-1}\,\delta_2^{\alpha})^{1+\eta}\left(\sum_{j=1}^{n}\delta_{j+1}^{\alpha}\,\delta_j^{\alpha}\right)^{-2(1+\eta)} = (\delta_1^{\alpha-1}\,\delta_2^{\alpha})^{1+\eta}\,\delta_2^{-2\alpha(1+\eta)}\,(\delta_1^{\alpha}+\delta_3^{\alpha})^{-2(1+\eta)}$$

$$=\delta_2^{-\alpha\,(1+\eta)}\times\left(\delta_1^{-(\alpha+1)\,(1+\eta)}+\delta_3^{-2\,\alpha\,(1+\eta)}\,\delta_1^{-(1-\alpha)\,(1+\eta)}\right),$$

and this quantity is not integrable for $\alpha > 0$, $\eta > 0$.

Remark 3.3 *For* $n = 1$, *one may use Corollary 2.2 in order to obtain an integration by parts formula.*
But for $n = 2$ *and* $n = 3$, *we are not able to handle the non degeneracy problem. In our numerical examples, we will use the noise coming from the amplitudes of the jumps in order to solve the problem for* $n = 2$ *and* $n = 3$.

3.3.1 Examples

• We consider the geometrical model:

$$dS_t = S_t\,(r\,dt + \alpha(t,a)\,dN(t,a))\,.$$

In this case $g(t,x) = x\,r$ and $c(t,a,x) = x\,\alpha(t,a)$. It follows that

$$q(t,a,x) = x\,\partial_t\alpha(t,a) + x\,r\,\alpha(t,a) + x\,r - r(x + x\,\alpha(t,a)) = x\,\partial_t\alpha(t,a)\,.$$

In particular, if α does not depend on the time, the model is degenerated from the point of view of the jump times. The non degeneracy condition reads

$$|\partial_t\alpha(t,a)| \geq \varepsilon\,.$$

On the other hand the condition $|1 + \partial_x c(t,a,x)| \geq \eta$ reads

$$|1 + \alpha(t,a)| \geq \eta\,.$$

• We consider now a Vasicek type model:

$$dS_t = S_t\,r\,dt + \alpha(t,a)\,dN(t,a)\,.$$

In this case $g(t,x) = x\,r$ and $c(t,a,x) = \alpha(t,a)$. It follows that

$$q(t,a,x) = \partial_t \alpha(t,a) + x\,r - r(x + \alpha(t,a)) = \partial_t \alpha(t,a) - r\,\alpha(t,a)\,.$$

Suppose that α does not depend on the time so that $\partial_t \alpha = 0$. Then the non degeneracy assumption reads

$$|\alpha(a)| \geq \varepsilon\,.$$

The condition $|1 + \partial_x c(t,a,x)| \geq \eta$ reads

$$|1 + \alpha(a)| \geq \eta\,.$$

3.4 Mixed calculus

In this section we briefly present the differential calculus with respect to both noises coming from the jump amplitudes and from the jump times. So the random variables will be $V_i = T_i$, $i = 1, \dots, n$ and $V_{n+i} = \Delta_i, i = 1, \dots, n$ and $\mathcal{G} = \sigma(J_t)$. We put together the results from the two previous sections (and we keep the notation there). We still assume hypothesis 3.1 and 3.2. The differential operators are

$$\begin{aligned} D_i S_t &= \partial_{u_i} s_t(u_1, \dots, u_n, \Delta_1(\omega), \dots, \Delta_n(\omega)), \quad i = 1, \dots, n \\ &= \partial_{a_{i-n}} s_t(T_1, \dots, T_n, \Delta_1, \dots, \Delta_n), \qquad i = n+1, \dots, 2n\,. \end{aligned}$$

We will use the weights defined in the previous sections, namely

$$\begin{aligned} \pi_i(\omega, u) &= (T_{i+1}(\omega) - u)^\alpha\,(u - T_{i-1}(\omega))^\alpha\,\mathbf{1}_{[T_{i-1}(\omega), T_{i+1}(\omega)]}(u)\,, \quad i = 1, \dots, n \\ \pi_i(y) = \pi(y) &= \sum_{p=1}^{k-1} (q_{p+1} - y)^\alpha\,(y - q_p)^\alpha\,\mathbf{1}_{(q_p, q_{p+1})}(y)\,, \quad i = n+1, \dots, 2n\,, \end{aligned}$$

where $\alpha \in (0, \frac{1}{2})$.

We have

$$\begin{aligned} L_i S_t &= -\left(\pi_i'(T_i)\,\partial_{u_i} s_t + \pi_i(T_i)\,\partial^2_{u_i} s_t\right) && \text{for } i = 1, \dots, n\,, \\ L_i S_t &= -\left(\pi(\Delta_i)\,\partial^2_{a_i} s_t + (\pi' + \pi\,\rho')(\Delta_i)\,\partial_{a_i} s_t\right) && \text{for } i = n+1, \dots, 2n\,. \end{aligned}$$

Finally, $LS_t = \sum_{i=1}^{2n} L_i S_t$. All these quantities may be computed using the formulas from the previous sections.

Theorem 3.1 *Suppose that hypothesis 3.1 and 3.2 hold true and*

$$i)\ |q(t,a,x)| \geq \varepsilon > 0\,,$$
$$ii)\ |\partial_a c(t,a,x)| \geq \varepsilon > 0$$
$$iii)\ |(1+\partial_x c)(t,a,x)| \geq \varepsilon > 0\,.$$

Then for every $n \geq 1$ and every continuously differentiable function ϕ which has linear growth we have

$$\mathrm{E}(\phi'(S_t)\partial_x S_t\,\mathbf{1}_{\{J_t=n\}}) = \mathrm{E}(\phi(S_t)\,H_n\,\mathbf{1}_{\{J_t=n\}})\,,$$

with, on $\{J_t = n\}$

$$\begin{aligned} H_n &:= H_n(S_t, \partial_x S_t) \\ &= \partial_x S_t\,\gamma_{S_t}\,LS_t - \gamma_{S_t} < DS_t, D(\partial_x S_t) >_\pi - \partial_x S_t < DS_t, D\gamma_{S_t} >_\pi\ . \end{aligned}$$

Proof. We write

$$\begin{aligned} \sigma_{S_t} &\geq \left(\pi_n(\omega,T_n)\ |\partial_{u_n} s_t|^2 + \pi(\Delta_n)\ |\partial_{a_n} s_t|^2\right)(T_1,\dots,T_n,\Delta_1,\dots,\Delta_n) \\ &\geq \varepsilon^2\ (\pi_n(\omega,T_n) + \pi(\Delta_n))\ , \end{aligned}$$

for some $\varepsilon > 0$. Then, using the same triks as in the proves of Propositions 3.2 and 3.1, one shows that the non degeneracy (11) condition holds true. ■

Remark 3.4 *Notice that the non degeneracy condition holds true for every n (including $n = 2$) because we may use the noises coming from the jump times and the jump amplitudes in the same time.*

4 Numerical results

4.1 Malliavin estimator

In this section, we compute the Delta of two European options: call option with payoff $\phi(x) = (x-K)_+$ and digitial option with payoff $\phi(x) = \mathbf{1}_{x \geq K}$. The asset $(S_t)_{t\geq 0}$ follows a one-dimensional pure jump diffusion process. We

use the notations from the beginning of section 3.
We deal with two different pure jump diffusion models. The first one is a Vasicek type model:

$$S_t = x - \int_0^t r\,(S_u - \alpha)\,du + \sum_{i=1}^{J_t} \sigma\,\Delta_i\,, \tag{27}$$

and the second one is a geometrical model:

$$S_t = x + \int_0^t r\,S_u\,du + \sigma \sum_{i=1}^{J_t} S_{T_i^-}\,\Delta_i\,. \tag{28}$$

In both models, we take $\Delta_i \sim \mathcal{N}(0,1)$, $i \geq 1$. That is for all $i \geq 1$, Δ_i has the dendity $p(x) = \frac{1}{\sqrt{2\,\pi}}\,e^{\rho(x)}$, with $\rho(x) = -\frac{x^2}{2}$. Notice that even if ρ is not bounded on $\mathbb{R}$, the integration by parts formula holds by a truncature argument.

Our aim is to compute $\partial_x \mathrm{E}(\phi(S_T))$ using the integration by parts formula derived in the previous sections. We write

$$\begin{aligned}\partial_x \mathrm{E}(\phi(S_T)) =& \mathrm{E}\left(\phi'(S_T)\,\partial_x S_T\right)\\ =& \mathrm{E}\left(\phi'(S_T)\,\partial_x S_T\,\mathbf{1}_{\{J_T=0\}}\right)\\ &+ \sum_{n=1}^{\infty} \mathrm{E}\left(\phi'(S_T)\,\partial_x S_T\,\mathbf{1}_{\{J_T=n\}}\right)\,,\end{aligned}$$

For $n \geq 1$, we use the integration by parts formula on $\{J_T = n\}$, and we obtain

$$\mathrm{E}\left(\phi'(S_T)\,\partial_x S_T\,\mathbf{1}_{\{J_T=n\}}\right) = \mathrm{E}\left(\phi(S_T)\,H_n\,\mathbf{1}_{\{J_T=n\}}\right)\,,$$

where H_n is a weight involving Malliavin derivatives of S_T and $\partial_x S_T$. Summing over $n = 1, 2, \ldots$, we obtain

$$\begin{aligned}\partial_x \mathrm{E}(\phi(S_T)) = & \mathrm{E}\left(\phi'(S_T)\,\partial_x S_T\,\mathbf{1}_{\{J_T=0\}}\right)\\ &+ \mathrm{E}\left(\phi(S_T)\,H_{J_T}(S_T, \partial_x S_T)\,\mathbf{1}_{\{J_T\geq 1\}}\right)\,.\end{aligned}$$

In order to compute the two terms in the right hand side of the above equality, we proceed as follows. On $\{J_T = 0\}$, there is no jump on $]0,T]$, thus S_T and $\partial_x S_T$ solve some deterministic integral equation. In the examples that we considered in this paper, the solution of these equations are explicit, so this term is explicitly known. We may use the finite difference method. For the computation of the second term, we use a Monte-Carlo algorithm. We simulate a sample $\big((T^k_n)_{n\in\mathbb{N}}, (\Delta^k_n)_{n\in\mathbb{N}}\big)$, $k = 1, \ldots, M$ of the times and the amplitudes of the jumps, and we compute the corresponding J^k_t, S^k_T, and $H^k_{J^k_T}$. Then we write

$$\mathrm{E}\left(\phi(S_T)\, H_{J_T}(S_T, \partial_x S_T)\, \mathbf{1}_{\{J_T\geq 1\}}\right) \simeq \frac{1}{M} \sum_{k=1}^{M} \phi(S^k_T)\, H^k_{J^k_T}\, \mathbf{1}_{\{J^k_T \geq 1\}}\,.$$

We compute now the Malliavin weights $H^k_{J_T}(S^k_T, \partial_x S^k_T)$ for our examples. One may use Lemma 3.1, but in the particular cases that we discuss here, we have explicit solutions. So direct computations are much easier.

• We first study the diffusion process defined by (27). We have an explicit expression of S_T on $\{J_T = n\}$:

$$S_T = x\, e^{-r\,T} + \alpha\,(1 - e^{-r\,T}) + \sigma \sum_{j=1}^{n} \Delta_j\, e^{-r\,(T-T_j)}\,. \tag{29}$$

We may use integration by parts with respect to the jump amplitudes, to the jump times or to both of them.

∗ Jump amplitudes: H_{J_T} has been calculated in [1]. Since Δ_i is Gaussian distributed for all i, the weight is $\pi(\omega, \Delta_i) = 1$, and on $\{J_T = n\}$

$$H_n(S_T, \partial_x S_T) = \frac{\sum_{j=1}^{n} e^{r\,T_j}\, \Delta_j}{\sigma \sum_{j=1}^{n} e^{2\,r\,T_j}}\,. \tag{30}$$

∗ Jump times: suppose that $n \geq 4$ and $J_T = n$. We use the weights $\pi_i(\omega, T_i) = (T_{i+1} - T_i)^{\alpha}\, (T_i - T_{i-1})^{\alpha}$ and we have $\pi'_i = \alpha\, \delta^{\alpha-1}_{i+1}\, \delta^{\alpha-1}_i\, (\delta_{i+1} - \delta_i)$, where $\delta_i = T_i - T_{i-1}$. Then

$$D_i S_T = \sigma\, \Delta_i\, r\, e^{-r\,(T-T_i)}\,,$$

and

$$L_i S_T = -\sigma\, r\, \Delta_i\, e^{-r\,(T-T_i)} \,\left(r\,\pi_i + \alpha\,(\delta_{i+1}\,\delta_i)^{\alpha-1}\,(\delta_{i+1}-\delta_i)\right)\,,$$

$$\sigma_{S_T} = \sum_{i=1}^{n} \pi_i\,(\sigma\, r)^2\,\Delta_i^2\, e^{-2\,r\,(T-T_i)}\,.$$

We denote by

$$\begin{aligned} A_j &= \alpha\,(\delta_{j+1}\,\delta_j)^{\alpha-1}\,\Delta_j^2\, e^{2\,r\,T_j}\,, \\ B_j &= \Delta_j^2\, e^{2\,r\,T_j}\,\left[2\,r\,\pi_j + \alpha\,(\delta_{j+1}\,\delta_j)^{\alpha-1}\,\,(\delta_{j+1}-\delta_j)\right]\,. \end{aligned}$$

Then

$$D_j \sigma_{S_T} = (\sigma\, r)^2\, e^{-2\,r\,T}\,(A_{j-1}\,\delta_{j-1} - A_{j+1}\,\delta_{j+2} + B_j)\,.$$

Moreover $\partial_x S_T = e^{-r\,T}$ so that $D_i \partial_x S_T = 0$ for all $i = 1, \ldots, n$.

We have now the expression of all the terms involved in H_n, we obtain

$$H_n = \frac{\sum_{i=1}^{n} \Delta_i\, e^{r\,T_i}\,\,(r\,\pi_i + \alpha\,(\delta_{i+1}\,\delta_i)^{\alpha-1}\,(\delta_{i+1}-\delta_i))}{\sigma\, r\,\hat{\sigma}} - \frac{\sum_{i=1}^{n} \pi_i\,\Delta_i\, e^{r\,T_i}\,\,(A_{i-1}\,\delta_{i-1} - A_{i+1}\,\delta_{i+2} + B_i)}{\sigma\, r\,\hat{\sigma}^2}\,, \quad (31)$$

where $\hat{\sigma} = \sum_{i=1}^{n} \pi_i\,\Delta_i^2\, e^{2\,r\,T_i}$.

For $n = 1$, 2, 3, we use integration by parts with respect to Δ_1 only. Then, similar computations give:

$$H_n = \frac{e^{-r\,T_1}}{\sigma\,\Delta_1}\,.$$

• We study the jump diffusion process defined by (28). On $\{J_T = n\}$, we have

$$S_T = x\, e^{r\,T}\,\prod_{j=1}^{n}(1+\sigma\,\Delta_j)\,.$$

We may not use integration by parts with respect to the jump times because S_T depends on $T_1, \ldots, T_k$ by means of J_t only. So we use integration by parts formula with respect the jump amplitudes only. On $\{J_T = n\}$, the Malliavin weight is in this case (see [1]):

$$H_n(S_T, \partial_x S_T) = \frac{B_\sigma}{\sigma\, x\, A_\sigma} + \frac{1}{x} - \frac{2\, C_\sigma}{x\, A_\sigma^2}\,, \qquad (32)$$

where $A_\sigma = \sum_{j=1}^{n} \frac{1}{(1+\sigma\,\Delta_j)^2}$, $B_\sigma = \sum_{j=1}^{n} \frac{\Delta_j}{(1+\sigma\,\Delta_j)}$ and $C_\sigma = \sum_{j=1}^{n} \frac{1}{(1+\sigma\,\Delta_j)^4}$.

5 Numerical experiments

In this section we present several numerical experiments in order to compare the Malliavin approach to the finite difference method. We use the Geometrical model and the Vasicek model and in the second model we use the Malliavin calculus with respect to the amplitudes of the jump and to the jump times. We also look at two types of payoff's: call options and digital options.
The comparison is illustrated by some graphs on one hand and by empirical variance tables on the other hand. In these tables we give the empirical variance of the two estimators denoted by Var Mall estimator and Var Diff estimator and we compare them. We also mention in our tables the value of the volatility σ that we use and the corresponding variance of the underlying, denoted by $Variance(S_t)$. We choose the parameter σ in the following way:

- For the geometrical model, the variance of S_t is

$$Variance(S_t) = x^2\, e^{2\,r\,t}\left(e^{\sigma^2\,\lambda\,t} - 1\right)\,.$$

  We take $\lambda = 1$, $r = 0.1$, $T = 5$ and $x = 100$. Then, for $\sigma \in [0.1, 0.6]$, $1393.69 \leq Variance(S_T) \leq 137264$.

- For the Vasicek type model, we have

$$Variance(S_t) = 2\,\alpha\, e^{-2\,r\,t}\,(x-\alpha) + \frac{\lambda\,\sigma^2}{2\,r}\left(1 - e^{-2\,r\,t}\right)\,.$$

  We take $\lambda = 1$, $r = 0.1$, $T = 5$, $\alpha = 10$ and $x = 100$. Then, for $\sigma \in [16, 50]$(notice that the practiciens use $\sigma = 20$ to 30 in the Vasicek model), $1471.3 \leq Variance(S_T) \leq 8563.69$.

In all our simulations we have used a variance reduction method based on localization (analogues to the one introduced in [12] and [11]). We use the following abreviations:

- AJ : Amplitude of the jumps
- JT : Jump times
- FD : Finite differences
- G : Geometrical model
- V : Vasicek model
- Call : Call option
- Dig : digital option.

Then (G/Dig/AJ) means that we are in the Geometrical model (G) with a digital option (Dig) and we use an algorithm based on the amplitudes of the jumps (AJ). (G/Dig/AJ) versus (G/Dig/FD) means that we compare these two estimators.

5.0.1 (G/Call/AJ) versus (G/Call/FD)

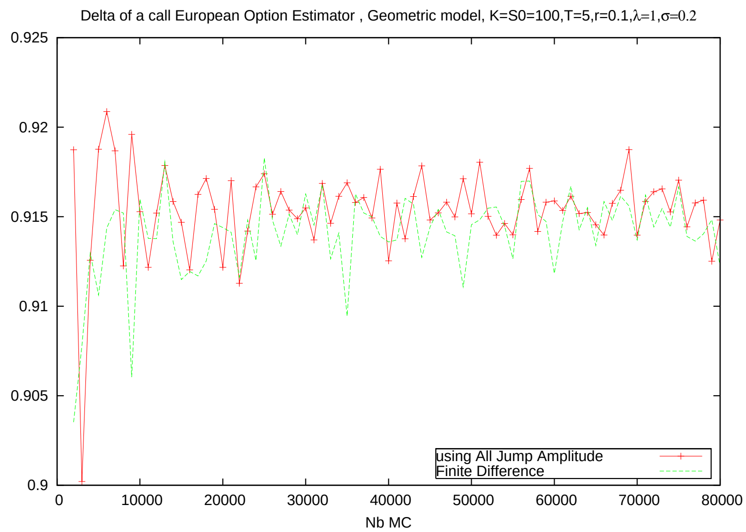

Figure 1: Geometrical model. Delta of an European call option using Malliavin calculus based on the jump amplitudes, and the finite difference method.

σ	$Variance(S_T)$	$VarMall$	$VarDiff$
0.1	1405.06	0.0717191	0.0697713
0.2	6183.72	0.345008	0.355809
0.3	16005.5	0.764091	0.752288
0.4	42590.8	1.33515	1.35804
0.5	69018.7	3.54353	2.87681
0.6	130425	4.65282	3.79088

Table 1: Variance of the Malliavin estimator of the delta and variance of the FD for Call option.

Both the graph and the variance table show that the two methods give comparable results.

5.0.2 (G/Dig/AJ) versus (G/Dig/FD)

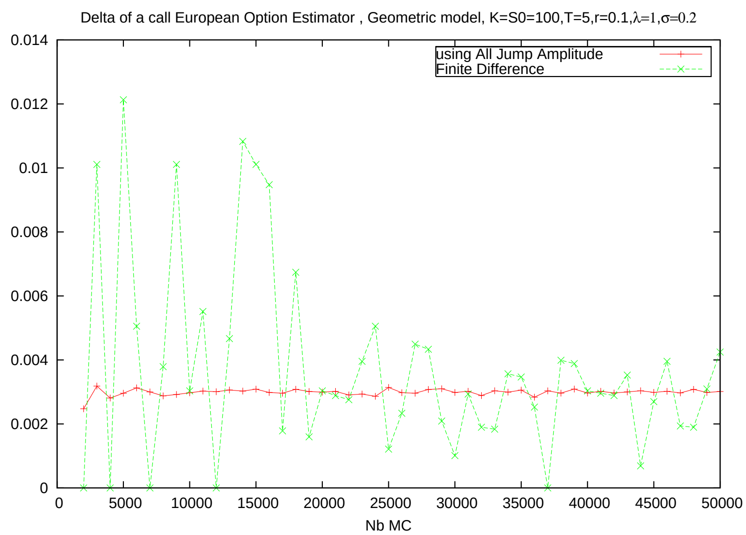

Figure 2: Geometrical model. Delta of an European digital option using Malliavin calculus based on the jump amplitudes, and finite difference method.

σ	$Variance(S_T)$	$VarMall$	$VarDiff$
0.1	1405.06	0.000263425	0.00102718
0.2	6183.72	0.000917207	0.00164801
0.3	16005.5	0.000885212	0.00117345
0.4	42590.8	0.000685313	0.0013196
0.5	69018.7	0.000531118	0.000917399
0.6	130425	0.000310461	0.0003307

Table 2: variance of the Malliavin estimator of the delta and variance of the FD for Digital option.

Both the graph and the variance table show that for a digital option the algorithm based on the Malliavin calculus is significantly better than the one based on the finite difference.

5.0.3 (V/Call/AJ) versus (V/Call/FD)

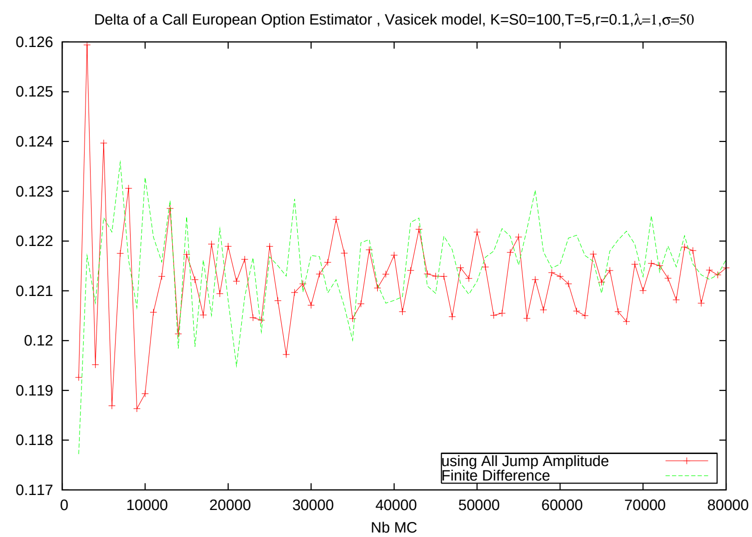

Figure 3: Vasicek model. Delta of an European Call option using Malliavin calculus based on the jump amplitudes, and finite difference method for a digital.

σ	$Variance(S_T)$	$VarMall$	$VarDiff$
15.8114	796.241	0.0106426	0.0300379
16.6667	897.577	0.0115955	0.0298567
17.6777	991.453	0.013123	0.0298904
18.8982	1134.11	0.0144516	0.0299574
20.4124	1313.42	0.0162378	0.029862
22.3607	1584.9	0.0178726	0.0298987
25	1967.53	0.0202055	0.0299007
28.8675	2604.22	0.0224265	0.0299651
35.3553	3961.31	0.0253757	0.0297775
50	7890.4	0.0287716	0.0299749

Table 3: Vasicek model. Variance of the Malliavin estimator of the delta and variance of the FD one for Call option.

Both the graph and the variance table show that the two methods give comparable results.

5.0.4 (V/Dig/AJ) versus (V/Dig/FD)

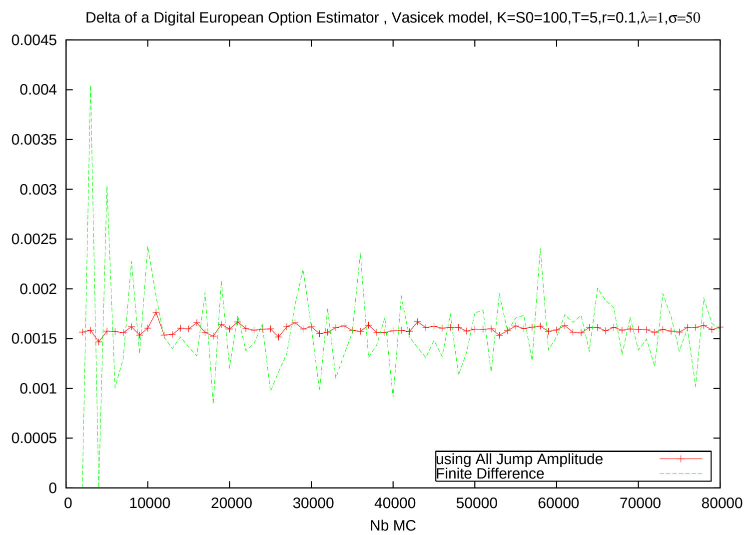

Figure 4: Vasicek model. Delta of an European Digital option using Malliavin calculus based on the jump amplitudes, and finite difference method for a digital option.

σ	$Variance(S_T)$	$VarMall$	$VarDiff$
15.8114	796.241	$7.18878e-005$	0.00514743
16.6667	897.577	$7.3629e-005$	0.00459619
17.6777	991.453	$7.85552e-005$	0.00496369
18.8982	1134.11	$8.14005e-005$	0.00477995
20.4124	1313.42	$8.1627e-005$	0.00386111
22.3607	1584.9	$8.06193e-005$	0.00496369
25	1967.53	$7.94341e-005$	0.0062497
28.8675	2604.22	$7.5835e-005$	0.00551488
35.3553	3961.31	$6.95225e-005$	0.00459619
50	7890.4	$5.64325e-005$	0.00533116

Table 4: Vasicek model. Variance of the Malliavin estimator of the delta and variance of the FD one for Digital option.

Both the graph and the variance table show that for a digital option the algorithm based on the Malliavin calculus is significantly better than the one based on the finite difference.

5.0.5 (V/Call/JT) versus (V/Call/FD)

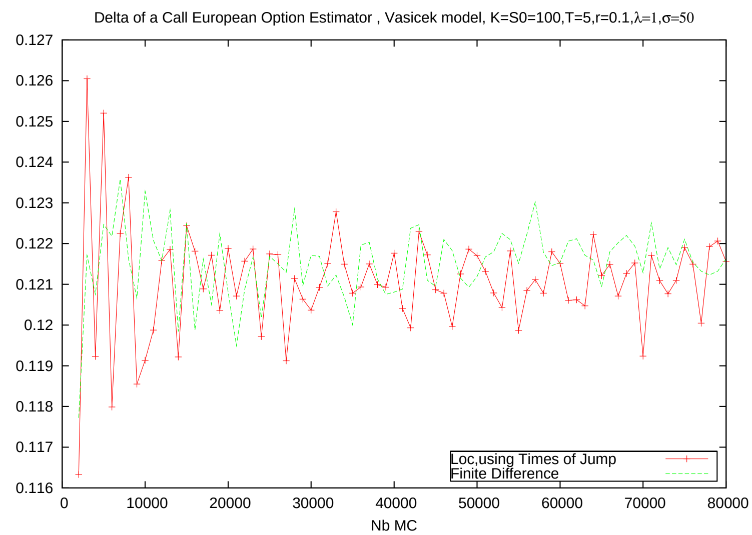

Figure 5: Vasicek model. Delta of an European Call option using Malliavin calculus based on the jump times, and finite difference method.

σ	$Variance(S_T)$	$VarMall$	$VarDiff$
15.8114	796.241	0.0285123	0.0300379
16.6667	897.577	0.0417219	0.0298567
17.6777	991.453	0.0400695	0.0298904
18.8982	1134.11	0.0410136	0.0299574
20.4124	1313.42	0.0433065	0.029862
22.3607	1584.9	0.0400481	0.0298987
25	1967.53	0.0407136	0.0299007
28.8675	2604.22	0.0362728	0.0299651
35.3553	3961.31	0.0343158	0.0297775
50	7890.4	0.0333298	0.0299749

Table 5: variance of the Malliavin Times of jump estimator of the delta and variance of the FD for Call option in the Vasicek model.

Both the graph and the variance table show that the two methods give comparable results.

5.0.6 (V/Dig/JT) versus (V/Dig/FD)

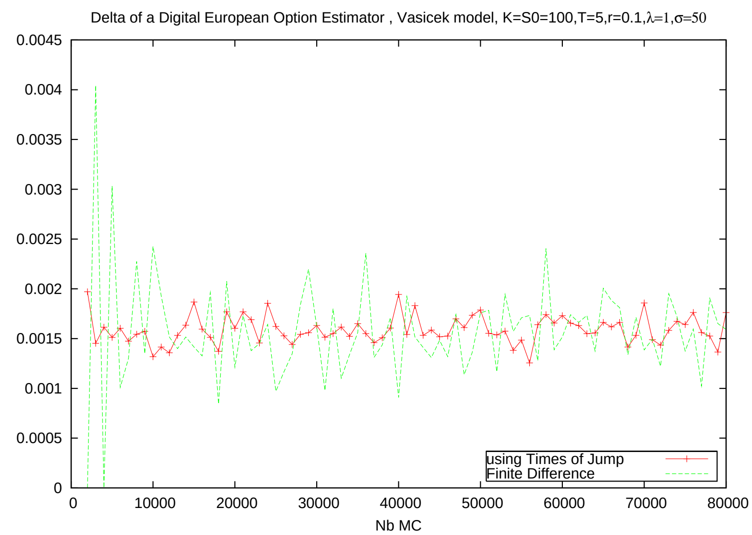

Figure 6: Vasicek model. Delta of an European Digital option using Malliavin calculus based on the jump times, and finite difference method.

σ	$Variance(S_T)$	$VarMall$	$VarDiff$
15.8114	796.241	0.00144622	0.00514743
16.6667	897.577	0.00254652	0.00459619
17.6777	991.453	0.0018011	0.00496369
18.8982	1134.11	0.0109864	0.00477995
20.4124	1313.42	0.00177648	0.00386111
22.3607	1584.9	0.00152777	0.00496369
25	1967.53	0.0013786	0.0062497
28.8675	2604.22	0.00100181	0.00551488
35.3553	3961.31	0.000617271	0.00459619
50	7890.4	0.000373802	0.00533116

Table 6: Vasicek model. Variance of the Malliavin Times of jump estimator of the delta and variance of the FD for Digital option.

Both the graph and the variance table show that for a digital option the algorithm based on the Malliavin calculus is significantly better than the one based on the finite difference.

5.0.7 (V/Call/AJ) versus (V/Call/JT) versus (V/Call/FD)

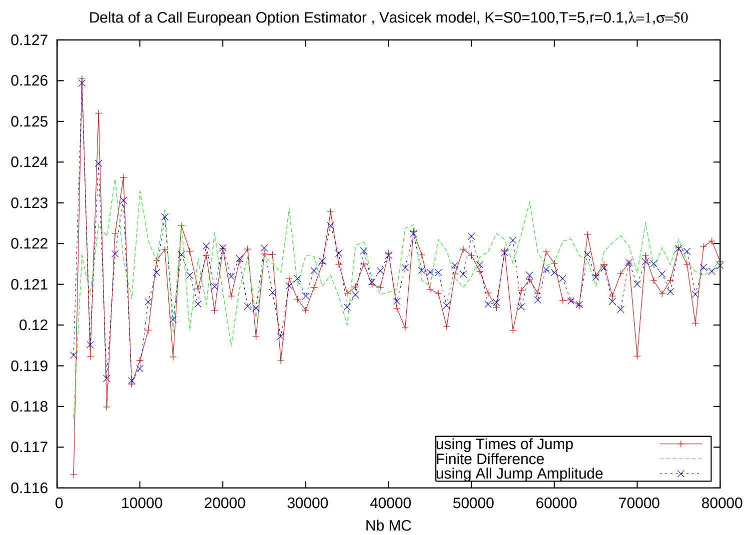

Figure 7: Vasicek model. Delta of an European Call option using Malliavin calculus based on the jump times, on the jump amplitudes, and finite difference method.

σ	$Variance(S_T)$	$VarMallJT$	$VarMallAJ$	$VarDiff$
15.8114	796.241	0.0285123	0.0106426	0.0300379
16.6667	897.577	0.0417219	0.0115955	0.0298567
17.6777	991.453	0.0400695	0.013123	0.0298904
18.8982	1134.11	0.0410136	0.0144516	0.0299574
20.4124	1313.42	0.0433065	0.0162378	0.029862
22.3607	1584.9	0.0400481	0.0178726	0.0298987
25	1967.53	0.0407136	0.0202055	0.0299007
28.8675	2604.22	0.0362728	0.0224265	0.0299651
35.3553	3961.31	0.0343158	0.0253757	0.0297775
50	7890.4	0.0333298	0.0287716	0.0299749

Table 7: variance of the Malliavin JT estimator, AJ estimator and of the FD for Call option in the Vasicek model.

Both the graph and the variance table show that the two methods give comparable results.

5.0.8 (V/Dig/AJ) versus (V/Dig/JT) versus (V/Dig/FD)

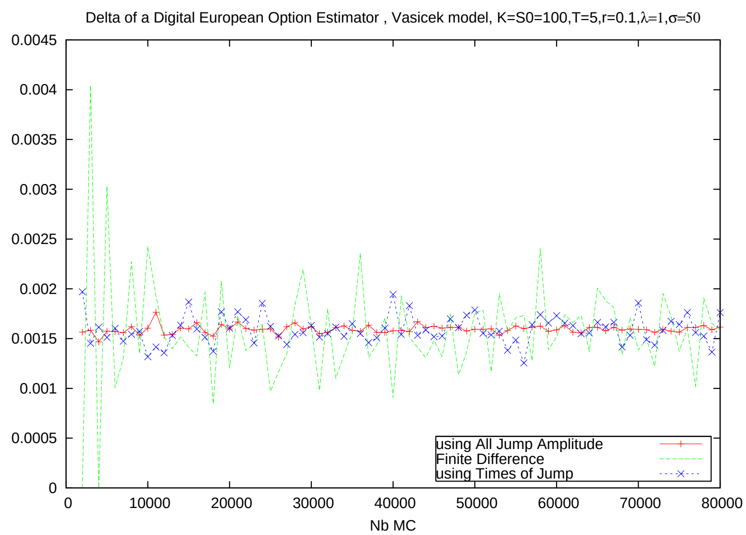

Figure 8: Vasicek model. Delta of an European Digital option using Malliavin calculus based on the jumpl amplitudes, on the jump times, and finite difference method.

σ	$Variance(S_T)$	$VarMallJT$	$VarMallAJ$	$VarDiff$
15.8114	796.241	0.00144622	$7.18878e-5$	0.00514743
16.6667	897.577	0.00254652	$7.3629e-5$	0.00459619
17.6777	991.453	0.0018011	$7.85552e-5$	0.00496369
18.8982	1134.11	0.0109864	$8.14005e-5$	0.00477995
20.4124	1313.42	0.00177648	$8.1627e-5$	0.00386111
22.3607	1584.9	0.00152777	$8.06193e-5$	0.00496369
25	1967.53	0.0013786	$7.94341e-5$	0.0062497
28.8675	2604.22	0.00100181	$7.5835e-5$	0.00551488
35.3553	3961.31	0.000617271	$6.95225e-5$	0.00459619
50	7890.4	0.000373802	$5.64325e-5$	0.00533116

Table 8: Vasicek model. Variance of the Malliavin JT estimator, AJ estimator and of the FD for Digital option.

Both the graph and the variance table show that for a digital option the algorithm based on the Malliavin calculus is significantly better than the one based on the finite difference.

5.1 conclusions

- For a smooth payoff (as the call) the algorithms based on the Malliavin calculus (with respect to the time moments or to the amplitudes of the jumps) give comparable results to the finite difference method.
- For a discontinuous payoff (as in the digital options), the algorithms based on the Malliavin calculus (with respect to the time moments or to the amplitudes of the jumps) give significantly better results than the finite difference method.

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Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399