Non-Symmetric Hall-Littlewood Polynomials

Francois Descouens, Alain Lascoux

Non-Symmetric Hall-Littlewood Polynomials

Francois Descouens Alain Lascoux

À Adriano Garsia, en toute amitié

Abstract

Using the action of the Yang-Baxter elements of the Hecke algebra on polynomials, we define two bases of polynomials in n variables. The Hall-Littlewood polynomials are a subfamily of one of them. For $q = 0$, these bases specialize into the two families of classical Key polynomials (i.e. Demazure characters for type A). We give a scalar product for which the two bases are adjoint of each other.

1 Introduction

We define two linear bases of the ring of polynomials in $x_1, \ldots, x_n$, with coefficients in q.

These polynomials, that we call *q-Key polynomials*, and denote $U_v, \widehat{U}_v$, $v \in \mathbb{N}^n$, specialize at $q = 0$ into key polynomials $K_v, \widehat{K}_v$. The polynomials U_v which are symmetrical in $x_1, \ldots, x_n$ are precisely the Hall-Littlewood polynomials P_λ, indexed by partitions $\lambda \in \mathfrak{Part}$, the relation between the two indices being $\lambda = [\lambda_1, \ldots, \lambda_n] = [v_n, \ldots, v_1]$.

Our main tool is the Hecke algebra $\mathcal{H}_n(q)$ of the symmetric group, acting on polynomials by deformation of divided differences. This algebra contains two adjoint bases of Yang-Baxter elements (Th. 2.1). The q-Key polynomials are the images of dominant monomials under these Yang-Baxter elements (Def. 3.1). These polynomials are clearly two linear bases of polynomials, since the transition matrix to monomials is uni-triangular.

We show in the last section that $\{U_v\}$ and $\{\widehat{U}_v\}$ are two adjoint bases with respect to a certain scalar product reminiscent of Weyl's scalar product on symmetric functions.

We have intensively used MuPAD (package `MuPAD-Combinat` [11]) and Maple (package `ACE` [10]).

2 The Hecke algebra $\mathcal{H}_n(q)$

Let $\mathcal{H}_n(q)$ be the Hecke algebra of the symmetric group $\mathfrak{S}_n$, with coefficients the rational functions in a parameter q. It has generators $T_1, \ldots, T_{n-1}$ satisfying the braid relations

$$\begin{cases} T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \\ T_iT_j = T_jT_i \quad (|j-i| > 1), \end{cases} \tag{1}$$

and the Hecke relations

$$(T_i + 1)(T_i - q) = 0 \ , \ 1 \leq i \leq n-1 \tag{2}$$

For a permutation σ in $\mathfrak{S}_n$, we denote by T_σ the element $T_\sigma = T_{i_1} \dots T_{i_p}$ where $(i_1, \dots, i_p)$ is any reduced decomposition of σ. The set $\{T_\sigma : \sigma \in \mathfrak{S}_n\}$ is a linear basis of $\mathcal{H}_n(q)$.

2.1 Yang-Baxter bases

Let $s_1, \dots, s_{n-1}$ denote the simple transpositions, $\ell(\sigma)$ denote the length of $\sigma \in \mathfrak{S}_n$, and let ω be the permutation of maximal length.

Given any set of indeterminates $\mathbf{u} = (u_1, \dots, u_n)$, let $\mathcal{H}_n(q)[u_1, \dots, u_n] = \mathcal{H}_n(q) \otimes \mathbb{C}[u_1, \dots, u_n]$.

One defines recursively a *Yang-Baxter basis* $(Y_\sigma^{\mathbf{u}})_{\sigma \in \mathfrak{S}_n}$, depending on $\mathbf{u}$, by

$$Y_{\sigma s_i}^{\mathbf{u}} = Y_\sigma^{\mathbf{u}} \left(T_i + \frac{1-q}{1 - u_{\sigma_{i+1}}/u_{\sigma_i}} \right), \quad \text{when } \ell(\sigma s_i) > l(\sigma) \, , \tag{3}$$

starting with $Y_{id}^{\mathbf{u}} = 1$.

Let φ be the anti-automorphism of $\mathcal{H}_n(q)[u_1, \dots, u_n]$ such that

$$\begin{cases} \varphi(T_\sigma) = T_{\sigma^{-1}}, \\ \varphi(u_i) = u_{n-i+1}. \end{cases}$$

We define a bilinear form $<\, , \,>$ on $\mathcal{H}_n(q)[u_1, \dots, u_n]$ by

$$< h_1 \, , \, h_2 > := \text{ coefficient of } T_\omega \text{ in } h_1 \cdot \varphi(h_2) \ . \tag{4}$$

The main result of [6, Th. 5.1] is the following duality property of Yang-Baxter bases.

Theorem 2.1 *For any set of parameters* $\mathbf{u} = (u_1, \dots, u_n)$*, the basis adjoint to* $(Y_\sigma^{\mathbf{u}})_{\sigma \in \mathfrak{S}_n}$ *with respect to* $<\, , \,>$ *is the basis* $(\widehat{Y}_\sigma^{\mathbf{u}})_{\sigma \in \mathfrak{S}_n} = (Y_\sigma^{\varphi(\mathbf{u})})_{\sigma \in \mathfrak{S}_n}$*. More precisely, one has*

$$\forall \ \sigma \, , \nu \ \in \mathfrak{S}_n \, , \quad < Y_\sigma^{\mathbf{u}} \, , \, \widehat{Y}_\nu^{\mathbf{u}} >= \delta_{\lambda, \ \nu\omega} \quad .$$

Let us fix from now on the parameters u to be $\mathbf{u} = (1, q, q^2, \dots, q^{n-1})$. Write $\mathcal{H}_n$ for $\mathcal{H}_n(q)[1, q, \dots, q^{n-1}]$.

In that case, the Yang-Baxter basis $(Y_\sigma)_{\sigma \in \mathfrak{S}_n}$ and its adjoint basis $(\widehat{Y}_\sigma)_{\sigma \in \mathfrak{S}_n}$ are defined recursively, starting with $Y_{id} = 1 = \widehat{Y}_{id}$, by

$$Y_{\sigma s_i} = Y_\sigma \ (T_i + 1/[k]_q) \quad \text{and } \widehat{Y}_{\sigma s_i} = \widehat{Y}_\sigma \ \left(T_i + q^{k-1}/[k]_q\right) \,) \, , \ \ell(\sigma s_i) > \ell(\sigma) \, , \tag{5}$$

with $k = \sigma_{i+1} - \sigma_i$ and $[k]_q = (1-q^k)/(1-q)$.

Notice that the maximal Yang-Baxter elements have another expression [2] :

$$Y_\omega = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma \ \text{ and } \ \widehat{Y}_\omega = \sum_{\sigma \in \mathfrak{S}_n} (-q)^{\ell(\sigma\omega)} \, T_\sigma \ .$$

Example 2.2 *For $\mathcal{H}_3$, the transition matrix between $\{Y_\sigma\}_{\sigma\in\mathfrak{S}_3}$ and $\{T_\sigma\}_{\sigma\in\mathfrak{S}_3}$ is*

$$\begin{array}{c|cccccc|}
123 & 1 & 1 & 1 & \frac{1}{q+1} & \frac{1}{q+1} & 1 \\
132 & \cdot & 1 & \cdot & 1 & \frac{1}{q+1} & 1 \\
213 & \cdot & \cdot & 1 & \frac{1}{q+1} & 1 & 1 \\
231 & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
312 & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
321 & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array} \;,$$

writing ' $\cdot$ ' for 0. Each column represents the expansion of some element Y_σ.

2.2 Action of $\mathcal{H}_n$ on polynomials

Let $\mathfrak{Pol}$ be the ring of polynomials in the variables $x_1, \ldots, x_n$ with coefficients the rational functions in q. We write monomials exponentially: $x^v = x_1^{v_1} \ldots x_n^{v_n}$, $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$. A monomial x^v is *dominant* if $v_1 \geq \ldots \geq v_n$.

We extend the natural order on partitions to elements of $\mathbb{Z}^n$ by

$$u \leq v \quad \text{iff} \quad \forall k > 0\,, \quad \sum_{i=k}^{n} (v_i - u_i) \geq 0\,.$$

For any polynomial P in $\mathfrak{Pol}$, we call *leading term* of P all the monomials (multiplied by their coefficients) which are maximal with respect to this partial order. This order is compatible with the right-to-left lexicographic order, that we shall also use. We also use the classiccal notation $\mathfrak{n}(v) = 0v_1 + 1v_2 + 2v_3 + \cdots + (n{-}1)v_n$.

Let i be an integer such that $1 \leq i \leq n-1$. As an operator on $\mathfrak{Pol}$, the simple transposition s_i acts by switching x_i and x_{i+1}, and we denote this action by $f \to f^{s_i}$. The i-th *divided difference* ∂_i and the i-th *isobaric divided difference* π_i, written on the right of the operand, are the following operators :

$$\partial_i : \; f \longmapsto f\,\partial_i := \frac{f - f^{s_i}}{x_i - x_{i+1}} \qquad , \qquad \pi_i : \; f \longmapsto f\,\pi_i := \frac{x_i f - x_{i+1} f^{s_i}}{x_i - x_{i+1}}\;.$$

The Hecke algebra $\mathcal{H}_n$ has a faithful representation as an algebra of operators on $\mathfrak{Pol}$ given by the following equivalent formulas [2, 8]

$$\left\{\begin{array}{lllllll}
T_i & = & \square_i - 1 & = & (x_i - qx_{i+1})\,\partial_i - 1 & = & (1 - qx_{i+1}/x_i)\pi_i - 1\,, \\
Y_{s_i} & = & \square_i & = & (x_i - qx_{i+1})\,\partial_i & = & (1 - qx_{i+1}/x_i)\pi_i\,, \\
\hat{Y}_{s_i} & = & \nabla_i & = & \square_i - (1+q) & = & \partial_i\,(x_{i+1} - qx_i)\,.
\end{array}\right.$$

The Hecke relations imply

$$\square_i^2 = (1+q)\square_i \quad , \quad \nabla_i^2 = -(1+q)\nabla_i \quad \text{and} \quad \square_i\nabla_i = \nabla_i\square_i = 0\;.$$

One easily checks that the operators $R_i(a,b)$ and $S_i(a,b)$ defined by

$$R_i(a,b) = \Box_i - q\frac{[b-a-1]_q}{[b-a]_q} \quad \text{and} \quad S_i(a,b) = \nabla_i + q\frac{[b-a-1]_q}{[b-a]_q}$$

satisfy the Yang-Baxter equation

$$R_i(a,b)\ R_{i+1}(a,c)\ R_i(b,c) = R_{i+1}(c,b)\ R_i(a,c)\ R_{i+1}(a,b)\,. \tag{6}$$

We have implicitely used these equations in the recursive definition of Yang-Baxter elements (5).

This realization comes from geometry [3], where the maximal Yang-Baxter elements are interpreted as Euler-Poincaré characteristic for the flag variety of $GL_n(\mathbb{C})$. This gives still another expression of the maximal Yang-Baxter elements :

$$Y_\omega = \prod_{1\le i<j\le n} (x_i - qx_j)\,\partial_\omega \qquad , \qquad \widehat{Y}_\omega = \partial_\omega \prod_{1\le i<j\le n} (x_j - qx_i)\,. \tag{7}$$

Example 2.3 *Let $\sigma = (3412) = s_2s_3s_1s_2$. The elements Y_{3412} and $\widehat{Y}_{3412}$ can be written*

$$\begin{aligned} Y_{3412} &= \Box_2\left(\Box_3 - \frac{q}{1+q}\right)\left(\Box_1 - \frac{q}{1+q}\right)\left(\Box_2 - \frac{q+q^2}{1+q+q^2}\right)\,, \\ \widehat{Y}_{3412} &= \nabla_2\left(\nabla_3 + \frac{q}{1+q}\right)\left(\nabla_1 + \frac{q}{1+q}\right)\left(\nabla_2 + \frac{q+q^2}{1+q+q^2}\right)\,. \end{aligned}$$

We shall now identify the images of dominant monomials under the maximal Yang-Baxter operators with Hall-Littlewood polynomials. Recall that there are two proportional families $\{P_\lambda\}$ and $\{Q_\lambda\}$ of Hall-Littlewood polynomials. Given a partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r] = (0^{m_0}, 1^{m_1}, \ldots, n^{m_n})$, with $m_0 = n - r = n - m_1 - \cdots - m_n$, then

$$Q_\lambda = \prod_{1\le i\le n} \prod_{j=1}^{m_i} (1-q^j)\,P_\lambda\,.$$

Let moreover $d_\lambda(q) = \prod_{0\le i\le n} \prod_{j=1}^{m_i} [j]_q$. The definition of Hall-Littlewood polynomials with raising operators [7],[9, III.2] can be rewritten, thanks to (7), as follows.

Proposition 2.4 *Let λ be a partition of n. Then one has*

$$x^\lambda\ Y_\omega\, d_\lambda(q)^{-1} = P_\lambda(x_1, \ldots, x_n; q) \tag{8}$$

The family of the Hall-Littlewood functions $\{Q_\lambda\}$ indexed by partitions can be extended into a family $\{Q_v : v \in \mathbb{Z}^n\}$, using the following relations due to Littlewood ([7], [9, II.2.Ex. 2])

$$Q_{(\ldots,u_i,u_{i+1},\ldots)} = -Q_{(\ldots,u_{i+1}-1,u_i+1,\ldots)} + q\,Q_{(\ldots,u_{i+1},u_i,\ldots)} + q\,Q_{(\ldots,u_i+1,u_{i+1}-1,\ldots)} \quad \text{if} \quad u_i < u_{i+1}, \tag{9}$$

$$Q_{(u_1,\dots,u_n)} = 0 \quad \text{if} \quad u_n < 0 \quad . \tag{10}$$

By iteration of the first relation, one can write any Q_u in terms of Hall-Littlewood functions indexed by decreasing vectors v such that $|v| = |u|$. Consequently, if u such that $|u| = 0$, Q_u must be proportionnal to $Q_{0\dots 0} = 1$, i.e. is a constant that one can note as the specialisation $Q_u(0)$ in $x_1 = 0 = \cdots = x_n$.

The final expansion of Q_u, after iterating (9) many times, is not easy to predict. In particular, one needs to know whether $Q_u \neq 0$. For that purpose, we shall isolate a distinguished term in the expansion of Q_u. Given a sum $\sum_{\lambda \in \mathfrak{Part}} c_\lambda(t) Q_\lambda$, call *top term* the image of the leading term $\sum c_\mu(t) Q_\mu$ after restricting each coefficient $c_\mu(t)$ to its term in highest degree in t.

Given $u \in \mathbb{Z}^n$, define recursively $\mathfrak{p}(u) \in \mathfrak{Part} \cup \{-\infty\}$ by

- if $u \not\geq [0,\dots,0]$ then $\mathfrak{p}(u) = -\infty$
- if $u_2 \geq u_3 \geq \cdots \geq u_n > 0$ then $\mathfrak{p}(u)$ is the maximal partition of length $\leq n$, of weight $|u|$ (eventual zero terminal parts are suppressed).
- $\mathfrak{p}(u) = \mathfrak{p}\big(u\, \mathfrak{p}([u_2,\dots,u_n])\big)$

Lemma 2.5 *Let $u \in \mathbb{Z}^n$. Then*

- *if $u \not\geq [0,\dots,0]$ then $Q_u = 0$,*
- *if $u \geq [0,\dots,0]$, let $v = \mathfrak{p}(u)$. Then $Q_u \neq 0$ and its leading term is $q^{\mathfrak{n}(u)-\mathfrak{n}(v)} Q_v$.*

Proof. Given any decomposition $u = u'.u''$, then one can apply (9) to u'' and write Q_u as a linear combination of terms $Q_{u'v}$ with v decreasing, with $|v| = |u''|$. Therefore, if $|u''| = 0$, then the last components of such v are negative, all $Q_{u'v}$ are 0, and $Q_u = 0$.

If $u \geq [0,\dots,0]$ and u is not a partition, write $u = [\dots,a,b,\dots]$, with a, b the rightmost increase in u. We apply relation (9), assuming the validity of lemma for the three terms in the RHS :

$$Q_{\dots,a,b,\dots} = -Q_{\dots,b-1,a+1,\dots} + qQ_{\dots,b,a,\dots} + qQ_{\dots,a+1,b-1,\dots}$$

Notice that the first two terms have not necessarily an index $\geq [0,\dots,0]$, but that $[\dots,a+1,b-1,\dots] \geq [0,\dots,0]$.

In any case, it is clear that $\mathfrak{p}([\dots,b-1,a+1,\dots]) = p_1 \leq v$, $\mathfrak{p}([\dots,b,a,\dots]) = p_2 \leq v$, and $\mathfrak{p}([\dots,a+1,b-1,\dots]) = v$.

Restricted to top terms, the expansion of the RHS in the basis Q_λ becomes

$$-\Big(\big(q^{\mathfrak{n}(u)+a+1-b-\mathfrak{n}(v)} + \cdots\big) Q_v\Big) + q\Big(\big(q^{\mathfrak{n}(u)+a-b-\mathfrak{n}(v)} + \cdots\big) Q_v\Big) + q\Big(\big(q^{\mathfrak{n}(u)-1-\mathfrak{n}(v)} + \cdots\big) Q_v\Big)\,,$$

where one or two of the first two terms may be replaced by 0, depending on the value of p_1, or p_2. In final, the top term of the RHS is $q^{\mathfrak{n}(u)-\mathfrak{n}(v)} Q_v$, as wanted. Q.E.D

Example 2.6 *For $v = [-2, 3, 2]$,*

$$Q_{2,3,2} = (q^3 - q^2)Q_3 + (q^5 + q^4 - q^3 - 2q^2 + q)Q_{21} + (q^4 - q^3 - q^2 + q)Q_{111}\,,$$

and the top term is $q^4 Q_{111}$, since $4 = (0(-2) + 1(3) + 2(2)) - (0(1) + 1(1) + 2(1))$ and $[1, 1, 1] > [2, 1]$, $[1, 1, 1] > [3]$. Notice that the coefficient of Q_{21} is of higher degree.

3 q-Key Polynomials

In this section, we show that the images of dominant monomials under the Yang-Baxter elements Y_σ (resp. $\widehat{Y}_\sigma$), $\sigma \in \mathfrak{S}_n$ constitute two bases of $\mathfrak{Pol}$, which specialize into the two families of Demazure characters.

We have already identified in the preceding section the images of dominant monomials under Y_ω to Hall-Littlewood polynomial, using the relation between Y_ω and ∂_ω. The other polynomials are new.

3.1 Two bases

The dimension of the linear span of the image of a monomial x^v under all permutations depends upon the stabilizer of v. We meet the same phenomenon when taking the images of a monomial under Yang-Baxter elements.

Let $\lambda = [\lambda_1, \ldots, \lambda_n]$ be a decreasing partition (adding eventual parts equal to 0). Denote its orbit under permutations of components by $\mathcal{O}(\lambda)$. Given any v in $\mathcal{O}(\lambda)$, let $\zeta(v)$ be the permutation of maximal length such that $\lambda\,\zeta(v) = v$ and $\eta(v)$ be the permutation of minimal length such that $\lambda\,\eta(v) = v$. These two permutations are representative of the same coset of $\mathfrak{S}_n$ modulo the stabilizer of λ.

Definition 3.1 *For all v in $\mathbb{N}^n$, the q-*Key polynomials *U_v and $\widehat{U}_v$ are the following polynomials :*

$$U_v(x; q) = \left(\frac{1}{d_\lambda(q)} x^\lambda\right) Y_{\zeta(v)} \qquad , \qquad \widehat{U}_v(x; q) = x^\lambda \widehat{Y}_{\eta(v)}\,,$$

where λ is the dominant reordering of v.

In particular, if v is (weakly) increasing, then $\zeta(v) = \omega$ and U_v is a Hall-Littlewood polynomial.

Lemma 3.2 *The leading term of U_v and $\widehat{U}_v$ is x^v. Consequently, the transition matrix between the U_v (resp. the $\widehat{U}_v$) and the monomials is upper unitriangular with respect to the right-to-left lexicographic order.*

Proof. Let k be an integer and u be a weight such that $u_k > u_{k+1}$. Suppose by induction that x^u is the leading term of U_u. Recall the the explicit action of $\square_k$ is (noting only the two variables x_k, x_{k+1})

$$
\begin{array}{rcl}
x^{\beta\alpha}\,\Box_k & = & x^{\beta\alpha} + (1-t)(x^{\beta-1,\alpha+1} + \cdots + x^{\alpha+1,\beta-1}) + x^{\alpha\beta}\,,\ \beta > \alpha \\
x^{\beta\beta}\,\Box_k & = & (1+t)x^{\beta\beta} \\
x^{\alpha\beta}\,\Box_k & = & tx^{\beta\alpha} + (t-1)(x^{\beta-1,\alpha+1} + \cdots + x^{\alpha+1,\beta-1}) + tx^{\alpha\beta}\,,\ \alpha < \beta\,.
\end{array}
$$

¿From these formulas, it is clear that for any constant c, the leading term of $x^u\,(\Box_k + c)$ is $(x^u)^{s_k}$, and, for any v such that $v < u$, all the monomials in $x^v\,(\Box_k + c)$ are strictly less (with respect to the partial order) than $(x^u)^{s_k}$. $\Box$

Example 3.3 *For $n = 3$, Figures 1 and 2 show the case of a regular dominant weight x^{210} and Figures 3 and 4 correspond to a case, x^{200}, where the stabilizer is not trivial. In this last case, the polynomials belonging to the family are framed, the extra polynomials denoted A, B do not belong to the basis.*

$$
\begin{tikzcd}
 & U_{210} = x^{210} \arrow[dl, "\Box_1"'] \arrow[dr, "\Box_2"] & \\
U_{120} = x^{120} + x^{210} \arrow[d, "\Box_2 - q/(1+q)"] & & U_{201} = x^{201} + x^{210} \arrow[d, "\Box_1 - q/(1+q)"] \\
U_{102} = x^{102} + (1-q)x^{111} + \frac{1}{1+q}x^{120} + x^{201} + \frac{1}{1+q}x^{210} \arrow[dr, "\Box_1"] & & U_{021} = x^{021} + (1-q)x^{111} + \frac{1}{1+q}x^{201} + x^{120} + \frac{1}{1+q}x^{210} \arrow[dl, "\Box_2"'] \\
 & U_{012} = +x^{102} + (-q-q+2)x^{111} + x^{120} + x^{201} + x^{210} &
\end{tikzcd}
$$

Figure 1: q-Key polynomials generated from x^{210}.

3.2 Specialization at $q = 0$

The specialization at $q = 0$ of the Hecke algebra is called the 0-*Hecke algebra*. The elementary Yang-Baxter elements specialize in that case into

$$
Y_{s_i} = T_i + 1 = \Box_i \quad \to \quad x_i\partial_i = \pi_i\ , \tag{11}
$$

$$
\hat{Y}_{s_i} = T_i = \nabla_i \quad \to \quad \partial_i x_{i+1} = \widehat{\pi}_i. \tag{12}
$$

$$
\begin{tikzcd}
 & \widehat{U}_{210} = x^{210} \arrow[dl, "\nabla_1"'] \arrow[dr, "\nabla_2"] & \\
\widehat{U}_{120} = x^{120} - qx^{210} \arrow[d, "\nabla_2+\frac{q}{1+q}"] & & \widehat{U}_{201} = x^{201} - qx^{210} \arrow[d, "\nabla_1+\frac{q}{1+q}"] \\
\widehat{U}_{102} = x^{102} + (1-q)x^{111} - qx^{201} - \frac{q^2}{q+1}x^{120} + \frac{q^3}{q+1}x^{210} \arrow[dr, "\nabla_1"] & & \widehat{U}_{021} = x^{021} + (1-q)x^{111} - qx^{120} - \frac{q^2}{q+1}x^{201} + \frac{q^3}{q+1}x^{210} \arrow[dl, "\nabla_2"'] \\
 & \widehat{U}_{012} = (1+q)B - q\widehat{K}_{120} &
\end{tikzcd}
$$

Figure 2: Dual q-Key polynomials generated from x^{210}.

$$
\begin{tikzcd}
 & x^{200}/(1+q) \arrow[dl, "\square_1"'] \arrow[dr, "\square_2"] & \\
A \arrow[d, "\square_2 - q/(1+q)"] & & \boxed{U_{200} = x^{200}} \arrow[d, "\square_1 - q/(1+q)"] \\
B \arrow[dr, "\square_1"] & & \boxed{U_{020} = x^{020} - qx^{200} + (1-q)x^{110}} \arrow[dl, "\square_2"'] \\
 & \boxed{U_{002} = (1-q)x^{011} + (1-q)x^{101} + x^{002} + \frac{q^2(q-1)}{q+1}x^{110} - \frac{q^2}{q+1}x^{020} - \frac{q^2}{1+q}x^{200}} &
\end{tikzcd}
$$

Figure 3: q-Key polynomials generated from $x^{200}/(1+q)$.

Definition 3.4 (Key polynomials) *Let $v \in \mathbb{N}^n$. The* Key polynomials *K_v and $\widehat{K}_v$ are*

$$\begin{tikzcd}
 & & \boxed{\widehat{U}_{200} = x^{200}} \arrow[dll, "\nabla_1"'] \arrow[d, "\nabla_2"] \\
\boxed{\widehat{U}_{020} = (1-q)x^{110} + x^{020} + \frac{1}{q+1}x^{200}} \arrow[d, "\nabla_2 + q/(1+q)"] & & 0 \arrow[d, "\nabla_1 + q/(1+q)"] \\
\boxed{\widehat{U}_{002} = (1-q)x^{011} + (1-q)x^{101} + x^{002} + (1-q)x^{110} + x^{020} + x^{200}} \arrow[dr, "\nabla_1"] & & 0 \arrow[dl, "\nabla_2"] \\
 & 0 &
\end{tikzcd}$$

Figure 4: Dual q-Key polynomials generated from x^{200}.

defined recursively, starting with $K_v = x^v = \widehat{K}_v$ *if* x^v *dominant, by*

$$K_{vs_i} = K_v\,\pi_i \qquad , \qquad \widehat{K}_{vs_i} = \widehat{K}_v\,\widehat{\pi}_i \ , \quad \text{for } i \text{ such that } v_i > v_{i+1}\,.$$

In particular, the subfamily (K_v) for v increasing, is the family of Schur functions in $x_1, \ldots, x_n$. Demazure [1] defined Key polynomials (using another terminology) for all the classical groups, and not only the type A_{n-1} which is our case.

Lemma 3.2 specializes into :

Lemma 3.5 *The transition matrix between the* U_v *and the* K_v *(resp. from* $\widehat{U}_v$ *to* $\widehat{K}_v$*) is upper unitriangular with respect to the lexicographic order.*

Example 3.6 *For* $n = 3$*, the transition matrix between* $\{U_v\}$ *and* $\{K_v\}$ *in weight 3 is (reading a column as the expansion of some* U_v*)*

$$\begin{array}{c}
300\\ 210\\ 201\\ 120\\ 111\\ 102\\ 030\\ 021\\ 012\\ 003
\end{array}
\left|\begin{array}{cccccccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{-q}{(q+1)} & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \frac{-q}{(q+1)} & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \frac{-q}{(q+1)} & -q & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & -q & \cdot & -q & -q\,(q+1) & q^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}\right| ,$$

and the transition matrix between $\{\widehat{U}_v\}$ and $\{\widehat{K}_v\}$ is

$$
\begin{array}{c}
300 \\ 210 \\ 201 \\ 120 \\ 111 \\ 102 \\ 030 \\ 021 \\ 012 \\ 003
\end{array}
\left|
\begin{array}{cccccccccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & -q & \cdot & \cdot & \frac{-q^2}{(q+1)} \\
\cdot & 1 & -q & -q & \cdot & \frac{q^3}{(q+1)} & -q & \frac{q^3}{(q+1)} & -q^3 & \frac{q^3}{(q+1)} \\
\cdot & \cdot & 1 & \cdot & \cdot & -q & \cdot & \frac{-q^2}{(q+1)} & q^2 & -q \\
\cdot & \cdot & \cdot & 1 & \cdot & \frac{-q^2}{(q+1)} & -q & -q & q^2 & \frac{q^3}{(q+1)} \\
\cdot & \cdot & \cdot & \cdot & 1 & -q & \cdot & -q & q\,(q+1) & q^2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & -q & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \frac{-q^2}{(q+1)} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -q \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{array}
\right| .
$$

4 Orthogonality properties for the q-Key polynomials

We show in this section that the q-Key polynomials U_v and $\widehat{U}_v$ are two adjoint bases with respect to a certain scalar product.

4.1 A scalar product on $\mathfrak{Pol}$

For any Laurent series $f = \sum_{i=k}^{\infty} f_i x^i$, we denote by $CT_x(f)$ the coefficient f_0.

Let

$$\Theta := \prod_{1 \le i < j \le n} \frac{1 - x_i/x_j}{1 - q x_i/x_j} \ .$$

Therefore, for any Laurent polynomial $f(x_1, \ldots, x_n)$, the expression

$$CT(f\,\Theta) := CT_{x_n}\left(CT_{x_{n-1}}\left(\ldots\left(CT_{x_1}\left(f\,\Theta\right)\right)\ldots\right)\right)$$

is well defined. Let us use it to define a bilinear form $(\ ,\ )_q$ on $\mathfrak{Pol}$ by

$$(f\ ,\ g)_q = CT\left(f\, g^{\clubsuit} \prod_{1 \le i < j \le n} \frac{1 - x_i/x_j}{1 - q x_i/x_j}\right) \tag{13}$$

where $\clubsuit$ is the automorphism defined by $x_i \longmapsto 1/x_{n+1-i}$ for $1 \le i \le n$.

Since Θ is invariant under $\clubsuit$, the form $(\ ,\ )_q$ is symmetrical. Under the specialization $q = 0$, the previous scalar product becomes

$$(f\ ,\ g) := (f\ ,\ g)\big|_{q=0} = CT\left(f\, g^{\clubsuit} \prod_{1 \le i < j \le n} (1 - x_i/x_j)\right) . \tag{14}$$

We can also write $(f\ ,\ g)_q = (f\ ,\ g\Omega)$ with $\Omega = \prod_{1 \le i < j \le n} (1 - q x_i/x_j)^{-1}$.

Notice that, interpreting Schur functions as characters of unitary groups, Weyl defined the scalar product of two symmetric functions f, g in n variables as the constant term of

$$\frac{1}{n!}\, f\, g^{\clubsuit} \prod_{i,j:\, i\neq j} (1 - x_i/x_j)\,.$$

Essentially, Weyl takes the square of the Vandermonde, while we are taking the quotient of the Vandermonde by the q-Vandermonde.

We now examine the compatibility of $\square_i$ and ∇_i with the scalar product.

Lemma 4.1 *For i such that $1 \le i \le n-1$, $\square_i$ (resp. ∇_i) is adjoint to $\square_{n-i}$ (resp. ∇_{n-i}) with respect to $(\ ,\ )_q$.*

Proof. Since π_i (resp. $\widehat{\pi}_i$) is adjoint to π_{n-i} (resp. $\widehat{\pi}_{n-i}$) with respect to $(\ ,\ )$ (see [5] for more details), we have

$$\begin{aligned}(f\square_i\,,\, g)_q &= (f\,,\, g\ \Omega\ \pi_{n-i}(1 - qx_{n-i+1}/x_{n-i}))\\ &= (f\,,\, g\ \frac{(1 - qx_{n-i+1}/x_{n-i})}{(1 - qx_{n-i+1}/x_{n-i})}\ \Omega\ \pi_{n-i}(1 - qx_{n-i+1}/x_{n-i}))\end{aligned}$$

Since the polynomial $\Omega/(1 - qx_{n-i+1}/x_{n-i})$ is symmetrical in the indeterminates x_{n-i} and x_{n-i+1}, it commutes with the action of π_{n-i}. Therefore

$$(f\square_i\,,\, g)_q = (f\,,\, g\ (1 - qx_{n-i+1}/x_{n-i})\ \pi_{n-i}\ \Omega) = (f\,,\, g\ \square_{n-i})_q\ .$$

This proves that $\square_i$ is adjoint to $\square_{n-i}$, and, equivalently, that ∇_i is adjoint to ∇_{n-i}. Q.E.D

We shall need to characterize whether the scalar product of two monomials vanishes or not. Notice that, by definition,

$$(x^u\,,\, x^v) = (x^{u-v\omega}\,,\, 1)\,,$$

so that one of the two monomials can be taken equal to 1.

Lemma 4.2 *For any $u \in \mathbb{Z}^n$, then $(x^u,\, 1)_q \neq 0$ iff $|u| = 0$ and $u \ge [0, \ldots 0]$. In that case, $(x^u,\, 1)_q = Q_u(0)$.*

Proof. Let us first show that the scalar products $(x^u\,,\, 1)_q$ satisfy the same relations (9) as the Hall-Littlewood functions Q_u.

Let k be a positive integer less than n. Write $x_k = y$, $x_{k+1} = z$. Any monomial x^v can be written $x^t\, y^a z^b$, with x^t of degree 0 in x_k, x_{k+1}. The product

$$x^t(y^a z^b + y^b z^a)(z - qy) \prod_{1\le i<j\le n} \frac{1 - x_i/x_j}{1 - qx_i/x_j}$$

is equal to

$$(y^a z^b + y^b z^a)(z - qy)\frac{1 - y/z}{1 - qy/z}\, F_1 = (y^a z^b + y^b z^a)(z - y)F_1$$

with F_1 symmetrical in y, z. The constant term $CT_{x_{k-1}} \dots CT_{x_1}(x^t(y^a z^b + y^b z^a)F_1) = F_2$ is still symmetric in x_k, x_{k+1}. Therefore

$$CT_y\Big(CT_z\big((z-y)F_2\big)\Big)$$

is null, and in final

$$CT\left(x^t(y^a z^b + y^b z^a)(z - qy) \prod_{1\le i<j\le n} \frac{1 - x_i/x_j}{1 - qx_i/x_j}\right) = 0\,.$$

This relation can be rewritten

$$(y^a z^{b+1} x^t, 1)_q + (y^{b+1} z^{a+1} x^t, 1)_q - q(y^{b+1} z^a x^t, 1)_q - q(y^{a+1} z^b x^t, 1)_q = 0\ ,$$

which is, indeed, relation (9).

On the other hand, if $u_n < 0$, then there is no term of degree 0 in x_n in $x^u \prod_{1\le i<j\le n}(1 - x_i/x_j)(1 - qx_i/x_j)^{-1}$, and $(x^u, 1) = 0$, so that rule (10) is also satisfied.

In consequence, the function $u \in \mathbb{Z}^n \to (x^u, 1)$ is determined by the values $(x^\lambda, 1)$, λ partition, as the function $u \in \mathbb{Z}^n \to Q_u$ is determined by its restriction to partitions. However, for degree reasons, $(x^\lambda, 1) = 0$ if $\lambda \neq 0$. Since $(x^0, 1) = 1$, one has in final that $(x^u, 1) = Q_u(0)$. Q.E.D

Example 4.3 *For $u = [1, 0, 3]$ and $v = [0, 1, 3]$,*

$$(x^{103}\,,\, x^{013})_q = (x^{-2,-1,3}\,,\, 1)_q = Q_{-2,-1,3}(0) = q^2(1-q)(1-q^2)\ .$$

4.2 Duality between $(U_v)_{v\in\mathbb{N}^n}$ and $(\widehat{U}_v)_{v\in\mathbb{N}^n}$

Using that $\square_i$ is adjoint to $\square_{n-i}$, we are going to prove in this section that U_v and $\widehat{U}_v$ are two adjoint bases of $\mathfrak{Pol}$ with respect to the scalar product $(\ ,\ )_q$.

We first need some technical lemmas, to allow an induction on the q-Key polynomials, starting with dominant weights.

Lemma 4.4 *Let i be an integer such that $1 \le i \le n{-}1$, let f_1, f_2, g_1 be three polynomials and b be a constant such that*

$$f_2 = f_1\,(\square_i + b)\ ,\ (f_1\,,\, g_1)_q = 0 \quad \textit{and} \quad (f_2\,,\, g_1)_q = 1\,.$$

Then the polynomial $g_2 = g_1(\nabla_{n-i} - b)$ is such that

$$(f_1\,,\, g_2)_q = 1\ ,\ (f_2\,,\, g_2)_q = 0\,.$$

Proof. Using that ∇_{n-i} is adjoint to $\Box_i$ and that $\Box_i\nabla_i = 0$, one has

$$(f_2\,,\,g_2)_q = (f_1\,(\Box_i + b)\,,\,g_1\,(\nabla_{n-i} - b))_q = (f_1\,(\Box_i + b)\,(\nabla_i - b)\,,\,g_1)_q$$
$$= (f_1(-b(1+q) - b^2), g_1)_q = 0\,.$$

Similarly, we have

$$\begin{aligned}(f_1\,,\,g_2)_q &= (f_1\,,\,g_1\,(\nabla_{n-i} - b))_q\\ &= (f_1\,,\,g_1\,(\Box_{n-i} - 1 - q - b))_q\\ &= (f_1\,(\Box_i + b - 1 - q - 2b)\,,\,g_1)_q = (f_2\,,\,g_1)_q = 1.\end{aligned}$$

Q.E.D

Corollary 4.5 *Let i be an integer such that $1 \leq i \leq n-1$, let V be a vector space such that $V = V^{'} \oplus < f_1, f_2 >$ with $f_2 = f_1(\Box_i + b)$ and $V^{'}$ stable under $\Box_i$, and let g_1 such that*

$$(f_1\,,\,g_1)_q = 0 \quad \textit{and} \quad (f_2\,,\,g_1)_q = 1 \quad \textit{and} \quad (v\,,\,g_1)_q = 0,\ \forall v \in V^{'}\ .$$

Then the element $g_2 = g_1(\nabla_{n-i} - b)$ is such that

$$(f_2\,,\,g_2)_q = 0 \quad \textit{and} \quad (f_1\,,\,g_2)_q = 1 \quad \textit{and} \quad (v\,,\,g_2)_q = 0,\ \forall v \in V^{'}\ .$$

Lemma 4.6 *Let u and λ be two dominant weights and v and μ two permutations of u and λ respectively. If $(x^v\,,\,x^\lambda) \neq 0$ and $(x^u\,,\,x^\mu) \neq 0$ then*

$$u = \lambda \quad , \quad v = \lambda\omega \quad \textit{and} \quad \mu = u\omega\,.$$

Proof. Using lemma 4.2 , the condition $(x^v, x^\lambda)_q \neq 0$ and $(x^u, x^\mu)_q \neq 0$ implies two systems of inegalities

$$\left\{\begin{array}{ccc} v_n & \geq & \lambda_1\,, \\ v_n + v_{n-1} & \geq & \lambda_1 + \lambda_2\,, \\ \vdots & \vdots & \vdots \\ v_n + \ldots + v_1 & \geq & \lambda_1 + \ldots + \lambda_n\,. \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ccc} \mu_n & \geq & u_1\,, \\ \mu_n + \mu_{n-1} & \geq & u_1 + u_2\,, \\ \vdots & \vdots & \vdots \\ \mu_n + \ldots + \mu_1 & \geq & u_1 + \ldots + u_n\,. \end{array}\right.$$

The first inequalities of the systems give $v_n \geq \lambda_1 \geq \mu_n \geq u_1 \geq v_n$. Consequently $u_1 = \lambda_1 = v_n = u_n$. By recursion, using the other inequalities, one gets the lemma. Q.E.D

Corollary 4.7 *Let v be a weight and λ a dominant weight. Then,*

$$(U_v, x^\lambda)_q = \delta_{v,\lambda\omega}$$

Proof. Let u be the decreasing reordering of v and σ the permutation such that $U_v = x^u Y_\sigma$. As the leading term of U_v is x^v and using lemma 4.6, we have that $(x^u Y_\sigma\,, x^\lambda)_q \neq 0$ implies $(x^v\,,\,x^\lambda)_q \neq 0$. By denoting $\triangle_\sigma$ the adjoint of Y_σ with respect to $(\,,\,)_q$, we have $(x^u Y_\sigma\,,\,x^\lambda)_q = (x^u\,,\,x^\lambda\triangle_\sigma)_q \neq 0$. As the leading term of $x^\lambda\triangle_\sigma$ is $x^{\lambda\sigma^{'}}$, where $\lambda\sigma^{'}$ is a permutation of λ, we obtain that $(x^u\,,\,x^{\lambda\sigma^{'}})_q \neq 0$. Using lemma 4.6 we conclude that $v = \lambda\omega$.

Q.E.D

Our main result is the following duality property between U_v and $\widehat{U}_v$.

Theorem 4.8 *The two sets of polynomials* $(U_v)_{v\in\mathbb{N}^n}$ *and* $(\widehat{U}_v)_{v\in\mathbb{N}^n}$ *are two adjoint bases of* $\mathfrak{Pol}$ *with respect to the scalar product* $(\ ,\ )_q$*. More precisely, they satisfy*

$$(U_v\ ,\ \widehat{U}_{u\omega})_q = \delta_{v,u}\ .$$

Proof. Let λ be a dominant weight and V the vector space spanned by the U_v for v in $\mathcal{O}(\lambda)$. The idea of the proof is to build by iteration the elements $(\widehat{U}_v)_{v\in\mathcal{O}(\lambda)}$ starting with $x^\lambda = \widehat{U}_\lambda$. By definition of the q-Key polynomials, it exists a constant b such that $U_{\lambda\omega} = U_{\lambda\omega s_1}\ (\square_1 + b)$. One can write the decomposition $V = V' \oplus < U_{\lambda\omega}\ , U_{\lambda\omega s_1} >$, with V' stable under the action of $\square_1$. Using the previous lemma, we have that $(U_{\lambda\omega}\ , x^\lambda)_q = (U_{\lambda\omega}\ , \widehat{U}_\lambda)_q = 1$ and $(U_{\lambda\omega\sigma_1}\ , x^\lambda)_q = (U_{\lambda\omega\sigma_1}\ , \widehat{U}_\lambda)_q = 0$. Consequently, by lemma 4.5, the function $x^\lambda(\nabla_{n-1} - b) = \widehat{U}_{\lambda s_1}$ satisfy the duality conditions

$$(U_{\lambda\omega}\ , \widehat{U}_{\lambda s_1})_q = 0 \quad , \quad (U_{\lambda\omega\sigma_1}\ , \widehat{U}_{\lambda s_1})_q = 1 \quad \text{and} \quad (v\ , \widehat{U}_{\lambda s_1})_q = 0 \quad \forall v \in V'\ .$$

By iteration, this proves that for all u, v, one has $(U_v\ , \widehat{U}_{u\omega})_q = \delta_{v,u}$. Q.E.D

This theorem implies that the space of symmetric functions and the linear span of dominant monomials are dual of each other, the Hall-Littlewood functions being the basis dual to dominant monomials.

We finally mention that in the case $q = 0$, one has a reproducing kernel, as stated by the following theorem of [4], which gives another implicit definition of the scalar product (,).

Theorem 4.9 *The two families of polynomials* $(K_v)_{v\in\mathbb{N}^n}$ *and* $(\widehat{K}_v)_{v\in\mathbb{N}^n}$ *satisfy the following Cauchy formula*

$$\sum_{u\in\mathbb{N}^n} K_u(x)\widehat{K}_{u\omega}(y) = \prod_{i+j\leq n+1} \frac{1}{1 - x_i y_j}\ . \tag{15}$$

References

[1] M. Demazure, *Une nouvelle formule des caractères*, Bull. Sci. Math. **98** (1974), 163-172.

[2] G. Duchamp, D. Krob, A. Lascoux, B. Leclerc, T. Scharf and J.-Y. Thibon, *Euler-Poincaré characteristic and polynomial representations of Iwahori-Hecke algebras*, Publ. RIMS Kyoto **31** (1995),179-201.

[3] A. Lascoux *About the y in the* χ_y*-characteristic of Hirzebruch*, Contemporary Mathematics **241**, (1999) 285–296.

[4] A. Lascoux. *Double Crystal graphs*, Studies in Memory of Issai Schur, Progress In Math. **210**, Birkhaüser(2003) 95–114.

[5] A. Lascoux *Symmetric functions & Combinatorial operators on polynomials*, CBMS/AMS Lectures Notes **99**, (2003).

[6] A. Lascoux, B. Leclerc and J.Y. Thibon. *Flag Varieties and the Yang-Baxter Equation*, Letters in Math. Phys., **40** (1997) 75–90.

[7] D.E. Littlewood. *On certain symmetric functions*, Proc. London. M.S. **43** (1961) 485–498.

[8] G. Lusztig. *Equivariant K-theory and representations of Hecke Algebras*, Proc. Amer. Math. Soc. **94** (1985), 337-342.

[9] I. G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, Oxford, 2nd edition (1995).

[10] Package ACE *http://phalanstere.univ-mlv.fr/ace*

[11] Package MuPAD-Combinat *http://mupad-combinat.sourceforge.net/*