Hyperdeterminants on semilattices

Jean-Gabriel Luque

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Abstract

We compute hyperdeterminants of hypermatrices whose indices belongs in a meet-semilattice and whose entries depend only of the greatest lower bound of the indices. One shows that an elementary expansion of such a polynomial allows to generalize a theorem of Lindström to higher-dimensional determinants. And we gave as an application generalizations of some results due to Lehmer, Li and Haukkanen.

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1 Introduction

Since the end of the nineteen century, it is known that some determinants, with entries depending only of the gcd of the indices, factorize. Readers interested in the story of the problem can refer to [13] and [2]. In 1876, Smith [15] evaluated the determinant of a GCD matrix whose entries belong to a factor closed set (*i.e.*, a set which contains all the factors of its elements) as a product of Euler's totient. The interest of this computation lies in its links with arithmetic functions [1] and in particular multiplicative functions (see [7, 8] for interesting remarks about the last notion). During the last century, many generalizations of Smith's theorem have been investigated. One of the

* Institut Gaspard Monge, Université de Marne-la-Vallée,
77454 Marne-la-Vallée cedex 2
France
email:Jean-Gabriel.Luque@univ-mlv.fr

ways to extend his result consists in changing the set of the indices of the matrices. In 1990, Li [10] gave the value of GCD determinant for an arbitrary set of indices. Beslin and Ligh [3] shown that such a determinant factorizes when the indices belongs to a gcd-closed set (*i.e.*, a set which contains the gcd of any pairs of its elements) as a product of certain functions evaluated in terms of Euler's totient. The fact that these determinants factorize can be seen as a corollary of a very elegant theorem due to Lindström [11] which evaluated the determinant of the GCD-matrix whose indices are chosen in a meet semilattice, *i.e.*, a poset such that each pair admits a greatest lower bound. Another way to generalize Smith's result consists in computing multidimensional analogous. In 1930 Lehmer gave [9] the first multi-indexed version of Smith's determinant. Other related computation are collected in [16, 17]. More recently, Haukkanen [12] generalized the results of Beslin and Ligh [3] and Li [10]) to hyperdeterminants.

We will see in Section 2 that the main trick for computing these multidimensional determinants consists in expanding it as a sum of (classical 2-way) determinants. In the aim to highlight this method, we apply it to a more general object $\mathrm{Det}_{\mathfrak{F}}$.
In Section 3, we recall shortly a classical technic and give a slight generalization of Lindström's Theorem. As a consequence, we give a multidimensional analogue of Lindström's Theorem.
In Section 4, one investigates minors of meet hypermatrices and generalizes two theorems due to Haukkanen [12].

2 Hyperdeterminants and $\mathfrak{F}$-determinants

The question of extending the notion of determinant to higher dimensional arrays has been raised by Cayley [5, 6] few after he introduced the modern notation as square arrays [4]. The simplest generalization is defined for a kth order tensor on an n-dimensional space $M = (M_{i_1,\cdots,i_k})_{1\leq i_1,\cdots,i_k\leq n}$ by the alternated sum

$$\mathrm{Det}M = \frac{1}{n!}\sum_{\sigma=(\sigma_1,\cdots,\sigma_k)\in\mathfrak{S}_n^k} \mathrm{sign}(\sigma)M^\sigma,$$

where $\mathrm{sign}(\sigma) = \mathrm{sign}(\sigma_1)\cdots\mathrm{sign}(\sigma_k)$, $M^\sigma = M_{\sigma_1(1)\ldots\sigma_k(1)}\cdots M_{\sigma_1(n)\ldots\sigma_k(n)}$ and $\mathfrak{S}_n$ is the symmetric group. A straightforward computation gives $\mathrm{Det}M = 0$

if k is odd.
For any k (even if k is odd), one defines the polynomial

$$\mathrm{Det}_1 M = \sum_{\sigma=(\mathrm{Id},\sigma_2,\cdots,\sigma_k)\in\mathfrak{S}_n^k} \mathrm{sign}(\sigma) M^\sigma.$$

When k is even the two notions coincide but for k odd, only Det_1 does not vanish. This is a special case of the "less-than-full-sign" determinant theory due to Rice [14].
Let us denote by $\mathfrak{F}$ a map from $\mathfrak{S}_n^{k-2}$ to a commutative ring. One defines a more general object, which will be called $\mathfrak{F}$-*determinant of* M by

$$\mathrm{Det}_{\mathfrak{F}}(M) = \sum_{\sigma=(\sigma_2,\cdots,\sigma_k)\in\mathfrak{S}_n^{k-1}} \mathrm{sign}(\sigma_2)\mathfrak{F}(\sigma_3,\ldots,\sigma_k)\prod_i M_{i\sigma_2(i)\ldots\sigma_k(i)}.$$

It exists an elementary identity which consists in expanding the $\mathfrak{F}$-determinant as a sum of $(n!)^{k-2}$ classical (2-way) determinants.

Lemma 2.1 *(Determinantal expansion)*
One has

$$\mathrm{Det}_{\mathfrak{F}} M = \sum_{\sigma_3,\ldots,\sigma_k} \mathfrak{F}(\sigma_3,\ldots,\sigma_k)\det(M^{\sigma_3,\ldots,\sigma_k}),$$

where $M^{\sigma_3,\ldots,\sigma_k}$ *denotes the* $n\times n$ *matrix such that* $M^{\sigma_3,\ldots,\sigma_k}_{i,j} = M_{i,j,\sigma_3(i),\ldots,\sigma_k(i)}$.

Proof – It suffices to remark that

$$\mathrm{Det}_{\mathfrak{F}} M = \sum_{\sigma_3,\ldots,\sigma_k} \mathfrak{F}(\sigma_3,\ldots,\sigma_k)\sum_{\sigma_2}\mathrm{sign}(\sigma_2)\prod_i M_{i\sigma_2(i)\sigma_3(i)\cdots\sigma_k(i)}$$

■

One of the most important property of hyperdeterminants is the invariance under the action of k copies of the special linear group. It is a very classical result which can be recover as a straightforward consequence of the following proposition.

Proposition 2.2 *The polynomial* $\mathrm{Det}_{\mathfrak{F}} M$ *is invariant under the action of linear group on* M *in the following sense*

$$\mathrm{Det}_{\mathfrak{F}} g.M = \det g \mathrm{Det}_{\mathfrak{F}} M, \tag{1}$$

where

$$g.M = \left(\sum_{1\le j_2\le n} g_{i_2 j_2} M_{i_1,j_2,i_3\cdots,i_k}\right)_{1\le i_1,\cdots,i_k\le n}.$$

Proof – By applying Lemma 2.1 to $g.M$, one gets

$$\begin{aligned}\mathrm{Det}_{\mathfrak{F}} g.M &= \sum_{\sigma_3,\ldots,\sigma_k} \mathfrak{F}(\sigma_3,\ldots,\sigma_k)\det(g.M^{\sigma_3,\ldots,\sigma_k}) \\ &= \sum_{\sigma_3,\ldots,\sigma_k} \mathfrak{F}(\sigma_3,\ldots,\sigma_k)\det g \det(M^{\sigma_3,\ldots,\sigma_k}) \\ &= \det g \mathrm{Det}_{\mathfrak{F}} M.\end{aligned}$$

■

3 Hyperdeterminants of meet hypermatrices

3.1 Meet semilattice

Consider a partially ordered finite set L so that every pairs $(x, y) \in L^2$ has a greatest lower bound denoted by $x \wedge y$. Such a poset is called a meet semilattice. One defines classically its ζ function by

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

Its Möbius function is the inverse of the zeta function and can be computed by the induction

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{x \leq z < y} \mu(z,y) & \text{if } x < y, \\ 0 & \text{in the other cases.} \end{cases}$$

If F and f verify the equality

$$F(x) = \sum_{y \leq x} f(y) = \sum_{y \in L} \zeta(y,x) f(y), \tag{2}$$

then, one has

$$f(x) = \sum_{y \in L} \mu(y,x) F(y) = (F \star \mu)(x) \tag{3}$$

where the symbol $\star$ means the convolution product.

3.2 Lindström Theorem

The factorization properties of the GCD determinants are the consequence of the semilattice structure of the integers with respect to divisibility and can be stated in a more general way. The manipulations of the identities (2) and (3) are the keys of the proof of Lindström's Theorem [11]. We recall its proof in a very slightly more general version.

For each $x \in L$, one considers a fixed element $z_x \leq x$. Let F_x be a function from L to $\mathbb{C}$ (or more generally to a commutative ring). Let M be the matrix defined by

$$M = (F_x(z_x \wedge y))_{x,y \in L}\,.$$

Remark that it suffices to define $F_x(z)$ only when $z \leq x$. In particular, one can suppose that $F_x(z) = F(z,x)$ is an incidence function, *i.e.*, $F(x,z) = 0$ unless $x \leq z$. One has

$$F_x(z_x \wedge y) = \sum_{z \in L} \zeta(z, z_x)\zeta(z,y) f_x(z),$$

where $f_x(z) = \sum_{y \in L} \mu(y,z) F_x(y)$. Hence, $\det M$ factorizes as the product

$$\det M = \det \Phi . \det Z,$$

where $\Phi = (\zeta(y, z_x) f_x(y))_{x,y \in L}$ and $Z = (\zeta(x,y))_{x,y \in L}$. As Φ and Z are triangular, $\det Z = 1$ and

$$\det \Phi = \prod_x \zeta(x, z_x) f_x(x) = \begin{cases} \prod_x f_x(y) & \text{if } z_x = x \text{ for each } x, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Then one obtains Lindström's Theorem.

Theorem 3.1 *(Lindström [11])*

$$\det\left(F_x(z_x \wedge y)\right)_{x,y \in L} = \begin{cases} \prod_x f_x(x) & \textit{if } z_x = x \textit{ for each } x, \\ 0 & \textit{otherwise.} \end{cases} \tag{5}$$

Note that, the original Lindstöm 's Theorem deals with the case where $z_x = x$ for each x. Furthermore, equality (4) generalizes a lemma of Cesaro.

Lemma 3.2 *(Cesaro)*
Denote by $\gcd_m(n) = \gcd(m,n)$. *One has,*

$$(\mu * (f \circ \gcd_m))(n) = \begin{cases} (f * \mu)(n) & \textit{if } m = n, \\ 0 & \textit{otherwise,} \end{cases} \tag{6}$$

where $*$ *is the Dirichlet convolution and* $\circ$ *is the composition of functions.*

3.3 Linström's Theorem for $\mathfrak{F}$-determinants

Lindstöm's Theorem can be extended to $\mathfrak{F}$-determinants.

Theorem 3.3 *(Lindström's theorem for $\mathfrak{F}$-determinants)*
If $L = \{x_1, \ldots, x_n\}$ denotes a meet semilattice, one has

$$\mathrm{Det}_{\mathfrak{F}}\left(F_{x_{i_1}}(z_{x_{i_1}} \wedge \cdots \wedge x_{i_k})\right) = \begin{cases} \mathfrak{F}(\mathrm{Id}, \cdots, \mathrm{Id}) \prod_x f_x(x) & \text{if } z_x = x \text{ for each } x \\ 0 & \text{otherwise} \end{cases}$$

Proof – Lemma 2.1 gives

$$\mathrm{Det}_{\mathfrak{F}}\left(F_{x_1}(z_{x_1} \wedge \cdots \wedge x_k)\right)_{x_1,\cdots,x_k \in L} = \sum_{\sigma_3,\ldots,\sigma_k} \mathfrak{F}(\sigma_3, \ldots, \sigma_k) \det\left(F_{x_i}(z_{x_i} \wedge x_j \wedge x_{\sigma_3(i)} \wedge \cdots \wedge x_{\sigma_k(i)}\right)_{i,j}$$

From Linström's Theorem (Theorem 3.1), one has

$$\det\left(F_{x_i}(z_{x_i} \wedge x_j \wedge x_{\sigma_3(i)} \wedge \cdots \wedge x_{\sigma_k(i)}\right) \neq 0$$

if and only if for each x_i one has

$$z_{x_i} \wedge x_{\sigma_3(i)} \wedge \cdots \wedge x_{\sigma_k}(i) = x_i.$$

Equivalently,

$$z_{x_i} \wedge x_{\sigma_3(i)} \wedge \cdots \wedge x_{\sigma_k}(i) \geq x_i,$$

for eacu i. Hence, $\sigma_3 = \cdots = \sigma_k = Id$ and $z_{x_i} = x_i$ for each i. The result follows. ∎

Example 3.4 *Consider the semilattice constituted with two elements $2 \geq 1$. The expansion of the $\mathfrak{F}$-determinant gives*

$$\begin{aligned} \mathrm{Det}_{\mathfrak{F}}(F_i(i \wedge j \wedge k \wedge l)) &= \mathfrak{F}(12,12) \begin{vmatrix} F_1(1) & F_1(1) \\ F_2(1) & F_2(2) \end{vmatrix} \\ &\quad + (\mathfrak{F}(12,21) + \mathfrak{F}(21,12) + \mathfrak{F}(21,21)) \begin{vmatrix} F_1(1) & F_1(1) \\ F_2(1) & F_2(1) \end{vmatrix} \\ &= \mathfrak{F}(12,12) F_1(1)(F_2(2) - F_2(1)) \\ &= \mathfrak{F}(12,12) f_1(2) f_2(2) \end{aligned}$$

4 Minors of meet Hypermatrices

4.1 Meet closed subsets

Consider a meet closed subset S of L (*i.e.*, a subset closed under the operation $\wedge$) and fix a linear extension $l = y_1 \dots y_n$ of S. Following the notations of [2], we denote by $x \trianglelefteq y_i$ the relation $x \leq y_i$ and $x \not\leq y_j$ for each $j < i$. Consider a pair of functions f and F verifying

$$F(y_i) = \sum_{\substack{x \leq y_i \\ x \in L}} f(x) \tag{7}$$

and set $\hat{f}(y_i) = \sum_{x \trianglelefteq y_i} f(x)$. One has the following lemma.

Lemma 4.1

$$F(y_i) = \sum_{y_k \leq y_i} \hat{f}(y_k). \tag{8}$$

Proof – Remarking that for each $x \in L$, it exists i such that $x \trianglelefteq y_i$ (it suffices to set $y_i = \min\{j | x \leq y_j\}$ and that $x \trianglelefteq y_i$ and $x \trianglelefteq y_j$ implies $i = j$, we have

$$F(y_i) = \sum_{\substack{x \leq y_i \\ x \in L}} f(x) = \sum_{y_k \leq y_i} \sum_{x \trianglelefteq y_k} f(x) = \sum_{y_k \leq y_i} \hat{f}(y_k).$$

■

Note that this identity appears in [2] (Theorem 4.1, p 7). Hence, using Theorem 3.3 and Lemma 4.1, one generalizes a result by Altinisik, Sagan and Tuglu ([2], Theorem 4.1 p 7).

Corollary 4.2

$$\mathrm{Det}_{\mathfrak{F}}(F_{y_{i_1}}(y_{i_1} \wedge \cdots \wedge y_{i_k})) = \mathfrak{F}(\mathrm{Id}, \cdots, \mathrm{Id}) \prod_{i=1}^{n} \left(\sum_{x_1 \trianglelefteq y_i} \sum_{x_2 \in L} \mu(x_1, x_2) F_{y_i}(x_2) \right).$$

Example 4.3 *Consider the semilattice L given by its Hasse diagram*

$$L = \begin{array}{ccc} 4 & & 5 \\ \uparrow & \nearrow & \uparrow \\ 2 & & 3 \\ \uparrow & \nearrow & \\ 1 & & \end{array}$$

where $i \to j$ means $i \leq j$. The sublattice generated by 2, 4 and 5,

$$S = \begin{array}{ccc} 4 & & 5 \\ \uparrow & \nearrow & \\ 2 & & \end{array}$$

is meet closed and

$$\begin{aligned} \hat{f}_2(2) &= f_2(2) + f_2(1) \\ \hat{f}_4(4) &= f_4(4) \\ \hat{f}_5(5) &= f_5(5) + f_5(3). \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Det}_{\mathfrak{F}}\left(F_i(i \wedge j \wedge k)\right)_{i,j,k \in S} &= \mathfrak{F}(123) \begin{vmatrix} F_2(2) & F_2(2) & F_2(2) \\ F_4(2) & F_4(4) & F_4(2) \\ F_5(2) & F_5(2) & F_5(5) \end{vmatrix} \\ &\quad + \mathfrak{F}(213) \begin{vmatrix} F_2(2) & F_2(2) & F_2(2) \\ F_4(2) & F_4(2) & F_4(2) \\ F_5(2) & F_5(2) & F_5(5) \end{vmatrix} \\ &\quad + \mathfrak{F}(321) \begin{vmatrix} F_2(2) & F_2(2) & F_2(2) \\ F_4(2) & F_4(2) & F_4(2) \\ F_5(2) & F_5(2) & F_5(2) \end{vmatrix} \\ &\quad + (\mathfrak{F}(132) + \mathfrak{F}(231) + \mathfrak{F}(312)) \begin{vmatrix} F_2(2) & F_2(2) & F_2(2) \\ F_4(2) & F_4(2) & F_4(2) \\ F_5(2) & F_5(2) & F_4(2) \end{vmatrix} \end{aligned}$$

The permutation 123 is the only one having a non zero contribution in this sum. Hence,

$$\begin{aligned} \mathrm{Det}_{\mathfrak{F}}\left(F_i(i \wedge j \wedge k)\right)_{i,j,k \in S} &= \mathfrak{F}(123) F_2(2)(F_4(4) - F_4(2))(F_5(5) - F_5(2)) \\ &= \mathfrak{F}(123)(f_2(2) + f_2(1)) f_4(4)(f_5(5) + f_5(3)) \\ &= \mathfrak{F}(123) \hat{f}_2(2) \hat{f}_4(4) \hat{f}_5(5) \end{aligned}$$

Remark 4.4 *If L is the semilattice structure of the integers with respect to divisibility. By the specialization $\mathfrak{F}(\sigma_3, \ldots, \sigma_k) = sign(\sigma_3) \ldots sign(\sigma_k)$, one recovers the computations of Lehmer [9] as a special case of Theorem 3.3 and the result of Haukkanen ([12] Theorem 1. p 56) from Corollary 4.2.*

4.2 Factor closed subsets

Let S be a factor closed subset of L. Then, $f = \hat{f}$ and

$$\mathrm{Det}_{\mathfrak{F}}\left(F_{x_1}(x_1 \wedge \cdots \wedge x_k)\right)_{x_1,\ldots,x_k \in S} = \mathfrak{F}(\mathrm{Id}, \cdots, \mathrm{Id}) \prod_x f_x(x). \qquad (9)$$

As special cases of equality (9), one recovers Lehmer's identities [9] and the original result of Smith [15].

4.3 General case

Let $X = \{x_1, \ldots, x_n\}$ be a subposet of a meet semi-lattice L. We will denote by $\overline{X} = \{x_1, \ldots, x_n, x_{n+1}, \ldots, x_m\}$ the smallest factor-closed subset of L containing X. The aim of this section consists in investigating the F-determinant

$$DF(X) := \mathrm{Det}_{\mathfrak{F}} \left(F_{x_{i_1}}(z_{x_{i_1}} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}) \right)_{1 \leq i_1, \ldots, i_k \leq n} \tag{10}$$

where for each $x \in X$, z_x denotes a fixed element of $\overline{X}$ such that $z \leq x$. As in the previous section, the result follows from the case $k = 2$ and Proposition 2.1.
Let us consider first the determinant $\det(F_{x_i}(z_{x_i} \wedge x_j))_{1 \leq i,j \leq n}$. The functions F_{x_i} can be chosen such that $F(x, y)$ is an incidence function. The set $\overline{X}$ being closed by factors, the functions f_x and $\hat{f}_x$ are equal. Hence,

$$F_{x_i}(z_{x_i} \wedge x_j) = \sum_{k=1}^{n+m} C_{x_i, x_k} \zeta(x_k, x_j) \tag{11}$$

where

$$C_{x,y} = f_x(y)\zeta(y, z_x) = \begin{cases} f_x(y) & \text{if } y \leq z_x \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

One has

Proposition 4.5

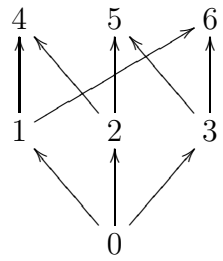
$$\det(F_{x_i}(x_i \wedge x_j)) = \sum_{1 \leq k_1 < \cdots < k_n \leq n+m} \det \left(C_{x_i, x_{k_j}} \right)_{1 \leq i,j \leq n} \det \left(\zeta(x_{k_i}, x_j) \right)_{1 \leq i,j \leq n}. \tag{13}$$

Proof – By multi-linearity, one obtains

$$\begin{aligned}
\det(F_{x_i}(x_i \wedge x_j)) &= \det\left(\sum_{k=1}^{n+m} C_{x_i,x_k}\zeta(x_k,x_j)\right) \\
&= \sum_{1\le k_1,\dots,k_n\le n+m} \det\left(C_{x_i,x_{k_j}}\zeta(x_{k_j},x_j)\right) \\
&= \sum_{1\le k_1,\dots,k_n\le n+m} \det\left(C_{x_i,x_{k_j}}\right)\prod_i \zeta(x_{k_i},x_i) \\
&= \sum_{1\le k_1<\cdots<k_n\le n+m} \det\left(C_{x_i,x_{k_j}}\right)_{1\le i,j\le n} \det\left(\zeta(x_{k_i},x_j)\right)_{1\le i,j\le n}
\end{aligned}$$

■

Example 4.6 *Let us consider the semilattice L whose Hasse diagram is*



and $X = \{4,5,6\}$. We set $z_4 = 1$, $z_5 = 2$ and $z_6 = 6$. Consider the determinant

$$\begin{aligned}
\det(F_i(z_i \wedge j \wedge k))_{i,j,k\in X} &= \begin{vmatrix} f_4(1)+f_4(0) & f_4(0) & f_4(1)+f_4(0) \\ f_5(2)+f_5(0) & f_5(2)+f_5(0) & f_5(0) \\ f_6(1)+f_6(0) & f_6(3)+f_6(0) & f_6(6)+f_6(3)+f_6(1)+f_6(0) \end{vmatrix} \\
&= -f_4(0)f_6(1)f_5(2) + f_4(1)f_{6\,0}f_5(2) + f_4(1)f_5(0)f_6(3) \\
&\quad + f_{4\,1}f_5(0)f_6(6) + f_4(0)f_5(2)f_{6\,3} + 2\,f_4(1)f_5(2)f_6(3) \\
&\quad + f_4(1)f_5(2)f_6(6)
\end{aligned}$$

Using the multilinearity of det*, one recovers the expression given by Propo-*

sition 4.5

$$\det(F_i(z_i\wedge j))=\left|\begin{array}{ccc} f_4(0) & f_4(1) & 0\\ f_5(0) & 0 & f_5(2)\\ f_6(0) & f_6(1) & 0\end{array}\right|\cdot\left|\begin{array}{ccc}1&1&0\\1&0&1\\1&1&0\end{array}\right|+\left|\begin{array}{ccc} f_4(0) & f_4(1) & 0\\ f_5(0) & 0 & 0\\ f_6(0) & f_6(1) & f_6(3)\end{array}\right|\cdot\left|\begin{array}{ccc}1&1&0\\1&0&0\\1&1&1\end{array}\right|$$
$$+\left|\begin{array}{ccc} f_4(0) & f_4(1) & 0\\ f_5(0) & 0 & 0\\ f_6(0) & f_6(1) & f_6(6)\end{array}\right|\cdot\left|\begin{array}{ccc}1&1&0\\1&0&0\\1&1&1\end{array}\right|+\left|\begin{array}{ccc} f_4(0) & 0 & 0\\ f_5(0) & f_5(2) & 0\\ f_6(0) & 0 & f_6(3)\end{array}\right|\cdot\left|\begin{array}{ccc}1&0&0\\1&1&0\\1&0&1\end{array}\right|$$
$$+\left|\begin{array}{ccc} f_4(1) & 0 & 0\\ 0 & f_5(2) & 0\\ f_6(1) & 0 & f_6(3)\end{array}\right|\cdot\left|\begin{array}{ccc}1&0&0\\0&1&0\\1&0&1\end{array}\right|+\left|\begin{array}{ccc} f_4(1) & 0 & 0\\ 0 & f_5(2) & 0\\ f_6(1) & 0 & f_6(6)\end{array}\right|\cdot\left|\begin{array}{ccc}1&0&0\\0&1&0\\1&0&1\end{array}\right|$$

More generally, one has a multi-indexed version of Proposition 4.5.

Theorem 4.7

$$DF(X)=\sum_{1\leq k_1<\cdots<k_n\leq n+m}\mathrm{Det}_{\mathfrak{F}}\left(f_{x_{i_1}}(x_{k_{i_2}})\zeta(x_{k_{i_2}},z_{x_{i_1}}\wedge x_{i_3}\wedge\cdots\wedge x_{i_k})\right)\times \times\det\left(\zeta(x_{k_i},x_j)\right) \tag{14}$$

Proof – We use Lemma 2.1 to expand $DF(X)$ and we obtain

$$DF(X)=\sum_{\sigma_1,\ldots,\sigma_k}\mathfrak{F}(\sigma_1,\ldots,\sigma_k)\det\left(F_{x_i}(z_{x_i}\wedge x_j\wedge x_{\sigma_3(i)}\wedge\cdots\wedge x_{\sigma_k(i)})\right).$$

Now, by Proposition 4.5, one gets

$$\begin{aligned}DF(X) &= \sum_{\sigma_1,\ldots,\sigma_k}\mathfrak{F}(\sigma_1,\ldots,\sigma_k)\sum_{1\leq k_1<\cdots<k_n\leq n+m}\det\left(\zeta(x_{k_i},x_j)\right)\times\\ &\quad\times\det\left(f_{x_i}(x_{k_j}\zeta(x_{k_j},z_{x_i}\wedge x_{\sigma_3(i)}\wedge\cdots\wedge x_{\sigma_k(i)}))\right)\\ &= \sum_{1\leq k_1<\cdots<k_n\leq n+m}\det\left(\zeta(x_{k_i},x_j)\right)\sum_{\sigma_1,\ldots,\sigma_k}\mathfrak{F}(\sigma_1,\ldots,\sigma_k)\times\\ &\quad\times\det\left(f_{x_i}(x_{k_j}\zeta(x_{k_j},z_{x_i}\wedge x_{\sigma_3(i)}\wedge\cdots\wedge x_{\sigma_k(i)}))\right)\\ &= \sum_{1\leq k_1<\cdots<k_n\leq n+m}\mathrm{Det}_{\mathfrak{F}}\left(f_{x_{i_1}}(x_{k_{i_2}})\zeta(x_{k_{i_2}},z_{x_{i_1}}\wedge x_{i_3}\wedge\cdots\wedge x_{i_k})\right)\times\\ &\quad\times\det\left(\zeta(x_{k_i},x_j)\right)\end{aligned}$$

This ends the proof. ∎

Let set $F_{x_1}=F_{x_2}=\cdots=F_{x_n}=F$ then for $1\leq k_1,k_2,\ldots,k_n\leq n$, we get

Corollary 4.8

$$\mathrm{Det}_{\mathfrak{F}}\left(F(z_{x_{i_1}} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k})\right) = \sum_{1 \leq k_1 < \cdots < k_n \leq n+m} \prod_i f(x_{k_i}) \times$$
$$\times \mathrm{Det}_{\mathfrak{F}}\left(\zeta(x_{k_{i_2}}, z_{x_{i_1}} \wedge x_{i_3} \wedge \cdots \wedge x_{i_k})\right) \det\left(\zeta(x_{k_i}, x_j)\right)$$

Proof – By applying the equality,

$$\mathrm{Det}_{\mathfrak{F}}\left(f(x_{k_{i_1}})\zeta(x_{k_{i_2}}, z_{x_{i_1}} \wedge x_{i_3} \wedge \cdots \wedge x_{i_k})\right) =$$
$$\sum_{\sigma_3,\ldots,\sigma_k} \mathfrak{F}(\sigma_3, \cdots, \sigma_k) \det\left(f(x_{k_i})\zeta(x_{k_j}, z_{x_i} \wedge x_{\sigma_1(i)} \wedge \cdots \wedge x_{\sigma_k(i)})\right) =$$
$$\prod_i f(x_{k_i}) \mathrm{Det}_{\mathfrak{F}}\left(\zeta(x_{k_{i_2}}, z_{x_{i_1}} \wedge x_{i_3} \wedge \cdots \wedge x_{i_k})\right).$$

to identity 14, one obtains the result. ∎

Remark 4.9 *Assume that L is the integer lattice. Then, if $z_x = x$ for each $x \in X$ and $\mathfrak{F}(\sigma_3, \cdots, \sigma_k) = sign(\sigma_3) \ldots sign(\sigma_k)$ in Corollary 4.8, one recovers Theorem 2 in [12]. Moreover, Proposition 4.5 generalizes the result of Li[10].*

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