New entropy estimates for the Oldroyd-B model, and related models

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7th March 2007

Abstract

This short note presents the derivation of a new *a priori* estimate for the Oldroyd-B model. Such an estimate may provide useful information when investigating the long-time behaviour of macro-macro models, and the stability of numerical schemes. We show how this estimate can be used as a guideline to derive new estimates for other macroscopic models, like the FENE-P model.

1 Introduction

We consider the Oldroyd-B model:

$$\mathrm{Re}\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{u}\right) = (1-\varepsilon)\Delta \boldsymbol{u} - \nabla p + \mathrm{div}\, \boldsymbol{\tau}, \tag{1}$$

$$\mathrm{div}\,(\boldsymbol{u}) = 0, \tag{2}$$

$$\frac{\partial \boldsymbol{\tau}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{\tau} = \nabla \boldsymbol{u} \boldsymbol{\tau} + \boldsymbol{\tau}(\nabla \boldsymbol{u})^T - \frac{1}{\mathrm{We}}\boldsymbol{\tau} + \frac{\varepsilon}{\mathrm{We}}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right), \tag{3}$$

where the Reynolds number $\mathrm{Re} > 0$, the Weissenberg number $\mathrm{We} > 0$ and $\varepsilon \in (0,1)$ are some non-dimensional numbers. We suppose that the space variable $\boldsymbol{x}$ lives in a bounded domain $\mathcal{D}$ of $\mathbb{R}^d$. This system is supplied with initial conditions on the velocity $\boldsymbol{u}$ and on the stress tensor $\boldsymbol{\tau}$. For simplicity, we assume no-slip boundary conditions on the velocity $\boldsymbol{u}$:

$$\boldsymbol{u} = 0 \text{ on } \partial\mathcal{D}. \tag{4}$$

We *suppose* that the initial data and the geometry are such that there exists a unique regular solution to (1)–(3) and our aim is to derive some *a priori* estimates on this solution.

Let us introduce the so-called conformation tensor $\boldsymbol{A} = \frac{\mathrm{We}}{\varepsilon}\boldsymbol{\tau} + \mathrm{Id}$. The partial differential equation (PDE) on $\boldsymbol{\tau}$ translates into the following PDE on $\boldsymbol{A}$:

$$\frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{A} = \nabla \boldsymbol{u} \boldsymbol{A} + \boldsymbol{A}(\nabla \boldsymbol{u})^T - \frac{1}{\mathrm{We}}\boldsymbol{A} + \frac{1}{\mathrm{We}}\mathrm{Id}. \tag{5}$$

One can check that if

$$\boldsymbol{A}(t=0) = \frac{\mathrm{We}}{\varepsilon}\boldsymbol{\tau}(t=0) + \mathrm{Id} \text{ is a positive definite symmetric matrix,} \tag{6}$$

then this property is propagated forward in time by (5) (and, in particular, $\boldsymbol{\tau}$ is symmetric). Assuming uniqueness of solution, this can be proven for example by using the probabilistic interpretation of $\boldsymbol{A}$ as a covariance matrix, as explained in Section 3. We will assume throughout this note that (6) is satisfied. Concerning the importance of positive-definiteness of $\boldsymbol{A}$, we refer for example to [7, Section 9.8.10] and also to the recent work [3, 4].

In Section 2, we recall how the classical *a priori* estimate for the Oldroyd-B model is derived. Next we show how it can be used to derive some bounds on the stress tensor, provided the initial condition satisfies $\det \boldsymbol{A}(t=0) > 1$. In Section 3, we establish a new estimate, which comes from an entropy estimate on the micro-macro model associated with the Oldroyd-B model (see [5]). This estimate provides bounds on $(\boldsymbol{u}, \boldsymbol{\tau})$ without any assumption on $\boldsymbol{\tau}(t=0)$ (apart from (6)). This new estimate could be useful to study the longtime behaviour of some macro-macro models, or to analyze the stability of some numerical schemes. Current research is directed towards clarifying this.

2 The classical estimate

Let us first introduce the kinetic energy:

$$E(t) = \frac{1}{2} \int_{\mathcal{D}} |\boldsymbol{u}|^2. \tag{7}$$

We easily obtain from (1)–(2):

$$\mathrm{Re}\frac{dE}{dt} = -(1-\varepsilon) \int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2 - \int_{\mathcal{D}} \boldsymbol{\tau} : \nabla \boldsymbol{u}, \tag{8}$$

where for two matrices A and B, we denote $A : B = A_{i,j} B_{i,j} = \mathrm{tr}\,(A^T B)$. On the other hand, taking the trace of the PDE (3) on $\boldsymbol{\tau}$ and integrating over $\mathcal{D}$, we get:

$$\frac{d}{dt} \int_{\mathcal{D}} \mathrm{tr}\, \boldsymbol{\tau} = 2 \int_{\mathcal{D}} \nabla \boldsymbol{u} : \boldsymbol{\tau} - \frac{1}{\mathrm{We}} \int_{\mathcal{D}} \mathrm{tr}\, \boldsymbol{\tau}.$$

We thus obtain the following estimate:

$$\begin{aligned} &\frac{d}{dt} \left(\frac{\mathrm{Re}}{2} \int_{\mathcal{D}} |\boldsymbol{u}|^2 + \frac{1}{2} \int_{\mathcal{D}} \mathrm{tr}\, \boldsymbol{\tau} \right) \\ &+(1-\varepsilon) \int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2 + \frac{1}{2\mathrm{We}} \int_{\mathcal{D}} \mathrm{tr}\, \boldsymbol{\tau} = 0. \end{aligned} \tag{9}$$

Remark 1 *In terms of $\boldsymbol{A}$, the energy estimate (9) writes:*

$$\begin{aligned} &\frac{d}{dt} \left(\frac{Re}{2} \int_{\mathcal{D}} |\boldsymbol{u}|^2 + \frac{\varepsilon}{2\,We} \int_{\mathcal{D}} \mathrm{tr}\, \boldsymbol{A} \right) \\ &+(1-\varepsilon) \int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2 + \frac{\varepsilon}{2\,We^2} \int_{\mathcal{D}} \mathrm{tr}\,(\boldsymbol{A} - Id) = 0. \end{aligned} \tag{10}$$

In Lemma 1 below, we prove that $\mathrm{tr}\, \boldsymbol{\tau}$ is positive if $\det \boldsymbol{A}(t=0) > 1$. This result combined with the estimate (9) thus yields some *a priori* bounds on $(\boldsymbol{u}, \tau)$ provided $\det(\boldsymbol{A})(t=0) > 1$. In particular, it shows that $\boldsymbol{u}$ and $\boldsymbol{\tau}$ go exponentially fast to 0 in the long time limit, using (9) and the Poincaré inequality: $\int_{\mathcal{D}} |\boldsymbol{u}|^2 \leq C \int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2$.

Lemma 1 *Let us assume that* $\det \boldsymbol{A}(t=0) > 1$. *Then, we have* $\forall t \geq 0$, $\det \boldsymbol{A}(t) > 1$ *and this implies that* $\mathrm{tr}\, \boldsymbol{\tau}(t) > 0$.

Proof: Using (5) and the Jacobi identity (which states that for any invertible matrix M depending smoothly on a parameter t, $\frac{d}{dt} \ln \det M = \mathrm{tr}\, \left(M^{-1} \frac{dM}{dt} \right))$, we have:

$$\frac{\partial \ln(\det \boldsymbol{A})}{\partial t} + \boldsymbol{u}.\nabla \ln(\det \boldsymbol{A}) = \frac{1}{\mathrm{We}} \mathrm{tr}\, \left(\boldsymbol{A}^{-1} - \mathrm{Id} \right). \tag{11}$$

Since for any symmetric positive matrix M of size $d \times d$,

$$(\det M)^{1/d} \leq (1/d)\mathrm{tr}\, M, \tag{12}$$

we obtain

$$\frac{\partial \ln(\det \boldsymbol{A})}{\partial t} + \boldsymbol{u}.\nabla \ln(\det \boldsymbol{A}) \geq \frac{d}{\mathrm{We}} \left((\det \boldsymbol{A})^{-1/d} - 1 \right),$$

which we can rewrite in terms of $y = (\det \boldsymbol{A})^{1/d}$:

$$\mathrm{We} \left(\frac{\partial y}{\partial t} + \boldsymbol{u}.\nabla y \right) \geq (1 - y). \tag{13}$$

This shows that $y > 1$ if $y(t = 0) > 1$, and thus that $\det \boldsymbol{A} > 1$ if $\det \boldsymbol{A}(t = 0) > 1$.

Indeed, using the characteristic method (by integrating the vector field $\boldsymbol{u}(t, \boldsymbol{x})$), one can rewrite (13) as

$$\mathrm{We} \frac{Dy}{Dt} \geq (1 - y).$$

Now, if y does not remain greater than 1, consider the first time t_0 such that $y(t_0) = 1$. We have on the one hand $\frac{Dy}{Dt}(t_0) < 0$ and, on the other hand $(1 - y(t_0)) = 0$. We reach a contradiction.

We thus have $\det \boldsymbol{A} > 1$ and therefore, using again (12), $\mathrm{tr}\, \boldsymbol{A} > d$. Since $\boldsymbol{\tau} = \frac{\varepsilon}{\mathrm{We}}(\boldsymbol{A} - \mathrm{Id})$, this is equivalent to $\mathrm{tr}\, \boldsymbol{\tau} > 0$. $\diamondsuit$

Remark 2 *If* $\det \boldsymbol{A}(t = 0) < 1$ *(which is the case if* $\mathrm{tr}\, \boldsymbol{\tau}(t = 0) < 0$*), Equation (13) shows that* $\det \boldsymbol{A}$ *grows along the characteristics as long as* $\det \boldsymbol{A} < 1$.

3 Entropy estimate

We now consider a micro-macro (or multiscale) formulation of the Oldroyd-B model and some estimates based on entropy, inspired from [5].

3.1 General derivation of the entropy estimate for micro-macro models

We consider the following system:

$$\left\{ \begin{array}{l} \mathrm{Re} \left(\dfrac{\partial \boldsymbol{u}}{\partial t}(t, \boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x}).\nabla \boldsymbol{u}(t, \boldsymbol{x}) \right) = (1 - \varepsilon)\Delta \boldsymbol{u}(t, \boldsymbol{x}) - \nabla p(t, \boldsymbol{x}) + \mathrm{div}\, \boldsymbol{\tau}(t, \boldsymbol{x}), \\ \mathrm{div}\, (\boldsymbol{u}(t, \boldsymbol{x})) = 0, \\ \boldsymbol{\tau}(t, \boldsymbol{x}) = \dfrac{\varepsilon}{\mathrm{We}} \left(\displaystyle\int_{\mathbf{R}^d} (\boldsymbol{X} \otimes \nabla \Pi(\boldsymbol{X})) \psi(t, \boldsymbol{x}, \boldsymbol{X})\, d\boldsymbol{X} - \mathrm{Id} \right), \\ \dfrac{\partial \psi}{\partial t}(t, \boldsymbol{x}, \boldsymbol{X}) + \boldsymbol{u}(t, \boldsymbol{x}).\nabla_{\boldsymbol{x}} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \\ \qquad = -\mathrm{div}_{\boldsymbol{X}} \left(\left(\nabla_{\boldsymbol{x}} \boldsymbol{u}(t, \boldsymbol{x}) \boldsymbol{X} - \dfrac{1}{2\mathrm{We}} \nabla \Pi(\boldsymbol{X}) \right) \psi(t, \boldsymbol{x}, \boldsymbol{X}) \right) + \dfrac{1}{2\mathrm{We}} \Delta_{\boldsymbol{X}} \psi(t, \boldsymbol{x}, \boldsymbol{X}). \end{array} \right. \tag{14}$$

This system is supplied with initial conditions on the velocity $\boldsymbol{u}$ and on the distribution ψ. We recall that we suppose no-slip boundary conditions (4) on the velocity $\boldsymbol{u}$. This system corresponds to a micro-macro model of polymeric fluids, the polymer being modelled by two beads linked by a spring with potential energy Π. The configurational variable $\boldsymbol{X} \in \mathbb{R}^d$ models the end-to-end vector of the polymer. For more details on the modelling, we refer to [1, 8].

Notice that we could rewrite the former system as a system coupling a PDE and a stochastic differential equation (SDE), replacing the last two equations by:

$$\boldsymbol{\tau}(t, \boldsymbol{x}) = \frac{\varepsilon}{\mathrm{We}} \Big(\mathbb{E} \left(\boldsymbol{X}_t(\boldsymbol{x}) \otimes \nabla \Pi(\boldsymbol{X}_t(\boldsymbol{x})) \right) - \mathrm{Id} \Big), \tag{15}$$

$$\begin{aligned} & d\boldsymbol{X}_t(\boldsymbol{x}) + \boldsymbol{u}(t, \boldsymbol{x}).\nabla_{\boldsymbol{x}} \boldsymbol{X}_t(\boldsymbol{x})\, dt \\ & = \left(\nabla_{\boldsymbol{x}} \boldsymbol{u}(t, \boldsymbol{x}) \boldsymbol{X}_t(\boldsymbol{x}) - \frac{1}{2\mathrm{We}} \nabla \Pi(\boldsymbol{X}_t(\boldsymbol{x})) \right) dt + \frac{1}{\sqrt{\mathrm{We}}} d\boldsymbol{W}_t. \end{aligned} \tag{16}$$

There, $\mathbb{E}$ denotes the expectation, $\boldsymbol{W}_t$ denotes a d-dimensional standard Brownian motion independent from the initial condition $(\boldsymbol{X}_0(\boldsymbol{x}))_{\boldsymbol{x}\in\mathcal{D}}$ which is such that, $\forall \boldsymbol{x} \in \mathcal{D}$, the law of $\boldsymbol{X}_0(\boldsymbol{x})$ is $\psi(0,\boldsymbol{x},\boldsymbol{X})\,d\boldsymbol{X}$.

Let us introduce the kinetic energy:

$$E(t) = \frac{1}{2}\int_{\mathcal{D}} |\boldsymbol{u}|^2. \tag{17}$$

We easily obtain:

$$\mathrm{Re}\frac{dE}{dt} = -(1-\varepsilon)\int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2 - \frac{\varepsilon}{\mathrm{We}}\int_{\mathcal{D}}\int_{\mathbf{R}^d} (\boldsymbol{X}\otimes\nabla\Pi(\boldsymbol{X})) : \nabla\boldsymbol{u}\,\psi. \tag{18}$$

We now introduce the entropy of the system, namely:

$$\begin{aligned} H(t) &= \int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi(t,\boldsymbol{x},\boldsymbol{X})\ln\left(\frac{\psi(t,\boldsymbol{x},\boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right), \\ &= \int_{\mathcal{D}}\int_{\mathbf{R}^d} \Pi(\boldsymbol{X})\psi(t,\boldsymbol{x},\boldsymbol{X}) + \int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi(t,\boldsymbol{x},\boldsymbol{X})\ln(\psi(t,\boldsymbol{x},\boldsymbol{X})) + C, \end{aligned} \tag{19}$$

with

$$\psi_\infty(\boldsymbol{X}) = \frac{\exp(-\Pi(\boldsymbol{X}))}{\int_{\mathbf{R}^d}\exp(-\Pi(\boldsymbol{X}))}, \tag{20}$$

and $C = \ln(\int_{\mathbf{R}^d}\exp(-\Pi(\boldsymbol{X})))|\mathcal{D}|$. The function H is actually the relative entropy of ψ with respect to the equilibrium distribution ψ_∞.

After some computations (see [5]), we obtain:

$$\frac{dH}{dt} = -\frac{1}{2\mathrm{We}}\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi\left|\nabla\ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 + \int_{\mathcal{D}}\int_{\mathbf{R}^d} (\boldsymbol{X}\otimes\nabla\Pi(\boldsymbol{X})) : \nabla\boldsymbol{u}\,\psi. \tag{21}$$

Therefore, introducing the free energy $F(t) = E(t) + \frac{\varepsilon}{\mathrm{We}}H(t)$ of the system, we have:

$$\boxed{\begin{aligned} &\frac{d}{dt}\left(\frac{\mathrm{Re}}{2}\int_{\mathcal{D}}|\boldsymbol{u}|^2 + \frac{\varepsilon}{\mathrm{We}}\int_{\mathcal{D}}\int_{\mathbf{R}^d}\psi\ln\left(\frac{\psi}{\psi_\infty}\right)\right) \\ &+(1-\varepsilon)\int_{\mathcal{D}}|\nabla\boldsymbol{u}|^2 + \frac{\varepsilon}{2\mathrm{We}^2}\int_{\mathcal{D}}\int_{\mathbf{R}^d}\psi\left|\nabla\ln\left(\frac{\psi}{\psi_\infty}\right)\right|^2 = 0. \end{aligned}} \tag{22}$$

Using a logarithmic Sobolev inequality with respect to ψ_∞ and a Poincaré inequality for $\boldsymbol{u}\in H^1_0(\mathcal{D})$, one can then obtain exponential convergence to equilibrium $\lim_{t\to\infty}(\boldsymbol{u},\psi) = (0,\psi_\infty)$ (see [5]). For some generalizations to the case $\boldsymbol{u}\neq 0$ on $\partial\mathcal{D}$, we refer to [5].

3.2 The Oldroyd-B case

Let us consider the Hookean dumbbell model, for which the potential Π of the entropic force is:

$$\Pi(\boldsymbol{X}) = \frac{||\boldsymbol{X}||^2}{2}. \tag{23}$$

By Itô's calculus, it is easy to derive from (16) that $\boldsymbol{A} = \mathbb{E}(\boldsymbol{X}_t\otimes\boldsymbol{X}_t)$ satisfies the following PDE:

$$\frac{\partial\boldsymbol{A}}{\partial t} + \boldsymbol{u}.\nabla\boldsymbol{A} = \nabla\boldsymbol{u}\boldsymbol{A} + \boldsymbol{A}(\nabla\boldsymbol{u})^T - \frac{1}{\mathrm{We}}\boldsymbol{A} + \frac{1}{\mathrm{We}}\mathrm{Id}. \tag{24}$$

This translates into the following PDE for $\boldsymbol{\tau} = \frac{\varepsilon}{\mathrm{We}}(\boldsymbol{A}-\mathrm{Id})$:

$$\frac{\partial\boldsymbol{\tau}}{\partial t} + \boldsymbol{u}.\nabla\boldsymbol{\tau} = \nabla\boldsymbol{u}\boldsymbol{\tau} + \boldsymbol{\tau}(\nabla\boldsymbol{u})^T - \frac{1}{\mathrm{We}}\boldsymbol{\tau} + \frac{\varepsilon}{\mathrm{We}}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right). \tag{25}$$

The Hookean dumbbell model is thus equivalent to the Oldroyd-B model (at least for regular enough solutions).

If $\psi(0, \boldsymbol{x}, .)$ is Gaussian (with zero mean), so is $\psi(t, \boldsymbol{x}, .)$:

$$\psi(t, \boldsymbol{x}, \boldsymbol{X}) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\boldsymbol{A})}} \exp\left(-\frac{\boldsymbol{X}^T\boldsymbol{A}^{-1}\boldsymbol{X}}{2}\right)$$

where $\boldsymbol{A} = \mathbb{E}(\boldsymbol{X}_t \otimes \boldsymbol{X}_t) = \int_{\mathbf{R}^d} \boldsymbol{X} \otimes \boldsymbol{X}\, \psi(t, \boldsymbol{x}, \boldsymbol{X})\, d\boldsymbol{X}$ denotes as above the covariance matrix of $\boldsymbol{X}_t$, which depends on time and also on the space variable $\boldsymbol{x}$. The covariance matrix $\boldsymbol{A}$ is symmetric and nonnegative. Moreover, since for almost all $t \geq 0$, $\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \ln\left(\frac{\psi(t,\boldsymbol{x},\boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right) < \infty$, then for almost all $t \geq 0$ and for almost all $\boldsymbol{x} \in \mathcal{D}$, $\boldsymbol{A}$ is positive.

The following explicit expression of the relative entropy can then be derived:

$$\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \ln\left(\frac{\psi(t, \boldsymbol{x}, \boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right) d\boldsymbol{X} = \int_{\mathcal{D}} \frac{1}{2}\left(-\ln(\det \boldsymbol{A}) - d + \operatorname{tr} \boldsymbol{A}\right).$$

On the other hand,

$$\int_{\mathcal{D}}\int_{\mathbf{R}^d} \psi(t, \boldsymbol{x}, \boldsymbol{X}) \left|\nabla_{\boldsymbol{X}} \ln\left(\frac{\psi(t, \boldsymbol{x}, \boldsymbol{X})}{\psi_\infty(\boldsymbol{X})}\right)\right|^2 d\boldsymbol{X} = \int_{\mathcal{D}} \operatorname{tr}\left((\mathrm{Id} - \boldsymbol{A}^{-1})^2\boldsymbol{A}\right).$$

Rewriting (22), we thus obtain the following estimate, in terms of $\boldsymbol{A}$:

$$\boxed{\begin{aligned}&\frac{d}{dt}\left(\frac{\mathrm{Re}}{2}\int_{\mathcal{D}} |\boldsymbol{u}|^2 + \frac{\varepsilon}{2\mathrm{We}}\int_{\mathcal{D}} \left(-\ln(\det \boldsymbol{A}) - d + \operatorname{tr} \boldsymbol{A}\right)\right)\\ &+(1-\varepsilon)\int_{\mathcal{D}} |\nabla \boldsymbol{u}|^2 + \frac{\varepsilon}{2\mathrm{We}^2}\int_{\mathcal{D}} \operatorname{tr}\left((\mathrm{Id} - \boldsymbol{A}^{-1})^2\boldsymbol{A}\right) = 0.\end{aligned}} \tag{26}$$

This is, in the specific case of Hookean dumbbells (that is Oldroyd-B model) the macroscopic version of (22).

Since $-\ln(\det(\boldsymbol{A})) - d + \operatorname{tr}(\boldsymbol{A}) \geq 0$, this energy estimate yields some *a priori* bounds on $(\boldsymbol{u}, \boldsymbol{A})$, and thus on $(\boldsymbol{u}, \boldsymbol{\tau})$. In sharp contrast to the classical estimate (9), it provides bounds on $(\boldsymbol{u}, \boldsymbol{\tau})$ without any assumption on $\boldsymbol{\tau}(t = 0)$ (apart from (6)). Using a Poincaré inequality and the fact[1] that, for any symmetric positive matrix M of size $d \times d$,

$$-\ln(\det M) - d + \operatorname{tr} M \leq \operatorname{tr}\left((\mathrm{Id} - M^{-1})^2 M\right)$$

exponential convergence to equilibrium ($\lim_{t\to\infty}(\boldsymbol{u}, \boldsymbol{A}) = (0, \mathrm{Id})$) can be obtained from (26).

Remark 3 *Notice that (26) can be schematically obtained as (10)*$-\dfrac{\varepsilon}{2\,We}\displaystyle\int_{\mathcal{D}}$ *(11).*

Remark 4 *If $\psi(0, \boldsymbol{x}, .)$ is not Gaussian, it is always possible to replace it by a Gaussian initial condition with the same mean and variance, so that the macroscopic quantities $(\boldsymbol{u}, p, \boldsymbol{A})$ would be the same for the two initial conditions.*

3.3 Application to related macroscopic models

The energy estimate (26) can be used as a guideline to derive energy estimates for other macroscopic models, even though they cannot be recast as a microscopic model of the form (14).

[1] which can be seen as the logarithmic Sobolev inequality for Gaussian random variables translated on their covariance matrices

Let us consider the example of the FENE-P model [9, 2], for which

$$\boldsymbol{\tau} = \frac{\varepsilon}{\mathrm{We}}\left(\frac{\boldsymbol{A}}{1-\mathrm{tr}\,(\boldsymbol{A})/b} - \mathrm{Id}\right), \tag{27}$$

$$\frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{u}.\nabla \boldsymbol{A} = \nabla \boldsymbol{u} \boldsymbol{A} + \boldsymbol{A}(\nabla \boldsymbol{u})^T - \frac{1}{\mathrm{We}}\frac{\boldsymbol{A}}{1-\mathrm{tr}\,(\boldsymbol{A})/b} + \frac{1}{\mathrm{We}}\mathrm{Id}. \tag{28}$$

For this model, we assume (6), and also that $\mathrm{tr}\,(\boldsymbol{A})(t=0) < b$, and this property is propagated forward in time by (28) (see [6]).

Using the same ideas as for the Oldroyd-B model, we consider the "entropy" $H(t) = -\ln(\det \boldsymbol{A}) - b\ln(1-\mathrm{tr}\,(\boldsymbol{A})/b)$, and we compute its time-derivative:

$$\frac{d}{dt}\int_{\mathcal{D}} -b\ln(1-\mathrm{tr}\,(\boldsymbol{A})/b) = 2\int_{\mathcal{D}} \frac{\nabla \boldsymbol{u} : \boldsymbol{A}}{1-\mathrm{tr}\,(\boldsymbol{A})/b} + \frac{1}{\mathrm{We}}\int_{\mathcal{D}}\left(-\frac{\mathrm{tr}\,(\boldsymbol{A})}{(1-\mathrm{tr}\,(\boldsymbol{A})/b)^2} + \frac{d}{1-\mathrm{tr}\,(\boldsymbol{A})/b}\right), \tag{29}$$

$$\frac{d}{dt}\int_{\mathcal{D}} \ln(\det(\boldsymbol{A})) = \frac{1}{\mathrm{We}}\int_{\mathcal{D}}\left(-\frac{d}{1-\mathrm{tr}\,(\boldsymbol{A})/b} + \mathrm{tr}\,(\boldsymbol{A}^{-1})\right). \tag{30}$$

Combining these expressions with (8), we obtain

$$\boxed{\begin{aligned}&\frac{d}{dt}\left(\frac{\mathrm{Re}}{2}\int_{\mathcal{D}}|\boldsymbol{u}|^2 + \frac{\varepsilon}{2\mathrm{We}}\int_{\mathcal{D}}\left(-\ln(\det \boldsymbol{A}) - b\ln(1-\mathrm{tr}\,(\boldsymbol{A})/b)\right)\right)\\ &+(1-\varepsilon)\int_{\mathcal{D}}|\nabla \boldsymbol{u}|^2 + \frac{\varepsilon}{2\mathrm{We}^2}\int_{\mathcal{D}}\left(\frac{\mathrm{tr}\,(\boldsymbol{A})}{(1-\mathrm{tr}\,(\boldsymbol{A})/b)^2} - \frac{2d}{1-\mathrm{tr}\,(\boldsymbol{A})/b} + \mathrm{tr}\,(\boldsymbol{A}^{-1})\right) = 0.\end{aligned}} \tag{31}$$

One can check that for any symmetric positive matrix M of size $d \times d$:

$$-\ln(\det(M)) - b\ln(1-\mathrm{tr}\,(M)/b) \geq -(b+d)\ln\left(\frac{b}{b+d}\right) \geq d \tag{32}$$

and that

$$-\ln(\det(M)) - b\ln(1-\mathrm{tr}\,(M)/b) + (b+d)\ln\left(\frac{b}{b+d}\right) \tag{33}$$

$$\leq \left(\frac{\mathrm{tr}\,(M)}{(1-\mathrm{tr}\,(M)/b)^2} - \frac{2d}{1-\mathrm{tr}\,(M)/b} + \mathrm{tr}\,(M^{-1})\right). \tag{34}$$

The proof of these inequalities is tedious and can be done by diagonalizing the matrix M.

Equation (32) shows that

$$\frac{\mathrm{Re}}{2}\int_{\mathcal{D}}|\boldsymbol{u}|^2 + \frac{\varepsilon}{2\mathrm{We}}\int_{\mathcal{D}}\left(-\ln(\det \boldsymbol{A}) - b\ln(1-\mathrm{tr}\,(\boldsymbol{A})/b) + (b+d)\ln\left(\frac{b}{b+d}\right)\right)$$

is a non-negative quantity, and thus that (31) indeed yields some *a priori* bounds on $(\boldsymbol{u}, \boldsymbol{A})$.

Equation (34) (which plays the role of the log-Sobolev inequality in the micro-macro models) shows that the estimate (31) can be used to prove exponential convergence to equilibrium.

References

[1] R.B. Bird, C.F. Curtiss, R.C. Armstrong, and O. Hassager. *Dynamics of polymeric liquids*, volume 2. Wiley Interscience, 1987.

[2] R.B. Bird, P.J. Dotson, and N.L. Johnson. Polymer solution rheology based on a finitely extensible bead-spring chain model. *J. Non-Newtonian Fluid Mech.*, 7:213–235, 1980. Errata: J. Non-Newtonian Fluid Mech., 8 (1981) 193.

[3] R. Fattal and R. Kupferman. Constitutive laws for the matrix-logarithm of the conformation tensor. *J. Non-Newtonian Fluid Mech.*, 123:281–285, 2004.

[4] R. Fattal and R. Kupferman. Time-dependent simulation of viscoelastic flows at high Weissenberg number using the log-conformation representation. *J. Non-Newtonian Fluid Mech.*, 126:23–37, 2005.

[5] B. Jourdain, C. Le Bris, T. Lelièvre, and F. Otto. Long-time asymptotics of a multiscale model for polymeric fluid flows. *Archive for Rational Mechanics and Analysis*, 181(1):97–148, 2006.

[6] B. Jourdain and T. Lelièvre. Convergence of a stochastic particle approximation of the stress tensor for the FENE-P model, 2004. CERMICS 2004-263 report.

[7] R. Keunings. *Fundamentals of Computer Modeling for Polymer Processing*, chapter Simulation of viscoelastic fluid flow, pages 402–470. Hanser, 1989.

[8] H.C. Öttinger. *Stochastic Processes in Polymeric Fluids*. Springer, 1995.

[9] A. Peterlin. Hydrodynamics of macromolecules in a velocity field with longitudinal gradient. *J. Polym. Sci. B*, 4:287–291, 1966.