Stochastic homogenization and random lattices

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Abstract

We present some variants of stochastic homogenization theory for scalar elliptic equations of the form $-\operatorname{div}\left[A\left(\dfrac{x}{\varepsilon},\omega\right)\nabla u(x,\omega)\right] = f$. These variants basically consist in defining stochastic coefficients $A\left(\frac{x}{\varepsilon},\omega\right)$ from stochastic deformations (using random diffeormorphisms) of the periodic setting, as announced in [4]. The settings we define are not covered by the existing theories. We also clarify the relation between this type of questions and our construction, performed in [3, 5], of the energy of, both deterministic and stochastic, microscopic infinite sets of points in interaction.

Résumé

Nous présentons dans cet article quelques variantes de la théorie de l'homogénéisation stochastique pour les équations elliptiques scalaires de la forme $-\operatorname{div}\left[A\left(\frac{x}{\varepsilon},\omega\right)\nabla u(x,\omega)\right] = f$. Ces variantes consistent essentiellement à définir les coefficients $A\left(\frac{x}{\varepsilon},\omega\right)$ comme déformations stochastiques (par des difféomorphismes aléatoires) de coefficients périodiques. Ce travail a été annoncé dans [4]. Les cas que nous définissons ainsi ne sont pas inclus dans les théories existantes de l'homogénéisation stochastique. Nous établissons également un lien entre ce type de problème et celui de définir l'énergie moyenne d'un système infini de particules, que nous avons traité dans [3] pour le cas déterministe, et dans [5] pour le cas stochastique.

Contents

1 Introduction

We study homogenization for scalar elliptic equations in divergence form with random coefficients:

$$-\operatorname{div}\left[A_\varepsilon(x,\omega)\,\nabla u(x,\omega)\right] = f. \tag{1.1}$$

In this context, the purpose of this article is two-fold.

First, we aim at slightly extending the usual ergodic stationary setting (see for instance [2, 7]) by considering specific cases of random coefficients $A_\varepsilon(x,\omega)$, mainly of the form

$$A_\varepsilon(x,\omega) = A\left(\frac{x}{\varepsilon},\omega\right), \tag{1.2}$$

not covered by the existing theories. These coefficients are typically obtained using random deformations of periodic coefficients. A prototypical case of such coefficients reads:

$$A\left(\frac{x}{\varepsilon},\omega\right) = A_{per}\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right)\right), \tag{1.3}$$

where A_{per} is $\mathbb{Z}^d$-periodic, and Φ is almost surely a diffeomorphism. Its gradient $\nabla\Phi$ is assumed stationary in the sense

$$\forall k\in\mathbb{Z}^d,\quad \nabla\Phi(x+k,\omega) = \nabla\Phi(x,\tau_k\omega) \text{ almost everywhere in } x, \text{ almost surely},$$

for a certain ergodic group action τ. Note that, although this sounds as a special case of existing theories, it is not. The above setting has been introduced in our previous work [4], and is recalled in details in Sections 1.2 and 1.3. Several variants along this general line are examined here, in Sections 2 and 5. We show that all these variants allow for *explicit* homogenization results. That is, we are able to prove that homogenization holds and identify the homogenized limit, using corrector problems, which are shown to be well-posed. A specific case (developed in Section 3) is that of a diffeomorphism Φ in (1.3) that is a "small" perturbation of Identity. Then, using a Taylor expansion with respect to a small parameter measuring the perturbation, we are able to show that this specific stochastic homogenization setting reduces to some particular, new, situation of *periodic* homogenization.

Our second purpose is to clarify the relation between the above questions of homogenization theory and our long term endeavour to define the energy of an infinite set of point particles in interaction, as exposed in [3, 5]. The reader is likely to be less familiar with that latter problem than with the classical homogenization problem. So, let us recall it briefly. More will be said in Section 4. If we are given an infinite set of points x_i, say interacting with the two-body potential $W(x_i - x_j)$, it is an easy exercise to define the notion of *energy per particle* of this assembly of particles when the x_i are periodically arranged. Some slight extensions of periodicity, such as quasi-periodicity, may also be treated. The construction also applies to energy models more sophisticated than the two-body interaction chosen here for simplicity of exposition. We will not enter the details of such questions, which have been the subject of many publications of ours (and others) in the past years. On the other hand, when the positions of the particles are more general, defining the energy per particle is a challenging question. In [3, 5], we addressed that latter question, respectively for some "general" deterministic sets of points, and for sets of random points. We will return to this in Section 4. The point was to determine the appropriate geometric properties that allow for defining the energy. It turns out that the properties we exhibited for that purpose have their counterpart in homogenization theory. This is what we are going to show in Sections 4 and 5 of the present work. In the language of homogenization, the positions x_i of the point particles may intuitively be thought of as the *obstacles*, or, equivalently, the vertices of the unit cells. More mathematically, the positions x_i may be used to define, using a construction introduced in [3], an appropriate algebra of functions, namely the smallest algebra, closed for some uniform norm on $\mathbb{R}^d$ (say L^∞), containing functions of the form

$$a(x) = \sum_{i \in \mathbb{N}} \varphi(x - x_i),$$

with, say $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Taking the entries A_{ij} of the matrix A in (1.2) in this algebra, one may then ask the question of the homogenization of (1.1) within this algebra. Using this construction, we establish a correspondence between the homogenization problem and the, apparently distant, problem of definition of energies for sets of point particles.

Let us also mention that the developments below, in particular those of Section 2, foremost Section 3, are likely to yield new, appropriate numerical strategies for stochastic homogenization in these particular settings.

The present work has been announced as references Ref. 3 and Ref. 4 in [4], as far as the variants of stochastic homogenization theory are concerned. This is the content of Sections 2 and 3. It has also been announced as reference Ref. 8 in [5], for what regards the definition of the energy for a large class of random sets of points. This is some of the material contained in Sections 4 and 5.

1.1 Elliptic homogenization theory

To begin with, let us recall some basic ingredients of elliptic homogenization theory. At least this will serve as a preliminary to set the notation. For this purpose, we argue in the canonical *periodic* setting (see for instance [2, 6, 7] for all the details). We thus consider, on a regular domain $\mathcal{D}$ in $\mathbb{R}^d$, the problem

$$\begin{cases} -\operatorname{div}\left[A_{per}\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right] = f \quad \text{in} \quad \mathcal{D}, \\ u^\varepsilon = 0 \quad \text{on} \quad \partial\mathcal{D}, \end{cases} \tag{1.4}$$

where the matrix A_{per} is symmetric and $\mathbb{Z}^d$-periodic. Let us emphasize that in the present section we will manipulate for simplicity *symmetric* matrices, but that our arguments may be adapted, in the usual way, to cover the cases of non symmetric matrices.

The problem (1.4) is the homogenization problem (1.1) for $A_\varepsilon(x) = A_{per}\left(\frac{x}{\varepsilon}\right)$. The associated corrector problem reads, for p fixed in $\mathbb{R}^d$,

$$\begin{cases} -\operatorname{div}\left(A_{per}(y)\left(p + \nabla w_p\right)\right) = 0, \\ w_p \text{ is } \mathbb{Z}^d\text{-periodic.} \end{cases} \tag{1.5}$$

It has a unique solution up to the addition of a constant. Then, the homogenized coefficients read

$$A^*_{ij} = \int_Q \left(e_i + \nabla w_{e_i}(y)\right)^T A_{per}(y)\left(e_j + \nabla w_{e_j}(y)\right) dy = \int_Q \left(e_i + \nabla w_{e_i}(y)\right)^T A_{per}(y) e_j dy, \tag{1.6}$$

where Q is the unit cube. As ε goes to zero, the solution u^ε to (1.4) converges to u^* solution to

$$\begin{cases} -\operatorname{div}\left[A^*\nabla u^*\right] = f \quad \text{in} \quad \mathcal{D}, \\ u^* = 0 \quad \text{on} \quad \partial\mathcal{D}, \end{cases} \tag{1.7}$$

The convergence holds in $L^2(\mathcal{D})$, and weakly in $H^1_0(\mathcal{D})$. The correctors w_{e_i} (for e_i the canonical vectors of $\mathbb{R}^d$) may then also be used to "correct" u^* in order to identify the behavior of u^ε in the strong topology $H^1_0(\mathcal{D})$. All this is well known.

As one of our purpose in this article is to perform such a homogenization procedure in a much more general setting, it is useful for us to recall now the main mathematical ingredients used in the proof of the above assertions regarding the convergence of u^ε and the existence and uniqueness of u^*. There are several approaches to do so. We will primarily argue on the so-called *energy method* of Murat and Tartar. Then we shall mention the *two-scale convergence* approach by (independently) Nguetseng and Allaire.

The energy method The *energy method*, also termed *method of oscillating test functions*, is due to Murat and Tartar [15, 17]. It is based on the principle of *compensated compactness*. Let us briefly sketch their proof. We do this in the periodic setting, but it should be borne in mind that the approach is not restricted to periodic setting and has been designed for more general settings. First, remark that the solution u_ε of (1.4) is bounded in H^1, and thus converges, up to an extraction, weakly in H^1. Similarly, $A(x/\varepsilon)\nabla u_\varepsilon$ converges weakly in L^2:

$$\nabla u_\varepsilon \rightharpoonup \nabla u_0 \text{ in } L^2, \tag{1.8}$$

$$A_{per}\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon \rightharpoonup r_0 \text{ in } L^2. \tag{1.9}$$

Then, the solution w_p to (1.5) satisfies

$$\nabla w_p\left(\frac{x}{\varepsilon}\right) \overset{*}{\rightharpoonup} \langle \nabla w_p\rangle = 0, \tag{1.10}$$

(where $\langle\cdot\rangle$ denotes the average in the periodic setting), thus:

$$p + \nabla w_p\left(\frac{x}{\varepsilon}\right) \overset{*}{\rightharpoonup} p \text{ in } L^\infty. \tag{1.11}$$

In addition,

$$A_{per}^T\left(\frac{x}{\varepsilon}\right)\left(p + \nabla w_p\left(\frac{x}{\varepsilon}\right)\right) \overset{*}{\rightharpoonup} (A^*)^T p \text{ in } L^\infty. \tag{1.12}$$

Next, pointing out that (1.8) and (1.11) are curl-free, and (1.9) and (1.12) are divergence-free, the compensated compactness principle [15, 17] (or, more precisely here, the celebrated *div-curl Lemma*) allows to pass to the limit in both sides of

$$\left[A_{per}^T\left(\frac{x}{\varepsilon}\right)\left(p + \nabla w_p\left(\frac{x}{\varepsilon}\right)\right)\right]^T \nabla u_\varepsilon = \left(p + \nabla w_p(\frac{x}{\varepsilon})\right)^T \left[A_{per}\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right],$$

getting

$$\left[(A^*)^T p\right]^T \nabla u_0 = p^T r_0.$$

All this is valid for any $p \in \mathbb{R}^d$. This, along with $-\operatorname{div}(r_0) = f$, gives the homogenized equation.

Looking back at the proof we have just outlined, we see that the main two ingredients have been

(i) the weak convergence of rescaled functions (for A_{per} and ∇w_p),

(ii) the well-posedness of the corrector problem (that is (1.5)),

along with the compensated compactness principle. At least formally, we may say that whenever we may define a setting for homogenization for which the above two properties (i)-(ii) are satisfied, we will be in position to apply the *energy method*, and perform an explicit homogenization of our equation. This will be exemplified by our argument in Section 1.3.

The two-scale convergence method. This method was first introduced by NGuetseng [10], and further developed by Allaire [1]. In contrast to the above energy method, it was originally introduced to deal with the periodic setting. In this setting, the crucial tool (which in some sense plays the role of the compensated compactness principle in the preceding method) is that any bounded sequence u_ε in H^1 satisfy the following convergences (up to extraction of a subsequence):

$$u_\varepsilon \rightharpoonup u_0 \text{ in } H^1,$$
$$\forall \xi \in L^2(\mathcal{D}, L^2_{\text{per}}(Q)), \quad \int_{\mathcal{D}} \nabla u_\varepsilon \xi\left(x, \frac{x}{\varepsilon}\right) dx \longrightarrow \int_{\mathcal{D}} \int_Q (\nabla u_0(x) + \nabla_y u_1(x,y)) \xi(x,y) dy dx,$$

for some $u_1 \in L^2(\mathcal{D}, H^1_{\text{per}}(Q))$. Using this result, the proof of homogenization goes as follows: we multiply the first line of (1.4) by $\varphi_0(x) + \varepsilon \varphi_1\left(x, \frac{x}{\varepsilon}\right)$, where $\varphi_0 \in H^1(\mathcal{D})$ and $\varphi_1 \in H^1(\mathcal{D}, H^1_{\text{per}}(Q))$, and use $\xi(x,y) = A(y)(\nabla \varphi_0(x) + \nabla_y \varphi_1(x,y))$ in the above convergence. This implies

$$\int_{\mathcal{D}} \int_Q (\nabla u_0(x) + \nabla_y u_1(x,y))\, A(y)(\nabla \varphi_0(x) + \nabla_y \varphi_1(x,y)) dy dx = \int_{\mathcal{D}} f \varphi_0. \tag{1.13}$$

It follows

$$-\operatorname{div}_y \left[A(y)(\nabla u_0(x) + \nabla_y u_1(x,y)) dy\right] = 0,$$

in Q with periodic boundary condition. This implies that

$$u_1(x,y) = \sum_{i=1}^d \frac{\partial u_0}{\partial x_i}(x) w_{e_i}(x).$$

Inserting this equation into (1.13) gives the homogenized problem.

Again, we see that the convergence of rescaled functions plays a key role, together with the definition of the corrector problem (in fact implicitly contained in (1.13)). This somehow shows that our general belief on the key ingredients hold true, at least formally.

As we pointed out above, the two-scale convergence method was at first designed to deal with the periodic setting. However, it was then developed further to deal with much more general cases (see [11, 12, 13, 14]), which provide a nice, rather technical, framework for putting the above formal considerations into mathematical terms. It remains that it intrinsically exploits the fact that we have two different scales: a micro scale, which we denote by ε, and a macro scale, which we set equal to 1. This explains the words "two-scale" convergence. In some sense, what depends on the micro scale is set on some unit cell (which is the unit cell of the periodic lattice in the periodic case), giving an "explicit" corrector equation. We thus have in this case a more explicit way of computing homogenized coefficients than with the energy method.

We will not overview the works [11, 12, 13, 14] in the present introductory section, because we will comment on them in Section 5.

1.2 Some stochastic settings

The present section introduces a discrete and a continuous stationary ergodic setting. Both settings will be used to define homogenization problems more general than the periodic one overviewed in the previous section.

1.2.1 Discrete setting

In what follows, $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space. For any random variable $X \in L^1(\Omega, d\mathbb{P})$, we denote by $\mathbb{E}(X) = \int_\Omega X(\omega) d\mathbb{P}(\omega)$ its expectation value. We fix $d \in \mathbb{N}^*$, and assume that the group $(\mathbb{Z}^d, +)$ acts on Ω. We denote by $(\tau_k)_{k\in\mathbb{Z}^d}$ this action, and assume that it preserves the measure $\mathbb{P}$, i.e

$$\forall k \in \mathbb{Z}^d, \quad \forall A \in \mathcal{F}, \quad \mathbb{P}(\tau_k A) = \mathbb{P}(A). \tag{1.14}$$

We assume that τ is *ergodic*, that is,

$$\forall A \in \mathcal{F}, \quad \big(\forall k \in \mathbb{Z}^d, \quad \tau_k A = A\big) \Rightarrow (\mathbb{P}(A) = 0 \,\text{or}\, 1). \tag{1.15}$$

In addition, we define the following notion of stationarity: any $F \in L^1_{\text{loc}}\left(\mathbb{R}^d, L^1(\Omega)\right)$ is said to be *stationary* if

$$\forall k \in \mathbb{Z}^d, \quad F(x+k,\omega) = F(x, \tau_k\omega) \text{ almost everywhere in } x, \text{ almost surely.} \tag{1.16}$$

In this setting, the ergodic theorem [8, 16] can be stated as follows:

Theorem 1.1 (Ergodic theorem, [8, 16]) *Let $F \in L^\infty\left(\mathbb{R}^d, L^1(\Omega)\right)$ be a stationary random variable in the sense of (1.16). For $k = (k_1, k_2, \dots k_d) \in \mathbb{R}^d$, we set $|k|_\infty = \sup_{1\le i\le d} |k_i|$. Then*

$$\frac{1}{(2N+1)^d} \sum_{|k|_\infty \le N} F(x, \tau_k\omega) \underset{N\to\infty}{\longrightarrow} \mathbb{E}\left(F(x,\cdot)\right) \quad \text{in } L^\infty(\mathbb{R}^d), \text{ almost surely.} \tag{1.17}$$

This implies that (here, Q is the unit cube)

$$F\left(\frac{x}{\varepsilon}, \omega\right) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \mathbb{E}\left(\int_Q F(x,\cdot)dx\right) \quad \text{in } L^\infty(\mathbb{R}^d), \text{ almost surely.} \tag{1.18}$$

1.2.2 Continuous setting

Alternately to the discrete setting of Subsection 1.2.1, it is possible to define a continuous ergodic setting as follows. The probability space is here again denoted by $(\Omega, \mathcal{F}, \mathbb{P})$, the expectation value is $\mathbb{E}$.

We fix $d \in \mathbb{N}^*$, and assume that the group $(\mathbb{R}^d, +)$ acts on Ω. We denote by $(\tau_x)_{x\in\mathbb{R}^d}$ this action, and assume that it preserves the measure $\mathbb{P}$, i.e

$$\forall y \in \mathbb{R}^d, \quad \forall A \in \mathcal{F}, \quad \mathbb{P}(\tau_y A) = \mathbb{P}(A). \tag{1.19}$$

We assume that τ is *ergodic*, that is,

$$\forall A \in \mathcal{F}, \quad \big(\forall x \in \mathbb{R}^d, \quad \tau_x A = A\big) \Rightarrow (\mathbb{P}(A) = 0 \quad \text{or} \quad 1). \tag{1.20}$$

Accordingly, we define the notion of stationarity as follows: $F \in L^1_{\text{loc}}\left(\mathbb{R}^d, L^1(\Omega)\right)$ is said to be *stationary* if

$$\forall y \in \mathbb{R}^d, \quad F(x+y,\omega) = F(x, \tau_y\omega) \quad \text{almost everywhere in } x, \text{ almost surely.} \tag{1.21}$$

To emphasize one difference (among many others) between the discrete setting of the previous section and the continuous one of the present section, let us simply mention the specific situation of a $\mathbb{Z}^d$-periodic function F. It is a particular case of (1.16), when F is assumed to be deterministic. In contrast, it is a particular case of (1.21), when F is *genuinely* random, Ω is the d dimensional torus and $\tau_x y \equiv x + y$.

Note also that none of the two settings is a particular case of the other.

In the present continuous setting, the ergodic theorem [8, 16] can be stated as follows:

Theorem 1.2 (Ergodic theorem, [8, 16]) *Let $F \in L^\infty\left(\mathbb{R}^d, L^1(\Omega)\right)$ be a stationary random variable in the sense of (1.21). Then*

$$\frac{1}{|B_R|}\int_{B_R} F(x,\tau_y\omega)dy \underset{N\to\infty}{\longrightarrow} \mathbb{E}(F(x,\cdot)) = \mathbb{E}(F) \quad \text{in } L^\infty(\mathbb{R}^d), \text{ almost surely.} \tag{1.22}$$

This implies that

$$F\left(\frac{x}{\varepsilon},\omega\right) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \mathbb{E}(F) \quad \text{in } L^\infty(\mathbb{R}^d), \text{ almost surely.} \tag{1.23}$$

1.3 Stochastic deformations for periodic homogenization

Let us fix $\mathcal{D}$ an open smooth and bounded subset of $\mathbb{R}^d$, and $A(y) = [A_{ij}(y)]$ a square matrix of size d, which is $\mathbb{Z}^d$-periodic, and satisfies the following hypotheses:

$$\exists\gamma > 0 \quad / \quad \forall\xi\in\mathbb{R}^d, \quad \xi^T A(y)\xi \geq \gamma|\xi|^2, \quad \text{almost everywhere in } y\in\mathbb{R}^d, \tag{1.24}$$

$$\forall i,j\in\{1,2,\ldots,d\}, \quad A_{ij}\in L^\infty\left(\mathbb{R}^d\right). \tag{1.25}$$

We also introduce a probability space $(\Omega,\mathcal{F},\mathbb{P})$, like in the previous section. A natural, and well-known, extension of the periodic setting recalled in Section 1.1 is the *stationary ergodic setting* (which is *continuous* in the sense defined in the previous section). Namely, the problem under consideration is then

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon},\omega\right)\nabla u\right) = f \quad \text{in} \quad \mathcal{D}, \\ u = 0 \quad \text{on} \quad \partial\mathcal{D}, \end{cases} \tag{1.26}$$

The matrix A is assumed stationary in the sense of (1.21), that is

$$\forall y\in\mathbb{R}^d, \quad A(x+y,\omega) = A(x,\tau_y\omega) \quad \text{almost everywhere in } x, \text{ almost surely,} \tag{1.27}$$

where the action τ is ergodic in the sense of (1.20). The problem under consideration is thus the homogenization problem (1.1) for

$$A_\varepsilon(x,\omega) = A\left(\frac{x}{\varepsilon},\omega\right). \tag{1.28}$$

Then the standard results of stochastic homogenization hold [2, 7]. They generalize the periodic results recalled in Subsection 1.1 (recall from Subsection 1.2 that periodicity is a special case of the above stochastic setting). We will not recall them here.

In [4], we have introduced a specific stochastic setting, which is *not* a particular case of (1.27). Let us recall the results obtained there. We fix some $\mathbb{Z}^d$-periodic matrix A_{per} and we consider the homogenization problem (1.1) for

$$A_\varepsilon(x,\omega) = A_{per}\left(\Phi^{-1}(\left(\frac{x}{\varepsilon},\omega\right)\right), \tag{1.29}$$

that is, we consider the following problem:

$$\begin{cases} -\operatorname{div}\left(A_{per}\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right)\right)\nabla u\right) = f \quad \text{in} \quad \mathcal{D}, \\ u = 0 \quad \text{on} \quad \partial\mathcal{D}, \end{cases} \tag{1.30}$$

where the function $\Phi(\cdot,\omega)$ is assumed to be a diffeomorphism from $\mathbb{R}^d$ to $\mathbb{R}^d$ for $\mathbb{P}$-almost every ω. The diffeomorphism is assumed to additionally satisfy

$$\underset{\omega\in\Omega,\, x\in\mathbb{R}^d}{\text{EssInf}}\left[\det(\nabla\Phi(x,\omega))\right] = \nu > 0, \tag{1.31}$$

$$\underset{\omega\in\Omega,\, x\in\mathbb{R}^d}{\text{EssSup}}\left(|\nabla\Phi(x,\omega)|\right) = M < \infty, \tag{1.32}$$

$$\nabla\Phi(x,\omega) \quad \text{is stationary in the sense of (1.16).} \tag{1.33}$$

Such a Φ will be called a *random stationary diffeomorphism.*

We proved in [4] the following results:

Theorem 1.3 *Let A be a square matrix which is $\mathbb{Z}^d$-periodic and satisfies (1.24)-(1.25) and Φ a random stationary diffeomorphism satisfying hypotheses (1.31)-(1.32)-(1.33). Then for any $p \in \mathbb{R}^d$, the system*

$$\begin{cases} -\operatorname{div}\left[A_{per}\left(\Phi^{-1}(y,\omega)\right)(p+\nabla w_p)\right]=0, \\ w_p(y,\omega)=\tilde{w}_p\left(\Phi^{-1}(y,\omega),\omega\right), \quad \nabla\tilde{w}_p \quad \textit{is stationary in the sense of (1.16)}, \\ \mathbb{E}\left(\int_{\Phi(Q,\cdot)}\nabla w_p(y,\cdot)dy\right)=0, \end{cases} \tag{1.34}$$

has a solution in $\left\{w \in L^2_{\text{loc}}(\mathbb{R}^d, L^2(\Omega)), \quad \nabla w \in L^2_{\text{unif}}(\mathbb{R}^d, L^2(\Omega))\right\}$. Moreover, this solution is unique up to the addition of a (random) constant.

Theorem 1.4 *Let $\mathcal{D}$ be a bounded smooth open subset of $\mathbb{R}^d$, and let $f \in H^{-1}(\mathcal{D})$. Let A and Φ satisfy the hypotheses of Theorem 1.3. Then the solution $u_\varepsilon(x,\omega)$ of (1.30) satisfies the following properties:*

(i) $u_\varepsilon(x,\omega)$ converges to some $u_0(x)$ strongly in $L^2(\mathcal{D})$ and weakly in $H^1(\mathcal{D})$, almost surely;

(ii) the function u_0 is the solution to the homogenized problem:

$$\begin{cases} -\operatorname{div}(A^*\nabla u)=f \quad \textit{in} \quad \mathcal{D}, \\ u=0 \quad \textit{on} \quad \partial\mathcal{D}. \end{cases} \tag{1.35}$$

In (1.35), the homogenized matrix A^* is defined by:

$$A^*_{ij}=\det\left(\mathbb{E}\left(\int_Q\nabla\Phi(z,\cdot)dz\right)\right)^{-1}\mathbb{E}\left(\int_{\Phi(Q,\cdot)}(e_i+\nabla w_{e_i}(y,\cdot))^T A_{per}\left(\Phi^{-1}(y,\cdot)\right)e_j\,dy\right), \tag{1.36}$$

where for any $p \in \mathbb{R}^d$, w_p is the corrector defined by (1.34).

In the theorems above, we have used the notation L^2_{unif} for the *uniform* L^2 space. Because spaces of this type will play a role throughout this article, let us recall the standard definition of $W^{k,p}_{\text{unif}}$ spaces:

Definition 1.5 *for $k \in \mathbb{N}$ and $p \in [1,\infty]$,*

$$W^{k,p}_{\text{unif}}=\left\{f \in W^{k,p}_{\text{loc}}, \quad \sup_{x\in\mathbb{R}^3}\|f\|_{W^{k,p}(B_1+x)}<+\infty\right\}, \qquad \textit{and} \quad \|f\|_{W^{k,p}_{\text{unif}}}=\sup_{x\in\mathbb{R}^3}\|f\|_{W^{k,p}(B_1+x)}. \tag{1.37}$$

In the case $k=0$, we set $L^p_{\text{unif}}=W^{0,p}_{\text{unif}}$, and in the case $p=2$, we set $H^k_{\text{unif}}=W^{k,2}_{\text{unif}}$.

We will also need in the following sections the notion of normalized integral:

Definition 1.6 *For any open subset $\mathcal{D} \subset \mathbb{R}^\ni$ of finite measure, for any $f \in L^1(\Omega)$, we define the normalized integral of f by*

$$⨍_{\mathcal{D}} f := \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} f(x)dx. \tag{1.38}$$

We now outline the main ideas of the proofs of these results. The details may be found in [4]. First, the following lemma is a direct consequence of (1.18):

Lemma 1.7 *Let Φ be a random stationary diffeomorphism from $\mathbb{R}^d$ to $\mathbb{R}^d$, which satisfies (1.31)-(1.32)-(1.33). Then*

$$\varepsilon\Phi\left(\frac{x}{\varepsilon},\omega\right)\underset{\varepsilon\to 0}{\longrightarrow}\mathbb{E}\left(\int_Q \nabla\Phi\right)x \text{ in } L^\infty_{\rm loc}(\mathbb{R}^d), \text{ almost surely.} \tag{1.39}$$

In addition, the following type of convergence is needed:

Lemma 1.8 *Let Φ be a random stationary diffeomorphism satisfying (1.31)-(1.32)-(1.33). Assume that $g\in L^\infty\left(\mathbb{R}^d, L^1(\Omega)\right)$ is a stationary function in the sense of (1.16). Then*

$$g\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right),\omega\right)\underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}}\det\left(\mathbb{E}\left(\int_Q \nabla\Phi(x,\cdot)dx\right)\right)^{-1}\mathbb{E}\left(\int_{\Phi(Q,\cdot)} g\left(\Phi^{-1}(x,\cdot),\cdot\right)dx\right) \quad in \quad L^\infty(\mathbb{R}^d), \tag{1.40}$$

almost surely.

Remark 1.9 *Taking $g=1$ in Lemma 1.8 yields*

$$\det\left(\mathbb{E}\left(\int_Q \nabla\Phi\right)\right)=\mathbb{E}\left(\int_Q \det\left(\nabla\Phi\right)\right).$$

This equality stems from (1.18) applied to $F=\nabla\Phi$ and the fact that the determinant is (in particular) continuous for the L^∞ weak-$*$ topology.

Given the above lemmas, let us give now a sketch of the proof of Theorems 1.3 and 1.4. As announced in the introduction, it is a simple adaptation of the *energy method* introduced by Murat and Tartar [15, 17].

The proof of Theorem 1.3 is performed applying Lax-Milgram lemma to prove the existence and uniqueness of the solution w_p^α of

$$\begin{cases} -\operatorname{div}\left[A\left(\Phi^{-1}(y,\omega)\right)(p+\nabla w_p)\right]+\alpha w_p=0, \\ w_p(y,\omega)=\tilde{w}_p\left(\Phi^{-1}(y,\omega),\omega\right), \quad \tilde{w}_p \quad \text{is stationary in the sense of (1.16),} \end{cases} \tag{1.41}$$

and then passing to the limit $\alpha\to 0$.

In order to prove Theorem 1.4, we introduce the test-function $v_p(y,\omega)=\tilde{v}_p\left(\Phi^{-1}(y,\omega),\omega\right)$ which is the solution to (1.34) with A^T instead of A, and define $g(y,\omega)=\left(\nabla\Phi(y,\omega)\right)^{-1}\nabla\tilde{v}_p(y,\omega)$. Applying Lemma 1.8, one finds that

$$\nabla v_p\left(\frac{x}{\varepsilon},\omega\right)\underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}}0 \quad \text{in} \quad L^\infty(\mathbb{R}^d), \quad \text{almost surely.}$$

Similar arguments allow to prove that

$$A\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right)\right)^T\left[\nabla v_p\left(\frac{x}{\varepsilon},\omega\right)+p\right]\underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}}(A^*)^T p \quad \text{in} \quad L^\infty(\mathbb{R}^d), \quad \text{almost surely.}$$

On the other hand, considering the solution u_ε of (1.30), one easily proves that it is bounded in $H^1(\mathcal{D})$, and thus converges (up to extracting a subsequence) strongly in L^2 and weakly in H^1 to some u_0. The function $A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}\right),\omega\right)\nabla u_\varepsilon(x,\omega)$ is bounded in L^2, and thus converges (up to extracting a subsequence) weakly in L^2 to some r_0. Then, applying the div-curl lemma (see [15, 17]) to the product $A\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right)\right)^T\left[\nabla v_p\left(\frac{x}{\varepsilon},\omega\right)+p\right]\nabla u_\varepsilon$ on the one hand, and on the other hand to the product $\left[\nabla v_p\left(\frac{x}{\varepsilon},\omega\right)+p\right]A\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right)\right)\nabla u_\varepsilon$, one finds that

$$r_0 p = (A^*\nabla u_0)\,p.$$

Since this is valid for any $p\in\mathbb{R}^d$ and $-\operatorname{div}(r_0)=f$, this implies (1.39).

2 Extensions

2.1 Stationary coefficients

As pointed out in [4], the above results are also valid in the case of a stationary (instead of periodic) matrix, that is the homogenization problem (1.1) for

$$A_\varepsilon(x,\omega)=A\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right),\omega\right), \tag{2.1}$$

with A a stationary ergodic matrix (in the sense of (1.27)).

We give here the corresponding results. The proofs are similar to those of Subsection 1.2.1 above, so we again skip them.

In the case of a discrete ergodic setting, using the same notations as in section 1.2.1, we have:

Theorem 2.1 *Let A be a square matrix which is stationary and satisfies (1.24)-(1.25) and Φ a random stationary diffeomorphism satisfying hypotheses (1.31)-(1.32)-(1.33). Then for any $p\in\mathbb{R}^d$, the system*

$$\begin{cases} -\operatorname{div}\left[A\left(\Phi^{-1}(y,\omega),\omega\right)(p+\nabla w_p)\right]=0, \\ w_p(y,\omega)=\tilde{w}_p\left(\Phi^{-1}(y,\omega),\omega\right), \quad \nabla\tilde{w}_p \quad \textit{is stationary in the sense of (1.16)}, \\ \mathbb{E}\left(\displaystyle\int_{\Phi(Q)}\nabla w_p(y,\cdot)dy\right)=0, \end{cases} \tag{2.2}$$

has a solution in $\left\{w\in L^2_{\rm loc}(\mathbb{R}^d,L^2(\Omega)),\quad \nabla w\in L^2_{\rm unif}(\mathbb{R}^d,L^2(\Omega))\right\}$. This solution is unique up to the addition of a (random) constant.

Theorem 2.2 *Let $\mathcal{D}$ be a bounded smooth open subset of $\mathbb{R}^d$, and let $f\in H^{-1}(\mathcal{D})$. Let A and Φ satisfy the hypotheses of Theorem 2.1. Then the solution $u_\varepsilon(x,\omega)$ of (1.30) satisfies the following properties:*

(i) $u_\varepsilon(x,\omega)$ converges to some $u_0(x)$ strongly in $L^2(\mathcal{D})$ and weakly in $H^1(\mathcal{D})$, almost surely;

(ii) the function u_0 is a solution to the homogenized problem:

$$\begin{cases} -\operatorname{div}\left(A^*\nabla u\right)=f \quad in \quad \mathcal{D}, \\ u=0 \quad on \quad \partial\mathcal{D}. \end{cases} \tag{2.3}$$

In (2.3), the homogenized matrix A^* is defined by:

$$A^*_{ij}=\det\left(\mathbb{E}\left(\int_Q\nabla\Phi(z,\cdot)dz\right)\right)^{-1}\mathbb{E}\left(\int_{\Phi(Q,\cdot)}\left(e_i+\nabla w_{e_i}(y,\cdot)\right)^T A\left(\Phi^{-1}(y,\cdot),\cdot\right)e_j\,dy\right), \tag{2.4}$$

where for any $p\in\mathbb{R}^d$, w_p is the corrector defined by the system (2.2).

2.2 Ergodic Continuous setting

In this subsection, we extend the results of Subsection 2.1 to the setting of Subsection 1.2.2. Here again, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. But now, τ is an action of $\mathbb{R}^d$, and not $\mathbb{Z}^d$, on Ω. We assume that this action preserves the measure $\mathbb{P}$ and satisfies (1.20).

The application Φ is here again a diffeomorphism satisfying (1.31) and (1.32), but we replace (1.33) by

$$\forall x \in \mathbb{R}^d, \quad \forall y \in \mathbb{R}^d, \quad \nabla\Phi(x, \tau_y \omega) = \nabla\Phi(x+y, \omega) \quad \text{almost surely.} \tag{2.5}$$

The setting is again that of (1.29), with a different notion of stationarity for $\nabla\Phi$ though. We have:

Theorem 2.3 *Let A be a stationary (in the sense of (1.21)) square matrix satisfying (1.24)-(1.25) and Φ a random stationary diffeomorphism satisfying hypotheses (1.31)-(1.32)-(2.5). Then for any $p \in \mathbb{R}^d$, the system*

$$\begin{cases} -\operatorname{div}\left[A\left(\Phi^{-1}(y,\omega),\omega\right)(p+\nabla w_p)\right] = 0, \\ w_p(y,\omega) = \tilde{w}_p\left(\Phi^{-1}(y,\omega),\omega\right), \quad \nabla\tilde{w}_p \quad \textit{is stationary in the sense of (1.21)}, \\ \mathbb{E}\left((\nabla w_p \circ \Phi)\det(\nabla\Phi)\right) = 0, \end{cases} \tag{2.6}$$

has a solution in $\left\{w \in L^2_{\text{loc}}(\mathbb{R}^d, L^2(\Omega)), \quad \nabla w \in L^2_{\text{unif}}(\mathbb{R}^d, L^2(\Omega))\right\}$. This solution is unique up to the addition of a (random) constant.

Theorem 2.4 *Let $\mathcal{D}$ be a bounded smooth open subset of $\mathbb{R}^d$, and let $f \in H^{-1}(\mathcal{D})$. Let A and Φ satisfy the hypotheses of Theorem 2.3. Then the solution $u_\varepsilon(x,\omega)$ of (1.30) satisfies the following properties:*

(i) $u_\varepsilon(x,\omega)$ converges to some $u_0(x)$ strongly in $L^2(\mathcal{D})$ and weakly in $H^1(\mathcal{D})$, almost surely;

(ii) the function u_0 is a solution to the homogenized problem:

$$\begin{cases} -\operatorname{div}(A^*\nabla u) = f \quad \textit{in} \quad \mathcal{D}, \\ u = 0 \quad \textit{on} \quad \partial\mathcal{D}. \end{cases} \tag{2.7}$$

In (2.7), the homogenized matrix A^* is defined by:

$$A^*_{ij} = \det\left(\mathbb{E}(\nabla\Phi)\right)^{-1}\mathbb{E}\left[\left(e_i + \nabla w_{e_i}\circ\Phi\right)^T A e_j \det(\nabla\Phi)\right], \tag{2.8}$$

where, for any $p \in \mathbb{R}^d$, w_p is the corrector defined by the system (2.6).

The proofs follow the same pattern as for the discrete case. Formally, one only needs to replace the notion of average of the discrete case, namely $\mathbb{E}\int_Q$, by that for the continuous case, that is, $\mathbb{E}$. For instance, the convergence analogous to (1.40) reads (assuming g is stationary)

$$g\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right),\omega\right) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \det\left(\mathbb{E}(\nabla\Phi)\right)^{-1}\mathbb{E}\left(g\det(\nabla\Phi)\right) \quad \text{in} \quad L^\infty(\mathbb{R}^d),$$

almost surely. Indeed, in the discrete setting,

$$\mathbb{E}\left(\int_{\Phi(Q,\cdot)} g\left(\Phi^{-1}(x,\cdot),\cdot\right)dx\right) = \mathbb{E}\left(\int_Q g(y,\cdot)\det(\nabla\Phi(y,\cdot))dy\right).$$

We therefore skip the proofs, which are a straightforward adaptation of our previous arguments, once the above formal analogy has been observed.

2.3 Problems with two different types of stationarity

A natural question is to try and carry out a similar analysis in the case of two different notion of stationarity, respectively for A and for $\nabla\Phi$? The answer is not clear to us, but we give in this subsection some information about a somewhat related, interesting, case.

Let us consider a $\mathbb{Z}^d$-periodic matrix A satisfying (1.24)-(1.25), and Φ a (deterministic) diffeomorphism satisfying (1.31) and (1.32), but with

$$\nabla\Phi \text{ is } \alpha\mathbb{Z}^d-\text{periodic}, \tag{2.9}$$

with $\alpha \notin \mathbb{Q}$. It is possible to carry out the same homogenization theory as above. Of course, this is a particular case of almost periodicity, which is treated for instance in [7]. However, the explicit feature of the present case seems interesting to us. For instance, proving a convergence result similar to Lemma 1.8 (which actually solves the homogenization problem in dimension one) requires the existence of

$$\lim_{\varepsilon\to 0} A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}\right)\right),$$

in L^∞-weak-$*$. In order to compute this limit, we proceed as follows: for any set open bounded subset B of $\mathbb{R}^d$, we have

$$\int_B A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}\right)\right)dx \quad = \quad \int_{\varepsilon\Phi\left(\frac{B}{\varepsilon}\right)} A\left(\frac{y}{\varepsilon}\right)\det\left(\nabla\Phi\left(\frac{y}{\varepsilon}\right)\right)dy.$$

Now, the function $\mathbf{1}_{\varepsilon\Phi\left(\frac{B}{\varepsilon}\right)}$ converges in $L^1(\mathbb{R}^d)$ to $\mathbf{1}_{⨍_{\alpha Q}\nabla\Phi B}$. Hence, in order to pass to the limit in the above formula, we need to investigate the L^∞-weak-$*$ limit of $A\left(\frac{y}{\varepsilon}\right)\det\left(\nabla\Phi\left(\frac{y}{\varepsilon}\right)\right)$. For this purpose, we write (here, $|i|_\infty = \max\{|i_k|, 1\le k\le d\}$)

$$\begin{aligned}\frac{1}{(2N+1)^d}\int_{(2N+1)Q} A(y)\det(\nabla\Phi(y))dy &= \frac{1}{(2N+1)^d}\sum_{i\in\mathbb{Z}^d,|i|_\infty\le N}\int_{Q+i} A(y)\det(\nabla\Phi(y))dy\\ &= \frac{1}{(2N+1)^d}\sum_{i\in\mathbb{Z}^d,|i|_\infty\le N}\int_{Q} A(y)\det(\nabla\Phi(y+i))dy.\end{aligned}$$

Setting $F(z) = \int_Q A(y)\det(\nabla\Phi(y+z))dy$, we see that F is $\alpha\mathbb{Z}^d$-periodic, which implies that

$$\frac{1}{(2N+1)^d}\sum_{i\in\mathbb{Z}^d,|i|_\infty\le N} F(i) \underset{N\to\infty}{\longrightarrow} ⨍_{\alpha Q} F,$$

where the normalized integral $⨍$ is defined by (1.38). As a consequence, we infer

$$A\left(\frac{y}{\varepsilon}\right)\det\left(\nabla\Phi\left(\frac{y}{\varepsilon}\right)\right) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \int_Q A ⨍_{\alpha Q}\det(\nabla\Phi). \tag{2.10}$$

This implies

$$A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}\right)\right) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \int_Q A.$$

The above argument does not only identify the limit of $A\left(\Phi^{-1}\left(\frac{x}{\varepsilon}\right)\right)$. Indeed, the convergence (2.10) gives a more general view of this convergence: we have an explicit form for the limit of product of functions with different period. This replaces Lemma 1.8 in the homogenization framework we are dealing with, and hence will allow to carry out the corresponding theory. We will not go further in this direction (see however Subsection 3.2 for a remark on the present setting).

Finally, note that the situation completely changes if $\alpha \in \mathbb{Q}$. Then, A and $\nabla\Phi$ share a periodic cell, which may be used to compute averages of the type (2.10), leading to the standard periodic homogenization theory.

3 Perturbations of identity

In this subsection, we return to the setting of Subsection 1.2.1, and consider the specific case when the diffeomorphism Φ is close to the identity. Then the matrix $A_{per}(\Phi^{-1}(y,\omega))$ is, formally, close to the periodic matrix $A_{per}(y)$. The expansion

$$\Phi(y,\omega) = y + \eta\Psi(y,\omega) + O(\eta^2), \tag{3.1}$$

with η small, being known, we now try and identify a similar development in powers of η for the homogenized coefficients (1.36). We will show in this section that such an expansion indeed exists, and that, furthermore, the computation of its coefficients, is much simpler than that of the homogenized matrix itself.

3.1 First order expansion

Formally expanding the solution w_p to the corrector equation (1.34) as

$$w_p(\Phi(x,\omega),\omega) = w_p^0 + \eta w_p^1 + O(\eta^2),$$

we see, identifying terms of identical order in η, and performing a tedious calculation that we skip here for brevity, that w_p^0 and w_p^1 respectively solve the following two problems. The zero order term w_p^0 is a solution to the periodic corrector problem (1.5), that is

$$\begin{cases} -\operatorname{div}\left(A_{per}(y)\left(p+\nabla w_p^0\right)\right) = 0, \\ w_p \text{ is } \mathbb{Z}^d\text{-periodic.} \end{cases} \tag{3.2}$$

On the other hand, w_p^1 is a solution to the following problem:

$$\begin{cases} -\operatorname{div}\left[A_{per}\left(\nabla w_p^1 - \nabla\Psi\nabla w_p^0\right) + \left(\nabla\Psi^T - (\operatorname{div}\Psi)Id\right)A_{per}\left(p+\nabla w_p^0\right)\right] = 0, \\ \mathbb{E}\left(\displaystyle\int_Q \nabla w_p^1\right) = \mathbb{E}\left(\displaystyle\int_Q (\nabla\Psi - (\operatorname{div}\Psi)Id)\nabla w_p^0\right). \end{cases} \tag{3.3}$$

The problem (3.3) is *a priori* stochastic in nature. However, taking the expectation value and setting

$$\overline{w}_p^1 = \mathbb{E}(w_p^1), \tag{3.4}$$

we have

$$\begin{cases} -\operatorname{div}\left[A_{per}\nabla\overline{w}_p^1\right] = \operatorname{div}\left[-A_{per}\left(\mathbb{E}(\nabla\Psi)\nabla w_p^0\right) + \left(\mathbb{E}(\nabla\Psi)^T - \mathbb{E}(\operatorname{div}\Psi))Id\right)A_{per}\left(p+\nabla w_p^0\right)\right], \\ \displaystyle\int_Q \nabla\overline{w}_p^1 = \int_Q (\mathbb{E}(\nabla\Psi) - \mathbb{E}(\operatorname{div}\Psi)Id)\nabla w_p^0. \end{cases} \tag{3.5}$$

The first point is, $\nabla\overline{w}_p^1$ is periodic since ∇w_p^1 is stationary, and it is the solution to (3.5), which involves only $\mathbb{E}(\nabla\Psi)$ and ∇w_p^0, both of which are periodic. Hence, $\nabla\overline{w}_p^1$ is the solution to a *periodic* problem. The existence and uniqueness of the solution to this problem is readily proved applying techniques similar to those for (3.3). Here again, see for instance [2], [6] or [7].

The second point is, only the knowledge of $\nabla\overline{w}_p^1$ (and not of ∇w_p^1 itself) is required for the calculation of the first order correction of the homogenized coefficient. Indeed, formally computing the expansion

$$A^*_{ij} = A^0_{ij} + \eta A^1_{ij} + O(\eta^2),$$

we have, after another tedious calculation we also skip here,

$$A^0_{ij} = \int_Q (e_i + \nabla w^0_{e_i})^T A_{per} e_j, \tag{3.6}$$

(as in (1.6)) and

$$A^1_{ij} = -\int_Q \mathbb{E}(\text{div }\Psi) A^0_{ij} + \int_Q (e_i + \nabla w^0_{e_i})^T A_{per} e_j \mathbb{E}(\text{div }\Psi) + \int_Q (\nabla \overline{w}^1_{e_i} - \mathbb{E}(\nabla\Psi)\nabla w^0_{e_i})^T A_{per} e_j. \tag{3.7}$$

As it is clear in the above formula, only $\mathbb{E}(\nabla\Psi)$ and $\nabla\overline{w}^1_{e_i}$ are needed in order to compute A^1_{ij}.

The determination of the homogenized coefficients for (1.30) is stochastic in nature. However, if we trust the above formulas (and we will see below they do hold true), this problem, in the specific case (3.1) reduces, at the first two orders in η, to the simpler solution to two periodic problems, namely (3.2) and (3.5). Both of them are of the same nature, basically corrector problems. Importantly, note that Ψ is only present through $\mathbb{E}(\nabla\Psi)$, both in (3.5) and in (3.7).

The question to know whether the same simplification (namely *periodic* replaces *stationary*) also holds true at higher orders in η will be examined at the end of this section. Let us now make precise the above formal expansions at the first order.

Let us now state the main results, and next prove them.

Proposition 3.1 *For any $\eta \in (0,1)$, let Φ_η be a stationary diffeomorphism (in the sense of (1.31)-(1.32)-(1.33)), and assume that*

$$\Phi_\eta(x,\omega) = x + \eta\Psi(x,\omega) + O(\eta^2), \tag{3.8}$$

in $C^1(\mathbb{R}^d, L^2(\Omega))$, with $\nabla\Psi$ stationary. Then, for any $p \in \mathbb{R}^d$, the solution $w^\eta_p = \tilde{w}^\eta_p \circ \Phi^{-1}$ to (1.34) satisfies

$$\nabla\tilde{w}^\eta_p(x,\omega) = \nabla w^0_p(x) + \eta\nabla w^1_p(x,\omega) + O(\eta^2) \quad \text{as } \eta \longrightarrow 0, \quad \text{in} \quad L^2(Q\times\Omega) - \text{weak}, \tag{3.9}$$

where w^0_p is the solution to (3.2), and w^1_p is the solution to (3.3).

Theorem 3.2 *Under the hypotheses of Proposition 3.1, consider a bounded open subset $\mathcal{D}$ of $\mathbb{R}^d$, and $f \in H^{-1}(\mathcal{D})$. Then, the solution $u_\varepsilon(x,\omega)$ of (1.30) satisfies the following properties*

(i) $u_\varepsilon(x,\omega)$ converges to some $u_0(x)$ strongly in $L^2(\mathcal{D})$ and weakly in $H^1(\mathcal{D})$, almost surely;

(ii) the function u_0 is a solution to the homogenized problem:

$$\begin{cases} -\text{div}\,(A^*\nabla u) = f & \text{in} \quad \mathcal{D}, \\ u = 0 & \text{on} \quad \partial\mathcal{D}. \end{cases} \tag{3.10}$$

In (1.35), the homogenized matrix A^* satisfies

$$A^*_{ij} = A^0_{ij} + \eta A^1_{ij} + O(\eta^2), \quad \text{as } \eta \longrightarrow 0, \tag{3.11}$$

where A^0_{ij} and A^1_{ij} are defined by (3.2) - (3.6), and by (3.5) - (3.7), respectively.

Remark 3.3 *The convergence (3.9) is only weak because the coefficients A are only in L^∞. Using stronger assumptions on A would yield a stronger convergence. For instance, if $A \in C^{0,\alpha}$ for some $\alpha > 0$, then it is possible to prove the strong convergence in (3.9).*

Proof of Proposition 3.1: The corrector $w_p^\eta = \tilde{w}_p^\eta \circ \Phi^{-1}$ satisfies (1.34), and $w_p^\eta \in L^2_{\rm loc}(\mathbb{R}^d, L^2(\Omega))$, and $\nabla w_p^\eta \in L^2_{\rm unif}(\mathbb{R}^d, L^2(\Omega))$. In addition, the last line of this system implies, using Lemma 1.8, that

$$\frac{\|w_p^\eta\|_{L^2(B_1+x)}}{1+|x|} \underset{|x|\to\infty}{\longrightarrow} 0, \quad \text{almost surely.} \tag{3.12}$$

Next, we define a cut-off function χ_R such that

$$\chi_R \in \mathcal{D}(\mathbb{R}^d), \quad \chi_R = 1 \text{ in } B_R, \quad \chi_R = 0 \text{ in } B_{R+1}^c, \quad \|\nabla \chi_R\|_{L^\infty(\mathbb{R}^d)} \leq 2. \tag{3.13}$$

We mutlipy the first line of (1.34) by $w_p^\eta \chi_R$, and integrate, finding

$$\int_{\mathbb{R}^d} \left[A_{per}\left(\Phi^{-1}(y)\right)\left(\nabla w_p^\eta + p\right)\right] \cdot \nabla w_p^\eta \chi_R = -\int_{\mathbb{R}^d} \left[A_{per}\left(\Phi^{-1}(y)\right)\left(\nabla w_p^\eta + p\right)\right] \cdot \nabla \chi_R w_p^\eta.$$

Using (3.12), one easily proves that the right-hand side of this equation is of order $o(R^d)$ as R goes to infinity. Hence,

$$\lim_{R\to\infty} \frac{1}{|B_R|} \int_{\mathbb{R}^d} \left[A_{per}\left(\Phi^{-1}(y)\right)\left(\nabla w_p^\eta + p\right)\right] \cdot \nabla w_p^\eta \chi_R = 0.$$

Using the ergodic theorem, this implies

$$\mathbb{E}\left(\int_{\Phi(Q)} |\nabla w_p^\eta|^2\right) = \mathbb{E}\left(\int_Q \left|(\nabla\Phi)^{-1}\nabla \tilde{w}_p^\eta\right|^2 \det(\nabla\Phi)\right) \leq C, \tag{3.14}$$

for some constant C that does not depend on η. As a consequence, it is possible to extract a subsequence η going to 0 such that ∇w_p^η weakly converges to some W_p^0. Now, differential operators being continuous with respect to the weak topology, W_p^0 is a gradient:

$$\nabla w_p^\eta \underset{\eta\to 0}{\rightharpoonup} \nabla w_p^0 \quad \text{in } L^2(Q, L^2(\Omega)).$$

With similar arguments, one proves that $\nabla \tilde{w}_p^\eta \rightharpoonup \nabla w_p^0$ in $L^2(Q\times\Omega)$. We then write the equation satisfied by ∇w_p^η, that is,

$$\forall \xi \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \left[A_{per}(x)\left(\left(\nabla\Phi(x)\right)^{-1}\nabla\tilde{w}_p^\eta(x) + p\right)\right] \cdot \nabla\xi(\Phi(x)) \det(\nabla\Phi(x)) dx = 0. \tag{3.15}$$

As η goes to zero, $(\nabla\Phi(x))^{-1}$ converges to Id in $L^\infty(\mathbb{R}^d)$, almost surely, and $\nabla\xi(\Phi(x))\det(\nabla\Phi(x))$ converges to $\nabla\xi(x)$ in $L^\infty(\mathbb{R}^d)$ almost surely. Hence, passing to the limit in (3.15), we have

$$\forall \xi \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \left[A_{per}(x)\left(\nabla w_p^0 + p\right)\right] \cdot \nabla\xi(x) dx = 0,$$

that is, the first line of (3.2). Next, as we did above, we pass to the limit in the third line of (1.34), finding

$$\int_Q \nabla w_p^0 = 0. \tag{3.16}$$

In addition, the fact that $\nabla\tilde{w}_p^\eta$ is stationary implies that ∇w_p^0 is $\mathbb{Z}^d$-periodic. This and (3.16) implies that w_p^0 itself is periodic. We thus have the first term of (3.9). We now turn to the second one. For this purpose, we first use (3.15), and setting $\theta = \xi \circ \Phi$, we infer

$$\forall \theta \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \left[A_{per}(x)\left((\nabla\Phi(x))^{-1}\nabla\tilde{w}_p^\eta(x) + p\right)\right] \cdot \left[(\nabla\Phi(x))^{-1}\nabla\theta(x)\right] \det(\nabla\Phi(x)) dx, \tag{3.17}$$

hence

$$- \operatorname{div} \left[\det(\nabla\Phi)(\nabla\Phi)^{-T} A_{per} \left((\nabla\Phi)^{-1}\nabla\tilde{w}_p^\eta + p \right) \right] = 0. \tag{3.18}$$

Next, using (3.8), we have, in $C^0(\mathbb{R}^d, L^2(\Omega))$,

$$\begin{cases} \nabla\Phi = Id + \eta\nabla\Psi + O(\eta^2), \\ (\nabla\Phi)^{-1} = Id - \eta\nabla\Psi + O(\eta^2), \\ \det(\nabla\Phi) = 1 + \eta \operatorname{div} \Psi + O(\eta^2), \end{cases} \tag{3.19}$$

Inserting these estimates in (3.17), we infer

$$- \operatorname{div} \left(A_{per}(\nabla\tilde{w}_p^\eta + p) \right) = \eta \operatorname{div} f_\eta, \tag{3.20}$$

where f_η is a stationary function such that $\|f\|_{L^2_{\text{unif}}(\mathbb{R}^d, L^2(\Omega))}$ is bounded independently of η. Hence, setting

$$v_p^\eta = \frac{\tilde{w}_p^\eta - w_p^0}{\eta},$$

and using (3.2), we see that

$$- \operatorname{div} \left(A_{per}\nabla v_p^\eta \right) = \operatorname{div}(f_\eta).$$

Using exactly the same argument as for ∇w_p^η, we have, using the fact that v_p^η satisfies (3.12),

$$\|\nabla v_p^\eta\|_{L^2(Q\times\Omega)} \leq C\|f_\eta\|_{L^2(Q\times\Omega)} \leq C', \tag{3.21}$$

where neither C nor C' depend on η. Hence, up to extracting a subsequence, we may find w_p^1 such that $\nabla w_p^\eta \in L^2(Q \times \Omega)$ and

$$\nabla v_p^\eta \underset{\eta\to 0}{\rightharpoonup} \nabla w_p^1.$$

We then return to (3.18), subtract the first line of (3.2) and divide by η, finding

$$- \operatorname{div} \left(A_{per}\nabla v_p^\eta + (\operatorname{div} \Psi) A_{per}(\nabla\tilde{w}_p^\eta + p) - \nabla\Psi^T A_{per}(\nabla\tilde{w}_p^\eta + p) - A_{per}\nabla\Psi\nabla\tilde{w}_p^\eta \right) = \eta \operatorname{div}(g_\eta),$$

where g_η is bounded in $L^2(Q \times \Omega)$ and is stationary. Hence, passing to the limit as η goes to 0, we find the first line of (3.3). On the other hand, developing the third line of (2.2), we have

$$\begin{aligned} 0 &= \mathbb{E}\left(\int_Q (\nabla\Phi)^{-1}\nabla\tilde{w}_p^\eta \det(\nabla\Phi) \right), \\ &= \eta\mathbb{E}\left(\int_Q (\operatorname{div} \Psi)\nabla\tilde{w}_p^\eta - \int_Q \nabla\Psi\nabla\tilde{w}_p^\eta \right) + \mathbb{E}\left(\int_Q \nabla\tilde{w}_p^\eta \right) + O(\eta^2), \\ &= \eta\mathbb{E}\left(\int_Q (\operatorname{div} \Psi)\nabla w_p^0 - \nabla\Psi\nabla w_p^0 + \nabla v_p^\eta \right) + O(\eta^2). \end{aligned}$$

Dividing by η and passing to the limit $\eta \to 0$, this gives the second line of (3.3). Finally, we point out that the solution to (3.3) is unique. Indeed, the difference γ of two solutions satisfies

$$\begin{cases} - \operatorname{div}(A_{per}\nabla\gamma) = 0, \\ \mathbb{E}\left(\int_Q \nabla\gamma \right) = 0, \quad \nabla\gamma \text{ is stationary,} \end{cases}$$

which implies that γ is constant. Hence, the whole sequence ∇v_p^η converges to ∇w_p^1. □

Proof of Theorem 3.2: In view of Theorem 1.4, we already know that (i) and (ii) are satisfied, with A^* given by (1.36). We now write it as

$$\begin{aligned} A^*_{ij} &= \det\left(\mathbb{E}\left(\int_Q \nabla\Phi(z,\cdot)dz\right)\right)^{-1} \mathbb{E}\left(\int_Q (e_i + \nabla w_{e_i}(\Phi(x)))^T A_{per}(x)\, e_j\, dx\right), \\ &= \det\left(\mathbb{E}\left(\int_Q \nabla\Phi(z,\cdot)dz\right)\right)^{-1} \mathbb{E}\left(\int_Q \left(e_i + (\nabla\Phi)^{-1}\nabla\tilde{w}^\eta_{e_i}\right)^T A_{per}(x) e_j dx\right). \end{aligned} \qquad (3.22)$$

We then insert (3.9) and (3.19) into (3.22) and find (3.11). □

Remark 3.4 *Here, we implicitly let $\varepsilon \longrightarrow 0$ first, and then let $\eta \longrightarrow 0$. It is possible to do the same computation the other way around, at least formally: first let η go to zero, with ε fixed, and then let ε go to zero. This would yield the same results. However, in this process, one needs to keep track of the dependance on η in the convergence $\varepsilon \longrightarrow 0$, which is much more technical then the method we use here.*

3.2 Remarks and extensions

We devote this subsection to some remarks on the previous proofs, along with some extensions.

Strong form of the equations First, we would like to mention it is also possible, at least formally, to carry out the same computation using a formulation of the equations in the strong sense. To be made rigorous, this alternate strategy requires proving a better convergence for the corrector, which is neither obvious, nor necessarily true. However, for the sake of illustration, we find it useful to outline the approach. Notably, the calculations are then significantly simpler, and somewhat more intuitive, than the calculations performed in the previous section.

We begin by expressing

$$-\operatorname{div}_x\left(A_{per}(\Phi^{-1}(x))(\nabla w^\eta_p(x) + p)\right) = -\operatorname{div}_x\left[\left[A_{per}\left((\nabla\Phi)^{-1}\nabla\tilde{w}^\eta_p + p\right)\right]\left(\Phi^{-1}(x)\right)\right],$$

where we used the fact that $\tilde{w}^\eta_p = w^\eta_p \circ \Phi$, hence $\nabla w^\eta_p = \nabla\left(\tilde{w}^\eta_p \circ \Phi^{-1}\right) = \left[(\nabla\Phi)^{-1}\nabla\tilde{w}^\eta_p\right] \circ \Phi^{-1}$, according to the chain rule. Next, we note that for any vector field G, we have, when $x = \Phi(z)$,

$$\operatorname{div}_x\left(G(\Phi^{-1}(x))\right) = 0 \iff \left[(\nabla\Phi)^{-T}\cdot\nabla_z\right]G(z) = 0,$$

where the operator $(\nabla\Phi)^{-T}\cdot\nabla_z$ is defined by $(\nabla\Phi)^{-T}\cdot\nabla_z G = (\nabla\Phi)^{-T}_{ki}\partial_{z_k}G_i$, with the convention of summation over repeated indices. Hence, the first line of (1.34) also reads

$$(\nabla\Phi)^{-T}\cdot\nabla_z\left[A_{per}(z)\left((\nabla\Phi)^{-1}(z)\nabla\tilde{w}^\eta_p(z) + p\right)\right] = 0,$$

which is easily seen to be equivalent to (3.18) using a weak formulation and a change of variables.

Then, the approach consists in directly expanding both Φ and $\tilde{w}_p$ in the above equation. All calculations performed, we obtain:

$$-\operatorname{div}\left[A_{per}\left(\nabla w^0_p + p\right)\right] = 0, \qquad (3.23)$$

$$-\operatorname{div}\left(A_{per}\nabla w^1_p\right) + \operatorname{div}\left[A_{per}\nabla\Psi\nabla w^0_p\right] + (\nabla\Psi)^T\cdot\nabla\left[A_{per}\left(\nabla w^0_p + p\right)\right] = 0, \qquad (3.24)$$

where, as above, we may take the expectation value and find a (periodic) elliptic equation set on $\overline{w}^1_p$. Equations (3.23) and (3.24) can be advantageously compared to the weak forms (3.2) and (3.3), showing that the results are formally identical. Moreover, using expressions (3.23) and (3.24), we of course obtain the values (3.6) and (3.7) of A^0 and A^1, respectively.

Development at higher orders The natural extension of the previous results is a *second-order* expansion of both the corrector and the homogenized coefficient. Indeed, if we assume that

$$\Phi(x,\omega) = x + \eta\Psi(x,\omega) + \eta^2\theta(x,\omega) + O(\eta^3)$$

in $C^2(\mathbb{R}^d, L^2(\Omega))$, then it is possible to carry out the same analysis as above, finding:

$$\nabla \tilde{w}_p^\eta = \nabla w_p^0 + \eta \nabla w_p^1 + \eta^2 \nabla w_p^2 + O(\eta^2),$$

and

$$A^*_{ij} = A^0_{ij} + \eta A^0_{ij} + \eta^2 A^2_{ij},$$

where ∇w_p^0, ∇w_p^1, A^0_{ij} and A^1_{ij} are the zero-order and first-order terms already identified. The second-order terms ∇w_p^2 and A^2_{ij} may be defined by equations similar to the "first-order" equations (3.3) and (3.7), although those are rather intricate. The second order term ∇w_p^2 is indeed solution to

$$\begin{aligned}
-\operatorname{div}\Big[A_{per}\left(\nabla w_p^2 - \nabla\Psi\nabla w_p^1 + ((\nabla\Psi)^2 - \nabla\theta)(\nabla w_p^0 + p)\right)\Big] = \\
\operatorname{div}\Big[\left((\operatorname{div}\Psi)Id - \nabla\Psi^T\right)A_{per}\left(\nabla w_p^1 - \nabla\Psi\nabla w_p^0\right)\Big] \\
+ \operatorname{div}\left[\left(\left(\operatorname{div}(\theta) + \frac{1}{2}D^2\det(\nabla\Psi)\right)Id - (\operatorname{div}\Psi)\nabla\Psi^T + \left(\nabla\Psi^T\right)^2 - \nabla\theta^T\right)A_{per}(\nabla w_p^0 + p)\right], \quad (3.25)
\end{aligned}$$

where $D^2\det(H) = \sum_{i\neq j} h_{ii}h_{jj} - h_{ij}h_{ji}$ is the second derivative of the determinant. However, the presence (for example) of the term $\operatorname{div}(A_{per}\nabla\Psi\nabla w_p^1)$ indicates that taking the expectation value of the equation will *a priori* not simplify into products of expectation values and periodic functions, as it was the case for the equation for w_p^1. Unless some specific form of Ψ is assumed, the formulae will not simplify. Or at least we have not been able to make them simpler. Here, one needs a genuinely stochastic computation to calculate w_p^2. Likewise, the expression of A^2_{ij} (that we omit here) requires the knowledge of ∇w_p^2 and $\nabla\Psi$ themselves, and not only of their expectation values.

Special cases for Ψ The expressions defining ∇w_p^1 and A^1_{ij} may be simpler for special cases of applications Ψ. For instance, if we impose $\mathbb{E}(\nabla\Psi) = \lambda(x)Id$, for some (periodic) $\lambda \in C^0(\mathbb{R})$, then the equation (3.5) simplifies into

$$\begin{cases}
-\operatorname{div}\left(A_{per}\left(\nabla\overline{w}_p^1 - \lambda\nabla w_p^0\right)\right) = (1-d)\nabla\lambda^T A_{per}\left(p + \nabla w_p^0\right), \\
\int_Q \nabla\overline{w}_p^1 = (1-d)\int_Q \lambda(x)\nabla w_p^0(x)dx,
\end{cases}$$

and (3.7) becomes

$$A^1_{ij} = d\int_Q\left(\lambda(x) - \int_Q\lambda\right)\left(e_i + \nabla w^0_{e_i}(x)\right)^T A_{per}(x)e_j dx + \int_Q\left(\nabla\overline{w}^1_{e_i}(x) - \lambda\nabla w^0_{e_i}\right)^T A_{per}(x)e_j.$$

These expressions are much simpler than (3.5) and (3.7). Note that if λ is constant, then the above equations imply $\mathbb{E}(\nabla w_p^1) = \lambda\nabla w_p^0$ and $A^1_{ij} = 0$. There is no correction at order η.

The one-dimensional case As is generically the case for homogenization theory, the one-dimensional situation enjoys very specific properties. It is often misleading by its simplicity (which rarely carries through to the higher dimensional situation), but it may also serves as a useful guideline.

For the question under examination here, we begin by observing that the first-order expansion we have performed in the general case takes a remarkably simple form in one dimension. The problem (3.5) then reads (here, $Q = (0,1)$)

$$\begin{cases} -\dfrac{d}{dx}\left[a_{per}\left(\dfrac{d\overline{w}^1}{dx} - \mathbb{E}\left(\dfrac{d\Psi}{dx}\right)\dfrac{dw^0}{dx}\right)\right] = 0, \\ \displaystyle\int_Q \dfrac{d\overline{w}^1}{dx} = 0, \end{cases}$$

where $\overline{w}^1 = \mathbb{E}(w^1)$. Since there is only one corrector in this case, which corresponds to $p = 1$, we have omitted the subscript p (note that the superscript 1 in w^1 corresponds to the order of the expansion in η). Likewise, we have

$$a^0 = \left(\int_Q a_{per}^{-1}\right)^{-1},$$

$$a^1 = \int_Q a_{per}\mathbb{E}\left(\frac{d\Psi}{dx}\right) - a^0\int_Q \mathbb{E}\left(\frac{d\Psi}{dx}\right) + \int_Q a_{per}\frac{d\overline{w}^1}{dx}.$$

But of course, it is well known that in the one-dimensional situation, the homogenized (scalar) coefficient a^* admits an explicit expression where the corrector may be eliminated, and only the original coefficient a_{per} appears. Indeed, since

$$a^* = \left(\frac{\int_Q \mathbb{E}\left(a_{per}^{-1}\frac{d\Phi}{dx}\right)}{\int_Q \mathbb{E}\left(\frac{d\Phi}{dx}\right)}\right)^{-1},$$

we may directly expand Φ there, and obtain a formula, *at all orders in* η, for a^*, without identifying the higher orders expansion on the corrector w. In particular, the first order term reads:

$$a^1 = -\int_Q \mathbb{E}\left(a_{per}^{-1}\frac{d\Psi}{dx}\right)\left(\int_Q a_{per}^{-1}\right)^{-2} + \left(\int_Q a_{per}^{-1}\right)^{-1}\int_Q \mathbb{E}\left(\frac{d\Psi}{dx}\right), \tag{3.26}$$

and the second order term:

$$\begin{aligned} a^2 \;=\; & \left(\int_Q a_{per}^{-1}\right)^{-1}\left[\int_Q \mathbb{E}\left(\frac{d\theta}{dx}\right) - \left(\int_Q a_{per}^{-1}\right)^{-1}\int_Q \mathbb{E}\left(a_{per}^{-1}\frac{d\Psi}{dx}\right)\int_Q \mathbb{E}\left(\frac{d\Psi}{dx}\right)\right. \\ & \left. +\frac{1}{2}\left(\int_Q a_{per}^{-1}\right)^{-2}\left(\int_Q \mathbb{E}\left(a_{per}^{-1}\frac{d\Psi}{dx}\right)\right)^2 - \left(\int_Q a_{per}^{-1}\right)^{-1}\int_Q \mathbb{E}\left(a_{per}^{-1}\frac{d\theta}{dx}\right)\right]. \end{aligned} \tag{3.27}$$

Two questions arise then. First, we may search for special situations when we may deduce, on the basis of the first and second order corrections (in η) to the periodic coefficient, that the homogenized coefficient a^* enjoys some qualitative property. A typical case is the situation where, like in Subsection 2.3, we apply the above expansion method to a_{per} periodic of period 1, and Ψ periodic of irrational period. Indeed, let us momentarily return to the setting of Subsection 2.3. It is easily seen that the formalism of the present section can be *mutatis mutandis* adapted to the situation of Subsection 2.3. The machinery of the Taylor expansion remains the same, and the proofs above are only slightly modified. It is of course even simpler in one dimension, which is our only concern here. We skip the details and concentrate on the result: if a_{per} is periodic of period 1, and if Ψ' and θ' are periodic of period $\alpha \notin \mathbb{Q}$, then the expectation values decouple. Thus, we have

$$\int_Q \mathbb{E}\left(a_{per}^{-1}\frac{d\Psi}{dx}\right) = \int_Q a_{per}^{-1} \fint \frac{d\Psi}{dx}, \quad \text{and} \quad \int_Q \mathbb{E}\left(a_{per}^{-1}\frac{d\theta}{dx}\right) = \int_Q a_{per}^{-1} \fint \frac{d\theta}{dx}.$$

We thus have

$$a^1 = 0, \quad a^2 = -\frac{1}{2}\left(\int_Q a_{per}^{-1}\right)^{-1}\left(\fint \frac{d\Psi}{dx}\right)^2,$$

which is negative as far as $\frac{d\Psi}{dx} \neq 0$. Hence, in this (very) particular case, the first order correction is zero, and the second order one is negative: the disorder added by the stochastic diffemorphism Φ to the periodic coefficient a_{per} decreases the homogenized coefficient in a deterministic way. Of course, this is a very special case, and we do not know if there is some general feature in this behavior. However, this result intuitively indicates that the same kind of decoupling may occur in the case of two different notions of stationarity, with some "independence" assumption between them.

A second question regards the calculation of the second order term for the corrector, even if such a calculation is not needed for the calculations of the second order correction w^2 to the homogenized coefficient, as we have just seen. In the higher dimensional case, we have seen that w_p^2 cannot be simply identified in average (that is, in expectation), in sharp contrast to $\overline{w}_p^1$ which *can* be identified through the solution to the periodic problem (3.5), without explicitly solving for w_p^1 in (3.3). Is it also the case in one dimension? A not too tedious calculation shows that the equation on w^2 reads:

$$-\frac{d}{dx}\left(a_{per}\frac{dw^2}{dx}\right) = -\frac{d}{dx}\left(a_{per}\frac{d\Psi}{dx}\frac{dw^1}{dx}\right) + \frac{d}{dx}\left[a_{per}\left(\left(\frac{d\Psi}{dx}\right)^2 - \frac{d\theta}{dx}\right)\left(\frac{dw^0}{dx}+1\right)\right]$$

We see that taking the expectation value does not yield any equation in closed form for $\overline{w}^2$ that would only depend on *averages* of $\dfrac{d\Phi}{dx}$. Determining w^2 itself, using $\dfrac{d\Phi}{dx}$ itself, is necessary.

Remark 3.5 *The above considerations indicate a special numerical strategy to tackle homogenization of the form (1.30), with Φ of the form (3.1). Indeed, if η is small enough, it is likely to be sufficient to compute the two first orders of the development of the homogenized coefficients in powers of η. Since these are periodic problems, it is much cheaper numerically then computing the full problem, which is stochastic.*

4 Random lattices as generic sets of points

We now turn to our second purpose in this article. We wish to relate the above questions of homogenization theory, with the question of defining energy per particle for infinite sets of points. The present section, together with our final section, Section 5, are devoted to this.

For this purpose, we recall in subsection 4.1 some results about the definition of the energy of *non periodic* infinite sets of particles. Under consideration are some appropriate geometric conditions that the set of points needs to satisfy in order for us to be able to define its energy. Next, in Subsection 4.2, we show how these geometric notions are, in particular, related to the stochastic setting we have introduced in Section 1.3, in the context of homogenization theory.

Section 5, will further address the relation between these two subjects. Settings in the vein of that of Section 1.3 will be examined. We will close the article with Subsection 5.3, which discusses some possible tracks for yet other extensions.

4.1 Deterministic microscopic energies

In [3], we considered an infinite set of points in $\mathbb{R}^d$, denoted by $\{X_i\}_{i\in\mathbb{N}}$, and gave some geometric properties allowing to define its average energy. More precisely, we proved that the following properties allowed to define the average energy of the infinite set of particles $\{X_i\}_{i\in\mathbb{N}}$ for a large class of models:

Definition 4.1 *We shall say that a set of points $\{X_i\}_{i\in\mathbb{N}}$ is admissible if it satisfies the following:*

(H1) $\sup_{x\in\mathbb{R}^d} \#\{i\in\mathbb{N} \quad / \quad |x-X_i|<1\} < +\infty;$

(H2) $\exists R>0$ *such that* $\inf_{x\in\mathbb{R}^d} \#\{i\in\mathbb{N} \quad / \quad |x-X_i|<R\} > 0;$

(H3) for any $n\in\mathbb{N}$, the following limit exists

$$\lim_{R\to\infty} \frac{1}{|B_R|} \sum_{X_{i_0}\in B_R} \cdots \sum_{X_{i_n}\in B_R} \delta_{(X_{i_0}-X_{i_1},\ldots X_{i_0}-X_{i_n})}(h_1,\ldots,h_n) = l^n(h_1,\ldots,h_n), \tag{4.1}$$

and is a non-negative uniformly locally bounded measure.

We use here the convention that if $n=0$, l^0 is the constant function equal to

$$l^0 = \lim_{R\to\infty} \frac{1}{|B_R|} \#\{i\in\mathbb{N} \quad / \quad X_i\in B_R\}. \tag{4.2}$$

Remark 4.2 *It is also possible to give a fully geometric characterization by replacing (H3) with the following property: $\forall n\in\mathbb{N}, \quad \forall(\delta_0,\delta_1,\ldots\delta_n)\in(\mathbb{R}^{+*})^{n+1}$, the following limit exists:*

$$f_n(\delta_0,h_1,\delta_1,h_2,\delta_2,\ldots,h_n,\delta_n) = \lim_{R\to\infty} \frac{1}{|B_R|} \#\Big\{(i_0,i_1,\ldots,i_n)\in\mathbb{N}^{n+1}, \\ |X_{i_0}|\le\delta_0 R, \quad |X_{i_0}-X_{i_1}-h_1|\le\delta_1,\ldots,|X_{i_0}-X_{i_n}-h_n|\le\delta_n\Big\}, \tag{4.3}$$

with convergence in $L^\infty(\mathbb{R}^n)$. The following equality then makes the link between (4.1) and (4.3):

$$f_n(\delta_0,h_1,\delta_1,h_2,\delta_2,\ldots,h_n,\delta_n) = |B_{\delta_0}| l^n\left[(h_1+B_{\delta_1})\times\cdots\times(h_n+B_{\delta_n})\right].$$

Intuitively, (H1) means there is no arbitrarily large cluster of particles, whereas (H2) means there is no arbitrarily large ball in $\mathbb{R}^d$ containing none of the X_i. The assumptions (H1)-(H2) are usually refered to as "Delaunay" hypotheses.

Assumption (H3) may be seen as a condition on n-body correlations. It is therefore rather natural in a context where we aim to define averages. However, the set of assumptions (H1)-(H2)-(H3) is for genericity. In some particular cases of simple models of energy, such as a two-body potential, there is no need for a condition on correlations of order higher than 2. In such a case, only (H1)-(H2) and (H3) for $n=0,1$ are needed for the definition of the energy per particle. For the energy per unit volume, it even sufficient to have (H1)-(H2) and (H3) for $n=1$. On the other hand, in the case of quantum models (such as Thomas-Fermi type theories), as was considered in [3], nonlinearities imply the need of (H3) for all $n\in\mathbb{N}$.

Remark 4.3 *None of the properties (H1), (H2) and (H3) implies another one, as is proved in [3].*

Given Definition 4.1, we introduced the corresponding functional spaces:

Definition 4.4 *Let $\{X_i\}_{i\in\mathbb{N}}$ be an admissible set, and denote by $\mathcal{A}(\{X_i\})$ the vector space generated by the functions of the form*

$$f(x) = \sum_{i_1\in\mathbb{N}} \sum_{i_2\in\mathbb{N}} \cdots \sum_{i_n\in\mathbb{N}} \varphi(x - X_{i_1}, x - X_{i_2}, \dots, x - X_{i_n}), \tag{4.4}$$

with $\varphi \in \mathcal{D}(\mathbb{R}^{3n})$. Then, for any $k \in \mathbb{N}$ and any $p \in [1, +\infty)$, we denote by $\mathcal{A}^{k,p}(\{X_i\})$, or simply $\mathcal{A}^{k,p}$ when there is no ambiguity, the closure of $\mathcal{A}(\{X_i\})$ for the norm $\|\cdot\|_{W^{k,p}_{\text{unif}}}$.

When $k = 0$, we use the notation $\mathcal{A}^p$ for $\mathcal{A}^{0,p}$. The closure of $\mathcal{A}$ for the norm $\|\cdot\|_{L^\infty(\mathbf{R}^3)}$ being a set of continuous functions, we will denote it by $\mathcal{A}_c$. We will call $\mathcal{A}^\infty$ the closure for the $L^\infty(\mathbb{R}^d)$ norm of the space of functions of the form (4.4), with $\varphi \in L^\infty(\mathbb{R}^d)$ having compact support.

Remark 4.5 *In the above definition, hypothesis (H2) is actually not needed. It was only needed in [3] to deal with a definition of N-body energies which are nonlocal (TFW model). This is not the case here.*

Note that $\mathcal{A}^{k,p}$ is the closure for the $W^{k,p}_{\text{unif}}$ norm of the algebra generated by functions of the form

$$f(x) = \sum_{i\in\mathbb{N}} \varphi(x - X_i), \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

Let us also point out that in the particular case of a periodic lattice $\{X_i\}_{i\in\mathbb{N}}$, $\mathcal{A}^{k,p}(\{X_i\}_{i\in\mathbb{N}})$ is the algebra of periodic functions with the appropriate period and regularity.

The point is, any function in the spaces $\mathcal{A}^{k,p}$ has an average:

Lemma 4.6 *Let $\{X_i\}_{i\in\mathbb{N}}$ be an admissible set of points. Then, for any $f \in \mathcal{A}^{k,p}$, the following limit exists:*

$$\langle f \rangle := \lim_{R\to\infty} \frac{1}{|B_R|} \int_{B_R} f.$$

In addition, in the special case of an f of the form (4.4), we have

$$\langle f \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^{d(n-1)}} \varphi(x, x - h_1, \dots, x - h_{n-1}) dl^{n-1}(h_1, \dots, h_{n-1}) dx. \tag{4.5}$$

4.2 Stochastic set of points

Being inspired by the stochastic setting introduced in Section 1.3, let us now consider a set of points $\{X_i\}_{i\in\mathbb{N}}$, which is the deformation of a periodic lattice by a stationary diffeomorphism Φ. More precisely, we assume that Φ satisfies (1.31)-(1.32)-(1.33), and we define

$$\forall i \in \mathbb{Z}^d, \quad X_i(\omega) = \Phi(i, \omega). \tag{4.6}$$

The relation between Definition 4.1 and the notion of stationary diffeomorphism is best illustrated by:

Proposition 4.7 *Let Φ be a stationary diffeomorphism, i.e a diffeomorphism satisfying (1.31)-(1.32)-(1.33). Let the set $\{X_i(\omega)\}_{i\in\mathbb{Z}^d}$ be defined by (4.6). Then, $\{X_i\}_{i\in\mathbb{Z}^d}$ satisfies (H1)-(H2)-(H3) of Definition 4.1, almost surely.*

Proof: First, it is clear that (1.31)-(1.32) imply that the first eigenvalue $\lambda_1(x,\omega)$ of $\nabla\Phi(x,\omega)\nabla\Phi(x,\omega)^T$ satisfies $\lambda_1(x,\omega) \geq \nu^2 / \left(M^{2d-1}\right)$ almost surely. Thus, we have:

$$\forall i \neq j, \quad |X_i - X_j| \geq \frac{\nu^2}{M^{d-1}} > 0,$$

almost surely. Hence, (H1) is satisfied almost surely.

Next, (1.32) implies that

$$\forall x \in \mathbb{R}^d,\ \forall i \in \mathbb{Z}^d, \quad |X_i - \Phi(x)| \leq M|i - x|,$$

almost surely. Hence,

$$\begin{aligned}
\inf_{x\in\mathbb{R}^d} \#\left\{i \in \mathbb{Z}^d, \quad |x - X_i| < R\right\} &= \inf_{x\in\mathbb{R}^d} \#\left\{i \in \mathbb{Z}^d, \quad |\Phi(x) - \Phi(i)| < R\right\} \\
&\geq \inf_{x\in\mathbb{R}^d} \#\left\{i \in \mathbb{Z}^d, \quad |x - i| < \frac{R}{M}\right\},
\end{aligned}$$

which is positive for some $R > 0$. Thus, (H2) is satisfied almost surely.

We now turn to (H3). We first deal with the case $n = 0$, then with the other cases. In order to prove (4.2), we write

$$\frac{\left|\Phi^{-1}(B_R)\right|}{|B_R|} \leq \frac{1}{|B_R|}\#\left\{i \in \mathbb{Z}^d \quad / \quad \Phi(i) \in B_R\right\} \leq \frac{\left|\Phi^{-1}(B_{R+1})\right|}{|B_R|}. \tag{4.7}$$

We have

$$\frac{\left|\Phi^{-1}(B_R)\right|}{|B_R|} = \frac{1}{|B_1|}\left|\frac{1}{R}\Phi^{-1}(RB_1)\right|.$$

We then point out that, using Lemma 1.7, the function $\mathbf{1}_{\varepsilon\Phi^{-1}\left(\frac{B_1}{\varepsilon}\right)}$ converges in L^1 to $\mathbf{1}_{\mathbb{E}\left(\int_Q \nabla\Phi\right)^{-1}B_1}$ as $\varepsilon \to 0$. Hence,

$$\lim_{R\to\infty} \frac{\left|\Phi^{-1}(B_R)\right|}{|B_R|} = \lim_{R\to\infty} \frac{\left|\Phi^{-1}(B_{R+1})\right|}{|B_R|} = \det\left(\mathbb{E}\left(\int_Q \nabla\Phi\right)\right)^{-1}.$$

Returning to (4.7), we thus find (4.2), with $l^0 = \det\left(\mathbb{E}\left(\int_Q \nabla\Phi\right)\right)^{-1}$. We next deal with the case $n \geq 1$. We set

$$l_R^n(h_1, \ldots, h_n) = \frac{1}{|B_R|}\sum_{X_{i_0}\in B_R} \cdots \sum_{X_{i_n}\in B_R} \delta_{(X_{i_0}-X_{i_1},\ldots X_{i_0}-X_{i_n})}(h_1, \ldots, h_n).$$

Given $\xi \in \mathcal{D}(\mathbb{R}^{nd})$, we have

$$\langle l_R^n, \xi\rangle = \frac{1}{|B_R|}\sum_{i_0\in\Phi^{-1}(B_R)\cap\mathbb{Z}^d} \cdots \sum_{i_n\in\Phi^{-1}(B_R)\cap\mathbb{Z}^d} \xi\left(\Phi(i_0) - \Phi(i_1), \ldots, \Phi(i_0) - \Phi(i_n)\right). \tag{4.8}$$

We then point out that since ξ has compact support, the sums over i_k for $k \geq 1$ are almost not changed if they are replaced by sums over the whole lattice $\mathbb{Z}^d$. Indeed, we have

$$\begin{aligned}
&\left|\sum_{i_0\in\Phi^{-1}(B_R)\cap\mathbb{Z}^d}\ \sum_{i_1\in\Phi^{-1}(B_R)^c\cap\mathbb{Z}^d}\ \sum_{i_2\in\Phi^{-1}(B_R)\cap\mathbb{Z}^d} \cdots \sum_{i_n\in\Phi^{-1}(B_R)\cap\mathbb{Z}^d} \xi\left(\Phi(i_0) - \Phi(i_1), \ldots, \Phi(i_0) - \Phi(i_n)\right)\right| \\
&\qquad \leq \sum_{i_0\in\Phi^{-1}(B_R\setminus B_{R-A})\cap\mathbb{Z}^d}\ \sum_{i_1\in B_A(i_0)}\ \sum_{i_2\in B_A(i_0)} \cdots \sum_{i_n\in B_A(i_0)} \|\xi\|_{L^\infty(\mathbb{R}^d)}, \\
&\qquad \leq CA^{dn}\#\left(\Phi^{-1}(B_R\setminus B_{R-A})\cap\mathbb{Z}^d\right),
\end{aligned}$$

where A is a constant depending on ξ and Φ, but not on R, and C is a constant depending only on d. Using Lemma 1.7, we have

$$\frac{1}{|B_R|}\#\left(\Phi^{-1}(B_R\setminus B_{R-A})\cap\mathbb{Z}^d\right) \leq \frac{|\Phi^{-1}(B_{R+1}\setminus B_{R-A})|}{|B_R|} \underset{R\to\infty}{\longrightarrow} 0 \quad \text{almost surely.}$$

Hence, the convergence of (4.8) amounts to the convergence of

$$S_R = \frac{1}{|B_R|} \sum_{i_0 \in \Phi^{-1}(B_R)\cap\mathbb{Z}^d} \sum_{i_1\in\mathbb{Z}^d} \cdots \sum_{i_n\mathbb{Z}^d} \xi\left(\Phi(i_0)-\Phi(i_1),\dots,\Phi(i_0)-\Phi(i_n)\right).$$

We thus set

$$F_{i_0} = \sum_{i_1\in\mathbb{Z}^d} \cdots \sum_{i_n\mathbb{Z}^d} \xi\left(\Phi(i_0)-\Phi(i_1),\dots,\Phi(i_0)-\Phi(i_n)\right),$$

which is easily seen to be a stationary sequence. In addition, $F_i \in L^\infty(\Omega)$. We then compare S_R with

$$T_R = \frac{1}{|B_R|} \sum_{i\in\mathbb{E}\left(\int_Q \nabla\Phi\right)^{-1} B_R\cap\mathbb{Z}^d} F_i.$$

We have

$$\begin{aligned}
|S_R - T_R| &\leq \frac{1}{|B_R|}\left(\sum_{i\in\left(\mathbb{E}\left(\int_Q\nabla\Phi\right)^{-1}B_R\setminus\Phi^{-1}(B_R)\right)\cap\mathbb{Z}^d} |F_i| + \sum_{i\in\left(\Phi^{-1}(B_R)\setminus\mathbb{E}\left(\int_Q\nabla\Phi\right)^{-1}B_R\right)\cap\mathbb{Z}^d} |F_i|\right)\\
&\leq \frac{\#\left[\left(\mathbb{E}\left(\int_Q\nabla\Phi\right)^{-1}B_R\setminus\Phi^{-1}(B_R)\right)\cap\mathbb{Z}^d\right]}{|B_R|}\|F_0\|_{L^\infty}\\
&\quad+\frac{\#\left[\left(\Phi^{-1}(B_R)\setminus\mathbb{E}\left(\int_Q\nabla\Phi\right)^{-1}B_R\right)\cap\mathbb{Z}^d\right]}{|B_R|}\|F_0\|_{L^\infty}\\
&\leq \frac{\left|\mathbb{E}\left(\int_Q\nabla\Phi\right)^{-1}B_R\setminus\Phi^{-1}(B_R)\right| + \left|\Phi^{-1}(B_R)\setminus\mathbb{E}\left(\int_Q\nabla\Phi\right)^{-1}B_R\right|}{|B_R|}\|F_0\|_{L^\infty}
\end{aligned}$$

Using Lemma 1.7 once again, we see that the right hand side converges to 0 as R goes to infinity, almost surely. To conclude the proof, we point out that

$$T_R = \frac{\left|\mathbb{E}\left(\int_Q\nabla\Phi\right)^{-1}B_R\right|}{|B_R|}\frac{1}{\left|R\mathbb{E}\left(\int_Q\nabla\Phi\right)^{-1}B_1\right|}\sum_{i\in R\mathbb{E}\left(\int_Q\nabla\Phi\right)^{-1}B_1\cap\mathbb{Z}^d} F_i.$$

Applying the ergodic theorem, we thus have

$$\lim_{R\to\infty} T_R = \det\left(\mathbb{E}\left(\int_Q\nabla\Phi\right)\right)^{-1}\mathbb{E}(F_0),$$

almoste surely. This concludes the proof, with

$$\langle l^n,\xi\rangle = \det\left(\mathbb{E}\left(\int_Q\nabla\Phi\right)\right)^{-1}\mathbb{E}\left(\sum_{i_1\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d}\xi\left(\Phi(0)-\Phi(i_1),\dots\Phi(0)-\Phi(i_n)\right)\right).$$

□

Remark 4.8 *The proof of Proposition 4.7 gives the expression of the measures l^n in terms of Φ:*

$$\forall \xi \in \mathcal{D}(\mathbb{R}^{nd}), \quad \langle l^n, \xi \rangle =$$
$$\det\left(\mathbb{E}\left(\int_Q \nabla\Phi\right)\right)^{-1} \mathbb{E}\left(\sum_{i_1\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d} \xi\left(\Phi(0)-\Phi(i_1),\dots\Phi(0)-\Phi(i_n)\right)\right). \quad (4.9)$$

5 Relation to homogenization theory

5.1 Deformation of a reference periodic lattice

As a consequence of Proposition 4.7, it is possible to construct, for $\mathbb{P}$-almost every ω, the algebras $\mathcal{A}^{k,p}(\{X_i\}_{i\in\mathbb{Z}^d})$. They satisfy Lemma 4.6. In particular, any $f \in \mathcal{A}^{k,p}(\{X_i\}_{i\in\mathbb{Z}^d})$ has an average. In addition, since the measures l^n are trivial random variables, the average in fact does not depend on ω. This is made precise in the following:

Proposition 5.1 *Let Φ be a stationary diffeomorphism, i.e a diffeomorphism satisfying (1.31)-(1.32)-(1.33). Let the set $\{X_i(\omega)\}_{i\in\mathbb{Z}^d}$ be defined by (4.6). Define $\mathcal{A}$ as the vector space generated by the functions of the form*

$$f(x,\omega) = \sum_{i_1\in\mathbb{Z}^d}\sum_{i_2\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d} \varphi(x - X_{i_1}(\omega), x - X_{i_2}(\omega), \dots, x - X_{i_n}(\omega)), \quad (5.1)$$

with $\varphi \in \mathcal{D}(\mathbb{R}^{3n})$. Denote by $\mathcal{A}^{k,p}$ the closure of $\mathcal{A}$ for the $W^{k,p}_{\text{unif}}\left(\mathbb{R}^d, L^1(\Omega)\right)$ norm. Then for any $f \in \mathcal{A}^{k,p}$, the following limit exists almost surely:

$$\langle f \rangle = \lim_{R\longrightarrow\infty} \frac{1}{|B_R|}\int_{B_R} f.$$

In addition, $\langle f \rangle$ does not depend on ω, and

$$f\left(\frac{x}{\varepsilon},\omega\right) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \langle f \rangle, \quad \textit{almost surely.} \quad (5.2)$$

Proof: As pointed out above, we only need to prove (5.2). It is possible to prove it using Proposition 4.7, but we will give a different proof directly relying on Lemma 1.8. Given $f \in \mathcal{A}$, we define

$$g(y,\omega) = f\left(\Phi(y,\omega),\omega\right),$$

so that $f(x,\omega) = g(\Phi^{-1}(x,\omega),\omega)$. We claim that g is stationary:

$$\begin{aligned} g(y,\tau_k\omega) &= f(\Phi(y,\tau_k\omega),\tau_k\omega) \\ &= \sum_{i_1\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d} \varphi\left(\Phi(y,\tau_k\omega)-\Phi(i_1,\tau_k\omega),\dots,\Phi(y,\tau_k\omega)-\Phi(i_n,\tau_k\omega)\right) \\ &= \sum_{i_1\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d} \varphi\left(\Phi(y+k,\omega)-\Phi(i_1+k,\omega),\dots,\Phi(y+k,\omega)-\Phi(i_n+k,\omega)\right) \\ &= f(\Phi(y+k,\omega),\omega) = g(y+k,\omega). \end{aligned}$$

We thus apply Lemma 1.8 and find that

$$f\left(\frac{x}{\varepsilon},\omega\right) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \det\left(\mathbb{E}\left(\int_Q \nabla\Phi\right)\right)^{-1}\mathbb{E}\left(\int_{\Phi(Q)} f\right) = \langle f \rangle, \quad (5.3)$$

almost surely. We then conclude by a density argument. $\square$

Note that formulae (4.5) and (5.3) give a seemingly different formula for the average $\langle f \rangle$ of f of the form (5.1). However, it is possible to recover (4.5) from (5.3) through the following computation:

$$\begin{aligned}
\mathbb{E}\left(\int_{\Phi(Q)} f\right) &= \mathbb{E}\left(\int_{\Phi(Q)} \sum_{i_1\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d} \varphi(x-\Phi(i_1),\ldots,x-\Phi(i_n))\right)\\
&= \sum_{i_1\in\mathbb{Z}^d}\mathbb{E}\left(\int_{\Phi(Q)-\Phi(i_1)} \sum_{i_2\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d} \varphi(x,x-\Phi(i_2)+\Phi(i_1),\ldots,x-\Phi(i_n)+\Phi(i_1))\right)\\
&= \sum_{i_1\in\mathbb{Z}^d}\mathbb{E}\left(\int_{\Phi(Q-i_1)-\Phi(0)} \sum_{i_2\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d} \varphi(x,x-\Phi(i_2-i_1)+\Phi(0),\ldots,x-\Phi(i_n-i_1)+\Phi(0))\right)\\
&= \mathbb{E}\left(\int_{\mathbb{R}^d} \sum_{j_2\in\mathbb{Z}^d}\cdots\sum_{j_n\in\mathbb{Z}^d} \varphi(x,x-\Phi(j_2)+\Phi(0),\ldots,x-\Phi(j_n)+\Phi(0))\right).
\end{aligned}$$

Hence, using (4.9), we find (4.5).

Remark 5.2 *The properties of the set of points $\{X_i(\omega)\}_{i\in\mathbb{Z}^d}$ are in fact much richer than simply satisfying (H1)-(H2)-(H3). Indeed, these hypotheses do not contain any form of translation invariance. On the contrary, the stationarity of $\nabla\Phi$ is a form of translation invariance. This is why the averages do not depend on ω. This is also why we have (5.2), which in general is not satisfied by $f \in \mathcal{A}^{k,p}$ as defined in Definitions 4.1 and 4.4.*

Remark 5.3 *In view of the proof of Proposition 5.1, we have the following property:*
$\forall f \in \mathcal{A}^{k,p}$, there exists $g \in W^{k,p}_{\text{unif}}\left(\mathbb{R}^d, L^1(\Omega)\right)$ such that g is stationary, i.e

$$\forall k\in\mathbb{Z}^d,\quad \forall x\in\mathbb{R}^d,\quad g(x+k,\omega)=g(x,\tau_k\omega),$$

almost surely, and

$$\forall x\in\mathbb{R}^d,\quad f(x,\omega)=g\left(\Phi^{-1}(x,\omega),\omega\right),$$

almost surely.

We are now in position to relate Subsection 4.2 and the homogenization setting discussed in Subsection 1.2.1. We recall Φ is a stationary diffeomorphism (i.e Φ satisfies (1.31)-(1.32)-(1.33)), and the set $\{X_i\}_{i\in\mathbb{Z}^d}$ is defined by (4.6), that is,

$$\forall i\in\mathbb{Z}^d,\quad X_i(\omega)=\Phi(i,\omega).$$

In addition, the algebras $\mathcal{A}^{k,p}$ are defined as in Proposition 5.1. Hence, if we consider a matrix $A \in \mathcal{A}^\infty(\{X_i\}_{i\in\mathbb{Z}^d})$, Remark 5.3 implies that* there exists a stationary matrix B such that

$$A(x,\omega)=B\left(\Phi^{-1}(x,\omega),\omega\right).$$

Consequently, Theorems 2.1 and 2.2 apply to the present case, giving:

Theorem 5.4 *Let $A \in \mathcal{A}^\infty(\{X_i\}_{i\in\mathbb{Z}^d})$ be a square matrix which satisfies (1.24)-(1.25). Then for any $p \in \mathbb{R}^d$, the system*

$$\begin{cases} -\operatorname{div}\left[A\left(y,\omega\right)\left(p+\nabla w_p\right)\right]=0, \\ w_p(y,\omega)=\tilde{w}_p\left(\Phi^{-1}(y,\omega),\omega\right), \quad \nabla\tilde{w}_p \quad \text{is stationary in the sense of (1.16)}, \\ \mathbb{E}\left(\int_{\Phi(Q)}\nabla w_p(y,\cdot)dy\right)=0, \end{cases} \tag{5.4}$$

has a solution in $\left\{w \in L^2_{\text{loc}}(\mathbb{R}^d, L^2(\Omega)), \quad \nabla w \in L^2_{\text{unif}}(\mathbb{R}^d, L^2(\Omega))\right\}$. In addition, this solution is unique up to the addition of a (random) constant.

Theorem 5.5 *Let $\mathcal{D}$ be a bounded smooth open subset of $\mathbb{R}^d$, and let $f \in H^{-1}(\mathcal{D})$. Let A satisfy the hypotheses of Theorem 5.4. Then the solution $u_\varepsilon(x,\omega)$ of*

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon},\omega\right)\nabla u\right)=f \quad \text{in} \quad \mathcal{D}, \\ u=0 \quad \text{on} \quad \mathcal{D} \end{cases} \tag{5.5}$$

satisfies the following properties:

(i) $u_\varepsilon(x,\omega)$ converges to some $u_0(x)$ strongly in $L^2(\mathcal{D})$ and weakly in $H^1(\mathcal{D})$, almost surely;

(ii) the function u_0 is a solution to the homogenized problem:

$$\begin{cases} -\operatorname{div}\left(A^*\nabla u\right)=f \quad \text{in} \quad \mathcal{D}, \\ u=0 \quad \text{on} \quad \partial\mathcal{D}. \end{cases} \tag{5.6}$$

In (2.3), the homogenized matrix A^* is defined by:

$$A^*_{ij}=\det\left(\mathbb{E}\left(\int_Q\nabla\Phi(z,\cdot)dz\right)\right)^{-1}\mathbb{E}\left(\int_{\Phi(Q,\cdot)}\left(e_i+\nabla w_{e_i}(y,\cdot)\right)^T A\left(y,\cdot\right)e_j\,dy\right), \tag{5.7}$$

where for any $p \in \mathbb{R}^d$, w_p is the corrector defined by the system (5.4).

5.2 Random lattices with stationary increments

As pointed out in [5], a natural property for a random lattice in order to define the corresponding average energy is the stationarity of its increments, rather than that of the atomic positions themselves. This is especially obvious in the case of two body interaction potentials, where the energy formally reads

$$\sum_{1\le i\neq j\le N} V(X_i-X_j).$$

This is the reason why, considering a random set of points

$$\ell=\left\{X_i(\omega), \quad i\in\mathbb{Z}^d\right\},$$

we now impose

$$\forall (i,j,k)\in\mathbb{Z}^d\times\mathbb{Z}^d\times\mathbb{Z}^d, \quad X_j(\tau_k\omega)-X_i(\tau_k\omega)=X_{j+k}(\omega)-X_{i+k}(\omega). \tag{5.8}$$

Our publication [5] was definitely focused on *stationary positions*, and it is time to examine *stationary increments.*

Our first point is that, under additional but rather natural conditions, the lattices (5.8) are of the type (4.6), with Φ satisfying (1.31)-(1.32)-(1.33).

Lemma 5.6 *Let $\ell = \{X_i(\omega), \quad i \in \mathbb{Z}^d\}$ be a random lattice satisfying (5.8). Assume in addition that*

$$\sup_{x\in\mathbb{R}^d} \# \left\{ i \in \mathbb{Z}^d, \quad |X_i - x| \le 1 \right\} \le M < +\infty \quad \textit{almost surely,} \tag{5.9}$$

and

$$\exists \delta > 0, \quad \forall i \in \mathbb{Z}^d, \quad \forall j \in \mathbb{Z}^d, \quad (X_i - X_j) \cdot (i - j) > \delta |i - j|^2 \quad \textit{almost surely.} \tag{5.10}$$

Then there exists a stationary diffeomorphism Φ (i.e satisfying (1.31)-(1.32)-(1.33)) such that

$$\forall i \in \mathbb{Z}^d, \quad \Phi(i, \omega) = X_i(\omega), \quad \textit{almost surely.}$$

Remark 5.7 *Note that assumptions (5.9) and (5.10) have a relation to the Delaunay assumptions (H1) and (H2) respectively. First, (5.9) is exactly assuming (H1) with some uniformity of the assumption for almost all ω. Likewise, (5.10) implies a "uniform" assumption, that is,*

$$\exists R > 0, \quad \inf_{x\in\mathbb{R}^d} \# \left\{ i \in \mathbb{Z}^d, \quad |X_i - x| \le R \right\} \ge \alpha > 0 \quad \textit{almost surely,} \tag{5.11}$$

even though the converse is not true.

Proof: We give the proof only in dimension one for the sake of clarity. In higher dimensions, the following argument is easily adapted.

We have:

$$\forall (i, j, k) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, \quad X_j(\tau_k\omega) - X_i(\tau_k\omega) = X_{j+k}(\omega) - X_{i+k}(\omega),$$

and (5.10) implies

$$\forall i \in \mathbb{Z}, \quad X_{i+1} - X_i \ge \delta \quad \text{almost surely.} \tag{5.12}$$

We now define Φ as follows: let $\rho \in \mathcal{D}(\mathbb{R})$ be such that

$$\rho \ge 0, \quad \mathrm{supp}(\rho) \subset (0, 1), \quad \int_{\mathbb{R}} \rho = 1,$$

and

$$\forall i \in \mathbb{Z}, \quad \forall x \in [i, i + 1], \quad \Phi(x, \omega) = \frac{\delta}{2}(x - i) + X_i(\omega) + \left(X_{i+1}(\omega) - X_i(\omega) - \frac{\delta}{2} \right) \int_0^{x-i} \rho(t)dt. \tag{5.13}$$

This application is easily seen to be a diffeomorphism. In addition,

$$\forall i \in \mathbb{Z}, \quad \forall x \in [i, i + 1], \quad \Phi'(x, \omega) = \frac{\delta}{2} + (X_{i+1}(\omega) - X_i(\omega))\, \rho(x - i),$$

which satisfies (1.31) because of (5.12). It satisfies (1.32) because of (5.9). Finally, (5.8) implies that Φ' is stationary.

In order to adapt this proof to the higher dimensional case, we replace (5.12)

$$\forall i \in \mathbb{Z}^d, \quad \forall j \in \mathbb{Z}^d, \quad (X_i - X_j) \cdot (i - j) \ge \delta |i - j|^2,$$

which is proved in the same way as above. Next, in order to construct Φ, we use on each cube $i + Q$ a polynomial transformation mapping i on X_i, $i + e_1$ on X_{i+e_1}, $i + e_2$ on X_{i+e_2}, etc... Then we use a regularization kernel ρ to ensure that the transformation is smooth in $\mathbb{R}^d$. $\square$

Lemma 5.6 implies that the theory of homogenization we have developed above for deformation of periodic settings readily applies to the present setting of lattices with stationary increments.

We conclude this subsection examining the following generalization. Is it possible to drop the assumption (5.10) of Lemma 5.6 (and possibly replace it with (5.11)), and still perform homogenization for the corresponding setting. It is not clear to us whether in the absence of this assumption the lattice may still be recasted as an adequate deformation of a periodic lattice (which would again allow to readily apply the existing theory). So the approach we choose, which might be useful for even more general purposes, is to redo the construction *ex nihilo*.

For this purpose, the first task is to define the corresponding algebra $\mathcal{A}$. The key ingredient for this is proving

$$\forall f \in \mathcal{A}, \quad f\left(\frac{x}{\varepsilon}, \omega\right) \overset{*}{\rightharpoonup} M(f),$$

for some scalar $M(f)$, which of course will be identified with the average of f.

We answer this question in Proposition 5.8 and Lemma 5.9 below. Both of them are stated in dimension one for the sake of clarity. Although we did not check the computations in higher dimensions, we do believe our results carry on to this latter case.

Proposition 5.8 *Let $\{X_i\}_{i\in\mathbb{Z}}$ be a random set of points satisfying (5.8). Assume that $\{X_i\}_{i\in\mathbb{Z}}$ satisfies (5.9) and (5.11). Then, hypotheses (H1), (H2) and (H3) of Definition 4.1 are satisfied. In addition, for any $f \in \mathcal{A}^p(\{X_i\}_{i\in\mathbb{Z}})$ (with $p \in [1, \infty]$), we have the following property*

$$\forall k \in \mathbb{Z}, \forall x \in \mathbb{R}, \quad f(x, \tau_k\omega) = f(x + X_k(\omega) - X_0(\tau_k\omega), \omega), \quad \textit{almost surely.} \tag{5.14}$$

Proof: Clearly, (5.9) implies (H1) and (5.11) implies (H2). We next turn to the proof of (H3). Applying the ergodic theorem to the stationary sequence $X_{i+1} - X_i$, we have

$$\frac{X_N - X_0}{N} \longrightarrow \mathbb{E}(X_1 - X_0) \quad \text{almost surely.} \tag{5.15}$$

In addition, (5.11) implies that $\mathbb{E}(X_1 - X_0) \neq 0$. Without loss of generality, we may assume that it is positive:

$$\mathbb{E}(X_1 - X_0) := L > 0. \tag{5.16}$$

Let us first prove the (H3) for $n = 1$: in such a case, we need to prove that

$$C_N = \frac{1}{N} \# \{i \in \mathbb{Z}, \quad X_i \in [0, N]\}$$

converges as N goes to infinity. We first point out that, with (5.15) and the fact that X_0 does not depend on N, C_N has the same limit as

$$\tilde{C}_N = \frac{1}{LN} \# \{i \in \mathbb{Z}, \quad X_i \in [X_0, X_N]\}.$$

Then, we write

$$\tilde{C}_N = \frac{1}{LN} \sum_{i=0}^{N-1} F_i(\omega),$$

where

$$F_i(\omega) = \begin{cases} \#\{k \in \mathbb{Z}, \quad X_k \in [X_i, X_{i+1})\}, & \text{if } X_i \le X_{i+1}, \\ -\#\{k \in \mathbb{Z}, \quad X_k \in [X_{i+1}, X_i)\}, & \text{if } X_{i+1} < X_i. \end{cases}$$

which is easily seen to be stationary. Hence, applying the ergodic theorem, we have

$$\begin{aligned} \lim_{N\to\infty} \frac{1}{N} \#\{i \in \mathbb{Z}, \quad X_i \in [0,N]\} &= \frac{1}{L}\mathbb{E}(F_i) \\ &= \frac{\mathbb{E}(F_0)}{\mathbb{E}(X_1 - X_0)}. \end{aligned}$$

This completes the proof in the case $n = 1$. For $n > 1$, we proceed exactly as in the proof of Proposition 4.7. We thus define

$$l_R^n = \frac{1}{|B_R|} \sum_{X_{i_0} \in B_R} \cdots \sum_{X_{i_n} \in B_R} \delta_{(X_{i_0} - X_{i_1}, \dots X_{i_0} - X_{i_n})}(h_1, \dots, h_n).$$

Given $\xi \in \mathcal{D}(\mathbb{R}^{nd})$, we have

$$\langle l_R^n, \xi \rangle = \frac{1}{|B_R|} \sum_{X_{i_0} \in B_R} \cdots \sum_{X_{i_n} B_R} \xi\left(X_{i_0} - X_{i_1}, \dots, X_{i_0} - X_{i_n}\right).$$

This sum is almost not changed if the sums in $i_1, i_2, \dots i_n$ are extended to all X_i. Thus,

$$\langle l_R^n, \xi \rangle = \frac{1}{|B_R|} \sum_{X_{i_0} \in B_R} \sum_{i_1 \in \mathbb{Z}} \cdots \sum_{i_n \in \mathbb{Z}} \xi\left(X_{i_0} - X_{i_1}, \dots, X_{i_0} - X_{i_n}\right) + o(1),$$

almost surely. We then notice that $F_{i_0} = \displaystyle\sum_{i_1 \in \mathbb{Z}} \cdots \sum_{i_n \in \mathbb{Z}} \xi\left(X_{i_0} - X_{i_1}, \dots, X_{i_0} - X_{i_n}\right)$ is a stationary sequence. Thus, applying the ergodic theorem, we conclude. Finally, (5.14) is easilly checked for $f \in \mathcal{A}(\{X_i\}_{i\in\mathbb{Z}})$, and is stable under convergence in L^p_{unif}. □

Next, we have the following Lemma:

Lemma 5.9 *Let $\{X_i\}_{i\in\mathbb{Z}}$ be a random set of points satisfying (5.8). Assume that $\{X_i\}_{i\in\mathbb{Z}}$ satisfies (5.9) and (5.11). Let $f \in L^1_{\text{unif}}\left(\mathbb{R}, L^1(\Omega)\right)$ satisfy (5.14). Then,*

$$f\left(\frac{x}{\varepsilon}, \omega\right) \overset{*}{\rightharpoonup} \mathbb{E}(X_1 - X_0)^{-1} \mathbb{E}\left(\int_{X_0}^{X_1} f(x,\omega)dx\right) \text{ in } L^\infty(\mathbb{R}), \text{ almost surely.} \tag{5.17}$$

Proof: Here again, we define $L = \mathbb{E}(X_1 - X_0)$ and assume that it is positive. We then compute

$$\frac{1}{NL}\int_0^{NL} f(x,\omega)dx = \frac{1}{NL}\left(\int_0^{X_0} f(x,\omega)dx - \int_{X_N}^{NL} f(x,\omega)dx\right) + \frac{1}{NL}\sum_{i=0}^{N-1}\int_{X_i}^{X_{i+1}} f(x,\omega)dx.$$

We bound the first two terms as follows:

$$\frac{1}{NL}\left|\int_0^{X_0} f(x,\omega)dx\right| \le C\frac{|X_0|}{NL}\|f\|_{L^1_{\text{unif}}(\mathbb{R})},$$

$$\frac{1}{NL}\left|\int_{X_N}^{NL} f(x,\omega)dx\right| \le C\frac{|X_N - NL|}{NL}\|f\|_{L^1_{\text{unif}}(\mathbb{R})},$$

which both converge to 0 almost surely. We then define

$$G_i = \int_{X_i}^{X_{i+1}} f(x,\omega)dx.$$

Due to (5.8) and (5.14), G_i is stationary. Hence, applying the ergodic theorem, we infer

$$\frac{1}{NL}\int_0^{NL} f(x,\omega)dx \longrightarrow \frac{1}{L}\mathbb{E}\left(\int_{X_0}^{X_1} f(x,\omega)dx\right).$$

This concludes the proof. □

On the basis of the above results, and the experience accumulated in the early sections of this article, it is now easy to perform the homogenization theory in the algebra constructed from Proposition 5.8.

Remark 5.10 *Note that setting*

$$y(k,\omega) = X_k(\omega) - X_0(\tau_k\omega) \tag{5.18}$$

we immediately have that y is a random variable with stationary increments, satisfying $y(0,\omega) \equiv 0$. Actually, only this structure is indeed utilized for the results above, and thus for homogenization. Therefore all this carries through to functions satisfying

$$\forall k \in \mathbb{Z}, \forall x \in \mathbb{R}, \quad f(x,\tau_k\omega) = f(x + y(k,\omega),\omega), \tag{5.19}$$

instead of (5.14).

Let us end this section with a summary of the logical links between the different notions of stationary lattices we have introduced in [5] and (partially) used here: here, we denote by $\ell(\omega)$ the infinite set $\{X_i\}_{i\in\mathbb{Z}^d}$, which is considered as a random variable as a whole. We then have four different types of stationary lattices $\ell(\omega)$:

(a) ℓ is such that $\forall k \in \mathbb{Z}^d$, $\ell(\tau_k\omega) = \ell(\omega) + k$;

(b) ℓ is such that $\ell = \Phi\left(\mathbb{Z}^d\right)$, with Φ a stationary diffeomorphism;

(c) ℓ is a lattice with stationary increments;

(d) ℓ is such that $\ell(\tau_k\omega) = \ell(\omega) + Y_k(\omega)$, where the sequence Y_k has stationary increments, and $Y_0 = 0$.

We then have the following implications:

$$\begin{array}{ccccc} (a) & \Rightarrow & (c) & \Leftarrow & (b) \\ & & \Downarrow & & \\ & & (d). & & \end{array}$$

Moreover, the implications $(b) \Rightarrow (a)$ and $(c) \Rightarrow (a)$ are clearly false. We have seen that $(c) \Rightarrow (b)$ only if we add the hypotheses (5.9) and (5.10) of Lemma 5.6, with a proof which is only valid in dimension one for now. Finally, we do not know if $(d) \Rightarrow (c)$.

5.3 Toward genericity

Here, we have proved Theorems 5.4 and 5.5 explicitly using the stationary feature of the points $\{X_i\}_{i\in\mathbb{Z}^d}$. This has in particular brought an additionnal structure "transverse in" ω', which has been very much employed above. This has already been emphasized in Remark 5.2. However, one may ask if any infinite set of points satisfying properties (H1)-(H2)-(H3) gives rise to algebras $\mathcal{A}^{k,p}$ which allow to carry out the homogenization procedure. Actually, the answer is yes, at least if (H3) is modified in order to include a form of translation invariance. For instance, one may replace (H3) by

(H3') for any $n \in \mathbb{N}$, the following limit exists

$$\lim_{\varepsilon\to 0} \mu^n\left(\frac{x}{\varepsilon}, h_1, \ldots, h_n\right) = \nu^n(h_1, \ldots h_n),$$

where

$$\mu^n(y, h_1, \ldots, h_n) = \sum_{i_0\in\mathbb{Z}^d}\sum_{i_1\in\mathbb{Z}^d}\cdots\sum_{i_n\in\mathbb{Z}^d} \delta_{(X_{i_0}, X_{i_0}-X_{i_1}, \ldots X_{i_0}-X_{i_n})}(x, h_1, h_2, \ldots h_n).$$

Of course, hypothesis (H3') is satisfied by any set of the form (4.6), where Φ is a stationary diffeomorphism. Under assumption (H3'), it is possible to prove that the corresponding algebras are particular cases of those considered by Nguetseng in [11], for which a homogenization procedure, in the same spirit as above, may be carried out.

Note that the difference between (H3) and (H3') is that the first one only deals with averages over balls centered at the origin (or equivalently at any point which is bounded independently of the radius R of the ball), whereas (H3') may be recast in the same kind of property as (H3), but with balls centered at any point x_R such that $|x_R| = O(R)$. In other words, we have the following three properties, for $f \in L^\infty(\mathbb{R}^d)$:

(a) $\displaystyle\lim_{R\to\infty} \frac{1}{|B_R|}\int_{B_R} f(x)dx := M_0$ exists;

(b) for any $x \in \mathbb{R}^d$, $\displaystyle\lim_{R\to\infty} \frac{1}{|B_R|}\int_{B_R+Rx} f(y)dy := M_1(x)$ exists;

(c) $\displaystyle f\left(\frac{x}{\varepsilon}\right) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} M_2(x)$ in $L^\infty(\mathbb{R}^d)$.

Properties (b) and (c) are equivalent, and $M_1(x) = \displaystyle⨍_{B_1+x} M_2(y)dy$, where $⨍$ denotes the normalized integral (see Definition 1.6). Both (b) and (c) imply (a), but the converse is not true. Moreover, we use here only the special case in which M_1, or equivalently M_2, is a constant function. Indeed, (H3) implies that any $f \in \mathcal{A}$ satisfies (a), whereas (H3') implies that any $f \in \mathcal{A}$ satisfies (c) (or equivalently (b)) for some constant function M_1.

Let us explain (very) briefly the content of [10]. The main hypotheses are the following: $\mathcal{A}$ is an algebra of functions such that any $f \in \mathcal{A}$ has an average $\langle f\rangle$ such that

$$f\left(\frac{x}{\varepsilon}\right) \underset{\varepsilon\to 0}{\overset{*}{\rightharpoonup}} \langle f\rangle.$$

In addition, $1 \in \mathcal{A}$ and $\mathcal{A}$ should be separable with respect to $\|\cdot\|_\infty$. Under these assumptions, it is proved in [10] that for any matrix $A \in \mathcal{A}$ which satisfies a deterministic version of (1.24) and (1.25), the problem

$$
\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u\right) = f & \text{in} \quad \mathcal{D}, \\ u = 0 \quad \text{on} \quad \partial\mathcal{D}, \end{cases} \tag{5.20}
$$

has a limit in the sense of homogenization theory. In other words, an adapted version of Theorems 1.3 and 1.4 is also valid in this case. The corrector problem is now deterministic and reads, for any $p \in \mathbb{R}^d$:

$$
\forall v \in \mathcal{A}, \quad \langle A(\nabla w_j + e_j)\nabla v\rangle = 0, \tag{5.21}
$$

with the condition $\langle \nabla w_j\rangle = 0$. The homogenized coefficients then read

$$
a^*_{ij} = \langle a_{ij} + a_{ik}\partial_k w_j\rangle,
$$

with summation of repeated indices.

In order to study the corrector problem (5.21), Nguetseng uses the natural C^*-algebra structure of $\mathcal{A}$, and in particular its spectrum $\Delta(\mathcal{A})$ and the associated Gelfand isomorphism. This set $\Delta(\mathcal{A})$ is compact, and plays the role here of the unit cell of the lattice in the periodic case. It provides an appropriate setting to apply the Lax-Milgram theorem, in order to prove that (5.21) is well posed.

However, the homogenization in [11] (and in particular the corrector problem (5.21)) is performed in a completely abstract setting, for which numerical computations seem difficult to handle. Hence, the question remains of finding a homogenization procedure making use of the "explicit" feature of the algebras $\mathcal{A}^{k,p}$ (in terms of $\{X_i\}_{i\in\mathbb{Z}^d}$). In particular, it could be interesting, at least from the perspective of the theory of elliptic PDEs, to study the corrector equation (5.21), written in the original ambient space $\mathbb{R}^d$, that is,

$$
\begin{cases} -\operatorname{div}\left[A(y)(\nabla w_p(y) + p)\right] = 0, \\ \nabla w_p \in \mathcal{A}, \quad \langle \nabla w_p\rangle = 0. \end{cases} \tag{5.22}
$$

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