Euler Scheme and Tempered Distributuions

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ABSTRACT. Given a smooth $\mathbb{R}^d$-valued diffusion $(X_t^x, t \in [0,1])$ starting at point x, we study how fast the Euler scheme $X_1^{n,x}$ with time step $1/n$ converges in law to the random variable X_1^x. Precisely, we look for which class of test functions f the approximate expectation $\mathbb{E}\left[f\left(X_1^{n,x}\right)\right]$ converges with speed $1/n$ to $\mathbb{E}\left[f\left(X_1^x\right)\right]$.

When f is smooth with polynomially growing derivatives or, under a uniform hypoellipticity condition for X, when f is only measurable and bounded, it is known that there exists a constant $C_1 f(x)$ such that

$$\mathbb{E}\left[f\left(X_1^{n,x}\right)\right] - \mathbb{E}\left[f\left(X_1^x\right)\right] = C_1 f(x)/n + O\left(1/n^2\right). \tag{1}$$

If X is uniformly elliptic, we expand this result to the case when f is a tempered distribution. In such a case, $\mathbb{E}\left[f\left(X_1^x\right)\right]$ (resp. $\mathbb{E}\left[f\left(X_1^{n,x}\right)\right]$) has to be understood as $\langle f, p(1,x,\cdot)\rangle$ (resp. $\langle f, p_n(1,x,\cdot)\rangle$) where $p(t,x,\cdot)$ (resp. $p_n(t,x,\cdot)$) is the density of X_t^x (resp. $X_t^{n,x}$). In particular, (1) is valid when f is a measurable function with polynomial growth, a Dirac mass or any derivative of a Dirac mass. We even show that (1) remains valid when f is a measurable function with exponential growth. Actually our results are symmetric in the two space variables x and y of the transition density and we prove that

$$\partial_x^\alpha \partial_y^\beta p_n(t,x,y) - \partial_x^\alpha \partial_y^\beta p(t,x,y) = \partial_x^\alpha \partial_y^\beta \pi(t,x,y)/n + r_n(t,x,y)$$

for a function $\partial_x^\alpha \partial_y^\beta \pi$ and a $O(1/n^2)$ remainder r_n which are shown to have gaussian tails and whose dependence on t is precised. We give applications to option pricing and hedging, proving numerical convergence rates for prices, deltas and gammas.

1. INTRODUCTION AND RESULTS

Let $d, r \geq 1$ be two integers. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which lives a r-dimensional Brownian motion B. We denote by $\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$ the filtration generated by B. Let us give two functions $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times r}$. We systematically use (column) vector and matrix notations, so that $b(x)$ should be thought of as a vector of size d and $\sigma(x)$ as a matrix of size $d \times r$. We denote transposition by a star and define a $d \times d$ matrix-valued function by putting $a = \sigma\sigma^*$. For a multiindex $\alpha \in \mathbb{N}^d$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$ is its length and ∂^α is the differential operator $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}$. Equipping $\mathbb{R}^d$ with the euclidian norm $\|\cdot\|$, we denote by

- $C_{\text{pol}}^\infty(\mathbb{R}^d)$ the set of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with polynomially growing derivatives of any order, i.e. such that for all $\alpha \in \mathbb{N}^d$, there exists $c \geq 0$ and $q \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$,

$$|\partial^\alpha f(x)| \leq c\left(1 + \|x\|^q\right), \tag{2}$$

- $C_b^\infty(\mathbb{R}^d)$ the set of infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with bounded derivatives of any order, i.e. such that $\partial^\alpha f \in L^\infty(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$.

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We shall make use of the following assumptions:

(A) For all $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, r\}$, b_i and $\sigma_{i,j}$ belong to $C^\infty_{\text{pol}}(\mathbb{R}^d)$ and have bounded first derivatives.

(B) For all $i \in \{1, \dots, d\}$ and $j \in \{1, \dots, r\}$, b_i and $\sigma_{i,j}$ belong to $C^\infty_b(\mathbb{R}^d)$.

(C) There exists $\eta > 0$ such that for all $x, \xi \in \mathbb{R}^d$, $\xi^* a(x) \xi \geq \eta \|\xi\|^2$.

(C) is known as the uniform ellipticity condition.

It is well known that, given $x \in \mathbb{R}$, the hypothesis (A) guarantees the existence and the $\mathbb{P}$-almost sure uniqueness of a solution $X^x = (X^x_t, t \geq 0)$ of the stochastic differential equation (SDE)

$$X^x_t = x + \int_0^t b(X^x_s)\, ds + \int_0^t \sigma(X^x_s)\, dB_s. \tag{3}$$

1.1. **Motivation.** Let us fix a time horizon $T > 0$. Without loss of generality, we can and do assume that $T = 1$. We try to estimate the law of X^x_1. To do so, the most natural idea is to approach X^x by its Euler scheme of order $n \geq 1$, say $X^{n,x} = (X^{n,x}_t, t \geq 0)$, defined as follows. We consider the regular subdivision $\mathfrak{S}_n = \{0 = t^n_0 < t^n_1 < \cdots < t^n_{n-1} < t^n_n = 1\}$ of the interval $[0, 1]$, i.e. $t^n_k = k/n$, and we put $X^{n,x}_0 = x$ and, for all $k \in \{0, \dots, n-1\}$ and $t \in [t^n_k, t^n_{k+1}]$,

$$X^{n,x}_t = X^{n,x}_{t^n_k} + b\left(X^{n,x}_{t^n_k}\right)(t - t^n_k) + \sigma\left(X^{n,x}_{t^n_k}\right)\left(B_t - B_{t^n_k}\right). \tag{4}$$

Then the random variable $X^{n,x}_1$ is exactly simulatable and should be close in law of X^x_1. Precisely, we measure the weak error between $X^{n,x}_1$ and X^x_1 by the quantities

$$\Delta^n_1 f(x) = \mathbb{E}\left[f\left(X^{n,x}_1\right)\right] - \mathbb{E}\left[f\left(X^x_1\right)\right]$$

and we try to find the largest space of test functions f for which, for each x, there exists a constant $C_1 f(x)$ such that

$$\Delta^n_1 f(x) = C_1 f(x)/n + O\left(1/n^2\right). \tag{5}$$

Practical interest of such an expansion has to be underlined (see, for instance, [7, 14]). When (5) holds, one can use the Euler scheme plus a Monte-Carlo method to estimate $\mathbb{E}\left[f\left(X^x_1\right)\right]$ and then, in a time of order nN, gets an error of order $1/\sqrt{N} + 1/n$, where N stands for the number of independants copies of $X^{n,x}_1$ generated by the Monte-Carlo procedure. Given a tolerance $\varepsilon \ll 1$, in order to minimize the time of calculus, one should then choose $N = O\left(n^2\right)$ and gets a result in a time of order $1/\varepsilon^3$.

One can even do better using Romberg's extrapolation technique: if one runs N independant copies $(X^{2n,x}_{i,1}, X^{n,x}_{i,1})$ of the couple $(X^{2n,x}_1, X^{n,x}_1)$, which still requires a time of order nN, then computing $\frac{1}{N}\sum_{i=1}^N (2f(X^{2n,x}_{i,1}) - f(X^{n,x}_{i,1}))$ one gets an estimate of $\mathbb{E}\left[f\left(X^x_1\right)\right]$ whose accuracy is of order $1/\sqrt{N} + 1/n^2$, since (5) implies that $\mathbb{E}[2f(X^{2n,x}_1) - f(X^{n,x}_1)] = \mathbb{E}[f(X^x_1)] + O(1/n^2)$. Given a tolerance $\varepsilon \ll 1$, one should now choose $N = O\left(n^4\right)$ and gets a result in a time of order $1/\varepsilon^{5/2}$.

1.2. **Previous results.** Using Itô expansions, D. Talay and L. Tubaro [14] have shown that (5) holds when $f \in C^\infty_{\text{pol}}(\mathbb{R}^d)$ under condition

(B') The b_i's and the $\sigma_{i,j}$'s are infinitely differentiable functions with bounded derivatives of any order ≥ 1.

Hypothesis (B') is almost (B) but in (B') the functions b_i and $\sigma_{i,j}$ are not supposed bounded themselves. Using Malliavin calculus, V. Bally and D. Talay [1] have extended this result to the case of measurable and bounded f's, with the extra hypothesis that X is uniformly hypoelliptic. If (C) holds, $X_1^{n,x}$ and X_1^x have densities, say $p_n(1,x,\cdot)$ and $p(1,x,\cdot)$ respectively (in this paper, densities are always taken with respect to the Lebesgue measure). Then, for each pair (x,y), the authors [2] get an expansion of the error on the density itself of the form

$$p_n(1,x,y) - p(1,x,y) = \pi(1,x,y)/n + \pi_n(1,x,y)/n^2. \tag{6}$$

They also show that the principal error term π and the remainder π_n have gaussian tails. Namely, they find constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $n \geq 1$ and $x, y \in \mathbb{R}^d$, $|\pi(1,x,y)| + |\pi_n(1,x,y)| \leq c_1 \exp(-c_2 \|x-y\|^2)$.

Besides, V. Konakov and E. Mammen [9] have proposed an analytical approach for this problem based on the so-called parametrix method. If (B) and (C) hold, for each pair (x,y), they get an expansion of arbitrary order j of $p_n(1,x,y)$ but whose terms depend on n:

$$p_n(1,x,y) - p(1,x,y) = \sum_{i=1}^{j-1} \pi_{n,i}(1,x,y)/n^i + O\left(1/n^j\right). \tag{7}$$

They also prove that the coefficients have gaussian tails, uniformly in n: for each i, they find constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $n \geq 1$ and $x, y \in \mathbb{R}^d$, $|\pi_{n,i}(1,x,y)| \leq c_1 \exp(-c_2 \|x-y\|^2)$. To do so, the authors use upper bounds on the partial derivatives of p - which they find in [4] - and prove analogous bounds on p_n's ones.

A link with generalized Watanabe distributions on Wiener's space is exhibited in [12]. For the general case of Lévy driven stochastic differential equations, (5) holds under regularity assumptions on f and integrability conditions on the Lévy process, see [7, 13]. The rate of convergence of the process $(X_t^{n,x} - X_t^x, t \in [0,1])$ is given in [5, 6]. As for the simulation of densities, see for instance [8].

1.3. **Purpose and method.** Equations (6) and (7) can be seen as expansions of $\Delta_1^n f(x) = \mathbb{E}\left[f\left(X_1^{n,x}\right)\right] - \mathbb{E}\left[f\left(X_1^x\right)\right]$ in the special case when $f = \delta_y$, the Dirac mass at point $y \in \mathbb{R}^d$. We aim at giving a precise sense to such quantities when f is any tempered distribution, and at proving that expansions in powers of $1/n$ remain valid in this extremely general setting. Moreover, we will derive expansions that are valid not only for $t = 1$, but also for any time $t \in (0,1]$, the stepsize $1/n$ being fixed, and we shall make explicit, in these expansions, the way the coefficients and the remainders depend on t, f and x.

To get these precise results, we shall place ourselves in a strong situation. Namely, we will assume infinite regularity and boundedness of the coefficients of the SDE (3), that is condition (B), and uniform ellipticity, that is condition (C). The reason for this is the following. Let us write $P_t f(x) = \mathbb{E}[f(X_t^x)]$ and $P_t^n f(x) = \mathbb{E}\left[f(X_t^{n,x})\right]$. We first expand $\Delta_t^n = P_t^n - P_t$ as an endomorphism of $C_{\mathrm{pol}}^\infty(\mathbb{R}^d)$, in powers of $1/n$. This can be done under nothing more than hypothesis (A), see Theorems 9 and 10 in Section 1.8. The coefficients in these expansions are operators of the form $\int_0^t P_s D P_{t-s}\, ds$ or $\sum_{t_k^n < t} P_{t_k^n}^n D P_{t-t_k^n}$, where D is a differential operator. Now, under (B) and (C), both X_t^x and $X_t^{n,x}$ have regular densities, say $p(t,x,\cdot)$ and $p_n(t,x,\cdot)$, with gaussian tails, as soon as $t > 0$, so that we may express these operators as integral operators on $\mathbb{R}^d$. For instance, for $f \in C_{\mathrm{pol}}^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$

and $0 < s < t$,

$$P_s D P_{t-s} f(x) = \int_{\mathbb{R}^d} p(s,x,z) D P_{t-s} f(z)\, dz$$
$$= \int_{\mathbb{R}^d} f(y) \left(\int_{\mathbb{R}^d} p(s,x,z) D(p(t-s,\cdot,y))(z)\, dz \right) dy.$$

Now the expansions read on the density itself, with coefficients of the form

$$\pi(t,x,y) = \int_0^t \int_{\mathbb{R}^d} p(s,x,z) D(p(t-s,\cdot,y))(z)\, dz ds, \tag{8}$$

$$\text{or} \quad \pi_n(t,x,y) = \sum_{t_k^n < t} \int_{\mathbb{R}^d} p_n(t_k^n,x,z) D(p(t-t_k^n,\cdot,y))(z)\, dz. \tag{9}$$

At this step, the key point is to prove that these coefficients, as well as any of their spatial derivatives, have gaussian tails (see Proposition 5). To do so, we split the above time integral (resp. sum) depending on whether s (resp. t_k^n) is small or large, and integrate by parts in the latter case. This is very similar to V. Bally and D. Talay's technique [1], but they use the Malliavin calculus integration by parts formula whereas we only use the genuine one. Then we use upper bounds on the partial derivatives of p and p_n, as is done in V. Konakov and E. Mammen's work [9]. Here the uniform ellipticity hypothesis is crucial: it provides upper bounds that have enough quality in t to allow us to conclude.

The same analysis, with a bit more work, can be done for the remainders. We then get *functional* expansions of the form

$$p_n - p = \pi/n + \pi_n/n^2 \qquad \text{or} \qquad p_n - p = \sum_{i=1}^{j} \pi_{n,i}/n^i \tag{10}$$

where π, π_n and the $\pi_{n,i}$'s and all their spatial derivatives have gaussian tails, uniformly in n. We then achieve to give a distributional sense to expansion (5) by a duality approach: any tempered distribution can be integrated or bracketed in the variable y with the expansions. Theorems 6, 7 and 8 provide precise statements, see Section 1.7.

1.4. **A first series of results.** Stating Theorems 6 and 8 requires a bit of preparation, namely defining appropriate functional spaces in which will live the coefficients π, π_n and $\pi_{n,i}$ in expansions (10). Before doing this, to encourage the reader, we would like to state a series of easy consequences of Theorem 6, including an application to financial markets. They will be proved in Section 1.7. The function π which appears in them is the principal *functional* error term. It is defined by (21)-(22). Note that analogous corollaries can be derived from Theorem 8 as well. The first result gives the rate of convergence of the spatial derivatives of the density:

Proposition 1. *Under (B) and (C), for all $\alpha, \beta \in \mathbb{N}^d$, there exists $c_1 \geq 0$ and $c_2 > 0$ such that for all $n \geq 1$, $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,*

$$\partial_x^\alpha \partial_y^\beta p_n(t,x,y) - \partial_x^\alpha \partial_y^\beta p(t,x,y) = \frac{1}{n}\, \partial_x^\alpha \partial_y^\beta \pi(t,x,y) + r_n(t,x,y)$$

and

$$|r_n(t,x,y)| \leq c_1 n^{-2} t^{-(|\alpha| + |\beta| + d + 4)/2} \exp\left(-c_2 \|x-y\|^2 / t \right).$$

The next proposition states that (5) is valid for measurable and polynomially growing f's:

Proposition 2. *Assume (B) and (C). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that there exists $c' \geq 0$ and $q \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$, $|f(x)| \leq c'(1 + \|x\|^q)$. Then there exists $c \geq 0$ such that for all $n \geq 1$, $t \in (0, 1]$ and $x \in \mathbb{R}^d$,*

$$\mathbb{E}[f(X_t^{n,x})] - \mathbb{E}[f(X_t^x)] = \frac{1}{n} \int_{\mathbb{R}^d} f(y)\pi(t, x, y)\, dy + r_n(t, x) \tag{11}$$

and

$$|r_n(t, x)| \leq cn^{-2}t^{-2} (1 + \|x\|^q).$$

As far as extending the class of f's for which (5) holds is concerned, we can even do better. Indeed, if for $\mu \in (0, 2)$ we denote by $\mathcal{E}_\mu$ the set of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that there exists $c_1, c_2 \geq 0$ such that for all $y \in \mathbb{R}^d$,

$$|f(y)| \leq c_1 \exp (c_2 \|y\|^\mu),$$

we have

Proposition 3. *Under (B) and (C), for all $\mu \in (0, 2)$ and $f \in \mathcal{E}_\mu$, there exists $c_1, c_2 \geq 0$ such that for all $n \geq 1$, $t \in (0, 1]$ and $x \in \mathbb{R}^d$, $f(X_t^x)$ and $f(X_t^{n,x})$ are integrable and*

$$\mathbb{E}[f(X_t^{n,x})] - \mathbb{E}[f(X_t^x)] = \frac{1}{n} \int_{\mathbb{R}^d} f(y)\pi(t, x, y)\, dy + r_n(t, x) \tag{12}$$

with

$$|r_n(t, x)| \leq c_1 n^{-2}t^{-2} \exp (c_2 \|x\|^\mu).$$

In particular, (5) remains true under (B) and (C) when $f \in \mathcal{E} = \cup_{\mu \in (0,2)} \mathcal{E}_\mu$. More generally, Theorem 6 leads to

Proposition 4. *Under (B) and (C), for all $\alpha \in \mathbb{N}^d$, $\mu \in (0, 2)$ and $f \in \mathcal{E}_\mu$, there exists $c_1, c_2 \geq 0$ such that for all $n \geq 1$, $t \in (0, 1]$ and $x \in \mathbb{R}^d$,*

$$\partial_x^\alpha \mathbb{E}[f(X_t^{n,x})] - \partial_x^\alpha \mathbb{E}[f(X_t^x)] = \frac{1}{n} \int_{\mathbb{R}^d} f(y)\partial_x^\alpha \pi(t, x, y)\, dy + r_n(t, x) \tag{13}$$

with

$$|r_n(t, x)| \leq c_1 n^{-2}t^{-(|\alpha|+4)/2} \exp (c_2 \|x\|^\mu).$$

This result can now be used in the context of financial markets.

1.5. **Application to option pricing and hedging.** Let $S^v = (S^{v,1}, \ldots, S^{v,d})$ be a basket of assets satisfying

$$\frac{dS_t^{v,i}}{S_t^{v,i}} = \mu_i(S_t^v)\, dt + \sum_{j=1}^r \sigma_{i,j}(S_t^v)\, dB_t^j, \qquad S_0^{v,i} = v^i > 0,$$

with $\mu, \sigma \in C_b^\infty(\mathbb{R}^d)$ and σ satisfying (C). Given a measurable and polynomially growing function ϕ, we try to estimate the price $\text{Price} = \mathbb{E}[\phi(S_t^v)]$, the deltas $\text{Delta}_i = \partial_v^{e_i}\mathbb{E}[\phi(S_t^v)]$ and the gammas $\text{Gamma}_{i,j} = \partial_v^{e_i+e_j}\mathbb{E}[\phi(S_t^v)]$ of the european option of maturity t and payoff ϕ ($(e_1, \ldots, e_d)$ is the canonical base of $\mathbb{R}^d$). To do so, let us set $x = \ln v$ (i.e. $x^i = \ln v^i$) and $X_t^{x,i} = \ln(S_t^{v,i})$. Then X is the solution of (3) with $b = \mu - \|\sigma\|^2/2 \in C_b^\infty(\mathbb{R}^d)$, where $\|\sigma\|_i^2(x) = \sum_{j=1}^r \sigma_{i,j}^2(x)$. If we set $\exp(x) = (\exp(x^1), \ldots, \exp(x^d))$ and $f(x) = \phi(\exp(x))$, we define a function $f \in \mathcal{E}_1$ and, since $\text{Price} = \mathbb{E}[f(X_t^x)]$, (12) leads to

$$\text{Price}^n - \text{Price} = C_t^{\text{Price}}\phi(v)/n + O\left(n^{-2}t^{-2} \exp (c_2 \|\ln v\|)\right),$$

where Price^n stands for the approximated price $\mathbb{E}[f(X_t^{n,x})]$ and

$$C_t^{\text{Price}}\phi(v) = \int_{(\mathbb{R}_+^*)^d} \phi(u) \frac{\pi(t, \ln v, \ln u)}{u_1 \cdots u_d}\, du.$$

Besides, if we set $\text{Delta}_i^n = \partial_v^{e_i} \mathbb{E}[f(X_t^{n,\ln v})]$ and $\text{Gamma}_{i,j}^n = \partial_v^{e_i+e_j} \mathbb{E}[f(X_t^{n,\ln v})]$, (13) shows that

$$\begin{aligned}
\text{Delta}^n - \text{Delta} &= C_t^{\text{Delta}}\phi(v)/n + O\left(n^{-2} t^{-5/2} \exp\left(c_2 \|\ln v\|\right)\right), \\
\text{Gamma}^n - \text{Gamma} &= C_t^{\text{Gamma}}\phi(v)/n + O\left(n^{-2} t^{-3} \exp\left(c_2 \|\ln v\|\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
C_t^{\text{Delta}}\phi(v)_i &= \frac{1}{v_i} \int_{(\mathbb{R}_+^*)^d} \phi(u) \frac{\partial_2^{e_i} \pi(t, \ln v, \ln u)}{u_1 \cdots u_d}\, du, \\
C_t^{\text{Gamma}}\phi(v)_{i,j} &= \frac{1}{v_i v_j} \int_{(\mathbb{R}_+^*)^d} \phi(u) \frac{\partial_2^{e_i+e_j} \pi(t, \ln v, \ln u) - \mathbf{1}_{\{i=j\}} \partial_2^{e_i} \pi(t, \ln v, \ln u)}{u_1 \cdots u_d}\, du.
\end{aligned}$$

Eventually we have proved that applying the Euler scheme of order n to the logarithm of the underlying leads to approximations of the price, the deltas and the gammas which converge to the true price, deltas and gammas with speed $1/n$, at least when the drift and volatility of the underlying satisfy (B) and (C), which in the context of financial markets seems not to be a restricting hypothesis. Note that the principal part of the error explodes as t tends to 0 as $t^{-1/2}$ for the prices, t^{-1} for the deltas and $t^{-3/2}$ for the gammas.

1.6. **Some functional spaces.** In order to state our main results (Proposition 5 and Theorems 6 and 8) precisely and shortly, let us introduce some families of functional spaces. Functional expansions like (10) will take place in such spaces. For $l \in \mathbb{Z}$, we first define $\mathcal{G}_l(\mathbb{R}^d)$ as the set of all measurable functions $\pi : (0,1] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

- for all $t \in (0,1]$, $\pi(t,\cdot,\cdot)$ is infinitely differentiable,
- for all $\alpha, \beta \in \mathbb{N}^d$, there exists two constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\left|\partial_x^\alpha \partial_y^\beta \pi(t,x,y)\right| \leq c_1 t^{-(|\alpha| + |\beta| + d + l)/2} \exp\left(-c_2 \|x-y\|^2 / t\right). \tag{14}$$

We say that a subset $\mathcal{B} \subset \mathcal{G}_l(\mathbb{R}^d)$ is bounded if, in (14), c_1 and c_2 can be chosen independently on $\pi \in \mathcal{B}$. We also introduce the space $\mathcal{G}(\mathbb{R}^d)$ defined in the same way as $\mathcal{G}_l(\mathbb{R}^d)$ with (14) replaced by the following two conditions:

$$\left|\partial_x^\alpha \partial_y^\beta \pi(t,x,y)\right| \leq c_1 t^{-(|\alpha| + |\beta| + d)/2} \exp\left(-c_2 \|x-y\|^2 / t\right), \tag{15}$$

$$\left|\partial_x^\alpha \left(\pi\left(t, x, x + y\sqrt{t}\right)\right)\right| \leq c_1 t^{-d/2} \exp\left(-c_2 \|y\|^2\right). \tag{16}$$

Note that we may always take the couple of constants (c_1, c_2) to be the same in both equations (15) and (16). Indeed, if they hold with two couples (c_1', c_2') and (c_1'', c_2''), they both hold with (c_1, c_2) if we take $c_1 = c_1' \vee c_1''$ and $c_2 = c_2' \wedge c_2''$. We say that a subset $\mathcal{B} \subset \mathcal{G}(\mathbb{R}^d)$ is bounded if, in (15) and (16), c_1 and c_2 can be chosen independently on $\pi \in \mathcal{B}$. Note that in equation (16), the upper bound keeps the same quality in t, namely $t^{-d/2}$, whatever the "number" α of times one differentiates the mapping $x \mapsto \pi\left(t, x, x + y\sqrt{t}\right)$. This will be crucial when proving Proposition 5.

It is convenient to extend these definitions to mappings that also depend on an intermediate time $s \in (0,t)$. To do so, let us denote by $\mathcal{T}_1$ the unit triangle $\{(s,t) \in \mathbb{R}^2 | 0 <$

$s < t \leq 1\}$ and, for $l \in \mathbb{Z}$, let us define $\mathcal{H}_l(\mathbb{R}^d)$ as the space of measurable functions $\rho : \mathcal{T}_1 \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ such that

- for all $(s,t) \in \mathcal{T}_1$, $\rho(s,t,\cdot,\cdot)$ is infinitely differentiable,
- for all $\alpha, \beta \in \mathbb{N}^d$, there exists two constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $(s,t) \in \mathcal{T}_1$ and $x, y \in \mathbb{R}^d$,

$$\left|\partial_x^\alpha \partial_y^\beta \rho(s,t,x,y)\right| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2} \exp\left(-c_2 \|x-y\|^2/t\right). \tag{17}$$

Again we say that a subset $\mathcal{B} \subset \mathcal{H}_l(\mathbb{R}^d)$ is bounded if, in (17), c_1 and c_2 can be chosen independently on $\rho \in \mathcal{B}$. We also introduce the space $\mathcal{H}(\mathbb{R}^d)$ which is defined in the same way as $\mathcal{H}_l(\mathbb{R}^d)$ with (17) replaced by

$$\left|\partial_x^\alpha \partial_y^\beta \rho(s,t,x,y)\right| \leq c_1 t^{-(|\alpha|+|\beta|+d)/2} \exp\left(-c_2 \|x-y\|^2/t\right), \tag{18}$$

$$\left|\partial_x^\alpha \left(\rho\left(s,t,x,x+y\sqrt{t}\right)\right)\right| \leq c_1 t^{-d/2} \exp\left(-c_2 \|y\|^2\right), \tag{19}$$

and we say that a subset $\mathcal{B} \subset \mathcal{H}(\mathbb{R}^d)$ is bounded if, in (18) and (19), c_1 and c_2 can be chosen independently on $\rho \in \mathcal{B}$. Again we may always choose the couple (c_1, c_2) to be the same in both equations (18) and (19). Note that the upper bounds in (17), (18) and (19) are exactly the same as the ones in (14), (15) and (16). In particular, they do not depend on s.

Eventually, for $\pi_1, \pi_2 \in \mathcal{G}(\mathbb{R}^d)$, $g \in C_b^\infty(\mathbb{R}^d)$ and $\gamma \in \mathbb{N}^d$, we define a function $\pi_1 *_{g,\gamma} \pi_2$ on $\mathcal{T}_1 \times \mathbb{R}^d \times \mathbb{R}^d$ by putting

$$(\pi_1 *_{g,\gamma} \pi_2)(s,t,x,y) = \int_{\mathbb{R}^d} g(z) \pi_1(s,x,z) \partial_2^\gamma \pi_2(t-s,z,y)\, dz.$$

Notation ∂_2 means differentiation with respect to the second argument, here z. Operation $*_{g,\gamma}$ is a space convolution which naturally appears when developping the differential operator D in equations (8) and (9).

1.7. **Main results.** We are now able to state our main results as follows.

Proposition 5. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bounded subsets of $\mathcal{G}(\mathbb{R}^d)$, $g \in C_b^\infty(\mathbb{R}^d)$ and $\gamma \in \mathbb{N}^d$. Then*

(i) *$\{\pi_1 *_{g,\gamma} \pi_2 | \pi_1 \in \mathcal{B}_1, \pi_2 \in \mathcal{B}_2\}$ is a bounded subset of $\mathcal{H}_{|\gamma|}(\mathbb{R}^d)$,*
(ii) *$\{\pi_1 *_{g,0} \pi_2 | \pi_1 \in \mathcal{B}_1, \pi_2 \in \mathcal{B}_2\}$ is a bounded subset of $\mathcal{H}(\mathbb{R}^d)$.*

Theorem 6. *Under (B) and (C),*

(i) *for all $t \in (0,1]$ and $x \in \mathbb{R}^d$, X_t^x has a density $p(t,x,\cdot)$ and $p \in \mathcal{G}(\mathbb{R}^d)$,*
(ii) *for all $t \in (0,1]$, $x \in \mathbb{R}^d$ and $n \geq 1$, $X_t^{n,x}$ has a density $p_n(t,x,\cdot)$ and $(p_n, n \geq 1)$ is a bounded sequence in $\mathcal{G}(\mathbb{R}^d)$,*
(iii) *there exists $\pi \in \mathcal{G}_1(\mathbb{R}^d)$ and a bounded sequence $(\pi_n, n \geq 1)$ in $\mathcal{G}_4(\mathbb{R}^d)$ such that for all $n \geq 1$,*

$$p_n - p = \pi/n + \pi_n/n^2. \tag{20}$$

These results are proved in Section 3.2. In Theorem 6, statement (i) is already known, see [4], Theorem 7, page 260, and statement (ii) has essentially been proved in [9]. As explained in Section 1.3, Proposition 5, together with these two statements, is the key to derive statement (iii).

The function π can be expressed in terms of p by

$$\pi(t,x,y) = \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} p(s,x,z) L_2^*(p(t-s,\cdot,y))(z)\, dzds, \tag{21}$$

where the differential operator L_2^* is explicitely given in terms of the functions a and b by

$$\begin{aligned} -L_2^* = & \sum_{i=1}^d \left(b\cdot\nabla b_i + \frac{1}{2}\operatorname{tr}\left(a\nabla^2 b_i\right)\right)\partial_i \\ & + \sum_{i,j=1}^d \left(\frac{1}{2} b\cdot\nabla a_{i,j} + a_j\cdot\nabla b_i + \frac{1}{4}\operatorname{tr}\left(a\nabla^2 a_{i,j}\right)\right)\partial_{ij} + \frac{1}{2}\sum_{i,j,k=1}^d a_k\cdot\nabla a_{i,j}\partial_{ijk}. \end{aligned} \tag{22}$$

Here, $\cdot$, a_k, tr, ∇ and ∇^2 respectively stand for the inner product in $\mathbb{R}^d$, the k-th column of a, the trace of a matrix, the gradient vector and the hessian matrix. In the case when $t=1$, (21) agrees with V. Bally and D. Talay's expression for π ([2], definition 2.2, page 100), but seems preferable because it does not involve differentiation with respect to t and makes explicit that the space differential operator L_2^* is of order less than 3, when V. Bally and D. Talay's operator $\mathcal{U}$ involves a fourth order differentiation in space.

We shall now prove that if X is elliptic the expansion (5) is valid in the very general case when f is a tempered distribution. Let us denote by $\mathcal{S}(\mathbb{R}^d)$ Schwartz's space, i.e. the space of infinitely differentiable functions $\varphi:\mathbb{R}^d\to\mathbb{R}$ such that $x\mapsto x^\alpha\partial^\beta\varphi(x)\in L^\infty(\mathbb{R}^d)$ for all $\alpha,\beta\in\mathbb{N}^d$ (x^α stands for $x_1^{\alpha_1}\cdots x_d^{\alpha_d}$), and let us denote by $\mathcal{S}'(\mathbb{R}^d)$ the space of tempered distributions. The seminorms $(N_q, q\in\mathbb{N})$ are defined on $\mathcal{S}(\mathbb{R}^d)$ by

$$N_q(\varphi) = \sum_{|\alpha|\le q, |\beta|\le q} \sup_{x\in\mathbb{R}^d}\left|x^\alpha\partial^\beta\varphi(x)\right|,$$

and the order $\#S$ of $S\in\mathcal{S}'(\mathbb{R}^d)$ is the smallest integer q such that there is a $c\ge 0$ such that $|\langle S,\varphi\rangle|\le cN_q(\varphi)$ for all $\varphi\in\mathcal{S}(\mathbb{R}^d)$. Note that whenever $\pi\in\mathcal{G}_l(\mathbb{R}^d)$, $\pi(t,x,\cdot)$ and $\pi(t,\cdot,y)$ belong to $\mathcal{S}(\mathbb{R}^d)$. More precisely, for $\mathcal{B}\subset\mathcal{G}_l(\mathbb{R}^d)$ bounded, there exists $c\ge 0$ such that for all $\pi\in\mathcal{B}$, $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,

$$N_q(\pi(t,x,\cdot)) \le ct^{-(d+l+q)/2}\left(1+\|x\|^q\right) \qquad \text{and} \qquad N_q(\pi(t,\cdot,y)) \le ct^{-(d+l+q)/2}\left(1+\|y\|^q\right).$$

Applying a tempered distribution S to (20), t and x or t and y being fixed, we immediately deduce from Theorem 6

Theorem 7. *Under (B) and (C), for all $S\in\mathcal{S}'(\mathbb{R}^d)$, there exists $c\ge 0$ such that for all $n\ge 1$, $t\in(0,1]$ and $x,y\in\mathbb{R}^d$,*

$$\begin{aligned} \langle S, p_n(t,x,\cdot)\rangle - \langle S, p(t,x,\cdot)\rangle &= \frac{1}{n}\langle S, \pi(t,x,\cdot)\rangle + r_n'(t,x), \\ \langle S, p_n(t,\cdot,y)\rangle - \langle S, p(t,\cdot,y)\rangle &= \frac{1}{n}\langle S, \pi(t,\cdot,y)\rangle + r_n''(t,y), \end{aligned}$$

and

$$\left|r_n'(t,x)\right| + \left|r_n''(t,x)\right| \le cn^{-2}t^{-(d+4+\#S)/2}\left(1+\|x\|^{\#S}\right).$$

Let us define $\mathbb{E}\left[S(Y)\right]$ by $\langle S, p_Y\rangle$ when $S\in\mathcal{S}'(\mathbb{R}^d)$ and Y is a random variable with density $p_Y\in\mathcal{S}(\mathbb{R}^d)$. Note that, when S is a measurable and polynomially growing function, this definition coincides with the usual expectation. We then have proved that, under (B) and (C), (5) is valid for f's being only tempered distributions, and not only for $t = 1$, but also for any time $t \in (0, 1]$, and we have even precised the way the $O(1/n^2)$ remainder depends on t, f and x. Precisely, this remainder grows slower than $\|x\|^{\#f}$ as x tends to infinity, and explodes slower than $t^{-(\#f+d+4)/2}$ as t tends to 0.

We can now prove the propositions stated in Section 1.4. Proposition 1 is immediate from Theorem 6. In the special case when S is a measurable and polynomially growing function, we get Proposition 2:

Proof of Proposition 2. Multiplying (20) by $f(y)$ and integrating in y leads to (11) with the remainder $r_n(t, x) = n^{-2} \int_{\mathbb{R}^d} f(y)\pi_n(t, x, y)\, dy$. Since $|f(y)| \le c'(1 + \|y\|^q)$ and $(\pi_n, n \ge 1)$ is bounded in $\mathcal{G}_4(\mathbb{R}^d)$, we can find $c_1 \ge 0$ and $c_2 > 0$ such that for all $n \ge 1$, $t \in (0, 1]$ and $x \in \mathbb{R}^d$, $|r_n(t, x)| \le c_1 n^{-2} t^{-(d+4)/2} \int_{\mathbb{R}^d} (1 + \|y\|^q) \exp(-c_2 \|x - y\|^2 / t)\, dy$. Setting $\zeta = (y - x)/\sqrt{t}$ leads to $|r_n(t, x)| \le c_1 n^{-2} t^{-2} \int_{\mathbb{R}^d} (1 + \|x + \zeta\sqrt{t}\|^q) \exp(-c_2 \|\zeta\|^2)\, d\zeta$. To complete the proof, it remains to observe that there exists $c \ge 0$ such that for all $t \in (0, 1]$ and $x, \zeta \in \mathbb{R}^d$, $\|x + \zeta\sqrt{t}\|^q \le c(\|x\|^q + \|\zeta\|^q)$. □

It is easy to adapt the preceding proof to get Proposition 3. In the same way, differentiating (5) α times in x, multiplying by $f(y)$ and integrating in y leads to Proposition 4.

Expansion (20) should be seen as an improvement of (6): it allows for infinite differentiation in x and y and also precises the way the coefficients explode when t tends to 0. We have an analogous improvement for expansion (7):

Theorem 8. *Under (B) and (C), for each $i \ge 1$, there exists a bounded family $(\pi_{n,i}, n \ge 1)$ in $\mathcal{G}_{2i-2}(\mathbb{R}^d)$ and two bounded families $(\pi'_{n,i}, n \ge 1)$ and $(\pi''_{n,i}, n \ge 1)$ in $\mathcal{G}_{2i}(\mathbb{R}^d)$ such that for all $j, n \ge 1$,*

$$p_n - p = \sum_{i=1}^{j-1} \frac{\pi_{n,i}}{n^i} + \sum_{i=2}^{j} (t - \lfloor nt \rfloor/n)^i\, \pi'_{n,i} + \frac{\pi''_{n,j}}{n^j}. \tag{23}$$

Here and in all the sequel we use the convention that a sum over an empty set is zero, and $\lfloor nt \rfloor$ denotes the greatest integer less than or equal to nt. Expressions involving $\lfloor nt \rfloor$ do not appear in (20) since they are hidden in the remainder. When $t = 1$ and no differentiation is applied neither in x nor in y, (23) boils down to the result of V. Konakov and E. Mammen [9]. Again note that (23) is much richer in the sense that it allows for infinite differentiation in space and also precises the dependence on t. Theorem 8 will also be proved in Section 3.2.

1.8. **A preliminary result.**

As explained in section 1.3, in order to prove point (iii) in Theorem 6, we first seek an expansion for the error operator

$$\Delta_t^n = P_t^n - P_t$$

where, for $f \in C^\infty_{\text{pol}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have set $P_t f(x) = \mathbb{E}[f(X_t^x)]$ and $P_t^n f(x) = \mathbb{E}\left[f(X_t^{n,x})\right]$. Precisely, we look for operators C_t and R_t^n such that $R_t^n = O(1/n^2)$ and $\Delta_t^n = C_t/n + R_t^n$. The following theorem, interesting in itself, is proved in Section 2. It can be seen as an improvement of [14]. It not only gives explicit formulas for $C_t f(x)$ and $R_t^n f(x)$ but also provides useful information about their dependencies on n, t, f and x.

Note that it does not require neither (B) nor (B') nor (C). In order to state it shortly, let us

- denote by $\mathcal{L}\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$ the space of endomorphisms of $C^\infty_{\text{pol}}(\mathbb{R}^d)$,
- say that a subset $\mathcal{B} \subset C^\infty_{\text{pol}}(\mathbb{R}^d)$ is bounded if, in (2), c and q can be chosen independently on $f \in \mathcal{B}$,
- say that $T \in \mathcal{L}\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$ is bounded if for all bounded $\mathcal{B} \subset C^\infty_{\text{pol}}(\mathbb{R}^d)$, $\{Tf | f \in \mathcal{B}\}$ is a bounded subset of $C^\infty_{\text{pol}}(\mathbb{R}^d)$,
- denote by $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$ the space of bounded endomorphisms of $C^\infty_{\text{pol}}(\mathbb{R}^d)$,
- say that a $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$-valued family $(T_i, i \in I)$ is bounded if for all bounded $\mathcal{B} \subset C^\infty_{\text{pol}}(\mathbb{R}^d)$, $\{T_i f | f \in \mathcal{B}, i \in I\}$ is a bounded subset of $C^\infty_{\text{pol}}(\mathbb{R}^d)$,
- say that $(T_i, i \in I)$ is a $O(h(i))$ family in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$ if the family $(h(i)^{-1} T_i, i \in I)$ is bounded.

It is already known that, under (A), $(P_t, t \in [0,1])$ is a bounded family in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$. A proof can de found in [11], Lemma 3.9, page 15. Using Lemma 25, this proof straightforwardly adapts uniformly in n so that $(P^n_t, t \in [0,1], n \geq 1)$ is also bounded in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$. We are now in the position to state the main result of the first step:

Theorem 9. *Under (A), $(\Delta^n_t, t \in [0,1], n \geq 1)$ is a $O(t/n)$ family in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$, and there exists a $O(t)$ process $(C_t, t \in [0,1])$ and a $O(1/n^2)$ family $(R^n_t, t \in [0,1], n \geq 1)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$ such that*

$$\Delta^n_t = C_t/n + R^n_t.$$

Moreover, C_t is explicitely given in terms of $(P_t, t \in [0,1])$ and of L^*_2 (see (22)) by

$$C_t = \frac{1}{2}\int_0^t P_s L^*_2 P_{t-s}\, ds. \tag{24}$$

Note that this theorem covers the result of D. Talay and L. Tubaro [14] since it implies that for any $f \in C^\infty_{\text{pol}}(\mathbb{R}^d)$ we can find a $q \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$, $t \in [0,1]$ and $n \geq 1$,

$$\Delta^n_t f(x) = C_t f(x)/n + O\left(\frac{1 + \|x\|^q}{n^2}\right).$$

It even improves it a bit since we see that this holds under nothing more than condition (A), whereas D. Talay and L. Tubaro state their result under the stronger condition (B'). Note also that if we restrict ourselves to times t belonging to the discretization grid $\mathfrak{S}_n$, we get a better control, of order $O(t/n^2)$, of the remainder, see Remark 14.

Instead of Theorem 9, in order to derive Theorem 8, we shall need

Theorem 10. *Under (A), there exists a sequence of differential operators $(L^*_j, j \geq 2)$, recursively defined by (27)-(28), and for each $i \geq 1$ a $O(t/n^i)$ family $R^i = (R^{n,i}_t, t \in [0,1], n \geq 1)$ in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$ such that for all $j \geq 1$,*

$$\Delta^n_t = \sum_{i=2}^{j} \frac{1}{i! n^i} \sum_{k=0}^{\lfloor nt \rfloor - 1} P^n_{t^n_k} L^*_i P_{t-t^n_k} + R^{n,j}_t + \sum_{i=2}^{j} \frac{(t - \lfloor nt \rfloor/n)^i}{i!} P^n_{\lfloor nt \rfloor/n} L^*_i P_{t - \lfloor nt \rfloor/n}. \tag{25}$$

Observe that the main term in (25) is

$$\frac{1}{n}\left(\frac{1}{2n}\sum_{k=0}^{\lfloor nt \rfloor - 1} P^n_{t^n_k} L^*_2 P_{t-t^n_k}\right) \approx \frac{C_t}{n},$$

and the remainder is of order $1/n^2$. Note also that if we restrict ourselves to times belonging to the discretization grid $\mathfrak{S}_n$, we get the following expansion in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$:

$$\Delta^n_{\lfloor nt\rfloor/n} = \sum_{i=2}^{j} \frac{1}{i!n^i} \sum_{k=0}^{\lfloor nt\rfloor-1} P^n_{t^n_k} L^*_i P_{\lfloor nt\rfloor/n - t^n_k} + O\left(\frac{t}{n^j}\right).$$

Theorem 10 is also proved in Section 2.

1.9. **Organization of the paper.** Section 2 deals with the expansion for the expectation: it is dedicated to the proofs of Theorems 9 and 10.

Section 3 is our second and final step. It is devoted to the proofs of Theorems 6 and 8. It begins with the proof of Proposition 5 concerning the space convolution $*_{g,\gamma}$ in $\mathcal{G}(\mathbb{R}^d)$.

Eventually, Section 4 is an appendix where we have gathered useful results on the Euler scheme and technical lemmas that are used in Sections 2 and 3.

2. First step: expansion for $\mathbb{E}\left[f\left(X^{n,x}_t\right)\right]$

In this section we seek to expand $\Delta^n_t f(x) = \mathbb{E}\left[f\left(X^{n,x}_t\right)\right] - \mathbb{E}\left[f\left(X^x_t\right)\right]$ in powers of the time step $1/n$ when f is a regular function, say $f \in C^\infty_{\text{pol}}(\mathbb{R}^d)$. The idea is the following. Recall the discussion preceding Theorem 9: under (A), both P_t and P^n_t are endomorphisms of $C^\infty_{\text{pol}}(\mathbb{R}^d)$. In $\mathcal{L}\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$ we then write

$$\Delta^n_t = P^n_t - P_t = \sum_{k=0}^{\lfloor nt\rfloor-1} P^n_{t^n_k} \Delta^n_{1/n} P_{t-t^n_{k+1}} + P^n_{\lfloor nt\rfloor/n} \Delta^n_{t-\lfloor nt\rfloor/n}. \tag{26}$$

There is a subtle point here: $(X^{n,x}_t, t \in [0,1])$ is not a Markov process, since the future of $X^{n,x}_t$ depends on the past value $X^{n,x}_{\lfloor nt\rfloor/n}$, see (4). Nevertheless, it is easy to check by conditioning on $\mathcal{F}_{t^n_k}$ that we have $P^n_{t^n_k} P^n_s = P^n_{t^n_k+s}$ for all $s \geq 0$ - but beware: this is different from $P^n_s P^n_{t^n_k}$ as soon as ns is not an integer.

Equation (26) leads us to expand Δ^n_t for small t, namely for $t \leq 1/n$. This naturally involves a series of differential operators as we shall now see.

2.1. **Operators associated with the Euler scheme.** Let us denote by L the infinitesimal generator of the diffusion X and by $(L^x, x \in \mathbb{R}^d)$ its tangent infinitesimal generator, i.e.

$$L = \sum_{i=1}^{d} b_i \partial^{e_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}\partial^{e_i+e_j} \qquad \text{and} \qquad L^x = \sum_{i=1}^{d} b_i(x) \partial^{e_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\partial^{e_i+e_j}.$$

We use the convention that L and L^x act on the variable y, so that, for instance, $L\psi(t,x,y)$ and $L^x\psi(t,x,y)$ respectively stand for $L\left(\psi(t,x,\cdot)\right)(y)$ and $L^x\left(\psi(t,x,\cdot)\right)(y)$. L^x is the infinitesimal generator of the Euler scheme $(X^{n,x}_t, t \in [0,1/n])$ starting from x, over the first discretization time interval: L^x is built from L in the same way as X^n is built from X, by freezing the drift b and the volatility σ to their initial value on discretization intervals. Besides, for each $x \in \mathbb{R}^d$ we define a sequence of differential operators $(L^x_j, j \in \mathbb{N})$ by putting $L^x_0 = I$ (the identity operator) and

$$L^x_{j+1} = L^x L^x_j - L^x_j L, \tag{27}$$

and we set

$$L^*_j f(x) = L^x_j f(x). \tag{28}$$

Observe that $L_1^* = 0$. Besides, L_2^* is given by (22) so that, under (A), $L_2^* \in \mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$ and there exists a family $(g^*_{2,\alpha}, 1 \le |\alpha| \le 3)$ in $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ such that

$$L_2^* = \sum_{1\le|\alpha|\le 3} g^*_{2,\alpha}\partial^\alpha. \tag{29}$$

L_2^* gives the exact principal error term in the expansion of Δ_t^n, see (24) and (21). L_j^* is the differential operator appearing in (25). It does not give the exact expansion in powers of $1/n$ but an approximated version, in the spirit of [9], since in (25) the coefficients depend on n - but should themselves be expanded in powers of $1/n$. See [7], equations (6.35) and (6.36), for an expression of the operators involved in the exact expansion.

Under (A), L and L^x belong to $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$ for each $x \in \mathbb{R}^d$, and, by induction, so does L_j^x. We can describe L_j^x more precisely. Indeed, defining the powers of an operator A by $A^0 = I$ and $A^{j+1} = AA^j$, inductions on j lead to $L_j^x = \sum_{i=0}^j (-1)^i \binom{j}{i} (L^x)^{j-i} L^i$ and to the existence of a family $(g_{j,\alpha}, h_{j,\alpha}, j \in \mathbb{N}^*, 1 \le |\alpha| \le 2j)$ in $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ such that

$$\forall x \in \mathbb{R}^d, \qquad (L^x)^j = \sum_{1\le|\alpha|\le 2j} g_{j,\alpha}(x)\partial^\alpha \qquad \text{and} \qquad L^j = \sum_{1\le|\alpha|\le 2j} h_{j,\alpha}\partial^\alpha.$$

Hence, for each $j \in \mathbb{N}^*$ one can find a family $(m_{j,\alpha}, 1 \le |\alpha| \le 2j)$ of integers and a family $(g_{j,\alpha,l}, h_{j,\alpha,l}, 1 \le |\alpha| \le 2j, 1 \le l \le m_{j,\alpha})$ in $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ such that for all $x \in \mathbb{R}^d$,

$$L_j^x = \sum_{1\le|\alpha|\le 2j} \left(\sum_{l=1}^{m_{j,\alpha}} g_{j,\alpha,l}(x) h_{j,\alpha,l} \right) \partial^\alpha. \tag{30}$$

Remark 11. Note that when (B) holds, the functions $g_{j,\alpha,l}$, $h_{j,\alpha,l}$ and $g^*_{2,\alpha}$ all belong to $C_b^\infty(\mathbb{R}^d)$ (in fact they are polynomial in b, σ and their derivatives).

We are now in the position to define a family of operators $\Phi^j = (\Phi^{n,j}_{s,t}, n \ge 1, 0 \le s \le t \le 1/n)$ as follows:

$$\forall f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d), \quad \Phi^{n,j}_{s,t} f(x) = \mathbb{E}\left[L_j^x P_{t-s} f\left(X_s^{n,x}\right)\right]. \tag{31}$$

Observe that $\Phi^{n,j}_{0,t} = L_j^* P_t$ and that, from (30),

$$\Phi^{n,j}_{s,t} = \sum_{1\le|\alpha|\le 2j} \sum_{l=1}^{m_{j,\alpha}} g_{j,\alpha,l} P_s^n (h_{j,\alpha,l}\partial^\alpha P_{t-s}). \tag{32}$$

Boundedness is a key property of this family:

Proposition 12. *Under (A), Φ^j is a bounded family in $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$.*

Proof. $(P_t, t \in [0,1])$ and $(P_t^n, t \in [0,1], n \ge 1)$ are bounded families in $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$, see the discussion preceding Theorem 9. Besides, multiplication by a function in $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and differentiation are bounded operators on $C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$. As a sum of compositions of bounded families in $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$, Φ^j is a bounded family in $\mathcal{L}_b\left(C^\infty_{\mathrm{pol}}(\mathbb{R}^d)\right)$. □

The family Φ^j naturally appears when we recusively use Itô's formula to expand Δ_t^n for small t, as we now explain.

2.2. **Itô expansions.** We recall (see [11], theorem 3.11, page 16) that for $f \in C^\infty_{\text{pol}}(\mathbb{R}^d)$, $(s,y) \mapsto P_{t-s}f(y)$ is infinitely differentiable on $[0,t] \times \mathbb{R}^d$ and

$$\forall (s,y) \in [0,t] \times \mathbb{R}^d, \qquad (\partial_s + L)P_{t-s}f(y) = 0. \tag{33}$$

Since ∂_s and L^x_j commute, (33) and the definition of L^x_j imply

$$(\partial_s + L^x)L^x_j P_{t-s} = (L^x L^x_j - L^x_j L)P_{t-s} = L^x_{j+1}P_{t-s}. \tag{34}$$

For a measurable family (A_s) in $\mathcal{L}_b\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$, we denote by $\int_{t_1}^{t_2} A_s\, ds$ the element of $\mathcal{L}\left(C^\infty_{\text{pol}}(\mathbb{R}^d)\right)$ which maps f to $x \mapsto \int_{t_1}^{t_2} A_s f(x)\, ds$. The following lemma states that $\Phi^{n,j+1}_{\cdot,t}$ is the derivative of $\Phi^{n,j}_{\cdot,t}$ on the interval $[0,t]$.

Lemma 13. *Under (A), for all $j \in \mathbb{N}$, $n \geq 1$ and $0 \leq s \leq t \leq 1/n$,*

$$\Phi^{n,j}_{s,t} = L^*_j P_t + \int_0^s \Phi^{n,j+1}_{s',t}\, ds'. \tag{35}$$

Proof. For $f \in C^\infty_{\text{pol}}(\mathbb{R}^d)$, $(s,y) \mapsto L^x_j P_{t-s}f(y)$ is infinitely differentiable on $[0,t] \times \mathbb{R}^d$ so that we can apply Itô's formula to it and to the semimartingale $X^{n,x}$ between 0 and s. Using (34) for the second equality, we get

$$\begin{aligned} &L^x_j P_{t-s} f\left(X^{n,x}_s\right) - L^x_j P_t f(x) - M_s \\ &\qquad = \int_0^s \left(\frac{\partial}{\partial s} + L^x\right) L^x_j P_{t-s'} f\left(X^{n,x}_{s'}\right)\, ds' = \int_{t^n_k}^s L^x_{j+1} P_{t-s'} f\left(X^{n,x}_{s'}\right)\, ds' \end{aligned}$$

where $M_s = \sum_{i=1}^d \sum_{j=1}^r \sigma_{i,j}(x) \int_0^s \partial^{e_i}\left(L^x_j P_{t-s'} f\left(X^{n,x}_{s'}\right)\right) dB^j_{s'}$. Since $\{L^x_j P_{t-s'} f | s' \in [0,t]\}$ is bounded in $C^\infty_{\text{pol}}(\mathbb{R}^d)$, (59) implies that $(M_s, s \in [0,t])$ is a square-integrable martingale and thus has zero mean. Hence, taking expectations and using (31) and Fubini's theorem, we have

$$\Phi^{n,j}_{s,t} f(x) - L^*_j P_t f(x) = \int_0^s \mathbb{E}\left[L^x_{j+1} P_{t-s'} f\left(X^{n,x}_{s'}\right)\right]\, ds' = \int_0^s \Phi^{n,j+1}_{s',t} f(x)\, ds',$$

which concludes the proof. $\square$

For $t \in [0, 1/n]$, since $\Delta^n_t = \Phi^{n,0}_{t,t} - \Phi^{n,0}_{0,t}$, by iterating (35) we get

$$\Delta^n_t = \sum_{i=2}^j \frac{t^i}{i!} L^*_i P_t + I^{n,j+1}_t, \tag{36}$$

where

$$I^{n,j+1}_t = \int_0^t \int_0^{s_1} \cdots \int_0^{s_j} \Phi^{n,j+1}_{s_{j+1},t}\, ds_{j+1} \cdots ds_2 ds_1. \tag{37}$$

The crucial point here is that, by construction, $L^*_1 = 0$ so that the sum in (36) begins with $i = 2$.

Injecting this in (26), we eventually get for all $t \in [0,1]$ and $n \geq 1$

$$\Delta^n_t = \sum_{i=2}^j \frac{1}{i! n^i} \sum_{k=0}^{\lfloor nt \rfloor - 1} P^n_{t^n_k} L^*_i P_{t-t^n_k} + R^{n,j}_t + \sum_{i=2}^j \frac{(t - \lfloor nt \rfloor/n)^i}{i!} P^n_{\lfloor nt \rfloor/n} L^*_i P_{t-\lfloor nt \rfloor/n}, \tag{38}$$

where

$$R_t^{n,j} = \sum_{k=0}^{\lfloor nt\rfloor -1} P^n_{t^n_k} I^{n,j+1}_{1/n} P_{t-t^n_{k+1}} + P^n_{\lfloor nt\rfloor/n} I^{n,j+1}_{t-\lfloor nt\rfloor/n}. \tag{39}$$

From Proposition 12, $(I_t^{n,j+1}, n \geq 1, t \in [0,1/n])$ is a $O(t^{j+1})$ family in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$. Recalling the boundedness of $(P_t, t \in [0,1])$ and $(P^n_t, t \in [0,1], n \geq 1)$, we get that the family $R^i = (R_t^{n,i}, t \in [0,1], n \geq 1)$ is $O(t/n^i)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$. Theorem 10 is thus proved. We are now in good position to prove Theorem 9.

2.3. **Proof of Theorem 9.** In the particular case when $j = 1$, (38) reads $\Delta^n_t = R^{n,1}_t$ so that we have proved that $(\Delta^n_t, t \in [0,1], n \geq 1)$ is $O(t/n)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$, which was the first statement of Theorem 9.

In the particular case when $j = 2$, if we set

$$C_t = \frac{1}{2}\int_0^t P_s L_2^* P_{t-s}\, ds, \tag{40}$$

$$A^n_{1,t} = \frac{1}{2n}\left(\frac{1}{n}\sum_{k=0}^{\lfloor nt\rfloor -1} P_{t^n_k} L_2^* P_{t-t^n_k} - \int_0^t P_s L_2^* P_{t-s}\, ds\right), \tag{41}$$

$$A^n_{2,t} = \frac{1}{2n^2}\sum_{k=0}^{\lfloor nt\rfloor -1}\left(P^n_{t^n_k} L_2^* P_{t-t^n_k} - P_{t^n_k} L_2^* P_{t-t^n_k}\right), \tag{42}$$

$$A^n_{3,t} = R^{n,2}_t + \frac{(t-\lfloor nt\rfloor/n)^2}{2} P^n_{\lfloor nt\rfloor/n} L_2^* P_{t-\lfloor nt\rfloor/n} \tag{43}$$

$$R^n_t = A^n_{1,t} + A^n_{2,t} + A^n_{3,t}, \tag{44}$$

equation (38) reads

$$\Delta^n_t = C_t/n + R^n_t. \tag{45}$$

As a composition of bounded families, $(P_s L_2^* P_{t-s}, 0 \leq s \leq t \leq 1)$ is a bounded family in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$, so that $(C_t, t \in [0,1])$ is $O(t)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$. It remains to prove that $(R^n_t, t \in [0,1], n \geq 1)$ is $O(1/n^2)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$. We have already proved that it is true of $(R^{n,2}_t, t \in [0,1], n \geq 1)$. It is obviously also true of $((t-\lfloor nt\rfloor/n)^2 P^n_{\lfloor nt\rfloor/n} L_2^* P_{t-\lfloor nt\rfloor/n}, t \in [0,1], n \geq 1)$, so that $(A^n_{3,t}, t \in [0,1], n \geq 1)$ is $O(1/n^2)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$.

For $(A^n_{1,t}, t \in [0,1], n \geq 1)$, observe that, if we set $L_3^{\#} = LL_2^* - L_2^* L \in \mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$, as $\partial_s P_s = LP_s = P_s L$, we have $\partial_s P_s L_2^* P_{t-s} = P_s L L_2^* P_{t-s} - P_s L_2^* L P_{t-s} = P_s L_3^{\#} P_{t-s}$. Hence the family $(P_{t^n_k} L_2^* P_{t-t^n_k} - P_s L_2^* P_{t-s}, t \in [0,1], n \geq 1, k \in \{0,\ldots,\lfloor nt\rfloor -1\}, s \in [t^n_k, t^n_{k+1}])$ satisfies

$$P_{t^n_k} L_2^* P_{t-t^n_k} - P_s L_2^* P_{t-s} = -\int_{t^n_k}^s P_u L_3^{\#} P_{t-u}\, du \tag{46}$$

and thus is $O(1/n)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$. As a consequence,

$$A^n_{1,t} = \frac{1}{2n}\sum_{k=0}^{\lfloor nt\rfloor -1}\int_{t^n_k}^{t^n_{k+1}}\left(P_{t^n_k} L_2^* P_{t-t^n_k} - P_s L_2^* P_{t-s}\right) ds - \frac{1}{2n}\int_{\lfloor nt\rfloor/n}^t P_s L_2^* P_{t-s}\, ds \tag{47}$$

is $O(1/n^2)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$.

As for $(A^n_{2,t}, t \in [0,1], n \geq 1)$, note that $P^n_{t^n_k} L^*_2 P_{t-t^n_k} - P_{t^n_k} L^*_2 P_{t-t^n_k} = \Delta^n_{t^n_k} L^*_2 P_{t-t^n_k}$. Since $(\Delta^n_t, t \in [0,1], n \geq 1)$ is $O(1/n)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$, so is the family $(P^n_{t^n_k} L^*_2 P_{t-t^n_k} - P_{t^n_k} L^*_2 P_{t-t^n_k}, t \in [0,1], n \geq 1, k \in \{0, \ldots, \lfloor nt \rfloor - 1\})$, as the composition of a bounded family by a $O(1/n)$ family in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$. This completes the proof of Theorem 9.

Remark 14. It is noteworthy that the family $(R'^n_t, t \in [0,1], n \geq 1)$ defined by

$$R'^n_t = R^n_t + \frac{1}{2n} \int_{\lfloor nt \rfloor/n}^t P_s L^*_2 P_{t-s}\, ds - \frac{(t - \lfloor nt \rfloor/n)^2}{2} P^n_{\lfloor nt \rfloor/n} L^*_2 P_{t-\lfloor nt \rfloor/n}$$

is $O(t/n^2)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$. In particular, $(R^n_{\lfloor nt \rfloor/n}, t \in [0,1], n \geq 1)$ is $O(t/n^2)$ in $\mathcal{L}_b\left(C^\infty_{\rm pol}(\mathbb{R}^d)\right)$.

3. Second step: expansion for the density of $X^{n,x}_t$

This section is devoted to the proofs of Theorems 6 and 8.

3.1. Space convolutions.

We begin by proving Proposition 5 which is the key argument. Recall the definitions of Section 1.6. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bounded subsets of $\mathcal{G}(\mathbb{R}^d)$, $g \in C^\infty_b(\mathbb{R}^d)$ and $\gamma \in \mathbb{N}^d$. We want to prove that

(i) $\{\pi_1 *_{g,\gamma} \pi_2 | \pi_1 \in \mathcal{B}_1, \pi_2 \in \mathcal{B}_2\}$ is a bounded subset of $\mathcal{H}_{|\gamma|}(\mathbb{R}^d)$,

(ii) $\{\pi_1 *_{g,0} \pi_2 | \pi_1 \in \mathcal{B}_1, \pi_2 \in \mathcal{B}_2\}$ is a bounded subset of $\mathcal{H}(\mathbb{R}^d)$.

The functions $\pi_1 *_{g,\gamma} \pi_2$ depend on (s,t,x,y). We shall proceed differently depending on s is small or large with respect to t. The main trick is to integrate by parts in the latter case, so that the derivatives should always rest on the regularizing part of the integral. This is analogous to V. Bally and D. Talay's use of Malliavin calculus integration by parts formula [1]. This is the reason why we partition the unit triangle $\mathcal{T}_1$ into $\mathcal{T}_1^- = \{(s,t) \in \mathcal{T}_1 | 0 < s \leq t/2\}$ and $\mathcal{T}_1^+ = \{(s,t) \in \mathcal{T}_1 | t/2 < s < t\}$, and, for $\epsilon = \pm$, we define $(\pi_1 *_{g,\gamma} \pi_2)_\epsilon(s,t,x,y) = \mathbf{1}_{\mathcal{T}_1^\epsilon}(s,t)\,(\pi_1 *_{g,\gamma} \pi_2)(s,t,x,y)$. We then have $\pi_1 *_{g,\gamma} \pi_2 = (\pi_1 *_{g,\gamma} \pi_2)_- + (\pi_1 *_{g,\gamma} \pi_2)_+$.

Before proving Proposition 5 and for the sake of clarity, let us state apart the following technical lemma, whose proof is a straightforward application of Lebesgue's dominated convergence theorem:

Lemma 15. *Let $l \in \mathbb{Z}$, $(\chi_i, i \in I)$ be a family of measurable functions mapping $\mathcal{T}_1 \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$ into $\mathbb{R}$ such that*

- *for all $i \in I$, $(s,t) \in \mathcal{T}_1$ and $\zeta \in \mathbb{R}^d$, $\chi_i(s,t,\cdot,\cdot,\zeta)$ is infinitely differentiable,*
- *for all $\alpha, \beta \in \mathbb{N}^d$, there exists two constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $i \in I$, $(s,t) \in \mathcal{T}_1$ and $x, y, \zeta \in \mathbb{R}^d$,*

$$\left|\partial^\alpha_x \partial^\beta_y \chi_i(s,t,x,y,\zeta)\right| \leq c_1 t^{-(|\alpha| + |\beta| + d + l)/2} \exp\left(-c_2 \|x-y\|^2/t - c_2 \|\zeta\|^2\right), \tag{48}$$

and let us define $\mathcal{I}(\chi_i)(s,t,x,y) = \int_{\mathbb{R}^d} \chi_i(s,t,x,y,\zeta)\, d\zeta$. Then $\{\mathcal{I}(\chi_i) | i \in I\}$ is a bounded subset of $\mathcal{H}_l(\mathbb{R}^d)$.

Proof of Proposition 5-(i). It is enough to show that both $\mathcal{B}_\epsilon \equiv \{(\pi_1 *_{g,\gamma} \pi_2)_\epsilon | \pi_1 \in \mathcal{B}_1, \pi_2 \in \mathcal{B}_2\}$ are bounded.

Step 1. Let us first treat $\mathcal{B}_-$, i.e. the case when s is small. After the change of variables $z = x + \zeta\sqrt{s}$, we get $(\pi_1 *_{g,\gamma} \pi_2)_- = \mathcal{I}(\chi^-_{\pi_1,\pi_2})$ with

$$\chi^-_{\pi_1,\pi_2}(s,t,x,y,\zeta) = \mathbf{1}_{\mathcal{T}_1^-}(s,t) s^{d/2} g(x + \zeta\sqrt{s}) \pi_1(s,x,x+\zeta\sqrt{s}) \partial^\gamma_2 \pi_2(t-s, x+\zeta\sqrt{s}, y).$$

It is enough to check that the family $\left(\chi^{-}_{\pi_1,\pi_2},(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2\right)$ satisfies the assumptions of Lemma 15 with $l=|\gamma|$. The first point is obvious. In order to check the second one, let us fix $\alpha,\beta\in\mathbb{N}^d$. According to Leibniz's formula, $\partial_x^\alpha\partial_y^\beta\chi_{\pi_1,\pi_2}(s,t,x,y,\zeta)$ can be written as a weighted sum of terms of the form

$$\chi^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta)=\mathbf{1}_{\mathcal{T}_1^-}(s,t)s^{d/2}\partial^{\alpha_1}g(x+\zeta\sqrt{s})$$
$$\partial_x^{\alpha_2}\left(\pi_1(s,x,x+\zeta\sqrt{s})\right)\partial_2^{\gamma+\alpha_3}\partial_3^\beta\pi_2(t-s,x+\zeta\sqrt{s},y),$$

with $|\alpha_1|+|\alpha_2|+|\alpha_3|=|\alpha|$, so that in order to check (48) it is enough to show that for each such $(\alpha_1,\alpha_2,\alpha_3)$ one can find $c_1\geq 0$ and $c_2>0$ such that for all $(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2$, $(s,t)\in\mathcal{T}_1$ and $x,y,\zeta\in\mathbb{R}^d$, $|\chi^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta)|$ is less than the r.h.s. of (48), with $l=|\gamma|$. Now, $\mathcal{B}_1$ and $\mathcal{B}_2$ are bounded subsets of $\mathcal{G}(\mathbb{R}^d)$ so that from (15)-(16) one can find $c_3,c_5\geq 0$ and $c_4>0$ such that for all $(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2$, $(s,t)\in\mathcal{T}_1$ and $x,y,\zeta\in\mathbb{R}^d$,

$$\left|\partial_x^{\alpha_2}\left(\pi_1(s,x,x+\zeta\sqrt{s})\right)\right|\leq c_3s^{-d/2}\exp\left(-c_4\left\|\zeta\right\|^2\right)$$

and

$$\mathbf{1}_{\mathcal{T}_1^-}(s,t)\left|\partial_2^{\gamma+\alpha_3}\partial_3^\beta\pi_2(t-s,x+\zeta\sqrt{s},y)\right|$$
$$\leq\mathbf{1}_{\mathcal{T}_1^-}(s,t)c_3(t-s)^{-(|\alpha_3|+|\beta|+|\gamma|+d)/2}\exp\left(-c_4\left\|x-y+\zeta\sqrt{s}\right\|^2/(t-s)\right)$$
$$\leq\mathbf{1}_{\mathcal{T}_1^-}(s,t)c_5t^{-(|\alpha|+|\beta|+|\gamma|+d)/2}\exp\left(-c_4\left\|x-y+\zeta\sqrt{s}\right\|^2/t\right)$$

where, for the last inequality, we have used the fact that when $(s,t)\in\mathcal{T}_1^-$, $t/2\leq t-s\leq t\leq 1$. Now, using the fact that $\left\|x-z\right\|^2\geq\left\|x\right\|^2/2-\left\|z\right\|^2$ for all $x,z\in\mathbb{R}^d$, we see that for all $(s,t)\in\mathcal{T}_1^-$, $\left\|\zeta\right\|^2+\left\|x-y+\zeta\sqrt{s}\right\|^2/t\geq(\left\|x-y\right\|^2/t+\left\|\zeta\right\|^2)/2$. Since $g\in C_b^\infty(\mathbb{R}^d)$, we can eventually find $c_1\geq 0$ and $c_2>0$ such that for all $(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2$, $(s,t)\in\mathcal{T}_1$ and $x,y,\zeta\in\mathbb{R}^d$,

$$\left|\chi^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta)\right|\leq c_1t^{-(|\alpha|+|\beta|+d+|\gamma|)/2}\exp\left(-c_2\left\|x-y\right\|^2/t-c_2\left\|\zeta\right\|^2\right),$$

which completes Step 1.

Step 2. Let us now treat $\mathcal{B}_+$, i.e. the case when s is large. After $|\gamma|$ integrations by parts, we have

$$(\pi_1*_{g,\gamma}\pi_2)_+(s,t,x,y)=\mathbf{1}_{\mathcal{T}_1^+}(s,t)\int_{\mathbb{R}^d}\partial_z^\gamma(g(z)\pi_1(s,x,z))\pi_2(t-s,z,y)\,dz.$$

Using Leibniz's formula and making the change of variables $z=y-\zeta\sqrt{t-s}$, we get that $(\pi_1*_{g,\gamma}\pi_2)_+$ is a weighted sum of terms of the form $\mathcal{I}(\chi^{+,\gamma_1,\gamma_2}_{\pi_1,\pi_2})$ with

$$\chi^{+,\gamma_1,\gamma_2}_{\pi_1,\pi_2}(s,t,x,y,\zeta)=\mathbf{1}_{\mathcal{T}_1^+}(s,t)(t-s)^{d/2}\partial^{\gamma_1}g(y-\zeta\sqrt{t-s})$$
$$\partial_3^{\gamma_2}\pi_1(s,x,y-\zeta\sqrt{t-s})\pi_2(t-s,y-\zeta\sqrt{t-s},y)$$

and $|\gamma_1|+|\gamma_2|=|\gamma|$, so that we are now in the position to apply the same arguments as in Step 1 and get that the family $(\chi^{+,\gamma_1,\gamma_2}_{\pi_1,\pi_2},(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2)$ satisfies the assumptions of Lemma 15 with $l=|\gamma|$, which completes the proof. □

Proof of Proposition 5-(ii). From (i), we know that $\{\pi_1*_{g,0}\pi_2|\pi_1\in\mathcal{B}_1,\pi_2\in\mathcal{B}_2\}$ is a bounded subset of $\mathcal{H}_0(\mathbb{R}^d)$. It remains to prove that (19) holds for $\rho=\pi_1*_{g,0}\pi_2$ with constants c_1 and c_2 which do not depend on $(\pi_1,\pi_2)\in\mathcal{B}_1\times\mathcal{B}_2$. As in the proof of

Proposition 5-(i), we treat $(\pi_1 *_{g,0} \pi_2)_-$ and $(\pi_1 *_{g,0} \pi_2)_+$ separately but analogously. That is, after integrating by parts, the term $(\pi_1 *_{g,0} \pi_2)_+$ can be treated in the same way as $(\pi_1 *_{g,0} \pi_2)_-$. Thus we shall only deal with the latter term. We have $(\pi_1 *_{g,0} \pi_2)_- = \mathcal{I}(\chi^-_{\pi_1,\pi_2})$ with

$$\chi^-_{\pi_1,\pi_2}(s,t,x,y,\zeta) = \mathbf{1}_{\mathcal{T}_1^-}(s,t) s^{d/2} g(x+\zeta\sqrt{s})\pi_1(s,x,x+\zeta\sqrt{s})\pi_2(t-s,x+\zeta\sqrt{s},y).$$

Then we write $\partial_x^\alpha \left(\chi^-_{\pi_1,\pi_2}\left(s,t,x,x+y\sqrt{t},\zeta\right)\right)$ as a weighted sum of terms of the form

$$\tilde{\chi}^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta) = \mathbf{1}_{\mathcal{T}_1^-}(s,t) s^{d/2} \partial^{\alpha_1} g(x+\zeta\sqrt{s}) \\ \partial_x^{\alpha_2}\left(\pi_1(s,x,x+\zeta\sqrt{s})\right) \partial_x^{\alpha_3}\left(\pi_2(t-s,x+\zeta\sqrt{s},x+y\sqrt{t})\right),$$

with $|\alpha_1|+|\alpha_2|+|\alpha_3| = |\alpha|$. Then we use (16) twice and the same arguments as in the preceding proof to get $c_1 \geq 0$ and $c_2 > 0$ such that for all $(\pi_1,\pi_2) \in \mathcal{B}_1 \times \mathcal{B}_2$, $(s,t) \in \mathcal{T}_1$ and $x,y,\zeta \in \mathbb{R}^d$, $|\tilde{\chi}^{-,\alpha_1,\alpha_2,\alpha_3}_{\pi_1,\pi_2}(s,t,x,y,\zeta)| \leq c_1 t^{-d/2} \exp(-c_2 \|y\|^2 - c_2 \|\zeta\|^2)$, and an obvious adaptation of Lemma 15 completes the proof. □

3.2. **Proof of Theorems 6 and 8.** In this section, we assume (B) and (C). We first want to prove Theorem 6. We recall that statement (i) is already known, see [4], theorem 7, page 260. The next lemma is statement (ii).

Lemma 16. *Under (B) and (C), for all $t \in (0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$, $X_t^{n,x}$ has a density $p_n(t,x,\cdot)$ and $(p_n, n \geq 1)$ is a bounded sequence in $\mathcal{G}(\mathbb{R}^d)$.*

Proof. It is known that for all $n \geq 1$, $k \in \{1,\dots,n\}$ and $x \in \mathbb{R}^d$, $X^{n,x}_{t_k^n}$ has a density $p_{n,k}(x,\cdot)$ such that $p_{n,k}$ is infinitely differentiable and satisfies (15)-(16) with $t = t_k^n$ and two constants c_1 and c_2 which do not depend on n and k (see the proof of theorem 1.1, page 278, in [9]). Since $\lfloor nt \rfloor / n \geq t/2$ for all $t \geq 1/n$, this shows that the sequence $(\tilde{p}_n, n \geq 1)$ defined by $\tilde{p}_n(t,x,y) = \mathbf{1}_{\{nt \geq 1\}} p_{n,\lfloor nt \rfloor}(x,y)$ is bounded in $\mathcal{G}(\mathbb{R}^d)$. If we denote by $\Gamma(t,x,\cdot)$ the density of $x + b(x)t + \sigma(x)B_t$ ($t \in (0,1]$), we observe that when $k \in \{1,\dots,n-1\}$ and $t \in (t_k^n, t_{k+1}^n)$, $X_t^{n,x}$ has the density $p_n(t,x,\cdot) = \int_{\mathbb{R}^d} p_{n,k}(x,z)\Gamma(t-t_k^n,z,\cdot)\,dz = (\tilde{p}_n *_{1,0} \Gamma)(t_k^n,t,x,\cdot)$. Hence, for all $t \in (0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$, $X_t^{n,x}$ has the density

$$p_n(t,x,\cdot) = \begin{cases} p_{n,k}(x,\cdot) & \text{if } \ t = t_k^n,\ k \in \{1,\dots,n\}, \\ \Gamma(t,x,\cdot) & \text{if } \ t \in (0,t_1^n), \\ (\tilde{p}_n *_{1,0} \Gamma)(t_k^n,t,x,\cdot) & \text{if } \ t \in (t_k^n,t_{k+1}^n),\ k \in \{1,\dots,n-1\}. \end{cases}$$

Observing that $\Gamma \in \mathcal{G}(\mathbb{R}^d)$ and applying Proposition 5-(ii), we get that $(p_n, n \geq 1)$ is a bounded sequence in $\mathcal{G}(\mathbb{R}^d)$. □

We shall now prove statement (iii) of Theorem 6. Recall (45). We want to make explicit C_t and R_t^n as integral operators on $\mathbb{R}^d$. To this end, note that, applying recursively Lebesgue's dominated convergence theorem, we have that for all $t \in (0,1]$, $f \in C^\infty_{\text{pol}}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$,

$$\partial^\alpha P_t f(x) = \int_{\mathbb{R}^d} f(y) \partial_2^\alpha p(t,x,y)\,dy. \tag{49}$$

The next lemma explicits C_t as an integral operator. The function π which appears there should be thought of as the kernel of C.

Lemma 17. *Under (B) and (C), there exists $\pi \in \mathcal{G}_1(\mathbb{R}^d)$, given by (21), such that for all $t \in (0,1]$, $f \in C^\infty_{\text{pol}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$C_t f(x) = \int_{\mathbb{R}^d} f(y)\pi(t,x,y)\,dy.$$

Proof. Using (40) for the first equality, (29) for the third one and (49) for the fourth one, we have

$$\begin{aligned}
2C_t f(x) &= \int_0^t P_s L_2^* P_{t-s} f(x)\,ds \\
&= \int_0^t \int_{\mathbb{R}^d} p(s,x,z) L_2^* P_{t-s} f(z)\,dzds \\
&= \sum_{1\le|\alpha|\le 3} \int_0^t \int_{\mathbb{R}^d} g^*_{2,\alpha}(z) p(s,x,z)\partial^\alpha P_{t-s} f(z)\,dzds \\
&= \sum_{1\le|\alpha|\le 3} \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y) g^*_{2,\alpha}(z) p(s,x,z)\partial_2^\alpha p(t-s,z,y)\,dydzds.
\end{aligned}$$

Using Fubini's theorem, we see that to complete the proof it is enough to show that the function π defined by

$$\pi(t,x,y) = \frac{1}{2}\sum_{1\le|\alpha|\le 3}\int_0^t (p *_{g^*_{2,\alpha},\alpha} p)(s,t,x,y)\,ds \tag{50}$$

belongs to $\mathcal{G}_1(\mathbb{R}^d)$. Now, $p \in \mathcal{G}(\mathbb{R}^d)$ and, from Remark 11, $g^*_{2,\alpha} \in C_b^\infty(\mathbb{R}^d)$ so that we can apply Proposition 5-(i): $p *_{g^*_{2,\alpha},\alpha} p \in \mathcal{H}_{|\alpha|}(\mathbb{R}^d)$. In particular, $\int_0^\cdot (p *_{g^*_{2,\alpha},\alpha} p)(s,\cdot,\cdot,\cdot)\,ds \in \mathcal{G}_{|\alpha|-2}(\mathbb{R}^d)$. Since $|\alpha| \le 3$ and by monotonicity of $(\mathcal{G}_l(\mathbb{R}^d), l \in \mathbb{Z})$, we finally get that $\pi \in \mathcal{G}_1(\mathbb{R}^d)$. To complete the proof, note that (50) can be rewritten as (21). □

We have a similar representation for $A^n_{1,t}$, recall (41). We say that a sequence $(\pi^n, n \ge 1)$ is $O(1/n^j)$ in $\mathcal{G}_l(\mathbb{R}^d)$ if $(n^j\pi^n, n \ge 1)$ is bounded in $\mathcal{G}_l(\mathbb{R}^d)$.

Lemma 18. *Under (B) and (C), there exists a $O(1/n^2)$ sequence $(\pi^n_1, n \ge 1)$ in $\mathcal{G}_3(\mathbb{R}^d)$ such that for all $t \in (0,1]$, $f \in C^\infty_{\text{pol}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$A^n_{1,t} f(x) = \int_{\mathbb{R}^d} f(y)\pi^n_1(t,x,y)\,dy.$$

Proof. Recall (46). From Remark 11, there is a family $(g^\#_{3,\alpha}, 1 \le |\alpha| \le 4)$ in $C_b^\infty(\mathbb{R}^d)$ such that $L_3^\# = \sum_{1\le|\alpha|\le 4} g^\#_{3,\alpha}\partial^\alpha$, so that, using (49), we have

$$\begin{aligned}
&(P_{t^n_k} L_2^* P_{t-t^n_k} - P_s L_2^* P_{t-s}) f(x) \\
&\qquad = -\sum_{1\le|\alpha|\le 4}\int_{t^n_k}^s \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y) g^\#_{3,\alpha}(z) p(u,x,z)\partial_2^\alpha p(t-u,z,y)\,dydzdu.
\end{aligned}$$

Using (47), we get $A^n_{1,t}f(x) = \int_{\mathbb{R}^d} f(y)\pi^n_1(t,x,y)\,dy$ with $\pi^n_1 = \pi^n_{1,1} + \pi^n_{1,2}$ and

$$\begin{aligned}
\pi^n_{1,1}(t,x,y) &= -\frac{1}{2n}\sum_{1\le|\alpha|\le 4}\sum_{k=0}^{\lfloor nt\rfloor-1}\int_{t^n_k}^{t^n_{k+1}}\int_{t^n_k}^{s}\left(p *_{g^{\#}_{3,\alpha},\alpha} p\right)(u,t,x,y)\,duds,\\
\pi^n_{1,2}(t,x,y) &= -\frac{1}{2n}\sum_{1\le|\alpha|\le 3}\int_{\lfloor nt\rfloor/n}^{t}(p *_{g^{*}_{2,\alpha},\alpha} p)(s,t,x,y)\,ds.
\end{aligned}$$

Now Proposition 5-(i) states that $p *_{g^{\#}_{3,\alpha},\alpha} p$ and $p *_{g^{*}_{2,\alpha},\alpha} p$ belong to $\mathcal{H}_{|\alpha|}(\mathbb{R}^d)$. Hence $(\int_{t^n_k}^{t^n_{k+1}}\int_{t^n_k}^{s}(p *_{g^{\#}_{3,\alpha},\alpha} p)(u,\cdot,\cdot,\cdot)\,duds, n\ge 1, k\in\{0,\dots,n-1\})$ is $O(1/n^2)$ in $\mathcal{G}_{|\alpha|}(\mathbb{R}^d)$ and $(\int_{\lfloor n\cdot\rfloor/n}^{\cdot}(p *_{g^{*}_{2,\alpha},\alpha} p)(s,\cdot,\cdot,\cdot)\,ds, n\ge 1)$ is $O(1/n)$ in $\mathcal{G}_{|\alpha|}(\mathbb{R}^d)$. As a consequence, $(\pi^n_{1,1}, n\ge 1)$ is $O(1/n^2)$ in $\mathcal{G}_2(\mathbb{R}^d)$ and $(\pi^n_{1,2}, n\ge 1)$ is $O(1/n^2)$ in $\mathcal{G}_3(\mathbb{R}^d)$. Eventually, $(\pi^n_1, n\ge 1)$ is $O(1/n^2)$ in $\mathcal{G}_3(\mathbb{R}^d)$. □

We shall now prove an analogous lemma for $A^n_{2,t}$.

Lemma 19. *Under (B) and (C), there exists a $O(1/n^2)$ sequence $(\pi^n_2, n\ge 1)$ in $\mathcal{G}_3(\mathbb{R}^d)$ such that for all $t\in(0,1]$, $f\in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$,*

$$A^n_{2,t}f(x) = \int_{\mathbb{R}^d} f(y)\pi^n_2(t,x,y)\,dy.$$

Proof. Since $P^n_{t^n_k}L^*_2P_{t-t^n_k} = P_{t^n_k}L^*_2P_{t-t^n_k}$ when $k=0$, (42) reads

$$\begin{aligned}
2n^2A^n_{2,t}f(x) &= \sum_{k=1}^{\lfloor nt\rfloor-1}(P^n_{t^n_k} - P_{t^n_k})L^*_2P_{t-t^n_k}f(x)\\
&= \sum_{k=1}^{\lfloor nt\rfloor-1}\int_{\mathbb{R}^d}(p_n-p)(t^n_k,x,z)L^*_2P_{t-t^n_k}f(z)\,dz\\
&= \sum_{1\le|\alpha|\le 3}\sum_{k=1}^{\lfloor nt\rfloor-1}\int_{\mathbb{R}^d}(p_n-p)(t^n_k,x,z)g^*_{2,\alpha}(z)\partial^\alpha P_{t-t^n_k}f(z)\,dz\\
&= \sum_{1\le|\alpha|\le 3}\sum_{k=1}^{\lfloor nt\rfloor-1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(p_n-p)(t^n_k,x,z)g^*_{2,\alpha}(z)f(y)\partial^\alpha_2 p(t-t^n_k,z,y)\,dydz
\end{aligned}$$

where we have used (29) for the third equality and (49) for the fourth one. From Remark 11, $g^*_{2,\alpha}\in C^\infty_b(\mathbb{R}^d)$ so that to complete the proof it is enough to show that whenever $g\in C^\infty_b(\mathbb{R}^d)$ and $\alpha\in\mathbb{N}^d$, the sequence $(\pi^n, n\ge 1)$ defined by

$$\pi^n(t,x,y) = \sum_{k=1}^{\lfloor nt\rfloor-1}\int_{\mathbb{R}^d}(p_n-p)(t^n_k,x,z)g(z)\partial^\alpha_2 p(t-t^n_k,z,y)dz = \sum_{k=1}^{\lfloor nt\rfloor-1}((p_n-p)*_{g,\alpha}p)(t^n_k,t,x,y)$$

is bounded in $\mathcal{G}_{|\alpha|}(\mathbb{R}^d)$. And to do so, it is enough to show that the sequence $(\rho^n_{t^n_k}, n\ge 2, k\in\{1,\dots,n-1\})$ defined by

$$\rho^n_{t^n_k}(t,x,y) = \mathbf{1}_{\mathcal{T}_1}(t^n_k,t)\left((p_n-p)*_{g,\alpha}p\right)(t^n_k,t,x,y)$$

is $O(1/n)$ in $\mathcal{G}_{|\alpha|+2}(\mathbb{R}^d)$. Let us write $\rho^{n,-}_{t^n_k}(t,x,y) = \mathbf{1}_{\mathcal{T}_1^-}(t^n_k,t)\rho^n_{t^n_k}(t,x,y)$ and $\rho^{n,+}_{t^n_k}(t,x,y) = \mathbf{1}_{\mathcal{T}_1^+}(t^n_k,t)\rho^n_{t^n_k}(t,x,y)$ so that $\rho^n_{t^n_k} = \rho^{n,-}_{t^n_k} + \rho^{n,+}_{t^n_k}$.

Let us first prove that $(\rho^{n,-}_{t^n_k}, n \geq 2, k \in \{1,\dots,n-1\})$ is $O(1/n)$ in $\mathcal{G}_{|\alpha|+2}(\mathbb{R}^d)$. The sequence $(\pi_{t^n_k}, n \geq 2, k \in \{1,\dots,n-1\})$ defined by $\pi_{t^n_k}(t,x,y) = \mathbf{1}_{\mathcal{T}_1^-}(t^n_k,t)g(x)\partial_2^\alpha p(t-t^n_k,x,y)$ is bounded in $\mathcal{G}_{|\alpha|}(\mathbb{R}^d)$, since $t - t^n_k \geq t/2$ when $(t^n_k,t) \in \mathcal{T}_1^-$. Now note that $\rho^{n,-}_{t^n_k} = P^n_{t^n_k}\pi_{t^n_k} - P_{t^n_k}\pi_{t^n_k} = \Delta^n_{t^n_k}\pi_{t^n_k}$ (see (51) in the appendix for the definition of $P^n_s\pi$, $P_s\pi$ and $\Delta^n_s\pi$ when $\pi \in \mathcal{G}_l(\mathbb{R}^d)$). Thus, from (38)-(39) and (37) applied with $j=1$,

$$\rho^{n,-}_{t^n_k} = \sum_{m=0}^{k-1} \int_0^{1/n} \int_0^{s_1} P^n_{t^n_m} \Phi^{n,2}_{s_2,1/n} P_{t^n_k - t^n_{m+1}} \pi_{t^n_k}\, ds_2 ds_1.$$

Proposition 24 in the appendix states that the family $(P^n_{t^n_m}\Phi^{n,2}_{s,1/n}P_{t^n_k-t^n_{m+1}}\pi_{t^n_k}, n \geq 2, k \in \{1,\dots,n-1\}, m \in \{1,\dots,k-1\}, s \in [0,1/n])$ is bounded in $\mathcal{G}_{|\alpha|+4}(\mathbb{R}^d)$. Since $k \leq \lfloor nt \rfloor$ when $(t^n_k,t) \in \mathcal{T}_1$, this implies that $(\rho^{n,-}_{t^n_k}, n \geq 2, k \in \{1,\dots,n-1\})$ is $O(1/n)$ in $\mathcal{G}_{|\alpha|+2}(\mathbb{R}^d)$.

Let us now prove the same for $\rho^{n,+}$. After $|\alpha|$ integrations by parts and after setting $z = y - \zeta\sqrt{t-s}$, we get that $((p_n - p) *_{g,\alpha} p)_+$ is a weighted sum of terms of the form $\mathcal{I}(\chi^{n,+}_{\alpha_1,\alpha_2})$ - see Lemma 15 - with

$$\chi^{n,+}_{\alpha_1,\alpha_2}(s,t,x,y,\zeta) = \mathbf{1}_{\mathcal{T}_1^+}(s,t)(t-s)^{d/2}\partial^{\alpha_1} g(y-\zeta\sqrt{t-s}) \\ \partial_3^{\alpha_2}(p_n - p)(s,x,y-\zeta\sqrt{t-s})p(t-s,y-\zeta\sqrt{t-s},y)$$

and $|\alpha_1| + |\alpha_2| = |\alpha|$. Now, from Corollary 22 in the appendix, $(p_n - p, n \geq 1)$ is $O(1/n)$ in $\mathcal{G}_2(\mathbb{R}^d)$ so that, using the same arguments as in Step 2 of the proof of Proposition 5-(i), we get that $((p_n - p) *_{g,\alpha} p)_+$ is $O(1/n)$ in $\mathcal{H}_{|\alpha|+2}(\mathbb{R}^d)$. Since $\rho^{n,+}_{t^n_k}(t,x,y) = \mathbf{1}_{\mathcal{T}_1^+}(t^n_k,t)((p_n - p) *_{g,\alpha} p)_+(t^n_k,t,x,y)$, we conclude that $(\rho^{n,+}_{t^n_k}, n \geq 2, k \in \{1,\dots,n-1\})$ is $O(1/n)$ in $\mathcal{G}_{|\alpha|+2}(\mathbb{R}^d)$. □

Lastly, starting from (43), Lemmas 21 and 23 with $j=2$ imply

Lemma 20. *Under (B) and (C), there exists a $O(1/n^2)$ sequence $(\pi^n_3, n \geq 1)$ in $\mathcal{G}_4(\mathbb{R}^d)$ such that for all $t \in (0,1]$, $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$A^n_{3,t}f(x) = \int_{\mathbb{R}^d} f(y)\pi^n_3(t,x,y)\,dy.$$

Statement (iii) of Theorem 6 is now proved: it follows from (45), (44) and Lemmas 17, 18, 19 and 20.

We now also have all the tools to prove Theorem 8. Indeed, note that (38) combined with Lemmas 21 and 23 imply that we have an expansion of arbitrary order j for $p_n - p$:

$$(p_n - p)(t,\cdot,\cdot) = \sum_{i=2}^{j} \frac{1}{i!n^i} \sum_{k=0}^{\lfloor nt \rfloor -1} \psi^{n,i}_{t^n_k}(t,\cdot,\cdot) + r^{n,j}(t,\cdot,\cdot) + \sum_{i=2}^{j} \frac{(t - \lfloor nt \rfloor/n)^i}{i!} \psi^{n,i}_{\lfloor nt \rfloor/n}(t,\cdot,\cdot).$$

Since $(r^{n,j}, n \geq 1)$ is $O(1/n^j)$ in $\mathcal{G}_{2j}(\mathbb{R}^d)$ and $(\psi^{n,j}_{t^n_k}, n \geq 1, k \in \{0,\dots,n\})$ is bounded in $\mathcal{G}_{2j}(\mathbb{R}^d)$, this gives (23) with $(\pi_{n,i}, n \geq 1)$ bounded in $\mathcal{G}_{2i-2}(\mathbb{R}^d)$ and $(\pi'_{n,i}, n \geq 1)$ and $(\pi''_{n,i}, n \geq 1)$ bounded in $\mathcal{G}_{2i}(\mathbb{R}^d)$.

4. Appendix

4.1. **Kernel of $R^{n,j}$.** Here we make explicit the kernel of the remainder $R_t^{n,j}$, recall (39):

Lemma 21. *Under (B) and (C), for each $j \in \mathbb{N}^*$, there exists a $O(1/n^j)$ sequence $(r^{n,j}, n \geq 1)$ in $\mathcal{G}_{2j}(\mathbb{R}^d)$ such that for all $t \in (0,1]$, $n \geq 1$, $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$R_t^{n,j} f(x) = \int_{\mathbb{R}^d} f(y) r^{n,j}(t,x,y)\, dy.$$

Proof. From (39) and (37), $R_t^{n,j} = R_{1,t}^{n,j} + R_{2,t}^{n,j}$ where

$$\begin{aligned} R_{1,t}^{n,j} &= \sum_{k=0}^{\lfloor nt \rfloor -1} \int_0^{1/n} \int_0^{s_1} \cdots \int_0^{s_j} P^n_{t_k^n} \Phi^{n,j+1}_{s_{j+1},1/n} P_{t-t^n_{k+1}}\, ds_{j+1} \cdots ds_2 ds_1, \\ R_{2,t}^{n,j} &= \int_0^{t-\lfloor nt \rfloor/n} \int_0^{s_1} \cdots \int_0^{s_j} P^n_{\lfloor nt \rfloor/n} \Phi^{n,j+1}_{s_{j+1},t-\lfloor nt \rfloor/n}\, ds_{j+1} \cdots ds_2 ds_1. \end{aligned}$$

Let us first deal with $R_{1,t}^{n,j}$. Using the fact that $k \geq 1$ for the first equality, (32) for the second one, the fact that $P_{1/n-s} P_{t-t^n_{k+1}} = P_{t-t^n_k-s}$ for the third one, and (49) and Fubini's theorem for the last one, we have for all $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $t \in [0,1]$, $n \geq 1$, $k \in \{1, \ldots, \lfloor nt \rfloor - 1\}$ and $s \in (0, 1/n)$,

$$\begin{aligned} & P^n_{t_k^n} \Phi^{n,j}_{s,1/n} P_{t-t^n_{k+1}} f(x) \\ &= \int_{\mathbb{R}^d} p_n(t_k^n, x, z_1) \Phi^{n,j}_{s,1/n} P_{t-t^n_{k+1}} f(z_1)\, dz_1 \\ &= \sum_{1 \leq |\alpha| \leq 2j} \sum_{l=1}^{m_{j,\alpha}} \int_{\mathbb{R}^d} p_n(t_k^n, x, z_1) g_{j,\alpha,l}(z_1) P^n_s (h_{j,\alpha,l} \partial^\alpha P_{1/n-s}) P_{t-t^n_{k+1}} f(z_1)\, dz_1 \\ &= \sum_{1 \leq |\alpha| \leq 2j} \sum_{l=1}^{m_{j,\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_n(t_k^n, x, z_1) g_{j,\alpha,l}(z_1) p_n(s, z_1, z_2) h_{j,\alpha,l}(z_2) \partial^\alpha P_{t-t^n_k-s} f(z_2)\, dz_2 dz_1 \\ &= \int_{\mathbb{R}^d} f(y) \varphi^{n,j}_{t_k^n}(s,t,x,y)\, dy \end{aligned}$$

where $\varphi^{n,j}_{t_k^n} = \sum_{1 \leq |\alpha| \leq 2j} \sum_{l=1}^{m_{j,\alpha}} \varphi^{n,j}_{t_k^n,\alpha,l}$ with

$$\begin{aligned} \varphi^{n,j}_{t_k^n,\alpha,l}(s,t,x,y) = \mathbf{1}_{(0,\frac{1}{n})}(s) \mathbf{1}_{[t^n_{k+1},1]}(t) & \\ \int_{\mathbb{R}^d} (p_n *_{g_{j,\alpha,l},0} p_n)(t_k^n, t_k^n + s, x, z_2) h_{j,\alpha,l}(z_2) \partial_2^\alpha p(t - t_k^n - s, z_2, y)\, dz_2. & \end{aligned}$$

Now, setting $q^{n,j}_{t_k^n,\alpha,l}(u,x,z) = \mathbf{1}_{(t_k^n,1]}(u)(p_n *_{g_{j,\alpha,l},0} p_n)(t_k^n, u, x, z)$, it follows from Proposition 5-(ii) that $(q^{n,j}_{t_k^n,\alpha,l}, n \geq 1, k \in \{1, \ldots, n\})$ is a bounded sequence in $\mathcal{G}(\mathbb{R}^d)$. Since $\varphi^{n,j}_{t_k^n,\alpha,l}(s,t,x,y) = \mathbf{1}_{(0,\frac{1}{n})}(s) \mathbf{1}_{[t^n_{k+1},1]}(t) (q^{n,j}_{t_k^n,\alpha,l} *_{h_{j,\alpha,l},\alpha} p)(t_k^n + s, t, x, y)$, Proposition 5-(i) shows that $(\varphi^{n,j}_{t_k^n,\alpha,l}, n \geq 1, k \in \{1, \ldots, n\})$ is bounded in $\mathcal{H}_{|\alpha|}(\mathbb{R}^d)$, so that $(\varphi^{n,j}_{t_k^n}, n \geq 1, k \in \{1, \ldots, n\})$ is bounded in $\mathcal{H}_{2j}(\mathbb{R}^d)$.

When $k = 0$, we have in the same way for all $f \in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$

$$\Phi^{n,j}_{s,1/n} P_{t-1/n} f(x) = \int_{\mathbb{R}^d} f(y) \varphi_0^{n,j}(s,t,x,y)\, dy$$

where $\varphi_0^{n,j} = \sum_{1\le|\alpha|\le 2j}\sum_{l=1}^{m_{j,\alpha}} \varphi_{0,\alpha,l}^{n,j}$ with

$$\varphi_{0,\alpha,l}^{n,j}(s,t,x,y) = \mathbf{1}_{(0,\frac{1}{n})}(s)\mathbf{1}_{[\frac{1}{n},1]}(t) g_{j,\alpha,l}(x)(p_n *_{h_{j,\alpha,l},\alpha} p)(s,t,x,y).$$

Again Proposition 5-(i) imply that $(\varphi_0^{n,j}, n\ge 1)$ is bounded in $\mathcal{H}_{2j}(\mathbb{R}^d)$.

Eventually, for all $f\in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$, we have $R_{1,t}^{n,j}f(x) = \int_{\mathbb{R}^d} f(y) r_1^{n,j}(t,x,y)\,dy$ with

$$r_1^{n,j}(t,x,y) = \sum_{k=0}^{\lfloor nt\rfloor -1}\int_0^{1/n}\int_0^{s_1}\cdots\int_0^{s_j}\varphi_{t_k^n}^{n,j+1}(s_{j+1},t,x,y)\,ds_{j+1}\cdots ds_2 ds_1,$$

and since the family $(\varphi_{t_k^n}^{n,j+1}, n\ge 1, k\in\{0,\dots,n\})$ is bounded in $\mathcal{H}_{2j+2}(\mathbb{R}^d)$, the sequence $(r_1^{n,j}, n\ge 1)$ is $O(1/n^j)$ in $\mathcal{G}_{2j}(\mathbb{R}^d)$.

As for $R_{2,t}^{n,j}$, similar arguments lead to

$$P^n_{\lfloor nt\rfloor/n}\Phi^{n,j}_{s,t-\lfloor nt\rfloor/n} f(x) = \int_{\mathbb{R}^d} f(y)\phi^{n,j}(s,t,x,y)\,dy$$

where $\phi^{n,j} = \sum_{1\le|\alpha|\le 2j}\sum_{l=1}^{m_{j,\alpha}} \phi_{\alpha,l}^{n,j}$ with

$$\phi_{\alpha,l}^{n,j}(s,t,x,y) = \mathbf{1}_{[\frac{1}{n},1]}(t)\mathbf{1}_{\left(0,t-\frac{\lfloor nt\rfloor}{n}\right)}(s)\int_{\mathbb{R}^d}(p_n *_{g_{j,\alpha,l},0} p_n)\left(\frac{\lfloor nt\rfloor}{n},\frac{\lfloor nt\rfloor}{n}+s,x,z_2\right)$$
$$h_{j,\alpha,l}(z_2)\partial_2^\alpha p\left(t-\frac{\lfloor nt\rfloor}{n}-s,z_2,y\right)\,dz_2 + \mathbf{1}_{\{0<s<t<\frac{1}{n}\}} g_{j,\alpha,l}(x)\left(p_n *_{h_{j,\alpha,l},\alpha} p\right)(s,t,x,y).$$

We can treat $\phi_{\alpha,l}^{n,j}$ exactly as we have treated $\varphi_{t_k^n,\alpha,l}^{n,j}$, and get that $(\phi^{n,j}, n\ge 1)$ is bounded in $\mathcal{H}_{2j}(\mathbb{R}^d)$, so that $R_{2,t}^{n,j}$ has a kernel $(r_2^{n,j}, n\ge 1)$ defined by

$$r_2^{n,j}(t,x,y) = \int_0^{t-\lfloor nt\rfloor/n}\int_0^{s_1}\cdots\int_0^{s_j}\phi^{n,j+1}(s_{j+1},t,x,y)\,ds_{j+1}\cdots ds_2 ds_1$$

which is $O(1/n^j)$ in $\mathcal{G}_{2j}(\mathbb{R}^d)$.

Eventually, putting $r^{n,j} = r_1^{n,j} + r_2^{n,j}$ completes the proof. □

In particular we have

Corollary 22. *Under (B) and (C),* $(p_n - p, n\ge 1)$ *is* $O(1/n)$ *in* $\mathcal{G}_2(\mathbb{R}^d)$.

Proof. From (38) applied with $j=1$ and Lemma 21, we have for all $f\in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} f(y)(p_n-p)(t,x,y)\,dy = \Delta_t^n f(x) = R_t^{n,1}f(x) = \int_{\mathbb{R}^d} f(y) r^{n,1}(t,x,y)\,dy$$

so that $p_n - p = r^{n,1}$, and Lemma 21 gives the result. □

Eventually, we have kernels for the operators $P^n_{t_k^n} L_j^* P_{t-t_k^n}$:

Lemma 23. *Under (B) and (C), for each* $j\in\mathbb{N}^*$*, there exists a bounded sequence* $(\psi_{t_k^n}^{n,j}, n\ge 1, k\in\{0,\dots,n\})$ *in* $\mathcal{G}_{2j}(\mathbb{R}^d)$ *such that for all* $t\in(0,1]$*,* $n\ge 1$*,* $k\in\{0,\dots,\lfloor nt\rfloor\}$*,* $f\in C^\infty_{\mathrm{pol}}(\mathbb{R}^d)$ *and* $x\in\mathbb{R}^d$*,*

$$P^n_{t_k^n} L_j^* P_{t-t_k^n} f(x) = \int_{\mathbb{R}^d} f(y)\psi_{t_k^n}^{n,j}(t,x,y)\,dy.$$

The proof is omitted since it copies the arguments of the proof of Lemma 21 - it is even a bit simpler.

4.2. **Operators on $\mathcal{G}_l(\mathbb{R}^d)$.** When $\pi \in \mathcal{G}_l(\mathbb{R}^d)$, $\pi(t,\cdot,y) \in L^\infty(\mathbb{R}^d)$ so that for $s \in [0,1]$ and $n \geq 1$ we can define two functions $P_s\pi$ and $P_s^n\pi$ on $(0,1] \times \mathbb{R}^d \times \mathbb{R}^d$ by $P_s\pi(t,\cdot,y) = \mathbf{1}_{\{s\leq t\}} P_s(\pi(t,\cdot,y))$ and $P_s^n\pi(t,\cdot,y) = \mathbf{1}_{\{s\leq t\}} P_s^n(\pi(t,\cdot,y))$, i.e.

$$P_s\pi(t,x,y) = \mathbf{1}_{\{s\leq t\}} \mathbb{E}\left[\pi\left(t, X_s^x, y\right)\right] \quad \text{and} \quad P_s^n\pi(t,x,y) = \mathbf{1}_{\{s\leq t\}} \mathbb{E}\left[\pi\left(t, X_s^{n,x}, y\right)\right]. \tag{51}$$

We also write $\Delta_s^n\pi = P_s^n\pi - P_s\pi$. For $j \in \mathbb{N}^*$ we denote by Φ^j the family $(\Phi_{s,1/n}^{n,j}, n \geq 1, s \in [0,1/n])$ of operators on $\mathcal{G}_l(\mathbb{R}^d)$ defined as in (31) by

$$\Phi_{s,1/n}^{n,j}\pi(t,x,y) = \mathbb{E}\left[L_j^x P_{1/n-s}\pi\left(t, X_s^{n,x}, y\right)\right],$$

i.e., using (30),

$$\Phi_{s,1/n}^{n,j} = \sum_{1\leq|\alpha|\leq 2j} \sum_{l=1}^{m_{j,\alpha}} g_{j,\alpha,l} P_s^n\left(h_{j,\alpha,l}\partial^\alpha P_{1/n-s}\right). \tag{52}$$

Denoting by $\mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d), \mathcal{G}_{l'}(\mathbb{R}^d))$ the space of all morphisms mapping any bounded subset of $\mathcal{G}_l(\mathbb{R}^d)$ into a bounded subset of $\mathcal{G}_{l'}(\mathbb{R}^d)$, we then have

Proposition 24. *Under (B) and (C), $(P_s, s \in [0,1])$ and $(P_s^n, s \in [0,1], n \geq 1)$ are bounded families in $\mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d))$, and Φ^j is a bounded family in $\mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d), \mathcal{G}_{l+2j}(\mathbb{R}^d))$.*

Proof. Let us first deal with (P_s). Let $\pi \in \mathcal{G}_l(\mathbb{R}^d)$. P_s is measurable. Moreover, Lebesgue's dominated convergence theorem shows that $P_s\pi(t,x,\cdot)$ is infinitely differentiable and that for all $\beta \in \mathbb{N}^d$

$$\partial_y^\beta P_s\pi(t,x,y) = \mathbf{1}_{\{s\leq t\}} \mathbb{E}\left[\partial_3^\beta \pi\left(t, X_s^x, y\right)\right].$$

Hypothesis (A) ensures that a version of X^x can be chosen such that for each $t \geq 0$, the map $x \mapsto X_t^x$ is infinitely differentiable (see, for example, [10]). Since $\partial_3^\beta\pi(t,\cdot,y) \in C_{\text{pol}}^\infty(\mathbb{R}^d)$, it follows from Theorem 3.14 page 16 in [11] that $\partial_y^\beta P_s\pi(t,\cdot,y)$ is infinitely differentiable and that for all $\alpha \in \mathbb{N}^d$ there exists universal polynomials $(\Pi_{\alpha,\mu}, |\mu| \leq |\alpha|)$ such that

$$\partial_x^\alpha \partial_y^\beta P_s\pi(t,x,y) = \mathbf{1}_{\{s\leq t\}} \sum_{|\mu|\leq|\alpha|} \mathbb{E}\left[\partial_2^\mu \partial_3^\beta \pi\left(t, X_s^x, y\right) \Pi_{\alpha,\mu}\left(\partial_x^\nu X_s^x, |\nu| \leq |\alpha|\right)\right] \tag{53}$$

with

$$\sup_{s\in[0,1],x\in\mathbb{R}^d} \mathbb{E}[\Pi_{\alpha,\mu}\left(\partial_x^\nu X_s^x, |\nu| \leq |\alpha|\right)^2] < \infty \tag{54}$$

for all $|\mu| \leq |\alpha|$. As a consequence, $P_s\pi(t,\cdot,\cdot)$ is infinitely differentiable and using Cauchy-Schwarz's inequality, (14) and (54), we see that for all bounded $\mathcal{B} \subset \mathcal{G}_l(R^d)$ and $\alpha, \beta \in \mathbb{N}^d$, there exists two constants $c_1 \geq 0$ and $c_2 > 0$ such that for all $\pi \in \mathcal{B}$, $s \in [0,1]$, $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\left|\partial_x^\alpha \partial_y^\beta P_s\pi(t,x,y)\right| \leq c_1 \mathbf{1}_{\{s\leq t\}} t^{-(|\alpha|+|\beta|+d+l)/2} \mathbb{E}\left[\exp\left(-c_2 \left\|X_x^s - y\right\|^2/t\right)\right]^{1/2}. \tag{55}$$

Now, partitioning Ω into $\{\|X_x^s - y\| \leq \|x-y\|/2\}$ and $\{\|X_x^s - y\| > \|x-y\|/2\}$, we have

$$\mathbb{E}\left[\exp\left(-c_2 \left\|X_x^s - y\right\|^2/t\right)\right] \leq \mathbb{P}\left(\|X_x^s - y\| \leq \|x-y\|/2\right) + \exp\left(-c_2 \|x-y\|^2/4t\right). \tag{56}$$

Using (16) for $p \in \mathcal{G}(\mathbb{R}^d)$ for the fourth inequality, we can find $c_3, c_5 \geq 0$ and $c_4, c_6 > 0$ such that for all $s \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbb{P}\left(\|X_s^x - y\| \leq \|x-y\|/2\right) &\leq \mathbb{P}\left(\|X_s^x - x\| \geq \|x-y\|/2\right) \\
&= \int_{\mathbb{R}^d} \mathbf{1}_{\{\|z-x\| \geq \|x-y\|/2\}} p(s,x,z)\, dz \\
&= \int_{\mathbb{R}^d} \mathbf{1}_{\{\|\xi\| \geq \|x-y\|/2\sqrt{s}\}} p(s,x,x+\xi\sqrt{s}) s^{d/2}\, d\xi \\
&\leq c_3 \int_{\mathbb{R}^d} \mathbf{1}_{\{\|\xi\| \geq \|x-y\|/2\sqrt{s}\}} \exp(-c_4 \|\xi\|^2)\, d\xi \\
&\leq c_5 \exp\left(-c_6 \|x-y\|^2/s\right). \qquad (57)
\end{aligned}$$

Eventually, from (56) and (57), we can find $c_7 \geq 0$ and $c_8 > 0$ such that for all $s \in [0,1]$, $t \in (0,1]$ and $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
\mathbf{1}_{\{s \leq t\}} \mathbb{E}\left[\exp\left(-c_2 \|X_x^s - y\|^2/t\right)\right] &\leq c_5 \exp\left(-c_6 \|x-y\|^2/t\right) + \exp\left(-c_2 \|x-y\|^2/4t\right) \\
&\leq c_7 \exp\left(-c_8 \|x-y\|^2/t\right). \qquad (58)
\end{aligned}$$

It is enough to inject (58) into (55) to complete the proof for (P_s).

This proof naturally extends to the case of (P_s^n). Indeed, (53) holds with (X^n, P^n) instead of (X, P). Moreover, from Lemma 26, (54) holds uniformly in n with X^n instead of X. Eventually, (57) holds with X^n instead of X, uniformly in n because $(p_n, n \geq 1)$ is bounded in $\mathcal{G}(\mathbb{R}^d)$.

As for Φ^j, it is enough to use (52), the boundedness of (P_s) and (P_s^n), Remark 11 and the facts that multiplication by a function in $\mathcal{B}$ belongs to $\mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d), \mathcal{G}_l(\mathbb{R}^d))$ and that $\partial_2^\alpha \in \mathcal{L}_b(\mathcal{G}_l(\mathbb{R}^d), \mathcal{G}_{l+|\alpha|}(\mathbb{R}^d))$. ☐

4.3. **Moments for the Euler scheme and its derivatives.** Let us assume (A). Then it is known that $X_t^{n,x}$ has bounded moments of any order and that for all $q \in \mathbb{N}$, one can find $c \geq 0$ such that for all $x \in \mathbb{R}^d$,

$$\sup_{t \in [0,1], n \geq 1} \mathbb{E}\left[\|X_t^{n,x}\|^q\right] \leq c\,(1 + \|x\|^q) \qquad (59)$$

(see [15]). From (4), $x \mapsto X_t^{n,x}$ is infinitely differentiable and we shall see that analogous upper bounds hold for its derivatives. Following [11], for $m \geq 1$, we denote by $X_t^{(m),n,x}$ the m-th derivative of $x \mapsto X_t^{n,x}$ at point x. It should be thought of as a $d \times d^m$ matrix. For instance, $X_t^{(1),n,x}$ is the jacobian matrix of $x \mapsto X_t^{n,x}$. Differentiating (4), we have

$$X_t^{(1),n,x} = I + \int_0^t b^{(1)}(X_{\lfloor ns \rfloor/n}^{n,x}) X_{\lfloor ns \rfloor/n}^{(1),n,x}\, ds + \sum_{j=1}^r \int_0^t \sigma_j^{(1)}(X_{\lfloor ns \rfloor/n}^{n,x}) X_{\lfloor ns \rfloor/n}^{(1),n,x}\, dB_s^j, \qquad (60)$$

where I stands for the identity matrix and σ_j is the j-th column of σ. Besides, by induction, there are for each $m \geq 2$ universal polynomials $P_{m,j}$, $j \in \{0, \dots, r\}$, such that

$$X_t^{(m),n,x} = \int_0^t b^{(1)}(X^{n,x}_{\lfloor ns \rfloor/n}) X^{(m),n,x}_{\lfloor ns \rfloor/n}\, ds + \sum_{j=1}^r \int_0^t \sigma_j^{(1)}(X^{n,x}_{\lfloor ns \rfloor/n}) X^{(m),n,x}_{\lfloor ns \rfloor/n}\, dB_s^j + \int_0^t Q^{n,x}_{m,0,\lfloor ns \rfloor/n}\, ds + \sum_{j=1}^r \int_0^t Q^{n,x}_{m,j,\lfloor ns \rfloor/n}\, dB_s^j, \tag{61}$$

where

$$\begin{cases} Q^{n,x}_{m,0,t} &= P_{m,0}(b^{(2)}(X_t^{n,x}), \dots, b^{(m)}(X_t^{n,x}), X_t^{(1),n,x}, \dots, X_t^{(m-1),n,x}), \\ Q^{n,x}_{m,j,t} &= P_{m,j}(\sigma_j^{(2)}(X_t^{n,x}), \dots, \sigma_j^{(m)}(X_t^{n,x}), X_t^{(1),n,x}, \dots, X_t^{(m-1),n,x}). \end{cases} \tag{62}$$

This is analogous to (1.8) page 4 in [11]. Then we have

Lemma 25. *Under (A), for all $m \geq 1$ and $q \in \mathbb{N}$, there exists $c \geq 0$ and $q' \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$,*

$$\sup_{t\in[0,1],n\geq 1} \mathbb{E}\left[\left\|X_t^{(m),n,x}\right\|^q\right] \leq c\left(1 + \|x\|^{q'}\right). \tag{63}$$

Proof. We give a proof by induction on m. Let us first assume that $m = 1$. Let $q \in \mathbb{N}$. From (60), and observing that (A) states that $b^{(1)}$ and all the $\sigma_j^{(1)}$ are bounded, Jensen's and Burkholder-Davis-Gundy's inequalities lead to the existence of $c \geq 0$ such that for all $t \in [0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}\left[\left\|X_t^{(1),n,x}\right\|^q\right] \leq c\left(1 + \int_0^t \mathbb{E}\left[\left\|X^{(1),n,x}_{\lfloor ns \rfloor/n}\right\|^q\right] ds\right).$$

Taking this inequality at time $\lfloor nt \rfloor/n$ and applying Gronwall's lemma, we get that

$$\sup_{t\in[0,1],n\geq 1,x\in\mathbb{R}^d} \mathbb{E}\left[\left\|X^{(1),n,x}_{\lfloor nt \rfloor/n}\right\|^q\right] < \infty.$$

From (4), one easily checks that the same holds at time t instead of $\lfloor nt \rfloor/n$, so that (63) holds for $m = 1$ with $q' = 0$.

Let us now assume that (63) holds for the $m-1$ first derivatives. Let $q \in \mathbb{N}$. From (61), and observing again that (A) states that $b^{(1)}$ and all the $\sigma_j^{(1)}$ are bounded, Jensen's and Burkholder-Davis-Gundy's inequalities lead to the existence of $c_1 \geq 0$ such that for all $t \in [0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$,

$$\mathbb{E}\left[\left\|X_t^{(m),n,x}\right\|^q\right] \leq c_1\left(\int_0^t \mathbb{E}\left[\left\|X^{(m),n,x}_{\lfloor ns \rfloor/n}\right\|^q\right] ds + \int_0^t \sum_{j=0}^r \mathbb{E}\left[\left\|Q^{n,x}_{m,j,\lfloor ns \rfloor/n}\right\|^q\right] ds\right). \tag{64}$$

Using (62), the induction hypothesis, (A) and (59), we find $c_2 \geq 0$ and $q' \in \mathbb{N}$ such that for all $s \in [0,1]$, $n \geq 1$ and $x \in \mathbb{R}^d$,

$$\sum_{j=0}^r \mathbb{E}\left[\left\|Q^{n,x}_{m,j,\lfloor ns \rfloor/n}\right\|^q\right] \leq c_2\left(1 + \|x\|^{q'}\right).$$

Thus, taking (64) at time $\lfloor nt \rfloor/n$ and applying Gronwall's lemma, we find $c \geq 0$ such that for all $x \in \mathbb{R}^d$,

$$\sup_{t\in[0,1],n\geq 1} \mathbb{E}\left[\left\|X^{(m),n,x}_{\lfloor nt \rfloor/n}\right\|^q\right] \leq c\left(1 + \|x\|^{q'}\right).$$

From (4), one easily checks that the same holds at time t instead of $\lfloor nt \rfloor / n$, which completes the proof. □

Observe that, under (B), the above proof holds with $q' = 0$ so that we have

Lemma 26. *Under (B), for all $m \geq 1$ and $q \in \mathbb{N}$,*

$$\sup_{t\in[0,1],n\geq 1,x\in\mathbb{R}^d} \mathbb{E}\left[\left\|X_t^{(m),n,x}\right\|^q\right] < \infty.$$

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