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New algorithm for solving variational problems in $W^{1,p}(\Omega)$ and $BV(\Omega)$: Application to image restoration

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Abstract: We propose a new unifying method for solving variational problems defined on the Sobolev spaces $W^{1,p}(\Omega)$ or on the space of functions of bounded variations $BV(\Omega)$ ($\Omega \subset R^N$). The method is based on a recent new characterization of these spaces by Bourgain, Brezis and Mironescu (2001), where norms can be approximated by a sequence of integral operators involving a differential quotient and a suitable sequence of radial mollifiers. We use this characterization to define a variational formulation, for which existence, uniqueness and convergence of the solution is proved. The proposed approximation is valid for any p and does not depend on the attach term. Implementation details are given and we show examples on the image restoration problem.

Key-words: Calculus of variation, functional analysis, Sobolev spaces, BV, variational approach

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Nouvel algorithme pour résoudre les problèmes variationels dans $W^{1,p}(\Omega)$ et $BV(\Omega)$: Applications en traitement d'image

Résumé : Nous proposons une nouvelle approche pour la résolution des problèmes variationnels définis sur les espaces de Sobolev $W^{1,p}(\Omega)$ ou sur l'espace des fonctions à variations bornées $BV(\Omega)$ ($\Omega \subset R^N$). La méthode est basée sur une caractérisation récente des ces espaces par Bourgain, Brezis et Mironescu (2001), où les normes peuvent être approchées par une suite d'opérateurs intégraux impliquant un quotient différentiel et une suite adaptée de noyaux radiaux. Nous utilisons cette caractérisation pour définir une formulation variationnelle, pour laquelle l'existence, l'unicité et la convergence de la solution sont démontrées. L'approximation proposée est valide pour tout p et ne dépend pas du terme d'attache aux données. Les détails d'implémentation sont donnés ainsi que des résultats sur l'exemple de la restauration d'images.

Mots-clés : Calcul des variations, analyse fonctionnelle, espaces de Sobolev, BV, approche variationnelle

1 Introduction

The goal of this work is to propose a new unifying method for solving variational problems defined on the Sobolev spaces $W^{1,p}(\Omega)$ or on the space of functions of bounded variations $BV(\Omega)$ of the form

$$\inf_{u \in W^{1,p}(\Omega)} F(u), \tag{1}$$

with

$$F(u) = \int_\Omega |\nabla u(x)|^p dx + \int_\Omega h(x, u(x))dx.$$

The method is based on a recent new characterization of these spaces by [5]. In [5] the authors showed that the Sobolev semi-norm of a function f can be approximated by a sequence of integral operators involving a differential quotient of f and a suitable sequence of radial mollifiers:

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(x-y)dxdy = K_{N,p} \int_\Omega |\nabla u|^p dx,$$

Here we show how this characterization can be used to approximate variational formulation (1) by defining the sequence of functionals

$$F_n(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p} \rho_n(x-y)dxdy + \int_\Omega h(x, u(x))\ dx.$$

We prove that the sequence of minimizers of F_n converges to the solution of the original variational formulation.

Our approximation is valid for any $p \geq 1$, so that the BV case is also covered by our method (thanks to results by Ponce [18]). Thus we propose an alternative to approximate variational problems defined on $BV(\Omega)$ which is not constrained by the fidelity attach term (see for instance [7]).

Numerically, to compute these minimizers, we use the associated Euler-Lagrange equation which is of integral type with a singular kernel. To discretize it, we propose a finite element-type model and we show some applications in the field of image restoration. Note that our method gives an approximation of the p-Laplacian for any $p \geq 1$, and in particular for high values of p.

This paper is organized as follows. We will first consider the case $p > 1$. Section 2 reminds the main results from [5] that we will use therein. In Section 3 we defined the approximated functional F_n and we show the existence and uniqueness of a solution u_n. In Section 4 we derive the Euler-Lagrange equation verified by u_n. In Section 5 we show that the sequence u_n tends to u the unique minimizer of original foumulation, as $n \to \infty$. Then in Section 6 we show how these results can be extended to the BV-case ($p = 1$). Finally, we show in Section 7 how this method can be implemented and we show some results in image restoration.

2 The Bourgain-Brezis-Mironescu result

Let us first recall the result of Bourgain Brezis and Mironescu [5].

Proposition 2.1 *Assume* $1 \le p < \infty$ *and* $u \in W^{1,p}(\Omega)$, *and let* $\rho \in L^1(R^N), \rho \ge 0$. *Then*

$$\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p}\rho(x-y)dxdy \le C\|u\|^p_{W^{1,p}}\|\rho\|_{L^1(R^N)} \tag{2}$$

where $\|u\|^p_{W^{1,p}}$ *denotes the (semi-)norm defined by* $\|u\|^p_{W^{1,p}} = \int_\Omega |\nabla u|^p dx$ *and* C *depends only on* p *and* Ω.

Now let us suppose that (ρ_n) is a sequence of radial mollifiers, i.e.,

$$\rho_n \ge 0, \quad \int_{R^N} \rho_n(|x|)dx = 1, \tag{3}$$

and for every $\delta > 0$, we assume that

$$\lim_{n\to\infty} \int_\delta^\infty \rho_n(r) r^{N-1} dr = 0. \tag{4}$$

With conditions (3) and (4) that we will assume in all this article, we have the following proposition.

Proposition 2.2 *If* $1 < p < \infty$ *and* $u \in W^{1,p}(\Omega)$, *then*

$$\lim_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy = K_{N,p}\|u\|^p_{W^{1,p}} \tag{5}$$

where $K_{N,p}$ *depends only on* p *and* N.

3 Approximation of variational problems on $\mathbf{W}^{1,p}(\Omega)$, $\mathbf{p} > 1$

We are going to apply Propositions 2.1 and 2.2 for solving general variational problems of the form:

$$\inf_{u \in W^{1,p}(\Omega)} F(u) \tag{6}$$

with

$$F(u) = \int_\Omega |\nabla u(x)|^p dx + \int_\Omega h(x, u(x))dx, u \in W^{1,p}(\Omega), \tag{7}$$

with some boundary conditions on $\partial\Omega$ (Dirichlet or Neumann conditions). An illustration will be provided in Section 7 for the image restoration problem. The method being the same for Dirichlet or Neumann boundary conditions we only present it for Dirichlet boundary conditions. We supppose that the function $x \to h(x, u(x))$ is well-defined for all $u \in L^p(\Omega)$. Following [5] we propose to approximate (6) by the following minimization problem

$$\inf_{u \in W^{1,p}(\Omega),\ u=\varphi \text{ on } \partial\Omega} F_n(u). \tag{8}$$

with

$$F_n(u) = \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^p} \rho_n(x-y) dxdy + \int_\Omega h(x, u(x))\ dx. \tag{9}$$

We would like to show that problem (8) admits a unique solution u_n. This question is not as simple as we could initially guess. Let $(u_n^l)_l$ be a minimizing sequence of (8) (n fixed), then by using results from [5] it is possible to bound u_n^l in $L^p(\Omega)$ uniformly with respect to l. Thus, up to a subsequence, $u_n^l \to u_n$ weakly. But a priori no bounds in $W^{1,p}(\Omega)$ are available. Therefore we do not know if u_n is in $W^{1,p}(\Omega)$ and unfortunately we cannot give any meaning to u_n on the boundary $\partial\Omega$. We will overcome this difficulty by proving a weaker result, namely that (8) admits a unique solution in $W^{s,p}(\Omega)$ for some $1/2 < s < 1$ under an additional assumption on the kernel $\rho_n(t)$.

In this section, we prove that the functional $F_n(u)$ is continuous from $W^{1,p}(\Omega)$ to R and then the existence and uniqueness of a minimizer u_n for (8) in $L^p(\Omega)$.

Note that in all this section and in the proofs, we will denote by C a universal constant that may be different from one line to the other. If the constant depends on n for example, it will be denoted by $C(n)$.

Let us state the first proposition concerning the continuity of functionals F_n.

Proposition 3.1 *Let us assume that the function $x \mapsto h(x, u(x))$ is in $L^1(\Omega)$ for all $u \in W^{1,p}(\Omega)$, then the functional $F_n(u)$ is continuous from $W^{1,p}(\Omega)$ to R.*

Proof We only have to check the continuity of the first term in $F_n(u)$, since the second term is automatically continuous thanks to a Krasnoselki's result (see [12]). From Proposition 2.1 we get for all u and v in $W^{1,p}(\Omega)$

$$\int_\Omega \int_\Omega \frac{|(u(x) - v(x)) - (u(y) - v(y))|^p}{|x-y|^p} \rho_n(x-y) dxdy \leq C \|u - v\|^p_{W^{1,p}} \|\rho_n\|_{L^1(R^N)}. \tag{10}$$

Let us define $u^n(x,y) = \frac{u(x)-u(y)}{|x-y|} \rho_n^{\frac{1}{p}}$ and the same for v. Then, from (10) and since $\|\rho_n\|_{L^1(R^N)} = 1$, we can rewrite (10)

$$\int_\Omega \int_\Omega |(u^n(x,y) - v^n(x,y)|^p dxdy \leq C \|u - v\|^p_{W^{1,p}}. \tag{11}$$

Since we always have in any Banach space $\|X-Y\| \geq |\|X\|-\|Y\||$, we get

$$|\|u^n\|_{L^p(\Omega)} - \|v^n\|_{L^p(\Omega)}| \leq C\|u-v\|_{W^{1,p}(\Omega)},$$

from which we deduce that the functional $u \mapsto \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy$ is continuous from $W^{1,p}(\Omega)$ to R. ■

Now, let us show that functional (9) admits a unique minimizer. It is clear by using again Proposition 2.1 and the fact that $\|\rho_n\|_{L^1(R^N)} = 1$, that we have for all v in $W^{1,p}(\Omega)$ with $v = \varphi$ on $\partial\Omega$

$$\inf_u F_n(u) \leq F_n(v) \leq C\|v\|^p_{W^{1,p}} + \int_\Omega h(x, v(x))\, dx,$$

from which we deduce that $\inf_u F_n(u)$ is bounded by a finite constant (independent of n).

Proposition 3.2 *Assume that $h \geq 0$, the function $x \mapsto h(x,u(x))$ is in $L^1(\Omega)$ for all u in $L^p(\Omega)$, h is convex with respect to its second argument and for each n, the function $t \mapsto \rho_n(t)$ is non-increasing. Then functional* (9) *admits a unique minimizer in $L^p(\Omega)$.*

Before proving this proposition, let us remind a technical lemma from Bourgain, Brezis et al. (Lemma 2 in [5]) that we will use in the proof of Proposition 3.2.

Lemma 3.1 *Let $g, k : (0,\delta) \to R_+$. Assume $g(t) \leq g(t/2)$, for $t \in (0,\delta)$, and that k is non-increasing. Then for all $M > 0$, there exists a constant $C(M) > 0$ such that*

$$\int_0^\delta t^{M-1} g(t)k(t)dt \geq C(M)\delta^{-M} \int_0^\delta t^{M-1} g(t)dt \int_0^\delta t^{M-1}k(t)dt \tag{12}$$

Proof *(of Proposition* 3.2*)* Let us consider a minimizing sequence u_n^l of $F_n(u)$ with $n > 0$ fixed. Since $h \geq 0$ and $\inf_u F_n(u)$ is bounded, then there exists a constant C such that

$$\int_\Omega \int_\Omega \frac{|u_n^l(x) - u_n^l(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy \leq C. \tag{13}$$

We are going to apply techniques borrowed from Brezis-Bourgain-Mironescu ([5], Theorem 4). Without loss of generality, we may assume that $\Omega = R^N$ and that the support of u_n^l is included in a ball B of diameter 1. This can be achieved by extending each function u_n^l by reflection across the boundary in a neighborhood of $\partial\Omega$. We may also assume the normalization condition $\int_\Omega u_n^l(x)dx = 0$ for all n and l. Let us define for each n, l, $t > 0$

$$E_n^l(t) = \int_{S^{N-1}} \int_{R^N} |u_n^l(x+tw) - u_n^l(x)|^p dxdw \tag{14}$$

where S^{N-1} denotes the unit sphere of R^N. An interesting property of E_n^l that we will use later is (the proof follows from the triangle inequality)

$$E_n^l(2t) \le 2^p E_n^l(t). \tag{15}$$

Straightforward changes of variables show that

$$\int_\Omega \int_\Omega \frac{|u_n^l(x) - u_n^l(y)|^p}{|x-y|^p} \rho_n(x-y)dxdy = \int_0^1 t^{N-1} \frac{E_n^l(t)}{t^p} \rho_n(t)dt,$$

thus (13) can be equivalently expressed as

$$\int_0^1 t^{N-1} \frac{E_n^l(t)}{t^p} \rho_n(t)dt \le C. \tag{16}$$

Now since we have supposed that u_n^l is of zero mean, we can write

$$u_n^l(x) = u_n^l(x) - \frac{1}{|B|} \int_B u_n^l(y)dy.$$

Thus

$$\int |u_n^l(x)|^p dx = \int \left| u_n^l(x) - \frac{1}{|B|} \int_B u_n^l(y)dy \right|^p dx = \frac{1}{|B|^p} \int \left| \int_B (u_n^l(x) - u_n^l(y)dy \right|^p dx,$$

and thanks to Holder inequality, there exists a constant C

$$\int |u_n^l(x)|^p dx \le C \int_{|h|\le 1} \Big(\int |u_n^l(x+h) - u_n^l(x)|^p \ dx \Big) dh = C \int_0^1 t^{N-1} E_n^l(t)dt \tag{17}$$

To conclude we apply Lemma 3.1 with $M = N$, $\delta = 1$, $k(t) = \rho_n(t)$ and $g(t) = \frac{E_n^l(t)}{t^p}$ (this choice is valid thanks to the hypotheses on ρ_n and property (15)). We obtain

$$\begin{aligned} \int_0^1 t^{N-1} \rho_n(t) \frac{E_n^l(t)}{t^p} dt &\ge C \int_0^1 t^{N-1} \rho_n(t)dt \int_0^1 t^{N-1} \frac{E_n^l(t)}{t^p} dt \\ &\ge C \int_0^1 t^{N-1} \rho_n(t)dt \int_0^1 t^{N-1} E_n^l(t)dt, \end{aligned} \tag{18}$$

where we have used in the last inequality the fact that $0 < t < 1$. Let us denote $d(n) = \int_0^1 t^{N-1} \rho_n(t)dt > 0$, we obtain thanks to (16), (17) and (18) that there exists a constant $C(n) > 0$ (but which is independent of l) such that

$$\left| u_n^l \right|_{L^p(\Omega)} \le C(n). \tag{19}$$

From equation (19), we deduce that, up to a subsequence, u_n^l tends weakly in $L^p(\Omega)$ to some $u_n \in L^p(\Omega)$ as $l \to +\infty$. Then, we deduce that the sequence $w_n^l(x,y) = u_n^l(x) - u_n^l(y)$ tends weakly in $L^p(\Omega \times \Omega)$ to $w_n(x,y) = u_n(x) - u_n(y)$. Since the functional

$$w \to \int\limits_\Omega \int\limits_\Omega |w(x,y)|^p \frac{\rho_n(|x-y|)}{|x-y|} dxdy$$

is non negative, convex and lower semi-continuous from $L^p(\Omega \times \Omega) \to \bar{R}$, we easily get

$$F_n(u_n) \leq \underline{\lim}_{l \to \infty} F_n(u_n^l) = \inf_u F_n(u),$$

where symbol $\underline{\lim}$ denotes the lower limit. Therefore u_n is a minimizer of F_n. Moreover it is unique since the function $t \mapsto |t|^p$ is strictly convex for $p > 1$. ■

We have obtained the existence of a minimizer on $L^p(\Omega)$ but the regularity of this space is not sufficient to give a meaning to the trace of u_n on the boundary $\partial\Omega$ and so to verify a Dirichlet boundary condition on $\partial\Omega$. Unfortunately it seems impossible to show that $u_n \in W^{1,p}(\Omega)$. However we prove here that u_n belongs to the Sobolev space $W^{s,p}(\Omega)$ with $1/2 < s < 1$ which is a sufficient regularity to define a trace on the boundary $\partial\Omega$ (cf [15, 1]). In this case, the trace is well-defined in the space $W^{s-1/2,p}(\partial\Omega)$.

The space $W^{s,p}(\Omega)$ can be characterized as $W^{1,p}(\Omega)$ by a differential quotient. For $0 < s < 1$ and $1 \leq p < \infty$, we define

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega); \frac{|u(x)-u(y)|}{|x-y|^{s+N/p}} \in L^p(\Omega \times \Omega) \right\},$$

endowed with the norm

$$|u|^p_{W^{s,p}(\Omega)} = \int_\Omega |u|^p dx + \int_\Omega \int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{sp+N}} dxdy.$$

Let us consider n fixed and let us denote by $C(n)$ a universal positive constant depending on n (i.e., $C(n)$ may be different from one line to the other). Let $(u_n^l)_l$ be a minimizing sequence of (8) so that

$$\int_\Omega \int_\Omega \frac{|u_n^l(x) - u_n^l(y)|^p}{|x-y|^p} \rho_n(x-y) dxdy \leq C(n). \tag{20}$$

Then we would like to prove that (20) implies

$$\int_\Omega \int_\Omega \frac{|u_n^l(x) - u_n^l(y)|^p}{|x-y|^{sp+N}} dxdy \leq C(n), \tag{21}$$

for some $1/2 < s < 1$ and some constant other constant $C(n)$, thus showing that u_n^l belongs to $W^{s,p}$.

Proposition 3.3 *Let q be a real number such that $\frac{p}{2} < q < p$ and $(p-1) \leq q$ and let us assume that ρ_n verifies (3)–(4) and also that conditions of Proposition 3.2 are fullfilled. Moreover let us suppose that the functions $t \to \rho_n(t)$ and $t \to t^{q+2-p}\rho_n(t)$ are non-increasing for $t \geq 0$, then $u_n^l \in W^{q/p,p}(\Omega)$ for all l.*

Proof Without loss of generality, let us prove Proposition 3.3 in the case $N = 2$. Equivalently, thanks to Definition (14) of E_n^l, we can rewrite (20) and (21) so that one needs to prove that

$$\int_0^1 t\frac{E_n^l(t)}{t^p}\rho_n(t)dt \leq C(n) \tag{22}$$

implies

$$\int_0^1 t\frac{E_n^l(t)}{t^{sp+2}}dt \leq C(n).$$

Let us apply Lemma 3.1 with $M = \delta = 1$, $g(t) = \frac{E_n^l(t)}{t^{q+1}}$, $k(t) = t^{q+2-p}\rho_n(t)$. Assuming the hypothese on $g(t)$ is true, Lemma 3.1 gives

$$\int_0^1 \frac{E_n^l(t)\rho_n(t)}{t^{p-1}}dt \geq C(M)\int_0^1 \frac{E_n^l(t)}{t^{q+1}}dt\int_0^1 t^{q+2-p}\rho_n(t)dt. \tag{23}$$

Therefore

$$\int_0^1 \frac{E_n^l(t)}{t^{q+1}}dt \leq \frac{1}{C(M)\int_0^1 t^{q+2-p}\rho_n(t)dt}\int_0^1 \frac{E_n^l(t)\rho_n(t)}{t^{p-1}}dt,$$

and according to (22), we get

$$\int_0^1 \frac{E_n^l(t)}{t^{q+1}}dt \leq \frac{C(n)/C(M)}{\int_0^1 t^{q+2-p}\rho_n(t)dt},$$

where the right-hand term is bounded independently of l. Thus $u_n^l \in W^{s,p}(\Omega)$ with $s = \frac{q}{p}$ and since we have supposed $\frac{p}{2} < q < p$ we have $\frac{1}{2} < s < 1$.

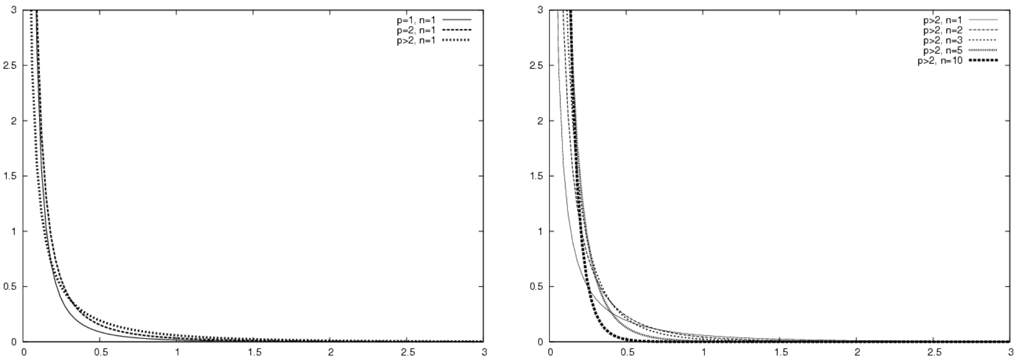
So it remains to show that function $g(t)$ verifies the hypothese of Lemma 3.1. We have to check $g(t) \leq g(t/2)$. Since $g(t) = \frac{E_n^l(t)}{t^{q+1}}$ then $g(t/2) = \frac{E_n^l(t/2)}{t^{q+1}}2^{q+1} \geq 2^{q+1-p}\frac{E_n^l(t)}{t^{q+1}} = 2^{q+1-p}g(t)$ (thanks to (14)). Thus we get $g(t/2) \geq g(t)$ if $q+1-p \geq 0$, i.e., if $q \geq (p-1)$. ■

Depending on p, one needs to find a function $\rho_n(t)$ so that $\rho_n(t)$ and $t^{q+2-p}\rho_n(t)$ are decreasing, and verify (3) and (4). Let us show that such ρ_n function exist. We define

$$\rho_n(t) = Cn^2\rho(nt) \quad \text{with} \quad C = \frac{1}{\int_{R^2}\rho(|x|)dx} \tag{24}$$

and, depending on the values of p, we propose the following functions (see Figure 1)

$$\rho(t) = \begin{cases} exp(-t)/t^{q+1} & \text{if } p = 1, \text{with } 0.5 < q < 1, \\ exp(-t)/t^{q} & \text{if } p = 2, \text{with } 1 < q < 2, \\ exp(-t)/t & \text{if } p > 2, \text{with } q = p - 1. \end{cases} \quad (25)$$



p varying n varying

Figure 1: Comparisons of functions ρ_n, for different values of p and n.

As a consequence, we have the following proposition

Proposition 3.4 *Let* $(u_n^l)_l$ *be a minimizing sequence of* (8)*. Let us suppose that* h *in the definition of* (6) *verifies the coercivity condition* $h(x,u) \geq a|u|^p + b$*, with* $a > 0$*. Then the sequence* $(u_n^l)_l$ *is bounded in* $W^{q/p,p}(\Omega)$ *uniformly with respect to* l*. Therefore, up to a subsequence,* u_n^l *tends weakly to* u_n *in* $W^{q/p,p}(\Omega)$ *(and strongly in* $L^p(\Omega)$*). Moreover if* $u_n^l = \varphi$ *on* $\partial\Omega$*, then by continuity of the trace operator, we have* $u_n = \varphi$ *on* $\partial\Omega$*. Thus* u_n *is the unique minimizer in* $W^{q/p,p}(\Omega)$ *of problem* (8)*.*

4 Euler-Lagrange Equation

Since u_n is a global minimizer of $F_n(u)$ it necessarily verifies $F_n^{'}(u_n) = 0$, i.e., an Euler-Lagrange equation. Euler-Lagrange equation is given in the following proposition.

Proposition 4.1 *If function* h *verifies for all* u *and a.e.* x*, an inequality of the form* $|\frac{\partial h(x,u)}{\partial u}| \leq l(x) + b|u|^{p-1}$ *for some function* $l(x) \in L^1(\Omega)$*,* $l(x) > 0$ *and some* $b > 0$*, then the* *unique minimizer u_n of $F_n(u)$ verifies for a.e. x*

$$2p\int_\Omega \frac{|u_n(x)-u_n(y)|^{p-2}}{|x-y|^p}(u_n(x)-u_n(y))\rho_n(x-y)dy+\frac{\partial h(x,u_n(x))}{\partial u}=0. \tag{26}$$

Proof Let us focus on the smoothing term and denote

$$E_n(u_n)=\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy,$$

and let us consider for all v in $W^{1,p}(\Omega)$, the differential quotient

$$D_v(t)=\frac{E_n(u_n+tv)-E_n(u_n)}{t}.$$

We have

$$D_v(t)=\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)+t(v(x)-v(y))|^p-|u_n(x)-u_n(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy.$$

Thanks to Taylor's formula, there exists $c(t,x,y)$ with $|c(t,x,y)-(u_n(x)-u_n(y))|<t|v(x)-v(y)|$ such that

$$D_v(t)=p\int_\Omega\int_\Omega \frac{(v(x)-v(y))c(t,x,y)|c(t,x,y)|^{p-2}}{|x-y|^p}\rho_n(x-y)dxdy.$$

Moreover, we have, as $t\to 0$

$$\frac{(v(x)-v(y))c(t,x,y)|c(t,x,y)|^{p-2}}{|x-y|^p}\rho_n(x-y)\to \frac{(v(x)-v(y))(u_n(x)-u_n(y))|u_n(x)-u_n(y)|^{p-2}}{|x-y|^p}\rho_n(x-y)$$

On the other hand

$$|c(t,x,y)|^{p-1}\le 2^p(|u_n(x)-u_n(y)|^{p-1}+|v(x)-v(y)|^{p-1}),$$

Thus

$$\Big|\frac{(v(x)-v(y))c(t,x,y)|c(t,x,y)|^{p-2}}{|x-y|^p}\rho_n(x-y)\Big|\le \frac{|v(x)-v(y)||u_n(x)-u_n(y)|^{p-1}}{|x-y|^p}\rho_n(x-y)+\frac{|v(x)-v(y)|^p}{|x-y|^p}\rho_n(x-y). \tag{27}$$

Let us discuss the integrability of the right-hand side terms denoted respectively by A and B. Second term B is bounded by an integrable function because $v \in W^{1,p}(\Omega)$ and thanks to Proposition 2.1. First term A gives

$$A = \frac{|v(x)-v(y)|}{|x-y|}\rho_n^{\frac{1}{p}}(x-y)\left|\frac{u_n(x)-u_n(y)}{|x-y|}\right|^{p-1}\rho_n^{\frac{p-1}{p}}(x-y),$$

where

$$\frac{|v(x)-v(y)|}{|x-y|}\rho_n^{\frac{1}{p}}(x-y)$$

is in $L^p(\Omega)$ since $v \in W^{1,p}(\Omega)$ and thanks to Proposition 2.1, and

$$\left|\frac{u_n(x)-u_n(y)}{|x-y|}\right|^{p-1}\rho_n^{\frac{p-1}{p}}(x-y)$$

is in $L^{\frac{p}{p-1}}(\Omega)$ since u_n is a minimizing sequence. So A is also bounded by an integrable function.

Therefore we can apply the Lebesgue's dominated convergence Theorem (n is fixed) and we get

$$\langle E_n'(u_n), v\rangle = p\int_\Omega \frac{|u_n(x)-u_n(y)|^{p-2}}{|x-y|^p}(v(x)-v(y))(u_n(x)-u_n(y))\rho_n(x-y)dy$$

The computation of the derivative of $\int_\Omega h(x,u(x))dx$ is classical. Thus the desired result (26) by remarking that the function $(x,y) \mapsto \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))}{|x-y|^p}$ is antisymmetric with respect to (x,y). ■

5 Study of the $\lim_{n\to\infty} u_n$

In Section 3 we proved the existence of a unique solution u_n for problem (8), with n fixed. Now, we are going to examine the asymptotic behaviour of (8) as $n \to \infty$. Throughout this section we will suppose hypotheses stated in Proposition 3.3 and 3.4. By definition of a minimizer, we have, for all $v \in W^{q/p,p}(\Omega)$ with $v = \varphi$ on $\partial\Omega$

$$F_n(u_n) \le F_n(v) = \int_\Omega\int_\Omega \frac{|v(x)-v(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy + \int_\Omega h(x,v(x))dx. \tag{28}$$

Thus by using (2.1) and the fact that $|\rho_n|_{L^1} = 1$ we deduce from (28) that $F_n(u_n)$ is bounded uniformly with respect to n. In particular, we get for some constant $C > 0$

$$\int_\Omega\int_\Omega \frac{|u_n(x)-u_n(y)|^p}{|x-y|^p}\rho_n(x-y)dxdy \le C.$$

By using the same technique as in Proposition 3.3, we still have that (u_n) is bounded in $W^{q/p,p}(\Omega)$. Therefore there exists u such that (up to a subsequence) $u_n \to u$ in $L^p(\Omega)$ and by continuity of the trace operator, we have $u = \varphi$ on $\partial\Omega$. Moreover, by applying the Theorem 4 from [5], we obtain that $u \in W^{1,p}(\Omega)$. We claim that u is the unique solution of problem (6), i.e., for all $v \in W^{1,p}(\Omega)$ with $v = \varphi$ on $\partial\Omega$:

$$\int_\Omega |\nabla u(x)|^p dx + \int_\Omega h(x, u(x))dx \leq \int_\Omega |\nabla v(x)|^p dx + \int_\Omega h(x, v(x))dx. \tag{29}$$

To prove (29) we refer to the paper by [18]. In this paper the author studies in the same spirit as in [5] new characterizations of Sobolev spaces and also of the space $BV(\Omega)$ of functions of bounded variations. The author considers more general differential quotients than the ones by [5], namely functionals of the form

$$E_n(u) = \int_\Omega \int_\Omega w\left(\frac{|u(x) - u(y)|}{|x - y|}\right) \rho_n(x - y) dx dy.$$

By studying the asymptotic behavior [18] obtained new characterizations of $W^{1,p}(\Omega)$ but also of $BV(\Omega)$. Among his interesting results he has proved, in the particular case $w(t) = |t|^p$ that $E_n(u)$ Γ-converge (up to a multiplicative constant) to $E(u) = \int_\Omega |\nabla u|^p dx$. Thus by applying general results from the Γ-convergence theory [9, 11], we have

Proposition 5.1 (i) *The sequence of functionals*

$$F_n(u) = E_n(u) + \int_\Omega h(x, u(x))dx$$

Γ-*converges (up to a multiplicative constant) to*

$$F(u) = E(u) + \int_\Omega h(x, u(x))dx.$$

(ii) *The sequence u_n of minimizers of $F_n(u)$, which is compact in $L^p(\Omega)$, converges to the unique minimizer of $F(u)$.*

6 Extension of previous results to the $BV(\Omega)$-case ($p = 1$)

Similar result as Proposition 2.2 holds if $p = 1$, see [18]. In this case we need to search for a solution for problem (6) in $BV(\Omega)$, the space of functions of bounded variations [2, 13]. In fact most of all proven results are still valid in this case with some adaptations. We do not reproduce here details of their proofs which rely upon the work by [18] who has generalized to $BV(\Omega)$, as said before, the results by [5] stated in the $W^{1,p}(\Omega)$ case. We only mention two points which are specific to the case $p = 1$.

The first point is that the proof of Proposition 3.2 does not apply in the case $p = 1$ since we cannot extract from a sequence bounded in $L^1(\Omega)$ a weakly converging subsequence. Thus we first have to show that a minimizing sequence u_n^l of $F_n(u)$ is bounded in the Sobolev space $W^{q,1}(\Omega)$, with $0.5 < q < 1$. To do that we use the same proof as in Proposition 3.3. Then, thanks to the 2D Rellich-Kondrachov Theorem $W^{q,1}(\Omega) \subset L^r(\Omega)$ with compact injection for $1 \leq r < \frac{2}{2-q}$ (note that if $0.5 < q < 1$ then $4/3 < \frac{2}{2-q} < 2$). Therefore, up to a subsequence, $u_n^l(x)$ tends, a.e., to some function $u_n(x)$. Then by using Fatou's lemma we get $F_n(u_n) \leq liminf_{l \to \infty} F_n(u_n^l)$, i.e., u_n is a minimizer of F_n.

The second point is that Euler-Lagrange equation (4.1) is no longer true in the case $p = 1$ since the function $t \to |t|$ is not differentiable. However it is subdifferentiable. Therefore equation (4.1) changes into an inclusion

$$0 \in \partial E_n(u_n) + \frac{\partial h}{\partial u}(x, u_n), \tag{30}$$

where $E_n(u) = \int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|} \rho_n(|x-y|dxdy$. Note that the subdifferential of $t \to |t|$ is the interval $[-1, +1]$.

7 Application to image restoration

The goal of this section is to show how the results presented in [5] can be used and implemented for a given classical problem: Image restoration. We detail how integral term can be discretized. Of course, different approaches could be possible to diminish the computational cost of the approach, but this is not the focus in this article (see remark in Section 7.2).

7.1 Variational formulation of image restoration

Let $u : \Omega \subset R^2 \to R$ be an original image describing a real scene, and let u_0 be the observed image of the same scene (i.e., a degradation of u). We assume that

$$u_0 = R\,u + \eta, \tag{31}$$

where η stands for a white additive Gaussian noise and where R is a linear operator representing the blur (usually a convolution). Given u_0, the problem is then to reconstruct u knowing (31). Supposing that η is a white Gaussian noise, and according to the maximum likelihood principle, we can find an approximation of u by solving the least-square problem

$$\inf_u \int_\Omega |u_0 - Ru|^2 \ dx,$$

where Ω is the domain of the image. However, this is well known to yield to an ill-posed problem [14, 4].

A classical way to overcome ill-posed minimization problems is to add a regularization term to the energy. This idea was introduced in 1977 by Tikhonov and Arsenin [20]. The authors proposed to consider the following minimization problem:

$$F(u) = \int_\Omega |u_0 - Ru|^2 \ dx + \lambda \int_\Omega |\nabla u|^2 \ dx. \tag{32}$$

The first term in $F(u)$ measures the fidelity to the data. The second is a smoothing term. In other words, we search for a u that best fits the data so that its gradient is low (so that noise will be removed). The parameter λ is a positive weighting constant. More generally, the smoothing term can be any $W^{1,p}$–norm. For $p = 1$ we have in fact a BV–norm which leads to discontinuous solutions (see [3] for a review).

So the idea is to consider the initial variational formulation (6), for a given n, with $h(x, u(x)) = |u_0 - Ru|^2$. Without loss of generality, we will assume in this article that the operator R is the identity operator.

7.2 Implementation details

Following Proposition 4.1, we need to find u verifying the Euler-Lagrange equation (26), writen $EL(u) = 0$ (p and n fixed). The classical way to solve it is to add a time variable and to consider the temporal scheme $\frac{\partial u}{\partial t} = EL(u)$, i.e., a gradient descent method. Discretizing in time, and starting from the initial condition $u^0(x) = u_0(x)\ \forall x \in \Omega$ (for example), we iterate

$$\begin{cases} u^{k+1}(x) = u^k(x) - \triangle t EL(u), \\ u^0(x) = u_0(x) \end{cases} \tag{33}$$

Taking into account the expression of the gradient, we have here

$$u^{k+1}(x) = u^k(x) + \triangle t\Big(- 2(u^k(x) - u_0(x)) - 2pI_{u^k}(x)\Big), \tag{34}$$

with

$$I_{u^k}(x) = \int_\Omega \frac{|u^k(x) - u^k(y)|^{p-2}}{|x-y|^p}(u^k(x) - u^k(y))\rho_n(|x-y|)dy, \tag{35}$$

where $\triangle t$ is the discrete time step and u^k is the image at time $k\triangle t$. We remind that the definition of ρ_n also depends on p (see equations (25)).

Remark Using classical gradient descent method (33) implies to choose very low values for the time step as p increases. This is a well–known issue for the minimization of convex problems where the Lipschitz constant of the gradient is high or infinite, which is the case when minimizing a $W^{1,p}$ norm (where Lipschitz constant is infinite). Lipschitz constant is harder to determine here for the approximated integral operator, but we observe that similar behavior occur. Empirically, we have to choose low time steps as p increases to obtain stability. To overcome this difficulty, other schemes have been proposed in litterature for convex problems, and we refer to [17] (Sectio 5.3) for more details. ■

Now the problem is to discretize in space the integral $I_{u^k}(x)$ (35) which has a singular kernel, not defined when $x = y$. Let us introduce the function $J_{u^k}(x, y)$ such that

$$I_{u^k}(x) = \int_\Omega \frac{J_{u^k}(x,y)}{|x-y|}dy, \tag{36}$$

with

$$J_{u^k}(x,y) = \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\rho_n(|x-y|)}{|x-y|^{p-1}}.$$

Because of the singularity, simple schemes using finite differences and integral approximations for example will fail. Here we propose to

- Discretise the space using a triangulation. We denote by $\mathcal{T}$ the familly of triangles covering Ω (see Figure 2).
- Interpolate linearly the function $J_{u_k}(x, y)$ on each triangle (x fixed).

- Find explicit expressions for the integral $J_{u_k}(x,y)/|x-y|$ of on each triangle. Note that this kind of estimation also appears for instance in electromagnetism problems such as MEG-EEG (see e.g., [10]) where one needs to estimate such singular integrals on a meshed domains (3D domains here).

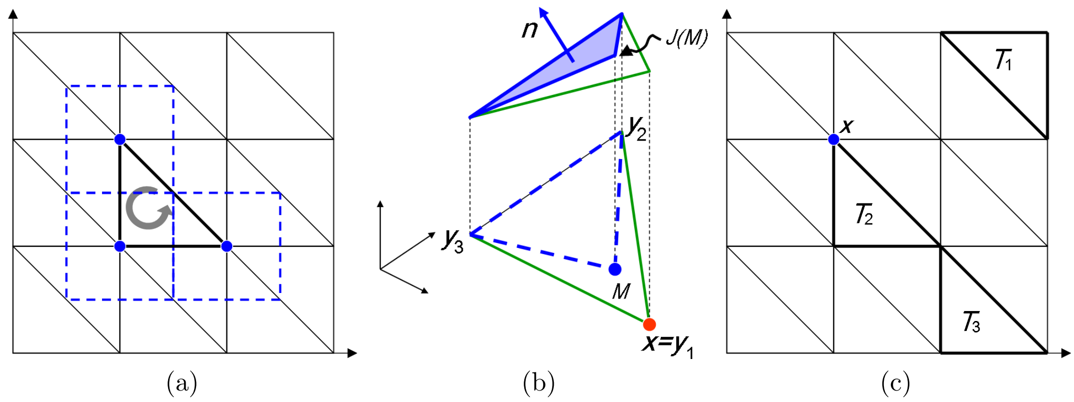


Figure 2: (a) Mesh definition. Pixels are represented by the blue dashed squares. The blue circles correspond to the centers of the pixels defining the nodes of the mesh. Four nodes define two triangles. (b) In the special case when x is a node, one needs also interpolation to define $J_{u^k}(x,y)$ at x. (c) Different cases depending on the situation of x with respect to T_i. Triangle T_1 has no edge aligned with x; For triangle T_2, x is one node; For T_3, x is aligned with one edge.

Let us now detail each step. First, integral (36) becomes:

$$I_{u^k}(x) = \sum_{T_i \in \mathcal{T}} \int_{T_i} \frac{J_{u^k}(x,y)}{|x-y|} dy. \tag{37}$$

Then, let us approximate $J_{u^k}(x,y)$ on each triangle by a linear interpolation. We assume that x is given and fixed. Given one triangle $T \in \mathcal{T}$, let us denote the three nodes of T by $\{y_i = (y_i^1, y_i^2)^T\}_{i=1..3}$, where the subscript indicates the component. Then we define $\{A_i\}_{i=1..3}$ the 3-D points

$$A_i = ((y_i)^T, J(x, y_i))^T.$$

Note that as soon as $x \neq y_i$, $J(x, y_i)$ is well defined. Otherwise, if x is in fact a node of T, for example y_1 (see Figure 2 (b)), we use a linear interpolation algorithm: We introduce one point $M \in \mathcal{T}$ close to y_1, estimate the value of J at this point, and deduce the value of $J(y_1)$ by interpolation.

So, given $\{A_i\}_{i=1..3}$ and any node y_k, we have

$$J(y) = J(x, y_k) - \frac{1}{n^3} \begin{pmatrix} n^1 \\ n^2 \end{pmatrix} (y - y_k). \tag{38}$$

With (38) we obtain:

$$\int_T \frac{J_{u^k}(x,y)}{|x-y|}dy = J(x,y_k)\int_T \frac{1}{|x-y|}dy - \frac{1}{n^3}\begin{pmatrix} n^1 \\ n^2 \end{pmatrix}\int_T \frac{(y-y_k)}{|x-y|}dy \tag{39}$$

$$= J(x,y_k)\int_T \frac{1}{|x-y|}dy - \frac{1}{n^3}\begin{pmatrix} n^1 \\ n^2 \end{pmatrix}\left[\int_T \frac{(y-x)}{|x-y|}dy + (x-y_k)\int_T \frac{1}{|x-y|}dy\right].$$

So, in order to know the integral over the triangle T, one needs only to estimate:

$$\int_T \frac{1}{|x-y|}dy \quad \text{and} \quad \int_T \frac{(y-x)}{|x-y|}dy. \tag{40}$$

If we introduce the distance function:

$$\text{Dist}(x,y) = |x-y| = \sqrt{(x^1-y^1)^2+(x^2-y^2)^2},$$

so that:

$$\nabla_y \text{Dist}(x,y) = \frac{y-x}{|x-y|},$$
$$\triangle_y \text{Dist}(x,y) = \frac{1}{\text{Dist}(x,y)},$$

then we have the following relations:

$$\int_T \frac{1}{|x-y|}dy = \int_T \triangle_y \text{Dist}(x,y)dy = \sum_{i=1,2}\int_{\partial T}\frac{\partial \text{Dist}}{\partial y^i}(x,y)N^i ds, \tag{41}$$

$$\int_T \frac{(y-x)}{|x-y|}dy = \int_T \nabla_y \text{Dist}(x,y)dy = \int_{\partial T}\text{Dist}(x,y)N ds, \tag{42}$$

where N is the normal to the edges of the triangle T. So we need to estimate the two kinds of integrals defined on the borders of the triangles. This can be done explicitly:

Lemma 7.1 *Let us consider a segment $S=(\alpha,\beta)$ of extremities $\alpha=(\alpha_1,\alpha_2)$, $\beta=(\beta_1,\beta_2)$, N the normal to this segment, and x a fixed given point. Let us define:*

$$\begin{array}{lll} a = |\alpha\beta|, & \delta = a^2b^2-c^2, & l_1 = c/\sqrt{\delta}, \\ b = |x\alpha|, & d = \vec{x\alpha}\cdot N, & l_2 = (a^2+c)/\sqrt{\delta}, \\ c = \vec{x\alpha}\cdot\vec{\alpha\beta}. & & \end{array}$$

Then we have:

$$\sum_{i=1,2}\int_S \frac{\partial \text{Dist}}{\partial y^i}(x,y)N^i ds = \begin{cases} 0 \text{ if } x \text{ is aligned with } S, \\ d(\text{asinh}(l_2)-\text{asinh}(l_1)) \text{ otherwise.}\end{cases} \tag{43}$$

With Lemma 7.1, one can estimate (41) and (42) and then (39). By summing over all the squares and for a given x, we obtain the estimation of the integral $I_{u^k}(x)$ (35), and we can iterate (34).

Remark In this remark, we show how the integral operator defined in (26) is related to a well-known class of integral-type smoothing operator: The neighborhood filters. In fact, in Proposition 4.1 we have shown that

$$\begin{aligned}\triangle_p u &\approx -\int_\Omega \frac{|u^k(x)-u^k(y)|^{p-2}}{|x-y|^p}(u(x)-u(y))\rho_n(|x-y|)dy \\ &\approx -\int_\Omega T(|u(x)-u(y)|)w_n(|x-y|)(u(x)-u(y))dy\end{aligned}$$

where

$$T(t)=|t|^{p-2} \quad \text{and} \quad w_n(t)=\frac{\rho_n(|t|)}{|t|^p}.$$

So we have

$$\triangle_p u \approx N(x)\Big(\frac{1}{N(x)}\int_\Omega T(|u(x)-u(y)|)w_n(|x-y|)u(y)dy - u(x)\Big), \tag{44}$$

where

$$N(x)=\int_\Omega T(|u(x)-u(y)|)w_n(|x-y|)dy. \tag{45}$$

Indeed, this result and the integral approximation have some interesting similarities with neighborhood filters (also called bilateral filter) in image processing. Neighborhood filtering is based on the idea that two pixels are close to each other not only if they occupy nearby spatial locations but also if they have some similarity in the photometric range. The formalization of this idea apparently goes back in the literature to Yaroslavsky [22], then Smith et al. [19] and Tomasi et al. [21].

A general neighborhood filtering can be described as follows. Let u be an image to be denoised and let $T_h : R^+ \to R^+$ and $w_n : R^+ \to R^+$ be two functions whose roles will be to enforce respectively photometric and geometric locality. Parameters h and n will measure the amount of filtering for the image u. The filtered image $u_{h,n}(x)$ at scale (h,n) is given by

$$F_{h,n}u(x)=\frac{1}{N(x)}\int_\Omega T_h(|u(y)-u(x)|)\ w_n(|x-y|)u(y)dy, \tag{46}$$

where $N(x)$ is a normalization factor

$$N(x)=\int_\Omega T_h(|u(y)-u(x)|)\ w_n(|x-y|)dy. \tag{47}$$

Of course many choices are possible for the kernels T_h and w_n so that classical nonlinear filters such as [19, 21] can be recovered. Classical choices are

$$T_h(t) = \exp\left(-\frac{t^2}{h^2}\right)$$

and

$$w_\rho(t) = \exp\left(-\frac{t^2}{\rho^2}\right) \quad \text{or} \quad w_\rho(t) = \chi_{B(x,\rho)}(t),$$

where $\chi_{B(x,\rho)}$ denotes the characteristic function of the ball of center x and radius ρ. With the former choice of w_ρ, we get the SUSAN filter [19] or the bilateral filter [21]:

$$S_{\rho,h}u(x) = \frac{1}{N(x)} \int_{R^2} \exp\left(-\frac{|u(y)-u(x)|^2}{h^2}\right) \exp\left(-\frac{|y-x|^2}{\rho^2}\right) u(y)dy.$$

With the latter choice of w_ρ, we recover the Yaroslavsky filter

$$Y_{\rho,h}u(x) = \frac{1}{N(x)} \int_{B(x,\rho)} \exp\left(-\frac{|u(y)-u(x)|^2}{h^2}\right) u(y)dy. \tag{48}$$

The SUSAN and Yaroslavsky filters have similar behaviors. Inside a homogeneous region, the gray level values slightly fluctuate because of the noise. Nearby sharp boundaries, between a dark and a bright region, both filters compute averages of pixels belonging to the same region as the reference pixel: edges are not blurred.

Interestingly, the estimation of the residue defined by

$$F_{h,n}u(x) - u(x),$$

is equal to some well-known diffusion operators (when scale parameters tend to zero), linear and nonlinear. We refer to work by Buades et al. [6] for more details. This kind of results is very similar to the approximation (44)–(45) and our approximation can be seen as a special case of the general formulation (46)–(47). However, note that kernels in our case are singular, whereas they are very smooth for SUSAN and Yaroslvsky filter.

Another interest of this analogy is also the way to implement the integral term. In Section 7.2 we propose an approach based on a triangulation of the domain (so sampling does not need to be homogeneous), doing some analytical estimations to handle the singularity. This kind of approximation is quite computationally expensive and one might think about alternative methods. In particular, several contributions exist to implement efficiently the bilateral filter (see e.g., [8]), so it would be interesting to investigate how thoses approaches can be extended to our case, when sampling is homogeneous. ■

7.3 Results

This section shows some results of gray-scale image diffusion. Our illustrations have three objectives. The first is to show that our approximation allows to recover results that can be obtained from the direct formulation in simple cases such as $p = 2$. The second is to show that this approximation is also a good way to approximate the minimization of the BV-norm (independently of the fidelity attach term). The third is to show that our approximation is also useful to minimize $W^{1,p}$-norms with high values of p, which is a challenging problem.

Figure 3 illustrates the minimization of the integral approximation on a synthetic noisy step image. We first recover that for $p = 1$, edges are well preserved, which corresponds to the minimization of a BV-norm. Also, for $p = 2$, we observe a predictible blurring effect, which corresponds to the minimization of a L^2-norm (i.e., a Laplacian operator in the partial differential equation).

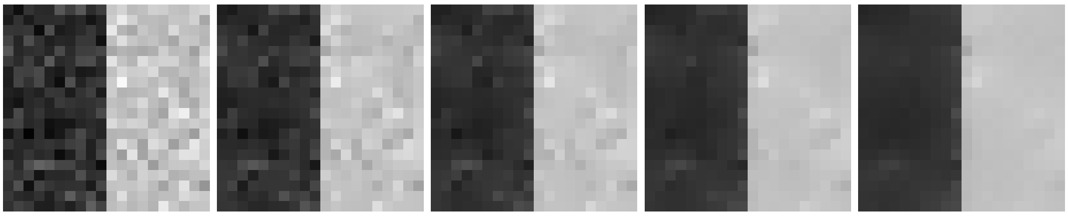
A step further, we propose some simulations with high values of p. The evolutions reveal some slightly different diffusion behaviours: For high p values, diffusion effect seem more important and focused near edges, and it needs more time to propagate homogeneously.

Figure 4 shows another example of result on a real noisy image for $p = 1$ where we recover expected results.

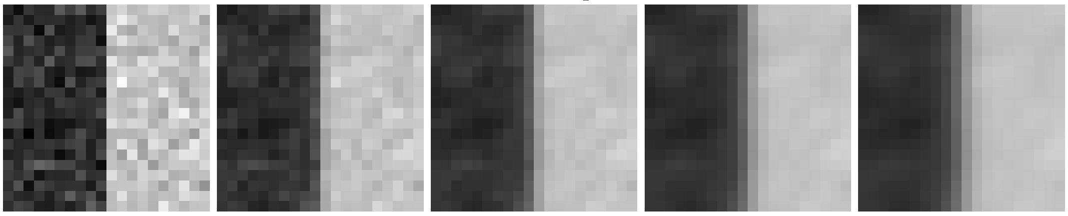
Remark Beside the problem of restoration, and since we are able to approximate $W^{1,p}$-norms for high p, we could also consider the problem of building absolutely minimizing Lipschitz extensions to a given function:

$$\frac{\partial f}{\partial t} = \triangle_\infty = \frac{d^2 f}{dg^2} \quad \text{where} \quad g = \frac{Df}{|Df|}, \tag{49}$$

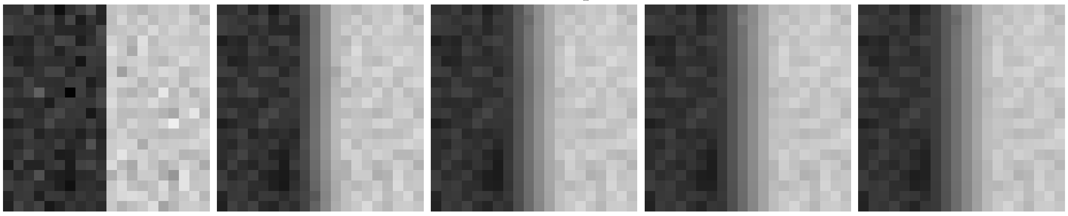
with Dirichlet boundary condition $f = f^*$ on the boundary of Ω (where $\triangle_\infty$ denotes the infinity Laplacian). In [16], Oberman proposes a well-posed convergent difference scheme for the infinity Laplacian equation. The author considers the function $h(x, y) = |x|^{4/3} - |y|^{4/3}$ proposed by Aronsson (see references therein), which is a an example of function absolutely minimizing (i.e., minimizing the L^∞ norm on every open, bounded subset of Ω) but not twice differentiable. The author evaluates his scheme by estimating the difference between h and the solution estimated by (49) with $f^* = h$. We verified that augmenting p make the solution be closer to the solution of the infinity Laplacian which was expected, and that augmenting n also gives a better approximation. This can be related to the approximation given by Oberman [16] where the author proves the convergence of his scheme when in fact the sampling becomes more precise. ■



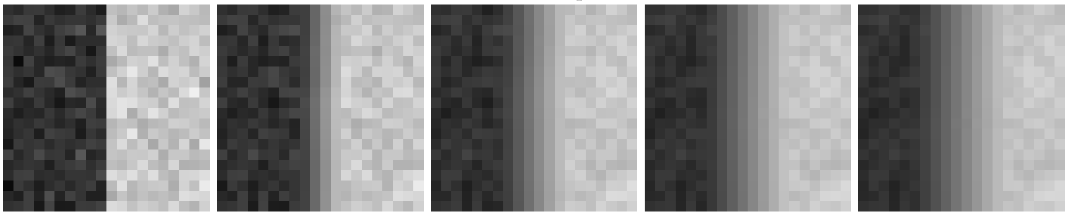
Evolution for $p = 1$



Evolution for $p = 2$



Evolution for $p = 20$



Evolution for $p = 40$

Figure 3: Example of evolutions with various values of p applied to a synthetic noisy images.

original noisy restored ($p = 1$)

Figure 4: Result on a real image.

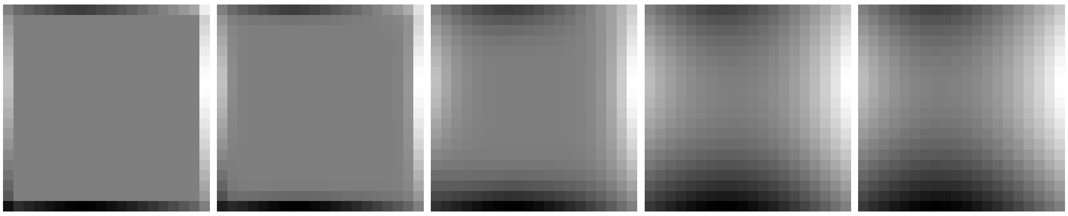


Figure 5: Example of inpainting with p-Laplacians with function $h(x,y) = |x|^{4/3} - |y|^{4/3}$ at the boundaries (see remark).

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