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Abstract. We give a combinatorial formula for the inverses of the alternating sums of free quasi-symmetric functions of the form $\mathbf{F}_{\omega(I)}$ where I runs over compositions with parts in a prescribed set C. This proves in particular three special cases (no restriction, even parts, and all parts equal to 2) which were conjectured by B. C. V. Ung in [Proc. FPSAC'98, Toronto].

1. Introduction

The algebra of Free Quasi-Symmetric Functions **FQSym** [5] is a graded algebra of noncommutative polynomials whose bases are parametrized by permutations. Under commutative image, it is mapped onto Gessel's algebra of quasi-symmetric functions, whence its name.

Quasi-symmetric functions generalize symmetric functions in a natural way, and many classical results admit quasi-symmetric extensions or analogs. However, very few results resembling symmetric series identities, like those of Schur or Littlewood (see, e.g., [11]) are known. In [15], B. C. V. Ung proves a quasi-symmetric analog of Schur's identity, and conjectures three further combinatorial inversions of quasi-symmetric series, which are even stated at the level of **FQSym**.

In this note, we prove a master identity, which consists in a combinatorial formula for the inverses of the alternating sums of free quasi-symmetric functions of the form $\mathbf{F}_{\omega(I)}$ where I runs over compositions with parts in a prescribed set C. Here $\mathbf{F}_\sigma$ denotes the standard basis of **FQSym** (mapped onto Gessel's fundamental basis), and $\omega(I)$ is the longest permutation with descent composition I. Ung's conjectures boil down to the following special cases : no restriction on the parts, even parts, and all parts equal to 2.

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2. Background and notations

2.1. **Free quasi-symmetric functions.** Let A be a totally ordered alphabet. Recall that the standardized $\mathrm{Std}(w)$ of a word $w \in A^*$ is the permutation obtained by iteratively scanning w from left to right, and labelling $1, 2, \ldots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. Alternatively, $\sigma = \mathrm{Std}(w)^{-1}$ can be characterized as the unique permutation of minimal length such that $w\sigma$ is a nondecreasing word. For example, $\mathrm{Std}(bbacab) = 341625$.

An elementary observation, which is at the basis of the constructions of [5], is that the noncommutative polynomials

$$\mathbf{G}_\sigma(A) = \sum_{w\in A^*;\mathrm{Std}(w)=\sigma} w \tag{1}$$

span a subalgebra of $\mathbb{Q}\langle A\rangle$. When A is infinite, this subalgebra admits a natural Hopf algebra structure, but this fact will not be needed here. This is **FQSym**, the algebra of *Free Quasi-Symmetric Functions.*

Let $\mathbf{F}_\sigma = \mathbf{G}_{\sigma^{-1}}$. The scalar product is defined by

$$\langle \mathbf{F}_\sigma\, ,\, \mathbf{G}_\tau \rangle = \delta_{\sigma,\tau}\,. \tag{2}$$

For a word w on the alphabet $\{1, 2, \ldots\}$, denote by $w[k]$ the word obtained by replacing each letter i by the integer $i + k$. If u and v are two words, with u of length k, one defines the *shifted concatenation*

$$u \bullet v = u \cdot (v[k]) \tag{3}$$

and the *shifted shuffle*

$$u \Cup v = u \sqcup\!\sqcup (v[k])\,. \tag{4}$$

where $u \sqcup\!\sqcup u'$ is the usual shuffle product on words.

The product formula in the $\mathbf{F}$ basis is

$$\mathbf{F}_\alpha \mathbf{F}_\beta = \sum_{\gamma\,\in\,\alpha \Cup \beta} \mathbf{F}_\gamma\,. \tag{5}$$

The sum of the inverses of the permutations occuring in $\alpha^{-1} \Cup \beta^{-1}$ is called *convolution* and denoted by $\alpha \star \beta$. Thus, **FQSym** provides a realization of the convolution algebra of permutations studied in [13, 12].

2.2. **Descent classes and compositions.** Recall that the descent set of a permutation σ is $D = \{i \,|\, \sigma(i) > \sigma(i+1)\}$. If $\sigma \in \mathfrak{S}_n$ has descent set $D = \{d_1 < \ldots < d_k\} \subseteq [n-1]$, the *descent composition* $I = C(\sigma)$ is the composition $I = (i_1, \ldots, i_{k+1})$ of n defined by $i_s = d_s - d_{s-1}$, where $d_0 = 0$ and $d_{k+1} = n$. The symbol $I \vDash n$ means that I is a composition of n, and $l(I)$ denotes the length of I.

The descent class $D_I = \{\sigma \in \mathfrak{S}_n \,|\, C(\sigma) = I\}$ has a unique element of minimal (resp. maximal) length denoted by $\alpha(I)$ (resp. $\omega(I)$). Actually, descent classes are intervals $D_I = [\alpha(I), \omega(I)]$ for the left weak order on $\mathfrak{S}_n$ (see, *e.g.*, [3]).

The mirror image of a word $w = a_1 a_2 \cdots a_m$ is $\bar{w} = a_m a_{m-1} \cdots a_1$. We shall use this notation for compositions and permutations as well.

Finally, the diameter of a descent class is the permutation

$$\mathrm{diam}(I) := \alpha(I)\omega(I)^{-1} = \alpha(I)\omega(\overline{I}). \tag{6}$$

2.3. **A multiplicative basis of FQSym.** The *left-shifted concatenation* of words is

$$u \blacktriangleright v = u[l] \cdot v \quad \text{if } u \in A^k,\ v \in A^l\,, \tag{7}$$

similar to the usual shifted concatenation $\bullet$, but with the shift on the first factor. The following basis is introduced in [4]:

$$\mathbf{S}^\sigma := \sum_{\tau \leq \sigma} \mathbf{G}_\tau \tag{8}$$

where $\leq$ is the left weak order. It has the property

$$\mathbf{S}^\sigma = \mathbf{S}^{\sigma_1}\mathbf{S}^{\sigma_2}\cdots\mathbf{S}^{\sigma_r} \tag{9}$$

whenever $\sigma = \sigma_1 \blacktriangleright \sigma_2 \blacktriangleright \cdots \blacktriangleright \sigma_r$.

The Moebius function of the left weak order is explicitely known [6, 2, 3], and gives in particular

$$\mathbf{G}_\sigma = \sum_{I \preceq C(\sigma^{-1})} (-1)^{l(I)-1} \mathbf{S}^{\alpha(I)\sigma}. \tag{10}$$

3. The main result

3.1. **Ung's conjectures.** In [15], Ung made the following conjectures. The inverses of the series

$$\begin{aligned} H_1 &= \sum_I (-1)^{\ell(I)} \mathbf{F}_{\omega(I)} \\ H_2 &= \sum_{n\geq 0} (-1)^n \mathbf{F}_{\omega(2^n)} \\ H_3 &= \sum_I (-1)^{\ell(I)} \mathbf{F}_{\omega(2I)} \end{aligned}$$

are as follows. For a permutation σ of shape I, let $\hat{\sigma} = \sigma\omega(I)^{-1}$. Then,

$$\begin{aligned} H_1^{-1} &= \sum_\alpha \mathbf{G}_{\hat{\alpha}} \\ H_2^{-1} &= \sum_\beta \mathbf{G}_{\hat{\beta}} \\ H_3^{-1} &= \sum_\gamma \mathbf{G}_{\hat{\gamma}} \end{aligned}$$

where α runs over all permutations, $\beta \in \mathfrak{S}_{2p}$ runs over permutations of shape 2^{2p}, and $\gamma \in \mathfrak{S}_{2p}$ runs over permutations with descent set contained in $\{2, 4, \ldots, 2p-2\}$.

Taking into account (8) and (6), we see that all three identities are of the form (11) below, with $E = \mathbb{N}^*$, $\{2\}$ and $2\,\mathbb{N}^*$, respectively.

3.2. Generalization.

Theorem 3.1. *Let E be any subset of $\mathbb{N}^*$. And let $C(E)$ be the set of all compositions with parts in this subset. Then*

$$\left(\sum_{I\in C(E)}(-1)^{l(I)}\mathbf{G}_{\omega(I)}\right)^{-1} = \sum_{K\in C(E)}\mathbf{S}^{\mathrm{diam}(K)}. \tag{11}$$

Proof – Thanks to (10), the statement to be proved is equivalent to

$$\left(\sum_{I\in C(E)}(-1)^{l(I)}\sum_{J\preceq\overline{I}}(-1)^{l(J)-1}\mathbf{S}^{\alpha(J)\omega(I)}\right)\left(\sum_{K\in C(E)}\mathbf{S}^{\alpha(K)\omega(\overline{K})}\right) = 1, \tag{12}$$

or, opening the parentheses,

$$\sum_{I,K\in C(E)}\sum_{J\preceq\overline{I}}(-1)^{l(I)+l(J)-1}\mathbf{S}^{\alpha(J)\omega(I)}\mathbf{S}^{\alpha(K)\omega(\overline{K})} = 1. \tag{13}$$

Now,

$$\mathbf{S}^{\alpha(J)\omega(I)}\mathbf{S}^{\alpha(K)\omega(\overline{K})} = \mathbf{S}^{\alpha(J')\omega(I')}, \tag{14}$$

where $I' = I\bullet\overline{K}$ and $J' = K\triangleright J$. Note that $J'\preceq\overline{I'}$ and that

$$(-1)^{l(I)+l(J)-1} = -(-1)^{(l(I')+l(J')-1}. \tag{15}$$

Now, given any non-empty permutation σ obtained as a product $\alpha(J)\omega(I)$ with $J\preceq\overline{I}$, it can be decomposed in exactly two ways as a product $\alpha(J)\omega(I)\blacktriangleright\alpha(K)\omega(\overline{K})$: either with $K=\emptyset$ or with $\alpha(K)\omega(\overline{K})$ corresponding to the last anticonnected permutation associated with the decomposition of σ into anticonnected permutations. This comes from the fact that $\alpha(J)\omega(I)$ (with $J\preceq\overline{I}$) is anticonnected iff $J=\overline{I}$.

Since the coefficients associated with these two decompositions are opposite, such a permutation does not occur in the final result. Hence the result reduces to the contribution of the empty permutation. ∎

4. Comments on Ung's other identities

In [15], Ung proves quasi-symmetric analogs of Schur's identity (for the sum of all Schur functions) and of Littlewood's identity (for its inverse). In fact, these analogs may be formulated without further work at the level of **FQSym**.

The first identity is

$$\sum_I F_I = \frac{1}{2}\left[\prod_i\frac{1+x_i}{1-x_i}-1\right] = \frac{1}{2}[\lambda_1(X)\sigma_1(X)-1] \tag{16}$$

where λ_1 (resp. σ_1) is the sum of the elementary (resp. complete) symmetric functions. Interpreting the right-hand side in the algebra of noncommutative symmetric functions, we have

$$\frac{1}{2}[\lambda_1(A)\sigma_1(A)-1] = \frac{1}{2}\left[\overleftarrow{\prod_i}(1+a_i)\overrightarrow{\prod_i}(1-a_i)^{-1}-1\right] = \sum_{n\geq 0} H_n \tag{17}$$

where

$$H_n = \sum_{k=0}^{n-1} R_{1^k,n-k}\,. \tag{18}$$

The commutative image of $R_{1^k,n-k}$ is the Schur function $s_{n-k,1^k}$, whose quasi-symmetric expansion is easily found to be

$$s_{n-k,1^k} = \sum_{I\vdash n,\ l(I)=k+1} F_I\,. \tag{19}$$

But $R_{1^k,n-k}$ can also be interpreted as an element of **FQSym**,

$$R_{1^k,n-k} = \sum_{C(\sigma^{-1})=(1^k,n-k)} \mathbf{F}_\sigma \tag{20}$$

so that (16) means that each descent class contains exactly one permutation whose inverse has a hook shape $(1^k, n-k)$.

The second identity is

$$\left(\sum_I F_I\right)^{-1} = 1 + \sum_{I\vDash 2n+1} (-1)^{n+1} c_I F_I \tag{21}$$

where c_I is the number of permutations of shape I whose inverse has shape (12^n). This formula is obtained by observing that the inverse of $H=\sum_n H_n$ is the noncommutative hyperbolic tangent of [7], that is

$$H^{-1} = 1 - \sum_{n\geq 0}(-1)^n R_{12^n}\,, \tag{22}$$

which can again be interpreted as an identity in **FQSym**

$$H^{-1} = 1 + \sum_{n\geq 0}(-1)^n \sum_{C(\sigma^{-1})=(12^n)} \mathbf{F}_\sigma\,. \tag{23}$$

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