w-Cycles in Surface Groups

J. I. MacColl

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Department of Mathematics
University College London
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I, Jo MacColl, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

For w an element in the fundamental group of a closed, orientable, hyperbolic surface Ω which is not a proper power, and Σ a surface immersing in Ω, we show that the number of distinct lifts of w to Σ is bounded above by $-\chi(\Sigma)$. In special cases which can be characterised by interdependencies of the lifts of w, we find a stronger bound, whereby the total degree of covering from curves in Σ representing the lifts to the curve representing w is also bounded above by $-\chi(\Sigma)$. This is achieved by a method we introduce for decomposing surfaces into pieces that behave similarly to graphs, and using them to estimate Euler characteristics using a stacking argument of the kind introduced by Louder and Wilton.

We then consider some consequences of these bounds for quotients of orientable surface groups by a single element. We demonstrate ways in which these groups behave analogously to one-relator groups; in particular, the ones with torsion are coherent (i.e. all finitely-generated subgroups have finite presentations), and those without torsion possess the related property of non-positive immersions as introduced by Wise.

Impact Statement

The results presented here are, at a base level, further exemplification of how those features of objects that we find visually pleasing or interesting can be given precise meaning and utilised. Surfaces are easy to conceptualise as they are described by visual features like genus and boundary components - here we discuss algebraic systems associated with surfaces, and how these exact features manifest as controls on those systems, showing that they obey specific and natural laws and prevent pathological behaviour from presenting in them. Having greatly enjoyed opportunities to engage with the public in demonstrating how fundamental concepts in group theory such as commutators can be realised in intuitive physical problems throughout my studies, it is my hope that an impact of this work can be to similarly aid in education of mathematics as a way to precisely express a natural, visual, concept. The method of stackings which we extend to surfaces here, as a geometric way to express how groups can be ordered, and to reduce the complexity of geometric problems to just the pieces of a stacking that can be seen from above or below, provides a good example of how a physically realisable system can impart an understanding of abstract ideas.

More specifically within research on geometric group theory, the tool of a rectangular decomposition developed here seems to be new, and a potentially useful way to reduce problems involving mappings of surfaces with boundary to those in which simpler graph-based arguments can be applied. Here for example they are used to show how dependencies in systems describing the adjunction of roots to subgroups of surface groups manifest in essentially the same way as they do for free groups. Coherence, which is proven here for one-relator surface groups with torsion, is also a useful property from an algorithmic perspective, implying finite termination of combinatorial algorithms to list all possible subgroups generated by any given finite set of group elements.

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Contents

1 Introduction

The aim of this thesis can broadly be stated as taking some results and methods in topological graph theory that have been developed over the last decade and showing that they also hold for closed orientable surfaces. Going from 1-dimensional to 2-dimensional objects is a natural direction of generalisation, and while graphs and free groups are comparatively well-understood, 2-complexes and their fundamental groups comprise too wide a class of objects for carrying aspects of graph theory over to all of them to be a realistic goal in general. As manifolds, surfaces inherently have nicer properties than generic 2-complexes, so it is more reasonable to make such an attempt. A key part of the method we use to prove our main theorem is to make use of the constraints on how 2-cells making up a surface can connect with each other in order to exhibit certain "graph-like" behaviours.

Beyond this practical reason for focussing on them, surfaces are of course interesting in their own right, having played important roles in the development of subjects across almost the entire range of pure mathematics. Even just focussing on the area of geometric group theory they were foundational, providing some of the first motivating examples of one-relator groups, hyperbolic groups abstracting the notion of negative curvature from classical geometry, and small-cancellation theory, illustrating the power of such abstractions in revealing behaviours of groups which would otherwise not be at all obvious from their concise combinatorial presentations [23, 55, 76, 62]. Furthermore, studying the action of a hyperbolic surface's mapping class group on its Teichmüller space of metrics — aside from its huge impact in understanding the geometry and topology of 3-manifolds — has helped provide a framework for the exchange of topological and algebraic information that arises in more abstract examples of groups acting on spaces, notably in the development of Culler-Vogtmann's Outer space for studying automorphisms of free groups, and later on automorphisms of other groups which have become important, such as right-angled Artin groups as a recent example [22, 60].

As influential as surfaces have been in the development of modern geometric group theory, there are still a variety of open questions relating to their fundamental groups and subgroups. It is interesting to attempt to answer these questions for their own sake, surfaces being such tangible objects that they are attractive to investigate. But it is also interesting for what deeper knowledge about the subgroup structure of their fundamental groups could possibly imply for any of those myriad research directions that grew out of the earlier geometric and topological studies of surfaces.

A lot of research into aspects of combinatorial group theory can be framed in relation to the classical decision problems formulated by Dehn [23] — the word, conjugacy and isomorphism problems — and related problems that have come from them, such as the generalised word problem or the subgroup membership problem. Although decidability is not much of a consideration in the work here, we observe that our main result can be viewed in terms of a quantification of the subgroup membership problem. We will give the precise meaning of the terms below, but essentially a "w-cycles theorem" gives a bound on how many ways a power of a group element w can appear in a given finitely-generated subgroup, purely dependent on the subgroup rank and not on w itself. The w-cycles theorem in free groups (Theorem 3.1) was proved in 2014 by Louder and Wilton [52], and independently Helfer and Wise [35], and our Theorem A gives the analogous result for surface groups.

A driving heuristic for the geometric approach to group theory has been to explain solutions to combinatorial problems for certain classes of groups as consequences of geometric and topological features of spaces that are intrinsically linked to those classes. The archetypal example of this approach comes from Gromov's work setting out a programme of research into hyperbolic groups [32], for which the understanding obtained by seeing groups as negatively-curved spaces has yielded solutions to Dehn's original three problems and far more. It was in an effort to find a similar geometric theory for the class of one-relator groups that Louder and Wilton have recently developed the idea of "2-complexes with negative immersions" [54], building on Wise's idea of non-positive immersions (Definition 1.4).

One of the more satisfying aspects of the methods used here is in the demonstration of Louder and Wilton's idea of stackings being applicable to surfaces. When they used graph stackings to prove the original w-cycles theorem, it was the starting point for a thread of ideas that has revealed a great deal more about subgroups of free groups and their one-relator quotients, with strong connections to the question of hyperbolicity of the latter class of groups. Although we do not yet have an analogue of the constructions related to negative immersions for surfaces, it is still interesting to note that one of the core methods, estimating Euler characteristics using height data from stackings, can also be made to work for surfaces. We obtain the beginnings of a geometric theory of one-relator quotients of surface groups, as captured by the properties of non-positive immersions and coherence.

1.1 Statement of Results

Here we state our main findings, together with the basic definitions required to state them. We give our central definition and Theorem A in the topological language that our proofs will go on to use, and defer further definitions and discussion of the context of these concepts to Part I.

Definition 1.1. Call a topological space *circular* if it admits a deformation retract onto S^1.
Let

$$f : Y \looparrowright X$$

be an immersion of compact spaces, and

$$w : S \looparrowright X$$

an immersion of a circular space into X. Let P be the pullback of Y and S along the maps f and w, and let $\mathbb{S}$ denote the collection of circular components of P. We call any component of $\mathbb{S}$ a *w-cycle in Y*.
Finally, call an immersion $A \looparrowright B$ of topological spaces *reducible* if B contains an interior point with at most one preimage in A.

Theorem A. *Let Ω be a closed connected orientable surface such that $\chi(\Omega) < 0$ (where χ denotes Euler characteristic). Fixing a choice of hyperbolic metric on Ω, let*

$$h : \Sigma \looparrowright \Omega$$

be a boundary-essential immersion of a compact connected surface into Ω such that the curves $h(\partial\Sigma)$ are geodesics in Ω. Let

$$w : S \looparrowright \Omega$$

be a covering map from an annulus S which does not factor properly through any other immersion of an annulus into Ω and with a core curve which is also sent to a geodesic in Ω. Let $\mathbb{S}$ be the set of w-cycles in Σ associated to the immersions w, f. Then the number of w-cycles in Σ is at most $-\chi(\Sigma)$ if the induced immersion $\mathbb{S} \looparrowright \Sigma$ is irreducible, and at most $1 - \chi(\Sigma)$ otherwise.

Our definition of w-cycle in Definition 1.1 is a generalised version of the one Wise gave for graphs in [79], which was motivated by his study of a property of groups which has yielded to investigation by topological methods in recent decades:

Definition 1.2. A group is *coherent* if all of its finitely-generated subgroups have finite presentations.

Finding a bound on the number of w-cycles in free groups was an attempt to show that one-relator groups are coherent, answering a question of Baumslag [5]. From the w-cycles theorem for graphs, Louder-Wilton [53] and independently Wise [80] were able to show that one-relator surface groups that contain torsion elements are coherent. Our work regarding w-cycles in surfaces gives new information on an analogous class of groups:

Definition 1.3. Let Ω be a closed connected surface, and g an element of its fundamental group. Denote by $\langle\langle g \rangle\rangle$ the normal closure of g in $\pi_1(\Omega)$. Then the quotient

$$\pi_1(\Omega)/\langle\langle g \rangle\rangle$$

is called a *one-relator surface group*. We may call it a *(non-)orientable one-relator surface group* reflecting the (non-)orientability of Ω.

These are of course two-relator groups, but, due to the special properties surfaces enjoy compared to generic 2-complexes, they warrant separate discussion. Various authors, in particular Hempel and Howie, have shown that the theory of one-relator surface groups mirrors that of one-relator groups in many fundamental ways [37, 41]. Our next result shows that this is also true of coherence:

Theorem B. *Orientable one-relator surface groups with torsion are coherent.*

Finally, we define another property introduced by Wise in relation to his investigation of coherence of one-relator groups:

Definition 1.4. A connected 2-complex X has *non-positive immersions* if, for any compact connected 2-complex Y immersing into X, either $\chi(Y) \leq 0$, or $\pi_1(Y)$ is trivial. We also say that a group has non-positive immersions if it has a presentation 2-complex with non-positive immersions.

The above theorems will imply:

Corollary C. *Every orientable one-relator surface group is either coherent, or has non-positive immersions.*

These two properties should not be viewed as a dichotomy but rather as part of a larger picture of the structure of one-relator surface groups — Wise conjectured that groups with non-positive immersions are coherent, with a recent and extensive list of evidence connecting these properties given in [81]. The idea of a geometric structure provided by non-positive immersions seems to give a great deal of insight into subgroup structure, with immersions describing how complexes representing subgroups of the fundamental group are able to sit inside the whole complex. From this point of view, information that non-positive immersions gives towards coherence is not surprising, given that it makes a statement about all finitely-generated subgroups.

Part I

Background

2 Foundations

In this section we will discuss some of the general concepts in geometric group theory that the ideas surrounding w-cycles grew out of — we will discuss the aspects more specific to w-cycles in § 3, but also try to stick to concepts and examples motivated by them here. We want to emphasise some of the techniques that have developed in the last 50 years to express group-theoretic ideas using topology, to motivate the constructions we will go on to make in Part II. We start by recalling some basic definitions and constructions that will be used ubiquitously in the later sections.

A 1-dimensional cell complex Γ is a graph, which we usually think of as being made up of a set of vertices, $V(\Gamma)$, and directed edges $E(\Gamma)$, with incidence functions ι, τ mapping edges to their initial and terminal vertices. The fundamental group of a connected graph Γ is a free group whose rank is given by

$$\mathrm{rk}\left(\pi_1(\Gamma)\right) = 1 - \chi(\Gamma)$$

where χ denotes the Euler characteristic, given for a general cell complex by the alternating sum of cardinalities of its sets of cells in each dimension, in this case

$$\chi(\Gamma) = |V(\Gamma)| - |E(\Gamma)|$$

An *immersion* f of topological spaces is a locally-injective continuous map, denoted $f : Y \looparrowright X$. If f is a combinatorial map of cell complexes (meaning it maps n-cells of Y homeomorphically to n-cells of X), this means it is an immersion if it is injective on links. Similarly, an immersion of manifolds is a smooth map which is injective on the tangent spaces at each point. The immersion f is called *essential* if it induces an injective homomorphism on fundamental groups.

Given a pair of immersions with common target $f_1 : Y_1 \looparrowright X$, $f_2 : Y_2 \looparrowright X$, the *pullback* of f_1 and f_2, which we denote $Y_1 \times_X Y_2$, is the universal object which forms the following commutative diagram:

$$\begin{tikzcd}
Y_1 \times_X Y_2 \arrow[r, "p_1"] \arrow[d, "p_2"] & Y_1 \arrow[d, "f_1"] \\
Y_2 \arrow[r, "f_2"] & X
\end{tikzcd}$$

and through which any pair of immersions from another space into Y_1 and Y_2 factors. For the spaces we consider, pullbacks of immersions can be given explicit topological structure as fibre products:

$$Y_1 \times_X Y_2 = \{(y_1, y_2) \in Y_1 \times Y_2 \mid f_1(y_1) = f_2(y_2) \in X\}$$

and the immersions p_1, p_2 from the commutative diagram are induced by the natural projection maps from $Y_1 \times Y_2$ to Y_1, Y_2.

Surfaces are smooth 2-dimensional manifolds, possibly with boundary components consisting of a disjoint union of embedded circles. They can always be given the structure of a 2-dimensional cell complex, but in general we won't canonically associate them with a set of vertices, edges and faces as we would a graph. We call surfaces *closed* if they are compact and have no boundary components. An immersion of smooth surfaces can only fail to be essential if the surface being immersed has a boundary component that is mapped to the boundary of a closed disc in the target — to emphasise this, essential immersions of surfaces may be called *boundary-essential*.

The classification of closed surfaces tells us that every closed surface can be obtained from the 2-sphere S^2 by taking connected sums with copies of the 2-torus T^2 and the real projective plane P^2. A surface is non-orientable if and only if it has at least one P^2 summand; we note that there is the standard relation for surfaces with three or more P^2 summands:

$$P^2 \# P^2 \# P^2 \simeq T^2 \# P^2$$

(where $\simeq$ denotes homeomorphism). Fixing a cell decomposition on any connected closed surface Σ, we can compute its Euler characteristic and categorise Σ as:

- *round*, when $\chi(\Sigma) > 0$, realised only by $\Sigma = S^2$ and P^2
- *flat*, when $\chi(\Sigma) = 0$, realised only by $\Sigma = T^2$ and $P^2 \# P^2 =: K$, the Klein bottle

- *hyperbolic*, when $\chi(\Sigma) < 0$, realised by all other connect-sums of T^2 and P^2

These names reflect the different types of curvatures that relate to the differential geometry of these surfaces, classically known to be constrained by Euler characteristic via the Gauss-Bonnet theorem. They also separate the complexity of the fundamental groups; in the round case:

$$\pi_1(S^2) = 1, \text{ the trivial group}, \qquad \pi_1(P^2) = \mathbb{Z}_2$$

in the flat case, we give the fundamental groups as combinatorial presentations in terms of generators and relations:

$$\pi_1(T^2) = \langle a, b \mid [a, b] \rangle \cong \mathbb{Z}^2, \qquad \pi_1(K) = \langle a, b \mid aba^{-1}b \rangle \cong \langle a', b' \mid (a')^2\,(b')^2 \rangle$$

where $\cong$ denotes group isomorphism, and $[a, b]$ denotes the commutator of a and b, that is, the element $aba^{-1}b^{-1}$. Fundamental groups of hyperbolic surfaces also have two different types of combinatorial presentation, according to whether or not they are orientable; for those that are orientable:

$$\pi_1\left(g(\#T^2)\right) = \left\langle a_1, b_1, \cdots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \right\rangle$$

and for non-orientable hyperbolic surfaces:

$$\pi_1\left(n(\#P^2)\right) = \left\langle a_1, \cdots, a_n \mid \prod_{i=1}^{n} a_i^2 \right\rangle$$

So almost all closed surfaces are hyperbolic, and in general when we talk about surface groups we are talking about groups of one of these latter two forms; where we want to be more specific, we may call them *orientable* and *non-orientable hyperbolic surface groups*, accordingly. For fundamental groups of surfaces with boundary, we can obtain presentations from those of the corresponding closed surfaces by adding a generator for each boundary component, and appending the product of these generators to the given relation. However, since surfaces with boundary retract onto graphs, we know that these give presentations of free groups, so we do not classify these as surface groups.

2.1 Stallings' Approach

Having introduced in § 1.1 the basic topological objects appearing in Theorem A, we now briefly discuss a "dictionary" relating them to fundamental concepts in group theory. In [73], Stallings laid out an approach for studying subgroups of free groups by looking at immersions of compact graphs, and it gave powerful ways to understand the theory of free groups. For instance, decidability of the subgroup membership problem, and Howson's theorem, that the intersection of two finitely-generated subgroups is again finitely-generated [42], followed immediately from Stallings' techniques and elementary topological arguments. His methods have informed the approach we take for studying w-cycles throughout this work, being heavily used also in the literature surrounding w-cycles and one-relator groups.

A starting point for these ideas is the familiar correspondence between subgroups of a space's fundamental group and coverings of that space. Coverings are local homeomorphisms, and so in particular are immersions. Generalising by dropping the requirement of local surjectivity, we can study more interesting ways that a group H can have a homomorphic image in a group G by specifying immersions $X_H \looparrowright X_G$, where X_H, X_G are appropriate spaces with fundamental groups H, G. Stallings showed that any map of compact graphs $\Gamma \to \Delta$ factors through an immersion $\Gamma' \looparrowright \Delta$, with Γ' obtained from Γ by a process of "folding edges" — that is, identifying edges e_1, $e_2 \in E(\Gamma)$ with the same image in Δ, and such that either $\iota(e_1) = \iota(e_2)$ or $\tau(e_1) = \tau(e_2)$ (or both). In this way, any finitely-generated subgroup of the free group $\pi_1(\Delta)$ can be represented by an immersion, simply by taking a wedge of circular graphs describing the paths of the subgroup generators in Δ and folding edges together. This represents the subgroup as the fundamental group of a graph whose local structure mimics that of Δ, therefore meaningfully showing how the subgroup sits inside $\pi_1(\Delta)$. This folding procedure has since been generalised to represent subgroups of fundamental groups of 2-complexes by immersions, see for instance [53, §4].

The next observation from Stallings' paper that we will make heavy use of is that topological pullbacks represent group intersections. More precisely, given two essential immersions of connected graphs $f_i : \Gamma_i \looparrowright \Delta$ $(i = 1, 2)$, then fixing a basepoint $b \in \Delta$ and basepoints $b_i \in f_i^{-1}(b)$, the connected component of the fibre product $\Gamma_1 \times_\Delta \Gamma_2$ containing the point represented by (b_1, b_2) will have fundamental group isomorphic to

$$f_{1*}\pi_1(\Gamma_1, b_1) \cap f_{2*}\pi_1(\Gamma_2, b_2) \leq \pi_1(\Delta, b)$$

since travelling round a loop in this component is the same as travelling round loops based at b_i in the Γ_i which project to the same loop in Δ. Making different choices of basepoints corresponds to taking conjugates — so, for instance, if we chose $b_1' \neq b_1 \in f_1^{-1}(b)$, there would be a path connecting b_1' to b_1 in Γ_1 which would map to a loop l based at b in Δ, and it follows that

$$\pi_1(\Gamma_1, b_1') = [l]\pi_1(\Gamma_1, b_1)[l]^{-1} \leq \pi_1(\Delta, b)$$

(where we use square brackets to denote the homotopy class of an immersed loop). But while these two subgroups are isomorphic, the pairs of paths based at b_1, b_2 in Γ_1, Γ_2 with the same image in Δ will generally be distinct from those pairs of paths based at b_1', b_2, and they will therefore correspond to a different connected component of $\Gamma_1 \times_\Delta \Gamma_2$. So, taking all the connected components of the pullback together gives the collection of conjugate intersections:

$$[l_1]\pi_1(\Gamma_1, b_1)[l_1]^{-1} \cap [l_2]\pi_1(\Gamma_2, b_2)[l_2]^{-1}, \qquad \forall [l_1],\ [l_2] \in \pi_1(\Delta, b)$$

which we can express more succinctly as

$$[l]\pi_1(\Gamma_1, b_1)[l]^{-1} \cap \pi_1(\Gamma_2, b_2), \qquad \forall [l] \in \pi_1(\Delta, b)$$

These observations can be extended to more complicated spaces than graphs, but some care must be taken to allow the statement that "pullbacks represent intersection" to still hold, since a single element of the fundamental group can have many different representatives in a space with more than one dimension. For a pair of connected surfaces Σ_1, Σ_2 equipped with immersions f_1, f_2 to a surface Ω equipped with a given metric, we can say that the pullback of the immersions has components whose fundamental groups are the intersection of $\pi_1(\Sigma_2)$ with various conjugates of $\pi_1(\Sigma_1)$, *provided* that the images of the f_i are geodesically convex with respect to the metric on Ω. This is the case in the hypotheses of Theorem A

So, if Definition 1.1 looked strange at first glance, we can now understand w-cycles for w an element of some group G as the generators of the intersection of a given subgroup of G with the cyclic subgroup $\langle w \rangle$. This will allow us to translate Theorem A into the group-theoretic statement Theorem A', after we discuss how the property of reducibility manifests in systems of equations on groups.

Remark 1. In the following sections we may interchangeably refer to w-cycles in both these topological and algebraic contexts. In our case of w-cycles in orientable surfaces, we fix the topological realisation of w to be an immersion from an open annulus which covers Ω as stated in Theorem A. The image of the core curve of this annulus generates $\langle w \rangle \leq \pi_1(\Omega)$, and fixing the whole covering annulus to take the pullback from allows these topological objects to be unambiguously identified with the algebraic analogues.

Example 2.1. Figure 1 shows a simple picture of the graphs associated to the rank 3 subgroup of F_2 generated by even length words, and indicates the paths traced in the folded graph for this subgroup by the w-cycles when $w = a^2 b$. The subgroup is index 2, and so in particular is normal, so Stallings' folding procedure (starting in the picture from the wedge of cycles representing the generating set $\{a^2, ab, ab^{-1}\}$) produces a regular degree 2 cover, but for general subgroups H we could only guarantee that we get a graph Γ_H immersing in Γ after folding.

For any index $n < \infty$ subgroup of $\pi_1(\Gamma, v)$, there will always be paths projecting to w between vertices of Γ_H in the preimage of v when Γ_H is constructed this way. As a result, it is easy to see that there can be at most

$$n \leq n \cdot (-\chi(\Gamma)) = -\chi(\Gamma_H)$$

w-cycles for such a subgroup ($\chi(\Gamma) < 0$ for any graph representing a non-trivial free group). This same reasoning can be applied to finite-index subgroups in any group which can be realised as the fundamental group of a cell complex of negative Euler characteristic (see Lemma 5.1), so in general we are interested in bounding the number of w-cycles in infinite-index subgroups, and will want to focus on the geometric characterisations of such subgroups.

2.2 Equations on Groups and Adjunction

The algebraic notion of a w-cycle that we gave above can be interpreted in terms of an older idea of solving systems of equations on groups, where essentially, we view the collection of conjugates of powers of some fixed $w \in G$ as defining

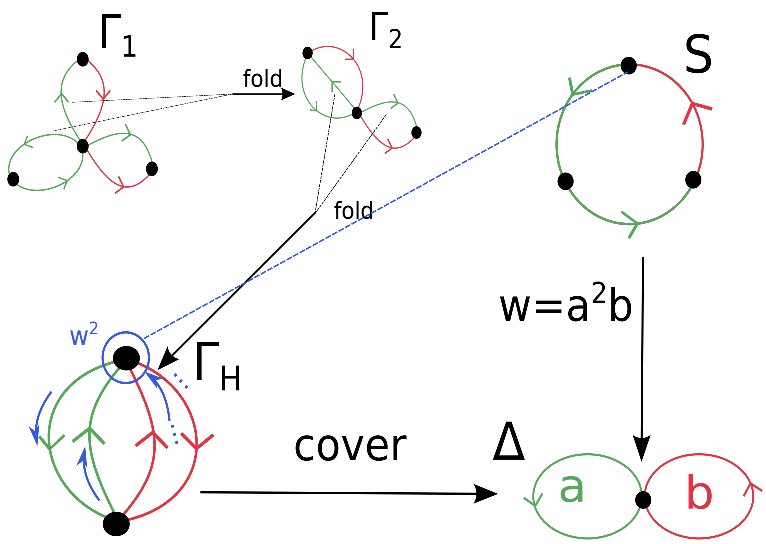

Figure 1: An illustration of Stallings' folding technique for the subgroup of even length words in the 2-generator free group $\pi_1(\Delta)$, and an immersion of a circle representing an element $w \in \pi_1(\Delta)$. The pullback of the pair of immersions into Δ consists of a circle double-covering S, determined by following whole copies of the edge sequence written around S, starting at either vertex of Γ_H (both giving the same w-cycle), until arriving back at the same vertex.

the form of solutions we would like to obtain, and seeking a generating system of solutions in the given subgroup H. This can also be viewed as a process of "adjoining roots" to elements of H (the w-cycles), a process whose natural topological realisation as a pushout we will touch upon here.

An equation on a group G is a declaration that G has elements satisfying some constraint on their combinatorial form; as Culler put it in [21], letting $\omega(x_1, \cdots, x_m)$ be a word in the free group F_m, the formal statement:

$$g = \omega$$

is an equation on G, for any $g \in G$. A solution to the equation is then any group homomorphism $\Phi : F_m \to G$ which maps ω to g. Many discoveries about properties of groups can be associated with attempts to show that generic systems of equations given in this way have solutions; for instance, this question of solvability motivated Howie's results about locally-indicable groups in [39], which lay the groundwork for the development of stackings.

Example 2.2. The Euclidean algorithm provides one of the earliest examples of solving equations on groups — in this case, free abelian groups. In a basic way, its underlying ideas relate to the process of finding w-cycles in free abelian groups. This is not a particularly interesting problem, but we will discuss it briefly here for completeness and illustrative purposes.

Start with the rank 1 case, $G := \mathbb{Z} \cong \langle t \rangle$. Then a subgroup H has the form $\langle s \rangle$ where $s = t^h$, and any $w \in G$ has the form $w = t^k$, for some $h, k \in \mathbb{Z}$. Existence of w-cycles in H is therefore equivalent to the existence of a pair of integers m and n such that:

$$w^m = s^n, \qquad \text{i.e. } mk = nh$$

We see that $\langle w \rangle \cap H = \langle t^l \rangle$ where l is the lowest common multiple of k and h. In other words, as long as neither w nor H are trivial, there is exactly 1=rk(H) w-cycle in H, namely t^l.

Now suppose $G = \mathbb{Z}^2 = \langle t \rangle \oplus \langle u \rangle$. Then we write

$$w = a \cdot t + b \cdot u$$

for some a, $b \in \mathbb{Z}$, and there are two possibilities for H — either it has the form

$$\langle j \cdot t + k \cdot u \rangle \qquad (\text{"line"})$$

for some non-zero vector $\overrightarrow{(j, k)}$ in $\mathbb{R}^2$, or

$$\langle j_1 \cdot t + k_1 \cdot u \rangle \oplus \langle j_2 \cdot t + k_2 \cdot u \rangle \qquad (\text{"lattice"})$$

for non-zero and linearly independent vectors $\overrightarrow{(j_1, k_1)}$, $\overrightarrow{(j_2, k_2)}$; see Figure 2. In the line case, it is easy to see that there are w-cycles in H if and only if the elements of $\langle w \rangle$ and H all lie on the same line in the (t, u)-plane, or equivalently, $\frac{a}{b} = \frac{j}{k}$. In this case, whenever there is a common multiple $am = jn$, we also have $bm = kn$, so we may choose m and n to produce simultaneously the lowest common multiples of (a, j) and (b, k) and thus obtain the single w-cycle

$$m(a \cdot t + b \cdot u) = n(j \cdot t + k \cdot u)$$

in H. Indeed this w-cycle must be the generator of the cyclic subgroup H. When the elements of H form a lattice in the plane, we find a w-cycle $m(a \cdot t + b \cdot u)$, by finding the lowest value integer m satisfying the linear system:

$$\begin{aligned} n_1 j_1 + n_2 j_2 &= ma \\ n_1 k_1 + n_2 k_2 &= mb \end{aligned}$$

for integers n_1, n_2. In both cases the number of w-cycles is bounded above by the rank of H. It is easy to generalise to arbitrarily high ranks of free abelian groups from this, but we can also simply observe that there is always at most one w-cycle in a subgroup of a free abelian group. Indeed, conjugating has no effect, and for any pair of integers m, n satisfying $m \cdot w$, $n \cdot w \in H$ we also have $\gcd(m, n) \cdot w \in H$.

We have included this special case of $\mathbb{Z}^2$, the fundamental group of the 2-torus, by way of the above example, as a first illustration of w-cycles removed from their original context of graphs. It provides the simplest example of an infinite closed surface group, and the only such example where all possible w-cycles

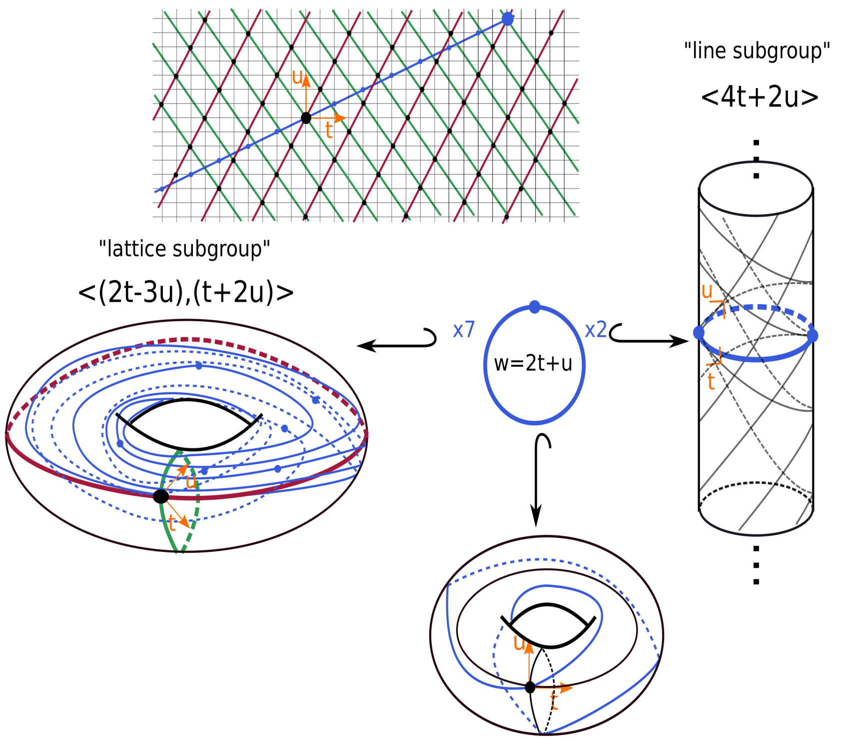

Figure 2: Visualising w-cycles for tori. At the top we have the universal cover, with the blue line a lift of w. It projects to the blue embedded curves in the intermediate covers, and these are core curves of the annular covers that we identify with w-cycles in surface groups (see Remark 1). For the “line subgroup” the w-cycle is exactly the intermediate cover shown; for the “lattice subgroup” it is immersed in the shown torus.

are easily visualised, as pictured in Figure 2. While the pictures associated to w-cycles in hyperbolic surface groups are not so different, their fundamental groups being neither free nor abelian makes finding a bound on the w-cycles a significantly different problem.

In [21], Culler observed that almost all successful attempts to solve equations in free groups up to that point were achieved for *quadratic* equations, i.e. those where the formal word $\omega \in F_m$ has each variable $x_1, \ldots, x_m$ appearing exactly twice. Standard forms for such words reduce them to products of commutators or products of squares, in other words, the relators for fundamental groups of orientable and non-orientable surfaces respectively. This motivated an approach to finding constraints on the number of solutions to quadratic equations in free groups by an analysis of which surfaces can continuously map into a graph, with the images of boundary components (which bound a disc attached along a quadratic path in the 1-skeleton of the surface) representing the solutions. This topological technique was powerful, and is still a core part of the modern study of commutator length in a range of groups, with Bavard drawing on it directly in the derivation of his famous duality theorem [7] (see also [14, 18]).

It also motivated Duncan-Howie in [25] to pose a decision problem, which they were able to show solvable for groups built from locally-indicable subgroups:

Question 2.3. *[25] Given a group G, an integer g and elements $w_1, \ldots, w_n \in G$, can we determine when the equation*

$$[x_1, y_1] \cdots [x_g, y_g] = \left(v_1 w_1 v_1^{-1}\right) \cdots \left(v_n w_n v_n^{-1}\right)$$

has a solution over G for some $v_1, \cdots, v_n \in G$?
Or, given a collection $[\gamma_1], \ldots, [\gamma_n]$ of based homotopy classes of loops in a connected topological space X, can the minimal genus g of a surface with n boundary components admitting a continuous map to X, such that the image of its boundary is in the homotopy class of $[\gamma_1] \cdots [\gamma_n] \in \pi_1(X)$, be determined algorithmically?

This *genus problem* immediately generalises two of the classical problems in combinatorial group theory:

- The *word problem* for a group G asks if we can determine whether a finite word w written in the generators of G is equal to the identity in G. This is equivalent to asking if a disc maps continuously to a presentation complex for G, with boundary circle mapping to the homotopy class of $w \in G$ — in other words, it asks if the case $g = 0$ in the genus problem for a single loop ($n = 1$) is solvable.

- The *conjugacy problem* asks whether two words w_1, w_2 are conjugate to each other in G. This will be true if loops representing the words cobound the image of an annulus in the presentation complex, so it is the genus problem with $g = 0$, $n = 2$.

It is worth observing then that attempts to study equations in surface groups would in a sense rule out these better-understood quadratic equations, since they are trivialised by the surface group relator. Elements satisfying a non-trivial quadratic relation in a surface group $\pi_1(\Sigma)$ will represent a generating set for the fundamental group of a closed surface which covers Σ regularly and with finite degree, and such a situation is generally easy to understand. Free subgroups of surface groups are nevertheless represented by surface mappings, and ideas that grew out of Culler's approach are still very relevant to our later sections.

When studying a system of equations over a group G given by words

$$\omega_1, \ldots, \omega_n \in F_m$$

a fundamental object is the universal solution group, given by the relative presentation:

$$G_{\underline{\omega}} = \frac{G * \langle x_1, \cdots, x_m \rangle}{\langle\langle \omega_1, \cdots, \omega_n \rangle\rangle}$$

There is a natural map induced by inclusion, $G \to G_{\underline{\omega}}$, and existence of solutions to the equations is then equivalent to injectivity of this map. When $n = 1$ and G is free, this relates to Magnus' famous Freiheitssatz [58] which states that any proper subset of the generators of a one-relator group generates a free subgroup. In [39, Theorem 4.3], a version of the Freiheitssatz for one-relator quotients of free products of locally-indicable groups is obtained by representing $G_{\underline{\omega}}$ as a 2-complex and investigating its topological structure using a tower argument of the kind we discuss in § 7.2.

These universal solution groups suggest a different type of question that we can ask, namely, when does a given group inject into a group obtained from it by adjoining new generators and relations? This is generally known as an *adjunction problem*, the exact nature of the problem depending on the form of the relators being adjoined — for our purposes, the problem of investigating w-cycles is linked with the problem of "adjoining roots" to subgroups. An early example of this kind of adjunction problem can be found in work of Lyndon, [57], who studied the equation

$$a^2b^2 = x^2$$

on the 2-generator free group $\langle a, b\rangle$ — he found that the only way the equation could hold for any a', b' in a free group is if a' and b' commute. As observed by Louder in [51], this means that the solution group

$$\frac{\langle a, b\rangle * \langle x\rangle}{\langle\langle a^2b^2x^{-2}\rangle\rangle}$$

of the above equation (the fundamental group of $3P^2$) in which a square root is adjoined to the element a^2b^2, cannot surject a non-cyclic free group, and in particular the two-generator group $\langle a, b\rangle$ does not inject.

The general problem of adjoining roots to elements in free groups can be described by a "graph of groups" construction. This assigns groups to the vertices of a graph, and to the edges isomorphic subgroups of their incident vertex groups, whose elements are then identified in the fundamental group of the graph of groups. So, if a vertex ν with cyclic group $\langle w\rangle$ is attached to a collection of vertices with groups H_i, the edge groups will be cyclic subgroups $\langle w^{n_{i,j}}\rangle$, and w is adjoined as the $n_{i,j}$-th root to collections of elements $v_{i,j} \in H_i$ (note that multiple edges can join ν to the same vertex, as long as none of the $v_{i,j}$ that it is adjoining roots to are conjugate). A recent result bounding the complexity of a system of adjoined roots in free groups is:

Theorem 2.4. *([54], Theorem 1.16) Let $H_1, \ldots, H_l$ be free groups, $\{\langle v_{i,j}\rangle\}$ a malnormal collection of non-trivial cyclic subgroups of H_i and $n_{i,j}$ positive integers. Let Δ be the associated graph of groups and let $f : \pi_1(\Delta) \to F$ be a surjective homomorphism to a free group with $f|_{H_i}$ injective for each i. Then*

$$rank(F) - 2 + \sum_{i,j} n_{i,j} \leq \sum_i (rank(H_i) - 1)$$

provided none of the H_i admits a free splitting with one of the $\langle v_{i,j} \rangle$ as a factor.

This was proved by studying an *adjunction space*, essentially the graph of groups viewed as a purely topological object. The idea of a "graph of spaces" replacing vertex and edge groups from a graph of groups with classifying spaces was not new, but Louder and Wilton studied its features as a 2-dimensional complex with what Wise calls a "VH-structure" [11] in more detail than a single graph decomposition shows. Taking graphs Γ_i that represent the collection of free groups H_i, we get a collection $\mathbb{S}$ of subdivided circles representing the $v_{i,j}$, immersing into Γ_i and also each covering a single circle S representing w with degree $n_{i,j}$. We obtain a 2-complex P by taking the product of $\mathbb{S}$ with an interval, and identifying the copies of $\mathbb{S}$ at either end with their images in the Γ_i and S. This P is the adjunction space, and for any free group which contains as subgroups the H_i with adjoined roots $v_{i,j} = w^{n_{i,j}}$, there is a corresponding graph Γ_Ω and an induced map $P \to \Gamma_\Omega$ extending the natural maps from P to $(\{\Gamma_i\}, S)$ and from $(\{\Gamma_i\}, S)$ to Γ_Ω to give the commutative diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & S \arrow[d, "w"] \\ \{\Gamma_i\} \arrow[r, "v_{i,j}"] & \Gamma_\Omega \end{tikzcd}$$

The collection $\mathbb{S}$ sits inside P as "horizontal slices", and if we compare this to the statement of our w-cycles Theorem A for surfaces, there is the dual situation, where, starting with *given* surface Ω and w, Σ with prescribed immersions to it, we realise $\mathbb{S}$ as the pullback of these data and bound the number of components of $\mathbb{S}$ by $-\chi(\Sigma)$. By realising the analogous objects for graphs as an adjunction space, Louder-Wilton were able to prove a more powerful result in the above theorem, effectively bounding the complexity of *any* graph Γ_Ω that can realise the adjunction arrangement by showing a dependency between the rank of $\pi_1(\Gamma_\Omega)$ and the sum of the ranks of the H_i and the $n_{i,j}$. Nevertheless, at the heart of the proof of this dependence theorem was a method to break the adjunction space down into simpler pieces and estimate the Euler characteristic of each using a slightly more sophisticated version of the stacking argument that we will use to prove Theorem A.

2.3 Independent Systems and Reducible Immersions

We now briefly focus on the notion of dependence for equations on groups, and how this manifests in the topological immersions we work with later. In

Theorem 2.4, the rank inequality held unless there was a free splitting of the subgroups H_i separating off the elements that were having roots adjoined. Such a splitting induces a free splitting of the universal solution group, in which case we say that the corresponding system of equations is independent. The idea of a "dependence theorem" is to find rank relationships on systems where it is not possible to split pieces off from each other.

We can compare with standard linear algebra where a rank-n vector space is generated by any system of n linearly independent vectors. In non-abelian groups relations between elements become much more complicated but the notion of independence as a quantifier for basis elements in some solution space still applies:

Definition 2.5. [54] Let G be a group and $X \subset G$ a set of elements generating a malnormal family of cyclic subgroups (that is, for each x, $y \in X$,

$$\langle x \rangle \cap \langle gyg^{-1} \rangle = \{1\}$$

unless $x = y$ and $g \in \langle x \rangle$). We say that X is *independent in* G if it contains an element x such that G admits a free splitting $G = G' * \langle x \rangle$, with each element of $X - \{x\}$ conjugate into G'. Otherwise we call X a *dependent* set in G. We may also refer to the collection of cyclic subgroups generated by the elements of X as dependent/independent as appropriate.

Since we represent subgroups as immersions of cell complexes it is convenient to have a characterisation of independence in topological terms. As a first example we consider the case of free groups represented by graphs.

Example 2.6. An element in a free group G can be represented by an immersion of a circle into a connected graph, $w : S \looparrowright \Gamma$. The element w represents a basis element in a generating set of G if Γ contains an edge e traversed exactly once by the image $w(S)$. Indeed, in this case the image of w contains a simple closed curve (which the once-covered edge e lies on) which can be identified with a generator of $\pi_1(\Omega) = G$, and there is an automorphism of G sending w to this generator. This can also be seen by Whitehead's algorithm [77]; fix a basis $x_1, \cdots, x_n$ of G represented by loops $\gamma_1, \cdots, \gamma_n$ in Γ. Expressing w as a product of these elements and their inverses, if e is covered only once in its image then e is contained in exactly one of the γ_i (as long as each x_i appears in w — otherwise w already visibly lies in a free factor of G to which we can restrict our attention), and the corresponding x_i appears exactly once in the basis' spelling of w. It follows that the Whitehead graph for the word w can be disconnected by removing the vertex labelled either by x_i or x_i^{-1}, as each of them contain only one edge connecting them to the rest of the graph.

More generally, a collection $\{w_1, \cdots, w_m\}$ is independent in G if Γ contains an edge traversed once by the union of the images of their associated immersions $w_i : S_i \looparrowright \Gamma$. In [52], the immersion of the union of circles is called *reducible* in this case.

Our definition of reducibility (Definition 1.1) is a generalisation of the one for graphs, applicable to our study of immersions of surfaces. As we will see in Proposition 7.7, when the map w' specifying w-cycles in Σ is an irreducible immersion of the components of $\mathbb{S}$, we get a stronger bound than $-\chi(\Sigma)$ as given by Theorem A. This is because the w-cycles correspond to a dependent system of adjunction equations in this case, and so we get a bound more in line with (although not as general as) Theorem 2.4. On the other hand, if we were to take a free subgroup of $\pi_1(\Omega)$ that is generated by a finite set of w-cycles, then the number of w-cycles is exactly the subgroup rank, in other words $1 - \chi(\Sigma)$ if $\Sigma \looparrowright \Omega$ is an immersion representing the subgroup. While Proposition 7.7 gives a more powerful bound, the most general form of our w-cycles theorem finds a bound on the number of w-cycles when this situation of complete independence is not realised; we can now state it using purely algebraic language. We let $\bar{r}(H)$ denote the reduced rank:

$$\bar{r}(H) := \max\{1, (\mathrm{rank}(H) - 1)\}$$

Theorem A'. *Let w be an element of an orientable hyperbolic surface group G such that $w \neq u^p$ for any $u \in G$, $p > 1$, let $H \leq G$ be a finitely-generated subgroup, and let $x_1, \cdots, x_k$ be a maximal set of representatives for the distinct double cosets in $H \backslash G / \langle w \rangle$ that satisfy*

$$H \cap x_i \langle w \rangle x_i^{-1} \neq \{1\}$$

Then, either $k \leq \bar{r}(H)$, or the x_i are independent in H.

3 w-Cycles

In this section we give background information on some aspects of geometric group theory and topology more closely linked to w-cycles. We aim to summarise some of the factors that motivated Wise to conjecture the following, original, w-cycles theorem for graphs, and some of the ideas behind its proof that we will later adapt to the setting of surfaces.

Theorem 3.1. *[52, 35] Let $f : \Gamma' \looparrowright \Gamma$ be an immersion of connected compact core graphs, and $w : S \looparrowright \Gamma$ an immersion of a circular graph which does not properly factor through any other map from a circle to Γ. Then the number of w-cycles in Γ' is at most $1 - \chi(\Gamma')$.*

Denoting the set of w-cycles in Γ' by $\mathbb{S}$, since this is a collection of circles, there is an induced covering map $\sigma : \mathbb{S} \to S$ of finite degree, as well as an immersion $w' : \mathbb{S} \looparrowright \Gamma'$. By [52, Theorem 1.2], when w' is irreducible, it is not only the cardinality of $\mathbb{S}$ but in fact the degree of σ which is bounded, by $-\chi(\Gamma')$. This bound on the covering degree from the w-cycles to the original representative of w was found by a reinterpretation of the following theorem of Duncan-Howie, who formulated it as an isoperimetric inequality that could be applied to solve instances of their genus problem (Question 2.3):

Theorem 3.2. *[25, Theorem 3.3] Let $h : \Sigma' \to \Sigma$ be an immersion of compact orientable surfaces, each with non-empty boundary, such that every component of $\partial\Sigma'$ maps to the homotopy class of some power of $w \in \pi_1(\Sigma)$, and let n denote the sum of these powers over all of $\partial\Sigma'$. Then*

$$n \leq -\chi(\Sigma')$$

Since the target surface Σ has boundary, this is a statement about free groups, which Louder-Wilton effectively translated to the language of graphs. The original version of Theorem 3.2 is actually more general, where the surfaces are decorated with some additional information letting them represent amalgamated products of *locally-indicable* groups — these are groups for which every non-trivial finitely-generated subgroup admits a homomorphism onto $\mathbb{Z}$. Free groups are locally-indicable, and the topological implications of local-indicability are crucially important to the proofs of Theorems 3.1 and 3.2 as we will describe later in this section. First, we will expand on our Definitions 1.2, 1.4, and why they motivated Theorem 3.1.

3.1 Coherence and Non-Positive Immersions

A one-relator group is realised topologically as a graph Γ with a single 2-cell attached along an immersion representing an element $w \in \pi_1(\Gamma)$. Subgroups of $\pi_1(\Gamma)/\langle\langle w \rangle\rangle$ are then represented by graphs immersing into Γ with 2-cells attached along w-cycles. Wise first introduced w-cycles in [79] as part of an ongoing effort to answer a question of Baumslag from the 1970s:

Question 3.3. *[5] Are all one-relator groups coherent?*

So far, there is no single geometrical notion that can be seen as characterising coherence. A related, but stronger, picture is given by the observation that fundamental groups of compact cell complexes are finitely-presented; so if all spaces with finitely-generated fundamental group immersing into a complex X are either compact or retract onto a compact subcomplex, then $\pi_1(X)$ will be coherent. This "compact core" approach was used by Scott in [68], and led to the proof that fundamental groups of 3-manifolds are coherent. However, many other groups for which compact cores cannot be constructed are known to be coherent — a detailed overview is given in [81].

Almost all approaches to proving coherence have been firmly rooted in geometric and topological techniques, with the key being an ability to describe all finitely-generated subgroups, and so the problem is naturally more tractable for groups that can be associated with nice geometric finiteness properties. The linear isoperimetric inequality satisfied by hyperbolic groups can be seen as such a property, a basic fact being that it implies finite-presentability (see for instance [2]), and therefore hyperbolic surface groups and free groups are immediately seen to be coherent, as indeed are any hyperbolic groups whose finitely-generated subgroups are again hyperbolic.

On the other hand, one of the first examples of an incoherent group is the direct product of free groups $F_2 \times F_2$. This group was originally shown to be incoherent by Baumslag, Boone and Neumann [6], although the best-known proof, generally attributed to Stallings, exhibits a surjective map $F_2 \times F_2 \twoheadrightarrow \mathbb{Z}$ whose kernel is finitely-generated but not finitely-presented. This method uses a result of Neumann, saying that the amalgamated product of two finitely-generated groups over an infinitely-generated subgroup is not finitely-presentable, and was generalised by Stallings to produce other examples of incoherence and related phenomena, see for instance [71]. This idea has recently been used extensively in work of Kropholler, Walsh and Vidussi to generalise the $F_2 \times F_2$ example, showing that semidirect products of hyperbolic surface groups with F_2 are also incoherent [45, 46]. Direct product structures obstruct the growth properties that characterise hyperbolic groups, so it is not surprising that the geometric ideas that prove coherence for certain hyperbolic groups do not carry over to products of them.

Similar to the absence of a single geometric concept capturing coherence, there is no complete geometric theory of one-relator groups. It is shown in [54] that the topological property of "negative immersions", a strengthening of Definition 1.4, strongly constrains the subgroup structure of those one-relator groups which possess it, and even further it is conjectured that it implies hyperbolicity. These properties appear to be closely linked to coherence, which motivated the following:

Conjecture 3.4. *[80, Conjecture 1.10] All groups with non-positive immersions are coherent.*

The basic intuition motivating this is that 2-cells make positive contributions to Euler characteristic, and represent relations in the fundamental groups of 2-complexes, so if we can control the number of 2-cells in immersed complexes representing subgroups, this should allow us to deduce finite-presentability. We note that the presence of torsion elements is an obstruction to a group G having non-positive immersions, since given such an element t of order p, the presentation complex for G has an immersion from the presentation complex for $\mathbb{Z}_p \cong \langle t \,|\, t^p \rangle$, which has Euler characteristic 1 and non-trivial fundamental group. However, we note that one-relator groups with torsion are also hyperbolic by Newman's spelling theorem [62], are coherent by [53, 80], and in general are "better-behaved" than the torsion-free class.

An immediate corollary of Theorem 3.1 was that the torsion-free one-relator groups have non-positive immersions, so we can see Conjecture 3.4 and these related developments as closing in on a resolution to Question 3.3. It is this line of ideas that our main results as stated in the introduction are intending to carry into the setting of one-relator surface groups. This forms an interesting emerging direction of study, where the investigation of coherence and attempts to find an underlying geometric theory as suggested by non-positive or negative immersions are mutually informing each other.

3.2 Stackings and Right-Orderings

The main innovation in Louder-Wilton's proof of Theorem 3.1 was their introduction of the idea of a stacking of the word w. In this section we introduce some of the ideas and terminology around stackings, ahead of our main construction and application of stackings in surfaces which we will treat more rigorously in Part II.

Definition 3.5. Let $\rho : \mathbb{S} \looparrowright X$ be an immersion of cell complexes, with each component of $\mathbb{S}$ circular. A *stacking* of ρ is a lift $\hat{\rho}$ to an embedding in the trivial line bundle over X — it yields the following commutative diagram, where π_X denotes the standard projection map to X:

$$\begin{tikzcd} & X \times \mathbb{R} \arrow[d, "\pi_X"] \\ \mathbb{S} \arrow[ur, hook, "\hat{\rho}"] \arrow[r, "\rho"] & X \end{tikzcd}$$

If $\pi_{\mathbb{R}}$ is projection to $\mathbb{R}$, then for each $x \in X$, $\rho^{-1}(x)$ consists of a discrete set of points in $\mathbb{S}$, which are each sent to distinct values by $\pi_{\mathbb{R}} \circ \hat{\rho}$. We collect the preimages sent to maximal and minimal values under this composition:

$$\mathcal{A}_{\hat{\rho}} := \{x \in \mathbb{S} \mid \forall y \neq x, [\rho(y) = \rho(x) \implies \pi_{\mathbb{R}} \circ \hat{\rho}(y) < \pi_{\mathbb{R}} \circ \hat{\rho}(x)]\}$$
$$\mathcal{B}_{\hat{\rho}} := \{x \in \mathbb{S} \mid \forall y \neq x, [\rho(y) = \rho(x) \implies \pi_{\mathbb{R}} \circ \hat{\rho}(y) > \pi_{\mathbb{R}} \circ \hat{\rho}(x)]\}$$

We call the connected components of $\mathcal{A}_{\hat{\rho}}$ (resp. $\mathcal{B}_{\hat{\rho}}$) *max-height (resp. min-height) regions* of the stacking, and the ones that are simply-connected we call *max-height (resp. min-height) strips*. We say that the stacking $\hat{\rho}$ is *good* if each component of $\mathbb{S}$ intersects both $\mathcal{A}_{\hat{\rho}}$ and $\mathcal{B}_{\hat{\rho}}$.

Good stackings let us compute the Euler characteristic of the underlying space. As an analogy, we can compare the idea of a stacking to a one-dimensional real vector field over the image of $\mathbb{S}$ in X, and the components of $\mathcal{A}_{\hat{\rho}}$ and $\mathcal{B}_{\hat{\rho}}$ in $\mathbb{S}$ as neighbourhoods of its critical values from which an index-type calculation is performed over the image $\rho(\mathbb{S}) \subset X$. Figure 3 illustrates the idea of computation from the maximal height regions of a stacking in the case of a graph. To start to get a picture of how we can extract information on Euler characteristic from good stackings, we note here the following elementary result:

Lemma 3.6. *If $\hat{\rho}$ is a good stacking of an irreducible immersion, then all of the max- and min-height regions are strips.*

Proof. Suppose $M \in \pi_0(\mathcal{A}_{\hat{\rho}})$ is a circular max-height region — M is then an entire component of $\mathbb{S}$. Since $\hat{\rho}$ is good, M also intersects $\mathcal{B}_{\hat{\rho}}$ in some open subset U. It follows from the definitions of $\mathcal{A}_{\hat{\rho}}$ and $\mathcal{B}_{\hat{\rho}}$ that the image of U in X is traversed exactly once by the image of ρ, but this contradicts irreducibility. So the max-height regions must be strips rather than circular, and by the same reasoning, the same holds for the min-height regions. □

This means that for a good choice of stacking, the max-height regions piece together to cover the image of ρ, *with each piece making a contribution of* -1 *to the Euler characteristic*, since they are contractible subsets of circular spaces with a pair of contractible compact subsets (which are not maximal among the preimages of their projection to X) removed from their boundaries. In the graph case [52], this is emphasised by Louder-Wilton's use of the term *open arc* to describe these regions; in a more general setting, these are replaced by open strips which retract to arcs. So, taking all these regions together, we count exactly (minus) the Euler characteristic of the image of ρ; we will make this idea precise for surfaces in Lemma 6.6.

The following theorem of Farrell suggests the group-theoretic property that corresponds to the existence of a stacking, namely, right-orderability:

Definition 3.7. A *total ordering* of a set X is a transitive, antisymmetric, irreflexive boolean relation defined on all pairs of elements of X. We call a total ordering $<$ on a group G a *right-ordering* if it is invariant under right-multiplication:

$$a < b \implies ag < bg, \quad \forall a,\, b,\, g \in G$$

If G possesses a right-ordering, we say it is *right-orderable*. We can similarly define *left-orderings* and *left-orderable* groups, and if $(G, <)$ is left- and right-orderable by the same total ordering $<$, we say it is *bi-orderable*.

Theorem 3.8. *[26, Theorem 2.3] A regular cover E of a Hausdorff, paracompact space X has right-orderable deck transformation group $\pi_1(X)/\pi_1(E)$ if and only if there is an embedding f of E into the product $X \times \mathbb{R}$ such that the following diagram commutes, where c denotes the covering map:*

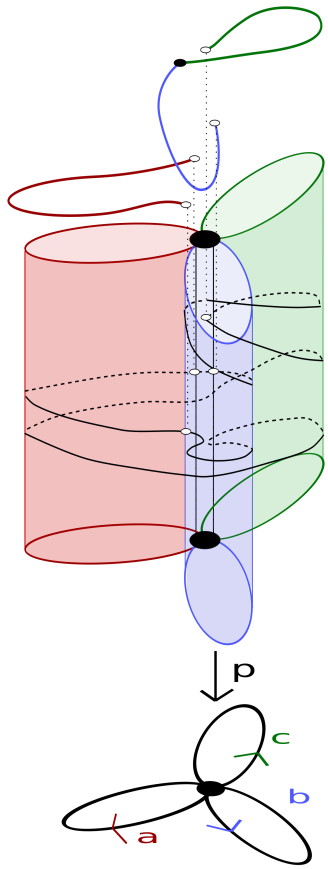

Figure 3: A stacking of the word $ab^{-1}c^2a^{-1}cb^{-1}$ in the rank 3 free group, represented by the graph with 1 vertex and 3 loop edges. The 2 maximal height arcs each consist of edge sequences with the terminal nodes missing (where they stop being maximal), and so their projections to the underlying graph contribute -1 to its Euler characteristic. When pieced together these arcs cover the whole graph of Euler characteristic -2.

$$\begin{tikzcd} & X \times \mathbb{R} \arrow[d, "\pi_X"] \\ E \arrow[ur, hook, "f"] \arrow[r, "c"] & X \end{tikzcd}$$

The theory of orderable groups has seen several interactions with low-dimensional topology in recent years, as detailed in [20] for example. However, although Farrell's theorem is older than many of those developments, the only application known to the author is a (non-constructive) proof for the existence of stackings associated to orderable quotients of fundamental groups of graphs and surfaces that we consider in this work. A useful characterisation of right-orderable groups is that they embed in $\mathrm{Homeo}^+(\mathbb{R})$, the group of order-preserving homeomorphisms of the real line. It is not clear who first formulated this property, the standard proof making use of a well-known construction of Cantor [16], which appears in the proof of Theorem 3.8.

The condition on w not factoring through any other immersions from circles in Theorem 3.1 may seem unusual at first glance. As stated in Theorem A' it just means that the element $w \in \pi_1(\Gamma)$ is not a proper power (we will often suppress the distinction between elements w and the immersions representing them), and we call such immersions *indivisible* to reflect this. In a one-relator group $G = F/\langle\langle w \rangle\rangle$, it is clear that if $w = u^n$ is a proper power then G contains torsion; Karrass-Magnus-Solitar showed that the opposite implication holds [44]. So, studying w-cycles for w indivisible in a free group is the same as studying properties of subgroups of the torsion-free one-relator groups. Restricting to torsion-free groups is important for the construction of stackings in light of Theorem 3.8 and the following easy obstruction to orderability:

Fact. If a group G contains any torsion elements, it cannot be right-orderable.

We noted above that torsion is also an obstruction to a group having non-positive immersions, and this is no coincidence. Although it is not immediately apparent that right-orderability should play a key role in the theory of w-cycles, it is less surprising when we consider their background as discussed in the previous subsection, and that non-positive immersions implies local-indicability by [80, Theorem 1.3]. Local-indicability is known to imply right-orderability by a theorem of Burns and Hale [13], and as a partial converse, bi-orderable groups are locally-indicable [47]. Torsion-free one-relator groups are locally-indicable, and hence right-orderable [12, 40]. Together with Farrell's theorem, this makes stackings a natural tool for proving Theorem 3.1.

Example 3.9. Letting R_2 be the "rose" graph, with a single vertex ν and 2 loop edges e_a, e_b in correspondence with a chosen generating set of a free group $F_2 = \langle a, b\rangle$, we can directly observe the equivalence between a right-ordering of $G := F_2/\langle\langle w\rangle\rangle$ and a stacking of an indivisible immersion to R_2 representing w. If n is the length of w, we let $v_i \in F_2$ be the ith prefix of w for $i = 0, \ldots, n$; so for $w = [a, b]$ we would have

$$v_0 = 1,\ v_1 = a,\ v_2 = ab,\ v_3 = aba^{-1},\ v_4 = aba^{-1}b^{-1}$$

Since w is not a power in F_2, we can fix a right-ordering on the one-relator group G. The n elements of G that the prefixes represent:

$$\bar{v}_0 = \bar{v}_n = 1,\ \bar{v}_1, \cdots, \bar{v}_{n-1}$$

are distinct [40, Proposition 3.3], and can therefore be put in increasing order. That is, there exists a permutation $\sigma \in S_n$ of the numbers $\{0, 1, \cdots, n-1\}$ such that

$$\bar{v}_{\sigma(0)} < \bar{v}_{\sigma(1)} < \cdots < \bar{v}_{\sigma(n-1)}$$

We now translate this ordering into a stacking — w is identified with an immersion

$$w : S \looparrowright R_2$$

where S is a circle subdivided by n vertices as shown in Figure 4, each of which is sent to the single vertex ν of R_2. Each v_i is represented by the arc μ_i consisting of the first i edges of S, which maps to a closed loop in R_2 under w, with $w(v_i)$ being obtained from $w(v_{i-1})$ by appending a traversal of one of the edges e_a, e_b in the appropriate direction. The ordering of the $\bar{v}_i$ defines an injection of the vertices,

$$w_0 : S^{(0)} \hookrightarrow \{\nu\} \times \mathbb{R} \subset R_2 \times \mathbb{R}$$

to be precise, we can set it so that

$$\tau(\mu_{\sigma(i)}) \xmapsto{w_0} (\nu, i)$$

where τ denotes the endpoint of each directed arc $\mu_i \subset S$. We now extend this map over the edges of S by paths that monotonically increase/decrease in the $\mathbb{R}$-fibres over the edges of R_2 according to whether or not $\bar{v}_i < \bar{v}_{i+1}$. We claim that this results in a stacking

$$\hat{w} : S \hookrightarrow R_2 \times \mathbb{R}$$

Indeed, suppose that two edges of S are mapped to crossing paths in $e_a \times \mathbb{R}$, and that these correspond to $a^{\pm 1}$ being appended to form the kth and lth prefixes of w. If we identify the vertices

$$\nu_i := \hat{w}\left(\tau(\mu_i)\right) \in \{\nu\} \times \mathbb{Z}$$

so that the relative position of ν_i in $\{\nu\} \times \mathbb{R}$ is the same as that of v_i in the ordering of the prefixes, we can easily identify the ways in which this can occur. In particular, if they lie in disjoint intervals of $\mathbb{R}$, e.g.

$$\nu_{k-1} < \nu_k < \nu_{l-1} < \nu_l$$

then the paths joining them in the image of $\hat{w}$ are also disjoint. If they are interlinked, e.g.

$$\nu_{k-1} < \nu_{l-1} < \nu_k < \nu_l$$

then the monotonically increasing paths joining the vertices cross if and only if they project down to traversals of e_a in opposite directions, meaning, say,

$$v_k = v_{k-1}a,\, v_l = v_{l-1}a^{-1}$$

But then such an intersection would mean we must have

$$v_{k-1} < v_{l-1} < v_k < v_l$$

and right-invariance of the ordering would then imply that we also have

$$v_{k-1}a = v_k < v_{l-1}a < v_k a < v_l a = v_{l-1}$$

which produces a contradiction, since the order of v_{l-1} and v_k has been switched by the right-multiplication. The other possible arrangements that could lead to self-intersections of $\hat{w}(S)$ are easily enumerated and found to produce contradictions in similar ways; we can also consider the case that the interval in the $\mathbb{R}$-fibre spanning ν_{k-1}, ν_k contains the one spanning ν_{l-1}, ν_l. Then, there would be an intersection no matter what the direction of edge traversals. In the subcase shown in Figure 4, we would have

$$v_{k-1} < v_l = v_{l-1}a < v_{l-1} < v_k = v_{k-1}a^{-1}$$

and right-multiplying this chain of inequalities by a gives a new chain containing

$$v_{l-1}a = v_l < v_{k-1} = v_k a$$

which is a reversal of their previous order, again contradicting right-invariance of $<$.

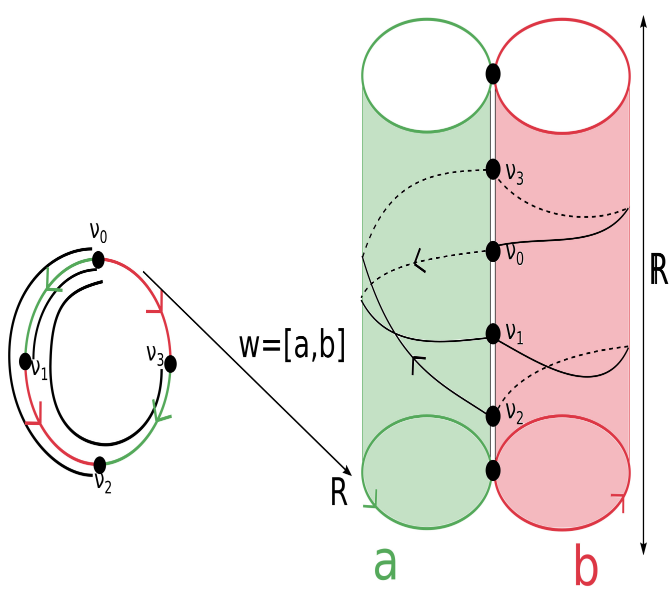

Figure 4: A non-example of a stacking of the commutator $[a, b] \in F_2$ — when the lifts of the edges of S to $R_2 \times \mathbb{R}$ intersect this way, the ordering of the prefixes of w as read by the vertex preimages cannot be right-invariant.

The observation of this example — that right-orderability is violated if the obvious lift fails to be an embedding — is similar to the one used to prove [25, Lemma 3.2]. We also note that the above example fixes an ordering and uses it directly, although such orderings are not really natural to work with in any sense, we simply know that local-indicability implies they exist. In § 7.2, we instead work with tower liftings to derive stackings, which are a somewhat more intuitive way of working with the complexes involved, and indeed were the tool used to prove that torsion-free one-relator groups are locally-indicable in the first place.

3.3 The Hanna Neumann Conjecture

As noted above, Helfer-Wise [35] also gave a proof of Theorem 3.1, which appeared at around the same time as [52]. We conclude our introduction to w-cycles by briefly discussing the ideas surrounding their proof; although we do not develop this viewpoint in the later sections, there are certainly interesting connections with the ideas discussed so far.

Importantly, their proof also relies on orderability properties of free groups and their torsion-free quotients. Where Louder-Wilton show that w-cycles contribute arcs lying at the top of a stacking, which in turn contribute to Euler characteristic, Helfer-Wise find a correspondence between the rank of any subgroup containing w-cycles, and "bridges" — edges which are maximal in an ordering inherited from the group elements, in the complexes they act on. The idea of bridges in graphs representing generators of orderable groups originated in the proof by Mineyev of Hanna Neumann's famous conjecture:

Theorem 3.10. *[61, 28] Let H, K be subgroups of a finitely-generated free group F and let $a_1, \ldots, a_n$ be a complete set of representatives of the double cosets $H\backslash F/K$. Then:*

$$\sum_{i=1}^{n} rank\left(H \cap a_i K a_i^{-1}\right) \leq \bar{r}(H)\bar{r}(K)$$

The original formulation of this conjecture was made in attempt to quantify Howson's theorem that rank$(H \cap K)$ is always finite, which as we noted in § 2, was trivialised by Stallings' introduction of folding techniques for graphs. It was remarkable that it took well over half a century for a proof to be found, and the independent proofs of Friedmann and Mineyev used machinery that was not at all developed at the time of the conjecture's formulation, as well as being seemingly completely different methods of proof. Dicks [24] would later give a simplified version of Mineyev's proof from which the methods of [35] can be more clearly traced, based only on Bass-Serre theory and separation properties of graphs.

The original formulation of the w-cycles conjecture in [79] was stated as a "rank-1 version" of the Hanna Neumann conjecture (HNC). This is more apparent in the algebraic formulation (compare with Theorem A'), giving an analogous result when one of the groups in the intersection is cyclic since neither the original nor strengthened HNC says anything non-trivial in that case. Moreover, Wise showed that truth of the HNC gave partial progress towards classifying which 2-complexes had non-positive immersions.

We have not pursued a "bridge-based" approach to proving the w-cycles theorem for surfaces, although it would not be surprising to find that such a proof exists. We would expect similar obstacles to arise in attempting to extend a graph-based argument over 2-cells as we detail in § 5 for the stacking argument, but it is not clear whether our technique of rectangular decomposition of the surfaces into graph-like pieces would help in this instance. We note that [35] also introduced "slim and bi-slim 2-complexes", similar to 2-complexes whose 2-cells have irreducible attaching maps, but whose definition emphasises desirable properties in their universal covers which relate to orderability of their fundamental groups, abstracting the ideas of the Dicks-Mineyev HNC proof. The connection between reducibility and slimness [35, §7] uses ideas that again are similar to the style of argument that we sketched in Example 3.9. Slim complexes were used to derive a version of non-positive immersions for one-relator groups with torsion, bearing some similarities to the arguments used in [53] and which we explore in § 9. In particular, the key inequality which we derive for one-relator surface groups in Lemma 9.3, seems to appear for one-relator groups in [35, Theorem 6.1].

4 One-Relator Surface Groups

Our introduction to w-cycles has revolved heavily around their relevance in understanding one-relator groups. It is easy enough to formulate the analogue of Theorem 3.1 for surface groups (that is, our Theorem A) from the point of view of the topological objects involved, but taking into consideration the group-theoretic ideas surrounding w-cycles, it is natural to also consider the analogue of one-relator groups for surfaces, as introduced in Definition 1.3. When we talk about one-relator surface groups, we will usually mean quotients of the fundamental groups of orientable hyperbolic surfaces, and will state when we are considering other types of surface.

The first notable appearance of one-relator surface groups came from Papakyriakopoulos' formulation of conjectures about torsion-free quotients of surface groups whose truth would imply truth of the Poincaré conjecture [64]. Since then, a lot of research surrounding these groups has followed a theme of finding parallels with the existing theory of one-relator groups. Of course this is exceptional behaviour when compared to arbitrary two-relator groups, corresponding to the special position that surfaces occupy among 2-complexes — in this section we try to highlight those aspects of this behaviour that will be relevant to our proof of coherence for one-relator surface groups with torsion in Part III.

4.1 Hempel's Trick

In reference to [37, Lemma 2.1, Theorem 2.2], Howie, in [41], uses the term "Hempel's trick" to mean a process that has allowed the study of one-relator surface groups to utilise the existing theory of one-relator groups. Given $G = \pi_1(\Omega)/\langle\langle w \rangle\rangle$, this is essentially achieved by finding a cover of Ω which has free fundamental group, and to which w lifts. Since there are many cyclic covers of a hyperbolic surface Ω (that is, coverings of Ω with deck groups isomorphic to $\mathbb{Z}$) which will accomplish this, this process is also naturally related to local-indicability.

Lemma 4.1. *([37], Lemma 2.1) Let w be a loop in a closed surface Ω with $\chi(\Omega) < 0$. Then there is a simple, non-separating loop γ in Ω with intersection number:*

$$\langle [w], [\gamma] \rangle = 0$$

where $\langle -, - \rangle$ denotes the integral intersection pairing on first homology,

$$H_1(\Omega; \mathbb{Z}) \times H_1(\Omega; \mathbb{Z}) \to \mathbb{Z}$$

For completeness, we repeat the brief proof here, noting only that we have removed the hypothesis that Ω is orientable, without changing the result or argument, but have added the assumption that $\chi(\Omega) < 0$, which is the case for all the surfaces we are interested in.

Proof. Since $\chi(\Omega) < 0$, we know

$$\operatorname{rank}(H_1(\Omega; \mathbb{Z})) \geq 2$$

and so we can find a pair of simple closed curves α, $\beta \in \pi_1(\Omega)$ representing generators of the first homology, and which intersect transversely in a single point. Then α, β cannot cobound an immersed surface, and nontrivial linear combinations satisfy

$$a[\alpha] + b[\beta] \neq 0 \in H_1(\Omega), \qquad (a, b \in \mathbb{Z})$$

In particular, such combinations can be represented by a non-separating simple closed curve (the (a, b)-slope on some connect-sum component of Ω homeomorphic to a torus or Klein bottle). We can now evaluate

$$\langle w, \alpha \rangle := i_\alpha \qquad \langle w, \beta \rangle := i_\beta$$

and find solutions (p, q) to the integer equation

$$pi_\alpha = -qi_\beta$$

Fixing (p, q) to be the smallest coprime solutions, we can now find a loop γ in the homotopy class of

$$\alpha^p \beta^q$$

satisfying the required properties. $\square$

Taking the dual viewpoint, it is easily seen that this is just an explicit way to construct a lift of w to a cyclic cover of Ω. Classes in $H^1(\Omega; \mathbb{Z})$ are represented by maps onto $\mathbb{Z}$ from $C_1(\Omega; \mathbb{Z})$, corresponding to regular cyclic coverings of Ω. For the loop γ we just constructed, the cohomology class dual to $[\gamma] \in H_1(\Omega; \mathbb{Z})$

then corresponds to the infinite cyclic cover Ω_∞ obtained by cutting Ω along γ and gluing copies of the resulting surface together by identifying boundary components. Its fundamental group is given by:

$$\ker\left[\langle -, [\gamma]\rangle : \pi_1(\Omega) \to (\pi_1(\Omega))^{\mathrm{Ab}} \to \mathbb{Z}\right]$$

By construction, the loop w lies in this kernel, and therefore lifts to Ω_∞. By [39, Lemma 3.1], every map of compact cell complexes has a maximal finite sequence of lifts over cyclic covers. Lemma 4.1 makes explicit that there is at least one non-trivial level to the lifting whenever the map is an immersion of a loop into a closed hyperbolic surface.

Now, Ω_∞ has infinitely-generated free fundamental group, and contains, for each $d \in \mathbb{Z}_{>0}$, copies of the compact surface Ω_d obtained by gluing together d copies of $\Omega - \gamma$ along their boundary components. Since w lifts to Ω_∞, there is a minimal d such that w lifts to $\Omega_d \subset \Omega_\infty$, and $\pi_1(\Omega)$ is then realised as an HNN extension of the free group $\pi_1(\Omega_d)$, with the fundamental groups of two of its subsurfaces isomorphic to Ω_{d-1} (which w does not sit inside) amalgamated. This provides the conditions for the style of "rewriting" argument popularised by Magnus and used to prove a great many results on one-relator groups (an overview of the technique is given in [56, Chp. 4]).

4.2 Torsion Properties

For our purposes, probably the most significant application of Hempel's trick was also the first one he gave, as it implies that torsion-free one-relator surface groups are right-orderable, and ultimately, that we can construct stackings of indivisible words in surface groups:

Theorem 4.2. *[37, Theorem 2.2] The following are equivalent:*

1. *The one-relator surface group $G = \pi_1(\Omega)/\langle\langle w\rangle\rangle$ is locally-indicable;*
2. *G is torsion-free; and*
3. *w is not a proper power of a non-trivial loop in Ω.*

In the case that a one-relator surface group does have torsion, it presents in the same way as in a one-relator group:

Lemma 4.3. *[41, Corollary 3.9] Let $w \in \pi_1(\Omega)$ be indivisible, and $n > 1$ an integer. Denote by $\bar{w}$ the image of w in the one-relator surface group with torsion:*

$$G = \pi_1(\Omega)/\langle\langle w^n \rangle\rangle$$

Then every torsion element of G lies in the normal closure $\langle\langle \bar{w} \rangle\rangle$.

The final property that we will make use of is that one-relator surface groups are virtually torsion-free, long known to hold for one-relator groups [27]. This will be important to construct "unwrapped covers" of their presentation 2-complexes in § 9, as was done for one-relator groups in [53]. We certainly do not believe that Lemma 4.5 is new, but having not found an explicit reference for it, we will give a quick argument here based on the above discussion, Fischer-Karrass-Solitar's proof for one-relator groups, and the following nice result of Allenby:

Theorem 4.4. *[1, §4] Let A, B be free groups, c some element of the free product $A * B$, and D the amalgamated product:*

$$D := A *_{\langle c \rangle} B$$

Then, for any non-trivial element $x \in D$ and any positive integer k, there exists a homomorphism φ from D onto a finite group Q such that $\varphi(x)$ has order exactly k in Q.

In particular, since hyperbolic surfaces all have non-trivial connect-sum decompositions, surface groups possess the property stated in the conclusion of Allenby's theorem, known as *potency.* Our desired property then follows easily:

Lemma 4.5. *(see also [3, §6]) Let $G = \pi_1(\Omega)/\langle\langle w^n \rangle\rangle$ be a one-relator surface group with torsion, where w is not a proper power. Then G is virtually torsion-free.*

Proof. We use Allenby's theorem to find a surjection to a finite group

$$\varphi : \pi_1(\Omega) \to Q$$

where $\varphi(w)$ has order n. Then φ descends to the cosets of $\langle\langle w^n \rangle\rangle$, defining a surjective homomorphism

$$\psi : G \to Q$$

whose kernel is finite-index in G. It remains to check that $\ker(\psi)$ is torsion-free, for which we appeal to Lemma 4.3. This tells us that any torsion element of $\ker(\psi)$ has the form $\left(g\bar{w}g^{-1}\right)^p$ for some $g \in G$, $p \in \mathbb{Z}$, with image under ψ equal to the identity in Q. But this means that $n|p$, so that the supposed torsion element in the kernel of f was trivial to begin with. $\square$

Remark 2. The results 4.2 and 4.3 that we have quoted in this section were only proved for orientable surfaces, but as we observed, Lemma 4.1 applies equally well to non-orientable surfaces, and so also should the consequent results about torsion. Moreover, Antolín-Dicks-Linnell give algebraic proofs of Hempel and Howie's results in [3] which apply equally to non-orientable one-relator surface groups, but we do not know of topological arguments covering them all yet.

Part II

Constructions on Surfaces

5 Parsing w-Cycles for Surfaces

While ultimately our proof of Theorem A will be in the spirit of Louder-Wilton's proof of Theorem 3.1, there are obstacles which arise when trying to carry their argument over directly to surfaces. Naïvely, we could fix the cell decomposition on a closed surface Ω corresponding to the standard one-relator presentation of its fundamental group, and express w as a loop in the graph $\Omega^{(1)}$. The fundamental group of $\Omega^{(1)}$ generates $\pi_1(\Omega)$, so of course we can represent any element of the group as an immersion of a circle $w : S \looparrowright \Omega^{(1)}$ and, as long as w is indivisible, use [52, Lemma 3.4] to lift this to an embedding in $\Omega^{(1)} \times \mathbb{R}$. Then [52, Lemma 2.4] shows that the number of open arcs of S lifted to maximal height is $-\chi(\Omega^{(1)}) = 1 - \chi(\Omega)$. However, this is insufficient to bound the number of w-cycles in an arbitrary subgroup of $\pi_1(\Omega)$, due to the nature of the 2-cells in the immersed surface Σ representing this subgroup.

Despite knowing the cell structure of Ω from the standard presentation of a compact surface, once this is fixed, our choice to represent w using only the 1-skeleton means we require an immersion $h : \Sigma \looparrowright \Omega$ which is combinatorial with respect to that cell structure, and we then have no control on the number of 2-cells in the induced cell structure of Σ which gives the subgroup presentation. In particular we do not know the degree of the covering map $\Sigma^{(2)} \to \Omega^{(2)}$ onto the 2-cell in Ω. So, while we can estimate the degree of the covering map $\sigma : \mathbb{S} \to S$ obtained by forming the fibre product $\Sigma^{(1)} \times_{\Omega^{(1)}} S$ and pulling back the stacking of w, obtaining $\deg(\sigma) \leq -\chi(\Sigma^{(1)})$, it could be the case that

$$-\chi(\Sigma) = -\left(\chi(\Sigma^{(1)}) + |\Sigma^{(2)}|\right) \ll -\chi(\Sigma^{(1)})$$

which does not give us the desired inequality of $\deg(\sigma) \leq -\chi(\Sigma)$.

To formulate our version of Theorem 3.1 for closed orientable surfaces, we have therefore used circular covering spaces to define w-cycles in Definition 1.1. This lets us replace an immersion of a circle into Ω — which as we have just seen will not be sufficient to bound the number of w-cycles with a stacking argument — with an immersion of a circular surface into Ω, which can carry the necessary information about the 2-cell structure. Since there is only one non-trivial homotopy class of indivisible closed curve in a circular space, we lose no generality in the group-theoretic results our topological methods produce when we replace immersions of circles with immersions of circular spaces. The only circular surfaces are annuli and Möbius bands, and since we restrict our results here to orientable hyperbolic surfaces, we will be principally concerned with immersions of annuli.

Throughout Part II, unless stated otherwise, we will assume that the circular space S in Theorem A is a non-compact annulus which infinitely covers Ω. Fixing a standard hyperbolic metric on Ω as in Theorem A, we denote by γ the core curve of S, so that $w(\gamma)$ is an immersed closed curve which we choose to be the unique geodesic in the homotopy class of $w \in \pi_1(\Omega)$. The covering can be understood in terms of the map from γ to this geodesic representative — a consistent choice of normal directions to γ, $w(\gamma)$ allows generic points of S to be mapped to Ω by considering where they sit in relation to γ. The data are related by the commutative diagram:

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\sigma"] \arrow[d, "w'"] & S \arrow[d, "w"] \\ \Sigma \arrow[r, "h"] & \Omega \end{tikzcd}$$

where we recall that the w-cycles are represented by the circular components $\mathbb{S}$ of the pullback of the immersions w, h, inducing the maps σ, w'. We will further assume that the immersion h maps all boundary components of Σ to geodesics in Ω with respect to the hyperbolic metric, so that we are in the setting of Theorem A.

Before examining the geometric and topological constructions that will allow us to prove Theorem A we will eliminate some special cases where such constructions are unnecessary.

5.1 Special Cases

For the moment, let

$$h : \Sigma \looparrowright \Omega$$

be any essential immersion of compact connected surfaces, and

$$w : S \looparrowright \Omega$$

a covering map from a circular surface.

If Ω has non-empty boundary then so must Σ, and furthermore Ω retracts onto a graph, say by a map

$$r : \Omega \twoheadrightarrow \Gamma_\Omega$$

whose fibres consist of compact arcs with endpoints on $\partial\Omega$, say I_p for each $p \in \Gamma_\Omega$. Each I_p has a discrete set $h^{-1}(I_p)$ of preimages in Σ, which are also intervals since h is an essential immersion (in particular, $h^{-1}(\partial\Omega) \subset \partial\Sigma$), and collapsing each of these intervals to a point then produces a graph Γ_Σ with induced immersion

$$h_* : \Gamma_\Sigma \looparrowright \Gamma_\Omega$$

We can also pass from the image of the geodesic $w(\gamma) \subset \Omega$ through the retraction r to an immersed circular graph $S \looparrowright \Gamma_\Omega$, at which point the situation is reduced to Theorem 3.1.

If Ω were a sphere or projective plane then its fundamental group has no non-trivial proper subgroups and the result is trivial. If Ω were a torus then its fundamental group is free abelian, so we know there is at most one w-cycle in Σ by the discussion in Example 2.2. In the case of the Klein bottle, the only proper subgroups are either cyclic or $\mathbb{Z} \oplus \mathbb{Z}$, so we can easily bound the number of w-cycles by similar considerations as for the torus. We therefore assume that Ω is a closed, orientable hyperbolic surface, so that we have the hypotheses of Theorem A; we discuss the case of non-orientable hyperbolic surfaces in §10.3.

Most obviously, if Σ is a disc, then it contains no w-cycles. If Σ is circular, it contains at most one w-cycle, exactly when a power of a conjugate of w generates $\pi_1(\Sigma) \leq \pi_1(\Omega)$, in which case it is described by a reducible immersion (in particular an embedding). These cover the cases when $\chi(\Sigma) \geq 0$ in Theorem A, so we assume $\chi(\Sigma) < 0$, and want to show that $-\chi(\Sigma)$ bounds the number of w-cycles when they are represented by irreducible immersions. This will essentially be achieved by Proposition 7.7, using the decompositions and stackings which we construct in the next two sections. In particular, we will always want to assume in these constructions that Σ has non-empty boundary.

Lemma 5.1. *Theorem A holds when Σ is closed.*

Proof. When $\partial\Sigma = \emptyset$, the immersion h is locally a homeomorphism from a neighbourhood of every point of Σ to its image in Ω. Since Σ is compact, it follows that h is a finite covering map; let d be the covering degree. Then there are at most d distinct lifts of $w(S)$ to Σ, and we have

$$\chi(\Sigma) = d\chi(\Omega) \leq -d,$$

(with the last inequality following from our assumption that $\chi(\Omega) < 0$). These lifts to Σ are exactly the w-cycles, so there at most

$$d \leq -\chi(\Sigma)$$

of them, as required. □

We will therefore assume from now on that Σ has non-empty boundary unless explicitly stated otherwise. This implies in particular that we are always considering finitely-generated subgroups of the surface group $\pi_1(\Omega)$ which are free and therefore infinite-index (by [43], for example). We may still assume that Σ is compact without loss of generality for the analogous algebraic statement, since any surface whose fundamental group is finitely-generated retracts to a compact core.

5.2 Geometric Structures on Surfaces

We need a way to decompose surfaces into pieces that are simple enough to allow for a "graph-like" argument using stackings to be applied (based on the discussion surrounding Lemma 3.6), but still see enough of the surfaces to compute their Euler characteristics precisely. We do not have a canonical cell decomposition on a given surface in the way that a graph is naturally identified with its combinatorial description as a set of vertices and edges. However, there is a rich theory of the geometric structures supported by hyperbolic surfaces that we can draw on to find the decompositions that will suit our purposes, coming from the ideas of measured laminations and foliations which have grown out of Thurston's work [75] in the last 50 years.

We will make use of certain aspects of this theory in the context of our w-cycles problem in §6. In particular, since we already know the w-cycles bound to hold for finite-index subgroups, we will need to deduce how these structures on closed surfaces appear when we pull them back to immersed surfaces with boundary. We recall the relevant concepts from the literature here — the general theory of laminations reaches far beyond what we will need for our constructions, overviews being given in numerous texts, such as [17] and [15]. Our main reference is Hatcher's paper [34], where he expresses key aspects of Thurston's theory using purely topological ideas.

A *geodesic lamination* Λ on the surface Ω is a non-empty disjoint union of simple geodesics comprising a closed subset of Ω. Each geodesic $\lambda \in \Lambda$ is called a *leaf* of the lamination, and a union of its leaves that is still closed in Ω is a *sublamination* of Λ. If Λ has no proper sublaminations, we say it is *minimal*. The closure of the components of $\Omega - \Lambda$ are called *principal regions*. If all of its principal regions are simply-connected, we say Λ is *filling*, and the principal regions in this case are ideal k-gons ($k \geq 3$), isometrically embedded from the universal cover $\tilde{\Omega} \simeq \mathbb{H}^2$, where the ideal vertices are identified with points on the bounding circle at infinity.

The simplest geodesic laminations are disjoint unions of simple closed curves in Ω which we call *multicurves*, each comprising a single leaf — multicurves are never filling, nor minimal unless they consist of a single curve. Multicurves can be extended to filling laminations using bi-infinite leaves whose ends spiral towards closed curves, but such laminations are not minimal. The laminations that will be useful to us are those that are both minimal and filling, consisting purely of bi-infinite leaves, each of which is dense in Λ, and throughout its traversal is adjacent to every principal region infinitely often. (Minimal and filling laminations are often called "ending laminations", and studied for the information they give on the geometry of 3-manifolds whose "ends" consist of hyperbolic surfaces [30, 74], but this is tangential to our use of these laminations.)

The set of laminations that Ω admits can be given the structure of a topological space, denoted $ML(\Omega)$. In [34, Proposition 1.5], Hatcher describes homeomorphisms

$$ML(\Omega) \simeq \mathbb{R}^{-3\chi(\Omega)} \qquad (\star)$$

by associating the laminations with measures on neighbourhoods transverse to "train tracks" on Ω supporting them. These train tracks are closed subsets of Ω that locally are just line segments except at finitely many branch points as shown in Figure 5. There are many maximal sets of train tracks on Ω (that is, sets of tracks that cut Ω into simply-connected regions), but each consists of a finite set of minimal sub-tracks, on which the branch-matching equations as indicated in Figure 5 are determined up to choice of $-3\chi(\Omega)$ real-valued parameters. After fixing a choice T of maximal track, a "measured lamination" on Ω is obtained by choosing values in $\mathbb{R}$ to assign to linear track segments such that the matching equation at each branch point of T is satisfied.

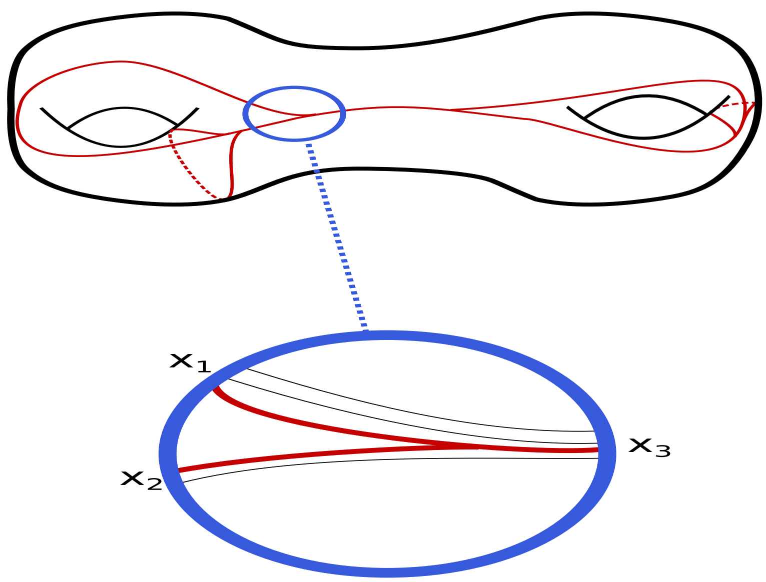

Figure 5: A train track on the closed genus 2 surface is shown in red, and the local model of branch points is shown underneath. Assigning values x_i to each linear track segment such that they are additive on convergent tracks specifies a measured lamination on Ω. For the model shown, this means we require $x_1 + x_2 = x_3$. We see that if each x_i is a positive integer, taking x_i arcs lying parallel to the corresponding tracks and matching throughout Ω produces a multicurve.

If we choose each of those assigned values x_i to be an integer, then we can take x_i geodesic arcs running parallel to the corresponding linear track segments, and the matching equations ensure that these can be joined at the branch points to produce a multicurve in Ω. As described in [34, §2], generic measured laminations on Ω are produced by replacing the $x_i \in \mathbb{Z}$ arcs with fibred neighbourhoods of measure $x_i \in \mathbb{R}$, where the measure is chosen on these neighbourhoods to extend the intersection number of a line segment transverse to the track with the finitely many geodesic arcs we had in the integer case. The rational points in $\mathbb{R}^{-3\chi(\Omega)}$ also canonically correspond to multicurves, those associated to the integer point obtained by multiplying each $x_i \in \mathbb{Q}$ by the lowest common multiple of their denominators. Points with irrational coordinates in $\mathbb{R}^{-3\chi(\Omega)}$ correspond to laminations in $ML(\Omega)$ without closed leaves, but are identified under the homeomorphism $(\star)$ with limits of sequences of those corresponding to rational points.

Hatcher also describes the projectivisation of $ML(\Omega)$, denoted $PL(\Omega)$, obtained by identifying any (non-empty) measured lamination with k parallel copies of itself for every $k \in \mathbb{R}_+$. $PL(\Omega)$ has a polyhedral structure and is homeomorphic to a $(-3\chi(\Omega) - 1)$-dimensional sphere, and a point in the cone in $ML(\Omega)$ over a top-dimensional face of this polyhedron corresponds to a filling lamination in Ω. This is all to say, we can consider the laminations on Ω which are minimal and filling to be generic among the laminations it admits, since a lamination without closed leaves which is not minimal must have at least two minimal sublaminations, corresponding to lower-dimensional faces of the polyhedron, and its complement cannot then be simply-connected. Further, a minimal filling lamination $\Lambda \in ML(\Omega)$ is identified with an irrational point in $\mathbb{R}^{-3\chi(\Omega)}$, and a sequence of rational points converging to it is then identified with a sequence of multicurves approximating the image of Λ in $PL(\Omega)$.

In the next subsection we will start to show how the properties of minimal and filling laminations will be useful for characterising w-cycles in subgroups of $\pi_1(\Omega)$. We will also apply the above idea of approximating such laminations by sequences of multicurves when we use the related concept of *foliations* on surfaces. These are decompositions of Ω into 1-dimensional subspaces, also called leaves (although generally not geodesics), so that a neighbourhood of a generic point in Ω is composed of a union of parallel arcs in leaves of the foliation, and countably many points are singularities modelled on k-pronged saddles ($k \geq 3$), where regular leaves parallel to pairs of prongs become tangent to each other. Many concepts in the theory of laminations can be mirrored in foliations — Levitt [48] describes a form of duality between laminations and foliations on surfaces, whereby leaves of foliations can be "straightened" into geodesics describing classes of leaves of laminations. In §6.1, we will explicitly construct a foliation based on the properties of laminations we are about to discuss, which will eventually give us a decomposition of the immersed surface Σ that a stacking argument can be applied to.

5.3 Pulling Back Laminations

In passing between the topological and algebraic statements of Theorems A and A', we can make use of the correspondence between subgroups of $\pi_1(\Omega)$ and regular coverings of Ω, which can be retracted onto compact cores when the subgroup is free. We can understand how laminations on Ω pull back to a surface Σ immersing in it by using this correspondence to identify Σ with a subsurface of a cover. As we will see in the proof of the following lemma, it is not difficult to understand how laminations on Ω lift over finite covers, but the cover corresponding to a free subgroup $\pi_1(\Sigma) \leq \pi_1(\Omega)$ is of course infinite-index.

We can however factor these infinite covers through finite ones in a way that will solve this problem, using an approach inspired by Hall's important theorem for free groups [33] which states that every finitely-generated subgroup of a free group is a free factor in a finite-index subgroup. Topologically, this implies that any immersion of compact graphs $\Gamma \looparrowright \Delta$ lifts to a finite cover of Δ where Γ is embedded. Wilton [78] generalised Hall's theorem to surface groups, in fact limit groups; we do not need the full strength of this result, but we note that it implies the property of subgroup-separability, or "LERF". In [69], Scott shows that surface groups are LERF, and as a preliminary step proves that the aforementioned topological implication of Hall's theorem also holds for immersions of compact surfaces. This result only relies on the fact that surface groups are residually-finite (RF), as shown by Baumslag [4] (see also [36]).

Remark 3. The remarks above apply regardless of whether Ω is orientable. Indeed, in [69], the main result is first shown for the case when Ω is the closed non-orientable surface of Euler characteristic -1 (the connect-sum of three projective planes), and then extended to other closed surfaces using the fact that they are all finite covers of $3P^2$. However, we do not yet have a proof of the next lemma where we apply Scott's technique to understand pullbacks of laminations on Ω, in the case that Ω is non-orientable.

As noted by Reynolds in the introduction of [66], Scott's results have consequences for pullbacks of minimal laminations along immersions of surfaces, which can be applied in the setting of Theorem A. Since filling laminations cut surfaces into simply-connected regions, our broad strategy will be to use them to derive cell decompositions. The following lemma will have implications about the nature of cell decompositions these laminations induce on Σ — essentially, preimages of leaves will be seen as spans of 2-cells of a particular form, and w-cycles in Σ will be encoded by their paths through the leaves. The basic idea is sketched in Figure 6, and we formalise it in the next section.

Lemma 5.2. *Let Ω, Σ, w be as in Theorem A, with $\partial\Sigma \neq \emptyset$. Let Λ be a minimal filling lamination on Ω. Then every leaf $\lambda \in \Lambda$ satisfies:*

1. *all intersections between λ and the collection of curves $w(\gamma)$, $h(\partial\Sigma)$ are transverse; and*
2. *all preimages $h^{-1}(\lambda)$ are compact arcs with endpoints on $\partial\Sigma$.*

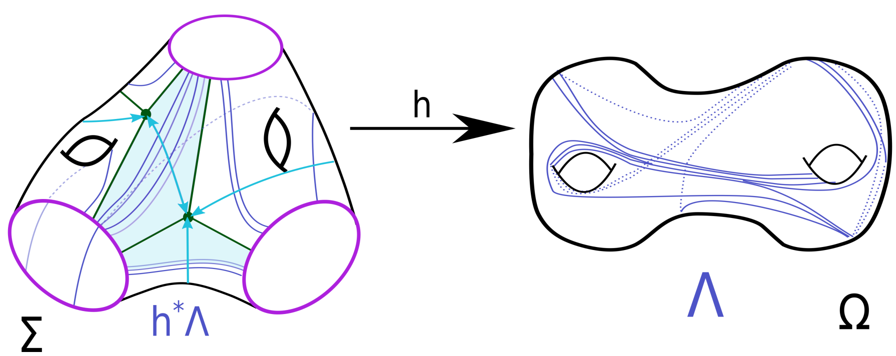

Figure 6: The preimages under h of leaves of the lamination Λ are arcs between boundary components, and all components of $(\Sigma - h^*\Lambda)$ are homeomorphic to discs. This results in those discs being subdivided into regions spanned by the pulled back leaves, and matching the regions up pairwise as the leaves are crossed produces 2-cells which are "rectangular" (see Definition 6.1).

Proof. Letting $H = h_*(\pi_1(\Sigma)) \leq \pi_1(\Omega)$, there is a corresponding non-compact, regular covering space $\Omega_H \xrightarrow{p_H} \Omega$, and a compact, essential subsurface $\Sigma' \subset \Omega_H$

homeomorphic to Σ whose image under p_H is exactly $h(\Sigma) \subset \Omega$. Then, since $\pi_1(\Omega)$ is RF, we can apply [69, Lemma 1.3] to find a finite-sheeted intermediate covering

$$\Omega_H \to \Omega_1 \xrightarrow{p_1} \Omega$$

such that the image of Σ' in Ω_1 is essentially-embedded, thereby identifying Σ with a surface embedded in the finite cover Ω_1. Then $\pi_1(\Omega_1)$ is a finite-index subgroup of $\pi_1(\Omega)$, and so contains a further finite-index subgroup which is normal in $\pi_1(\Omega)$ (to be exact, we can take this to be the intersection of conjugates of $\pi_1(\Omega_1)$ by a complete set of coset representatives). In other words, there is a further finite-sheeted cover

$$\Omega_2 \xrightarrow{p_2} \Omega_1 \xrightarrow{p_1} \Omega$$

such that

$$p := p_2 \circ p_1$$

is a regular covering map. Since such covers are easily understood, we will deduce minimality of the lifted lamination $\Lambda_2 := p^{-1}\Lambda$ in Ω_2, and thereby deduce the same for $\Lambda_1 := p_1^{-1}\Lambda$. It is immediate that since Λ is filling and p essential, the complementary regions $\Omega_i - \Lambda_i$ are all homeomorphic to discs, for $i = 1, 2$. Taking a regular neighbourhood of Λ in Ω, we can reconstruct Ω by attaching to it finitely many closed discs $D_1, \ldots, D_k$. If Λ_2 were not minimal, we could similarly take a neighbourhood of each of its minimal sublaminations and obtain disjoint essential subsurfaces of Ω_2, by attaching lifts of the D_i to these neighbourhoods. Since Λ itself is minimal, each minimal sublamination of Λ_2 covers the entirety of Λ, and it follows that each of those essential subsurfaces is homeomorphic to a union of fundamental domains for the cover p. But then Ω_2 would be formed by gluing the boundaries of the subsurfaces along regions that map into $\Omega - \Lambda$, which is a contradiction as such regions must be discs. So Λ_2 instead consists of a single minimal component, and so too does its continuous image $p_2(\Lambda_2) = \Lambda_1$ in Ω_1.

Now, restricting to Σ embedded in Ω_1, if Σ were to contain any infinite half-leaf of the minimal lamination Λ_1, since this leaf is dense in Λ_1, then Σ would also have to contain the entirety of Λ_1, disjoint from its boundary. This forces any boundary components of Σ to lie in complementary regions of the lifted lamination, and therefore bound discs in Ω_1, but this contradicts the fact that

Σ is the image of the essential subsurface $\Sigma' \subset \Omega_H$ under a covering map. In other words, we could only have non-compact leaves in Σ if $\Sigma = \Omega_1$ meaning our subgroup $H \leq \pi_1(\Omega)$ was finite-index to begin with. Since Σ has boundary it is therefore a proper subset of Ω_1, and we then know that Λ_1 restricted to Σ must consist of compact arcs between its boundary components.

We have now seen that the second condition is satisfied. Since h is an immersion and $h(\partial\Sigma)$ and the leaves of Λ are distinct geodesics in Ω (by the hypotheses of Theorem A), all intersections of $h(\partial\Sigma)$ with Λ are transverse. Any Λ can be viewed in the space $ML(\Omega)$ as the limit of a sequence of multicurves comprised of simple closed geodesics whose lengths increase unboundedly as they converge to Λ. Since the closed geodesic representative $w(\gamma)$ has fixed length in Ω, it follows that after finitely many steps any sequence of multicurves converging to Λ does not contain $w(\gamma)$, so that all of the intersections of $w(\gamma)$ with such multicurves, as well as with Λ, are transverse. $\square$

6 Rectangular Decompositions

In this section we introduce a specific type of cell decomposition for compact surfaces with boundary. Knowing that we have this type of cell structure will allow us to perform an Euler characteristic computation from a combinatorial stacking as was done for graphs.

Definition 6.1. Let Σ be a compact surface with non-empty boundary. We call a cell decomposition of Σ *rectangular* if:

1. its 1-skeleton is the union of $\partial\Sigma$ with a finite forest Φ whose leaf vertices are exactly the 0-cells lying on $\partial\Sigma$;

2. each 2-cell contains exactly two edges of $\partial\Sigma$ in its boundary.

We call Φ the *interior 1-skeleton*, and the 2-cells *rectangles*, each having its two "parallel" arcs of $\partial\Sigma$ spanned by arcs (linear sequences of edges) in a pair of components of Φ. For each connected component T of Φ, we define its valency $\Delta(T)$ as the total number of distinct 2-cells its edges are incident to.

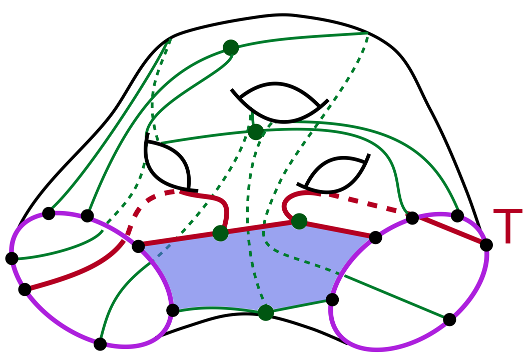

Figure 7: Sketch of a rectangular decomposition on a surface Σ. The rectangular 2-cell highlighted blue spans arcs on distinct components of $\partial\Sigma$, and the component T of the interior 1-skeleton highlighted red has valency $\Delta(T) = 4$.

Example 6.2. It is not difficult to construct a rectangular decomposition for any orientable compact surface with boundary (other than the disc D^2). For instance an annulus has a rectangular decomposition induced by a pair of disjoint edges both joining the two distinct boundary components. Similarly a disc with 3 holes can be decomposed into rectangles as shown in the upper-left of Figure 8. The decomposition shown there generalises to discs with any number n of holes, by making the vertex in the centre of the disc n-valent, connecting to each of the holes, and the other parts of the interior 1-skeleton connecting pairs of holes to each other as well as to the outer boundary of the disc staying essentially the same.

Any compact surface with boundary can be obtained from a disc with some positive number of holes by adding in 1-handles that connect pairs of boundary components. The bottom-right of Figure 8 shows how, given a rectangular decomposition on such a disc as shown, it can be extended to a surface with genus (and fewer boundary components) by adding another edge to the existing interior 1-skeleton, and one crossing the core of the 1-handle attached. Rather than these generic decompositions, we will go on in the rest of this section to find ones whose rectangles carry meaningful information about the w-cycles in Σ.

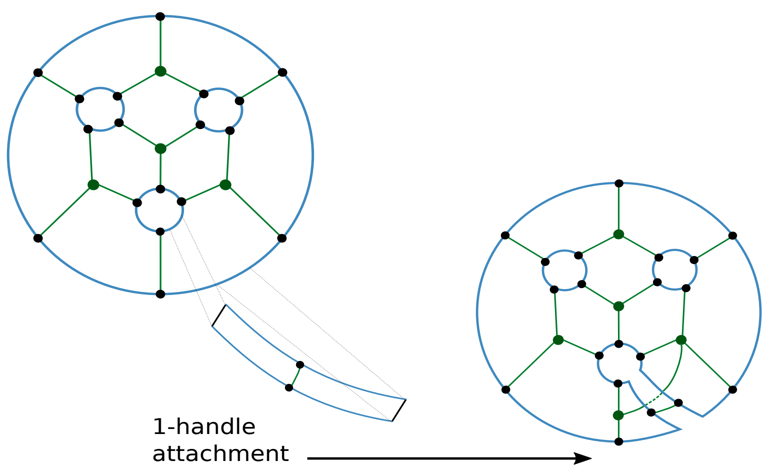

Figure 8: A disc with 3 holes can be decomposed into rectangles, with arcs in boundary components (blue) spanned by pairs of arcs in the tree components of the interior 1-skeleton (green). A method for extending the rectangular decomposition over handle attachments is shown.

Rectangular decompositions provide our desired graph-like description of Σ, with each tree T of the interior 1-skeleton (including its vertices) corresponding to a single vertex in a graph, and the rectangular 2-cells behaving like thickened edges. Note that each such T has Euler characteristic equal to 1, and the rectangular regions (minus their boundary arcs lying in the interior 1-skeleton, but including the arcs on $\partial\Sigma$) contribute -1 to the total Euler characteristic, being homeomorphic to a disc with 2 disjoint closed arcs removed from its boundary; so the components of a rectangular decomposition make the same contributions to Euler characteristic as do vertices and edges in a graph.

We observe at this point that all the 2-cells in such a decomposition have the same structure, as rectangles spanning two (non-adjacent) arcs in the boundary of the surface. It follows that the interior 1-skeleton contains all the data of the decomposition. We will go on to obtain rectangular decompositions from pulling back the structure of foliations on a closed surface which, analogously, can be specified just from the data of their singular leaves.

6.1 Foliations and Rectangles from Long Geodesics

Property (2) of Lemma 5.2 suggests how we can use a lamination Λ to derive a decomposition on Σ satisfying Definition 6.1. Intuitively, we can think of forming rectangles made up of parallel copies of the arcs of $h^{-1}\Lambda$, and the interior 1-skeleton will contain the singularities which result when parallel copies of arcs mapping to different principal regions meet; see Figure 6. In this subsection we make this process exact using foliations, which will also help provide a framework for our construction in §7.1.

Fix a minimal filling geodesic lamination Λ on the orientable surface Ω which has length function l on curves induced by its hyperbolic metric. Recall from our discussion in §5.2 that Λ consists of uncountably many bi-infinite geodesics, and can be considered as a limit of points in the "rational" subspace of $ML(\Omega) \cong \mathbb{R}^{-3\chi(\Omega)}$ consisting of (geodesic) multicurves. As such, we now fix a sequence of multicurves $\{\mathcal{C}_i\}_{i\in\mathbb{Z}}$ in Ω converging geometrically to Λ. In particular, we assume that the lengths of these curves increase monotonically and unboundedly with i. Given $r \in \mathbb{R}$, we denote by i_r the first integer such that:

$$l(c) > r, \forall c \in \pi_0(\mathcal{C}_{i_r}) \subset \Omega$$

Compare this with Lemma 5.2, which tells us that in the limit as $\mathcal{C}_i \to \Lambda$, the preimages under h of the components of the $\mathcal{C}_i$ converge to finite length arcs between components of $\partial\Sigma$. We can lift l to $\tilde{l}$ on the degree d cover Ω_1 from the proof of Lemma 5.2, measuring lengths of curve segments within each fundamental domain just as we would in Ω. Restricting $\tilde{l}$ to $\Sigma \subset \Omega_1$, we then have a well-defined supremum

$$M = \sup\{\tilde{l}(\tilde{\lambda}) \mid \tilde{\lambda} \text{ a component of } h^{-1}\Lambda\}$$

Then for all $i > i_M$, the preimage

$$\tilde{\mathcal{C}}_i := h^{-1}\mathcal{C}_i$$

must similarly consist of compact arcs with endpoints on $\partial\Sigma$. Indeed, since $\Lambda \subset \Omega$, $\Lambda_1 \subset \Omega_1$ are both minimal laminations, any leaf λ has a unique lift $\tilde{\lambda}$, and the points of these two leaves are related by a length-preserving bijection, so when i is sufficiently large, we can observe similar behaviour for the components of $\mathcal{C}_i$, $\tilde{\mathcal{C}}_i$ as we would for the leaves of Λ, Λ_1. The preimage of each component c of $\mathcal{C}_i$ is a simple closed curve $\tilde{c} \subset \Omega_1$ covering c at most d times. So if $l(c) > M$, then $\tilde{l}(\tilde{c})$ also exceeds the supremum M for each such preimage $\tilde{c}$ in $\tilde{\mathcal{C}}_i$, and since we can assume $\tilde{c} \cap \Sigma$ lies arbitrarily close to components of $h^{-1}\Lambda$, its restriction to Σ also consists of a disjoint union of arcs spanning components of $\partial\Sigma$.

Now, we may assume each multicurve is maximal in the sequence (i.e. cannot have another disjoint, geodesic, simple closed curve added to it and still form part of a sequence converging to Λ). Since the limit Λ is filling, after a finite number of steps, say I, the sequence stabilises to consist entirely of multicurves that cut Ω into pieces which have no topology other than their boundary components. More precisely, since $\Omega - \Lambda$ is a union of discs, if, for some i, a component of $\Omega - \mathcal{C}_i$ contains an essential simple closed curve s in its interior then this component cannot be a disc, therefore not contained in a principal region and so arcs of leaves of Λ cut s. This means that for some $j > i$, $\mathcal{C}_j$ contains an arc lying sufficiently close to such an arc of Λ to also cut s, so that

$$s \not\subset \Omega - \mathcal{C}_j$$

So after I steps, the components of $\Omega - \mathcal{C}_I$ can only contain simple closed curves which are either nullhomotopic or homotopic to products of their boundary components (i.e. copies of the components of $\mathcal{C}_I$). This means the components of $\Omega - \mathcal{C}_I$ are homeomorphic to spheres with some number of discs removed.

The number of such components, and the number of boundary components of each, is determined by the number of ideal polygons in $\Omega - \Lambda$, and how many sides each has.

We can now choose any i greater than the maximum of i_M and I, and fix

$$\mathcal{C} := \mathcal{C}_i$$

a collection of simple closed curves cutting Ω into a collection of spheres with various numbers of discs removed. In summary, translating Lemma 5.2 into a statement about a multicurve that sufficiently closely approximates Λ produces:

Lemma 6.3. *There exists a multicurve $\mathcal{C} \subset \Omega$ such that:*

1. *$\Omega - \mathcal{C}$ is homeomorphic to a disjoint union of spheres each with at least 3 discs removed; and*
2. *$h^{-1}\mathcal{C}$ is a disjoint union of compact arcs with endpoints on $\partial\Sigma$.*

With a choice of $\mathcal{C}$ now fixed, we consider some component:

$$X \simeq \overline{S^2 - \sqcup_{i=1}^{k} D_i}$$

of the closure of $\Omega - \mathcal{C}$, where each $D_i \simeq D^2$, and $k \geq 3$. Taking parallel copies of the boundary components of X, we can produce a foliation of X with two k-pronged singularities. The smooth leaves are all circles and the singular leaf is a graph containing the 2 singularities as vertices, and k edges joining the prongs. Call the foliation induced on each component X in this way $\mathcal{F}^X$, and denote the singular graph:

$$\Gamma_X := \mathcal{F}^X_{\text{sing}}, \qquad \forall X \in \pi_0(\Omega - \mathcal{C})$$

We can view X as the union of k annuli, each with one of its boundary components identified with a component of ∂X, and the other with a simple loop in Γ_X, and these annuli are naturally foliated by parallel copies of such loops. See Figure 9.

Since Ω is orientable, each boundary component $c \in \partial X$ represents a 2-sided curve in Ω, and there is some other boundary component $c' \in \partial X'$ (where X' could either be X or another component of $\Omega - \mathcal{C}$) which is identified with c when

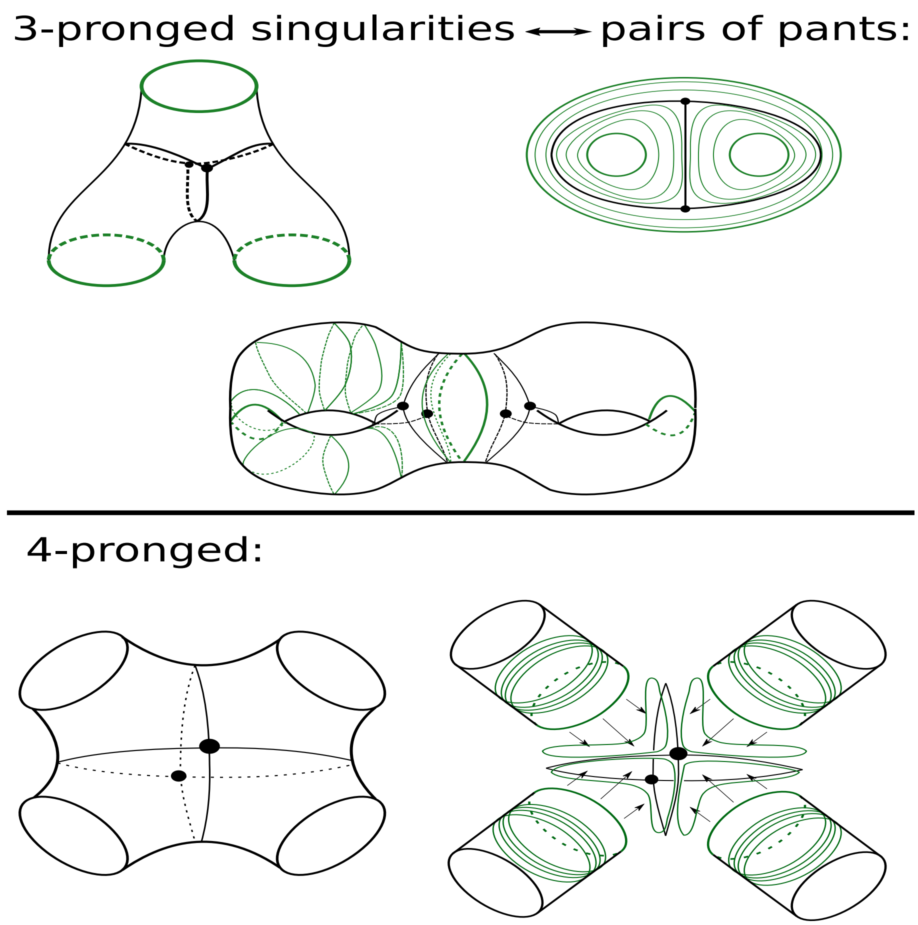

Figure 9: At the top we show two views of a “pair of pants”, $\overline{S^2 - \sqcup_{i=1}^{3} D_i}$. By taking parallel copies of the boundary components of the pants and matching the boundary components in pairs we form a foliated genus 2 surface. The smooth leaves are shown in green, and their isotopy classes converge on the black singular leaves. Below, we show a 4-holed sphere X decomposed as a union of 4 foliated annuli, producing a foliation $\mathcal{F}^X$ with two 4-pronged singularities.

recovering Ω. The annuli forming X, X' that had c, c' as boundary components then join to form a single foliated annulus whose boundary components are identified with distinct simple loops in Γ_X, $\Gamma_{X'}$. This extends the foliations $\mathcal{F}^X$ over all components of $\Omega - \mathcal{C}$, producing a foliation on the whole of Ω, which we call $\mathcal{F}^{\mathcal{C}}$. Then the collection of singular leaves,

$$\mathcal{F}^{\mathcal{C}}{}_{\text{sing}} = \coprod_{X \in \pi_0(\Omega - \mathcal{C})} \Gamma_X$$

cuts Ω into a disjoint union of annuli.

We now come to our reason for constructing these foliations:

Lemma 6.4. *Let $\mathcal{C}$ be a multicurve in Ω such that:*

1. *$\Omega - \mathcal{C}$ is a disjoint union of spheres each with at least 3 discs removed*
2. *$h^{-1}\mathcal{C}$ is a disjoint union of compact arcs with endpoints on $\partial\Sigma$.*

If $\mathcal{F}^{\mathcal{C}}$ is the filling foliation on Ω induced by $\mathcal{C}$, then

$$h^{-1}\left(\mathcal{F}^{\mathcal{C}}{}_{sing}\right)$$

comprises the interior 1-skeleton of a rectangular decomposition of Σ.

Proof. The immersion h preserves the local surface structure away from the boundary; in particular, the smooth leaves of $\mathcal{F}^{\mathcal{C}}$ have smooth preimage in Σ. Smooth leaves in Ω are all isotopic to one of the curves in $\mathcal{C}$, so by our hypotheses, the smooth leaf preimages are all compact arcs with endpoints on $\partial\Sigma$.

We similarly consider the singular leaves $\Gamma_X \subset \mathcal{F}^{\mathcal{C}}$, for X the components of $\Omega - \mathcal{C}$ as above. Each Γ_X consists of 2 vertices of the same valency $k \geq 3$, containing k different simple loops isotopic to classes of smooth leaves in $\mathcal{F}^X$. So, since none of those loops have closed curve preimages in Σ, the $h^{-1}(\Gamma_X)$ consist of disjoint unions of trees in the interior of Σ, where we add leaf vertices at the intersection points of the $h^{-1}(\Gamma_X)$ with $\partial\Sigma$. These leaf vertices together with the k-valent preimages of the singular vertices, $h^{-1}\left(\Gamma_X^{(0)}\right)$, form the 0-skeleton of our decomposition of Σ. The edges of the trees will form the interior 1-skeleton, so we now denote

$$\Phi := h^{-1}\left(\mathcal{F}^{\mathcal{C}}{}_{\text{sing}}\right)$$

The rest of the 1-skeleton consists of the edges of the components of $\partial\Sigma$, subdivided by the leaf vertices of Φ.

We easily verify that the 2-dimensional regions obtained by cutting Σ along $\Phi \cup \partial\Sigma$ are all simply-connected, so that we really have obtained a cell decomposition. Indeed, suppose that one of the regions contains a non-contractible curve c; since h is π_1-injective, c maps to a non-contractible curve in $\Omega - {\mathcal{F}^{\mathcal{C}}}_{\text{sing}}$. But the only non-contractible curves in $\Omega - {\mathcal{F}^{\mathcal{C}}}_{\text{sing}}$ are homotopic to products of curves in $\mathcal{C}$ by construction, so this contradicts our hypotheses, and no such c can exist.

It remains to check that the 2-cells of Σ have the rectangular structure specified by Definition 6.1. The boundary of each 2-cell consists of an alternating sequence of arcs contained in components of Φ, and arcs contained in components of $\partial\Sigma$, and we just need to confirm that there are just two of each type of arc, describing the spans of a rectangle. This simply follows from the fact that all the smooth leaves in any given component of $\Omega - \mathcal{F}^{\mathcal{C}}_{\text{sing}}$ are isotopic, implying that the arcs between components of $\partial\Sigma$ which comprise the interior of any given 2-cell of Σ are all isotopic through such arcs. If any 2-cell contained 3 or more arcs of $\partial\Sigma$ in its boundary, its interior would have to contain arcs of $h^{-1}\mathcal{C}$ spanning different pairs of arcs of $\partial\Sigma$, and so would contain more than one isotopy class of arcs spanning boundary components. $\square$

Finally, we have:

Proposition 6.5. *Let Ω, Σ, w be as in Theorem A, with $\partial\Sigma \neq \emptyset$. There is a rectangular decomposition of Σ, whose interior 1-skeleton is transverse to the core curves of the w-cycles in Σ.*

Proof. That the decomposition exists follows from the fact that Ω admits minimal filling laminations and Lemma 5.2, as Lemma 6.3 then gives a multicurve $\mathcal{C}$ satisfying the hypotheses of Lemma 6.4. Lemma 5.2 also allows us to assume $w(\gamma)$ is transverse to the leaves of an arbitrary minimal filling lamination Λ, as well as to any multicurve approximating Λ sufficiently closely. Since Ω is foliated by leaves parallel to $\mathcal{C}$, we can therefore arrange all intersections of $w(\gamma)$ with ${\mathcal{F}^{\mathcal{C}}}_{\text{sing}}$ to be transverse. Since the interior 1-skeleton of the rectangular decomposition on Σ is $h^{-1}\left({\mathcal{F}^{\mathcal{C}}}_{\text{sing}}\right)$ and the core curves of the w-cycles in Σ lie in $h^{-1}\left(w(\gamma)\right)$ (and since h is an immersion), all of their intersections are also transverse. $\square$

The statement regarding transversality to the w-cycles effectively means that the w-cycles are carried by circular sequences of rectangles, directed along paths in Σ determined by the pair of boundary components that they are moving between at any given point in their traversal. This property will be used in the next section, when we want to establish a correspondence between the w-cycles and the max-height regions of a stacking of the rectangles, in order to apply Lemma 6.6.

6.2 Structure of Components of $\mathbb{S}$

Before showing how we use rectangular decompositions to compute $\chi(\Sigma)$ from a stacking of the map $w' : \mathbb{S} \to \Sigma$ describing the w-cycles, we briefly discuss here the cell structure that we fix on the components of $\mathbb{S}$ to make w' a combinatorial map, given a rectangular decomposition of Σ. The circular components of the pullback $\Sigma \times_\Omega S$ are all homeomorphic to annuli with countably many closed arcs removed from their boundary curves. The core curve generating each component's fundamental group covers γ, the core curve of S, finitely under the induced map $\sigma : \mathbb{S} \to S$.

We visualise the components of the pullback by first thinking of the universal cover of Σ, which is the hyperbolic plane minus countably many bigons, bounded by segments of the ideal boundary and lines with endpoints on this boundary, each line covering some component of $\partial\Sigma$. Each component of $\mathbb{S}$ is an intermediate cover between the universal cover and Σ. Since they are circular, we can visualise them by moving outwards orthogonally from the core curves in $\sigma^{-1}(\gamma)$, and as we move out we see a similar recurring pattern of lifts of components of $\partial\Sigma$ as in the universal cover; see the component illustrated in the top-left of Figure 10.

The components of $\mathbb{S}$ are non-compact but we are only interested in them insofar as they describe the w-cycles in Σ, and this information can be captured by restricting to compact subsets of each component. To make this precise in our setting, we fix a rectangular decomposition on Σ with interior 1-skeleton Φ, and we pull it back to $\mathbb{S}$. That is, for any given component $\psi \in \pi_0(\mathbb{S})$, we fix a cell decomposition of ψ with skeleta:

$$\psi^{(i)} := (w')^{-1}\left(\Sigma^{(i)}\right), \qquad 0 \leq i \leq 2$$

(since the components of the pullback cover Σ, this is indeed a cell decomposition, with infinitely many cells, each mapping homeomorphically to cells of Σ). Now the intersection of ψ with $\sigma^{-1}(\gamma)$ is contained in finitely many cells in this decomposition. We delete the interiors of all 2-cells except for those intersecting $\sigma^{-1}(\gamma)$, as well as all components of $(w')^{-1}(\Phi) \subset \psi^{(1)}$ which contain no edge adjacent to such a 2-cell, to obtain a compact circular 2-complex representing each w-cycle. We have taken care to keep enough of $\psi^{(1)}$ intact from the original pullback so that entire components of the interior 1-skeleton from the decomposition of Σ lift to the truncated components and not just subsets of them — in general this stops the truncated components from being surfaces, instead they are "circular surfaces with trees hanging from the boundary", but we want to keep these trees for a technical point in the proof of Lemma 6.6. From this point on, when we refer to the components of $\mathbb{S}$ we will assume that we have restricted to these compact cores, unless stated otherwise.

Of course lifting the decomposition to $\mathbb{S}$ produces a cell structure that mimics the one on Σ, but we take a little care to note that it does not strictly lift to a rectangular decomposition on the truncated components in the sense of Definition 6.1, even if we were to delete the extra trees attached to it. This is because after we truncate as shown in Figure 10, the boundary of each of these components will consist not just of edges in the preimage of $\partial\Sigma$, but also edges in the preimage of the interior 1-skeleton of Σ that were in the boundary of exactly one of the deleted 2-cells from the whole pullback. The 2-cells essentially have a rectangular structure, but instead of having exactly 2 edges of $\partial\mathbb{S}$ in their boundary, they have 2 subdivided boundary arcs possibly with extra trees attached, and the analogue of the interior 1-skeleton of these circular components is just a disjoint union of single edges crossed by the core curve.

Ultimately it will only be important that our decomposition of Σ satisfies Definition 6.1 for our Euler characteristic computation in the next subsection. Since w' is a combinatorial map with respect to that decomposition, we do not need to alter the cell structure on the components of $\mathbb{S}$ any further.

6.3 Computing with Stackings of Rectangles

To justify introducing this specific form of decomposition, we now show that surface stackings which respect rectangular decompositions allow us to mimic

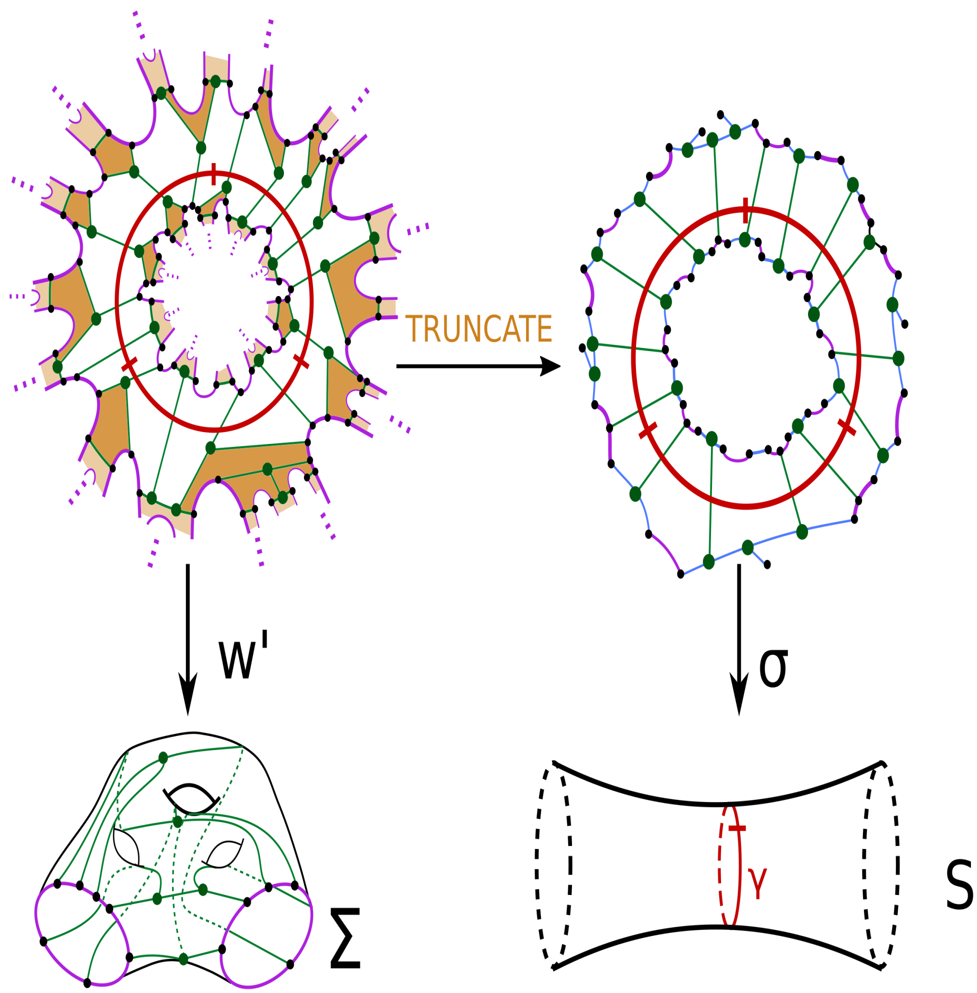

Figure 10: A component of $\mathbb{S}$ covering the core of S 3 times. We can truncate the component to a compact core, by deleting the 2-cells that don't intersect $\sigma^{-1}(\gamma)$, i.e. the regions highlighted orange here. In the truncated component, segments of the boundary that are in the preimage of $\partial\Sigma$ are purple, and the rest of the boundary, which is in the preimage of the interior 1-skeleton of Σ, is blue. This figure shows a special case where all trees of Φ are trivalent.

the argument of [52, Lemma 2.4], in order to compute the Euler characteristic of the image surface.

Lemma 6.6. *Let $\mathbb{S}$ be a disjoint union of compact circular 2-complexes, Σ a compact surface with non-empty boundary, and*

$$\rho : \mathbb{S} \twoheadrightarrow \Sigma$$

a surjective immersion which has a stacking

$$\hat{\rho} : \mathbb{S} \hookrightarrow \Sigma \times \mathbb{R}$$

Suppose also that Σ has a rectangular decomposition with interior 1-skeleton Φ, with respect to which ρ is combinatorial and maps the preimages of components of Φ onto the components of Φ homeomorphically. Then $-\chi(\Sigma)$ is equal to the number of max-height strips of $\hat{\rho}$.

Proof. We compute $\chi(\Sigma)$ from the rectangular decomposition: all vertices and non-boundary edges are contained in the set Φ of connected components of the interior 1-skeleton, and each of these trees has 1 more vertex than it has edges. This gives

$$\left|\Sigma^{(0)}\right| - \left|\Sigma^{(1)}\right| = |\Phi| - \left|(\partial\Sigma)^{(1)}\right|$$

(we are using $|\Phi|$ here to denote the number of connected components of the forest Φ). Since each rectangular face of the decomposition can be specified by the pair of edges in the boundary that it spans, we also have

$$\left|(\partial\Sigma)^{(1)}\right| = 2\left|\Sigma^{(2)}\right|$$

giving

$$\chi(\Sigma) = \left|\Sigma^{(0)}\right| - \left|\Sigma^{(1)}\right| + \left|\Sigma^{(2)}\right| = |\Phi| - \left|\Sigma^{(2)}\right|$$

Each rectangular face similarly specifies a pair of arcs in components of Φ, so that

$$\sum_{T\in\Phi} \Delta(T) = 2\left|\Sigma^{(2)}\right|$$

from which we obtain

$$\chi(\Sigma) = |\Phi| - \frac{1}{2}\sum_{T\in\Phi}\Delta(T) = \frac{1}{2}\sum_{T\in\Phi}(2 - \Delta(T))$$

The result will be proven by comparing this formula for $\chi(\Sigma)$ in terms of the data of the decomposition, with the data of our stacking of the surjective combinatorial map ρ. Consider any tree $T \in \Phi$, and the intersection of its $\Delta(T)$ incident 2-cells with $\rho(\mathcal{A}_{\hat{\rho}})$. Since the components of $\mathcal{A}_{\hat{\rho}}$ are open in $\mathbb{S}$, and ρ surjective, it must be the case that two of the incident 2-cells to T are contained in the image of a single max-height region crossing an edge of T, and the interiors of the remaining $\Delta(T) - 2$ are in the images of other max-height strips terminating at edges of T; see Figure 11. This gives a bijection between ends of the max-height strips and subsets of the rectangles incident at each tree, so that the total number of strips is counted by

$$\frac{1}{2} \sum_{T \in \Phi} (\Delta(T) - 2) = -\chi(\Sigma)$$

$\square$

Remark 4. Essentially the same argument shows that $-\chi(\Sigma)$ is equal to the number of min-height strips, from which we observe that these two quantities are equal.

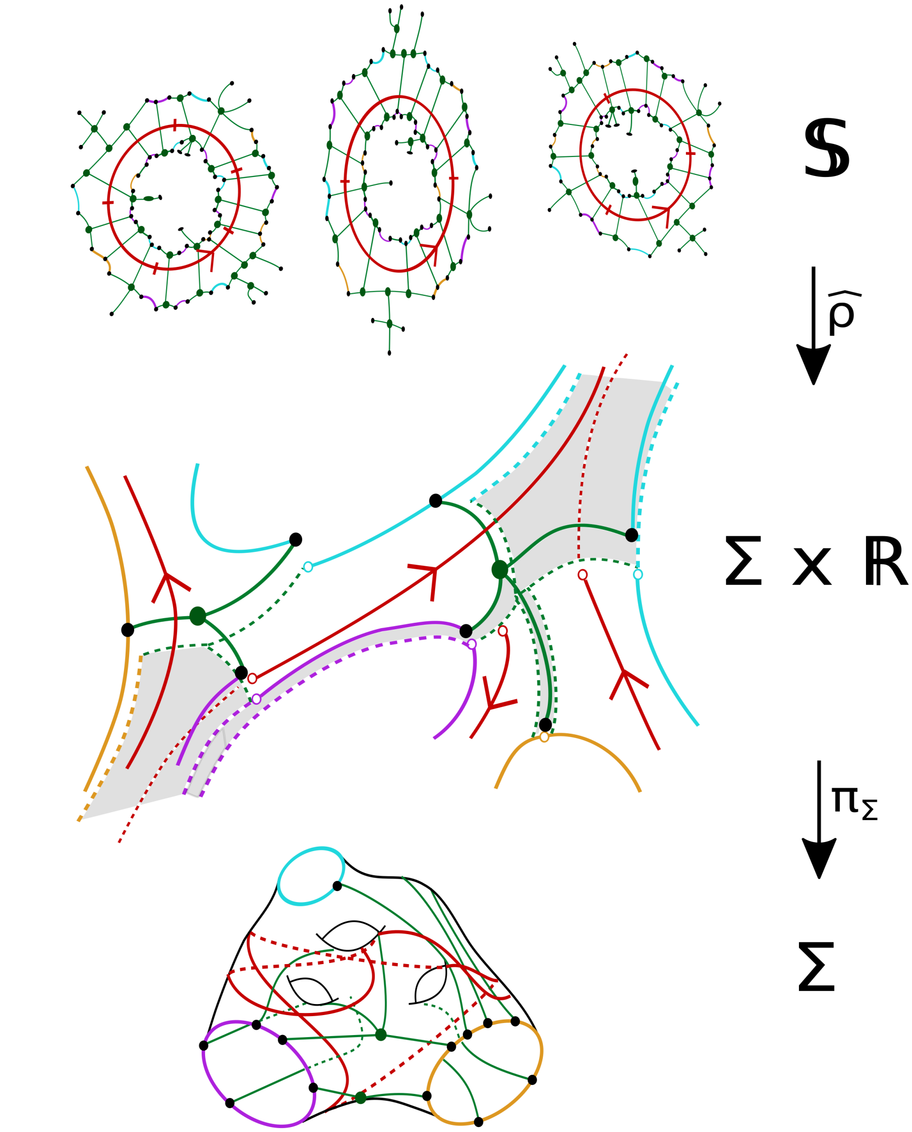

Figure 11: Self-intersections of $\rho(\mathbb{S})$ in Σ come from rectangles which get sent to distinct heights in $\Sigma \times \mathbb{R}$ by the stacking $\hat{\rho}$. The middle level in this sketch is zoomed in on part of $\Sigma \times \mathbb{R}$, with Σ stretched (compared to how it is drawn on the bottom) to emphasise the rectangular shape of the 2-cells carrying the cores of components of $\mathbb{S}$, and shading/dashed lines used to indicate subsets of the image of $\mathbb{S} - \mathcal{A}_{\hat{\rho}}$. Each component T of the interior 1-skeleton has a unique preimage $\tilde{T}$ in $\mathcal{A}_{\hat{\rho}}$ — one of the edges of $\tilde{T}$ is adjacent to a pair of rectangles in a single max-height region, and the remaining $(\Delta(T) - 2)$ rectangles in contact with T have preimages in $\mathcal{A}_{\hat{\rho}}$ with $\mathbb{R}$ coordinates smaller than those of $\tilde{T}$, and therefore containing terminal edges of max-height strips.

7 Surface Stackings

We will use stackings of the components of $\mathbb{S}$ which respect the rectangular decomposition of Σ as discussed above to prove Theorem A. To show that such stackings exist, we apply in this section the arguments of [39, Lemma 3.1] and [52, Lemmas 3.3, 3.4] to the setting of surfaces. Since it will be useful in proving Proposition 7.7, we will obtain the stacking of $\mathbb{S}$ in $\Sigma \times \mathbb{R}$ as a pullback of a stacking in $\Omega \times \mathbb{R}$ of the circular surface S whose core curve γ maps to the geodesic representative of w in Ω.

Finding such a stacking is equivalent to putting discrete orderings on the preimage sets of points in Ω under w that remain consistent with each other as S is traversed, and this can be achieved by finding a sequence of covers of Ω to which w lifts, and whose deck transformation group at each stage is isomorphic to $\mathbb{Z}$. This will induce the orderings we need on point preimages from the standard ordering of the integers, since points of S with the same image in Ω will eventually lift to distinct points in the covers, and these points will then be related by the deck transformations; this idea relates to our discussion in §3.2. To guarantee that the sequence of lifts terminates, we require a finite amount of data in the space we initially map from, so we begin by finding a suitable compact subset of S to which we will restrict w.

Remark 5. As stated above, for the proof of Theorem A we will be assuming that Ω is orientable, and the circular surface S an annulus. However, if there was a proof of Proposition 6.5 that could guarantee a rectangular decomposition carrying the w-cycles in Σ induced from some filling foliation when Ω is non-orientable, the method we describe in this section for obtaining a stacking with respect to this decomposition would work equally well. We will discuss the minor differences that would arise from non-orientability in §10.3.

7.1 Constructing the Core

We will construct our compact core annulus, which we denote $S_c \subset S$, so that it is naturally equipped with a cell structure that makes w into a cellular map from S_c to Ω. Fixing a filling foliation $\mathcal{F}^{\mathcal{C}}$ as in Lemma 6.4, the singular leaves cut Ω into a disjoint union of annuli. We can extend the graphs forming $\mathcal{F}^{\mathcal{C}}_{\text{sing}}$ to be the 1-skeleton of a cell decomposition by adding an edge to each annulus with an endpoint on a vertex of the singular leaf at each of its boundary circles.

To make this precise, we now consider $w^{-1}\left(\mathcal{F}^{\mathcal{C}}_{\text{sing}}\right)$, the pullback of the singular leaves to S. Since $w(\gamma)$ is transverse to the leaves, $w^{-1}\left(\mathcal{F}^{\mathcal{C}}_{\text{sing}}\right)$ contains no closed curves, and so consists of a disjoint union of infinite trees embedded in S, each one covering one of the graphs Γ_X comprising $\mathcal{F}^{\mathcal{C}}_{\text{sing}}$. The core curve γ passes through a single edge in a finite number of these trees, as shown in Figure 12. These edges are met in a cyclic sequence by γ, and between any two of them γ lies in a non-compact strip which is bounded by an "outer" sequence of edges in the trees it is passing between, and which covers one of the circular components of $\Omega - \mathcal{F}^{\mathcal{C}}_{\text{sing}}$ infinitely under the immersion w.

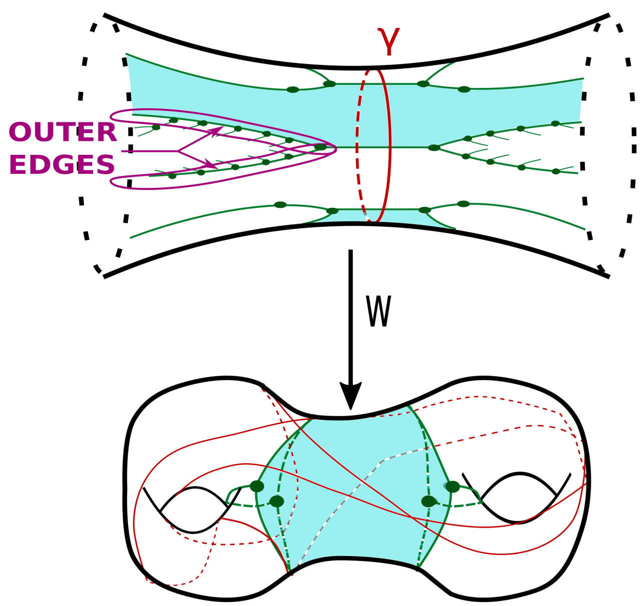

Figure 12: Schematic illustrating the structure of S when $\mathcal{F}^{\mathcal{C}}$ is induced by a pants decomposition of Ω. The singular leaves and their pullback are shown in green, and core curve γ shown in red. The blue annulus in Ω, made up of isotopic smooth leaves, has preimage consisting of non-compact strips in S, and the boundary edges of these strips consist of sequences of edges in the trees comprising $w^{-1}\left(\mathcal{F}^{\mathcal{C}}_{\text{sing}}\right)$.

The 2-cells of S_c will eventually be constructed by truncating each of these strips to obtain a specific compact neighbourhood of γ. Since we will want to pull back our stacking of S_c to the components of $\mathbb{S}$, we need to ensure that this neighbourhood is sufficiently large to contain the images under σ of all the 2-cells of $\mathbb{S}$. So consider now a single component:

$$a \in \pi_0 \left(\Omega - \mathcal{F}^{\mathcal{C}}_{\text{sing}} \right)$$

The preimage $h^{-1}(a)$ consists of a finite union of the rectangular 2-cells of Σ, and each of these 2-cells immerses in a, their intersection with $\partial\Sigma$ mapping to a pair of arcs with endpoints on $\mathcal{F}^{\mathcal{C}}_{\text{sing}}$. In S, the union (in general not disjoint) of the preimages of these arcs from all the 2-cells of $h^{-1}(a)$ produces a cyclic sequence which we see infinite periodic repetitions of in each of the strips comprising $w^{-1}(a)$; see Figure 13.

Taking the image of a 2-cell of $\mathbb{S}$ under σ, it will be contained in one of these strips in $S - w^{-1}\left(\mathcal{F}^{\mathcal{C}}_{\text{sing}}\right)$, with each of its edges that lie in $(w')^{-1}(\partial\Sigma)$ mapping to one of the periodically repeating arcs, one on either side of γ. Each of these arcs in the image of σ has endpoints lying either on a vertex or between a pair of vertices on some component of $w^{-1}\left(\mathcal{F}^{\mathcal{C}}_{\text{sing}}\right)$, and we can roughly describe the "width" of the 2-cell by the number of vertices lying on this component between the arc endpoints.

We now consider all of the (finitely many) images of 2-cells of $\mathbb{S}$ that lie in strips of $w^{-1}(a)$, and identify the corresponding arc endpoint at the furthest distance from γ as counted by number of vertices on the strip boundary. Once we have identified this endpoint, we continue outward from the core curve along the strip boundary on until we reach the next vertex which we will call v_a, and starting from v_a, count some positive integer number n_a of vertices between it and γ. There may be in fact be several choices of v_a — all that matters is that no rectangles of $(h \circ w')^{-1}(a) \subset \mathbb{S}$ have image in S extending further than n_a vertices away from the core curve; see Figure 14.

We add an edge joining v_a to the vertex on the opposite singular leaf making up the strip boundary, on the same side of γ as v_a, and also n_a vertices away from γ. We call the image in Ω of this edge e_a — if a is an annulus with boundary components identified with loops in the graphs Γ_X, $\Gamma_{X'}$ (it is possible $X = X'$, but the loops will then be distinct), then e_a stretches across a to join one of the

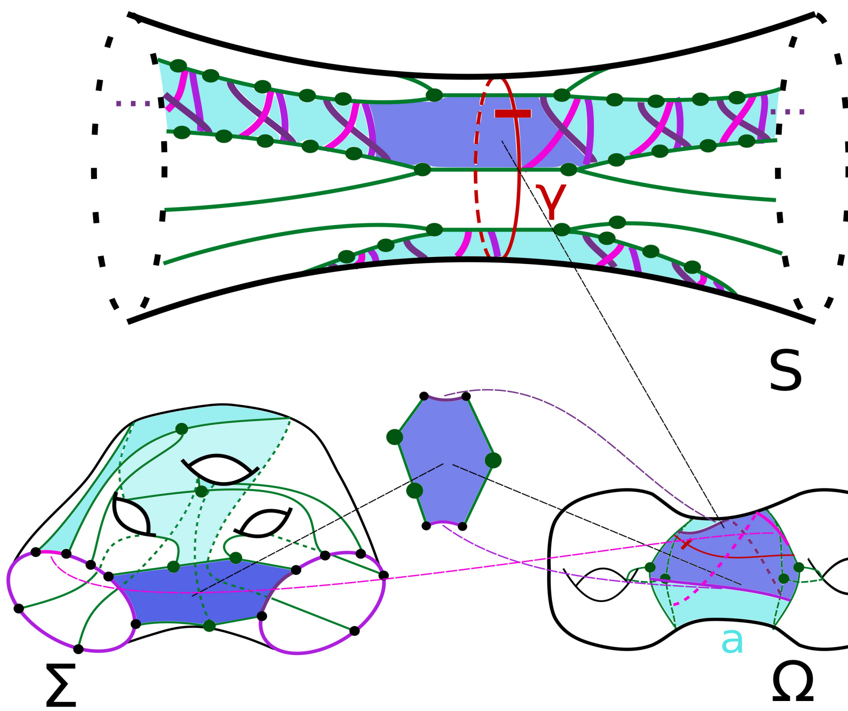

Figure 13: The regions shaded blue indicate a chosen component $a \in \pi_0\left(\Omega - \mathcal{F}^{\mathcal{C}}_{\mathrm{sing}}\right)$ and its preimages in Σ and S, and the darker regions correspond to a specific rectangular 2-cell in Σ, with image in a and one of its preimages in S highlighted. The images of the purple arcs in the boundary of this rectangle cut across a, as do those from all other pairs of arcs in $h^{-1}(a) \cap \partial\Sigma$ (another one is indicated in pink). Lifting these arcs from Ω to S, we see a sequence of arcs in each strip of $w^{-1}(a)$ which repeats periodically as we move outwards from γ. On each of the strip boundary components shown in green, moving past 2 lifts of the singular vertices roughly corresponds to a full rotation around the core of a in Ω.

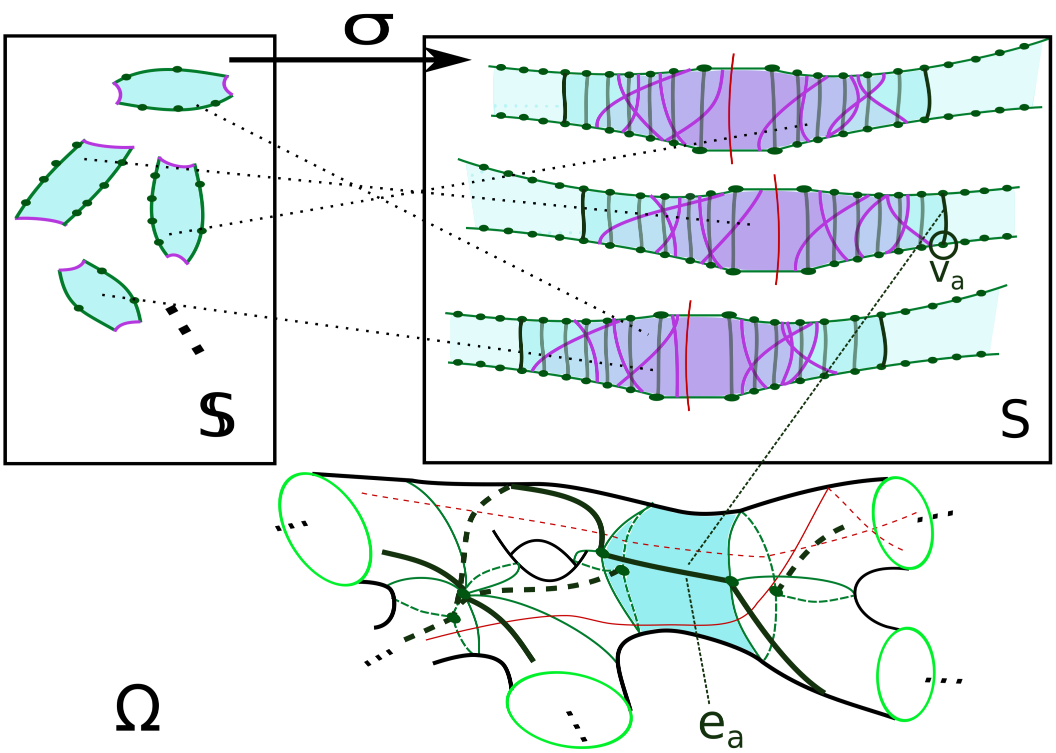

Figure 14: Strips making up $w^{-1}(a)$ lie in S, where a is the component of $\left(\Omega - \mathcal{F}^{\mathcal{C}}{}_{\text{sing}}\right)$ highlighted blue, and rectangular 2-cells of $\mathbb{S}$ map into these strips. We have little control over the subset of each strip spanned around the core curve (red) by the images of the rectangles, but we can count how many vertices we have to travel out along each strip boundary from the core until we have gone past the union of all of them. In this sketch, $n_a = 7$, and a choice of v_a is indicated. We can then add n_a lifts of e_a on either side of the core in each strip and delete everything further out to get a subdivided compact subset of $w^{-1}(a)$ that includes the images of all relevant rectangles.

vertices of Γ_X to one of those of $\Gamma_{X'}$. When Ω is cut along the graph

$$\Gamma_X \cup e_a \cup \Gamma_{X'}$$

the result is that a is cut into a single disc. Doing this in each component defines the 2-cells, and therefore the entire cell decomposition that we want to fix on Ω.

The 1-skeleton of $w^{-1}(a) \subset S_c$ is then specified by all lifts of e_a in each strip of $w^{-1}(a)$ with endpoints on the first n_a vertices away from γ on both sides. We delete any edges and vertices of $w^{-1}\left(\mathcal{F}^{\mathcal{C}}_{\text{sing}}\right)$ which are not adjacent to one of these lifts of e_a. The 2-cells of $S_c \cap w^{-1}(a)$ are now specified in the obvious way, by adding one to fill each region in the strips between all of the lifts of e_a that have been included in the 1-skeleton — each 2-cell we add maps to $(a - e_a)$ homeomorphically under w.

Once we have done this for each $a \in \pi_0\left(\Omega - \mathcal{F}^{\mathcal{C}}_{\text{sing}}\right)$, the embedded graphs of $\mathcal{F}^{\mathcal{C}}_{\text{sing}}$ together with the collection of edges e_a comprise the 1-skeleton of a cell decomposition for Ω, with 2-cells being the complementary regions $a - e_a$. The immersion w restricted to S_c is now combinatorial with respect to this cell structure on Ω. We also have a canonical inclusion map for the image under σ of each 2-cell of $\mathbb{S}$ into S_c, although this map is not combinatorial.

7.2 The Loo-Roll Lemma for Surfaces

We will now construct a stacking of the core circular surface S_c in $\Omega \times \mathbb{R}$, based on [52, §3]. As we discussed in § 3.2, these methods are strongly connected to right-orderability, with the existence of a stacking being equivalent to right-orderability of the one-relator surface group $\pi_1(\Omega)/\langle\langle w \rangle\rangle$. This manifests in a process of "tower-lifting" of maps, originating in Papakyriakopoulos' methods for proving the loop and sphere theorems for 3-manifolds [65] (see also [72]).

Definition 7.1. [39] Let $f : Z \to X$ be a combinatorial map between compact, connected cell complexes. A *cyclic tower lifting* of f consists of the following data:

1. A finite sequence of combinatorial maps between connected cell complexes:

$$Y_n \xhookrightarrow{i_n} X_n \xrightarrow{p_n}\!\!\!\!\rightarrow Y_{n-1} \xhookrightarrow{i_{n-1}} X_{n-1} \xrightarrow{p_{n-1}}\!\!\!\!\rightarrow \cdots Y_0 \xhookrightarrow{i_0} X_0 := X$$

where each i_j is an inclusion map and each p_j a covering space map for a cover of Y_{j-1} with deck group isomorphic to $\mathbb{Z}$.

2. A collection of cellular maps $f_j : Z \to X_j$ for $1 \leq j \leq n$ such that

$$Y_j = f_j\,(Z) \subset X_j,$$

and

$$Y_0 = f\,(Z) \subset X,$$

and such that the following diagram commutes:

$$\begin{tikzcd}
& Y_n \arrow[d, "i_n"] \\
& X_n \arrow[d, "p_n"] \\
& \vdots \\
& Y_0 \arrow[d, "i_0"] \\
Z \arrow[r, "f"] \arrow[ur, two heads, "f"] \arrow[uuuur, two heads, bend left, "f_n"] \arrow[uuur, two heads, bend left] & X
\end{tikzcd}$$

We call such a lifting *maximal* if f does not lift to any proper cyclic cover of Y_n.

In the rest of the section we discuss the arguments used to show the existence firstly of maximal cyclic tower liftings and then stackings for w-cycles in surfaces; our core circular surface S_c with the cell structure described in the previous section will play the role of Z, and Ω the role of X, in Definition 7.1. First, we present the following lemma to make explicit the structure of the top level of a tower lifting of a circular complex.

Lemma 7.2. *Let S be a circular surface, and $f : S \looparrowright \Omega$ an essential combinatorial immersion to a closed surface with a maximal cyclic tower lifting:*

$$S \xrightarrow{f_n} Y_n \xrightarrow{i_n} X_n \xrightarrow{p_n} \cdots Y_0 \xrightarrow{i_0} \Omega$$

Then $Y_n = f_n(S)$ is circular.

Proof. Every cover of Ω that has infinite deck group must have free fundamental group, so either $Y_n = Y_0 = \Omega$ and there is no non-trivial lifting, or all the X_i, Y_i have free fundamental groups (except for $X_0 = \Omega$). The former possibility is ruled out by the first application of Hempel's trick discussed in § 4.1. In the latter case, if Y_n is not circular then $\pi_1(Y_n)$ is a free group of rank at least 2, so $\langle w_n \rangle$ is a proper subgroup, whose normal closure is contained in the kernel of a homomorphism

$$\pi_1(Y_n) \twoheadrightarrow \mathbb{Z}$$

(where $w_n = [f_n(S)] \in \pi_1(Y_n)$). However, this implies that there is a lift of f_n over a cyclic cover of Y_n (again, compare with the discussion in § 4.1), contradicting the maximality of the tower, so Y_n must be circular.

$\square$

A maximal cyclic tower lifting of $w : S_c \looparrowright \Omega$ exists by induction on the difference in size of 0-skeleta between S_c and its images at each step. If $|S_c^{(0)}| = |Y_0^{(0)}|$, then w would already be a π_1-surjection onto its image in Ω, so could not lift to any proper connected cyclic cover of Y_0, and we would vacuously have a maximal cyclic tower lifting to begin with (but this is not the case by Lemma 4.1). Suppose now that we have a sequence of complexes

$$Y_m \xhookrightarrow{i_m} X_m \overset{p_m}{\twoheadrightarrow} Y_{m-1} \xhookrightarrow{i_{m-1}} X_{m-1} \overset{p_{m-1}}{\twoheadrightarrow} \cdots Y_0 \xhookrightarrow{i_0} \Omega$$

providing cyclic tower lifts of w, and that w lifts to a proper, connected, cyclic cover of Y_m, so that

$$|S_c^{(0)}| - |Y_m^{(0)}| > 0$$

and we have a commutative diagram:

$$\begin{tikzcd}
& Y_{m+1} \arrow[d, "i_{m+1}"] \\
& X_{m+1} \arrow[d, "p_{m+1}"] \\
S_c \arrow[uur, bend left, "w_{m+1}"] \arrow[ur] \arrow[r, "w_m"] & Y_m
\end{tikzcd}$$

Then $|Y_{m+1}^{(0)}| \geq |Y_m^{(0)}|$, since $p_{m+1} \circ i_{m+1}$ is surjective. In fact this inequality is strict, as otherwise the image of S_c in X_{m+1} would be a homeomorphic copy of its image in X_m, and X_{m+1} would fail to be a proper covering of $Y_m = w_m(S_c)$.

So we have

$$0 \leq |S_c^{(0)}| - |Y_{m+1}^{(0)}| < |S_c^{(0)}| - |Y_m^{(0)}|$$

and since the quantity $|S_c^{(0)}| - |Y_j^{(0)}|$ can never become negative, we see that the process of finding successive lifts of w to cyclic covers of its images must terminate.

We now convert our maximal cyclic tower lifting into a stacking of w. This is done by first finding a stacking of the image of S_c at the top level of the tower — that is, an embedding

$$\hat{w}_n : S_c \hookrightarrow Y_n \times \mathbb{R}$$

such that $\pi_{Y_n} \circ \hat{w}_n = w_n$ — and then showing that this stacking can be pushed down each level of the tower inductively, until we have the required embedding in $\Omega \times \mathbb{R}$.

To start, we have that by Lemma 7.2 $Y_n = w_n (S_c)$ is circular. Now we can use our assumption of indivisibility of w to observe that, since w factors as

$$(i_0 \circ p_1 \circ \cdots \circ i_n) \circ w_n : S_c \to Y_n \to \Omega,$$

then w_n cannot be a proper covering map of circular surfaces, and Y_n is simply a 1-to-1 copy of S_c embedded in X_n.

So w_n is already injective as a map into $Y_n \subset X_n$, giving the stacking $\hat{w}_n$ of S_c into both $Y_n \times \mathbb{R}$ and $X_n \times \mathbb{R}$. Similarly, all the maps i_j, $0 \leq j \leq n$ are injective, so any stacking $\hat{w}_j$ of w_j into $Y_j \times \mathbb{R}$ immediately extends to:

$$\begin{tikzcd} & X_j \times \mathbb{R} \arrow[d, "\pi_{X_j}"] \\ S_c \arrow[ur, hook, "\hat{w}_j"] \arrow[r, "w_j"'] & X_j \end{tikzcd}$$

(we have slightly abused notation here to denote $(i_j \times \mathrm{id}_{\mathbb{R}}) \circ \hat{w}_j$ by $\hat{w}_j$, extending our map into $X_j \times \mathbb{R}$ in the obvious way). It therefore remains to show that stackings can be pushed down through the cyclic covering maps p_j.

Consider then a stacking

$$\hat{w}_j : S_c \hookrightarrow X_j \times \mathbb{R}$$

and covering map

$$p_j : X_j \twoheadrightarrow Y_{j-1}$$

Since we are working with cyclic covers, we have a right-action of

$$\mathbb{Z} \cong \pi_1(Y_{j-1})/\pi_1(X_j)$$

on X_j by deck transformations, and there is also the standard action of $\mathbb{Z}$ on $\mathbb{R}$ by translations. So we can take the quotient of the trivial bundle $X_j \times \mathbb{R}$ by this $\mathbb{Z}$-action on each factor (the diagonal action of $\mathbb{Z}$ on $X_j \times \mathbb{R}$), obtaining a bundle:

$$B_j := \frac{X_j \times \mathbb{R}}{\{(x,r) \sim (x \cdot g, g \cdot r),\, \forall x \in X_j,\, r \in \mathbb{R},\, g \in \pi_1(Y_{j-1})\}}$$

where the action of $\pi_1(Y_{j-1})$ by path-lifting to X_j takes each x to some point in the preimage set of $p_j(x)$, which is isomorphic as a set to $\mathbb{Z}$, and where $g \cdot r = \bar{g} + r$, $\bar{g}$ denoting the image of g in the identification of the quotient $\pi_1(Y_{j-1})/\pi_1(X_j)$ with $\mathbb{Z}$. (To see why B_j is well-defined as a bundle, we can appeal to the fact that the surface group $\pi_1(\Omega)$ is right-orderable (shown by Boyer-Rolfsen-Wiest to hold for all hyperbolic surfaces in [9]) and so all the subgroups $\pi_1(Y_{j-1})$ are too, implying that they act faithfully on $\mathbb{R}$ by orientation-preserving homeomorphisms. It is then standard that the diagonal action produces a bundle over Y_{j-1} from this data, see for instance [29].)

When we consider some point $x \in X_j$ and the image of the fibre $\{x\} \times \mathbb{R}$ in B_j, we see that any point in the fibre is identified with integer translations of the same point in the various fibres $\{x \cdot g\} \times \mathbb{R}$, all of these sitting over the same point in Y_{j-1}. Another way of looking at this is, if we consider a point $y \in Y_{j-1}$ and the fibres over all points of $p_j^{-1}(y)$ together, each of these fibres can be split into unit intervals, and each interval on a given fibre is identified with a unique interval on every other fibre, resulting in a single fibre of equivalence classes $\{y\} \times \mathbb{R} \subset B_j$. So B_j is identified with a trivial bundle $Y_{j-1} \times \mathbb{R}$, although the quotient map

$$q_j : X_j \times \mathbb{R} \to B_j$$

that we used to define it does not preserve the fibres in the natural way. Indeed, restricting to just the image of $X_j \times (0,1)$ in B_j, we see the different fundamental domains for Y_{j-1} that comprise X_j all sitting over Y_{j-1} but in distinct unit intervals of the fibre, since the fundamental domains are all related by distinct deck transformations, equivalent to integer shifts.

In this way we obtain an embedding not just of the image of S_c, but all of $X_j \times (0,1)$ in $Y_{j-1} \times \mathbb{R}$ simply by restricting to its image under q_j, and this will allow us to push our stacking of w_j down the tower. Indeed, letting

$$\psi : \mathbb{R} \xrightarrow{\simeq} (0,1)$$

be a fixed orientation-preserving homeomorphism, we obtain a sequence of embeddings:

$$S_c \xhookrightarrow{\hat{w}_n} X_n \times \mathbb{R} \xhookrightarrow{\mathrm{id}_{X_n} \times \psi} X_n \times (0,1) \xhookrightarrow{q_n|_{X_n \times (0,1)}} Y_{n-1} \times \mathbb{R}$$

and our stacking $\hat{w}_{n-1} : S_c \hookrightarrow Y_{n-1} \times \mathbb{R}$ is therefore defined as the composition of these embeddings. By pushing our trivial stacking of $\hat{w}_n$ down the whole sequence of tower lifts in this way, we obtain the required stacking

$$\hat{w} : S_c \hookrightarrow \Omega \times \mathbb{R}$$

In summary, we have demonstrated:

Proposition 7.3. *(adapted from: [39, Lemma 3.1], [52, Lemma 3.4]) Suppose that $w \in \pi_1(\Omega)$ is not a proper power in a surface group represented by an annular cover $S \twoheadrightarrow \Omega$. Then there is a stacking of the core surface $S_c \subset S$ in $\Omega \times \mathbb{R}$, that is, an embedding $\hat{w}$ making the following diagram commute:*

$$\begin{tikzcd} & \Omega \times \mathbb{R} \arrow[d, "\pi_\Omega"] \\ S_c \arrow[ur, hook, "\hat{w}"] \arrow[r, "w"'] & \Omega \end{tikzcd}$$

(where π_Ω is the natural projection map).

7.3 Stacking the w-Cycles

The next step is to pull back our stacking of w in Ω to obtain a stacking of the w-cycles in Σ, which will allow us to relate the number of w-cycles to $\chi(\Sigma)$ using Lemma 6.6. We point out again that although w' is a combinatorial map with respect to the rectangular decomposition of Σ, the maps σ and h are not. Our derivation of a stacking of the w-cycles will not depend on the intermediate levels of the tower lifting discussed in the proof above, just relying on the fact that the map $\hat{w} : S_c \hookrightarrow \Omega \times \mathbb{R}$ itself is an embedding. So, since we chose the core S_c such that each cell of $\mathbb{S}$ has well-defined image in it under σ, we can forget about the cell decomposition used to derive $\hat{w}$ and obtain our stacking of w' as a map that respects the rectangular decomposition instead, as required for Lemma 6.6. The following three lemmas all apply to the setting of Theorem A, where we have a fixed stacking as given by Proposition 7.3:

Lemma 7.4. *The stacking $\hat{w}$ pulls back to a stacking $\hat{w}'$ of $\mathbb{S}$ in $\Sigma \times \mathbb{R}$.*

Proof. The desired map will come naturally when we consider the structure inherited by pulling the trivial $\mathbb{R}$-bundle over Ω back along h to give the diagram:

$$\begin{tikzcd} \Sigma \times \mathbb{R} \arrow[r, "\bar{h}"] \arrow[d, "\pi_\Sigma"] & \Omega \times \mathbb{R} \arrow[d, "\pi_\Omega"] \\ \Sigma \arrow[r, "h"] & \Omega \end{tikzcd}$$

where

$$\bar{h} := h \times \mathrm{id}_{\mathbb{R}}$$

Indeed, we already have maps w' and $\hat{w} \circ \sigma$ sending $\mathbb{S}$ to Σ and $\Omega \times \mathbb{R}$ respectively, so the universal property of the pullback implies that they both factor through a map to the bundle $\Sigma \times \mathbb{R}$, and we will (pre-emptively) call this map $\hat{w}'$. At this point we have the following commutative diagram telling us that $\hat{w}'$ is a lift of w',

$$\begin{tikzcd} & \Sigma \times \mathbb{R} \arrow[d, "\pi_\Sigma"] \\ \mathbb{S} \arrow[ur, "\hat{w}'"] \arrow[r, "w'"'] & \Sigma \end{tikzcd}$$

and so it remains to check that $\hat{w}'$ is injective. To this end, suppose we have points x, $y \in \mathbb{S}$ such that $\hat{w}'(x) = \hat{w}'(y)$. Then, since $\bar{h} \circ \hat{w}' = \hat{w} \circ \sigma$ and $\hat{w}$ is injective,

$$\sigma(x) = \sigma(y) \in S_c$$

But by the definition of $\mathbb{S}$ as a collection of components of the pullback of the maps h and w, x and y are determined by a pair of points in the direct product $\Sigma \times S_c$. Since $w' = \pi_\Sigma \circ \hat{w}'$, they project to the same point in Σ, and as we just observed they also project to the same point in S_c. Therefore $x = y$, confirming that $\hat{w}'$ is an embedding of $\mathbb{S}$ in $\Sigma \times \mathbb{R}$ lifting w'. $\square$

Next, we show the the maximal and minimal height data of the stacking $\hat{w}$ is respected when we pull it back:

Lemma 7.5. *We have inclusions of sets $\sigma^{-1}\left(\mathcal{A}_{\hat{w}}\right) \subseteq \mathcal{A}_{\hat{w}'}$, and $\sigma^{-1}\left(\mathcal{B}_{\hat{w}}\right) \subseteq \mathcal{B}_{\hat{w}'}$.*

Proof. We show the contrapositive; suppose we have a point $p \in \mathbb{S} - \mathcal{A}_{\hat{w}'}$, and let $x := w'(p) \in \Sigma$. Then there exists another point $q \in (w')^{-1}(x) \cap \mathcal{A}_{\hat{w}'}$, and we denote by $\bar{x}$ the image in Ω:

$$\bar{x} := w \circ \sigma(p) = w \circ \sigma(q) = h(x)$$

Since p and q are distinct points in the pullback of h and w, they must project to different points under σ, say $\bar{p}$, $\bar{q} \in S$ respectively, and these points are mapped to distinct heights among the preimages of $\bar{x}$ by $\hat{w}$ — it suffices then to show that

$$\pi_{\mathbb{R}} \circ \hat{w}(\bar{p}) < \pi_{\mathbb{R}} \circ \hat{w}(\bar{q})$$

so that $p \notin \sigma^{-1}\left(\mathcal{A}_{\hat{w}}\right)$. This is a direct consequence of the facts that $\hat{w}'$ is obtained via the pullback of w' and $\hat{w} \circ \sigma$,

$$w'(p) = w'(q),$$

and

$$\pi_{\mathbb{R}} \circ \hat{w}'(p) < \pi_{\mathbb{R}} \circ \hat{w}'(q)$$

Indeed, by the commutativity of:

$$\begin{tikzcd}
\mathbb{S} \arrow[dr, hook, "\hat{w}'"] \arrow[drr, bend left, "\hat{w}\circ\sigma"] \arrow[ddr, bend right, "w'"'] & & \\
& \Sigma \times \mathbb{R} \arrow[r, "\bar{h}"] \arrow[d, "\pi_\Sigma"] & \Omega \times \mathbb{R} \arrow[d, "\pi_\Omega"] \\
& \Sigma \arrow[r, "h"] & \Omega
\end{tikzcd}$$

we have $\hat{w} \circ \sigma = \bar{h} \circ \hat{w}'$, so

$$\pi_{\mathbb{R}} \circ \hat{w}(\bar{p}) = \pi_{\mathbb{R}}\left(\bar{h} \circ \hat{w}'(p)\right) = \pi_{\mathbb{R}} \circ \hat{w}'(p)$$

since $\bar{h} = h \times \mathrm{id}_{\mathbb{R}}$, and likewise $\pi_{\mathbb{R}} \circ \hat{w}(\bar{q}) = \pi_{\mathbb{R}} \circ \hat{w}'(q)$, giving the required inequality.

So, if $p \notin \mathcal{A}_{\hat{w}'}$, then $\sigma(p) \notin \mathcal{A}_{\hat{w}}$, proving the first inclusion; the inclusion for $\mathcal{B}$ follows similarly. $\square$

Lemma 7.6. *The stackings $\hat{w}$, $\hat{w}'$ are good.*

Proof. This is an immediate consequence of Lemma 7.5 and Definition 3.5. Since S_c is connected, $\hat{w}$ is automatically a good stacking, and therefore so is $\hat{w}'$, since each component of $\mathbb{S}$ covers all of $\mathcal{A}_{\hat{w}}$ and $\mathcal{B}_{\hat{w}}$ at least once under σ. $\square$

Finally, we come to the main technical result that will be used to prove both Theorems A and B. It is our analogue of [52, Theorem 1.2], and gives a control on how many elements in a free subgroup of an orientable surface group can have a root simultaneously adjoined in terms of the subgroup rank.

Proposition 7.7. *Let Ω be a closed orientable hyperbolic surface and $w \in \pi_1(\Omega)$ an indivisible element with corresponding annular cover $w : S \looparrowright \Omega$, and let γ be the core curve of S. Let $h : \Sigma \looparrowright \Omega$ be an essential immersion from a compact surface with non-empty boundary, and $\mathbb{S}$ the set of w-cycles in Σ, with induced map $\sigma : \mathbb{S} \to S$, which restricts to a degree D cover from their core curves to γ. Then, either*

$$D \leq -\chi(\Sigma),$$

or the pullback immersion $w' : \mathbb{S} \looparrowright \Sigma$ is reducible.

Proof. Fix a rectangular decomposition of Σ according to Proposition 6.5 and a stacking of w' according to Proposition 7.3 and Lemma 7.4. Since w is a covering map to Ω and w' is a pullback of w restricted to S_c which contains images of all the 2-cells of Σ, w' is surjective, so Lemma 6.6 asserts that $-\chi(\Sigma)$ is the number of max-height strips of the stacking $\hat{w}'$. Then, if this number is less than D, there must be an open disc $U \subset S$ which intersects γ, and a pair of open discs in a single max-height strip ξ of $\mathbb{S}$ which both map to U homeomorphically under σ.

The image $\sigma(\xi)$ therefore completely covers γ, since every 2-cell of $\mathbb{S}$ intersects its component's core curve. In particular, $\sigma(\xi)$ intersects $\mathcal{B}_{\hat{w}}$ by Lemma 7.6. By Lemma 7.5, there is an open disc in $\xi \cap \mathcal{B}_{\hat{w}'}$, and therefore in $\mathcal{A}_{\hat{w}'} \cap \mathcal{B}_{\hat{w}'}$, and by definition (3.5), this is only possible if the image in Σ of this disc is covered exactly once by w'. So, if w' is not reducible, $D \leq -\chi(\Sigma)$. $\square$

Part III

Results

We are now ready to prove the our main results A, B and C stated in the introduction. §8 is the proof of the main theorem, the w-cycles bound for indivisible loops in orientable surfaces, for which we developed the tools of rectangular decompositions and stackings in Part II. This will follow quickly from our last Proposition 7.7. Then in §9, we prove the coherence of orientable one-relator surface groups with torsion. This is another application of Proposition 7.7 analogous to Louder-Wilton's argument for one-relator groups, although with some changes to make it applicable to our setting of 2-complexes "built on surfaces". We will then end by considering some other possible applications of our results in relation to the surrounding areas of geometric group theory — notably the proof of Corollary C and the geometric structure that non-positive immersions begins to suggest we can put on one-relator surface groups. We will also discuss the missing step that would extend our results to non-orientable surfaces, and finally, how rectangular decompositions and Proposition 7.7 could potentially be used in the study of stable commutator length. The discussion in the final three subsections of §10 is mostly speculative at this point, but we feel it is worth pointing out how the tools we have developed could be applied in future work.

8 Proof of Theorem A

Given the hypotheses of Theorem A, we have the diagram describing the w-cycles:

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\sigma"] \arrow[d, "w'"] & S \arrow[d, "w"] \\ \Sigma \arrow[r, "h"] & \Omega \end{tikzcd}$$

By our treatment of the special cases in §5.1, we may reduce to the hypotheses of Proposition 7.7. Then, since each w-cycle contributes at least 1 to the degree of the covering map σ, if the pullback immersion w' is irreducible Proposition 7.7 tells us the number of w-cycles is at most $-\chi(\Sigma)$.

Suppose then that w' is reducible, and that we are using the rectangular decomposition of Σ given by Proposition 6.5. Then Σ contains a rectangular 2-cell R with at most one preimage in $\mathbb{S}$. We may assume R has a preimage by restricting to the image $w'(\mathbb{S}) \subseteq \Sigma$, which we now denote by Σ^w — indeed, we observe that Σ^w may be obtained from Σ by deleting rectangular 2-cells with no preimage in $\mathbb{S}$ as shown in Figure 15. Each such deletion raises Euler characteristic by 1, so it will suffice to show that the number of w-cycles is at most $-\chi(\Sigma^w) \leq -\chi(\Sigma)$.

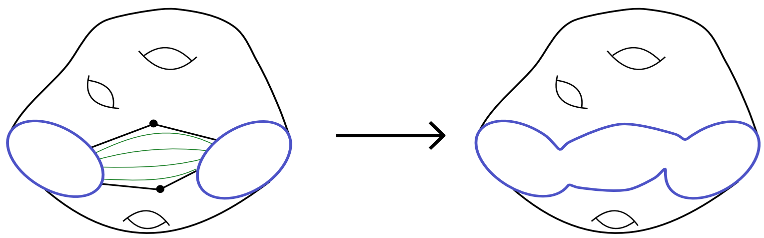

Figure 15: Deleting a rectangular 2-cell in the case that it spans two distinct components of $\partial\Sigma$. In the other case one boundary component splits into two but genus is reduced, so in both cases the Euler characteristic increases by 1.

Now we consider the component s of $\mathbb{S}$ containing the unique preimage of R. Deleting R from Σ^w means that s, as a component of the pullback $\Sigma \times_\Omega A$, becomes a contractible strip rather than a circular 2-complex while the other components of $\mathbb{S}$ are unaffected. Therefore we can pass to the surface

$$\Sigma_1^w := \Sigma^w - \mathring{R}$$

with immersion $w_1' := w'|_{\mathbb{S}-s}$ of the remaining w-cycles. This satisfies both:

1. $-\chi(\Sigma_1^w) = -\chi(\Sigma^w) - 1$; and
2. there is one fewer w-cycle in Σ_1^w than there is in Σ^w.

If the immersion w_1' is reducible, then it contains a rectangle with one preimage among the w-cycles in Σ_1^w, so we can repeat this process to obtain a surface Σ_2^w, again lowering both the quantities ($-\chi$ and $\#w$-cycles) by 1. Since Σ is compact and the rectangles are non-separating, iterating this process will eventually leave us with a surface Σ_n^w which is either contractible, or has an irreducible immersion w_n' of its w-cycles. In the former case, since properties

(1) and (2) above hold for each deletion carried out, we have that

$$-1 = -\chi(\Sigma_n^w) = -\chi(\Sigma^w) - \#\{w\text{-cycles in } \Sigma\}$$

thus satisfying the bound of $1 - \chi(\Sigma)$ for the total number of w-cycles. In the latter case, we appeal to Proposition 7.7 as in the case of w' irreducible above to see that the number of w-cycles in Σ_n^w is at most $-\chi(\Sigma_n^w)$, and then observe that Σ contains n additional w-cycles, and:

$$-\chi(\Sigma^w) = -\chi(\Sigma_n^w) + n$$

□

Remark 6. We note that in the case of irreducible w-cycles in infinite-index subgroups dealt with in Proposition 7.7, the bound we attain is stronger than the full generality of Theorem A, and strictly stronger unless each such cycle covers $w(A)$ in Ω with degree 1. The methods used in Lemma 5.1 and the above proof allow us to show that other w-cycles contribute at most 1 to $-\chi(\Sigma)$, and at least 1 to the degree of σ, but we cannot currently say more about the total degree of σ in the general case.

9 Proof of Theorem B

In this section we prove the coherence of orientable one-relator surface groups with torsion. The basic idea of the proof is to study presentation 2-complexes for the groups involved, realised in this case as surfaces with discs attached along curves representing the additional relators (that is, the relators which did not come from the product of commutators corresponding to the surface structure). Attaching a disc along a curve representing w gives a complex representing a quotient of the closed surface group, and the bound obtained in Proposition 7.7 on the number of w-cycles in certain subgroups provides a bound on the number of such discs in the presentation complex of those subgroups, which can be used to derive a finite presentation.

Louder and Wilton made note of the portability of their methods, in particular the potential for Euler characteristic computations via stackings to be applied to groups other than the standard one-relator groups. As observed in [53, §5], the key ingredients to showing coherence of quotients of a group by a proper power of a single element are:

- an unwrapped cover of the quotient group's presentation complex; and
- a good ("branched") stacking of the root of the relator

The stackings we have constructed for surfaces are not exactly the same as their branched stackings, but they serve the same purpose, and existence of unwrapped covers can be seen as a topological consequence of a group being virtually torsion-free, which Allenby's Theorem 4.4 implies for one-relator surface groups. So, Lemma 4.5 and Proposition 7.3 give heuristic justification for our proof of coherence, with the full details in the following subsections.

9.1 Unwrapped Covers

The diagram relating the 2-complexes representing one-relator surface groups and particular subgroups is essentially the same as the one at the heart of our construction in §7:

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\sigma} & S \\ \downarrow{\scriptstyle w'} & & \downarrow{\scriptstyle w} \\ \Sigma & \xrightarrow{h} & \Omega \end{array}$$

Here we still have h a boundary-essential immersion representing a subgroup $\pi_1(\Sigma) \leq \pi_1(\Omega)$, inducing a homomorphism h_* on the fundamental groups of these surfaces. But we can now view the vertical maps also as describing attaching maps for 2-cells — if S is an annulus as before, we can attach a disc to Ω by taking the image of the core curve of S as its boundary, resulting in a 2-complex X whose fundamental group is $\pi_1(\Omega)/\langle\langle w \rangle\rangle$. Similarly w' can be seen as the coproduct of attaching maps for a 2-complex Y, a collection of discs attached to Σ along the images of the core curves of $\mathbb{S}$ (which could now represent just a subset of the w-cycles in Σ). Commutativity of the diagram indicates how h extends to a map $Y \to X$, with the discs glued onto Σ covering the one glued onto Ω, and so h_* likewise extends to represent a homomorphism into a one-relator surface group.

Our methods will only allow us to prove coherence in the case where our one-relator surface group has torsion; the proof of Lemma 9.3 shows why, but it is essentially because of the extra information we can gain by viewing this extension of h_* as a factor of a map to the associated torsion-free one-relator surface group. We fix a group of the form:

$$G = \pi_1(\Omega)/\langle\langle w^n \rangle\rangle, \qquad n > 1$$

where we assume that Ω is orientable and hyperbolic, and w is indivisible in $\pi_1(\Omega)$. In this case the attaching map

$$S \xrightarrow{w} \Omega$$

only accurately describes a presentation complex X for G if the 2-cell attached by w includes a cone point of order n. We call the coned disc D_n, and note that allowing the inclusion of it makes X an *orbicomplex* (since G is the orbifold fundamental group of X), rather than a genuine 2-complex. In general $H = \pi_1(Y)$ then has torsion elements too, and, beyond the knowledge that they must be divisors of n, we do not have any control a priori on the order of cone points of the discs glued onto Σ.

However, in the case that H is torsion-free (so all of the discs attached by w' are the nonsingular disc D^2), we could use Proposition 7.7 to bound, in terms of $\chi(\Sigma)$ and n, the degree of the overall covering map to the disc attached to Ω.

We will be able simplify to this special case using the fact that all one-relator surface groups are virtually torsion-free. We fix the normal, torsion-free finite-index subgroup $G_0 = \ker(\psi)$ as found in the proof of Lemma 4.5. Then, given any finitely-generated subgroup $H \leq G$, we have the exact sequence:

$$G_0 \cap H \hookrightarrow H \twoheadrightarrow H/G_0$$

describing H as a finite extension of $G_0 \cap H$. Tautologically, the finite group H/G_0 has a finite presentation, so if we could guarantee a finite presentation of $H \cap G_0$ we could combine them to obtain a finite presentation of H. A similar argument shows that, since G is finitely-presented, so is any finite-index subgroup H (this also follows from the Reidemeister-Schreier method [59, Chp. 2, Corollary 2.8]), so we need only consider infinite-index subgroups.

We will now use Lemma 4.5 to construct the *unwrapped cover* of our orbicomplex X which will later be a useful intermediate space to factor maps representing finitely-generated torsion-free subgroups through. First, we obtain a 2-complex X' with the same fundamental group $G = \pi_1(X)$ by attaching a disc D to Ω along a curve representing w^n. Despite being a genuine 2-complex, we don't work with X' directly in the first place as we will want to bound the number of relators in subgroups by considering the covering maps from the boundaries of 2-cells representing them to a curve representing w, not w^n. The torsion-free finite index subgroup $G_0 \leq G$ corresponds to a covering of 2-complexes,

$$X'_0 \to X'$$

The finite quotient $Q = G/G_0$ is the deck group of this cover, so we know that w acts as an order n deck transformation on the cells of X'_0. Each 2-cell covering D in X'_0 is attached along a curve representing $w^n \in \pi_1(\Omega)$, and the action of w by path-lifting results in an order n rotation, fixing each of these curves setwise. So every such curve must bound n distinct 2-cells in X'_0 which are related by the action of w, partitioning the preimages of D into families of cardinality n. Define a 2-complex X_0 with fundamental group G_0 by retracting each of these families onto a single representative 2-cell; this satisfies:

Theorem 9.1. *([53, Theorem 2.2], see also [10]) X_0 is a finite-sheeted cover of the orbicomplex X, called its* unwrapped cover.

9.2 w-2-Cells

An important point in the proof of Lemma 4.3 was Hempel's trick as discussed in §4, but it also requires analysis of the topology of classifying spaces for one-relator surface groups constructed by Howie. In particular, [41, Theorem 3.5] shows that the orbicomplex

$$X = \Omega \cup_w D_n$$

we introduced above is aspherical, and therefore a classifying space for G. Since we are assuming H is torsion-free, it will have a classifying space which is a genuine 2-complex Y, and there will then be a naturally-induced map of classifying spaces,

$$f : Y \looparrowright X$$

which maps some 2-cells of Y into Ω, and others to $D_n \subset X$ via the map which is a degree n cover, regular away from the cone point (modelled by the map $z \mapsto z^n$ in the unit disc of the complex plane). We will find it useful to make the following distinction:

Definition 9.2. Let Y be any compact 2-complex and $f : Y \looparrowright X$ an immersion to the presentation orbicomplex for $G = \pi_1(\Omega)/\langle\langle w^n \rangle\rangle$ as above. We call the 2-cells in Y which are attached along curves that cover, under f, a curve in Ω representing w the *w-2-cells in Y*. We call the other 2-cells in Y *surface 2-cells.*

We will obtain bounds on the number of w-2-cells in the complexes representing subgroups of G using Proposition 7.7. A *free edge* in a 2-complex is a 1-cell not incident to any 2-cells. Generally, a free face is is any 2-cell whose boundary contains an edge crossed exactly once by the union of the attaching maps of all the 2-cells (this is how it is defined in [53]), but since we are interested in 2-complexes built by attaching w-2-cells to a surface with boundary, we use a slightly altered definition. In this setting, we say that a w-2-cell is a *free face* if either:

- its boundary contains an edge of Y not incident to any surface 2-cells and it is a free face in the traditional sense; or
- its boundary is contained in the boundaries of surface 2-cells and there exists an arc with endpoints on boundary components of immersed surfaces in Y, meeting the w-2-cell boundary exactly once.

The effect is the same for our purposes — free faces can be collapsed via a homotopy equivalence, relating to our earlier discussion of reducible immersions and dependency, as the next lemma shows. Furthermore, if no w-2-cells are free faces in the sense described here, then those whose boundaries lie on surface 2-cells can have their attaching map altered by a homotopy so that every point in the boundary components of the immersed surfaces in Y is incident to a w-2-cell, and this implies that Y has no free faces in the traditional sense. This process is reversible, so the definitions are in fact equivalent, we have simply chosen to phrase it this way here to emphasise that only w-2-cells can be free faces, and not cells on the boundary of surfaces they are attached to.

Lemma 9.3. *(adapted from [53, Corollary 3.2]) Let Σ be a connected compact surface with non-empty boundary, and Z a 2-complex obtained by attaching additional 2-cells to Σ. Suppose that there is an immersion $f : Z \looparrowright X$ from Z to the presentation orbicomplex for $G = \pi_1(\Omega)/\langle\langle w^n \rangle\rangle$ with Ω orientable and hyperbolic, w indivisible and $n > 1$. If Z has no free faces and the image of $f|_{Z-\Sigma}$ is contained in $X - \Omega$, then the total number of w-2-cells in Z is at most:*

$$\frac{-\chi(Z)}{n-1}$$

Proof. Since Z has no free faces, the attaching maps for the w-2-cells on Σ can be identified with irreducible immersions from a collection $\mathbb{S}$ of annular covers of Σ, obtained from the pullback of $w : S \looparrowright \Omega$ and $f|_\Sigma$ as we considered in Part II. Indeed, since the core curve of each component of $\mathbb{S}$ is at each point running between a pair of arcs in lifts of components of $\partial\Sigma$ (as illustrated in Figure 10), there must be at least two points in $\mathbb{S}$ covering any point in Σ to ensure that no arcs between components of $\partial\Sigma_i$ cross the boundary of a w-2-cell only once. We can now apply Proposition 7.7, telling us that the degree of the map f restricted to the boundaries of the w-2-cells, as a cover of the boundary of $D_n \subset X$, is at most

$$-\chi(\Sigma) = \#\{w\text{-2-cells}\} - \chi(Z)$$

We denote this restriction by f_w, and note that we also have

$$\deg f_w = n \cdot \#\{w\text{-2-cells}\}$$

since each w-2-cell of Y covers $D_n \subset X$ with degree n. Therefore,

$$
\begin{aligned}
n \cdot \#\{w\text{-2-cells}\} &\leq \#\{w\text{-2-cells}\} - \chi(Z) \\
\implies \#\{w\text{-2-cells}\} &\leq \frac{-\chi(Z)}{(n-1)}
\end{aligned}
$$

as required. □

9.3 Folding Subgroups

Stallings' folding technique that we discussed in § 2.1 introduced the necessary topological ideas to represent finitely-generated subgroups of free groups as immersions of compact graphs, and the technique has since been extended to fundamental groups of 2-complexes [53, 54]. Since folding is only defined for genuine 2-complexes, and not orbicomplexes, we will focus on folding immersions to the unwrapped cover X_0 from Theorem 9.1. In [53, §4], a folding procedure for studying subgroups of one-relator groups is laid out, which we can also apply to one-relator surface groups. First, an application of Scott's lemma [68, Lemma 2.2] provides 2-complexes mapping into X_0 whose images surject each free factor of a given subgroup. If this immersion is not essential, then there are discs representing additional subgroup relators that can be attached, and the resulting 2-complex can then be folded, first by folding edges in the 1-skeleton, and then identifying 2-cells attached along the same folded paths and with the same image in X_0. We note that this preserves the distinction between w-2-cells and surface 2-cells.

Using such 2-complexes to represent subgroups of one-relator surface groups with torsion, and then bounding their complexity using Lemma 9.3, lets us prove coherence:

Proof of Theorem B. Let H be a finitely-generated subgroup of a one-relator surface group

$$G = \pi_1(\Omega)/\langle\langle w^n \rangle\rangle$$

with Ω orientable, $n > 1$, w indivisible. Since finite extensions of finitely-presented groups are again finitely-presented, we can assume that H is contained in the torsion-free finite-index subgroup $G_0 \leq G$, and also that H is infinite-index in G_0. We will also assume for now that H is freely indecomposable, which allows us to apply [53, Theorem 4.2]. Letting X_0 be the unwrapped cover from Theorem 9.1, this gives us an immersion of 2-complexes $f : Y \looparrowright X_0$, where Y is compact, connected and aspherical with no free faces or edges, and such that $f_*\pi_1(Y)$ is conjugate to H in G.

We can now proceed as in [53, Lemma 4.3], obtaining a sequence of π_1-surjective immersions from 2-complexes without free faces,

$$Y_0 \looparrowright Y_1 \looparrowright \cdots Y_i \looparrowright \cdots X_0$$

by attaching reduced disc diagrams one by one to Y that represent conjugacy classes of elements of the kernel of f_* and then folding the resulting complexes until the induced map to X_0 is an immersion, by [53, Lemma 4.1]. The direct limit of the $\pi_1(Y_i)$ is isomorphic to H by construction; the process is equivalent to finding a reduced presentation written in generators of G, with relators all conjugate to powers of w^n.

The 2-complex Y_0 is obtained by taking the wedge of Y with an interval, so that the image of $\pi_1(Y_0)$ in G is exactly H, not just conjugate to it. The Y_i may therefore have some edges that are not incident to surface 2-cells, but we note that since the w-2-cells are attached along loops representing indivisible elements in their fundamental groups, any given w-2-cell could only cross such an edge once, and since they immerse in X_0, distinct w-2-cells cannot be attached to them. Therefore any w-2-cells attached to such edges would be free faces, so we can assume that any edges not adjacent to surface 2-cells are in fact free, and also non-separating, since we assume H is freely-indecomposable.

The proof of the theorem can be obtained quickly from the following:

Fact. (†) There is a uniform bound on the number of cells making up any of the Y_i in the sequence of immersions.

Indeed, given this fact, since X_0 is finite this means there are only finitely many types of combinatorial immersions from any Y_i into X_0. But the sequence gives infinitely many factorisations:

$$Y_i \looparrowright Y_j \looparrowright X_0, \qquad \forall j > i$$

so one of the maps obtained by first passing from Y_i to Y_j and then applying the immersion $Y_j \looparrowright X_0$ must be the same as applying $Y_i \looparrowright X_0$ directly, and in fact there must be an i such that the immersion $Y_i \looparrowright X_0$ repeats infinitely often. But this means there is an infinite subsequence:

$$\{Y_{\sigma(i)}\} \subset \{Y_i\} \qquad (\sigma : \mathbb{Z} \to \mathbb{Z} \text{ strictly increasing})$$

such that every map $Y_{\sigma(i)} \looparrowright Y_{\sigma(i+1)}$ obtained by composition along the original sequence of immersions is a homeomorphism. Therefore

$$H \cong \varinjlim \pi_1(Y_{\sigma(i)}) = \pi_1(Y_{\sigma(1)})$$

meaning H has a finite presentation with loops in $Y^{(1)}_{\sigma(1)}$ as generators and the 2-cells of $Y_{\sigma(1)}$ as relators. This proves that the torsion-free, freely-indecomposable, finitely-generated subgroups of G are finitely-presented.

We then also have finite-presentability for the case where H is torsion-free and infinite-index but with multiple free factors. Consider the Grushko decomposition of H into free factors:

$$H = H_1 * \cdots * H_k * F$$

where each H_i is freely-indecomposable, and F is a free group. Since each H_i (and F) is finitely-presentable, so too is H.

Proof of Fact (†) The w-2-cells in Y_0 are attached to a compact (not necessarily connected) surface with boundary immersed in Y. It follows that there is a set of 2-complexes $\{Z_k\}$ each of the form of Z in Lemma 9.3, and an immersion

$$\coprod_k Z_k \looparrowright Y$$

whose image contains all of the w-2-cells in Y. We have that $\chi(Y) = \chi(Y_0)$ is no greater than the sum of the $\chi(Z_k)$, so by Lemma 9.3, the number of w-2-cells in Y is bounded above by

$$\sum_k -\chi(Z_k) \leq -\chi(Y)$$

Further, since Y is aspherical, we have

$$\chi(Y) = 1 - b_1(Y) + b_2(Y) \implies -\chi(Y) \leq b_1(Y) = \text{rank}\left(\pi_1(Y)^{\text{Ab}}\right)$$

so the number of w-2-cells can be bounded by the rank of $\pi_1(Y)$, which we will denote by R. The same observations can be applied to each Y_i in the sequence, but since the subsequent 2-complexes correspond to adding relators and not generators to the fundamental group, the number of w-2-cells is in fact bounded by R uniformly in each Y_i.

We now consider a decomposition of the Y_i into subcomplexes:

- Y_i^w the union of the w-2-cells and the smallest essentially-embedded subsurfaces in Y_i containing their boundaries, and
- $\Delta_i = Y_i - Y_i^w$.

This gives Y_i a graph of spaces structure, with "w-vertices" corresponding to the connected components of Y_i^w, and "Δ-vertices" corresponding to core surfaces embedded in Y_i with no w-2-cells attached; the free edges and vertices connected only to free edges are embedded unchanged in the underlying graph Γ_i. The edge spaces corresponding to edges of Γ_i which are not free in Y_i are annuli connecting boundary components of the core surfaces in w- or Δ-vertex spaces. Note that the graph components of Y_i where all edges are free are images of those in Y_0, so the total number of edges and vertices making them up can be uniformly bounded by that of Y_0, say G.

The total number of w-vertices in Γ_i is bounded above by R. The non-free edges adjacent to them correspond to annuli with at least one boundary component homotopic into loops consisting of sequences of arcs in the boundaries of w-2-cells. Since all such loops are disjoint, and w-2-cells in Y_i map homeomorphically to the ones in X_0, the number of edges adjacent to w-vertices is bounded above by $(R{\cdot}L{+}G)$, where L is the maximal boundary length of a w-2-cell in X_0.

Any w-vertex space is made up of w-2-cells, the surface 2-cells adjacent to their boundaries, and discs made up of surface 2-cells in Y_i whose boundaries are sequences of arcs in the boundaries of those w-adjacent surface 2-cells. There are at most RL edges in the boundary of w-2-cells, and at most 2 surface 2-cells incident to any such edge, so at most $2RL$ w-adjacent surface 2-cells. If M is the maximal boundary length of a surface 2-cell (which is bounded as they map homeomorphically to surface 2-cells in X_0), the length of the boundary of any of the discs that make up the rest of the w-vertex spaces is bounded above by $2RLM$. The discs are images of discs in Y_0 contained in its underlying immersed surface Σ_0, which can be equipped with a hyperbolic metric such that distances between points on surface 2-cells are non-increasing, and lengths of simple closed curves preserved, throughout the sequence of immersions. The hyperbolic metric satisfies a linear isoperimetric inequality (see for instance [8]), so there is a constant $K > 0$ such that the number of 2-cells making up the non-w-adjacent disc regions in the w-vertex spaces in any Y_i is uniformly bounded above by $2KRLM$. Overall, this gives a uniform bound on the total number of 2-cells, and therefore also 1- and 0-cells, in any w-vertex space.

The remaining Δ-vertices and edges connecting them in Γ_i correspond to connected surfaces embedded in Y_i. Considering how the graph of spaces changes along the sequence of immersions, new w-vertices can be created as discs corresponding to relators of H are attached, either filling or splitting up the previous Δ-vertex spaces. However the Δ-vertex spaces in Y_i are always images of subsets of Δ-vertex spaces in Y_0, and so the total number of cells in any Δ-vertex space in Y_i is bounded by the maximal such number in Y_0. The same applies to the annuli that pass to edges connecting a pair of exclusively Δ-vertices in Γ_i.

Finally, we must find a uniform bound on the number of cells forming the annuli that pass to edges adjacent to w-vertices in Γ_i. Since the number of cells in every vertex space, and hence the number making up the boundary of these annuli, is bounded, we just need to rule out the possibility that the number of cells between boundary components can be arbitrarily large among the Y_i. The annuli in Y_i are homeomorphic images of annuli in Y_0 (although they don't need to appear as edge spaces for Γ_0). As such, we can again use the hyperbolic metric on the surface Σ_0, which implies the existence of a $\delta > 0$ such that geodesic triangles are δ-slim. Constructing such a triangle with two vertices x, y diametrically opposed on one boundary component of an annulus a and the third vertex z on the opposite boundary component, we have that if the side (x, z) has length greater than δ, then every point on it further than δ from x is δ-close to a point on (y, z). The longer a is compared to δ, the sooner (y, z) must converge to (x, z), and so if a could be arbitrarily long it would have to have an arbitrarily short geodesic core curve, but the minimal length of a simple closed geodesic in Σ_0 is bounded away from 0 by the systole, $\mathrm{sys}(\Sigma_0) > 0$. There is therefore a fixed D, the maximal distance between points on opposite boundary components of edge spaces for edges in Γ_i adjacent to w-vertices, only realised for annuli containing a core curve of length $\mathrm{sys}(\Sigma_0)$. The surface 2-cells making up any of these annuli have a uniform minimal diameter, so there is a uniform bound on the number of 2-cells forming any one of them. As we observed above there are at most $(RL+G)$ of them to begin with, giving a bound on the number of 2-cells in annuli connected to w-vertex spaces throughout the sequence of immersions.

$\square$

10 Further Directions

The final result that we stated in § 1.1, whose proof we give in the next subsection, is essentially just a combination of the previous two theorems, finishing an analogue for surface groups of Louder, Wilton, Helfer and Wise's original work on w-cycles in free groups.

10.1 Proof of Corollary C

By Theorem B, we need only consider torsion free one-relator surface groups,

$$G = \pi_1(\Omega)/\langle\langle w \rangle\rangle$$

where as usual, Ω is orientable and hyperbolic and w is indivisible. Let X be the presentation 2-complex

$$X = \Omega \cup_w D^2$$

for G, and

$$f : Y \looparrowright X$$

an immersion from a compact connected 2-complex. In the degenerate case where Y is a graph, since it is connected either $\chi(Y) \leq 0$ or Y is a tree, hence contractible. Otherwise, we will assume that Y has no free faces, since they could be collapsed without changing $\chi(Y)$, resulting in a 2-complex with induced immersion to X that f factors through. So, similar to the proof of Theorem B, Y is made up of w-2-cells, regions of surface 2-cells, and free edges. The regions of surface 2-cells are the image of a finite set of compact surfaces $\{\Sigma_i\}$ which are immersed in Y,

$$h : \coprod_i \Sigma_i \looparrowright Y$$

and the Euler characteristic of the image of h in Y is at most $\sum_i \chi(\Sigma_i)$, since vertices or compact arcs in the boundary components of the surfaces may be identified in Y.

Since there are no free faces, the w-2-cells are attached along w-cycles in a subset of the Σ_i, and these w-cycles are mapped into the Σ_i by irreducible immersions. Since G is torsion-free, the w-2-cells are homeomorphic to the disc attached along w in Ω, and we can apply Theorem A directly to see that the number of w-2-cells attached to any of the Σ_i supporting them is at most $-\chi(\Sigma_i)$. Since

$$-\chi(\Sigma_i) \leq -\chi\left(h(\Sigma_i)\right)$$

we have that the image of each such Σ_i together with the w-2-cells attached to it makes a non-positive contribution to $\chi(Y)$. Note also that if any Σ_i is closed, it is a finite cover of Ω, so $f|_{\Sigma_i}$ is a local homeomorphism and there can be no other surface components or free edges in Y. These finite covers are hyperbolic as Ω is, so there is nothing left to prove.

It remains then, in the case that all the Σ_i have boundary, to show that the images of the Σ_i that do not support w-2-cells, as well as the graph components made up of free edges, in Y, do not make a positive contribution to $\chi(Y)$. But this is immediate, since any such Σ_i has Euler characteristic at most 1, when it is a disc, and since $h(\Sigma_i)$ is connected to the rest of Y via one or more vertices or compact arcs in its boundary, the contribution to $\chi(Y)$ coming exclusively from Σ_i is at most 0. The same holds for the graph components — individually they have characteristic at most 1, but attaching them to the rest of Y means they contribute at most 0. $\square$

As observed above, torsion obstructs non-positive immersions, but if Conjecture 3.4 is true then coherence of all one-relator groups and orientable one-relator surface groups will be known, answering the question that motivated Wise's definition of w-cycles. Beyond w-cycles then, we can ask what else the property of non-positive immersions implies, and, since our methods so far have followed Louder-Wilton's, we are led to consider how their theory of negative immersions could be developed for surfaces.

10.2 Towards a Dependence Theorem for Surface Groups

Proposition 7.7 bounds the total degree in a dependent system of equations describing the adjunction of an element in an orientable surface group to a collection of elements in a given subgroup. Dependence results such as Theorem 2.4 bound the rank of any free group surjected by a graph of groups that describes an element being simultaneously adjoined as a root to a collection of elements in free groups. Our results here do not apply to such a general situation; one rather artificial setting in which Theorem A may be applied would be if we knew a given group element $w \in G$ had a collection of w-cycles in $H \leq G$ exceeding the rank of H, then G could not be an orientable surface group, but it would be nice to have a more general statement about the dependency aspect of w-cycles for surfaces.

In Part II, to bound the number of w-cycles in a given subgroup of $\pi_1(\Omega)$, we went through a process of *pulling back* certain topological structures that we knew existed on the orientable surface Ω, and analysed how they would appear on the annulus representing w and immersed compact surface Σ. In Theorem 2.4 (which was actually proved via the purely topological statement [54, Theorem 2.21]) similar ideas are involved, but — aside from the fact that the objects involved are graphs instead of surfaces — an important difference lies in how its proof matches structures between a circle S and collection of graphs $\{\Gamma_i\}$ to form an adjunction space and *one-relator pushout* from these data, a graph through which all maps from the adjunction space to a target graph Γ_Ω must factor. By considering how the structures of the Γ_i and the circle S are encoded in the pushout, Louder-Wilton estimated its Euler characteristic using stackings, and this is what gives their dependence theorem.

While we now know how to compute Euler characteristics of surfaces by stacking circular 2-complexes running through them, integrating such data into a "one-relator surface pushout" is still not within grasp, and so neither is a dependence theorem analogous to Theorem 2.4 for surfaces. More specifically, an adjunction space describing the data of such a dependence theorem would involve some circular 2-complexes $\mathbb{S}$ being used to glue subsets of some collection of compact surfaces $\{\Sigma_i\}$ around the annular cover S representing w. For Σ_i with non-empty boundary, our method for relating $-\chi(\Sigma_i)$ to the degree of the map to S requires that we use a rectangular decomposition of Σ, with respect to which the map from $\mathbb{S}$ is combinatorial.

The images of the rectangles in S will overlap, as indicated in Figure 14; this didn't stop us from pulling back a stacking of w to a stacking of the w-cycles, since we could restrict S to a core which contained the images of all the rectangles, and simply stack our S_c using the map to Ω which was combinatorial with respect to a separate cell structure. But it does not seem clear how an adjunction space containing these overlapping rectangles could be given a single cell structure capturing all this information simultaneously, especially since our rectangular decompositions are built around boundary components in an essential way, and the idea of a one-relator pushout would be to factor through a combinatorial map to a surface Ω with no boundary. It would seem more reasonable that a cohesive cell structure could be obtained if the Σ_i could be replaced with non-compact surfaces with the same fundamental group, but in that case we do not have a method for computing Euler characteristic as we do in Lemma 6.6.

We believe that a precise combinatorial description of such an adjunction space would allow Proposition 7.7 to be upgraded to an analogue of Theorem 2.4 for orientable surfaces. Theorem 2.4 was used to find powerful constraints on the subgroup structure of one-relator groups [54, Theorem 1.5]. It would be interesting to know if one-relator surface groups could be shown to mirror this structure with a dependence theorem, given that many of the other core tools used in [54] have found analogues for surface groups ([49, 50]).

10.3 Non-Orientable Surfaces

It is slightly regrettable that we do not have proofs of our main results in the case that Ω is a non-orientable hyperbolic surface. We do not see any obvious reason why the bound we obtained on the number of w-cycles should not still apply, and indeed made note of points in our constructions that could be applied unchanged to non-orientable surfaces (Remarks 2, 3 and 5). Still, we have not been able to guarantee the existence of a rectangular decomposition on any surface Σ immersing into Ω which will carry the w-cycles, as we obtained for orientable Ω in Proposition 6.5.

The obstruction to our method for obtaining rectangular decompositions for non-orientable Ω lies in the fact that the lamination space $ML(\Omega)$ has some notable differences in this case as described in [67, 31], with multicurves that contain a finite collection of 1-sided curves playing a special role. In contrast to laminations with no closed leaves being easily found in orientable surfaces, Hatcher showed that such laminations are never dense in the lamination space of a non-orientable surface. This creates a problem in trying to prove the analogues of Lemmas 5.2 and 6.3, since if sequences of multicurves generically converge towards laminations that contain isolated 1-sided curves of fixed length, we do not have a guarantee that their pullbacks to Σ consist only of arcs between boundary components.

If some other method were found to obtain a version of Proposition 6.5 for non-orientable surfaces, we believe that the rest of the work we have done in proving Theorem A would still apply. In fact, we can outline how our method of pulling back a foliation of Ω by parallel copies of a multicurve as in Lemmas 6.3 and 6.4 would still work if a multicurve $\mathcal{C}$ satisfying their conditions can be found in Ω, even if $\mathcal{C}$ contains some simple 1-sided curves.

Any 1-sided curves of $\mathcal{C}$ would be core curves of Möbius bands embedded in Ω, so cutting along these curves turns those bands into annuli, with both boundary components double-covering the 1-sided curve. These annuli are identified with the ones that make up the components X of $\Omega - \mathcal{C}$ as discussed in §6.1, with one end glued to a simple loop in a Γ_X. To recover Ω, instead of gluing the corresponding component, say c, of ∂X to another boundary component of $\Omega - \mathcal{C}$, c is glued onto itself to form a cross-cap. When this is done, the foliation $\mathcal{F}^X$ extends to the non-orientable surface

$$X \cup_c M$$

where M is a Möbius band, and c is identified with its boundary curve. Foliating in this way makes the 1-sided curves in $\mathcal{C}$ into distinguished leaves, half the length of the 2-sided leaves converging on them ("non-primitive 2-sided curves" in Papadopoulos-Penner's terminology [63]), but still smooth.

When using the foliation $\mathcal{F}^{\mathcal{C}}$ to construct our core circular surface S_c as in §7.1 (note that S could now be a Möbius strip), the components of $\Omega - \mathcal{F}^{\mathcal{C}}{}_{\text{sing}}$ are still circular, and have preimages in S consisting of strips, so very little is different. When extending $\mathcal{F}^{\mathcal{C}}{}_{\text{sing}}$ to the 1-skeleton of a cell decomposition of Ω, there are fewer choices for a Möbius band component of $\Omega - \mathcal{F}^{\mathcal{C}}{}_{\text{sing}}$, as we always connect the two vertices on the singular leaf of $\mathcal{F}^{\mathcal{C}}$ containing its boundary circle with an edge that crosses its core curve. Our procedure for determining how large a neighbourhood we need to take around the core curve in each strip would still work, as γ separates each strip forming S into two parts, even if S as a whole is 1-sided. Nothing in §7.2 is specific to orientable surfaces, so, if we were given a multicurve filling a non-orientable Ω, transverse to $w(\gamma)$ and pulling back to a collection of compact arcs between boundary components of Σ, we should obtain Propositions 7.3 and 7.7 as before.

So, even if it is impossible to avoid 1-sided simple closed curves in the multicurves we would like to use to decompose Ω, in general it doesn't seem unreasonable that we could find some that allow essentially the same methods we used in Part II to work, and this would be an interesting question to investigate. The most likely exception to this would be the case where Ω is the connected sum of three projective planes. For any identification of this Ω with a once-punctured torus whose boundary is attached to the boundary of a Möbius band, the core of this band plays a distinguished role as the only curve c such that $\Omega - c$ is orientable [67, Lemma 2.1]. We cannot guarantee that this c has no closed pullback to any surface Σ immersing in Ω, and unlike surfaces with higher genus, there are no longer one-sided curves that we could hope to replace c with in the type of "elementary move" described in [63]. Moreover, we note that the fundamental group of this surface is also special among the closed surface groups as it is the only one with a square root adjoined to a single commutator. As we mentioned in § 2.2, it can have no dependent systems of more than one equation, and as another consequence, it the only hyperbolic surface group which is not residually free. Taking into account our discussion in the following subsection, this makes us even more sceptical that w-cycles would behave the same way in $3P^2$.

10.4 Stable Commutator Length

The study of commutator length in groups, going back at least as far as [21], relates to the work we have presented here, as computing commutator length can be naturally phrased as a question about maps from orientable surfaces with boundary, and trying to find the minimal absolute value of Euler characteristic of those surfaces (as in Question 2.3, for instance).

Definition 10.1. Given a group element $x \in G$, the *commutator length*, $\mathrm{cl}(x)$ is defined as the minimal integer g such that x can be expressed as a product of g commutators,

$$x = [a_1, b_1] \cdots [a_g, b_n] \qquad a_1,\, b_1,\, \ldots,\, a_n,\, b_n \in G$$

(by convention $cl(x) = \infty$ if no such g exists). Equivalently, $cl(x)$ is the minimal genus of a surface Σ equipped with a continuous map f to a presentation complex

X for G such that the image of the boundary of Σ represents x, i.e.

$$f_* [\partial\Sigma] = x \in \pi_1(X)$$

We call such a surface Σ a *realising surface* for x.
We also define the *stable commutator length*:

$$\mathrm{scl}(x) := \lim_{n\to\infty} \frac{\mathrm{cl}(x^n)}{n}$$

While extremely difficult to compute in almost all groups, studies into scl have yielded a surprising number of deep connections to other ideas in algebra and topology, as detailed in Calegari's book [14]. Theorem 3.2 can be viewed as an early result on scl in free groups, before much of the current viewpoint on the subject was developed — in that setting, $\pi_1(\Sigma)$ is a free group F, and if there is a map from a surface Σ' with b boundary components to Σ whose image restricted to $\partial\Sigma'$ represents some power $w^n \in F$, it says that:

$$n \leq -\chi(\Sigma') = 2\,\mathrm{genus}(\Sigma') + (b-2)$$

We can assume that as n increases, the number of boundary components of realising surfaces for w^n becomes negligible in comparison — in fact by a process of joining boundary components with 1-handles, as in the proof of [14, Proposition 2.13], we do not lose any generality in assuming $b = 1$. In other words, the commutator length of w^n is proportional to $\frac{n}{2}$, implying that for any element w of the commutator $[F, F]$, we have

$$\mathrm{scl}(w) \geq \frac{1}{2}$$

By the convention in Definition 10.1, as well as the fact that cl is sub-additive (that is, $\mathrm{cl}(g^n) \leq n\mathrm{cl}(g)$, implying that "$\lim_{n\to\infty}$" can be replaced with "$\inf_{n\in\mathbb{Z}_+}$" in the definition of scl), this tells us that each element of a free group has stable commutator length either 0 or uniformly bounded below by $\frac{1}{2}$, a feature known as a *gap in the scl spectrum* of $\frac{1}{2}$. Spectral gaps in scl have been found for various groups commonly studied in geometric group theory, including many recent developments by Heuer and Chen [18, 19, 38].

One particularly interesting result, [38, Theorem 6.3], together with Bavard's duality theorem [7], finds that groups with presentations as amalgamated products:

$$G = A *_C B$$

have a $\frac{1}{2}$-gap in their scl spectra when the sets A/C and B/C of left cosets possess left-invariant orderings. In particular this applies to orientable surface groups, although a $\frac{1}{2}$-gap was already known to exist for them, and indeed for any residually free group, by combining Theorem 3.2 with the fact that commutator length is non-increasing under homomorphisms. As discussed in §3.2, Duncan-Howie's methods also rely on such orderability properties, and indeed this was the basis for our derivation of stackings in §7. Since it follows for free from that construction, we include here a more direct topological proof of the $\frac{1}{2}$-gap in scl spectrum for orientable surface groups.

The only difference that we really need to consider from the general setting of Proposition 7.7 to apply it to commutator length is that we are interested in lifts of powers of w from Ω to Σ that are all homotopic into components of $\partial\Sigma$, since Σ is to be a realising surface for some power of w. This results in components of the pullback with a slight cosmetic difference to the generic structure illustrated in Figure 10. As shown in Figure 16, the core circular 2-complexes representing the w-cycles each have a distinguished boundary circle, which we denote by ∂_w, that finitely covers both a component of $\partial\Sigma$, as well as the core curve γ of S, so we have two covering maps of circles, generally of different degrees:

$$\begin{array}{ccc} \partial_w\mathbb{S} & \xrightarrow{\sigma} & \gamma \\ \big\downarrow{\scriptstyle w'} & & \big\downarrow{\scriptstyle w} \\ \partial\Sigma & \xrightarrow{h} & \Omega \end{array}$$

Lemma 10.2. *Let w be indivisible in the surface group $\pi_1(\Omega)$, and suppose*

$$h : \Sigma \to \Omega$$

is an immersion from an orientable compact surface with non-empty boundary, such that, for each boundary component $c \in \pi_0(\partial\Sigma)$,

$$h_*([c]) = w^{n_c} \in \pi_1(\Omega)$$

for some $n_c \in \mathbb{Z}$, and let the sum of the n_c be n, so that

$$h_* ([\partial\Sigma]) = w^n$$

Then

$$n \leq -\chi(\Sigma)$$

Proof. Recall the immersion w' induced on the w-cycles,

$$w' : \mathbb{S} \to \Sigma, \quad h \circ w' = w \circ \sigma$$

If w' is irreducible, the result is just a rewording of Proposition 7.7 — each boundary component is a w-cycle, contributing n_c to deg $\sigma \geq n$ (it could be that deg $\sigma > n$, when there are additional w-cycles independent of the boundary components, but we can ignore these for the purpose of computing commutator length). What's more, the restricted form of the w-cycles in this case makes it apparent that w' is indeed irreducible: let R be any rectangle in a decomposition of Σ as given by Proposition 6.5, then we want R to have at least two preimages in the truncated components $\mathbb{S}$ of the pullback of h and w. There are two edges of $\partial\Sigma$ in the boundary of R, and since there is a component of $\mathbb{S}$ covering each component of $\partial\Sigma$ at least once, both of these edges have at least one rectangle covering them in their respective boundary components' traversals (note that even if the two edges come from the same boundary component, the same reasoning shows they still must be covered twice in each traversal). See Figure 16.

□

As an immediate corollary, we can reprove:

Theorem 10.3. *[25, 38] The fundamental group of any closed orientable hyperbolic surface Ω has a $\frac{1}{2}$-gap in its scl spectrum.*

Proof. This is the same reasoning as outlined above. If $w^n \in \pi_1(\Omega)$ lies in $[\pi_1(\Omega), \pi_1(\Omega)]$, then it can be realised by a compact orientable surface Σ with genus g and one boundary component mapping into Ω by an immersion h. We have

$$\chi(\Sigma) = 2 - 2g - 1$$

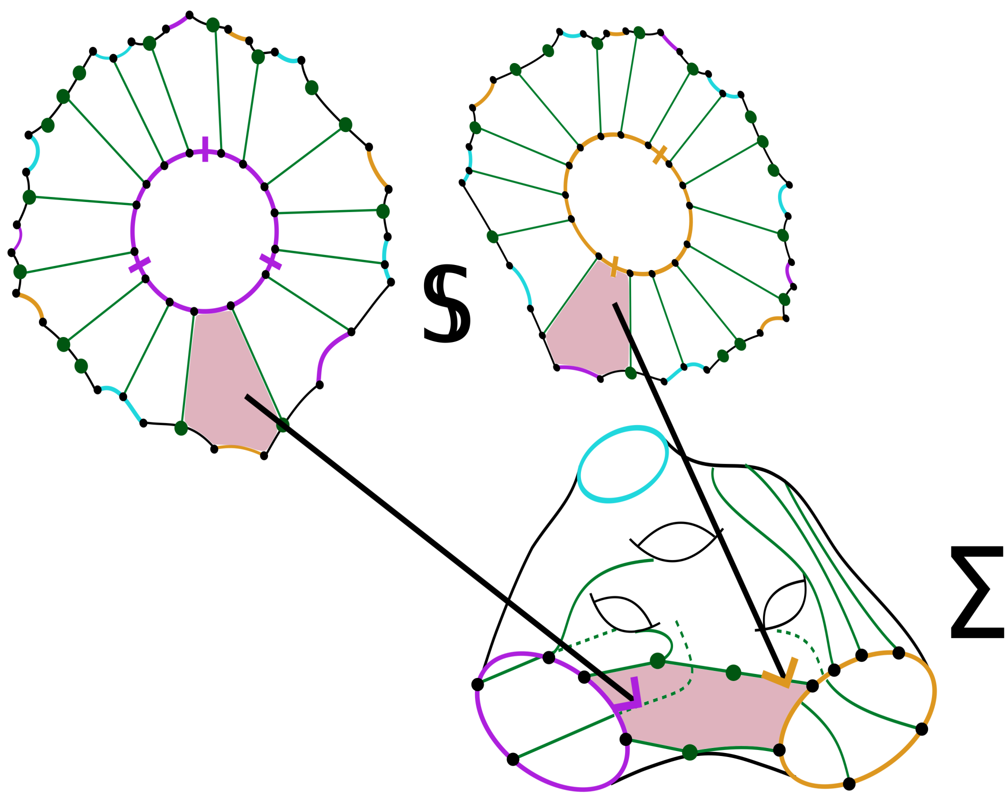

Figure 16: If w^n is in the commutator of $\pi_1(\Omega)$, there is a map from a surface Σ with all boundary components representing w-cycles. Here we show the components of $\mathbb{S}$ whose boundaries ∂_w cover the purple and yellow boundary components of Σ under the pullback immersion w' — each must contain a preimage of the red rectangular 2-cell which has a purple arc and a yellow arc in its boundary. Since the same occurs for every rectangle in a decomposition of Σ carrying the w-cycles, we see that w' is irreducible.

so, using Lemma 10.2,

$$\mathrm{cl}(w^n) = g = \frac{2 - 1 - \chi(\Sigma)}{2} \geq \frac{n}{2}$$

Therefore, in the limit we get

$$\mathrm{scl}(w) = \lim_{n\to\infty} \frac{\mathrm{cl}(w^n)}{n} \geq \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2}$$

for all w with non-trivial commutator length, as required. □

It is possible that a similar approach could be used to study commutator length using surface mappings to 2-complexes other than closed surfaces, but this is conjectural at this point. In particular, the groups whose scl-spectra we wish to investigate would need to have presentation complexes which can be equipped with some topological structure that pulls back to give a rectangular decomposition on Σ, in analogy with our constructions in Part II, and also admit stackings of their indivisible elements (likely meaning they would have to have locally-indicable quotients). Reynolds' paper [66], which we drew on for our proof of Lemma 5.2 to deduce the existence of compatible rectangular decompositions from geodesic laminations, showed that analogues of such laminations exist in $\mathbb{R}$-trees, so we might be able to derive rectangular decompositions on realising surfaces for elements of groups acting on $\mathbb{R}$-trees in a similar way. But groups acting freely on $\mathbb{R}$-trees are already known to be residually free by the constructions of Sela [70], and we do not yet have any general method for obtaining a rectangular decomposition on a realising surface when the target complex is not also a surface. On the other hand, there are still many open questions about scl in closed surface groups, for instance whether it is rational, where an approach using rectangular decompositions may be useful.

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