Contributions to the logit assignment model

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ABSTRACT

In the past, research in traffic assignment modeling has been directed primarily towards the deterministic model. Alternative, more behavioral principles were thought to be too demanding computationally.

This paper presents two mathematical contributions that enable one to solve a logit assignment model with flow-dependent travel times at a reduced cost. First, a convergence test for Fisk's minimization program is introduced, based on a duality gap principle. Second, a new definition of Dial's STOCH fixed-time logit assignment procedure is given, in which the set of available paths is defined only once and the computations are re-interpreted.

A numerical experiment indicates that these tools make the logit assignment model very competitive compared to the procedures conventionally used to solve the deterministic model.

KEYWORDS

Road Transportation; Traffic Assignment Model; Logit; Optimization

1. INTRODUCTION

Traffic assignment is the fourth and final step in the conventional travel demand forecasting scheme; by partitioning the origin-destination trip rates between several paths, the assignment program attempts to duplicate the vehicular flows on the network.

Most assignment models assume that travellers behave rationally. The most well-known assignment principle is that of Wardrop (1952): that every

traveller strives to maximize the utility derived from his transportation choices, in other words to minimize his generalized travel time. Thus, a user-optimal equilibrium is achieved when no traveller may decrease his travel time by unilaterally switching paths.

To account for errors in trip-makers' perception of travel time, Daganzo and Sheffi (1977) defined the stochastic user principle, according to which every trip-maker strives to minimize his/her stochastic generalized travel time. This rule allows for partitioning origin-destination trip-rates between several alternative paths, even if their true travel times differ from each other.

Two stochastic models are of particular interest: the logit model (Dial, 1971) and the probit model (Abraham, 1961; Burrell, 1968; Daganzo and Sheffi, 1977). The latter, though behaviorally more appealing, is impractical because only Monte-Carlo procedures are available, unless all paths can be identified. The logit model however, is endowed with both an extremely efficient fixed time assignment procedure (Dial's STOCH2), as well as a convex minimization formulation with a closed-form objective function (Fisk, 1980).

Nevertheless, computational difficulties have prevented the logit model from enjoying more widespread use. Among other drawbacks, Fisk's objective function was thought difficult to evaluate. Only recently have heuristic methods been developed (Chen and Alfa, 1991, and Damberg et al.,1992).

In this article, we present two developments which make computation of a logit user equilibrium competitive with its deterministic counterpart. First, we design a theoretically-sound convergence test for an equilibrium algorithm like the Method of Successive Averages; then it is possible to check whether an equilibrium has been reached. Second, we modify the definition of the set of available paths in Dial's STOCH2: this procedure is problematic if crudely implemented within an equilibration scheme, as the path set is likely to change from one iteration to the next. We put forward some changes that remedy this flaw.

The organization of the paper is the following: Section 2 states the problem in a formal way. Section 3 introduces the convergence test for Fisk's model. In Section 4, we derive a definition of efficient paths that does not depend on congestion phenomena; it is inspired from Dial's STOCH2, and a related path loading procedure is provided, wherein it is easy to compute all the terms in

Fisk's objective function. In Section 5, a numerical experiment is carried out to demonstrate that the Method of Successive Averages, combined with the proposed tools, is indeed a very efficient algorithm when applied to the logit model. Section 6 concludes and suggests some further developments. We note that all proofs of the assertions presented here can be found in (Leurent, 1994), in which elastic demand and capacity constraints are also considered, and a dual solution scheme is proposed.

2.　PROBLEM FORMULATION AND MODELING NEEDS

2.1　Logit equilibrium model

Let *r-s* be an origin-destination pair with traffic flow q_{rs}, θ a non-negative parameter, k a path from r to s with deterministic travel time T_{rs}^k and flow f_{rs}^k. In the logit assignment model (Dial, 1971), it is assumed that the path flow f_{rs}^k is proportional to a negative exponential function of the travel time T_{rs}^k:

$$f_{rs}^k = q_{rs} \frac{\exp(-\theta T_{rs}^k)}{\sum_k \exp(-\theta T_{rs}^k)}.$$　(1)

Then it is automatically ensured that:

$$q_{rs} = \sum_k f_{rs}^k.$$　(2)

The travel time of path k is related to the travel times T_a of the links a that belong to it via

$$T_{rs}^k = \sum_{a \in k} T_a = \sum_a \delta_{rs}^{ak} T_a.$$　(3)

where $\delta_{rs}^{ak} = 1$ if $a \in k$, or 0 if not.

Let x_a be the traffic flow on link a:

$$x_a = \sum_{rsk} \delta_{rs}^{ak} f_{rs}^k.$$　(4)

Let finally t_a be the travel time function of link a (assumed to be continuous and non-decreasing):

$$T_a = t_a(x_a).$$　(5)

Then eqns. (1)-(5) define a logit-based equilibrium. Figure 1 illustrates a logit

split between two paths.

<div align="center">

Fig. 1. Proportion of travellers that choose path 1
as a function of θ and the time difference T_2-T_1 (binary case).

</div>

2.2 Fisk's minimization program

Fisk (1980) characterized the logit equilibrium with variable travel times as the unique solution to the following convex minimization program (6):

$$\min_{\mathbf{f}} \quad J_L(\mathbf{f}) = \sum_a \int_0^{x_a} t_a(x)\, dx + \frac{1}{\theta} \sum_{rsk} f_{rs}^k \log(\frac{f_{rs}^k}{q_{rs}}) \tag{6}$$

subject to (2) and (4) and of course to $f_{rs}^k \geq 0$.

In (6) we replaced Fisk's $\sum_{rsk} f_{rs}^k \log(f_{rs}^k)$ with $\sum_{rsk} f_{rs}^k \log(f_{rs}^k / q_{rs})$ to facilitate the understanding of the relationship between (6) and the computations in the STOCH algorithm. This does not alter the existence and uniqueness results obtained by Fisk.

Fisk did not address a crucial question: how should the available paths be defined? In Beckmann's deterministic model (1956), all existing acyclic paths may be considered; but in a logit model a specific definition is required, since the conventional shortest path routines do not automatically find suboptimal paths.

In Dial's paper (1971), two alternative definitions of efficient paths are provided, namely STOCH and STOCH2. But these definitions are consistent only with respect to fixed travel times (ie. with constant functions t_a in eqn. (6)), and cannot be used in a variable-time program. A definition of available paths that is consistent with flow dependent travel times will be provided in Section 4. First, we address equilibration issues.

2.3 The Method of Successive Averages

Powell and Sheffi (1982) proved the convergence of the Method of Successive Averages (MSA) applied to minimization programs as Fisk's (provided that the definition of available paths cannot vary).

Let us define a Fixed-Time Assignment (FTA) as a path loading procedure that partitions the OD flow according to the logit rule, based on a given set of

available paths. An FTA yields a solution to (6) with constant travel time functions and a given set of utilized paths.

The MSA equilibration algorithm is comprised of four steps.

Step 0: Initialization.

o Set iteration counter $n := 0$.

o Choose a sequence $\alpha^{(k)}$ of real numbers such that ($0 \le \alpha^{(k)} \le 1$), ($\Sigma \alpha^{(k)} = +\infty$) and ($\Sigma \alpha^{(k)2} < +\infty$).

o Find an initial feasible flow pattern $x_a^{(0)} = x_a(\mathbf{f}^{(0)})$. It may be obtained through an FTA based on link times $t_a^{(-1)} := t_a(0)$.

Step 1: Link Travel Time Update.

o Set $t_a^{(n)} := t_a(x_a^{(n)})$.

Step 2: Direction Finding.

o Compute an FTA of traffic of all O-D pairs, based on link travel times $t_a^{(n)}$: this yieds a path flow solution $\mathbf{g}^{(n)}$ and also an auxiliary arc flow pattern $y_a^{(n)} = x_a(\mathbf{g}^{(n)})$.

Step 3: Link Flow Update.

o Set $x_a^{(n+1)} = x_a(\mathbf{f}^{(n+1)}) := x_a^{(n)} + \alpha^{(n)}(y_a^{(n)} - x_a^{(n)})$.

Step 4: Convergence Test.

o Apply a convergence test: either a maximum number of iterations, or a test on the maximum value (over the arcs a of the network) of the change in $\sum_{k=1}^{n} \alpha^{(k)} x_a^{(k)} / \sum_{k=1}^{n} \alpha^{(k)}$ from the previous iteration n-1 to the current one n.

If test is satisfied, then terminate; else increment the iteration counter $n := n+1$ and go to step 1.

The MSA has been widely applied to solve Fisk's program. However, the definition of efficient paths has not been adequately addressed. Thomas (1991) wrote that "it seems likely that methods which incorporate definitions of acceptable paths similar to those of Dial and Gunnarsson are intrinsically non-convergent, though in practice users often claim them to be satisfactory in that respect". In the following Section, we provide a theoretically-sound convergence

test for the equilibration algorithm, that will be of use together with a formal definition of the efficient paths, as will be given in Section 4.

3.   A CONVERGENCE TEST FOR THE LOGIT MODEL

We first consider the issue of designing a theoretically-sound convergence test for an application of the MSA to Fisk's program. It is based on a duality gap principle inspired from the deterministic model.

3.1   The duality gap principle in the deterministic model

In the deterministic case, where only those paths whose travel times are minimal are used, the objective function reduces to $J_D(\mathbf{f}) = \sum_a \int_0^{x_a(\mathbf{f})} t_a(x)\,dx$.
The usual convergence test is to evaluate a duality gap between the objective function $J_D(\mathbf{f}^{(n+1)})$ and a lower bound estimate:

$$J_D(\mathbf{f}^{(n)}) + \nabla J_D(\mathbf{f}^{(n)}).(\mathbf{g}^{(n)} - \mathbf{f}^{(n)})$$

where $\mathbf{g}^{(n)}$ is obtained in the Step 2 of the MSA (or equivalently of the Frank-Wolfe's method). Thus, the duality gap is given by:

$$\mathrm{DG}_D^{(n)} = \sum_a t_a^{(n)}(x_a(\mathbf{f}^{(n+1)}) - x_a(\mathbf{g}^{(n)})) = \sum_{rsk} f_{rs}^{k(n+1)}(T_{rs}^{k(n)} - \min_k T_{rs}^{k(n)})$$

The duality gap $\mathrm{DG}_D^{(n)}$ is always positive, except at equilibrium at which point it is zero. Hence, a convergence test involves checking whether DG is close to zero.

3.2   A convergence test for Fisk's model

We suggest applying the duality gap principle to the logit model. Let us denote the entropic part of the logit objective function as:

$$J_E(\mathbf{f}) = J_L(\mathbf{f}) - J_D(\mathbf{f}) = \tfrac{1}{\theta} \sum_{rsk} f_{rs}^k \log(\frac{f_{rs}^k}{q_{rs}}). \tag{7}$$

Then the flow vector $\mathbf{g}^{(n)}$ considered in Step 2 of the MSA is the unique solution to the following auxiliary program:

$$\min_{\mathbf{g}} \; J_{\mathbf{f}^{(n)}}(\mathbf{g}) = J_D(\mathbf{f}^{(n)}) + \nabla J_D(\mathbf{f}^{(n)}).(\mathbf{g} - \mathbf{f}^{(n)}) + J_E(\mathbf{g}). \tag{8}$$

The   duality   gap   associated   with   the   logit   objective   function   is

$\mathrm{DG}_L^{(n)} = \mathrm{J}_L(\mathbf{f}^{(n+1)}) - \mathrm{LBE}^{(n)}$, where the lower bound estimate $\mathrm{LBE}^{(n)}$ is defined as $\mathrm{J}_D(\mathbf{f}^{(n)}) + \nabla\mathrm{J}_D(\mathbf{f}^{(n)}).(\mathbf{g}^{(n)} - \mathbf{f}^{(n)}) + \mathrm{J}_E(\mathbf{g}^{(n)})$.

When applying the MSA algorithm to the logit model, it is in general not possible to compute $\mathrm{J}_E(\mathbf{f})$, unless all paths are identified. However, for some models like the one that will be described in Section 4, it is easy to compute $\mathrm{J}_E(\mathbf{g})$.

The trick is to evaluate the duality gap with respect to $\mathbf{g}^{(n)}$ and not with respect to $\mathbf{f}^{(n+1)}$. We also suggest the following convergence test, based on functions related to $\mathbf{g}^{(n)}$ rather than to $\mathbf{f}^{(n+1)}$:

if $\mathrm{J}_L(\mathbf{g}^{(n)}) - \mathrm{LBE}^{(n)} \le \varepsilon(\left|\mathrm{J}_L(\mathbf{g}^{(n)})\right| + \left|\mathrm{LBE}^{(n)}\right|)$, then terminate and let $\mathbf{g}^{(n)}$

be the solution to the minimization program (6), else return to Step 1.

If true, the test gives a vector that solves the minimization program, based on the convexity of J_L. Conversely, if the path flow vector $\mathbf{f}^*$ solves the program, then auxiliary vector $\mathbf{g}^*$ that corresponds to $\mathbf{f}^*$ is in fact equal to it and thus the convergence test is satisfied (Leurent, 1994).

Remark that if only a relative measure $\mathrm{J}_L - \mathrm{LBE}$ of the duality gap is needed, then it is not necessary to compute J_E: the test can reduce to check if $\mathrm{J}_L(\mathbf{g}^{(n)}) - \mathrm{LBE}^{(n)} \le \varepsilon$, in other words to check if $\mathrm{J}_D(\mathbf{g}^{(n)}) - \mathrm{J}_D(\mathbf{f}^{(n)}) - \nabla\mathrm{J}_D(\mathbf{f}^{(n)}).(\mathbf{g}^{(n)} - \mathbf{f}^{(n)}) \le \varepsilon$.

4. DEVELOPMENT OF THE STOCH3 PROCEDURE

The results obtained so far apply to any set of utilized paths under the sole constraint that no path may include more than once a given node. We now define a set of efficient paths that enable one to benefit from the efficiency of Dial's STOCH2.

Most previous logit assignment models have used Dial's second definition of efficient paths, according to which "a path is efficient (reasonable) if every link in it has its initial node closer to the origin than is its final node". The word "closer" refers to the travel time measured from the origin with respect to a current travel time vector that may change from one iteration to the next. Therefore there was no use trying to compute an objective function for the logit assignment model.

Three problems had to be tackled:

- to restrict Dial's set of efficient paths so as to limit its size and for each reasonable path not to be much longer than the shortest one.

- to stabilize the definition of efficient paths so that it depends neither on congestion nor on the iteration number; and

- to find a way to compute the entropic part of the objective function, so as to measure the convergence rate.

Subsection 4.1 deals with the first two questions, based on previous work by Tobin (1977) as regards the first question. Subsection 4.2 introduces the STOCH3 procedure, which offers a practical way to perform a fixed-time logit assignment on the efficient paths defined in Subsection 4.1. Subsection 4.3 describes a way to evaluate $J_E(\mathbf{g})$ in the STOCH3 model.

4.1   Definition of a stable set of efficient, not-too-long paths

A path is called "STOCH3-efficient" (or reasonable, or available) if it does not include the same node more than once, if every link has its initial node closer to the origin than its final node, if every link is "reasonable enough" compared to a reference shortest path.

More precisely, let:

- T_a^0 be a reference generalized travel cost for link a;

- $C_r^0(n)$ be a reference shortest generalized travel cost from origin r to node n, based on the link costs T_a^0;

- h_r^a be a maximum "elongation ratio" for link a wrt. the origin r;

- B_a, E_a be respectively the beginning and end nodes of link a.

Definition 1: a path k from origin r to destination s is STOCH3-efficient iff

- it does not comprise more than once a given node;

- $C_r^0(E_a) > C_r^0(B_a)$ $\forall a \in k$;

- $(1+h_r^a)(C_r^0(E_a) - C_r^0(B_a)) \geq T_a^0$, with $h_r^a \geq 0$, $\forall a \in k$.

A link a that satisfies the two last conditions is called STOCH3-reasonable wrt. origin r.

The last condition in Def. 1 limits the number of efficient paths by limiting their total reference generalized travel cost: defining $H_r = \max_a h_r^a$, summing over all links a that are incident to an efficient path k yields that:

$$Length(k) = \sum_{a \in k} T_a^0 \leq (1 + H_r)(C_r^0(s) - C_r^0(r)) = (1 + H_r)\min_{k'} Length(k')$$

Conversely, if k satisfies $Length(k) \leq (1 + H_r)\min_{k'} Length(k')$, it may not be efficient since the two first conditions in Def. 1 must hold as well.

Def. 1 is inspired from Dial's specification STOCH2 (Dial, 1971), as regards the second condition, and from Tobin (1977) as regards the third. Our own contribution is to impose fixed reference travel costs, thus ensuring a stable definition of the efficient paths whatever the congestion phenomena.

4.2 The STOCH3 procedure

Recall that in the STOCH3 procedure it is necessary to consider, on the one hand, the reference generalized travel costs to enumerate the available paths, and, on the other hand, the "actual" travel times according to which the OD flows are partitioned between the paths.

Program variables

n node with reference travel cost $C_r^0(n)$ from origin r .

$O_r(i)$ the i-th node in the order of increasing access cost $C_r^0(n)$ from r.

Ω_r^a indicator variable = 1 if link a is reasonable from r and 0 otherwise.

T_a current travel time on link a.

$A(a)$ impedance of link a.

$W_A(a)$ link weight that accounts for the importance of a in contributing to a reasonable path.

$W_N(n)$ node weight.

$X_A(a)$ flow on link a from the current origin r.

$X_N(n)$ flow passing through node n from the current origin r.

$F(a)$ total current flow on link a (over all origins).

Index r can be omitted when writing variables A, W_A, W_N, X_A and X_N, as these variables do not need to be stored after dealing with origin r.

Algorithm STOCH3

Step 0. Overall preliminaries: calculation of reasonable paths.

o From every origin node r, compute the shortest paths to all nodes n, based on the reference link travel costs T_a^0, yielding the reference access costs $C_r^0(n)$ and a labelling $O_r(i)$ of the nodes n in the order of increasing access cost from r. For each link a, set $\Omega_r^a := 1$ if $(1 + h_r^a)(C_r^0(E_a) - C_r^0(B_a)) \geq T_a^0 > 0$, $\Omega_r^a := 0$ otherwise.

Step 1. Preliminaries for a standard iteration.

o Initialize the total link flows variables $F(a)$ to 0.

o Set the link impedances $A(a) := \exp(-\theta T_a)$.

Steps 2, 3 and 4 are to be run for each origin node r.

Step 2. Forward pass.

o Set all $W_A(a)$ and $W_N(n)$ to 0. Set $W_N(r) := 1$.

o For each node n taken in the order of increasing reference cost $C_r^0(n)$ (the i-th node to be considered is indicated by $O_r(i)$), for each link a with beginning node $B_a = n$, if $\Omega_r^a = 1$ then compute $W_A(a) := A(a)W_N(n)$ and add $W_A(a)$ to $W_N(E_a)$, else do nothing.

Step 3. Backward pass.

o For each node n, set $X_N(n) := q_{rn}$ if n is a destination node, 0 otherwise.

o For each node n taken in the order of decreasing reference cost $C_r^0(n)$ (use the labelling $O_r(i)$ in decreasing order), for each link a with end node $E_a = n$, if $\Omega_r^a = 1$ then compute $X_A(a) := X_N(n)W_A(a)/W_N(E_a)$ and add $X_A(a)$ to $X_N(B_a)$, else set $X_A(a) := 0$.

Step 4. Contribution to total link-flows.

o $\forall a$, $F(a) := F(a) + X_A(a)$

At the end of the procedure, the vector $\mathbf{F}$ gives the fixed time logit assignment based on link travel times T_a.

4.3 Computation of the entropic part of the objective function in the STOCH3 model

It is shown in (Leurent, 1994) that, at the end of the forward pass from origin r, it holds that

$$W_N(s) = \sum_k \exp(-\theta T_{rs}^k).\qquad(10)$$

As $g_{rs}^k = q_{rs} \exp(-\theta T_{rs}^k) / \sum_k \exp(-\theta T_{rs}^k)$, we get

$$\tfrac{1}{\theta} \sum_k g_{rs}^k \log(\frac{g_{rs}^k}{q_{rs}}) = -\sum_k g_{rs}^k T_{rs}^k - \frac{q_{rs}}{\theta} \log(\sum_k \exp(-\theta T_{rs}^k))$$

and by summing over all origin-destination pairs *r-s*,

$$J_E(\mathbf{g}) = \tfrac{1}{\theta} \sum_{rsk} g_{rs}^k \log(\frac{g_{rs}^k}{q_{rs}}) = -\sum_a x_a(\mathbf{g}).T_a - \tfrac{1}{\theta} \sum_{rs} q_{rs} \log(\sum_k \exp(-\theta T_{rs}^k))$$

Then, the convergence test designed in Section 3 can be applied to the STOCH3 set of available paths.

5. COMPUTATIONAL EVIDENCE

In this section, we carry out a numerical example to compare the performance of the STOCH3 logit model using the MSA, with that of the deterministic model using both the Frank-Wolfe algorithm and the MSA.

5.1 The case study

The application is related to the western part of the Paris metropolitan area during the evening peak period, with a typical trip travel time of one hour. The test network is composed of 2,000 directed links. There are 141 origin and destination zones.

The dispersion parameter θ is set to 0.233 mn^{-1}, so that when two routes compete with each other, the first one with a travel time five minutes shorter than the second one, approximately three out of four drivers choose the first road. As only the rate of convergence is of interest here, the elongation ratios h_r^a are set to +•; note that from previous surveys they may be set to $h_r^a := 1.6$ for interurban studies (USAP, 1992) or $h_r^a \in [1.3; 1.5]$ for urban studies

(Tagliacozzo and Pirzio, 1973).

5.2   Results

Figure 2 depicts the performance of the three algorithms, showing the evolution of $\log\left|\frac{X^{(n)}}{J^*} - 1\right|$ where:

- in the logit model, J^* is the optimal value of the objective function in (6), and $X^{(n)}$ is the value of $J_L(\mathbf{g}^{(n)})$. In the MSA, the step size $\alpha^{(n)}$ is set to $\frac{1}{4+n/10}$.

- for the deterministic model, J^* is the optimal value of the deterministic objective function, and $X^{(n)}$ is the value of $J_D(\mathbf{f}^{(n+1)})$. In the MSA, the step size $\alpha^{(n)}$ is set to $\frac{1}{1+n}$.

<p align="center">Fig. 2. Convergence rates of the three algorithms.</p>

The convergence rate is much better in the case of the logit model, notably because the descent direction includes information about all of the available paths, not only about the shortest path in each iteration.

6   COMMENTS AND CONCLUSION

6.1   IVHS implications

In an IVHS context, the logit model may be of particular interest for assessing the level of information provided to motorists by a route guidance system (Van Vliet et al. 1990). One way to evaluate the effects of such a system is to model two classes of motorists, the first equipped with a route guidance device and characterized by a large dispersion parameter θ, and the other class of non-equipped drivers characterized by a small θ.

6.2   Model extensions

In (Leurent, 1994), the case of elastic demand and capacity constraints is addressed. A dual solution scheme is also introduced, but for large scale applications it is not efficient.

The computational efficiency of the MSA applied to the logit assignment model facilitates the following possible extensions of the model:

- diagonalization schemes, for example with travel time functions that depend on flows of several links. It is easy to derive a variational inequality formulation of (6).

- simultaneous models that capture more than one step in the conventional transportation planning process.

6.3   About path identification

It is useful to identify paths. The STOCH procedure is a way to consider all available paths at a reduced cost. We believe that our numerical experiment demonstrates, above all, that path-based equilibration algorithms are much more efficient than link-based algorithms. This conclusion is also supported by recent work by Schittenhelm (1990) and Larsson and Patriksson (1992), among others.

Algorithms that identify paths should better address more behavioral models. In a fixed-time path loading procedure like STOCH, the origin-destination flow is partitioned between the paths according to a behavioral rule. Other available behavioral rules are the probit model (Abraham, 1961; Burrell, 1968; see Daganzo and Sheffi, 1977, and Powell and Sheffi, 1982, for a mathematical foundation), and the bicriterion, cost-versus-time model (Marche, 1973; see Leurent, 1993, for a mathematical foundation). By applying a behavioral rule, we by-pass the need to search for an effective step-size in the descent. It is thus remarkable that, by the identification of paths, the computational process is greatly facilitated, especially in the case of behavioral models.

ACKNOWLEDGMENTS

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Fig. 1. Proportion of travellers that choose path 1
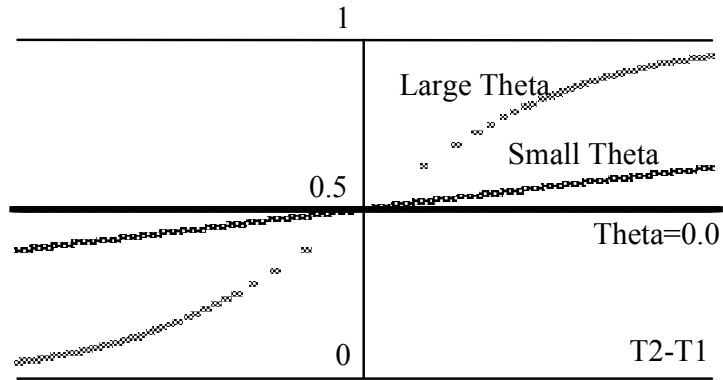as a function of θ and the time difference T_2-T_1 (binary case).

Fig. 2. Convergence rates of the three algorithms.