The Theory and Practice of a Dual Criteria Assignment Model with a continuously distributed Value-of-Time

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THE THEORY AND PRACTICE OF A DUAL CRITERIA ASSIGNMENT MODEL WITH A CONTINUOUSLY DISTRIBUTED VALUE-OF-TIME

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ABSTRACT

A dual criteria assignment model enables an analyst to represent disaggregate trade-offs between two cost criteria in the trip-makers' route choices, e.g. time and price in the cost vs. time model in which a continuously distributed value-of-time (VOT) is assumed.

This paper develops the theory and practice of dual criteria assignment. First, the economic background of the model is set out: it enables the analyst to represent various route choice behaviors, variable demand, several user classes, flow-dependent travel time and capacity constraints. Second, the mathematical framework is introduced; due to a special transformation, the model is cast into a variational inequality which under some assumptions reduces to a convex minimisation program. Third, solution algorithms are introduced and compared. Fourth, econometric tools are provided to estimate the distribution of the VOT and to evaluate the uncertainty about the predicted revenue of a toll road, arising from the uncertainty about the distribution of the VOT.

It is shown that these tools ensure the practicality of a dual criteria assignment in a medium-sized application.

INTRODUCTION

Context

How do trip-makers choose their routes on a transportation network? In most traffic assignment models, it is assumed that every trip-maker attempts to minimize his/her own generalized travel time, either deterministic (Wardrop 1952; Beckmann 1956) or stochastic (which means that a random error is included in the travel time; see Abraham 1961, Burrell 1968, Dial 1971, Daganzo and Sheffi 1977).

To give a stonger behavioral basis to the modeling of route choice, it has long been felt necessary to differentiate the trip-makers with respect to demographic or socio-economic

F.M. Leurent (1996)
in Transportation and Traffic Theory, Lesort J.B. (ed), 455-477. Pergamon, Exeter, England.

factors: such a differentiation is deterministic and leads to multiple user classes models; each class may exhibit a deterministic behavior (e.g. Dafermos 1972) or a stochastic one (e.g. Daganzo 1983). In France, the modeling of the choice between alternatives characterized by their attributes in terms of both price and time has been addressed by means of the cost vs. time model, which differentiates the trip-makers with respect to the ratio of travel time to money i.e. the value-of-time (VOT) (Marche, 1973; Abraham and Blanchet, 1973; Morellet and Julien, 1990).

Basic principle of the dual criteria model

If there are only two alternative paths, the first one cheaper but slower and the second one faster but more expensive, people with high VOT would choose the second path whereas people with low VOT would be satisfied with the first one. Taking a French interurban mode-choice example, the first path may be thought of as train and the second path as plane.

The frontier VOT $\hat{v}$ between the two paths is such that it equalizes their generalized times: $T_1 + P_1/\hat{v} = T_2 + P_2/\hat{v}$ hence $\hat{v} = (P_2 - P_1)/(T_1 - T_2)$. Trip-makers with VOT $v_i \le \hat{v}$ choose the slow, inexpensive path 1, while trip-makers with VOT $v_i > \hat{v}$ choose the fast, costly path 2.

Given the statistical distribution of the VOT across the trip-makers' population from its cumulative distribution function H (i.e. the proportion of people with VOT in $[x; x + \mathrm{d}x]$ is $\mathrm{dH}(x) = \mathrm{H}(x+\mathrm{d}x) - \mathrm{H}(x)$), the market share of the first, slow but inexpensive path is $\mathrm{H}(\hat{v})$, and the market share of the second, fast but costly path is $1 - \mathrm{H}(\hat{v})$.

Purpose and structure of the paper

The purpose of this paper is to present a comprehensive methodology for dual criteria assignment, and to investigate its practicality. The methodology encompasses first the economic representation of demand- or supply-related behaviors, then the mathematical formulation and solution algorithms, finally econometric tools for sensitivity analysis and error propagation.

The remainder of the paper is comprised of five parts.

Section 1 addresses the economic representation of demand- and supply-related behaviors: available modeling components are a continuously distributed trade-off between two cost criteria in the evaluation of a path travel impedance, flow-dependent travel times due to congestion effects and capacity constraints, elasticity of demand to the level-of-service defined as the mean generalized travel time, multiple user classes.

In section 2, the modeling components are cast into a mathematical formulation; a dual criteria equilibrium is defined which may be characterized as a solution to a non linear complementarity problem, to a variational inequality problem or to a convex minimisation program if the travel time functions are symmetric.

Section 3 introduces algorithms to solve the variational inequality. The side constraints are dealt with using an augmented Lagrangian scheme, in which each iteration consists of computing an unconstrained equilibrium. The unconstrained equilibrium may be computed by means of one of the following algorithms: a Method of Successive Averages (MSA) without path enumeration, an MSA with path enumeration, or a Procedure of Equalization by Transfer (PET) also requiring path enumeration. The algorithms are compared on a medium size application (2,000 links network and 140 origin-destination zones); it is shown that the equalization procedure makes the dual criteria assignment model as practicable as the single criterion assignment model of Beckmann (1956).

Section 4 provides econometric tools to make use of the probabilistic assumptions in the dual criteria model: first a maximum likelihood method is given to estimate the distribution of the VOT; second a method to propagate the exogenous error through the model is provided.

In section 5, we discuss and develop the economic representation in dual criteria assignment: a generalized trade-off function, stochastic dual criteria assignment, two flow-dependent criteria.

1. ECONOMIC REPRESENTATION

This section extends the economic representation introduced by Leurent (1993a). We define some notation (sub-section 1.1) and the concept of efficient path (sub-section 1.2). Then the assumptions on the travel times (sub-section 1.3) and the elasticity of demand (sub-section 1.4) are presented and illustrated.

1.1 Basic notation

Let *i* be a trip-maker with VOT v_i. Let *r* be an origin zone and *s* a destination zone. Let q_{rs} be the trip rate associated with the O-D pair *r-s* and $\mathrm{D}_{rs}(S)$ the O-D demand function, where *S* is a generalized travel time to be defined later. $\mathrm{H}_{rs}(v)$ is the cumulative distribution function for the VOT of the trip-makers on the O-D pair.

Let *k* be a path from *r* to *s*, with flow f_{rs}^k, travel time T_{rs}^k and travel price P_{rs}^k. It holds that $q_{rs} = \sum_k f_{rs}^k$. To a trip-maker with VOT *v*, the generalized time of travel on path *k* is $\mathrm{G}_{rs}^k(v) := \mathrm{T}_{rs}^k + \mathrm{P}_{rs}^k / v$.

The vector $\mathbf{f} = [f_{rs}^k]_{rsk}$ is called the path flow pattern.

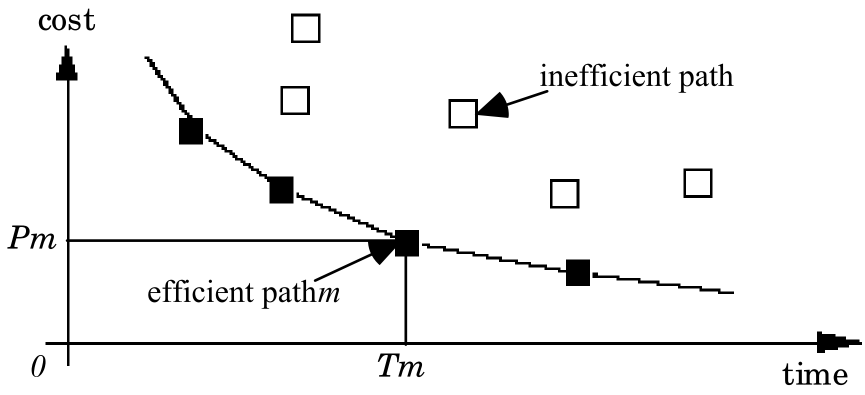
Let *a* be a network link (= arc) with flow x_a. Let a link path incidence indicator $\delta_{rs}^{ak} := 1$ if link *a* is incident to path *k* or 0 otherwise. So $x_a = \sum_{rsk} \delta_{rs}^{ak} f_{rs}^k$.

1.2 Efficient paths

Let us call a path "efficient" if there exists some positive VOT for which the path ensures a minimum generalized travel time (figure 1).

In the dual criteria model, only efficient paths may be assigned positive flows. If we rank the $\overline{m}$ efficient paths with respect to increasing prices, then the *m*-th efficient path is travelled on by trip-makers with VOT v_i belonging to $v_i \in [\hat{v}_{rs}^{m-1}; \hat{v}_{rs}^m]$ where $\hat{v}_{rs}^m$ is the frontier VOT between efficient paths *m* and *m*+1, defined as $\hat{v}_{rs}^m = (\mathrm{P}_{rs}^{m+1} - \mathrm{P}_{rs}^m) / (\mathrm{T}_{rs}^m - \mathrm{T}_{rs}^{m+1})$ since it equalizes the generalized travel times.

Fig. 1. Identification of the efficient paths.



Assuming a total trip rate of q_{rs}, the m-th efficient path is assigned a flow equal to $q_{rs}\int_{\hat{v}_{rs}^{m-1}}^{\hat{v}_{rs}^{m}} \mathrm{dH}_{rs}(v)$. Note that for consistency with respect to the first and last efficient alternatives, in the latter case the upper bound must be $\hat{v}_{rs}^{\overline{m}} := +\infty$, and in the former case the lower bound must be $\hat{v}_{rs}^{0} := 0$.

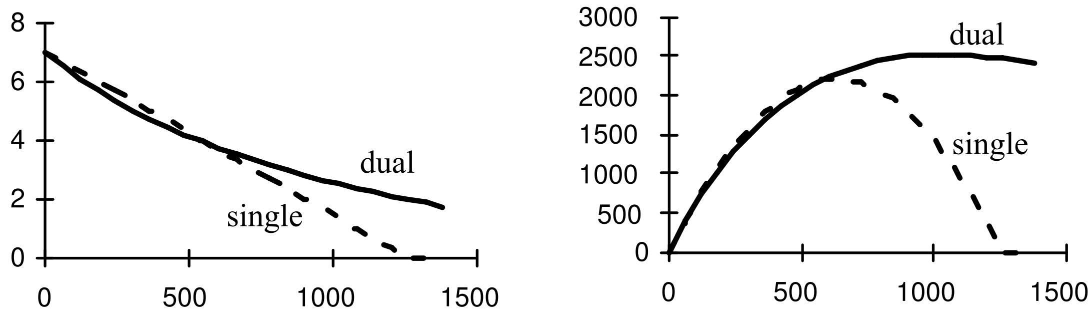
1.3 On the formation of travel times

Several factors determine the travel time of a road: geometric factors, operating conditions, the traffic level, and specific conditions (weather, accidents…). We assume that the travel time on link a depends on the path flow pattern **f** via travel time function $\mathrm{t}_a(\mathbf{f})$. We assume further that the flow is subject to conditions $\mathrm{z}_b(\mathbf{f}) \le 0$ called side constraints; if such a side constraint is binding $(\mathrm{z}_b(\mathbf{f}) = 0)$, then, associating to it a multiplier $w_b \ge 0$ because of the network conditions, we add a penalty $w_b \frac{\partial \mathrm{z}_b}{\partial f_{rs}^k}$ to the travel time of each path k. For instance, we may define a local capacity constraint $x_a(\mathbf{f}) - C_a \le 0$ for link a; if it is binding then the travel time of a path k will be penalized by $w_a \frac{\partial (x_a - C_a)}{\partial f_{rs}^k} = w_a \delta_{rs}^{ak}$.

Thus, the travel time of a path k is formulated as $\mathrm{T}_{rs}^k = \left(\sum_a \delta_{rs}^{ak}\, \mathrm{t}_a\right) + \left(\sum_b w_b \frac{\partial \mathrm{z}_b}{\partial f_{rs}^k}\right)$.

Let us now illustrate the way in which a dual criteria model differs from the single criterion model of Beckmann. We consider an elementary network made of two links between a single origin-destination pair. The travel time function is $\mathrm{t}_1 = 5 + 2x_1$ on the first link and $\mathrm{t}_2 = 4 + x_2$ on the second; a toll of P_2 is levied on the second link whereas $\mathrm{P}_1 = 0$. Assume for now that the trip rate is fixed to $q = 10 = x_1 + x_2$ and that there is no capacity constraint. Then the traffic and the revenue of the second road may be computed as a function of P_2 for the single criterion model and for the dual criteria model (figure 2). In the dual criteria case we consider a log-normal distribution of the VOT, with mean M = 60 and 0.6 as the standard deviation of its natural logarithm. In the single criterion case the mean VOT is the same and there is no dispersion.

Fig. 2. Traffic (a) and revenue (b) on the toll road.



At $P_2 = 0$ the two models yield the same result; they remain close at low values of P_2 but differ considerably at high values. Note that the revenue obtained according to the single criterion model decreases sharply with P_2.

The use of capacity constraints can make the simulation more realistic. Let us now assume that the flow on the second road is limited to 5. Then at low values of P_2 a model that does not account for capacity constraints overestimates both the traffic and the revenue of the toll road. A simultaneous modeling of capacity constraints and time vs. money trade-offs enables one to test and compare various network control strategies.

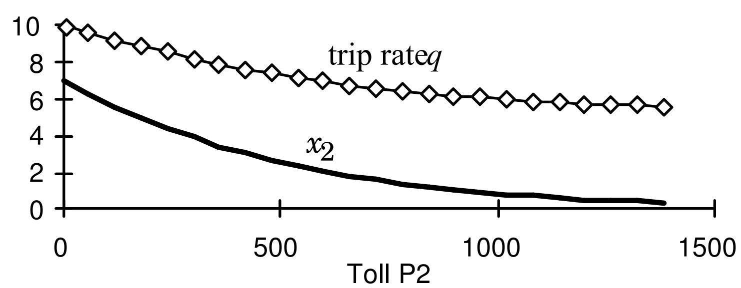
1.4 Allowing for elastic demand

Elastic demand allows one to model the fact that a change in supply entails either an increase or a decrease in the number of trips, depending on whether the change is an improvement or a decrease in quality. We also assume that the actual trip rate q_{rs} is a decreasing function D_{rs} with respect to the mean generalized travel time S_{rs}: $q_{rs} = D_{rs}(S_{rs})$.

In the case of the dual criteria model, we assume an aggregate measure of the mean generalized travel time: $S_{rs} = \int \{\min_k G_{rs}^k(v)\}\, dH_{rs}(v)$.

Let us consider again the small example (in the unconstrained case) and assume for now that $q = 21 - S$. Hence at $P_2 = 0$, $q = 10$. As P_2 increases, the total O-D flow and, most notably, the flow on the toll road, decrease (compare figure 3 to figure 2).

Fig. 3. Trip rate and toll road traffic, variable demand case.



1.5 Comments

We have provided a basic economic representation for a dual criteria assignment: even if the trip-makers are differentiated by their trade-off between two cost criteria, they still are dealt with as if a single class, notably as regards their contribution to the traffic flow. If there are reasons to differentiate the trip-makers in other dimensions than the VOT, for instance with respect to the roadway ability of heavy vehicles vs. cars or to a trade-off between time and distance, then a possible solution is to consider multiple user classes, each one with a dual criteria behavior or not.

To adapt the notation introduced so far to multiple user classes, a possible way is to assume that the notation *r-s* corresponds now to a user class index *r* and an O-D pair index *s*, and to change the index a into *a-r* so as to consider class link flows x_{ar} and class link times t_{ar}.

2. MATHEMATICAL FORMULATION

The originality of the dual criteria model lies in the continuous differentiation and in the definition of a mean level-of-service. As regards the other components, mathematical tools are taken from Beckmann (1956) as regards the user optimum principle, the elasticity of demand and the travel time function, from Dafermos (1982) as regards the differentiation between a discrete number of user classes, from Larsson and Patriksson (1994) as regards side constraints. We shall now add to these tools a special treatment of dual criteria differentiation, introduced by Leurent (1993a) and presented here in a hopefully clearer and more intuitive way.

To that end, we first introduce a set of definitions and propositions to define a dual criteria equilibrium (sub-section 2.1); the proofs of all the propositions are given in Appendix C. Then we characterize the equilibrium as a solution to several standard mathematical problems (sub-section 2.2). Finally we establish some existence and uniqueness results (sub-section 2.3).

2.1 Monetary expense classes of paths; efficiency sets; supply-related frontier VOTs

The continuous distribution of the VOT induces an infinity of distinct behaviors: how can this complexity be reduced? The problem may be redefined with path related variables, hence with a finite number of variables, by linking implicitly each VOT to a path that is efficient for it. This is performed by selecting the efficient paths by means of conditions on the ME classes of paths.

Defintion D1, Monetary Expense Classes of Paths. For every OD pair *r-s* we divide the paths into the equivalency classes of the equivalency relationship R_{rs} : $k\,\mathrm{R}_{rs}\,l$ iff $\mathrm{P}_{rs}^{k}=\mathrm{P}_{rs}^{l}$. We call these monetary expense (ME) classes of paths.

We index the ME classes with respect to increasing prices from 1 to $\overline{m}_{rs}$. We define $\mathrm{M}_{rs}(k)$ as the class index of the path *(r-s-)*k and Δ_{rs}^{ki} as an indicator variable := 1 if $i=\mathrm{M}_{rs}(k)$ or 0 otherwise. Let $\overline{\mathrm{P}}_{rs}^{m}$ be the price on the path(s) of the *m*-th class, $\overline{\mathrm{T}}_{rs}^{m}$ the minimum travel time proper across the paths of this class. Let $q_{rs}^{m}:=\sum_{k}\Delta_{rs}^{ki}f_{rs}^{k}$ be the traffic flow on the paths of the *m*-th ME class from *r* to *s*, and $Q_{rs}^{m}:=\sum_{i\leq m}q_{rs}^{i}$ the traffic flow from *r* to *s* on the paths whose prices are less than or equal to those of the *m*-th class. It also holds that $q_{rs}=\sum_{i}q_{rs}^{i}=Q_{rs}^{\overline{m}_{rs}}$. We define $q_{rs}^{0}=Q_{rs}^{0}:=0$.

The minimum generalised travel time experienced by a trip-maker with VOT *v* on the paths of the *m*-th monetary expense class is $\overline{\mathrm{G}}_{rs}^{m}(v):=\overline{\mathrm{T}}_{rs}^{m}+\overline{\mathrm{P}}_{rs}^{m}/v$.

We now make the important assumption that H_{rs} increases strictly on $\Omega_{rs} := \{v\ ;\ \mathrm{H}_{rs}(v) \in]0;1[\}$; hence H_{rs}^{-1} increases strictly on $\mathrm{H}_{rs}(\Omega_{rs})$. Ω_{rs} is the set of "useful" VOTs, i.e. of the VOTs v such that $\mathrm{dH}_{rs}(v) > 0$.

Definition 2, Efficiency Sets. On a given O-D pair r-s, the efficiency set of the ME class m is the set of the useful VOTs such that m is efficient for them: $\mathrm{E}_{rs}^{m} := \{v \in \Omega_{rs}\ ;\ \overline{\mathrm{G}}_{rs}^{m}(v) = \min_{\ell} \overline{\mathrm{G}}_{rs}^{\ell}(v)\}$.

From their construction $\forall rs,\ \Omega_{rs} = \bigcup_m \mathrm{E}_{rs}^{m}$. $\mathrm{E}_{rs}^{m} = \varnothing$ if there is no useful VOT for which m is efficient. E_{rs}^{m} is either empty or convex, because for any $v_0, v_1 \in \mathrm{E}_{rs}^{m}$ we have that $v_0\, \overline{\mathrm{G}}_{rs}^{m}(v_0) = \min_n v_0\, \overline{\mathrm{G}}_{rs}^{n}(v_0)$ and $v_1\, \overline{\mathrm{G}}_{rs}^{m}(v_1) = \min_n v_1\, \overline{\mathrm{G}}_{rs}^{n}(v_1)$, hence $\forall \alpha \in [0;1]$ with $v_\alpha = \alpha v_0 + (1-\alpha) v_1$, as $v_\alpha\, \overline{\mathrm{G}}_{rs}^{n}(v_\alpha) = \alpha v_0\, \overline{\mathrm{G}}_{rs}^{n}(v_0) + (1-\alpha) v_1 \overline{\mathrm{G}}_{rs}^{n}(v_1)$, it must hold that $v_\alpha\, \overline{\mathrm{G}}_{rs}^{m}(v_\alpha) \le v_\alpha\, \overline{\mathrm{G}}_{rs}^{n}(v_\alpha)$ hence $v_\alpha \in \mathrm{E}_{rs}^{m}$.

$\mathrm{E}_{rs}^{m} \cap \mathrm{E}_{rs}^{n}$ with $n \ne m$ is either empty or a singleton: note that both E_{rs}^{m} and E_{rs}^{n} are empty or intervals (from the convexity); when both are non empty, it holds that $\forall u \in \mathrm{E}_{rs}^{m}$, $\overline{\mathrm{G}}_{rs}^{m}(u) \le \overline{\mathrm{G}}_{rs}^{n}(u)$ hence $(\overline{\mathrm{P}}_{rs}^{m} - \overline{\mathrm{P}}_{rs}^{n}) / u \le \overline{\mathrm{T}}_{rs}^{n} - \overline{\mathrm{T}}_{rs}^{m}$, whereas $\forall v \in \mathrm{E}_{rs}^{n}$, $\overline{\mathrm{G}}_{rs}^{m}(v) \ge \overline{\mathrm{G}}_{rs}^{n}(v)$ hence $(\overline{\mathrm{P}}_{rs}^{m} - \overline{\mathrm{P}}_{rs}^{n}) / v \ge \overline{\mathrm{T}}_{rs}^{n} - \overline{\mathrm{T}}_{rs}^{m}$. Combining terms we obtain that

$$\forall u \in \mathrm{E}_{rs}^{m},\ \forall v \in \mathrm{E}_{rs}^{n},\ (\overline{\mathrm{P}}_{rs}^{m} - \overline{\mathrm{P}}_{rs}^{n}) / u \le \overline{\mathrm{T}}_{rs}^{n} - \overline{\mathrm{T}}_{rs}^{m} \le (\overline{\mathrm{P}}_{rs}^{m} - \overline{\mathrm{P}}_{rs}^{n}) / v. \tag{1}$$

If $n > m$ hence $\overline{\mathrm{P}}_{rs}^{n} > \overline{\mathrm{P}}_{rs}^{m}$, we deduce that $u \le v$ hence $\sup \mathrm{E}_{rs}^{m} \le \inf \mathrm{E}_{rs}^{n}$ which limits the intersection to at most one point.

Definition D3, supply-related frontier VOT. The supply-related frontier VOT of the ME class m is defined as $\hat{v}_{rs}^{m} = \sup \mathrm{E}_{rs}^{m}$ if $\mathrm{E}_{rs}^{m} \ne \varnothing$ or 0 if $\mathrm{E}_{rs}^{m} = \varnothing$. It is positive if and only if $\mathrm{E}_{rs}^{m} \ne \varnothing$, hence if and only if there is some useful VOT at which point m is efficient.

Proposition P1, properties of the supply-related frontier VOT. Between two ME classes m and n such that $\mathrm{E}_{rs}^{m} \ne \varnothing$, $\mathrm{E}_{rs}^{n} \ne \varnothing$, $\overline{\mathrm{P}}_{rs}^{m} < \overline{\mathrm{P}}_{rs}^{n}$ and $\mathrm{E}_{rs}^{i} = \varnothing\ \forall i \in]m;n[$, it holds that $\overline{\mathrm{T}}_{rs}^{m} > \overline{\mathrm{T}}_{rs}^{n}$ and $\hat{v}_{rs}^{m} = (\overline{\mathrm{P}}_{rs}^{n} - \overline{\mathrm{P}}_{rs}^{m}) / (\overline{\mathrm{T}}_{rs}^{m} - \overline{\mathrm{T}}_{rs}^{n}) = \inf \mathrm{E}_{rs}^{n}$.

Definition D4, ME class flow pattern compatible with the efficiency sets. On the O-D pair r-s, an ME class flow pattern $\mathbf{q} = [q_{rs}^{m}]_m$ is compatible with the efficiency sets E_{rs}^{m} when $q_{rs}^{m} \ge 0$ and $\sum_m q_{rs}^{m} = q_{rs}$ and $q_{rs}^{m} = q_{rs} \int_{\mathrm{E}_{rs}^{m}} \mathrm{dH}_{rs}(v)$.

The sets E_{rs}^{m} depend on supply conditions $\{(\overline{\mathrm{P}}_{rs}^{m}; \overline{\mathrm{T}}_{rs}^{m})\}_m$ and on Ω_{rs}. An ME class flow pattern is compatible if the proportions of the total O-D flow assigned to the ME classes correspond to an assignment of each useful VOT to an ME class which provides it with a minimum generalized travel time, hence to a situation where all trip-makers behave rationally: if $\forall v \in \Omega_{rs}$, v is assigned to an m such that $\overline{\mathrm{G}}_{rs}^{m}(v) \le \min_n \overline{\mathrm{G}}_{rs}^{n}(v)$, then all the VOTs in E_{rs}^{m} are assigned to m and to no other class (except possibly at frontier points) hence $q_{rs}^{m} = q_{rs} \int_{\mathrm{E}_{rs}^{m}} \mathrm{dH}_{rs}(v)$ which implies the compatibility. Conversely, the compatibility of an

ME class flow pattern yields no information about the economic rationality of the trip-makers of each VOT: it <u>does not imply a knowledge</u> of which ME class is chosen by each VOT, it only ensures that the class proportions of the O-D flow are the same as if every trip-maker behaved rationally.

<u>Proposition P2, characterization of compatibility</u>. The class flow pattern $\mathbf{q} = [q_{rs}^m]_m$ such that $q_{rs}^m \geq 0$ and $\sum_m q_{rs}^m = q_{rs}$ is compatible with efficiency sets E_{rs}^m if and only if $\forall n$, $Q_{rs}^n = q_{rs}\,\mathrm{H}_{rs}(\hat{v}_{rs}^{e(n)})$, where $e(n)$ is the efficient ME class with the largest index less than or equal to *n*.

Let us comment on P2 which serves to equilibrate supply ($\hat{v}_{rs}^{e(n)}$) and demand (q_{rs}^m, q_{rs}, H_{rs}). The notion of a compatible pattern reduces the infinite dimensional problem of the optimal ME class choice to a finite dimensional equilibrium problem. Note that P2 addresses the case of a class flow pattern, not of a path flow pattern: to ensure the choice of an optimal path, one has to impose one more condition within each ME class which possesses more than one path.

How may the conditions in P2 be cast into a standard mathematical problem? Let $\hat{v}_{rs}^0 := \inf \Omega_{rs}$ and define demand-related frontier VOTs as $\breve{v}_{rs}^m := \mathrm{H}_{rs}^{-1}(Q_{rs}^m / q_{rs})$; let also $c(i)$ be the efficient ME class with the smallest index strictly greater than *i*. Then the set of equations in P2 is equivalent to $\{\breve{v}_{rs}^n = \hat{v}_{rs}^{e(n)}\ \forall n\}$. This yields $\breve{v}_{rs}^n = (\overline{\mathrm{P}}_{rs}^{c(n)} - \overline{\mathrm{P}}_{rs}^n) / (\overline{\mathrm{T}}_{rs}^n - \overline{\mathrm{T}}_{rs}^{c(n)})$ for each efficient ME class except the last one $\bar{e} = e(\overline{m}_{rs})$, hence $\overline{\mathrm{T}}_{rs}^n + (\overline{\mathrm{P}}_{rs}^n - \overline{\mathrm{P}}_{rs}^{c(n)}) / \breve{v}_{rs}^n = \overline{\mathrm{T}}_{rs}^{c(n)}$; we obtain $\overline{\mathrm{T}}_{rs}^n + \sum_{l\ \mathrm{efficient}=n}^{e(\bar{e}-1)} (\overline{\mathrm{P}}_{rs}^l - \overline{\mathrm{P}}_{rs}^{c(l)}) / \breve{v}_{rs}^l = \overline{\mathrm{T}}_{rs}^{\bar{e}}$ by adding $(\overline{\mathrm{P}}_{rs}^l - \overline{\mathrm{P}}_{rs}^{c(l)}) / \breve{v}_{rs}^l$ to the left-hand side and $\overline{\mathrm{T}}_{rs}^{c(l)} - \overline{\mathrm{T}}_{rs}^l$ to the right-hand side (hence there is still equality).

The next step is $\overline{\mathrm{T}}_{rs}^n + \sum_{l=n}^{\overline{m}_{rs}-1} (\overline{\mathrm{P}}_{rs}^l - \overline{\mathrm{P}}_{rs}^{l+1}) / \breve{v}_{rs}^l = \overline{\mathrm{T}}_{rs}^{\bar{e}} + (\overline{\mathrm{P}}_{rs}^{\bar{e}} - \overline{\mathrm{P}}_{rs}^{\overline{m}_{rs}}) / \breve{v}_{rs}^{\bar{e}}$ because $\sum_{m=l}^{c(l)} (\overline{\mathrm{P}}_{rs}^m - \overline{\mathrm{P}}_{rs}^{m+1}) / \breve{v}_{rs}^l = (\overline{\mathrm{P}}_{rs}^l - \overline{\mathrm{P}}_{rs}^{c(l)}) / \breve{v}_{rs}^l$ since from $e(m) = l$ we have that $\breve{v}_{rs}^m = \breve{v}_{rs}^l$.

It may be noticed that the terms on the left-hand side apparently do not depend on the efficiency but are functions of the class index *n*. These functions must be equal between all efficient classes, including $\bar{e}$. Consequently, when compatibility holds, the impedance functions must be equal for all the ME classes *n* with positive flow $q_{rs}^n > 0$.

<u>Definition D5, impedance function of an ME class</u>. The impedance of an ME class *n* is defined as $\bar{\mathrm{I}}_{rs}^n(\mathbf{f};\mathbf{w}) = \overline{\mathrm{T}}_{rs}^n + \sum_{l=n}^{\overline{m}_{rs}-1} (\overline{\mathrm{P}}_{rs}^l - \overline{\mathrm{P}}_{rs}^{l+1}) / \breve{v}_{rs}^l$.

<u>Proposition P3</u>. The class flow pattern $\mathbf{q} = [q_{rs}^m]_m$ where $q_{rs}^m \geq 0$ is compatible with the efficiency sets E_{rs}^m if and only if $\forall n$, $q_{rs}^n > 0 \Rightarrow \bar{\mathrm{I}}_{rs}^n = \min_m \bar{\mathrm{I}}_{rs}^m$.

P3 is analogous to the definition of a Wardropian equilibrium: we can also formulate the search for a compatible class flow pattern (i.e. for a dual criteria equilibrium between classes) as a variational inequality (Smith, 1979). Hence a suitable mathematical formulation is available to deal with a continuous differentiation of the VOT. The mathematical formulations that are commonly used to deal with flow-dependent travel times, variable demand and side constraints may then be used.

2.2 Characterization of a dual criteria equilibrium

Definition D6, dual criteria equilibrium. A dual criteria equilibrium is a bi-tuple $(\mathbf{f};\mathbf{w})$ where $\mathbf{f}$ is a vector of path flows $f_{rs}^{k} \geq 0$ and $\mathbf{w}$ is a vector of multipliers w_b related to the side constraints $z_b(\mathbf{f}) \leq 0$, such that:

(i) Formation of the travel times: $T_{rs}^{k}(\mathbf{f};\mathbf{w}) = \left(\sum_a \delta_{rs}^{ak}\, t_a(\mathbf{f})\right) + \left(\sum_b w_b \frac{\partial z_b(\mathbf{f})}{\partial f_{rs}^{k}}\right)$ with $w_b \geq 0$, $z_b(\mathbf{f}) \leq 0$ and $w_b z_b(\mathbf{f}) = 0$.

(ii) Supply-demand exchange: $q_{rs} = D_{rs}\left(\int \{\min_k G_{rs}^{k}(v)\}\, dH_{rs}(v)\right)$.

(iii) Optimizing behavior of the consumers. for each O-D pair *r-s*, within each ME class *m*, it holds that $\forall k \in m,\ f_{rs}^{k} > 0 \Rightarrow T_{rs}^{k} = \overline{T}_{rs}^{m}$, and between the ME classes the class flow pattern $[q_{rs}^{m}]_m$ derived from $\mathbf{f}$ by $q_{rs}^{m} := \sum_k \Delta_{rs}^{km} f_{rs}^{k}$ is compatible with efficiency sets E_{rs}^{m} derived from the supply conditions $\{(\overline{P}_{rs}^{m};\overline{T}_{rs}^{m})\}_m$, using the definition $\overline{T}_{rs}^{m} := \min_{k \in m} T_{rs}^{k}$.

From P2, at a dual criteria equilibrium, the path flow pattern looks as if each VOT was assigned first to an ME class which would supply it with a minimum generalized travel time, and then inside that class to a path with a minimum travel time.

Definition D7, path impedance function. Let $F_{rs}(x) := \int_0^x \frac{dt}{H_{rs}^{-1}(t)}$. The impedance function of path *k* is defined as $I_{rs}^{k}(\mathbf{f};\mathbf{w}) = T_{rs}^{k} + \sum_{i=M_{rs}(k)}^{\overline{m}_{rs}-1} \frac{\overline{P}_{rs}^{i} - \overline{P}_{rs}^{i+1}}{H_{rs}^{-1}(Q_{rs}^{i}/q_{rs})}$

$$+\left[-D_{rs}^{-1}(q_{rs}) + \overline{P}_{rs}^{\overline{m}_{rs}}\, F_{rs}(1) + \sum_{i=1}^{\overline{m}_{rs}-1} (\overline{P}_{rs}^{i} - \overline{P}_{rs}^{i+1})\left(F_{rs}\left(\frac{Q_{rs}^{i}}{q_{rs}}\right) - \frac{Q_{rs}^{i}/q_{rs}}{H_{rs}^{-1}(Q_{rs}^{i}/q_{rs})}\right)\right],$$

which depends on $(\mathbf{f};\mathbf{w})$ through T_{rs}^{k} and the Q_{rs}^{i}.

Denoting $M_{rs}(k)$ by *m*, we have that $I_{rs}^{k} = \overline{I}_{rs}^{m} + T_{rs}^{k} - \overline{T}_{rs}^{m} - B_{rs}$ where B_{rs} can be interpreted as the impedance of a fictive ME class created to address the elasticity of demand using an excess-demand formulation: it can be broken into $B_{rs} = \overline{I}'_{rs} + D_{rs}^{-1} - S'_{rs}$ where $S'_{rs} = \sum_n \overline{T}_{rs}^{n} \frac{q_{rs}^{n}}{q_{rs}} + \overline{P}_{rs}^{n}[F_{rs}(\frac{Q_{rs}^{n}}{q_{rs}}) - F_{rs}(\frac{Q_{rs}^{n-1}}{q_{rs}})] = \sum_n \int_{\breve{v}_{rs}^{n-1}}^{\breve{v}_{rs}^{n}} \overline{G}_{rs}^{n}(v)\, dH_{rs}(v)$ would be equal to the mean level-of-service $S_{rs} := \int \{\min_n \overline{G}_{rs}^{n}(v)\}\, dH_{rs}(v)$ if we had that $\forall v \in [\breve{v}_{rs}^{n-1};\breve{v}_{rs}^{n}]$, $\overline{G}_{rs}^{n}(v) = \min_m \overline{G}_{rs}^{m}(v)$, and $\overline{I}'_{rs} = (\sum_n \overline{T}_{rs}^{n} \frac{q_{rs}^{n}}{q_{rs}}) + \sum_{l=1}^{\overline{m}_{rs}-1} \frac{(\overline{P}_{rs}^{l} - \overline{P}_{rs}^{l+1})}{\breve{v}_{rs}^{l}} \frac{Q_{rs}^{l}}{q_{rs}}$ would be equal to $I_{rs} := \min_n \overline{I}_{rs}^{n} = \sum_n \frac{q_{rs}^{n}}{q_{rs}} \overline{I}_{rs}^{n}$ under the same assumption.

Theorem T1, characterization of a dual criteria equilibrium. The bi-tuple $(\mathbf{f};\mathbf{w})$ is a dual criteria equilibrium if and only if:

(i) $\mathrm{T}_{rs}^{k}(\mathbf{f};\mathbf{w}) = \left(\sum_a \delta_{rs}^{ak}\, \mathrm{t}_a(\mathbf{f})\right) + (\sum_b w_b \frac{\partial \mathrm{z}_b(\mathbf{f})}{\partial f_{rs}^{k}})$, $w_b \geq 0$, $\mathrm{z}_b(\mathbf{f}) \leq 0$ and $w_b \mathrm{z}_b(\mathbf{f}) = 0$.

(ii) $\forall r-s-k,\ f_{rs}^{k} > 0 \Rightarrow \mathrm{I}_{rs}^{k} = \min_l \mathrm{I}_{rs}^{l}$.

(iii) $\forall r-s,\ q_{rs} > 0 \Rightarrow q_{rs} = \mathrm{D}_{rs}\left(\int \{\min_k \mathrm{G}_{rs}^{k}(v)\}\, \mathrm{dH}_{rs}(v)\right)$.

Proof: It is sufficient to show that D6 (iii) is equivalent to T1 (ii). D6 (iii) implies that for a loaded path *k*, its ME class *m* is such that $\mathrm{T}_{rs}^{k} = \overline{\mathrm{T}}_{rs}^{m}$ and $q_{rs}^{m} > 0$ hence $\overline{\mathrm{I}}_{rs}^{m} = \min_n (\overline{\mathrm{I}}_{rs}^{n})$: hence $\mathrm{I}_{rs}^{k} = \overline{\mathrm{I}}_{rs}^{m} - \mathrm{B}_{rs} = \min_n (\overline{\mathrm{I}}_{rs}^{n}) - \mathrm{B}_{rs} = \min_n (\overline{\mathrm{I}}_{rs}^{n} - \mathrm{B}_{rs}) \leq \min_l \mathrm{I}_{rs}^{l}$ which yields T1 (ii).

Proof of the converse: Let us consider *m* such that $q_{rs}^{m} > 0$. Taking in *m* a path *k* with $f_{rs}^{k} > 0$, we have that $\overline{\mathrm{I}}_{rs}^{m} - \mathrm{B}_{rs} = \mathrm{I}_{rs}^{k} = \min_l (\mathrm{T}_{rs}^{l} - \overline{\mathrm{T}}_{rs}^{\mathrm{M}_{rs}(l)} + \overline{\mathrm{I}}_{rs}^{\mathrm{M}_{rs}(l)} - \mathrm{B}_{rs})$ hence $\overline{\mathrm{I}}_{rs}^{m} \leq \min_n \overline{\mathrm{I}}_{rs}^{n}$ hence the class flow pattern is compatible. Furthermore, $\mathrm{I}_{rs}^{k} = \min_l \mathrm{I}_{rs}^{l} \leq \min_{l \in m} \mathrm{I}_{rs}^{l}$ hence $\mathrm{T}_{rs}^{k} = \overline{\mathrm{T}}_{rs}^{m}$.

Let us now consider the mapping $\mathbf{V}$: $(\Re^{+})^{\mathrm{N}} \to \Re^{\mathrm{N}}$ (where N sums up the dimensions of $\mathbf{f}$ and $\mathbf{w}$), $(\mathbf{f};\mathbf{w}) \mapsto \mathbf{V}(\mathbf{f};\mathbf{w}) = ([\mathrm{I}_{rs}^{k}(\mathbf{f};\mathbf{w})]_{rsk};[-\mathrm{z}_b(\mathbf{f})]_b)$.

Theorem T2, non linear complementarity problem for the dual criteria equilibrium. A bi-tuple $(\mathbf{f};\mathbf{w})$ is a dual criteria equilibrium if and only if it solves the problem "Find $(\mathbf{f};\mathbf{w}) \geq \mathbf{0}$ such that $\mathbf{V}(\mathbf{f};\mathbf{w}) \geq \mathbf{0}$ and $\mathbf{V}(\mathbf{f};\mathbf{w}) \cdot (\mathbf{f};\mathbf{w}) = \mathbf{0}$".

The problem can be reformulated as: (i) $\mathrm{I}_{rs}^{k} \geq 0$, (ii) $\mathrm{z}_b \leq 0$, (iii) $f_{rs}^{k}\, \mathrm{I}_{rs}^{k} = 0$ and (iv) $w_b \mathrm{z}_b = 0$.

T1 (i) $\Leftrightarrow$ T2 (ii) and (iv) since that $w_b \geq 0$ is assumed in D6 as in T2. T1 (ii) implies the compatibility hence we obtain $\mathrm{B}_{rs} = \mathrm{I}_{rs} + \mathrm{D}_{rs}^{-1} - S_{rs}$ which yields that $\min_k \mathrm{I}_{rs}^{k} = 0$ if T1 (iii) holds. Conversely, T2 (i) and (iii) imply that if $f_{rs}^{k} > 0$ then $\mathrm{I}_{rs}^{k} = 0 \leq \min_l \mathrm{I}_{rs}^{l}$ hence T1 (ii) and (iii).

Theorem T3, variational inequality problem for the dual criteria equilibrium. $(\tilde{\mathbf{f}};\tilde{\mathbf{w}}) \geq \mathbf{0}$ is a dual criteria equilibrium if and only if it solves the variational inequality problem (VIP): "$\forall (\mathbf{f};\mathbf{w}) \geq \mathbf{0},\ \mathbf{V}(\tilde{\mathbf{f}};\tilde{\mathbf{w}}) \cdot (\mathbf{f} - \tilde{\mathbf{f}};\mathbf{w} - \tilde{\mathbf{w}}) \geq 0$".

As the VIP is in $(\Re^{+})^{\mathrm{N}}$, T3 is equivalent to T2.

Let us finally define $\mathrm{J}_{\mathrm{bic}}(\mathbf{f}) := \sum_{rs} q_{rs} [\sum_{i=1}^{\overline{m}_{rs}} \overline{\mathrm{P}}_{rs}^{i} (\mathrm{F}_{rs}(\frac{Q_{rs}^{i}}{q_{rs}}) - \mathrm{F}_{rs}(\frac{Q_{rs}^{i-1}}{q_{rs}}))]$: Leurent (1993a) showed that $\frac{\partial}{\partial f_{rs}^{k}} \mathrm{J}_{\mathrm{bic}}(\mathbf{f}) = \mathrm{I}_{rs}^{k} - \mathrm{T}_{rs}^{k} + \mathrm{D}_{rs}^{-1}(q_{rs})$.

Theorem T4, optimization program for a dual criteria equilibrium in the symmetric case. If the travel time functions $\mathrm{t}_a(\mathbf{f})$ derive from a potential function $\mathrm{J}_{\mathrm{t}}(\mathbf{f})$, then a path flow pattern $\mathbf{f} \geq \mathbf{0}$ contributes to a dual criteria equilibrium if and only if it solves the optimization program $\min_{\mathbf{f}\geq\mathbf{0}} \mathrm{J}(\mathbf{f}) = \mathrm{J}_{\mathrm{t}}(\mathbf{f}) + \mathrm{J}_{\mathrm{bic}}(\mathbf{f}) - \sum_{rs} \int_0^{q_{rs}} \mathrm{D}_{rs}^{-1}(u)\, \mathrm{d}u$ subject to side constraints $\mathrm{z}_b(\mathbf{f}) \leq 0$. If $\mathrm{t}_a(\mathbf{f}) = \mathrm{t}_a(x_a)$ then $\mathrm{J}_{\mathrm{t}}(\mathbf{f}) = \sum_a \int_0^{x_a} \mathrm{t}_a(u)\, \mathrm{d}u$.

Proof. $\mathrm{J}(\mathbf{f}) = £(\mathbf{f};\mathbf{0})$ where the Lagrangian £ is defined as $£(\mathbf{f};\mathbf{w}) = \mathrm{J}(\mathbf{f}) + \sum_b z_b . w_b$: its saddle points are the solutions of the VIP in T3, i.e. the dual criteria equilibria.

2.3 Existence and uniqueness results

Theorem T5, existence. If travel time functions t_a are continuous, constraining functions z_b are continuous and differentiable with continuous derivatives, functions H_{rs}^{-1} are continuous, D_{rs} are bounded and D_{rs}^{-1} continuous, and if there is a feasible path flow pattern $\mathbf{f}_0 \geq 0$ such that $\mathrm{z}_b(\mathbf{f}_0) \leq 0\ \forall b$, then there exists a dual criteria equilibrium.

Proof. Under these assumptions the mapping **V** is continuous and its admissible set is non empty and bounded: hence there is a solution to the VIP in T3.

Theorem T6, uniqueness. Under the assumptions in T4 and T5, if furthermore travel time functions t_a and inverse demand functions D_{rs}^{-1} decrease, then at a dual criteria equilibrium $(\tilde{\mathbf{f}};\tilde{\mathbf{w}})$ it holds that:

(i) if t_a is strictly increasing, then the link flow $x_a(\tilde{\mathbf{f}})$ is unique.

(ii) if D_{rs}^{-1} is strictly decreasing, then the O-D flow $q_{rs}(\tilde{\mathbf{f}})$ is unique.

(iii) the ME classes proportions q_{rs}^m / q_{rs} are unique.

Proof. Under the assumptions made in T6, J is convex. If t_a is strictly increasing, then J is strictly convex with respect to x_a. If D_{rs}^{-1} is strictly decreasing, then J is strictly convex with respect to q_{rs}. T6 (iii) results from the strict convexity of $\mathrm{J}_{\mathrm{bic}}$ which was shown by Leurent (1993a).

To sum up, the results applicable to the single criterion model of Beckmann hold for the dual criteria model, which is also endowed with an additional property: the uniqueness of the ME class proportions at an equilibrium.

3. ALGORITHMS

We shall first review the convergent procedures available to solve the variational inequality problem in T3 (i.e. to compute a dual criteria equilibrium) in the symmetric, unconstrained case (sub-section 3.1). Then we shall provide an augmented Lagrangian method to address the constrained case (sub-section 3.2). The asymmetric case may be addressed by means of a diagonalization algorithm (Florian and Spiess, 1982) or of a fixed-point method (Dafermos, 1980): these procedures converge if the mapping **V** is strongly monotone and both consist of iteratively solving symmetric problems.

3.1 The symmetric, unconstrained case

Leurent (1993a) put forward a Method of Successive Averages (MSA) which does not require path enumeration since it relies on randomisation. A path storing version of the MSA can also be implemented (Leurent 1993b). A Frank-Wolfe procedure was put forward by Marcotte and Zhu (1994) to address the fixed-demand case without enumerating paths. Leurent (1995b) applied the Procedure of Equalization-by-Transfer (PET) to the dual criteria model with elastic demand. The PET is based on the following principle: for a given O-D pair, the flow is transferred from the loaded path with the longest travel time to the shortest path in such a way as to equalize journey times or remove flow from the long path. This is performed first for one pair of paths and then for others, until the travel times on the paths used reach equilibrium thereby ensuring that (identified) unused paths are not quicker. All O-D pairs are processed in this way and the algorithm iterates until convergence. To address the dual criteria model, the travel times must be replaced by impedance functions.

We compared the two MSAs and the PET for the dual criteria model; we also tested the PET and the MSA against the Frank-Wolfe algorithm for the single criterion model with mean VOT same as in the dual criteria case but no dispersion of the VOT. The test was run on a portable PC with 120 MHz Pentium.

The application involved the west part of the Paris metropolitan area during the evening peak period, with a typical trip travel time of one hour. The test network consisted of 2,000 arcs and there were 141 origin and destination zones. Demand was assumed to be elastic with demand function $q = q_0(S/S_0)^{-0.6}$. A log-normal distribution of the VOT was assumed in the dual criteria case, with mean M = 75 FF/h and 0.6 as the standard deviation of its natural logarithm. The convergence was measured using two Root Mean Square Error functions of the link flows:

- first a relative RMSE: $\mathrm{Rel}(\mathrm{step}\, n) := \sqrt{\frac{1}{\bar{a}} \sum_a (x_a^{(n)} - \bar{x}_a^{(n)})^2}$ where $\bar{a}$ is the number of links, $x_a^{(n)}$ is the flow on link a at the end of the n-th iteration and $\bar{x}_a^{(n)} = \frac{1}{n} \sum_{k=0}^{n-1} x_a^{(k)}$.
- second an absolute RMSE: $\mathrm{Abs}(\mathrm{step}\, n) := \sqrt{\frac{1}{\bar{a}} \sum_a (x_a^{(n)} - x_a^{\infty})^2}$ where x_a^{∞} is the equilibrium link flow on link a obtained from a previous run of an algorithm.

Figure 4 displays the evolution of Rel (part a) and Abs (part b) in the case of the single criterion model, addressed using either Frank-Wolfe or the MSA or the PET. Part (a) suggests that the PET is more efficient with respect to the number of iterations; hence the Abs criteria plotted in part (b) are evaluated using the link flows obtained at the end of the PET. We notice that the difference between the PET and the two algorithms that do not identify the paths is much larger than that between the MSA and Frank-Wolfe.

Fig. 4. The case of the single criterion model.

a) Evolution of Rel (logarithmic scale). b) Evolution of Abs (logarithmic scale).

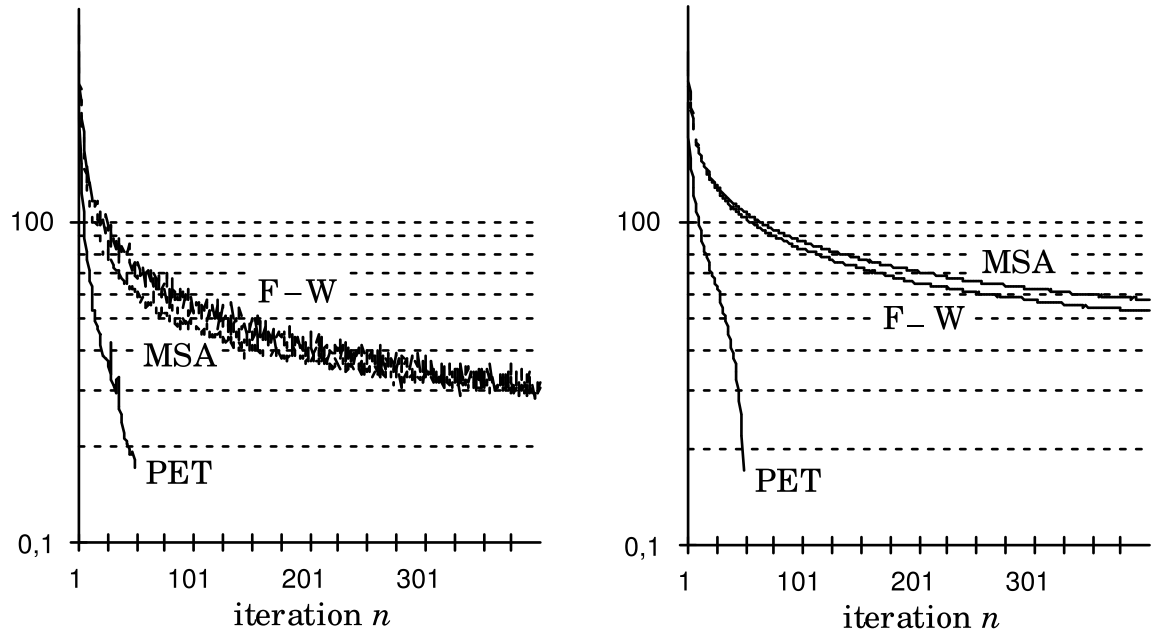


Figure 5 displays the evolution of Rel (part a) and Abs (part b) in the case of the dual criteria model, addressed using either the MSA with path enumeration or the MSA without path enumeration or the PET. From part (a), the PET is still the most efficient algorithm, we also used its results to evaluate the absolute RMSE in part (b). Surprisingly enough, the convergence criteria Rel and Abs do not enable us to discern an advantage of the MSA with path enumeration over the MSA without path enumeration.

Fig. 5. The case of the dual criteria model.

a) Evolution of Rel (logarithmic scale). b) Evolution of Abs (logarithmic scale).

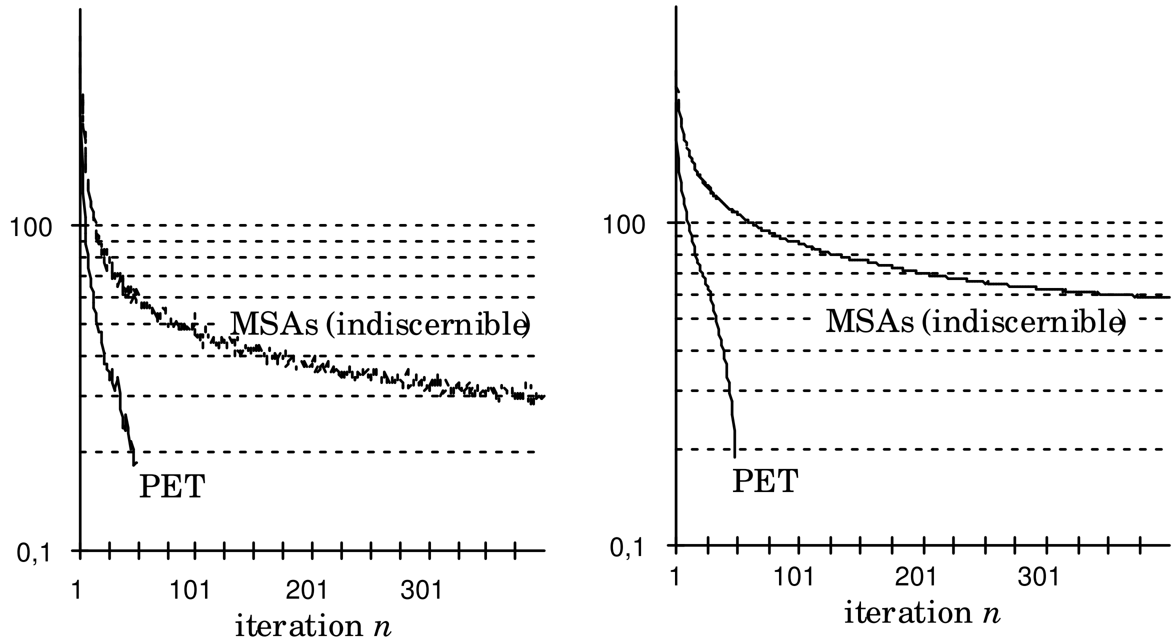


Table 1 reports the running times per iteration. One iteration of a method identifying paths is more time consuming than one that does not require path storing. We note that an iteration of the PET in the dual criteria case is not much more demanding than in that of the single criterion case.

Tab.1. Running times per iteration.

Single criterion model		Dual criteria model	
Algorithm	RTI (mn)	Algorithm	RTI (mn)
Frank-Wolfe	0.5	MSA with path enumeration	2.8
MSA	0.3	MSA without path storing	0.4
PET	3.2	PET	4.1

The overall conclusion of this medium size experiment is that the dual criteria model can be solved at a computational cost competitive with that needed to solve the single criterion model using Frank-Wolfe.

3.2 An augmented Lagrangian method to deal with the symmetric, constrained case

To incorporate side constraints, one possible approach is to compute the saddle points of an augmented Lagrangian designed in such a way that its saddle points are the same as those of the original Lagrangian (see the proof of T4). An augmented Lagrangian scheme consists of iteratively solving unconstrained optimization programs in which the dual variables (the multipliers w_b) are given and only the primal variables (the path flows) are allowed to vary; at the end of each master iteration the dual variables are updated for the convergence test and (eventually) the next iteration.

In the case of link capacity constraints $x_a(\mathbf{f}) - C_a \leq 0$, an augmented Lagrangian is $\mathrm{L_A}(\mathbf{f};\mathbf{w};\tau) := \mathrm{J}(\mathbf{f}) + \frac{1}{2\tau}\sum_a (\max\{0; w_a + \tau(x_a(\mathbf{f}) - C_a)\}^2 - {w_a}^2)$ where $\tau > 0$. Defining ad hoc travel time functions $\mathrm{t}_a^{[n]}(u) := \mathrm{t}_a(u) + \max\{0; w_a^{[n]} + \tau(u - C_a)\}$, the n-th master iteration consists of solving the problem $\min_{\mathbf{f}} \mathrm{J}^{[n]}(\mathbf{f})$ subject to $\mathbf{f} \geq \mathbf{0}$, where

$$\mathrm{J}^{[n]}(\mathbf{f}) := \mathrm{L_A}(\mathbf{f};\mathbf{w}^{[n]};\tau) = \left(\sum_a \int_0^{x_a} \mathrm{t}_a^{[n]}(u)\,\mathrm{d}u\right) + \mathrm{J_{bic}}(\mathbf{f}) - \sum_{rs}\int_0^{q_{rs}} \mathrm{D}_{rs}^{-1}(u)\,\mathrm{d}u\,.$$

Based on the solution $\mathbf{f}^{[n]}$ to the n-th problem, the multipliers are updated via $w_a^{[n+1]} := \max\{0\,;\, w_a^{[n]} + \rho(x_a(\mathbf{f}^{[n]}) - C_a)\}$ where $\rho > 0$. An overall convergence test is to check if $\mathrm{C}(n) = \left\| \mathbf{w}^{[n+1]} - \mathbf{w}^{[n]} \right\| \leq \varepsilon$.

We tested the method with the network described in sub-section 3.1 using the practical capacities as the limit capacities. The values of the parameters were $\tau = \rho = .05$, $\varepsilon = 5$ veh/h. Convergence was attained after 25 master iterations for both the single and the dual criteria models. In the case of the dual criteria model, from $\mathbf{w}^{[0]} = 0$, we obtained C(1) = 35.7; C(5) = 13.8; C(10) = 9.2; C(20) = 5.8. Each master iteration consisted of three (inner) iterations of the PET: the paths obtained at the end of a master iteration were used at the beginning of the next one, so the convergence in each application of the PET was very fast after the first 5 master iterations.

4. TOOLS FOR ECONOMETRIC ANALYSIS

We have shown the practicality of the dual criteria model, due to its mathematical characterization and path identification. We shall now investigate its relevance for the econometric analysis of a traffic forecast. To that end, we shall first indicate a maximum likelihood method to estimate the exogenous error on the VOT (sub-section 4.1). Then we shall look at how the dual criteria model propagates the exogenous (that is, from outside the assignment model) error on the VOT when forecasting the revenue of a toll road, by restating the principle of error propagation (sub-section 4.2) and dealing with a small example (sub-section 4.3).

4.1 A maximum likelihood estimator of the VOT

To measure the exogenous error related to the VOT, one has to measure the estimation error on the parameters of its distribution. The maximum likelihood method can be used to estimate a VOT distribution, using disaggregate, individual data (Leurent, 1995c). Assume that, to trip-maker *i*, the generalized cost of travelling along path *k* is

$$\mathrm{G}_k(i) = \left\{\sum_b r_{i,b} . X_{b,k}\right\} + v_i \, \mathrm{T}_k = \mathrm{P}_k + v_i \, \mathrm{T}_k$$

where *b* is an attribute index, $X_{b,k}$ the value of the *b*-th attribute along path *k* (an attribute may be the monetary expense, or distance, or discomfort etc, but it cannot be the travel time), $r_{i,b}$ are trade-off coefficients, v_i is the VOT and T_k is the travel time along path *k*.

There are $\bar{b} + \bar{d}$ parameters to be estimated:

- the $\bar{b}$ coefficients $r_{i,b}$,

- the $\bar{d}$ parameters of the distribution of the VOT, hereafter assumed to follow a log-normal distribution characterized by means of the mean μ and standard-deviation σ of its logarithm (in this case $\bar{d} = 2$).

We assume that, on a given O-D pair, the alternative paths *k* were surveyed so that their flows f_k are known. The proposed estimation method relies on the maximisation of the log-likelihood of the sample, evaluated as

$$\mathrm{L}(\theta) = \ln \prod_i \pi_{k_i}(\theta) = \sum_k f_k \ln(\pi_k(\theta))$$

where $\theta = \{\theta_n\}_n = \{r_b\}_b \cup \{\mu ; \sigma\}$ denotes the vector of parameters, $\pi_k(\theta)$ is the modelled probability of travelling on path *k*, and k_i is the observed, chosen path of trip-maker *i*.

If we assume that the alternative paths are ranked in order of increasing travel prices and that they are all efficient, then

$$\pi_k(\theta) = \mathrm{H}_{\mu,\sigma}(U_k) - \mathrm{H}_{\mu,\sigma}(L_k)$$

where U_k and L_k are the supply-related frontier VOTs between alternatives *k* and *k*+1, and *k*-1 and *k*, respectively.

Maximisation of $\mathrm{L}(\theta)$ can be performed using a first- or second- order descent algorithm: it is easy to calculate the partial derivatives of L with respect to the θ_n from the partial derivatives of the π_k with respect to the θ_n (see Appendix A). From the value of the Hessian of L at the solution point, we can estimate the variance-covariance matrix of the estimators.

Table 2 provides an application with Revealed Preference data: three O-D pairs in the west of France, each with two competing paths, were surveyed during one day in 1993. The travel times T_k were measured, and the travel prices P_k include the toll if applicable and the travel costs. A log-normal distribution of the VOT was estimated, yielding $\hat{\mu}$ = 4.284, $\hat{\sigma}$ = 0.348, $\hat{\sigma}_{\hat{\mu}}$ = 0.0045, $\hat{\sigma}_{\hat{\sigma}}$ = 0.01, $\mathrm{Cov}(\hat{\mu};\hat{\sigma})/\hat{\mu}\hat{\sigma} = -7e-6$, hence an estimated mean VOT of 77.1 FF/hour.

Tab. 2. Modeled vs. observed flows in the Angers-Nantes O-D survey.

Origin-destination pair		P_k (FF)	T_k (h)	veh/day observed	veh/day modeled
Nantes-Angers	Free highway N23	65.4	0.967	1500	1522
	Toll motorway A11	86.7	0.617	3500	3477
Nantes-Ancenis	Free highway N23	25.4	0.40	2600	2621
	Toll motorway A11	33.9	0.30	1300	1279
Angers-Ancenis	Free highway N23	40.1	0.567	500	456
	Toll motorway A11	50.8	0.417	450	494

4.2 Principle of error propagation

Our aim is to measure the error on the output vector Y of a model, given the exogenous error ε_X on the input vector X. In the case of a model with a dependence expressed straightforwardly as Y = F(X) with a differentiable F, for small ε_X it holds that $\varepsilon_Y = (\nabla_X \mathrm{F})\varepsilon_X$, from which the mean and deviation of ε_Y can be deduced, given the distribution of ε_X.

In the case of an implicit model, i.e. $\mathrm{F}(\mathrm{X},\mathrm{Y}) = 0$, if $\nabla_Y \mathrm{F}$ can be inverted, then it holds that $\varepsilon_Y = [\nabla_Y \mathrm{F}]^{-1}[-\nabla_X \mathrm{F}]\varepsilon_X$ (Tobin 1986). If the relationship F is obtained by differentiating a convex function J, then $\mathrm{F} = \nabla_X \mathrm{J}$ hence

$$\varepsilon_Y = [\nabla_Y \nabla_Y \mathrm{J}]^{-1}[-\nabla_X \nabla_Y \mathrm{J}]\varepsilon_X$$

in which $\nabla_Y \mathrm{J}$ denotes the gradient of J with respect to Y, $\nabla_Y \nabla_Y \mathrm{J}$ the Hessian of J with respect to Y, and $\nabla_X \nabla_Y \mathrm{J}$ the matrix of the partial derivatives of $\nabla_Y \mathrm{J}$ with respect to X.

In the case of the dual criteria model, the input data set X includes the demand functions D_{rs}, the VOT (from its distribution H_{rs}), the paths, and the link travel times t_a and side constraints z_b that contribute to the path travel times.

4.3 A two-link example

We applied the formula restated in sub-section 4.2 to the dual criteria model. For a two-link network it is also possible to derive closed-form expressions (Leurent 1994).

Let us consider the following network: two parallel roads that compete between a single origin and a single destination. The first link is a freeway (F), while the other one is a toll motorway (T). Let:

- T_F and T_T be the travel times on the two roads, assumed to be normally distributed following respectively $\mathrm{N}(\overline{T}_F;\sigma_{T_F})$ and $\mathrm{N}(\overline{T}_T;\sigma_{T_T})$.

- q the trip rate, distributed $\mathrm{N}(\overline{q};\sigma_q)$.
- M and σ respectively the mean VOT and the standard deviation of the natural logarithm of the VOT, distributed $\mathrm{N}(\overline{M};\sigma_M)$ and $\mathrm{N}(\overline{\sigma};\sigma_\sigma)$.
- p the fare on the toll road.

Assuming that $\Delta T = T_F - T_T > 0$ and denoting by $\hat{v} = \frac{p}{\Delta T}$ the frontier VOT between the two travel alternatives, the flow on the toll road is $f_T = q(1-\mathrm{H}(\hat{v}))$, yielding a revenue of $R = pq(1-\mathrm{H}(\hat{v}))$.

We can compute the revenue elasticities with respect to the input variables $\mathrm{X} = \{\Delta T, q, M, \sigma\}$ in three different dual criteria modeling contexts: first with fixed times and a fixed trip rate, second with variable times but a fixed trip rate, third with variable times and a variable trip rate.

Let us consider numerical values of $\Delta\overline{T} = 0.2145\,\mathrm{h}$, $p = 15\mathrm{FF}$, $\overline{q} = 3000\,\mathrm{veh/h}$, $\overline{M} = 60\mathrm{FF/h}$ and $\overline{\sigma} = 0.6$. Assume that all the input variables are independent, and that the exogenous errors are $\frac{\sigma_q}{\overline{q}} = \frac{\sigma_\sigma}{\overline{\sigma}} = \frac{\sigma_M}{\overline{M}} = 10\%$ and $\frac{\sigma_{\Delta T}}{\Delta\overline{T}} = 15\%$ in the fixed time case but is reduced to 5% in the variable time cases (taking into account the dependence of the travel time with respect to traffic flow diminishes the residual error on the travel time). Then the propagated error on the revenue can be evaluated in each of the three cases (table 3).

Tab. 3. Numerical results of error propagation.

Elasticities of f_T to an input variable	fixed times, fixed trip rate	variable times, fixed trip rate	variable times, variable trip rate
$\varepsilon_{\Delta T}$	1.97	0.59	0.63
ε_q	1.00	0.94	0.76
ε_M	1.97	0.59	0.63
ε_σ	-0.037	-0.016	-0.003
$\frac{\sigma_{f_T}}{\overline{f_T}} = \frac{\sigma_R}{\overline{R}}$	37%	12%	10%

We notice that a more sophisticated model enables one to reduce the propagated exogenous error, by explicitly accounting for the variability and dispersion that were at first only implicitly addressed. This reduction in residual variability corresponds rigorously to the reduction in the residual variance in an analysis-of-variance when additional explanatory variables are introduced.

The dual criteria model, through the probabilistic assumptions on the VOT, enables one to make statistical inference on traffic and revenue forecasts. In this way, it allows practitioners to simulate toll roads.

5. ELABORATION OF THE ECONOMIC REPRESENTATION

In this section, we provide several refinements to the economic representation in a dual criteria model: demand-related (a generalized trade-off function, stochastic dual criteria model) and supply-related (two path-dependent criteria). Finally we present some directions for future research.

5.1 More general trade-off functions

So far we have specified the path generalized travel time as

$$\mathrm{G}_{rs}^{k}(v) = \mathrm{T}_{rs}^{k} + \mathrm{P}_{rs}^{k} / v$$

A more general dual criteria model can be based on

$$\mathrm{G}_{rs}^{k}(v) = \mathrm{G}(\mathrm{T}_{rs}^{k}; \mathrm{P}_{rs}^{k}; v)$$

but this expression is not practical. An intermediate stage is to consider

$$\mathrm{G}_{rs}^{k}(v) = \mathrm{T}_{rs}^{k} + \mathrm{P}_{rs}^{k}\, \tau_{rs}(v)$$

where τ_{rs} is a trade-off function and *v* an attribute of the trip-maker, characterized by its cumulative distribution function $\mathrm{H}_{rs}(v)$.

The treatment of such a dual criteria model is adapted from that of section 1 by:

- replacing $\mathrm{G}_{rs}^{k}(v)$ with $\mathrm{T}_{rs}^{k} + \mathrm{P}_{rs}^{k}\, \tau_{rs}(v)$, notably in the mean level-of-service $S_{rs} = \int \{\min_k \mathrm{G}_{rs}^{k}(v)\} \mathrm{dH}_{rs}(v)$.

- replacing the supply-related frontier VOTs $\hat{v}_{rs}^{m}$ with $\hat{\tau}_{rs}^{m}$, or with $\tau_{rs}(\hat{v}_{rs}^{m})$ if τ_{rs} is monotonic and invertible.

- in the impedance functions $\bar{\mathrm{I}}_{rs}^{m}$ and I_{rs}^{k} and in the dual criteria equilibrium conditions, replacing $1/\mathrm{H}_{rs}^{-1}$ by $\tau_{rs} \circ \mathrm{H}_{rs}^{-1}$.

- defining $\mathrm{F}_{rs}(u)$ as $\int_0^u (\tau_{rs} \circ \mathrm{H}_{rs}^{-1})(t)\mathrm{d}t$ if τ_{rs} is non-increasing, or if τ_{rs} is non-decreasing as $-\int_0^u (\tau_{rs} \circ \mathrm{H}_{rs}^{-1})(1-t)\mathrm{d}t$. If τ_{rs} is not monotonic, then the CDF of τ should be considered and dealt with as in the basic dual criteria model, rather than dealing with the CDF of *v*.

5.2 Stochastic behavior

Another way to differentiate the trip-makers in dimensions other than the VOT is to take these other dimensions into account as a random phenomenon, which is the basic assumption made in stochastic assignment. The dual criteria model can be made stochastic by considering

$$\mathrm{G}_{rs}^{k}(v) = \mathrm{T}_{rs}^{k} + \mathrm{P}_{rs}^{k}\, \tau_{rs}(v) + \varepsilon_{rs}^{k}(\omega)$$

where $\varepsilon_{rs}^{k}(\omega)$ is a random variable. Then, probabilistic assumptions need to be made on the joint distribution of v_{rs} and the ε_{rs}^{k}. The simplest solution is to assume that the $\{\varepsilon_{rs}^{k}\}_k$ and

v_{rs} are independent. An appealing alternative, if the ε_{rs}^{k} are distributed Multi-Variate Normal, would be to consider a log-normal v_{rs} and assume that $\left\{\{\varepsilon_{rs}^{k}\}_k ; \ln v_{rs}\right\}$ is distributed MVN.

However, inclusion of stochastic behavior into the dual criteria model precludes most of the mathematical tools worked out to solve the deterministic dual criteria model; only the MSA without path enumeration remains available as a practical algorithm.

5.3 The case of two flow-dependent criteria

If both the travel time and the travel price depend on the traffic conditions, then the definition of efficient paths and monetary expense classes of paths may change over the set of the feasible path flow vectors. More precisely, these definitions become relative to the prevailing path flow vector; in other words a path flow vector may be a dual criteria equilibrium with respect to itself. The following notions and results are preserved:

• the monetary expense classes of paths.

• the frontier VOTs.

• the impedance functions $\bar{\mathrm{I}}_{rs}^{m}$ and I_{rs}^{k} still exist, though conditional to the path flow vector.

• the VIP formulation is still available; if the travel costs vary continuously with respect to the flow vector, then under the assumptions in T5, the $\bar{\mathrm{I}}_{rs}^{m}$ and I_{rs}^{k} are still continuous, hence there still exists at least one dual criteria equilibrium.

But the convex programming formulation is lost and the algorithms become heuristic.

5.4 State-of-the-art and future research

The extensions introduced so far may of course be combined. Leurent (1995a) designed a multi-purpose assignment package which enables one to combine the basic dual criteria model as defined in sections 1 to 4, multiple user classes, and stochastic behavior. Leurent (1995c) addressed the case of two flow-dependent criteria, concluding that the MSA with path enumeration is the most convenient algorithm.

Many contributions are still required: notably to address the uniqueness issue in asymmetric assignment, to develop an estimation method for the stochastic dual criteria model (it would be especially useful in the case of a probit dual criteria assignment), and of course to implement non stationary, dynamic dual criteria models.

CONCLUDING COMMENTS

Three main comments are in order:

First, we have provided a comprehensive methodology for dual criteria assignment, from its mathematical characterization to its statistical estimation and sensitivity analysis, including an efficient and accurate path identifying algorithm. This makes the dual criteria model at least as appealing as any other assignment model, from a technical point of view.

Second, the dual criteria model enables the practitioner to make efficient use of information about the trip-makers' trade-offs between time and money; this information can be used to forecast the mean traffic and revenue, as well as their possible variations.

Third, the practicality of path identification in the context of assignment models calls for empirical research about which paths are selected by the trip-makers, and how can we make automatic procedures to reproduce the processes of path identification. This issue was addressed in Leurent (1995c) in the interurban case, concluding that a stochastic dual criteria assignment looks like a desirable modeling solution, as some identified paths are not efficient in the sense of a deterministic dual criteria model.

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APPENDIX A. FORMULAE FOR MAXIMUM LIKELIHOOD ESTIMATION

The partial derivatives of the log-likelihood function L with respect to the parameters θ_m can be derived from those of the probabilities π_k using the formulae:

$$\frac{\partial \mathrm{L}}{\partial \theta_m} = q\sum_k \frac{P_k}{\pi_k}\frac{\partial \pi_k}{\partial \theta_m}$$

$$\frac{\partial^2 \mathrm{L}}{\partial \theta_m\,\partial \theta_n} = q\sum_k \frac{P_k}{\pi_k}\left\{\frac{\partial^2 \pi_k}{\partial \theta_m\,\partial \theta_n} - \frac{1}{\pi_k}\frac{\partial \pi_k}{\partial \theta_m}\frac{\partial \pi_k}{\partial \theta_n}\right\}.$$

By construct, the probabilities $\pi_m(\{r_a\};\mu;\sigma)$ depend on μ and σ only through the function $\mathrm{H}_{\mu,\sigma}$, and on the coefficients r_a only through the supply-related frontier VOTs U_m and L_m formulated as

$$\hat{v}_m^{m+n} := \frac{\mathrm{P}^{m+n} - \mathrm{P}^m}{\mathrm{T}^m - \mathrm{T}^{m+n}}.$$

By the linearity of differentiation

$$\frac{\partial}{\partial \mu}\pi_m = \frac{\partial}{\partial \mu}\mathrm{H}_{\mu,\sigma}(U_m) - \frac{\partial}{\partial \mu}\mathrm{H}_{\mu,\sigma}(L_m)$$

and in a general fashion, $\forall \mathrm{D} \in \left\{\frac{\partial}{\partial \mu};\frac{\partial}{\partial \sigma};\frac{\partial^2}{\partial \mu \partial \sigma};\frac{\partial^2}{\partial \mu^2};\frac{\partial^2}{\partial \sigma^2}\right\}$,

$$\mathrm{D}\,\pi_m = \mathrm{DH}_{\mu,\sigma}(U_m) - \mathrm{DH}_{\mu,\sigma}(L_m).$$

As regards the partial derivatives with respect to the r_a parameters, assuming that all the paths in a given monetary expense class share the same attributes, i.e. $X_{a,k} = X_{a,m(k)}$, we note from the definition of a frontier VOT that

$$\frac{\partial}{\partial r_a}\hat{v}_m^{m+n} = \frac{X_{a,m+n} - X_{a,m}}{\mathrm{T}_m - \mathrm{T}_{m+n}} \text{ and } \frac{\partial^2}{\partial r_a \partial r_b}\hat{v}_m^{m+n} = 0 .$$

Hence

$$\frac{\partial \pi_m}{\partial r_a} = \frac{\partial\, \mathrm{H}_{\mu,\sigma}(U_m)}{\partial x}\frac{\partial U_m}{\partial r_a} - \frac{\partial\, \mathrm{H}_{\mu,\sigma}(L_m)}{\partial x}\frac{\partial L_m}{\partial r_a}$$

$$\frac{\partial^2 \pi_m}{\partial r_a \partial r_b} = \frac{\partial^2\, \mathrm{H}_{\mu,\sigma}(U_m)}{\partial x^2}\frac{\partial U_m}{\partial r_a}\frac{\partial U_m}{\partial r_b} - \frac{\partial^2\, \mathrm{H}_{\mu,\sigma}(L_m)}{\partial x^2}\frac{\partial L_m}{\partial r_a}\frac{\partial L_m}{\partial r_b}$$

$$\frac{\partial^2 \pi_m}{\partial r_a \partial \mu} = \frac{\partial^2\, \mathrm{H}_{\mu,\sigma}(U_m)}{\partial \mu\, \partial x}\frac{\partial U_m}{\partial r_a} - \frac{\partial^2\, \mathrm{H}_{\mu,\sigma}(L_m)}{\partial \mu\, \partial x}\frac{\partial L_m}{\partial r_a}$$

$$\frac{\partial^2 \pi_m}{\partial r_a \partial \sigma} = \frac{\partial^2\, \mathrm{H}_{\mu,\sigma}(U_m)}{\partial \sigma \partial x}\frac{\partial U_m}{\partial r_a} - \frac{\partial^2\, \mathrm{H}_{\mu,\sigma}(L_m)}{\partial \sigma \partial x}\frac{\partial L_m}{\partial r_a} .$$

APPENDIX B. THE CASE OF A LOG-NORMAL VOT

The dual criteria model can be applied with a log-normally distributed VOT x; if its natural logarithm has a mean μ and a standard deviation σ, then $\mathrm{H}(x) = \Phi(\frac{\ln(x)-\mu}{\sigma})$ where Φ is the CDF of a normal distribution with mean 0 and standard deviation 1. Let ϕ be the probability density function of this normal distribution, i.e. $\phi(t) = \exp(-t^2/2)/\sqrt{2\pi}$. Denote $\dfrac{\ln(x)-\mu}{\sigma}$ by t_x. Then the VOT-related functions of interest in dual criteria assignment can be evaluated as:

$$\mathrm{H}^{-1}(y) = \exp\left(\mu + \sigma\Phi^{-1}(y)\right)$$

$$\mathrm{F}(x) = \int_0^x \frac{1}{\mathrm{H}^{-1}(u)}\mathrm{d}u = \exp(\frac{\sigma^2}{2} - \mu)\Phi\left(\Phi^{-1}(x) - \sigma\right)$$

$$\partial\begin{bmatrix} /\partial x \\ /\partial \mu \\ /\partial \sigma \end{bmatrix}\mathrm{H}(x) = \frac{\phi(t_x)}{\sigma}\begin{bmatrix} 1/x \\ -1 \\ -t_x \end{bmatrix}$$

$$\partial^2\begin{bmatrix} /\partial x^2 & /\partial x \partial \mu & /\partial x \partial \sigma \\ \times & /\partial \mu^2 & /\partial \mu \partial \sigma \\ \times & \times & /\partial \sigma^2 \end{bmatrix}\mathrm{H}(x) = \frac{\phi(t_x)}{\sigma^2}\begin{bmatrix} -(\sigma+t_x)/x^2 & t_x/x & -(1-{t_x}^2)/x \\ \times & -t_x & 1-{t_x}^2 \\ \times & \times & t_x.(2-{t_x}^2) \end{bmatrix}.$$

APPENDIX C. PROOF OF PROPOSITIONS P1, P2, P3

Proof of P1: From (1) $\exists u \in \mathrm{E}_{rs}^{m}$ hence $(\overline{\mathrm{P}}_{rs}^{n} - \overline{\mathrm{P}}_{rs}^{m}) / u > 0$ and $0 < \overline{\mathrm{T}}_{rs}^{m} - \overline{\mathrm{T}}_{rs}^{n}$. Let $v_0 := (\overline{\mathrm{P}}_{rs}^{n} - \overline{\mathrm{P}}_{rs}^{m}) / (\overline{\mathrm{T}}_{rs}^{m} - \overline{\mathrm{T}}_{rs}^{n})$: from (1) we know that $v_0 \in [\sup \mathrm{E}_{rs}^{m}; \inf \mathrm{E}_{rs}^{n}]$. It must hold that $\sup \mathrm{E}_{rs}^{m} = \inf \mathrm{E}_{rs}^{n}$ because if not each $v \in]\sup \mathrm{E}_{rs}^{m}; \inf \mathrm{E}_{rs}^{n}[$ would be in an E_{rs}^{i} "between" E_{rs}^{m} and E_{rs}^{n}, but this contradicts our initial assumption. So

$$\hat{v}_{rs}^{m} = \sup \mathrm{E}_{rs}^{m} = \inf \mathrm{E}_{rs}^{n} = (\overline{\mathrm{P}}_{rs}^{n} - \overline{\mathrm{P}}_{rs}^{m}) / (\overline{\mathrm{T}}_{rs}^{m} - \overline{\mathrm{T}}_{rs}^{n}).$$

Proof of P2: When compatibility holds, for each efficient ME class n we have that $Q_{rs}^{n} = \sum_{m \,\text{efficient} \leq n} q_{rs}^{m} = \sum_{m \,\text{efficient} \leq n} q_{rs} \int_{\mathrm{E}_{rs}^{m}} \mathrm{dH}_{rs}(v) = q_{rs}\, \mathrm{H}_{rs}(\hat{v}_{rs}^{n})$ whereas $e(n) = n$ holds by assumption on n. If n is not efficient, since inefficient classes are empty from the compatibility relation, then it holds that $Q_{rs}^{n} = Q_{rs}^{e(n)}$ hence $Q_{rs}^{n} = q_{rs}\, \mathrm{H}_{rs}(\hat{v}_{rs}^{e(n)})$, since $e(n)$ is efficient.

Proof of the converse: the condition $q_{rs}^{m} \geq 0$ results from the fact that both H_{rs} and the sequence $n \mapsto \hat{v}_{rs}^{e(n)}$ increase; the condition $\sum_m q_{rs}^{m} = q_{rs}$ results from $Q_{rs}^{\overline{m}_{rs}} = q_{rs}\, \mathrm{H}_{rs}(\hat{v}_{rs}^{e(\overline{m}_{rs})}) = q_{rs}\, \mathrm{H}_{rs}(\mathrm{H}_{rs}^{-1}(1)) = q_{rs}$. If n is inefficient, then $e(n) = e(n-1)$, hence $Q_{rs}^{n-1} = Q_{rs}^{n}$ and $q_{rs}^{n} = Q_{rs}^{n} - Q_{rs}^{n-1} = 0$. If n is efficient, then $q_{rs} \int_{\mathrm{E}_{rs}^{m}} \mathrm{dH}_{rs}(v) = q_{rs}\, \mathrm{H}_{rs}(\hat{v}_{rs}^{n}) - q_{rs}\, \mathrm{H}_{rs}(\hat{v}_{rs}^{e(n-1)}) = Q_{rs}^{n} - Q_{rs}^{n-1}$ because of the assumed property, hence $= q_{rs}^{n}$ which yields the compatibility.

Proof of P3: On deriving the impedance functions from P2, we have shown that the impedances of the classes with positive flows are equal. Let us now show that a class with flow zero has an impedance superior or equal to that of the closest loaded class: thus the proof of the property will be complete. Consider an unloaded class j: if there are loaded classes, we can choose the (m-th) one that satisfies $Q_j = Q_m$. $\breve{v}_{rs}^{m} = \hat{v}_{rs}^{m}$ hence $\breve{v}_{rs}^{m} \in \mathrm{E}_{rs}^{m}$ so $\overline{\mathrm{G}}_{rs}^{m}(\breve{v}_{rs}^{m}) = \min_n \overline{\mathrm{G}}_{rs}^{n}(\breve{v}_{rs}^{m}) \leq \overline{\mathrm{G}}_{rs}^{j}(\breve{v}_{rs}^{m}) = \overline{\mathrm{T}}_{rs}^{j} + \overline{\mathrm{P}}_{rs}^{j} / \breve{v}_{rs}^{j}$ since $Q_j = Q_m$. Then $\overline{\mathrm{I}}_{rs}^{m} \leq \overline{\mathrm{I}}_{rs}^{j}$ because we can add and collapse the possibly intermediate terms $(\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / \breve{v}_{rs}^{i}$ with $m < i < j$ if $\overline{\mathrm{P}}_{rs}^{m} < \overline{\mathrm{P}}_{rs}^{j}$, or $j < i < m$ if $\overline{\mathrm{P}}_{rs}^{m} > \overline{\mathrm{P}}_{rs}^{j}$.

Proof of the converse: We shall prove compatibility by demonstrating that the class flow pattern $[q_{rs}^{m}]_m$ corresponds to an efficient assignment of each VOT in Ω_{rs}. Let $v \in \Omega_{rs}$: as $\Omega_{rs} \subset \bigcup_m [\breve{v}_{rs}^{m-1}; \breve{v}_{rs}^{m}]$, there exists an ME class m such that $v \in [\breve{v}_{rs}^{m-1}; \breve{v}_{rs}^{m}]$ and $\breve{v}_{rs}^{m-1} < \breve{v}_{rs}^{m}$. Let us show that m supplies v with a minimum generalized travel time. As regards $n \leq m$ we have that $\overline{\mathrm{I}}_{rs}^{n} - \overline{\mathrm{I}}_{rs}^{m} = \overline{\mathrm{T}}_{rs}^{n} - \overline{\mathrm{T}}_{rs}^{m} + \sum_{i=n}^{m-1} (\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / \breve{v}_{rs}^{i}$; for $n \leq i < m$ it holds that $\breve{v}_{rs}^{i} \leq \breve{v}_{rs}^{m-1} \leq v$ hence $(\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / \breve{v}_{rs}^{i} \leq (\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / v$. By summation we obtain that $\sum_{i=n}^{m-1} (\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / \breve{v}_{rs}^{i} \leq \sum_{i=n}^{m-1} (\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / v = (\overline{\mathrm{P}}_{rs}^{n} - \overline{\mathrm{P}}_{rs}^{m}) / v$. $q_{rs}^{m} > 0 \Rightarrow 0 \leq \overline{\mathrm{I}}_{rs}^{n} - \overline{\mathrm{I}}_{rs}^{m}$ hence $0 \leq \overline{\mathrm{T}}_{rs}^{n} - \overline{\mathrm{T}}_{rs}^{m} + (\overline{\mathrm{P}}_{rs}^{n} - \overline{\mathrm{P}}_{rs}^{m}) / v$ which implies that $\overline{\mathrm{G}}_{rs}^{m}(v) \leq \overline{\mathrm{G}}_{rs}^{n}(v)$. In a similar fashion, as regards $n \geq m$, it holds that $\breve{v}_{rs}^{i} \geq \breve{v}_{rs}^{m-1} \geq v$ for

$n \geq i \geq m$, hence $-(\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / \breve{v}_{rs}^{i} \leq -(\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / v$ which we sum over *i* from *m* to *n*-1 to obtain $-\sum_{i=n}^{m-1}(\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / \breve{v}_{rs}^{i} \leq -\sum_{i=n}^{m-1}(\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / v \ = (\overline{\mathrm{P}}_{rs}^{n} - \overline{\mathrm{P}}_{rs}^{m}) / v$. From the definition of the impedances $\bar{\mathrm{I}}_{rs}^{n} - \bar{\mathrm{I}}_{rs}^{m} = \overline{\mathrm{T}}_{rs}^{n} - \overline{\mathrm{T}}_{rs}^{m} - \sum_{i=m}^{n-1}(\overline{\mathrm{P}}_{rs}^{i} - \overline{\mathrm{P}}_{rs}^{i+1}) / \breve{v}_{rs}^{i}$, and as it is assumed that $0 \leq \bar{\mathrm{I}}_{rs}^{n} - \bar{\mathrm{I}}_{rs}^{m}$, we obtain $\overline{\mathrm{G}}_{rs}^{m}(v) \leq \overline{\mathrm{G}}_{rs}^{n}(v)$.