GLOBAL EXISTENCE RESULTS AND UNIQUENESS FOR DISLOCATION EQUATIONS

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ABSTRACT. We are interested in nonlocal Eikonal Equations arising in the study of the dynamics of dislocations lines in crystals. For these nonlocal but also non monotone equations, only the existence and uniqueness of Lipschitz and local-in-time solutions were available in some particular cases. In this paper, we propose a definition of weak solutions for which we are able to prove the existence for all time. Then we discuss the uniqueness of such solutions in several situations, both in the monotone and non monotone case.

1. INTRODUCTION

In this article we are interested in the dynamics of defects in crystals, called dislocations. The dynamics of these dislocations is the main microscopic explanation of the macroscopic behaviour of metallic crystals (see for instance the physical monographs Nabarro [24], Hirth and Lothe [19], or Lardner [21] for a mathematical presentation). A dislocation is a line moving in a crystallographic plane, called a slip plane. The typical length of such a dislocation line is of the order of $10^{-6}m$. Its dynamics is given by a normal velocity proportional to the Peach-Koehler force acting on this line.

This Peach-Koehler force may have two possible contributions: the first one is the self-force created by the elastic field generated by the dislocation line itself (i.e. this self-force is a nonlocal function of the shape of the dislocation line); the second one is the force created by everything exterior to the dislocation line, like the exterior stress applied on the material, or the force created by other defects. In this paper,

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we study a particular model introduced in Rodney et al. [27].

More precisely, if, at time t, the dislocation line is the boundary of an open set $\Omega_t \subset \mathbb{R}^N$ with $N = 2$ for the physical application, the normal velocity to the set Ω_t is given by

$$(1) \qquad\qquad V_n = c_0 \star \mathbb{1}_{\overline{\Omega}_t} + c_1$$

where $\mathbb{1}_{\overline{\Omega}_t}(x)$ is the indicator function of the set $\overline{\Omega}_t$, which is equal to 1 if $x \in \overline{\Omega}_t$ and equal to 0 otherwise. The function $c_0(x, t)$ is a kernel which only depends on the physical properties of the crystal and on the choice of the dislocation line whose we follow the evolution. In the special case of application to dislocations, the kernel c_0 does not depend on time, but to keep a general setting we allow here a dependence on the time variable. Here $\star$ denotes the convolution is space, namely

$$(2) \qquad (c_0(\cdot, t) \star \mathbb{1}_{\overline{\Omega}_t})(x) = \int_{\mathbb{R}^N} c_0(x - y, t) \mathbb{1}_{\overline{\Omega}_t}(y) dy,$$

and this term appears to be the Peach-Koehler self-force created by the dislocation itself, while $c_1(x, t)$ is an additional contribution to the velocity, created by everything exterior to the dislocation line. We refer to Alvarez et al. [3] for a detailed presentation and a derivation of this model.

We proceed as in the level-set approach to derive an equation for the dislocation line. We replace the evolution of a set Ω_t (the strong solution), by the evolution of a function u such that $\Omega_t = \{u(\cdot, t) > 0\}$. Roughly speaking the dislocation line is represented by the zero level-set of the function u which solves the following equation

$$(3)$$
$$\begin{cases} \dfrac{\partial u}{\partial t} = (c_0(\cdot, t) \star \mathbb{1}_{\{u(\cdot, t) \geq 0\}}(x) + c_1(x, t)) |Du| & \text{in} \quad \mathbb{R}^N \times (0, T) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

where (2) now reads

$$(4) \qquad c_0(\cdot, t) \star \mathbb{1}_{\{u(\cdot, t) \geq 0\}}(x) = \int_{\mathbb{R}^N} c_0(x - y, t) \mathbb{1}_{\{u(\cdot, t) \geq 0\}}(y) dy.$$

Note that (3) is not really a level-set equation since it is not invariant under nondecreasing changes of functions $u \to \varphi(u)$ where φ is nondecreasing. As noticed by Slepčev [28], the natural level-set equation should be (11), see Section 1.2.

Although equation (3) seems very simple, there are only a few known results. Under suitable assumptions on the initial data and on c_0, c_1, the existence and uniqueness of the solution is known in two particular cases: either for short time (see [3]), or for all time under the additional assumption that $V_n \geq 0$, which is for instance always satisfied for c_1 satisfying $c_1(x,t) \geq |c_0(\cdot,t)|_{L^1(\mathbb{R}^N)}$ (see [2], [12] or [5] for a level-set formulation).

In the general case, the existence for all time of solutions to Equation (3) is not known and, in particular, in the case when the kernel c_0 has negative values; indeed, in this case, the front propagation problem (3) does not satisfy any monotonicity property (preservation of inclusions) and therefore, even if a level-set type equation can be derived, viscosity solutions' theory cannot be used readily. At this point, it is worth pointing out that a key property in the level-set approach is the comparison principle for viscosity solutions which is almost equivalent to this monotonicity property (See for instance Giga's monograph [18]). On the other hand, one may try to use partly viscosity solutions' theory together with some other approximation and/or compactness arguments to prove at least the existence of weak solutions (in a suitable sense). But here also the bad sign of the kernel creates difficulties since one cannot use readily the classical half-relaxed limits techniques to pass to the limit in the approximate problems. Additional arguments are needed to obtain weak solutions.

The aim of this paper is to describe a general approach of these dislocations' dynamics, based on the level-set approach, which allows us to introduce a suitable notion of weak solutions, to prove the existence of these weak solutions for all time and to analyse the uniqueness (or non-uniqueness) of these solutions.

1.1. **Weak solutions of the dislocation equation.** We introduce the following definition of weak solutions, which uses itself the definition of L^1-viscosity solutions, recalled in Appendix A.

Definition 1.1. (Classical and weak solutions)
For any $T > 0$, we say that a function $u \in W^{1,\infty}(\mathbb{R}^N \times [0,T))$ is a weak solution of equation (3) on the time interval $[0,T)$, if there is some measurable map $\chi : \mathbb{R}^N \times (0,T) \to [0,1]$ such that u is a L^1-viscosity solution of

$$
(5) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = \bar{c}(x,t)|Du| & in \quad \mathbb{R}^N \times (0,T) \\ u(\cdot,0) = u_0 & in \ \mathbb{R}^N \ , \end{cases}
$$

where

(6)
$$\bar{c}(x,t) = c_0(\cdot,t) \star \chi(\cdot,t)(x) + c_1(x,t)$$

and

(7)
$$\mathbb{1}_{\{u(\cdot,t)>0\}}(x) \leq \chi(x,t) \leq \mathbb{1}_{\{u(\cdot,t)\geq 0\}}(x),$$

for almost all $(x,t) \in \mathbb{R}^N \times [0,T]$. We say that u is a classical solution of equation (3) if u is a weak solution to (5) and if

(8)
$$\mathbb{1}_{\{u(\cdot,t)>0\}}(x) = \mathbb{1}_{\{u(\cdot,t)\geq 0\}}(x)$$

for almost all $(x,t) \in \mathbb{R}^N \times [0,T]$.

Note that we have $\chi(x,t) = \mathbb{1}_{\{u(\cdot,t)>0\}}(x) = \mathbb{1}_{\{u(\cdot,t)\geq 0\}}(x)$ for almost all $(x,t) \in \mathbb{R}^N \times [0,T]$ for classical solutions.

To state our first existence result, we use the following assumptions

(H0) $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, $-1 \leq u_0 \leq 1$ and there exists $R_0 > 0$ such that $u_0(x) \equiv -1$ for $|x| \geq R_0$,

(H1) $c_0 \in C([0,T]; L^1(\mathbb{R}^N))$, $D_x c_0 \in L^\infty([0,T]; L^1(\mathbb{R}^N))$, $c_1 \in C(\mathbb{R}^N \times [0,T))$ and there exists constants M_1, L_1 such that, for any $x,y \in \mathbb{R}^N$ and $t \in [0,T]$

$$|c_1(x,t)| \leq M_1 \quad \text{and} \quad |c_1(x,t) - c_1(y,t)| \leq L_1|x-y|.$$

In the sequel, we denote by M_0, L_0, constants such that, for any (or almost every) $t \in [0,T)$, we have

$$|c_0(\cdot,t)|_{L^1(\mathbb{R}^N)} \leq M_0 \quad \text{and} \quad |D_x c_0(\cdot,t)|_{L^1(\mathbb{R}^N)} \leq L_0.$$

Our first main result is the following.

Theorem 1.2. (Existence of weak solutions)
*Under assumptions **(H0)**-**(H1)**, for any $T > 0$ and for any initial data u_0, there exists a weak solution of equation (3) on the time interval $[0,T)$ in the sense of Definition 1.1.*

Our second main result states that a weak solution is a classical one if the evolving set is expanding and if the following additional condition is fulfilled

(H2) c_1 and c_0 satisfy **(H1)** and there exists constants m_0, N_1 and a positive function $N_0 \in L^1(\mathbb{R}^N)$ such that, for any $x,h \in \mathbb{R}^N$, $t \in [0,T)$, we have

$$|c_0(x,t)| \leq m_0,$$
$$|c_1(x+h,t) + c_1(x-h,t) - 2c_1(x,t)| \leq N_1|h|^2,$$

$$|c_0(x + h, t) + c_0(x - h, t) - 2c_0(x, t)| \leq N_0(x)|h|^2.$$

Theorem 1.3. (Some links between weak solutions and classical continuous viscosity solutions and uniqueness results)
Assume **(H0)**-**(H1)** *and suppose that there is some* $\delta \geq 0$ *such that, for all measurable map* $\chi : \mathbb{R}^N \times (0, T) \rightarrow [0, 1]$,

$$(9) \quad for \ all \ (x, t) \in \mathbb{R}^N \times [0, T], \quad c_0(\cdot, t) \star \chi(\cdot, t)(x) + c_1(x, t) \geq \delta,$$

and that the initial data u_0 *satisfies (in the viscosity sense)*

$$(10) \qquad\qquad - |u_0| - |Du_0| \leq -\eta_0 \quad in \ \mathbb{R}^N \ ,$$

for some $\eta_0 > 0$. *Then any weak solution* u *of (3) in the sense of Definition 1.1, is a classical continuous viscosity solution of (3). This solution is unique if* **(H2)** *holds and*
  (i) either $\delta > 0$,
  (ii) or $\delta = 0$ *and* u_0 *is semiconvex, i.e. satisfies for some constant* $C > 0$:

$$u_0(x + h) + u_0(x - h) - 2u_0(x) \geq -C|h|^2, \quad \forall x, h \in \mathbb{R}^N.$$

Assumption (9) ensures that the velocity V_n in (1) is positive for positive δ. Of course, we can state similar results in the case of negative velocity. Assumption (10) means that u_0 is a viscosity subsolution of $-|v(x)| - |Dv(x)| + \eta_0 \leq 0$. When u_0 is C^1, it follows that the gradient of u_0 does not vanish on the set $\{u_0 = 0\}$ (see [22] for details). Point (ii) of the theorem is the main result of [2, 5]. We also point out that, with adapted proofs, only a bound from below could be required in **(H2)** on $c_1(x+h, t) + c_1(x-h, t) - 2c_1(x, t)$ and $c_0(x+h, t) + c_0(x-h, t) - 2c_0(x, t)$.

Remark 1.1. In particular Theorem 1.3 implies uniqueness in the case $c_0 \geq 0$ and $c_1 \equiv 0$. The general study of nonnegative kernels is provided below (see Theorem 1.5 and Remark 1.3).

1.2. **Nonnegative kernel** $c_0 \geq 0$. In the special case where the kernel c_0 is non-negative, an inclusion principle for the dislocations lines, or equivalently a comparison principle for the functions of the level-set formulations is expected (cf. Cardaliaguet [11] and Slepčev [28]).
   Moreover, in the classical level-set approach, all the level-sets of u should have the same type of normal velocity and Slepčev [28] remarked that a formulation with a nonlocal term of the form $\{u(\cdot, t) \geq u(x, t)\}$ is more appropriate. Therefore it is natural to start studying the following

equation (which replaces equation (3))
(11)
$$\begin{cases} \dfrac{\partial u}{\partial t} = (c_0(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t) \geq u(x,t)\}}(x) + c_1(x,t))|Du| & \text{in} \quad \mathbb{R}^N \times (0,T) , \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N , \end{cases}$$

where $\star$ denotes the convolution in space as in (4).

The precise meaning of a viscosity solution of (11) is given in Definition 5.1.

In this context, assumption **(H0)** can be weakened into the following condition which allows to consider unbounded evolving sets

(H0') $u_0 \in BUC(\mathbb{R}^N)$.

Our main result for this equation is

Theorem 1.4. (Existence and uniqueness)
*Assume that $c_0 \geq 0$ on $\mathbb{R}^N \times [0,T]$ and that **(H0')**-**(H1)** hold. Then there exists a unique viscosity solution u of (11).*

Remark 1.2. The comparison principle for this equation (see Theorem 5.2) is a generalization of [28, Theorem 2.3]: indeed, in [28], everything takes place in a fixed bounded set whereas here one has to deal with unbounded sets. See also [15] for related results.

Now we turn to the connections with weak solutions. To do so, if u is the unique continuous solution of (11) given by Theorem 1.4, we introduce the functions $\rho^+, \rho^- : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ defined by

$$\rho^+ := \mathbb{1}_{\{u \geq 0\}} \quad \text{and} \quad \rho^- := \mathbb{1}_{\{u > 0\}} .$$

Our result is the

Theorem 1.5. (Maximal and minimal weak solutions)
Under the assumptions of Theorem 1.4, the maximal and minimal weak solutions of (3) are the continuous functions v^+, v^- which are the unique L^1-viscosity solutions of the equations

$$(12) \qquad \begin{cases} \dfrac{\partial v^\pm}{\partial t} = c[\rho^\pm](x,t)|Dv^\pm| & \text{in} \quad \mathbb{R}^N \times (0,T) , \\ v^\pm(x,0) = u_0(x) & \text{in} \quad \mathbb{R}^N , \end{cases}$$

where

$$c[\rho](x,t) := c_0(\cdot, t) \star \rho(\cdot, t)(x) + c_1(x,t) \quad \text{in} \quad \mathbb{R}^N \times (0,T) .$$

The functions $v^\pm$ satisfy $\{v^+(\cdot, t) \geq 0\} = \{u(\cdot, t) \geq 0\}$ and $\{v^-(\cdot, t) > 0\} = \{u(\cdot, t) > 0\}$, where u is the solution of (11).

Moreover, if the set $\{u(\cdot,t) = 0\}$ has a zero-Lebesgue measure for almost all $t \in (0,T)$, then problem (3) has a unique weak solution which is also a classical one.

Remark 1.3.
1. Theorem 1.5 shows that, in the case when $c_0 \geq 0$, Slepčev's approach allows to identify the maximal and minimal weak solutions as being associated to $\rho^{\pm}$.
2. Equalities $\{v^-(\cdot,t) \geq 0\} = \{u(\cdot,t) \geq 0\}$ and $\{v^+(\cdot,t) > 0\} = \{u(\cdot,t) > 0\}$ do not hold in general (see for instance example 3.1 in Section 3).
3. If the set $\{v^{\pm}(\cdot,t) = 0\}$ develops an interior, a dramatic loss of uniqueness for the weak solution of (3) may occur. This is illustrated by Example 3.1 below, where we are able to built infinitely many solutions after the onset of fattening.
4. We have uniqueness for (3) if $\{u(\cdot,t) = 0\}$ has a zero-Lebesgue measure for almost all $t \in (0,T)$. This condition is fulfilled when, for instance, $c[\rho] \geq 0$ holds for any indicator function ρ and (10) holds (see also Remark 1.1).

1.3. **Organization of the paper.** In Section 2, we recall basic results for the classical Eikonal Equation which are used throughout the paper. In Section 3, we prove the existence of weak solutions for equation (3), namely Theorem 1.2 and give a counter-example to the uniqueness in general. Let us mention that this first part of the paper, even if it requires rather deep results of viscosity solutions theory, is of a general interest for a wide audience and can be read without having an expertise in this theory since one just need to apply the results which are anyway rather natural. In Section 4, we prove Theorem 1.3 in the case of expanding dislocations. The arguments we use here are far more involved from a technical point of view: in particular we need some fine estimates of the perimeter of the evolving sets. In Section 5, we study the Slepčev formulation in the case of non-negative kernels, and prove Theorems 1.4 and 1.5. In the spirit, this section is closely related to the classical level-set approach but is more technical. Finally, for sake of completeness, we recall in Appendix A the Definition of L^1-viscosity solutions and a new stability result proved by Barles in [4].

2. Some basic results for the classical (local) eikonal equation

We want to recall in this section some basic results on the level-set equation

$$(13) \qquad \begin{cases} \dfrac{\partial v}{\partial t} = a(x,t)|Dv| & \text{in } \mathbb{R}^N \times (0,T) \\[2mm] v(\cdot,0) = u_0 & \text{in } \mathbb{R}^N , \end{cases}$$

where $T > 0$ and $a : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ is, at least, a continuous function.

We provide some classical estimates on the solutions to (13) when a satisfies suitable assumptions. Our result is the following.

Theorem 2.1. *If u_0 satisfies **(H0)** and a satisfies the assumptions of c_1 in **(H1)**, then Equation (13) has a unique continuous solution v which is Lipschitz continuous in $\mathbb{R}^N \times [0,T]$ and which satisfies*

(i) $-1 \le v \le 1$ in $\mathbb{R}^N \times (0,T)$, $v(x,t) \equiv -1$ for $|x| \ge R_0 + M_1 t$,
(ii) $|Dv(\cdot,t)|_\infty \le |Du_0|_\infty \, e^{L_1 t}$,
(iii) $|v_t(\cdot,t)|_\infty \le M_1 |Du_0|_\infty \, e^{L_1 t}$.

We skip the very classical proof of Theorem 2.1; we just point out that the first point comes from the comparison result for (13) and the "finite speed of propagation property" (See Crandall & Lions [14]), while the second one is a basic gradient estimate (see for example Ley [22]) and the last one comes directly from the fact that the equation is satisfied almost everywhere.

The main consequence of this result is that the solution remains in a compact subset of the Banach space $(C(\mathbb{R}^N \times [0,T]), |\cdot|_\infty)$ as long as u_0 and a satisfies **(H0)-(H1)** with fixed constants.

Let us introduce the following

Definition 2.2. (Interior ball property)
We say that a closed set $K \subset \mathbb{R}^N$ has an interior ball property of radius $r > 0$, if for any $x \in K$, there exists $p \in \mathbb{R}^N \setminus \{0\}$ such that $B(x - r\frac{p}{|p|}, r) \subset K$.

We will also use the following result, due to Cannarsa and Frankowska [10], the proof of which is given in Appendix B for sake of completeness.

Lemma 2.3. (Interior ball regularization)
*Suppose **(H0)** and that a satisfies the assumptions of c_1 in **(H1)-(H2)** and there exists a constant $\delta > 0$ such that*

$$c_1 \ge \delta > 0 \quad on \quad \mathbb{R}^N \times [0,T].$$

Then there exists a constant γ (depending in particular on $\delta > 0$ and T and on the other constants of the problem) such that for the solution v of (13), the set $\{v(\cdot, t) \geq 0\}$ has an interior ball property of radius $r_t \geq \gamma t$ for $t \in (0, T)$.

3. EXISTENCE OF WEAK SOLUTIONS FOR EQUATION (3)

We aim at solving equation (3), i.e.

$$\begin{cases} \dfrac{\partial u}{\partial t} = (c_0(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t) \geq 0\}}(x) + c_1(x,t))|Du| & \text{in} \quad \mathbb{R}^N \times (0, T) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N, \end{cases}$$

proving Theorem 1.2, which states the existence of weak solutions as introduced in Definition 1.1.

A key difficulty to solve (3) comes from the fact that, in this kind of level-set equations, one may face the so-called "non-empty interior difficulty", i.e. that the 0–level-set of the solution is "fat" which may mean either that it has a non-empty interior or a non-zero Lebesgue measure. Clearly, in both cases, $\mathbb{1}_{\{u(\cdot,t) \geq 0\}}$ is different from $\mathbb{1}_{\{u(\cdot,t) > 0\}}$ and this leads to rather bad stability properties for (3) and therefore to difficulties to prove the existence of a solution (and even more for the uniqueness). The notion of weak solution (5)-(6)-(7) emphasizes this difficulty. On the contrary, if $\bar{c}(x, t) \geq 0$ in $\mathbb{R}^N \times [0, T]$, it is known that the "non-empty interior difficulty" cannot happen (See Barles, Soner and Souganidis [6] and Ley [22]) and we recover a more classical formulation. We discuss this question in the next section, as well as some uniqueness issues for our weak solutions. Let us finally note that weak solutions for (3) satisfy the following inequalities:

Proposition 3.1. *Let u be a weak solution to (3). Then u also satisfies in the L^1-sense*

(14)
$$\frac{\partial u}{\partial t} \leq \left(c_0^+(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t) \geq 0\}}(x) - c_0^-(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t) > 0\}}(x) + c_1(x,t) \right) |Du|,$$

(15)
$$\frac{\partial u}{\partial t} \geq \left(c_0^+(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t) > 0\}}(x) - c_0^-(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t) \geq 0\}}(x) + c_1(x,t) \right) |Du|,$$

in $\mathbb{R}^N \times (0, T)$, where $c_0^+ = \max(0, c_0)$ and $c_0^- = \max(0, -c_0)$.

Proof of Proposition 3.1.
Let $\bar{c}$ be associated with u as in (5)-(6)-(7). Then we have

$$\bar{c}(x, t) \geq c_0^+(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t) > 0\}}(x) - c_0^-(\cdot, t) \star \mathbb{1}_{\{u(\cdot,t) \geq 0\}}(x) + c_1(x, t)$$

for every $x \in \mathbb{R}^N$ and almost every $t \in (0, T)$. We note that the right-hand side of the inequality is lower-semicontinuous. Following Lions and Perthame [23], u then solves (15) in the usual viscosity sense. The proof of (14) can be achieved in a similar way. $\square$

Proof of Theorem 1.2.

1. Introduction of a perturbated equation. First we are going to solve the equation

$$(16) \quad \frac{\partial u}{\partial t} = (c_0(\cdot, t) \star \psi_\varepsilon(u(\cdot, t))(x) + c_1(x, t)) |Du| \quad \text{in } \mathbb{R}^N \times (0, T) ,$$

where $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a sequence of continuous functions such that $\psi_\varepsilon(t) \equiv 0$ for $t \leq -\varepsilon$, $\psi_\varepsilon(t) \equiv 1$ for $t \geq 0$ and ψ_ε is an affine function on $[-\varepsilon, 0]$.

We aim at applying Schauder's fixed point Theorem to a suitable map. We note that an alternative proof could be given by using techniques developed by Alibaud in [1].

2. Definition of a map $\mathcal{T}$. We introduce the convex and compact (by Ascoli's Theorem) subset

$$X = \{ u \in C(\mathbb{R}^N \times [0, T]) : u \equiv -1 \text{ in } \mathbb{R}^N \backslash B(0, R_0 + MT),$$
$$|Du|, |u_t|/M \leq |Du_0|_\infty e^{LT} \}$$

of $(C(\mathbb{R}^N \times [0, T]), |\cdot|_\infty)$ for $M = M_0 + M_1$ and $L = L_0 + L_1$, and the map $\mathcal{T} : X \to X$ defined by : if $u \in C(\mathbb{R}^N \times [0, T])$, then $\mathcal{T}(u)$ is the unique solution v of (13) for

$$c_\varepsilon(x, t) = c_0(\cdot, t) \star \psi_\varepsilon(u(\cdot, t))(x) + c_1(x, t)$$
$$= \int_{\mathbb{R}^N} c_0(x - z, t) \psi_\varepsilon(u(z, t)) dz + c_1(x, t).$$

This definition is justified by the fact that, under assumption **(H1)** on c_1 and c_0, c_ε satisfies **(H1)** with fixed constants $M = M_0 + M_1$ and $L = L_0 + L_1$; indeed M is a bound on $\sup_{[0,T]} |c_0(\cdot, t)|_{L^1} + M_1$ while L is estimated by the following calculation: for all $x, y \in \mathbb{R}^N$, $t \in [0, T]$ and $u \in X$, we have

$$(17) \quad c_\varepsilon(x, t) - c_\varepsilon(y, t)$$
$$= \int_{\mathbb{R}^N} (c_0(x - z, t) - c_0(y - z, t)) \psi_\varepsilon(u(z, t)) dz + c_1(x, t) - c_1(y, t)$$
$$\leq \int_{\mathbb{R}^N} |c_0(x - z, t) - c_0(y - z, t)| dz + |c_1(x, t) - c_1(y, t)|$$
$$\leq (L_0 + L_1)|x - y|,$$

since $0 \leq \psi_\varepsilon \leq 1$.

Finally, under assumption **(H0)-(H1)**, for any $u \in X$, the results of Theorem 2.1 apply to (16) which imply that $\mathcal{T}(u) \in X$. It follows that $\mathcal{T}$ is well-defined.

3. Application of Schauder's fixed point Theorem to $\mathcal{T}$. The map $\mathcal{T}$ is continuous since ψ_ε is continuous, by using the classical stability result for viscosity solutions (see for instance (30) in Section 4). Therefore $\mathcal{T}$ has a fixed point u_ε which is bounded in $W^{1,\infty}(\mathbb{R}^N \times [0,T])$ uniformly with respect to ε (since M and L are independent of ε).

4. Convergence of the fixed point when $\varepsilon \to 0$. From Ascoli's Theorem, we extract a subsequence $(u_{\varepsilon'})_{\varepsilon'}$ which converges locally uniformly to a function denoted by u (in fact globally since the $u_{\varepsilon'}$ are equal to -1 outside a fixed compact subset).

The functions $\chi_{\varepsilon'} := \psi_{\varepsilon'}(u_{\varepsilon'})$ satisfy $0 \leq \chi_{\varepsilon'} \leq 1$. Therefore we can extract a subsequence—still denoted $(\chi_{\varepsilon'})$—which converges weakly$-*$ in $L^\infty_{\text{loc}}(\mathbb{R}^N \times [0,T])$ to some function $\chi : \mathbb{R}^N \times (0,T) \to [0,1]$. Therefore, for all $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N \times [0,T])$,

$$(18) \qquad \int_0^T \int_{\mathbb{R}^N} \varphi \chi_{\varepsilon'} dx dt \to \int_0^T \int_{\mathbb{R}^N} \varphi \chi dx dt.$$

From Fatou's lemma, if φ is nonnegative, it follows

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}^N} \varphi(x,t) \chi(x,t) dx dt \;&\leq\; \int_0^T \int_{\mathbb{R}^N} \varphi(x,t) \limsup_{\varepsilon' \to 0} \chi_{\varepsilon'}(x,t) dx dt \\
&\leq\; \int_0^T \int_{\mathbb{R}^N} \varphi(x,t) \limsup_{\varepsilon' \to 0, x' \to x, t' \to t} \chi_{\varepsilon'}(x',t') dx dt \\
&\leq\; \int_0^T \int_{\mathbb{R}^N} \varphi(x,t) \mathbb{1}_{\{u(\cdot,t) \geq 0\}}(x) dx dt.
\end{aligned}
$$

Since the previous inequalities hold for any nonnegative $\varphi \in L^1_{\text{loc}}(\mathbb{R}^N \times [0,T])$, we obtain that, for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$,

$$\chi(x,t) \leq \mathbb{1}_{\{u(\cdot,t) \geq 0\}}(x).$$

Similarly we get

$$\mathbb{1}_{\{u(\cdot,t) > 0\}}(x) \leq \chi(x,t).$$

Furthermore, setting $c_{\varepsilon'} = c_0 \star \chi_{\varepsilon'} + c_1$, from (18), we have, for all $(x,t) \in \mathbb{R}^N \times [0,T]$,

$$\int_0^t c_{\varepsilon'}(x,s)ds = \int_0^t \int_{\mathbb{R}^N} c_0(x-y,s)\chi_{\varepsilon'}(y,s)dyds + \int_0^t c_1(x,s)ds$$

$$\rightarrow \int_0^t \bar{c}(x,s)ds,$$

where $\bar{c}(x,t) = c_0(\cdot,t) \star \chi(\cdot,t)(x) + c_1(x,t)$. The above convergence is pointwise but, noticing that $c_{\varepsilon'}$ satisfies (**H3**) (with $M := M_0 + M_1$ and $L := L_0 + L_1$) and using Remark 5.2, we can apply the stability Theorem 5.5 given in Appendix A. We obtain that u is L^1-viscosity solution to (5) with $\bar{c}$ satisfying (6)-(7). $\qquad\square$

The following example is inspired from [6].

Example 3.1. (Counter-example to the uniqueness of weak solutions) Let us consider, in dimension $N = 1$, the following equation of type (3),

(19) $\qquad \begin{cases} \dfrac{\partial U}{\partial t} = (1 \star \mathbb{1}_{\{U(\cdot,t)\geq 0\}}(x) + c_1(t))|DU| & \text{in} \quad \mathbb{R} \times (0,2] \\ U(\cdot,0) = u_0 & \text{in } \mathbb{R}, \end{cases}$

where we set $c_0(x,t) := 1$, $c_1(x,t) := c_1(t) = 2(t-1)(2-t)$ and $u_0(x) = 1 - |x|$. Note that $1 \star \mathbb{1}_A = \mathcal{L}^1(A)$ for any measurable set $A \subset \mathbb{R}$, where $\mathcal{L}^1(A)$ is the Lebesgue measure on $\mathbb{R}$.

We start by solving auxiliary problems for time in $[0,1]$ and $[1,2]$ in order to produce a family of solutions for the original problem in $[0,2]$.
1. *Construction of a solution for $0 \leq t \leq 1$.* The function $x_1(t) = (t-1)^2$ is the solution of the ode

$$\dot{x}_1(t) = c_1(t) + 2x_1(t) \text{ for } 0 \leq t \leq 1, \quad \text{and } x(0) = 1,$$

(note that $\dot{x}_1 \leq 0$ in $[0,1]$). Consider

(20) $\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \dot{x}_1(t)\left|\dfrac{\partial u}{\partial x}\right| & \text{in } \mathbb{R} \times (0,1], \\ u(\cdot,0) = u_0 & \text{in } \mathbb{R}. \end{cases}$

There exists a unique continuous viscosity solution u of (20). Looking for u under the form $u(x,t) = v(x,\Gamma(t))$ with $\Gamma(0) = 0$, we obtain that v satisfies

$$\frac{\partial v}{\partial t}\dot{\Gamma}(t) = \dot{x}_1(t)\left|\frac{\partial v}{\partial x}\right|.$$

Choosing $\Gamma(t) = -x_1(t) + 1$, we get that v is the solution of

$$\begin{cases} \dfrac{\partial v}{\partial t} = -\left|\dfrac{\partial v}{\partial x}\right| & \text{in } \mathbb{R} \times (0, 1], \\ v(\cdot, 0) = u_0 & \text{in } \mathbb{R}. \end{cases}$$

By Oleinik-Lax formula, $v(x, t) = \inf\limits_{|x-y| \leq t} u_0(y)$. Since u_0 is even, we have, for all $(x, t) \in \mathbb{R} \times [0, 1]$,

$$u(x, t) = \inf\limits_{|x-y| \leq \Gamma(t)} u_0(y) = u_0(|x| + \Gamma(t)) = u_0(|x| - x_1(t) + 1).$$

Therefore, for $0 \leq t \leq 1$,
(21)
$$\{u(\cdot, t) > 0\} = (-x_1(t), x_1(t)) \quad \text{and} \quad \{u(\cdot, t) \geq 0\} = [-x_1(t), x_1(t)].$$

We will see in Step 3 that u is a solution of (19) in $[0, 1]$.

2. *Construction of solutions for $1 \leq t \leq 2$.* Consider now, for any measurable function $0 \leq \gamma(t) \leq 1$, the unique solution y_γ of the ode

(22) $\dot{y}_\gamma(t) = c_1(t) + 2\gamma(t)y_\gamma(t)$ for $1 \leq t \leq 2$, and $y_\gamma(1) = 0$.

By comparison, we have $0 \leq y_0(t) \leq y_\gamma(t) \leq y_1(t)$ for $1 \leq t \leq 2$, where y_0, y_1 are the solutions of (22) obtained with $\gamma(t) \equiv 0, 1$. In particular, it follows that $\dot{y}_\gamma \geq 0$ in $[1, 2]$. Consider

$$\begin{cases} \dfrac{\partial u_\gamma}{\partial t} = \dot{y}_\gamma(t)\left|\dfrac{\partial u_\gamma}{\partial x}\right| & \text{in } \mathbb{R} \times (1, 2], \\ u_\gamma(\cdot, 1) = u(\cdot, 1) & \text{in } \mathbb{R}, \end{cases}$$

where u is the solution of (20). Again, this problem has a unique continuous viscosity solution u_γ and setting $\Gamma_\gamma(t) = y_\gamma(t) \geq 0$ for $1 \leq t \leq 2$, we obtain that v_γ defined by $v_\gamma(x, \Gamma_\gamma(t)) = u_\gamma(x, t)$ is the unique continuous viscosity solution of

$$\begin{cases} \dfrac{\partial v_\gamma}{\partial t} = \left|\dfrac{\partial v_\gamma}{\partial x}\right| & \text{in } \mathbb{R} \times (0, \Gamma_\gamma(2)], \\ v_\gamma(\cdot, 0) = u(\cdot, 1) & \text{in } \mathbb{R}. \end{cases}$$

Therefore, for all $(x, t) \in \mathbb{R} \times [1, 2]$, we have

$$u_\gamma(x, t) = \sup\limits_{|x-y| \leq y_\gamma(t)} u(y, 1) = \begin{cases} 0 & \text{if } |x| \leq y_\gamma(t), \\ u(|x| - y_\gamma(t), 1) & \text{otherwise.} \end{cases}$$

(Note that $u(-x, t) = u(x, t)$ since u_0 is even and, since $u(\cdot, 1) \leq 0$, by the maximum principle, we have $u_\gamma \leq 0$ in $\mathbb{R} \times [1, 2]$.) It follows that, for all $1 \leq t \leq 2$,
(23)
$$\{u_\gamma(\cdot, t) > 0\} = \emptyset \quad \text{and} \quad \{u_\gamma(\cdot, t) \geq 0\} = \{u_\gamma(\cdot, t) = 0\} = [-y_\gamma(t), y_\gamma(t)].$$

3. There are several weak solutions of (19). Set, for $0 \leq \gamma(t) \leq 1$,

$$
\begin{aligned}
c_\gamma(t) &= c_1(t) + 2x_1(t), & U_\gamma(x,t) &= u(x,t) & \text{if } (x,t) \in \mathbb{R} \times [0,1], \\
c_\gamma(t) &= c_1(t) + 2\gamma(t)y_\gamma(t), & U_\gamma(x,t) &= u_\gamma(x,t) & \text{if } (x,t) \in \mathbb{R} \times [1,2].
\end{aligned}
$$

Then, from Steps 1 and 2, U_γ is the unique continuous viscosity solution of

$$
(24) \qquad
\begin{cases}
\dfrac{\partial U_\gamma}{\partial t} = c_\gamma(t) \left| \dfrac{\partial U_\gamma}{\partial x} \right| & \text{in } \mathbb{R} \times (0,2], \\
U_\gamma(\cdot, 0) = u_0 & \text{in } \mathbb{R}.
\end{cases}
$$

Taking $\chi_\gamma(\cdot, t) = \gamma(t) \mathbb{1}_{[-y_\gamma(t), y_\gamma(t)]}$ for $1 \leq t \leq 2$, from (21) and (23), we have

$$
\mathbb{1}_{\{U_\gamma(\cdot,t) > 0\}} \leq \chi_\gamma(\cdot, t) \leq \mathbb{1}_{\{U_\gamma(\cdot,t) \geq 0\}},
$$

(see Figure 1). It follows that all the U_γ's, for measurable $0 \leq \gamma(t) \leq 1$, are all weak solutions of (19) so we do not have uniqueness and the set of solutions is quite large.
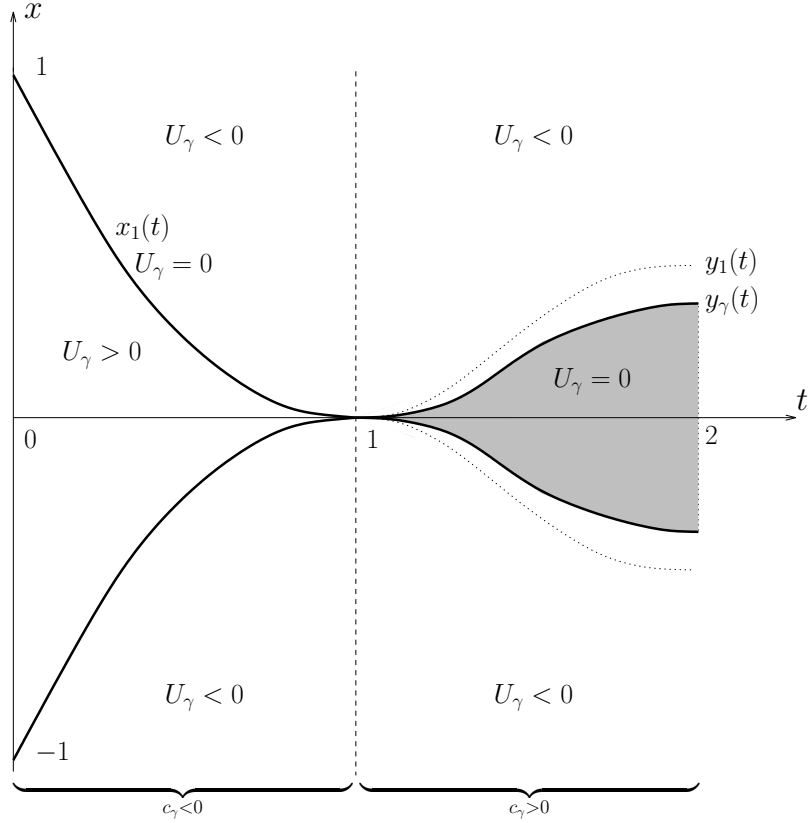

FIGURE 1.   *Fattening phenomenon for the functions U_γ.*

Let us complete this counter-example by pointing out:
(i) as in [6], non uniqueness comes from the fattening phenomenon for the front which is due to the fact that c_γ in (24) changes its sign at $t = 1$. It is even possible to build an autonomous counter-example up to start with a front with several connected components;
(ii) $c_0 \geq 0$, and therefore it complements also the results of Section 5; indeed the unique solution u of (11) has the same 0–level-set than U_1 (obtained with $\gamma(t) \equiv 1$) and, with the notations of Theorem 1.5, $\rho^+ = \mathbb{1}_{\{U^1(\cdot,t)\geq 0\}}$ and $\rho^- = \mathbb{1}_{\{U^1(\cdot,t)>0\}}$. In particular, for $t \geq 1$,

$$\{v^+(\cdot, t) \geq 0\} = \{U^1(\cdot, t) \geq 0\} = [-y_1(t), y_1(t)]$$

and

$$\{v^-(\cdot, t) > 0\} = \{U^1(\cdot, t) > 0\} = \emptyset.$$

Finally, we note that there is no strong solutions since (8) is obviously never satisfied;
(iii) $c_0 = 1$ does not satisfies **(H1)** but because of the finite speed of propagation property, it is possible to keep the same solution on a large ball in space and for $t \in (0, T)$, if we replace c_0 by a function with compact support in space such that $c_0(x,t) = 1$ for $|x| \leq R$ with R large enough. By this way, it is possible for c_0 to satisfy **(H1)**.

4. Uniqueness results for weak solutions of (3)

Uniqueness of weak solutions of (3) is false in general as shown in the counter-example of the previous section for sign changing velocities c_1. This is in particular related to the "fattening phenomenon". In [2] and [5] the authors proved that there is a unique "classical" viscosity solution for (3) under the assumptions that the initial set $\{u(\cdot, 0) \geq 0\}$ has the "interior sphere property" and that $c_1(\cdot, t) \geq |c_0(\cdot, t)|_{L^1}$ for any $t \geq 0$—condition which ensures that the velocity $\bar{c}$ is nonnegative. By "classical" continuous viscosity solutions we mean that $t \to \mathbb{1}_{\{u(\cdot,t)\geq 0\}}$ is continuous in L^1, which entails that $(x, t) \mapsto \bar{c}(x, t)$ is continuous, and that (3) holds in the usual viscosity sense.

Here we prove Theorem 1.3. If the condition $c_1(\cdot, t) \geq |c_0(\cdot, t)|_{L^1}$ is satisfied, then weak solutions are viscosity solutions. We also prove that the weak solution is unique if we suppose moreover either that the initial condition has the interior sphere property, or that the strict inequality $c_1(\cdot, t) > |c_0(\cdot, t)|_{L^1}$ holds for any $t \geq 0$.

Proof of Theorem 1.3.
1. Weak solutions are classical continuous viscosity solutions. Let u

be a weak solution and let $\bar{c}$ be associated with u as in Definition 1.1. Then, for any $x \in \mathbb{R}^N$ and for almost all $t \in [0, T]$, we have

$$\bar{c}(x, t) \geq c_1(x, t) + c_0^+(\cdot, t) \star \mathbb{1}_{\{u(\cdot, t) > 0\}}(x) - c_0^-(\cdot, t) \star \mathbb{1}_{\{u(\cdot, t) \geq 0\}}(x)$$

$$\geq c_1(x, t) - |c_0^-(\cdot, t)|_{L^1}$$

$$\geq \delta \geq 0 .$$

From [22, Theorem 4.2], there exists a constant η which depends on T such that (10) implies

$$(25) \qquad -|u| - |Du| \leq -\eta \quad \text{on } \mathbb{R}^N \times (0, T) ,$$

when we assume moreover that $\bar{c}$ is continuous. In our case where $\bar{c}$ is not assumed continuous in time, (10) follows from the L^1-stability result Theorem 5.5 where we approximate $\bar{c}$ by a continuous function, and from the usual stability for L^1-viscosity subsolutions.

Let us note that from the proof of [5, Corollary 2.5], we have in the viscosity sense

$$(26) \qquad -|u(\cdot, t)| - |Du(\cdot, t)| \leq -\eta \quad \text{on } \mathbb{R}^N, \text{ for every } t \in (0, T) .$$

Following [5, Corollary 2.5], we get that, for every $t \in (0, T)$, the 0–level-set of $u(\cdot, t)$ has a zero Lebesgue measure. Then we deduce that

$$\chi(x, t) = \mathbb{1}_{\{u(\cdot, t) \geq 0\}}(x), \quad \text{for a.e. } x \in \mathbb{R}^N, \text{ for all } t \in (0, T)$$

which (with (7)) entails that

$$\bar{c}(x, t) = c_1 + c_0 \star \mathbb{1}_{\{u(\cdot, t) \geq 0\}}$$

for any (x, t). Moreover $t \mapsto \mathbb{1}_{\{u(\cdot, t) \geq 0\}}$ is also continuous in L^1, and then $\bar{c}$ is continuous. Therefore u is a classical viscosity solution of (3).

2. Uniqueness when u_0 is semiconvex (Part (ii)). If we assume that (9) and (10) hold and that u_0 is semiconvex, then weak solutions are viscosity solutions, and we can apply the uniqueness result for viscosity solutions given in [5], namely Theorem 4.2 (which remains true under our assumptions) which requires in particular semiconvexity of the velocity, see assumption **(H2)**.

3. A Gronwall type inequality (Part (i)). From now on we assume that $\delta > 0$ and we aim at proving that the solution to (3) is unique. Let u_1, u_2 be two solutions. We set

$$\rho_i = \mathbb{1}_{\{u_i(\cdot, t) \geq 0\}} \qquad \text{and} \qquad \bar{c}_i(x, t) = c_0 \star \rho_i + c_1 \qquad \text{for } i = 1, 2.$$

We want to prove in a first step the following Gronwall type inequality for any t sufficiently small:

(27)
$$|\rho_1(\cdot, t) - \rho_2(\cdot, t)|_{L^1}$$
$$\leq\; C\left[\mathrm{per}(\{u_1(\cdot, t) \geq 0\}) + \mathrm{per}(\{u_2(\cdot, t) \geq 0\})\right] \int_0^t |\rho_1(\cdot, s) - \rho_2(\cdot, s)|_{L^1} ds$$

where C is a constant depending on the constants of the problem, where $\mathrm{per}(\{u_i(\cdot, t) \geq 0\})$ is the $\mathcal{H}^{N-1}$ measure of the set $\partial\{u_i(\cdot, t) \geq 0\})$ (for $i = 1, 2$).

We have

(28)
$$|\rho_1(\cdot, t) - \rho_2(\cdot, t)|_{L^1} \leq \mathcal{L}^N(\{-\alpha_t \leq u_1(\cdot, t) < 0\}) + \mathcal{L}^N(\{-\alpha_t \leq u_2(\cdot, t) < 0\}) \,,$$

where $\mathcal{L}^N$ is the Lebesgue measure in $\mathbb{R}^N$ and

(29) $$\alpha_t = \sup_{s \in [0,t]} |(u_1 - u_2)(\cdot, s)|_\infty \qquad \text{for any } t \in (0, T).$$

In order to estimate the right-hand side of inequality (28), as in the proof of [5, Theorem 4.2], we need a lower-gradient bound as well as a semiconvexity property for u_1 and u_2. We already know from step 1 that $\bar{c}_i$ is continuous for $i = 1, 2$.

Let us start to estimate the right-hand side of (28). From the "stability estimates" on the solutions with respect to variations of the velocity (see [5, Lemma 2.2]), we have

(30) $$\alpha_t \leq |Du_0|_\infty e^{Lt} \int_0^t |(\bar{c}_1 - \bar{c}_2)(\cdot, s)|_\infty ds \,,$$

where $L = L_0 + L_1$. Therefore

(31) $$\alpha_t \leq m_0 \, |Du_0|_\infty e^{Lt} \int_0^t |(\rho_1 - \rho_2)(\cdot, s)|_{L^1} ds \,.$$

where the constant m_0 is given in **(H2)**. In particular, since the $\rho_i(\cdot, t)$ are continuous in L^1 and equal at time $t = 0$ for $i = 1, 2$, we have $\alpha_t/t \to 0$ as $t \to 0^+$.

From now on, we mimick the proof of [5, Proposition 4.5]. Using the lower-gradient bound (25) for $u_i(\cdot, t)$ combined with the increase principle (see [5, Lemma 2.3]), we obtain for $\alpha_t < \eta/2$ that

$$\{-\alpha_t \leq u_i(\cdot, t) < 0\} \subset \{u_i(\cdot, t) \geq 0\} + (2\alpha_t/\eta)\overline{B}(0, 1) \,,$$

for $i = 1, 2$. From the interior ball regularization Lemma 2.3, the set $\{u_i(\cdot, t) \geq 0\}$ satisfies for $t \in (0, T)$ the interior ball property of radius

$r_t = \eta t / C_0$. Applying [2, Lemma 2.5 and 2.6], we obtain for $\sigma_t = 2\alpha_t/\eta$ that

$$\mathcal{L}^N(\{-\alpha_t \leq u_i(\cdot, t) < 0\}) \leq \quad \mathcal{L}^N\left(\left(\{u_i(\cdot, t) \geq 0\} + \sigma_t \overline{B}(0,1)\right) \backslash \{u_i(\cdot, t) \geq 0\}\right)$$

$$\leq \quad \frac{r_t}{N}\left[\left(1 + \frac{\sigma_t}{r_t}\right)^N - 1\right] \mathrm{per}(\{u_i(\cdot, t) \geq 0\})$$

$$\leq \quad \frac{2^N \alpha_t}{\eta} \mathrm{per}(\{u_i(\cdot, t) \geq 0\})$$

(using $(1+a)^N - 1 \leq aN(1+a)^{N-1}$ for $a \geq 0$) for $t \in [0, \tau]$ where $0 < \tau \leq T$ is defined by

$$(32) \qquad \tau = \sup\{t > 0 : \alpha_t < \frac{\eta}{2} \text{ and } 2\frac{C_0}{\eta^2}\frac{\alpha_t}{t} \leq 1\}.$$

Putting together (31), (28) and the previous inequality proves (27).

4. Uniqueness when $\delta > 0$ (Part (i)). We now complete the uniqueness proof under the assumption $\delta > 0$. For this we first show that $\rho_1 = \rho_2$ in $[0, \tau]$. In order to apply Gronwall Lemma to the L^1-estimate (27) obtained in Step 3, it is enough to prove that the functions $t \mapsto \mathrm{per}(\{u_i(\cdot, t) \geq 0\})$ belong to L^1. For this, let us set

$$w_i(x) = \inf\{t \geq 0 \; : \; u_i(x, t) \geq 0\} \;.$$

Since u_i solves the eikonal equation $(u_i)_t = \bar{c}_i(x, t)|Du_i|$, from classical representation formulae, we have

$$\{u_i(\cdot, t) \geq 0\} \quad = \quad \{x \; : \; \exists y(\cdot), \; |\dot{y}(s)| \leq c(y(s), s), \; 0 \leq s \leq t,$$
$$u_0(y(0)) \geq 0 \text{ and } y(t) = x\}.$$

Therefore

$$w_i(x) \quad = \quad \inf\{t \geq 0 \; : \; \exists y(\cdot), \; |\dot{y}(s)| \leq c(y(s), s), \; 0 \leq s \leq t,$$
$$u_0(y(0)) \geq 0 \text{ and } y(t) = x\}.$$

Applying the dynamic programming principle, since $\bar{c}_i \geq \delta > 0$, we obtain that w_i is Lipschitz continuous and is a viscosity solution of the autonomous equation $\bar{c}_i(x, w_i(x))|Dw_i(x)| = 1$. Note that $\{u_i(\cdot, t) \geq 0\} = \{w_i \leq t\}$. In particular, by Theorem 2.1 (i), $\{w_i \leq t\} \subset B(0, R_0+$

$(M_0 + M_1)T)$ is bounded for any t. From the coarea formula, we have

$$\int_0^t \text{per}(\{u_i(\cdot, s) \geq 0\})ds = \int_0^t \text{per}(\{w_i \leq s\})ds$$

$$= \int_{\{w_i \leq t\}} |Dw_i(x)|dx \ ,$$

which is finite since w_i is Lipschitz continuous. Therefore we have proved that $t \mapsto \text{per}(\{u_i(\cdot, t) \geq 0\})$ belongs to $L^1([0, \tau])$, which entails from Gronwall Lemma that $\rho_1 = \rho_2$ in $[0, \tau]$ since $\rho_1(\cdot, 0) = \rho_2(\cdot, 0)$. Hence $\bar{c}_1 = \bar{c}_2$ and $u_1 = u_2$ in $[0, \tau]$. From the definition of α_t and τ, (see (29) and (32)), necessarily $\tau = T$. It completes the proof.   □

5. Nonnegative kernel c_0 and Slepčev formulation for the nonlocal term

In this section, we deal with nonnegative kernels $c_0 \geq 0$. In this monotone framework, inclusion principle for evolving sets and comparison for solutions to the dislocation equation are expected (see Cardaliaguet [11] for related results). We start by studying the right level-set equation using a Slepčev formulation with the convolution term using all the level-sets $\{u(\cdot, t) \geq u(x, t)\}$ instead of only one level-set $\{u(\cdot, t) \geq 0\}$. This choice is motivated by the good stability properties of the Slepčev formulation.

The equation we are concerned with is

(33) $$\begin{cases} \dfrac{\partial u}{\partial t} = c^+[u](x, t)|Du| & \text{in } \mathbb{R}^N \times (0, T) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N \ , \end{cases}$$

where the nonlocal velocity is

(34)
$$c^+[u](x, t) = c_1(x, t) + c_0(\cdot, t) \star \mathbb{1}_{\{u(\cdot, t) \geq u(x, t)\}}(x)$$

$$= c_1(x, t) + \int_{\mathbb{R}^N} c_0(x - z, t) \mathbb{1}_{\{u(\cdot, t) \geq u(x, t)\}}(z)dz$$

and the additional velocity c_1 has no particular sign.

We denote

$$c^-[u](x, t) = c_1(x, t) + \int_{\mathbb{R}^N} c_0(x - z, t) \mathbb{1}_{\{u(\cdot, t) > u(x, t)\}}(z)dz.$$

We recall the notion of viscosity solutions for (33) as it appears in [28].

Definition 5.1. (Slepčev viscosity solutions)
An upper-semicontinuous function $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ is a viscosity subsolution of (33) if, for any $\varphi \in C^1(\mathbb{R}^N \times [0, T])$, for any maximum point $(\bar{x}, \bar{t})$ of $u - \varphi$, if $\bar{t} > 0$ then

$$\frac{\partial \varphi}{\partial t}(\bar{x}, \bar{t}) \leq c^+[u](\bar{x}, \bar{t}) |D\varphi(\bar{x}, \bar{t})|$$

and $u(\bar{x}, 0) \leq u_0(\bar{x})$ if $\bar{t} = 0$.

A lower-semicontinuous function $u : \mathbb{R}^N \times [0, T] \to \mathbb{R}$ is a viscosity supersolution of (33) if, for any $\varphi \in C^1(\mathbb{R}^N \times [0, T])$, for any minimum point $(\bar{x}, \bar{t})$ of $u - \varphi$, if $\bar{t} > 0$ then

$$\frac{\partial \varphi}{\partial t}(\bar{x}, \bar{t}) \geq c^-[u](\bar{x}, \bar{t}) |D\varphi(\bar{x}, \bar{t})|$$

and $u(\bar{x}, 0) \geq u_0(\bar{x})$ if $\bar{t} = 0$.

A locally bounded function is a viscosity solution of (33) if its upper-semicontinuous enveloppe is subsolution and its lower-semicontinuous envelope is supersolution of (33).

Note that for the supersolution, we require the viscosity inequality with c^- instead of c^+. It is the definition providing the expected stability results (see [28]).

Theorem 5.2. (Comparison principle)
Assume (H0'), and that the kernel $c_0 \geq 0$ and c_1 satisfy (H1). Let u (respectively v) be a bounded upper-semicontinuous subsolution (respectively a bounded lower semicontinuous supersolution) of (33). Then $u \leq v$ in $\mathbb{R}^N \times [0, T]$.

Remark 5.1. We could deal with second-order terms in (33) (for instance we can add the mean curvature to the velocity (1)). See Forcadel [17] and Srour [29] for related results.

Before giving the proof of Theorem 5.2, let us note the following consequence.

Proof of Theorem 1.4. The uniqueness of a continuous viscosity solution to (33) is an immediate consequence of Theorem 5.2. Then existence is proved by Perron's method using classical arguments (see for instance [16, Theorem 1.2]), so we skip the details.   □

Proof of Theorem 5.2.
1. The test-function. Since $u - v$ is a bounded upper-semicontinuous

function, for any $\varepsilon, \eta, \alpha > 0$ and $K = 2(L_0 + L_1) \geq 0$, the supremum

$$M_{\varepsilon,\eta,\alpha} = \sup_{(x,y,t)\in(\mathbb{R}^N)^2\times[0,T]} \{u(x,t) - v(y,t) - \mathrm{e}^{Kt}(\frac{|x-y|^2}{\varepsilon^2} + \alpha|x|^2 + \alpha|y|^2) - \eta t\}$$

is finite and achieved at a point $(\bar{x}, \bar{y}, \bar{t})$. Classical arguments show that

$$\liminf_{\varepsilon,\eta,\alpha\to 0} M_{\varepsilon,\eta,\alpha} = \sup_{\mathbb{R}^N\times[0,T]} \{u - v\}$$

and that

(35) $$\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2}, \alpha|\bar{x}|^2, \alpha|\bar{y}|^2 \leq M_\infty,$$

where $M_\infty = |u|_\infty + |v|_\infty$.

2. Viscosity inequalities when $\bar{t} > 0$. Writing the viscosity inequalities for the subsolution u and the supersolution v, we obtain
(36)
$$K\mathrm{e}^{K\bar{t}}(\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} + \alpha|\bar{x}|^2 + \alpha|\bar{y}|^2) + \eta \leq c^+[u](\bar{x}, \bar{t})|\bar{p} + \bar{q}_x| - c^-[v](\bar{y}, \bar{t})|\bar{p} - \bar{q}_y|$$

where $\bar{p} = 2e^{K\bar{t}}(\bar{x} - \bar{y})/\varepsilon^2$, $\bar{q}_x = 2e^{K\bar{t}}\alpha\bar{x}$ and $\bar{q}_y = 2e^{K\bar{t}}\alpha\bar{y}$. We point out a difficulty to obtain this inequality: in general, one gets it by doubling the time variable first and then by passing to the limit in the time penalization. This is not straightforward here because of the dependence with respect to time of the nonlocal terms. But the stability arguments of the Slepčev formulation take care of this difficulty.

3. Difference between $\{u(\cdot, \bar{t}) \geq u(\bar{x}, \bar{t})\}$ and $\{v(\cdot, \bar{t}) > v(\bar{y}, \bar{t})\}$. We have

(37) $$\{u(\cdot, \bar{t}) \geq u(\bar{x}, \bar{t})\} \subset \{v(\cdot, \bar{t}) > v(\bar{y}, \bar{t})\} \cup \mathcal{E},$$

where $\mathcal{E} = \{u(\cdot, \bar{t}) \geq u(\bar{x}, \bar{t})\} \cap \{v(\cdot, \bar{t}) \leq v(\bar{y}, \bar{t})\}$. If $x \in \mathcal{E}$, then $u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t}) \leq u(x, \bar{t}) - v(x, \bar{t})$. But from the definition of $M_{\varepsilon,\eta,\alpha}$,

$$u(x, \bar{t}) - v(x, \bar{t}) - \mathrm{e}^{K\bar{t}}2\alpha|x|^2 - \eta\bar{t}$$
$$\leq u(\bar{x}, \bar{t}) - v(\bar{y}, \bar{t}) - \mathrm{e}^{K\bar{t}}(\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} + \alpha|\bar{x}|^2 + \alpha|\bar{y}|^2) - \eta\bar{t}.$$

It follows that

$$\mathcal{E} \subset \{x \in \mathbb{R}^N : |x|^2 \geq \frac{1}{2}(|\bar{x}|^2 + |\bar{y}|^2) + \frac{|\bar{x} - \bar{y}|^2}{2\alpha\varepsilon^2}\}.$$

4. Upper bound for $c^+[u](\bar{x}, \bar{t})$. We have

$$(38) \quad c^+[u](\bar{x}, \bar{t}) = \int_{\mathbb{R}^N} c_0(\bar{x} - z, \bar{t}) \mathbb{1}_{\{u(\cdot, \bar{t}) \geq u(\bar{x}, \bar{t})\}}(z) dz + c_1(\bar{x}, \bar{t})$$

$$\leq \int_{\mathbb{R}^N} (c_0(\bar{x} - z, \bar{t}) - c_0(\bar{y} - z, \bar{t})) \mathbb{1}_{\{u(\cdot, \bar{t}) \geq u(\bar{x}, \bar{t})\}}(z) dz$$

$$+ \int_{\mathbb{R}^N} c_0(\bar{y} - z, \bar{t}) \mathbb{1}_{\{u(\cdot, \bar{t}) \geq u(\bar{x}, \bar{t})\}}(z) dz + c_1(\bar{x}, \bar{t}).$$

Using that $c_0 \geq 0$ and (37), we obtain

$$\int_{\mathbb{R}^N} c_0(\bar{y} - z, \bar{t}) \mathbb{1}_{\{u(\cdot, \bar{t}) \geq u(\bar{x}, \bar{t})\}}(z) dz \leq \int_{\{v(\cdot, \bar{t}) > v(\bar{y}, \bar{t})\} \cup \mathcal{E}} c_0(\bar{y} - z, \bar{t}) dz.$$

From (38), we get

$$(39) \qquad c^+[u](\bar{x}, \bar{t}) \quad \leq \quad c^-[v](\bar{y}, \bar{t}) + \mathcal{I}_1 + \mathcal{I}_2 + c_1(\bar{x}, \bar{t}) - c_1(\bar{y}, \bar{t}),$$

where

$$\mathcal{I}_1 = \int_{\mathbb{R}^N} (c_0(\bar{x} - z, \bar{t}) - c_0(\bar{y} - z, \bar{t})) \mathbb{1}_{\{u(\cdot, \bar{t}) \geq u(\bar{x}, \bar{t})\}}(z) dz$$

and

$$\mathcal{I}_2 = \int_{\mathcal{E}} c_0(\bar{y} - z, \bar{t}) dz.$$

5. Estimate of $\mathcal{I}_1$ using **(H1)**. We have

$$c_0(\bar{x} - z, t) - c_0(\bar{y} - z, t) = \int_0^1 D_x c_0((1 - \lambda)(\bar{y} - z) + \lambda(\bar{x} - z), \bar{t})(\bar{x} - \bar{y}) d\lambda.$$

It follows

$$|\mathcal{I}_1| \quad \leq \quad \int_{\mathbb{R}^N} \int_0^1 |D_x c_0((1 - \lambda)(\bar{y} - z) + \lambda(\bar{x} - z), \bar{t})| |\bar{x} - \bar{y}| d\lambda dz$$

$$\leq \quad |D_x c_0(\cdot, \bar{t})|_{L^1} |\bar{x} - \bar{y}|$$

$$(40) \qquad \leq \quad L_0 |\bar{x} - \bar{y}|.$$

6. Estimate of the right-hand side of inequality (36). Noticing that $|\bar{p}||\bar{x} - \bar{y}| = 2e^{K\bar{t}}|\bar{x} - \bar{y}|^2/\varepsilon^2$ and using (40), (39) and **(H1)**, we have

$$c^+[u](\bar{x}, \bar{t})|\bar{p} + \bar{q}_x| - c^-[v](\bar{y}, \bar{t})|\bar{p} - \bar{q}_y|$$

$$\leq (c^-[v](\bar{y}, \bar{t}) + \mathcal{I}_1 + \mathcal{I}_2 + c_1(\bar{x}, \bar{t}) - c_1(\bar{y}, \bar{t}))|\bar{p} + \bar{q}_x| - c^-[v](\bar{y}, \bar{t})|\bar{p} - \bar{q}_y|$$

$$\leq |c^-[v](\bar{y}, \bar{t})| |\bar{q}_x + \bar{q}_y| + (L_0 + L_1)|\bar{x} - \bar{y}||\bar{p} + \bar{q}_x| + \mathcal{I}_2|\bar{p} + \bar{q}_x|$$

$$\leq (M_0 + M_1)(|\bar{q}_x| + |\bar{q}_y|) + 2e^{K\bar{t}}(L_0 + L_1)\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2}$$

$$+ (L_0 + L_1)|\bar{x} - \bar{y}||\bar{q}_x| + \mathcal{I}_2(|\bar{q}_x| + 2e^{K\bar{t}}\frac{|\bar{x} - \bar{y}|}{\varepsilon^2}).$$

Since $|\bar{q}_x|, |\bar{q}_y| \to 0$ as $\alpha \to 0$ (see (35)) and $\mathcal{I}_2$ is bounded by $|c_0(\cdot, \bar{t})|_{L^1} \leq L_0$, there exists a modulus $m_\varepsilon(\alpha) \to 0$ as $\alpha \to 0$ such that (36) becomes

$$K e^{K\bar{t}} (\frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} + \alpha |\bar{x}|^2 + \alpha |\bar{y}|^2) + \eta$$

$$\leq \quad m_\varepsilon(\alpha) + 2(L_0 + L_1) e^{K\bar{t}} \frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} + 2 \mathcal{I}_2 e^{K\bar{t}} \frac{|\bar{x} - \bar{y}|}{\varepsilon^2}.$$

Recalling that we chose $K \geq 2(L_0 + L_1)$, we finally obtain

$$(41) \qquad 0 < \eta \leq m_\varepsilon(\alpha) + 2 \mathcal{I}_2 e^{K\bar{t}} \frac{|\bar{x} - \bar{y}|}{\varepsilon^2}.$$

7. Limit when $\alpha \to 0$. First, suppose that

$$(42) \qquad \frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} \to 0 \quad \text{as } \alpha \to 0.$$

It follows that $|\bar{x} - \bar{y}| \to 0$ as $\alpha \to 0$. Passing to the limit in (41), we obtain a contradiction. Therefore, (42) cannot hold and, up to extract a subsequence, there exists $\delta > 0$ such that

$$(43) \qquad \frac{|\bar{x} - \bar{y}|^2}{\varepsilon^2} \geq \delta > 0 \quad \text{for } \alpha > 0 \text{ small enough.}$$

From (41) and (35), we get

$$(44) \qquad \eta \leq \limsup_{\alpha \to 0} 2 \mathcal{I}_2 e^{K\bar{t}} \frac{|\bar{x} - \bar{y}|}{\varepsilon^2} \leq \frac{2 e^{K\bar{t}} M_\infty^{1/2}}{\varepsilon} \limsup_{\alpha \to 0} \mathcal{I}_2.$$

To obtain a contradiction, it suffices to show that $\limsup_{\alpha \to 0} \mathcal{I}_2 = 0$.

8. Convergence of $\mathcal{I}_2$ to 0 when $\alpha \to 0$. By a change of variable, we have

$$\mathcal{I}_2 = \int_{\mathcal{E}} c_0(\bar{y} - z, \bar{t}) dz \leq \int_{\bar{\mathcal{E}}} c_0(z, \bar{t}) dz,$$

where

$$\bar{\mathcal{E}} = \{x \in \mathbb{R}^N : |x - \bar{y}|^2 \geq \frac{1}{2}(|\bar{x}|^2 + |\bar{y}|^2) + \frac{|\bar{x} - \bar{y}|^2}{2\alpha \varepsilon^2}\}.$$

Since $|c_0(\cdot, \bar{t})|_{L^1} \leq L_0$, to prove that $\mathcal{I}_2 \to 0$, it suffices to show that $\bar{\mathcal{E}} \subset \mathbb{R}^N \backslash B(0, R_\alpha)$ with $R_\alpha \to +\infty$. From (43), if $x \in \bar{\mathcal{E}}$, then

$$\begin{aligned} |x|^2 &\geq \quad -2|x||\bar{y}| + \frac{1}{2}|\bar{x}|^2 - \frac{1}{2}|\bar{y}|^2 + \frac{\delta}{2\alpha} \\ &\geq \quad -2|x||\bar{y}| - |\bar{y}||\bar{x} - \bar{y}| + \frac{\delta}{2\alpha} \\ &\geq \quad \frac{1}{2\alpha}(\delta - 2(C^2 + 2C|x|)\sqrt{\alpha}), \end{aligned}$$

since by (35), there exists $C > 0$ such that $|\bar{x}|, |\bar{y}| \leq C/\sqrt{\alpha}$ and $|\bar{x}-\bar{y}| \leq C$. It follows that

$$|x| \geq \frac{1}{\sqrt{\alpha}} \left(-C + \sqrt{C^2 + \delta/2 - C^2\sqrt{\alpha}} \right) := R_\alpha \xrightarrow[\alpha \to 0]{} +\infty.$$

9. End of the proof. Finally, for every ε, if $\alpha = \alpha_\varepsilon$ is small enough, the supremum $M_{\varepsilon,\eta,\alpha}$ is necessarily achieved for $\bar{t} = 0$. It follows

$$M_{\varepsilon,\eta,\alpha} \leq u(\bar{x},0) - v(\bar{y},0) - \frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} \leq u_0(\bar{x},0) - u_0(\bar{y},0) - \frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}.$$

Since, u_0 is uniformly continuous, for all $\rho > 0$, there exists $C_\rho > 0$ such that

$$M_{\varepsilon,\eta,\alpha} \leq \rho + C_\rho|\bar{x}-\bar{y}| - \frac{|\bar{x}-\bar{y}|^2}{\varepsilon^2} \leq \rho + \frac{C_\rho^2\varepsilon^2}{4}.$$

Passing to the limits $\varepsilon \to 0$ and then $\rho, \alpha, \eta \to 0$, we obtain that $\sup\{u - v\} \leq 0$ ☐

Now we turn to the connections with *discontinuous solutions* and weak solutions, which are closely connected. To do so, if u is the unique continuous solution of (33) given by Theorem 1.4, we recall that we use the notations

$$\rho^+ := \mathbb{1}_{\{u \geq 0\}}, \quad \rho^- := \mathbb{1}_{\{u > 0\}} \quad \text{and} \quad c[\rho](x,t) = c_0(\cdot,t) \star \rho(\cdot,t)(x) + c_1(x,t).$$

Proof of Theorem 1.5.

1. Claim: Under the assumptions of Theorem 1.5, the functions ρ^+ and ρ^- are L^1-viscosity solutions of the equation

$$(45) \qquad \begin{cases} \rho_t = c[\rho]|D\rho| & in \ \mathbb{R}^N \times [0,T) , \\ \rho(x,0) = \mathbb{1}_{\{u_0 \geq 0\}} & in \ \mathbb{R}^N . \end{cases}$$

We consider two sequences of smooth nondecreasing functions $(\psi_\alpha)_\alpha$, $(\psi^\alpha)_\alpha$, taking values in $[0,1]$, such that, for any $s \in \mathbb{R}$

$$\psi_\alpha(s) \leq \mathbb{1}_{\{x>0\}}(s) \leq \mathbb{1}_{\{x \geq 0\}}(s) \leq \psi^\alpha(s) ,$$

and such that, as $\alpha \to 0$, $\psi_\alpha \uparrow \mathbb{1}_{\{x>0\}}$, $\psi^\alpha \downarrow \mathbb{1}_{\{x \geq 0\}}$.

We first remark that u satisfies, in the sense of Definition 5.1,

$$\begin{aligned} u_t &\leq c[\psi^\alpha(u(\cdot,t) - u(x,t))]|Du| & \text{in } \mathbb{R}^N \times (0,T) , \\ u_t &\geq c[\psi_\alpha(u(\cdot,t) - u(x,t))]|Du| & \text{in } \mathbb{R}^N \times (0,T) \end{aligned}$$

since u is a continuous solution of (33), $c^+[u](x,t) \leq c[\psi^\alpha(u(\cdot,t) - u(x,t))]$ and $c^-[u](x,t) \geq c[\psi_\alpha(u(\cdot,t) - u(x,t))]$. The point for doing that is that the functions $c[\psi^\alpha(u(\cdot,t) - u(x,t))]$, $c[\psi_\alpha(u(\cdot,t) - u(x,t))]$ are now continuous in x and t.

Then we show that ρ^+, ρ^- satisfy the same inequalities, the functions $c[\psi^\alpha(u(\cdot, t) - u(x, t))]$, $c[\psi_\alpha(u(\cdot, t) - u(x, t))]$ being considered as fixed functions (in other words, we forget that they depend on u). In fact, we just provide the proof in details for ρ^+, the one for ρ^- being analogous. Following the proof of [6], we set

$$u_\varepsilon(x, t) := \frac{1}{2}\left(1 + \tanh\left(\varepsilon^{-1}(u(x, t) + \varepsilon^{1/2})\right)\right) .$$

Noticing that $u_\varepsilon = \phi_\varepsilon(u)$ for an increasing function ϕ_ε, we have that the function u_ε still satisfies the two above inequalities. It is easy to see that

$$\rho^+ = \limsup{}^* u_\varepsilon \quad \text{and} \quad (\rho^+)_* = \liminf{}_* u_\varepsilon ,$$

and the half-relaxed limits method indeed shows that

$$\begin{aligned}
(\rho^+)_t^* &\leq c[\psi^\alpha(u(\cdot, t) - u(x, t))]|D(\rho^+)^*| \quad \text{in } \mathbb{R}^N \times (0, T) , \\
((\rho^+)_*)_t &\geq c[\psi_\alpha(u(\cdot, t) - u(x, t))]|D(\rho^+)_*| \quad \text{in } \mathbb{R}^N \times (0, T) .
\end{aligned}$$

The next step consists in remarking that the viscosity sub and supersolutions inequalities for ρ^+ are obviously satisfied in the complementary of $\partial\{u \geq 0\}$ since ρ^+ is locally constant there and therefore it is a classical solution of the problem. The only nontrivial viscosity sub and supersolutions inequalities we have to check are at points $(x, t) \in \partial\{u \geq 0\}$, i.e. such that $u(x, t) = 0$ since u is continuous. For such points, as $\alpha \to 0$,

$$c[\psi^\alpha(u(\cdot, t) - u(x, t))] \to c[\rho^+](x, t) = c[(\rho^+)^*](x, t)$$

since ρ^+ is upper-semicontinuous, and

$$c[\psi_\alpha(u(\cdot, t) - u(x, t))] \to c[\rho^-](x, t).$$

The stability result for equations with a L^1-dependence in time yields the inequalities

$$\begin{aligned}
(\rho^+)_t^* &\leq c[(\rho^+)^*]|D(\rho^+)^*| \quad \text{in } \mathbb{R}^N \times (0, T) , \\
(46) \qquad ((\rho^+)_*)_t &\geq c[\rho^-]|D(\rho^+)_*| \quad \text{in } \mathbb{R}^N \times (0, T) .
\end{aligned}$$

The second inequality is weaker than the one we claim : to obtain $c[(\rho^+)_*]$ instead of $c[\rho^-]$, we have to play with the different level-sets of u : for $\beta > 0$ small, we set $\rho_\beta^+ = \mathbb{1}_{\{u \geq -\beta\}}$. Since u is a solution of (33) and $\psi_{\alpha,\beta} := \psi_\alpha(\cdot + \beta)$ is nondecreasing, then $\psi_{\alpha,\beta}(u)$ is a (continuous) supersolution of

$$(\psi_{\alpha,\beta}(u))_t \geq c^-[\psi_{\alpha,\beta}(u)]|D\psi_{\alpha,\beta}(u)| \quad \text{in } \mathbb{R}^N \times (0, T) .$$

By stability we get, as $\alpha \to 0$,

$$((\rho_\beta^+)_*)_t \geq c^-[(\rho_\beta^+)_*]|D(\rho_\beta^+)_*| \quad \text{in } \mathbb{R}^N \times (0, T) .$$

But, for $(x, t) \in \partial\{u \geq -\beta\}$,
(47)
$$c^-[(\rho_\beta^+)_*](x, t) = c[\mathbb{1}_{\{(\rho_\beta^+)_*(\cdot, t)>0\}}](x, t) = c[(\rho_\beta^+)_*](x, t) \geq c[\rho^+](x, t).$$

It follows that $(\rho_\beta^+)_*$ is a supersolution of the Eikonal Equation with $c[\rho^+](x, t)$ (as before, the only nontrivial inequalities we have to check are on $\partial\{u \geq -\beta\}$ and they are true because of (47). Letting β tends to 0 and using that $(\rho^+)_* = \liminf_* (\rho_\beta^+)_*$, we obtain the expected inequality (even something better since (46) holds actually with $c[\rho^+](x, t)$). In particular, we get that ρ^+ is a solution of

(48)
$$\begin{cases} \rho_t = c[\rho^+]|D\rho| & \text{in } \mathbb{R}^N \times [0, T) , \\ \rho(x, 0) = \mathbb{1}_{\{u_0 \geq 0\}} & \text{in } \mathbb{R}^N . \end{cases}$$

And the proof of the claim is complete.

2. The functions $v^\pm$ are weak solutions of (3). Let us start with the "+" case. We first remark that the existence and uniqueness of v^+ follows from the standard theory for equations with a L^1-dependence in time (see Appendix A).

To prove that v^+ is a weak solution of (3), it remains to prove that (7) holds. It is sufficient to show that

(49)
$$\{v^+(\cdot, t) > 0\} \subset \{u(\cdot, t) \geq 0\} \subset \{v^+(\cdot, t) \geq 0\} .$$

We use again the functions ψ_α, ψ^α introduced above. We remark that

$$\psi_\alpha(u_0) \leq \rho^+(x, 0) \leq \psi^\alpha(u_0) \quad \text{in } \mathbb{R}^N .$$

Moreover v^+ and ρ^+ are solutions of the same equation, namely (48) with $c[\rho^+]$ which is considered as a fixed function, and so are $\psi_\alpha(v^+)$ and $\psi^\alpha(v^+)$ because the equation is geometric. Therefore, a standard comparison result implies

$$\psi_\alpha(v^+) \leq (\rho^+)_* \leq \rho^+ = (\rho^+)^* \leq \psi^\alpha(v^+) \quad \text{in } \mathbb{R}^N \times [0, T) .$$

And letting α tends to 0, these inequalities imply (49).

We can prove the symmetric result for v^-, the only difference is that inclusion (49) has to be replaced by

(50)
$$\{v^-(\cdot, t) > 0\} \subset \{u(\cdot, t) > 0\} \subset \{v^-(\cdot, t) \geq 0\} .$$

3. Claim: If v is a weak solution of (3), then $\mathbb{1}_{\{v(\cdot, t) \geq 0\}}$ is a L^1-subsolution of (45). From Proposition 3.1 and since $c_0 \geq 0$, v satisfies in the L^1-sense,

(51)
$$v_t \leq c[\mathbb{1}_{\{v(\cdot, t) \geq 0\}}]|Dv| \quad \text{in } \mathbb{R}^N \times [0, T) .$$

By similar arguments as we used above, the function $1\!\!1_{\{v(\cdot,t)\geq 0\}}$ satisfies the same inequality which gives the result.

4. The function ρ^+ is the maximal L^1-subsolution of (45). Let w be a L^1 (upper-semicontinuous) subsolution of (45). First we have $w \leq 1$ in $\mathbb{R}^N \times [0, T)$ by comparison with the constant supersolution 1 for the equation with $c[w]$ fixed. By considering $\max(w, 0)$ we may assume that $0 \leq w \leq 1$ in $\mathbb{R}^N \times [0, T)$. By similar arguments as we already used in Step 1, we can show that $1\!\!1_{\{w(\cdot,t)>0\}}$ is also a L^1-subsolution of (45); thus we can assume that w is a characteristic function.

Then we remark that w is also a subsolution of (33) : indeed, again, the only nontrivial viscosity inequalities are on the boundary of the set $\{w = 1\}$ and if (x, t) is such a point we have $w(x, t) = 1$ because w is upper-semicontinuous and $w = 1\!\!1_{\{w(\cdot,t)\geq w(x,t)\}}$. Since u is a solution of the geometric equation (33), $\psi^\alpha(u)$ is still a solution which satisfies $\psi^\alpha(u)(x, 0) \geq 1\!\!1_{\{u_0 \geq 0\}} \geq w(x, 0)$ in $\mathbb{R}^N$. By Theorem 5.2 we obtain

$$w \leq \psi^\alpha(u) \quad \text{in } \mathbb{R}^N \times [0, T) .$$

And letting α tend to 0 provides $w \leq \rho^+$ which proves that ρ^+ is the maximal subsolution of (45).

5. The function v^+ is the maximal weak solution of (3). Let v be a weak solution of (3). From Steps 3 and 4, we get $1\!\!1_{\{v(\cdot,t)\geq 0\}} \leq \rho^+(\cdot, t)$ in $\mathbb{R}^N \times [0, T)$ and (51) implies

$$v_t \leq c[\rho^+]|Dv| \quad \text{in } \mathbb{R}^N \times [0, T) .$$

Therefore v is a subsolution of (12) and by standard comparison result, this leads to $v \leq v^+$ in $\mathbb{R}^N \times [0, T)$, which proves the result.

6. We have $\{v^+(\cdot, t) \geq 0\} = \{u(\cdot, t) \geq 0\}$ and $\{v^-(\cdot, t) > 0\} = \{u(\cdot, t) > 0\}$. From Step 5, we get $1\!\!1_{\{v^+(\cdot,t)\geq 0\}} \leq \rho^+(\cdot, t) = 1\!\!1_{\{u(\cdot,t)\geq 0\}}$. The conclusion follows for v^+ using (49). The inclusion for v^- uses symmetric arguments.

7. Uniqueness when $\{u(\cdot, t) = 0\}$ has Lebesegue measure 0. If $\mathcal{L}^N(\{u(\cdot, t) = 0\}) = 0$, then $c[\rho^+] = c[\rho^-]$. Hence $v^+ = v^-$ is the unique weak solution of (3) and it is obviously a classical one. $\qquad \square$

Appendix A: a stability result for Eikonal Equations with L^1-dependence in time

The aim of this appendix is to provide a self-contained presentation of a stability result for viscosity solutions of Eikonal Equations with

L^1-dependences in time which handles the case of weak convergence of the equations, instead of the classical strong L^1 convergence. This stability result is a particular case of a general stability result proved by Barles in [4].

For $T > 0$, we are interested in solutions of the following equation

$$(52) \qquad \begin{cases} \dfrac{\partial v}{\partial t} = \bar{c}(x,t)|Dv| & \text{dans} \quad \mathbb{R}^N \times (0,T) \\[2mm] v(\cdot, 0) = u_0 & \text{dans } \mathbb{R}^N, \end{cases}$$

where the velocity $\bar{c} : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ is defined for almost every $t \in (0,T)$. We also assume that $\bar{c}$ satisfies
(H3) The function $\bar{c}$ is continuous with respect to $x \in \mathbb{R}^N$ and measurable in t. For all $x, y \in \mathbb{R}^N$ and almost all $t \in [0,T]$,

$$|\bar{c}(x,t)| \leq M \quad \text{and} \quad |\bar{c}(x,t) - \bar{c}(y,t)| \leq L|x - y|.$$

Let us underline that we do not assume any continuity in time of $\bar{c}$. We recall the following (under assumption **(H0)**)

Definition 5.3. (L^1-viscosity solutions)
An upper-semicontinuous (respectively lower-semicontinuous) function v on $\mathbb{R}^N \times [0,T]$ is a L^1-viscosity subsolution (respectively supersolution) of (52), if

$$v(0,\cdot) \leq u_0 \quad (\text{respectively} \quad v(0,\cdot) \geq u_0),$$

*and if for every $(x_0, t_0) \in \mathbb{R}^N \times [0,T]$, $b \in L^1(0,T)$, $\varphi \in C^\infty(\mathbb{R}^N \times (0,T))$ and continuous function $G : \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \to \mathbb{R}$ such that
(i) the function*

$$(x,t) \longmapsto v(x,t) - \int_0^t b(s)ds - \varphi(x,t)$$

*has a local maximum (respectively minimum) at (x_0, t_0) over $\mathbb{R}^N \times (0,T)$ and such that
(ii) for almost every $t \in (0,T)$ in some neighborhood of t_0 and for every (x,p) in some neighborhood of (x_0, p_0) with $p_0 = \nabla\varphi(x_0, t_0)$, we have*

$$\bar{c}(x,t)|p| - b(t) \leq G(x,t,p) \quad (\text{respectively} \quad \bar{c}(x,t)|p| - b(t) \geq G(x,t,p))$$

then

$$\frac{\partial\varphi}{\partial t}(x_0, t_0) \leq G(x_0, t_0, p_0) \quad (\text{respectively} \quad \frac{\partial\varphi}{\partial t}(x_0, t_0) \geq G(x_0, t_0, p_0)).$$

Finally we say that a locally bounded function v defined on $\mathbb{R}^N \times [0,T]$ is a L^1-viscosity solution of (52), if its upper-semicontinuous (respectively lower-semicontinuous) envelope is a L^1-viscosity subsolution (respectively supersolution).

Let us recall that viscosity solutions in the L^1-sense were introduced in Ishii's paper [20]. We refer to D. Nunziante [25, 26] and M. Bourgoing [7, 8] for a complete presentation of the theory.

Then we have the following result

Theorem 5.4. (Existence and uniqueness)
For any $T > 0$, under assumptions **(H0)** *and* **(H3)**, *there exists a unique L^1-viscosity solution to (52).*

Finally, let us consider the solutions v^ε to the following equation

$$(53) \qquad \begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} = \bar{c}^\varepsilon(x,t)|Dv^\varepsilon| & \text{in } \mathbb{R}^N \times (0,T), \\ v^\varepsilon(\cdot,0) = u_0 & \text{in } \mathbb{R}^N . \end{cases}$$

Then we have the following

Theorem 5.5. (L^1-stability, [4])
Under assumption **(H0)**, *let us assume that the velocity $\bar{c}^\varepsilon$ satisfies* **(H3)** *(with some constants M, L independent of ε). Let us consider the L^1-viscosity solution v^ε to (53). Assume that v^ε converges locally uniformly to a function v and, for all $x \in \mathbb{R}^N$,*

$$(54) \quad \int_0^t \bar{c}^\varepsilon(x,s)ds \to \int_0^t \bar{c}(x,s)ds \quad \textit{locally uniformly in } (0,T).$$

Then v is a L^1-viscosity solution of (52).

Remark 5.2. Theorem 5.5 is stated as in [4] but note that, under **(H3)**, assumption (54) is automatically satisfied as soon as the convergence is merely pointwise. Indeed, since

$$\left| \int_0^t \bar{c}^\varepsilon(x,s)ds \right| \leq MT \;\; \text{and} \;\; \left| \int_0^t \bar{c}^\varepsilon(x,s)ds - \int_0^{t'} \bar{c}^\varepsilon(x,s)ds \right| \leq M|t-t'|,$$

from Ascoli's Theorem, the convergence is uniform.

Appendix B: Interior ball regularization (proof of Lemma 2.3)

The proof of this result can be adapted from those of Cannarsa and Frankowska [10] or [2] (see also [9] for related perimeter estimate for general equations). For the sake of completeness, we give a proof close to the one of [10] (this latter holds for much more general, but time-independent, dynamics). The unique (and small) contribution of this part amounts to explain how this proof can be simplified in the particular case of dynamics of the form (55) and to point out that the time-dependence is not an issue for the results of [10] to hold.

We first prove that the reachable set for controlled dynamics of the form

(55) $\dot{x}(t) = c(x(t), t)u(t), \quad u \in L^\infty([0, T], \overline{B}(0, 1))$

enjoys the interior ball property for positive time. We assume that $c : [0, T] \times \mathbb{R}^N \to \mathbb{R}$ satisfies, for any $x, y \in \mathbb{R}^N$ and $t \in [0, T]$,

$$\begin{cases} (i) & c \text{ is Borel measurable,} \\ & \text{differentiable with respect to the space variable for a.e. time,} \\ (ii) & |c(x, t) - c(y, t)| \leq L_1 |x - y|, \\ (iii) & |D_x c(x, t) - D_x c(y, t)| \leq N_1 |x - y|, \\ (iv) & M_1 \geq c(x, t) \geq \delta > 0, \end{cases}$$

where $L_1, N_1 \geq 0$ et $M_1, \delta > 0$ are given constants. Let $K_0 \subset \mathbb{R}^N$ be the initial set. We define the reachable set $\mathcal{R}(t)$ from K_0 for (55) at time t by:

$$\mathcal{R}(t) = \{x(t) , \ x(\cdot) \text{ solution to (55) with } x(0) \in K_0\} .$$

It is known that $\mathcal{R}(t)$ is a closed subset of $\mathbb{R}^N$. Let y_0 be an extremal solution on the time interval $[0, T]$, i.e., a solution of (55) such that

$$y_0(0) \in K_0 \quad \text{and} \quad y_0(T) \in \partial\mathcal{R}(T).$$

From the Pontryagin Maximum Principle for extremal trajectories (see for instance [13]), there is some adjoint function $p_0 : [0, T] \to \mathbb{R}^N \setminus \{0\}$ such that (y_0, p_0) is a solution to:

$$\begin{cases} \dot{y}_0(t) = c(y_0(t), t)\dfrac{p_0(t)}{|p_0(t)|}, \\ -\dot{p}_0(t) = D_x c(y_0(t), t)|p_0(t)|. \end{cases}$$

Since the system is positively homogeneous with respect to p, we can assume, without loss of generality, that $|p_0(T)| = 1$ and we set $\theta_0 := p_0(T)$.

Let P be the matrix valued solution to

$$\begin{cases} \dot{P}(t) = \dfrac{p_0(t)}{|p_0(t)|}[D_x c(y_0(t), t)]^* P(t), \\ P(T) = Id. \end{cases}$$

A straightforward computation shows that $P^*(t)p_0(t) = \theta_0$ for any $t \in [0, T]$.

Let us fix some parameter $\gamma > 0$ to be chosen later, $\theta \in B(0, 1)$ and let us set, for all $t \in [0, T]$,

$$y_\theta(t) = y_0(t) - \gamma t P(t)(\theta_0 - \theta).$$

Our aim is to show that y_θ is a solution to (55). Indeed we have

$$
\begin{aligned}
|\dot{y}_\theta|^2 &= \left| c(y_0, t) \frac{p_0}{|p_0|} - \gamma P(\theta_0 - \theta) - \gamma t \frac{p_0}{|p_0|} D_x c^* P(\theta_0 - \theta) \right|^2 \\
&= |c(y_\theta, t)|^2 - 2\gamma c(y_0, t) \left\langle \frac{p_0}{|p_0|}, P(\theta_0 - \theta) \right\rangle \\
&\quad + |c(y_0, t)|^2 - |c(y_\theta, t)|^2 - 2\gamma t c(y_0, t) \left\langle \frac{p_0}{|p_0|}, \frac{p_0}{|p_0|} D_x c^* P(\theta_0 - \theta) \right\rangle \\
&\quad + \gamma^2 \left| P(\theta_0 - \theta) + t \frac{p_0}{|p_0|} D_x c^* P(\theta_0 - \theta) \right|^2 \\
&\leq |c(y_\theta, t)|^2 - 2\gamma \frac{c(y_0, t)}{|p_0|} \langle \theta_0, \theta_0 - \theta \rangle \quad \text{(because } P^* p_0 = \theta_0) \\
&\quad + c^2(y_0, t) - c^2(y_\theta, t) - \langle D_x(c^2)(y_0, t), y_0 - y_\theta \rangle + \gamma^2 M |\theta_0 - \theta|^2 \\
&\leq |c(y_\theta, t)|^2 - \gamma \frac{\delta}{|p_0|} |\theta_0 - \theta|^2 \ + \ M_1' |y_0 - y_\theta|^2 + \gamma^2 M |\theta_0 - \theta|^2 \\
&\leq |c(y_\theta, t)|^2 - \gamma \frac{\delta}{|p_0|} |\theta_0 - \theta|^2 \ + \ \gamma^2 M' |\theta_0 - \theta|^2,
\end{aligned}
$$

with $M_1' = L_1^2 + M_1 N_1$, and where M and M' only depend on T, L_1, N_1, M_1 because $|p_0(t)|$ is bounded from below by a constant depending only on T, L_1. Hence, for γ sufficiently small, y_θ is a solution of (55) starting from $y_0(0) \in K_0$ and therefore $y_\theta(T) \in \mathcal{R}(T)$.

Finally $\mathcal{R}(T)$ contains all the $y_\theta(T)$ for $\theta \in B(0,1)$, i.e., the ball centered at $y_0(T) - \gamma T \theta_0$ and of radius γT (since $P(T) = Id$).

We apply the previous result with $c = c_1$ and $K_0 = \{v(\cdot, 0) \geq 0\}$. Then $\{v(\cdot, t) \geq 0\} = \mathcal{R}(t)$ for all $t > 0$.

We end with a remark: in the statement of Lemma 2.3, c_1 is assumed to be continuous in time. As we have seen, it is not necessary, c_1 can be merely measurable in time up to consider the L^1-solution v of (13) as recalled in Appendix A. $\square$

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