Short time existence and uniqueness in Hölder spaces for the 2D dynamics of dislocation densities

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Abstract

In this paper, we study the model of Groma and Balogh [11] describing the dynamics of dislocation densities. This is a two-dimensional model where the dislocation densities satisfy a system of two transport equations. The velocity vector field is the shear stress in the material solving the equations of elasticity. This shear stress can be related to Riesz transforms of the dislocation densities. Basing on some commutator estimates type, we show that this model has a unique local-in-time solution corresponding to any initial datum in the space $C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for $r > 1$ and $1 < p < +\infty$, where $C^r(\mathbb{R}^2)$ is the Hlder-Zygmund space.

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Key words: Cauchy's problem, non-linear transport equations, non-local transport equations, system of hyperbolic equations, Riesz transform, Hlder-Zygmund space, dynamics of dislocation densities.

1 Introduction

1.1 Physical motivation and presentation of the model

Real crystals show certain defects in the organization of their crystalline structure, called dislocations. These defects were introduced in the Thirties by Taylor [21], Orowan [18] and Polanyi [19] as the principal explanation of plastic deformation of materials at the microscopic scale. Dislocations can move under the action of exterior stresses applied to the material.

Groma and Balogh in [11] considered the particular case where these defects are parallel lines in the three-dimensional space, that can be viewed as points in a plane considering

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their cross-section.

In this model we consider two types of "edge dislocations" in the plane (x_1, x_2). Typically, for a given velocity field, those dislocations of type $(+)$ propagate in the direction $+\vec{e_1}$ where $\vec{e_1} = (1, 0)$ is the Burgers vector, while those of type $(-)$ propagate in the direction $-\vec{e_1}$. We refer the reader to the book of Hith and Lothe [13], for a detailled description of the classical notion in physics of edge dislocations and of the Burgers vector associated to these dislocations.

In [11] Groma and Balogh have considered the case of densities of dislocations. More precisely, this 2-D system is given by the following coupled non-local and non-linear transport equations (see Cannone et al. [4, Section 2] for more modeling details):

$$\begin{cases} \dfrac{\partial \rho^+}{\partial t}(x,t) + u\dfrac{\partial \rho^+}{\partial x_1}(x,t) = 0 \quad \text{on } \mathbb{R}^2 \times (0,T), \\ \\ \dfrac{\partial \rho^-}{\partial t}(x,t) - u\dfrac{\partial \rho^-}{\partial x_1}(x,t) = 0 \quad \text{on } \mathbb{R}^2 \times (0,T), \\ \\ u = R_1^2 R_2^2(\rho^+ - \rho^-). \end{cases} \tag{1.1}$$

The unknowns of this system are the two scalar functions ρ^+ and ρ^- at the time t and the position $x = (x_1, x_2)$, that we denote for simplification by $\rho^\pm$. This term correspond to the plastic deformations in a crystal. Its derivative in the x_1-direction $\dfrac{\partial \rho^\pm}{\partial x_1}$ represents the dislocation densities of type $(\pm)$. Physically, these quantities are non-negative. The function u is the velocity vector field which is equal to the shear stress in the material, solving the equations of elasticity. The operators R_1 (resp. R_2) are the $2D$ Riesz transform associated to x_1 (resp. x_2). More precisely, the Fourier transform of these $2D$ Riesz transforms R_1 and R_2 are given by

$$\widehat{R_k f}(\xi) = \frac{\xi_k}{|\xi|}\hat{f}(\xi) \quad \text{for} \quad \xi \in \mathbb{R}^2, \quad k = 1, 2.$$

The goal of this work is to establish local existence and uniqueness result of the solution of (1.1) when the initial datum

$$\rho^\pm(x_1, x_2, t = 0) = \rho_0^\pm(x_1, x_2) = \bar{\rho}_0^\pm(x_1, x_2) + Lx_1, \quad L \in \mathbb{R} \tag{1.2}$$

with $\bar{\rho}_0^\pm \in C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, for $r > 1$, $p \in (1, +\infty)$, where $C^r(\mathbb{R}^2)$ is the Hlder-Zygmund space defined in Section 2. The choice $L > 0$ guarantee the possibility to choose $\bar{\rho}_0^\pm \in L^p(\mathbb{R}^2)$ such that the assumption is compatible with the non-negativity of $\dfrac{\partial \rho_0^\pm}{\partial x_1}$. In a particular case where the initial datum is increasing, the global existence of a solution was proved by Cannone et al. [4], using especially an entropy inequality satisfies by the dislocation densities. The fundamental issue of uniqueness for global solutions remains open.

In a particular sub-case of model (1.1) where the dislocation densities depend on a single variable $x_1 + x_2$, the existence and uniqueness of a Lipschitz solution was proved by El Hajj et al. in [10] in the framework of viscosity solutions. Also the existence and uniqueness of a strong solution in $W^{1,2}_{loc}(\mathbb{R} \times [0, +\infty))$ was proved by El Hajj [9] in the framework of Sobolev spaces. For a similar model describing moreover boundary layer effects (see Groma, Csikor, Zaiser [12]), we refer the reader to Ibrahim [14] where a result of existence and uniqueness is established, using the framework of viscosity solutions and also entropy solution for nonlinear hyperbolic equations.

Our study of the dynamics of dislocation densities in a special geometry is related to the more general dynamics of dislocation lines. We refer the interested reader to the work of Alvarez et al. [1], for a local existence and uniqueness of some non-local Hamilton-Jacobi equation. We also refer to Barles et al. [2] for some long-time existence results.

1.2 Main results

We shall show that the system (1.1) possesses a unique local-in-time solution for any initial datum satisfy (1.2) such that $\bar{\rho}_0^{\pm} \in C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, for $r > 1$ and for $p \in (1, +\infty)$. This functional setting allows us to control the velocity field u in terms of $\rho^+ - \rho^-$ (see the third line of (1.1)). As we wrote it before, the velocity u is related to $\rho^+ - \rho^-$ through the two-dimensional Riesz transforms R_1, R_2. Riesz transforms do not map $C^r(\mathbb{R}^2)$ into $C^r(\mathbb{R}^2)$, but they are bounded on $C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$, for $r \in [0, +\infty)$ and for $p \in (1, +\infty)$, as we will see later.

For notational convenience, we define the space $Y_{r,p}$, for $r \in [0, +\infty)$ and $p \geq 1$ as follows

$$Y_{r,p} = \{ f = (f_1, f_2) \text{ such that } f_k \in C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2), \text{ for } k = 1, 2 \},$$

where $C^r(\mathbb{R}^2)$ is the inhomogeneous Hlder-Zygmund space (see Section 2, for more precise definition). This space is a Banach space endowed with the following norm: for $f = (f_1, f_2)$

$$\|f\|_{r,p} = \max_{k=1,2} (\|f_k\|_{C^r}) + \max_{k=1,2} (\|f_k\|_{L^p}).$$

In order to avoid technical difficulties, we first consider (see Theorem 1.1) the case $L = 0$. Then (see Theorem 1.2) we treat the general case $L \in \mathbb{R}$.

Theorem 1.1 (Local existence and uniqueness, case $L = 0$)
Consider the initial data

$$\rho_0 = (\rho_0^+, \rho_0^-) \in Y_{r,p}. \tag{1.3}$$

If $r > 1$ and $p \in (1, +\infty)$, then (1.1) has a unique solution $\rho = (\rho^+, \rho^-) \in L^\infty([0,T]; Y_{r,p})$, where the time $T > 0$ depends only on $\|\rho_0\|_{r,p}$. Moreover, the solution ρ satisfies

$$\rho \in Lip([0,T]; Y_{r-1,p}).$$

In order to prove this theorem, we strongly use the fact that the Riesz transforms are continuous on $C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ for $r \in [0, +\infty)$, $p \in (1, +\infty)$. This result ensures that the velocity vector field remains bounded on $C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. Using this property and some commutator estimates, we can prove that there exists some $T > 0$ such that the solution ρ^n of an approached system of (1.1) (see system (4.27) in Subsection 4.2) is uniformly bounded in $L^\infty([0,T]; Y_{r,p})$ for $r > 1$, $1 < p < +\infty$. Finally, we show that the sequence of the approximate solutions ρ^n is a Cauchy sequence in $L^\infty([0,T]; Y_{r-1,p})$, which gives the local existence and uniqueness of the solution of (1.1).

The next theorem treats the general case $L \in \mathbb{R}$.

Theorem 1.2 (Local existence and uniqueness, case $L \in \mathbb{R}$)
Consider the equation (1.1) corresponding to initial data (1.2), where $L \in \mathbb{R}$ and $\bar{\rho}_0 = (\bar{\rho}_0^+, \bar{\rho}_0^-) \in Y_{r,p}$. If $r > 1$ and $1 < p < +\infty$, then (1.1) has a unique solution $\rho = (\rho^+, \rho^-) \in L^\infty([0,T]; Y_{r,p})$, where the time $T > 0$ depends only on L and $\|\bar{\rho}_0\|_{r,p}$. Moreover,

$$\rho^\pm(x_1, x_2, t) = \bar{\rho}^\pm(x_1, x_2, t) + Lx_1,$$

where

$$\bar{\rho} = (\bar{\rho}^+, \bar{\rho}^-) \in Lip([0,T]; Y_{r-1,p}).$$

Remark 1.3 *If at the initial time we have $\dfrac{\partial \rho^\pm}{\partial x_1}(\cdot, \cdot, t = 0) \geq 0$ two positive quantities, then this remains true for $0 \leq t \leq T$, i.e., $\dfrac{\partial \rho^\pm}{\partial x_1} \geq 0$ for all $(x,t) \in \mathbb{R} \times [0,T]$.*

Related to our analysis in the present paper, we get the following theorem as a by-product.

Theorem 1.4 (Global existence and uniqueness for linear transport equations)
Take $g_0 \in C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ and $v = (v^1, v^2) \in L^\infty([0,T); Y_{r,p})$ for all $T > 0$, $r > 1$ and $1 < p < +\infty$. Then, there exists a unique solution

$$g \in L^\infty([0,T); C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)) \cap Lip([0,T); C^{r-1}(\mathbb{R}^2) \cap L^p(\mathbb{R}^2))$$

of the linear transport equation

$$
\begin{cases}
\dfrac{\partial g}{\partial t} + v \cdot \nabla g = 0 \quad on \quad \mathbb{R}^2 \times (0,T) \\
\\
g(x,0) = g_0(x) \qquad on \quad \mathbb{R}^2.
\end{cases}
\tag{1.4}
$$

1.3 Organization of the paper

This paper is organized as follows. In Section 2, we recall the characterization of Hlder spaces and gather several important estimates. In particular, the boundedness of Riesz transforms on $C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$ is established. Section 3 presents two key commutator estimates (Lemma 3.1). Finally in Section 4, we prove Theorem 1.4 and a basic *a priori* estimate. Then, thanks to this *a priori* estimate, we give in Subsections 4.2 and 4.3 the proofs of Theorems 1.1 and 1.2 respectively.

2 Some results on Hlder-Zygmund spaces

This is a preparatory section in which we recall some results on Hlder-Zygmund spaces, and gather several estimates that will be used in the subsequent sections. A major part of the following results can be found in Meyer [16] and Meyer, Coifman [17].
We start with a dyadic decomposition of $\mathbb{R}^d$, where $d > 0$ is an integer. To this end, we take an arbitrary radial function $\chi \in C_0^\infty(\mathbb{R}^d)$, such that

$$
supp\,\chi \subset \left\{\xi : |\xi| \le \frac{4}{3}\right\}, \quad \chi \equiv 1 \text{ for } |\xi| \le \frac{3}{4}, \quad \|\chi\|_{L^1} = 1.
$$

It is a classical result that, for $\phi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$, we have $\phi \in C_0^\infty(\mathbb{R}^d)$ and

$$
supp\,\phi \subset \left\{\xi : \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\},
$$

$$
\chi(\xi) + \sum_{j \ge 0} \phi(2^{-j}\xi) = 1, \quad \text{for all } \xi \in \mathbb{R}^d.
$$

For the purpose of isolating different Fourier frequencies, define the operators Δ_i for $i \in \mathbb{Z}$ as follows:

$$
\Delta_i f =
\begin{cases}
0 & \text{if } i \le -2, \\
\\
\chi(D) f = \displaystyle\int \check{\chi}(y) f(x-y)\,dy & \text{if } i = -1, \\
\\
\phi(2^{-i}D) f = 2^{id} \displaystyle\int \check{\phi}(2^i y) f(x-y)\,dy & \text{if } i \ge 0,
\end{cases}
\tag{2.5}
$$

where $\check{\chi}$ and $\check{\phi}$ are the inverse Fourier transforms of χ and ϕ, respectively.

For $i \in \mathbb{Z}$, S_i is the sum of Δ_j with $j \le i-1$, i.e.

$$S_i f = \Delta_{-1} f + \Delta_0 f + \Delta_1 f + \cdots + \Delta_{i-1} f = 2^{id} \int \check{\chi}(2^i y) f(x-y) dy.$$

It can be shown for any tempered distribution f that $S_i f \to f$ in the distributional sense, as $i \to \infty$.

For any $r \in \mathbb{R}$ and $p, q \in [1, \infty]$, the inhomogeneous Besov space $B^r_{p,q}(\mathbb{R}^d)$ consists of all tempered distributions f such that the sequence $\{2^{jr} \|\Delta_j f\|_{L^p}\}_{j \in \mathbb{Z}}$ belongs to $l^q(\mathbb{Z})$. When both p and q are equal to ∞, the Besov space $B^r_{p,q}(\mathbb{R}^d)$ reduces to the inhomogeneous Hlder-Zygmund space $C^r(\mathbb{R}^d)$, i.e. $B^r_{\infty,\infty}(\mathbb{R}^d) = C^r(\mathbb{R}^d)$. More explicitly, $C^r(\mathbb{R}^d)$ with $r \in \mathbb{R}$ contains any function f satisfying

$$\|f\|_{C^r} = \sup_{j \in \mathbb{Z}} 2^{jr} \|\Delta_j f\|_{L^\infty} < \infty. \tag{2.6}$$

It is easy to check that $C^r(\mathbb{R}^d)$ endowed with the norm defined in (2.6) is a Banach space.

For $r \ge 0$, $C^r(\mathbb{R}^d)$ is closely related to the classical Hlder space $\tilde{C}^r(\mathbb{R}^d)$ equipped with the norm

$$\|f\|_{\tilde{C}^r} = \sum_{|\beta| \le [r]} \left\|\partial^\beta f\right\|_{L^\infty} + \sup_{x \ne y} \frac{\left|\partial^{[r]} f(x) - \partial^{[r]} f(y)\right|}{|x-y|^{r-[r]}}. \tag{2.7}$$

In fact, if r is not an integer, then the norm (2.6) and (2.7) are equivalent, and $C^r(\mathbb{R}^d) = \tilde{C}^r(\mathbb{R}^d)$. The proof for this equivalence is classical and can be found in Chemin [7]. When r is an integer, say $r = k$, $\tilde{C}^k(\mathbb{R}^d)$ is the space of bounded functions with bounded j-th derivatives for any $j \le k$. In particular, $\tilde{C}^1(\mathbb{R}^d)$ contains the usual Lipschitz functions and is sometimes denoted by $Lip(\mathbb{R}^d)$. As a consequence of Bernstein's Lemma (stated below), $\tilde{C}^r(\mathbb{R}^d)$ is a subspace of $C^r(\mathbb{R}^d)$. Explicit examples can be constructed to show that such an inclusion is genuine. In addition, according to Proposition 2.2, $\tilde{C}^r(\mathbb{R}^d)$ includes $C^{r+\varepsilon}(\mathbb{R}^d)$ for any $\varepsilon > 0$. In summary, for any integer $k \ge 0$ and $\varepsilon > 0$,

$$C^{k+\varepsilon}(\mathbb{R}^d) \subset \tilde{C}^k(\mathbb{R}^d) \subset C^k(\mathbb{R}^d).$$

Proposition 2.1 (Bernstein's Lemma, *Meyer [16]* **)**
Let $d > 0$ be an integer and $\alpha_1 > \alpha_2 > 0$ be two real numbers.
1) If $1 \le p \le q \le \infty$ and supp $\hat{f} \subset \left\{\xi \in \mathbb{R}^d : |\xi| \le \alpha_1 2^j\right\}$, then

$$\max_{|\alpha|=k} \|\partial^\alpha f\|_{L^q} \le C 2^{jk + d(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p},$$

where $C > 0$ is a constant depending only on k and α_1.
2) If $1 \le p \le \infty$ and supp $\hat{f} \subset \left\{\xi \in \mathbb{R}^d : \alpha_1 2^j \le |\xi| \le \alpha_2 2^j\right\}$, then

$$C^{-1} 2^{jk} \|f\|_{L^p} \le \max_{|\alpha|=k} \|\partial^\alpha f\|_{L^p} \le C 2^{jk} \|f\|_{L^p},$$

where $C > 0$ is a constant depending only on k, α_1 and α_2.

Proposition 2.2 (Logarithmic Sobolev inequalities in Hlder-Zygmund space, *Kozono et al. [15, Th 2.1]*)
Let $d > 0$ be an integer. There exists a constant $C = C(d)$ such that for any $\varepsilon > 0$ and $f \in C^\varepsilon(\mathbb{R}^d)$, we have

$$\|f\|_{L^\infty} \leq \frac{C}{\varepsilon} \|f\|_{C^0} \log\left(e + \frac{\|f\|_{C^\varepsilon}}{\|f\|_{C^0}}\right) \leq \frac{C}{\varepsilon} \|f\|_{C^\varepsilon}. \tag{2.8}$$

In the system (1.1), the velocity field u is determined by $\rho^+ - \rho^-$ through the $2D$ Riesz transforms. These Riesz transforms do not map a $C^r(\mathbb{R}^d)$ Hlder-Zygmund space to itself, but their action on $C^r(\mathbb{R}^d)$ is indeed bounded in $C^r(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for $p \in (1, +\infty)$ (see Proposition 2.4). We first recall a general result concerning the boudedness of Fourier multiplier operators on Hlder spaces.

Proposition 2.3 (Fourier multiplier operators on Hlder spaces, *Meyer [16]*)
Let $d > 0$ be an integer and F be an infinitely differentiable function on $\mathbb{R}^d$. Assume that for some $R > 0$ and $m \in \mathbb{R}$, we have

$$F(\lambda\xi) = \lambda^m F(\xi)$$

for any $\xi \in \mathbb{R}^d$ with $|\xi| > R$ and $\lambda \leq 1$. Then the Fourier multiplier operator $F(D)$ maps continuously $C^r(\mathbb{R}^d)$ into $C^{r-m}(\mathbb{R}^d)$ for any $r \in \mathbb{R}$.

Proposition 2.4 (Boundedness of Riesz transforms on $C^r(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$)
Let $r \in \mathbb{R}$ and $p \in (1, +\infty)$. Then there exists a positive constant C depending only on r and p such that

$$\|R_k f\|_{C^r} \leq C \|f\|_{C^r \cap L^p}$$

where $k = 1, 2$.

Proof of Proposition 2.4:
Using the operator Δ_{-1} defined in (2.5), we divide $R_k f$ into two parts,

$$R_k f = \Delta_{-1} R_k f + (1 - \Delta_{-1}) R_k f. \tag{2.9}$$

Since *supp* $\chi(\xi) \cap$ *supp* $\phi(2^{-j}\xi) = \emptyset$ for $j \geq 1$, the operator $\Delta_j \Delta_{-1} = 0$ when $j \geq 1$. Thus, according to (2.6),

$$\begin{aligned} \|\Delta_{-1} R_k f\|_{C^r} &= \sup\nolimits_{j\in\mathbb{Z}} 2^{jr} \|\Delta_j \Delta_{-1} R_k f\|_{L^\infty} \\ &= \max\left[2^{-r} \|\Delta_{-1}\Delta_{-1} R_k f\|_{L^\infty}, \|\Delta_0 \Delta_{-1} R_k f\|_{L^\infty}\right] \\ &\leq \max\left[1, 2^{-r}\right] \|\Delta_{-1} R_k f\|_{L^\infty}. \end{aligned}$$

Let q be the conjugate of p, namely $\frac{1}{p} + \frac{1}{q} = 1$. It then follow, since Riesz transforms are bounded on $L^p(\mathbb{R}^d)$, that for all $p \in (1, +\infty)$:

$$\begin{aligned} \|\Delta_{-1} R_k f\|_{C^r} &\le \max[1, 2^{-r}] \|\check{\chi} * R_k f\|_{L^\infty} \\ &\le \max[1, 2^{-r}] \|\check{\chi}\|_{L^q} \|R_k f\|_{L^p} \\ &= C \|f\|_{L^p} \end{aligned}$$

where C is a constant depending only on r and p. To estimate the second part in (2.9), we apply Proposition 2.3 with $F(\xi) = (1 - \chi(\xi))\dfrac{\xi_k}{|\xi|}$ and $m = 0$, and hence we conclude that it maps $C^r(\mathbb{R}^d)$ into $C^r(\mathbb{R}^d)$. This gives the proof of Proposition 2.4. $\square$

Finally, we recall the notion of Bony's paraproduct (see Bony [3]). The usual product uv of two functions u and v can be decomposed into three parts. More precisely, using $v = \sum_{j \in \mathbb{Z}} \Delta_j v$, $u = \sum_{j \in \mathbb{Z}} \Delta_j u$ and

$$\Delta_j \Delta_k v = 0 \text{ if } |j - k| \ge 1, \quad \Delta_j (S_{k-1} v \Delta_k v) = 0 \text{ if } |j - k| \ge 5,$$

we can write

$$uv = T_u v + T_v u + R(u, v),$$

where

$$T_u v = \sum_{j \ge 1} S_{j-1}(u) \Delta_j v, \qquad R(u, v) = \sum_{|i-j| \le 1} \Delta_i u \Delta_j v.$$

We remark that the previous decomposition allows one to distinguish different types of terms in the product of uv. The Fourier frequencies of u and v in $T_u v$ and $T_v u$ are separated from each other while those of the terms in $R(u, v)$ are close to each other. Using this decomposition, one can show that

$$\|uv\|_{C^s} \le \|u\|_{C^s} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|v\|_{C^s} \quad \text{for} \quad s > 0. \tag{2.10}$$

For the prove of (2.10) see Chen [8, Prop. 5.1].

3 Two commutator estimates

Two major commutator estimates are stated and proved in this section. We remark that this commutator estimates was often used to resolve the Navier-Stokes equations (see for instance Cannone et al. [5], [6]). Here, we apply these techniques on our system (1.1).

***Lemma* 3.1** (L^∞ **commutator estimates)**
Let $j \geq -1$ be an integer and $r > 0$. Then, for some absolute constant C, we have
1)

$$\left\| \left[u\frac{\partial}{\partial x_\alpha}, \Delta_j \right] f \right\|_{L^\infty} \leq C2^{-jr} \left(\left\| \frac{\partial f}{\partial x_\alpha} \right\|_{L^\infty} \|u\|_{C^r} + \|\nabla u\|_{L^\infty} \|f\|_{C^r} \right), \quad \textit{for} \quad \alpha = 1, 2,$$

2)

$$\left\| \left[u\frac{\partial}{\partial x_\alpha}, \Delta_j \right] f \right\|_{L^\infty} \leq C2^{-jr} \left(\|f\|_{L^\infty} \|u\|_{C^{r+1}} + \|\nabla u\|_{L^\infty} \|f\|_{C^r} \right), \quad \textit{for} \quad \alpha = 1, 2,$$

where the bracket $[\ ,\ ]$ *represents the commutator, namely*

$$\left[u\frac{\partial}{\partial x_\alpha}, \Delta_j \right] f = u\frac{\partial}{\partial x_\alpha}(\Delta_j f) - \Delta_j \left(u\frac{\partial f}{\partial x_\alpha} \right), \quad \textit{for} \quad \alpha = 1, 2. \tag{3.11}$$

Proof of Lemma 3.1:
Proof of 1): Using the paraproduct notations T and R, we decompose $\left[u\frac{\partial}{\partial x_\alpha}, \Delta_j \right] f$, for $\alpha = 1, 2$, into five parts,

$$\left[u\frac{\partial}{\partial x_\alpha}, \Delta_j \right] \rho = I_1 + I_2 + I_3 + I_4 + I_5,$$

where

$$I_1 = \left[T_u\frac{\partial}{\partial x_\alpha}, \Delta_j \right] f = T_u \left(\frac{\partial}{\partial x_\alpha}(\Delta_j f) \right) - \Delta_j \left(T_u\frac{\partial f}{\partial x_\alpha} \right),$$
$$I_2 = -\Delta_j T_{\frac{\partial f}{\partial x_\alpha}} u,$$
$$I_3 = T_{\frac{\partial(\Delta_j f)}{\partial x_\alpha}} u,$$
$$I_4 = R\left(u, \frac{\partial(\Delta_j f)}{\partial x_\alpha} \right),$$
$$I_5 = -\Delta_j R\left(u, \frac{\partial f}{\partial x_\alpha} \right).$$

Back to the definition of T, we can write

$$\begin{aligned} I_1 &= \sum_{k\geq 1} S_{k-1}(u)\Delta_j \frac{\partial(\Delta_k f)}{\partial x_\alpha} - \Delta_j \left(\sum_{k\geq 1} S_{k-1}(u)\Delta_k \frac{\partial f}{\partial x_\alpha} \right) \\ &= \sum_{k\geq 1} \left[S_{k-1}(u)\Delta_j \frac{\partial(\Delta_k f)}{\partial x_\alpha} - \Delta_j S_{k-1}(u) \frac{\partial(\Delta_k f)}{\partial x_\alpha} \right]. \end{aligned} \tag{3.12}$$

Since $\Delta_j\Delta_k = 0$ for $|j-k| > 1$ and

$$supp\left[S_{k-1}(u)\widehat{\frac{\partial(\Delta_k f)}{\partial x_\alpha}}\right] \subset \left\{\xi : \frac{1}{3}2^{k-2} \le |\xi| \le \frac{5}{3}2^{k+1}\right\},$$

the sum in (3.12) only involves those terms with k satisfying $|j-k| \le 4$. We only take $j \ge 0$ since the case $j=-1$ can be handled similarly. Applying the definition of Δ_j in (2.5), we obtain

$$\begin{aligned} I_1 &= \sum_{|j-k|\le 4} 2^{jd} \int \check{\phi}(2^j(x-y))\left[S_{k-1}(u(x)) - S_{k-1}(u(y))\right] \frac{\partial(\Delta_k f)}{\partial x_\alpha}(y)dy \\ &= \sum_{|j-k|\le 4} \int \check{\phi}(y)\left[S_{k-1}(u(x)) - S_{k-1}(u(x-2^{-j}y))\right] \frac{\partial(\Delta_k f)}{\partial x_\alpha}(x-2^{-j}y)dy. \end{aligned}$$

Using the fact that $\check{\phi} \in S(\mathbb{R}^d)$ and S_j are continuous from L^∞ onto itself, we get for $r \in \mathbb{R}$ and an absolute constant C:

$$\begin{aligned} \|I_1\|_{L^\infty} &\le C2^{-j}\|\nabla u\|_{L^\infty} \left\|\frac{\partial(\Delta_j f)}{\partial x_\alpha}\right\|_{L^\infty} \\ &\le C\|\nabla u\|_{L^\infty} \|\Delta_j f\|_{L^\infty} \\ &\le C2^{-jr}\|\nabla u\|_{L^\infty} \|f\|_{C^r}, \end{aligned} \tag{3.13}$$

where we have used Proposition 2.1 in the second inequality. To estimate I_2 and I_3, we first write them as

$$I_2 = -\sum_{|j-k|\le 4} \Delta_j\left(S_{k-1}\left(\frac{\partial f}{\partial x_\alpha}\right)\Delta_k u\right), \qquad I_3 = \sum_{|j-k|\le 4} S_{k-1}\left(\frac{\partial(\Delta_j f)}{\partial x_\alpha}\right)\Delta_k u.$$

Similarly, only terms with k satisfying $|j-k| \le 4$ are considered in the above sums. Thus, since Δ_j and S_j are continuous from L^∞ onto itself, we have for $r \in \mathbb{R}$:

$$\|I_2\|_{L^\infty} \le C\|\Delta_j u\|_{L^\infty} \left\|S_j \frac{\partial f}{\partial x_\alpha}\right\|_{L^\infty} \le C2^{-jr}\|u\|_{C^r} \left\|\frac{\partial f}{\partial x_\alpha}\right\|_{L^\infty}, \tag{3.14}$$

$$\|I_3\|_{L^\infty} \le C\|\Delta_j u\|_{L^\infty} \left\|S_j \Delta_j \frac{\partial f}{\partial x_\alpha}\right\|_{L^\infty} \le C2^{-jr}\|u\|_{C^r} \left\|\frac{\partial f}{\partial x_\alpha}\right\|_{L^\infty}, \tag{3.15}$$

where the $C's$ in the above inequalities are absolute constants. From the definition of R, we have

$$I_4 = \sum_{|k_1-k_2|\le 1} \left(\Delta_{k_1}(u)\Delta_{k_2}\left(\frac{\partial(\Delta_j f)}{\partial x_\alpha}\right)\right).$$

Obviously, only a finite number of terms involved in the above sums are non-zeros. Then,

$$\|I_4\|_{L^\infty} \leq C\|\Delta_j u\|_{L^\infty} \left\|\Delta_j \frac{\partial f}{\partial x_\alpha}\right\|_{L^\infty} \leq C2^{-jr}\|u\|_{C^r} \left\|\frac{\partial f}{\partial x_\alpha}\right\|_{L^\infty}. \tag{3.16}$$

Note that from the definition of R and Δ_j, $j \geq -1$, we can write I_5 as

$$I_5 = -\sum_{k\geq j-3} \sum_{k_1=k-1}^{k+1} \Delta_j \left(\Delta_k(u)\Delta_{k_1}\left(\frac{\partial f}{\partial x_\alpha}\right)\right).$$

Therefore, for an absolute constant C, we have:

$$\begin{aligned} \|I_5\|_{L^\infty} &\leq C \sum_{k\geq j-3} \sum_{k_1=k-1}^{k+1} \|\Delta_k u\|_{L^\infty} \left\|\Delta_{k_1} \frac{\partial f}{\partial x_\alpha}\right\|_{L^\infty} \\ &\leq C\left\|\frac{\partial f}{\partial x_\alpha}\right\|_{L^\infty} \|u\|_{C^r} \sum_{k\geq j-3} 2^{-kr} \\ &\leq C2^{-jr}\left\|\frac{\partial f}{\partial x_\alpha}\right\|_{L^\infty} \|u\|_{C^r}. \end{aligned} \tag{3.17}$$

Gathering the estimates in (3.13)-(3.17), we establish the desired inequality in 1).

Proof of 2): As in the proof of 1), we decompose $\left[u\frac{\partial}{\partial x_\alpha}, \Delta_j\right] f$ as the sum of I_1, I_2, I_3, I_4 and I_5. The estimate on I_1 remains untouched, while different bounds are needed for I_2, I_3, I_4 and I_5. Indeed, for $j \geq 1$:

$$\begin{aligned} \|I_2\|_{L^\infty} &\leq C\|\Delta_j u\|_{L^\infty} \left\|\frac{\partial S_{j-1} f}{\partial x_\alpha}\right\|_{L^\infty} \\ &\leq C2^j\|\Delta_j u\|_{L^\infty} \|S_{j-1} f\|_{L^\infty} \\ &\leq C2^{-jr}\|u\|_{C^{r+1}} \|f\|_{L^\infty}, \end{aligned} \tag{3.18}$$

where we have used Proposition 2.1 in the second inequality. I_3 and I_4 can be similarly estimated as I_2:

$$\begin{aligned} \|I_3\|_{L^\infty} &\leq C\left\|\frac{\partial \Delta_j f}{\partial x_\alpha}\right\|_{L^\infty} \|\Delta_j u\|_{L^\infty} \leq C2^j\|\Delta_j u\|_{L^\infty} \|\Delta_j f\|_{L^\infty} \\ &\leq C2^{-jr}\|u\|_{C^{r+1}} \|f\|_{L^\infty}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} \|I_4\|_{L^\infty} &\leq C\|\Delta_j u\|_{L^\infty} \left\|\frac{\partial \Delta_j f}{\partial x_\alpha}\right\|_{L^\infty} \leq C2^j\|\Delta_j u\|_{L^\infty} \|\Delta_j f\|_{L^\infty} \\ &\leq C2^{-jr}\|u\|_{C^{r+1}} \|f\|_{L^\infty}. \end{aligned} \tag{3.20}$$

Finally, we have

$$
\begin{aligned}
\|I_5\|_{L^\infty} &\le C \sum_{k\ge j-3} \sum_{k_1=k-1}^{k+1} \|\Delta_k u\|_{L^\infty} \left\| \frac{\partial \Delta_{k_1} f}{\partial x_\alpha} \right\|_{L^\infty} \\
&\le C \sum_{k\ge j-3} \sum_{k_1=k-1}^{k+1} 2^{k_1} \|\Delta_k u\|_{L^\infty} \, \|\Delta_{k_1} f\|_{L^\infty} \\
&\le C \|f\|_{L^\infty} \sum_{k\ge j-3} 2^k \|\Delta_k u\|_{L^\infty} \\
&\le C \|f\|_{L^\infty} \, \|u\|_{C^{r+1}} \sum_{k\ge j-3} 2^{-kr} \\
&\le C 2^{-jr} \, \|f\|_{L^\infty} \, \|u\|_{C^{r+1}}.
\end{aligned} \tag{3.21}
$$

Combining (3.18)-(3.21) yields 2).

□

4 Local existence and uniqueness results

This section is devoted to the proofs of Theorems 1.1 and 1.2. For the sake of a clear presentation, we divide it into three subsections. In the first subsection, we show a basic *a priori* estimate and we prove Theorem 1.4. With the aid of this estimate, we prove Theorems 1.1 and 1.2 in the next subsections.

4.1 An *a priori* estimate

Proposition 4.1 (*A priori* **estimate)**
Let $r > 1$ and $p > 1$. For all $T > 0$, $\rho_0 = (\rho_0^+, \rho_0^-) \in Y_{r,q}$ and $u \in L^\infty([0,T); C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2))$, there exists a unique solution $\rho = (\rho^+, \rho^-) \in L^\infty([0,T); Y_{r,p})$ of the following system of linear transport equations

$$
\frac{\partial \rho^\pm}{\partial t} \pm u \frac{\partial \rho^\pm}{\partial x_1} = 0 \tag{4.22}
$$

Moreover, for all $t \in [0,T]$, we have

$$
\|\rho(\cdot,t)\|_{r,p} \le \|\rho_0\|_{r,p} \, exp \left(C \int_0^t \|u(\cdot,\tau)\|_{C^r \cap L^p} \, d\tau \right)
$$

where $C > 0$ is a constant depending only on r and p.

Proof of Proposition 4.1:
From the fact that $u(\cdot,t)\in C^r(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$, for $t\in[0,T]$, we can define the flow map $X^{\pm}(\cdot,t)$ satisfying

$$\begin{cases} \dfrac{\partial X^{\pm}(x,t)}{\partial t} = \pm\bar{u}(X(x,t),t), \quad \text{where} \quad \bar{u}=(u,0), \\ \\ X^{\pm}(x,0)=x. \end{cases} \tag{4.23}$$

By the characteristics method, we know that, if $(X^{\pm})^{-1}$ is the inverse function of $X^{\pm}$ with respect to x, then $\rho^{\pm}(x,t)=\rho_0^{\pm}((X^{\pm})^{-1}(x,t))$ is the unique solution of system (4.22) (see Serre [20] for more details).

Let $j\geq -1$. Applying the operator Δ_j to both sides of the system (1.1) yields

$$\frac{\partial \Delta_j\rho^{\pm}}{\partial t} \pm u\frac{\partial \Delta_j\rho^{\pm}}{\partial x_1} = \pm\left[u\frac{\partial}{\partial x_1},\Delta_j\right]\rho^{\pm},$$

where $\left[u\dfrac{\partial}{\partial x_1},\Delta_j\right]\rho^{\pm}$ is defined in (3.11). This equation can be rewritten in the following form

$$\Delta_j\rho^{\pm}(x,t)=\Delta_j\rho_0^{\pm}((X^{\pm})^{-1}(x,t)) \pm \int_0^t \left[u\frac{\partial}{\partial x_1},\Delta_j\right]\rho^{\pm}(X^{\pm}((X^{\pm})^{-1}(x,t),s),s)ds.$$

Taking the L^∞-norm of both sides of this equality, we get:

$$\left\|\Delta_j\rho^{\pm}(\cdot,t)\right\|_{L^\infty} \leq \left\|\Delta_j\rho_0^{\pm}\right\|_{L^\infty} + \int_0^t \left\|\left[u\frac{\partial}{\partial x_1},\Delta_j\right]\rho^{\pm}(\cdot,s)\right\|_{L^\infty} ds.$$

Applying Lemma 3.1 (1), we obtain

$$\left\|\rho^{\pm}(\cdot,t)\right\|_{C^r} \leq \left\|\rho_0^{\pm}\right\|_{C^r} + C\int_0^t \left(\left\|\frac{\partial\rho^{\pm}}{\partial x_1}(\cdot,s)\right\|_{L^\infty} \|u(\cdot,s)\|_{C^r} + \|\nabla u(\cdot,s)\|_{L^\infty}\left\|\rho^{\pm}(\cdot,s)\right\|_{C^r}\right) ds.$$

According to (2.8), we know that for $r>1$ and a constant $C=C(r)>0$, we have:

$$\left\|\frac{\partial\rho^{\pm}}{\partial x_1}\right\|_{L^\infty} \leq C\left\|\rho^{\pm}\right\|_{C^1}\log\left(e+\frac{\|\rho^{\pm}\|_{C^r}}{\|\rho^{\pm}\|_{C^1}}\right) \leq C\left\|\rho^{\pm}\right\|_{C^r}.$$

In a similar way, we can obtain $\|\nabla u\|_{L^\infty}\leq C\|u\|_{C^r}$. Therefore, for $C=C(r)>0$,

$$\begin{aligned} \|\rho^{\pm}(\cdot,t)\|_{C^r} &\leq \left\|\rho_0^{\pm}\right\|_{C^r} + C\int_0^t \|u(\cdot,s)\|_{C^r}\left\|\rho^{\pm}(\cdot,s)\right\|_{C^r} ds \\ &\leq \max_{\pm}\left(\left\|\rho_0^{\pm}\right\|_{C^r}\right) + C\int_0^t \|u(\cdot,s)\|_{C^r}\|\rho(\cdot,s)\|_{r,p} ds. \end{aligned}$$

Moreover, integrating in time the system (1.1), we get the following L^p estimate:

$$\begin{aligned} \|\rho^{\pm}(\cdot,t)\|_{L^p} &\le \left\|\rho_0^{\pm}\right\|_{L^p} + \int_0^t \|u(\cdot,s)\|_{L^p} \left\|\frac{\partial \rho^{\pm}}{\partial x_1}(\cdot,s)\right\|_{L^\infty} ds \\ &\le \left\|\rho_0^{\pm}\right\|_{L^p} + C\int_0^t \|u(\cdot,s)\|_{L^p} \|\rho(\cdot,s)\|_{r,p}\, ds, \end{aligned}$$

where we have used Hlder inequality in the first line and (2.8) in the second line. Now, adding the two previous inequalities, we obtain

$$\|\rho(\cdot,t)\|_{r,p} \le \|\rho_0\|_{r,p} + C\int_0^t \|u(\cdot,s)\|_{C^r\cap L^p} \|\rho(\cdot,s)\|_{r,p}\, ds,$$

where $\|\cdot\|_{C^r\cap L^p} = \|\cdot\|_{C^r} + \|\cdot\|_{L^p}$. By Gronwall's Lemma and Proposition 2.4,

$$\|\rho(\cdot,t)\|_{r,p} \le \|\rho_0\|_{r,p}\, exp\left(C\int_0^t \|u(\cdot,s)\|_{C^r\cap L^p}\, ds\right). \tag{4.24}$$

Which completes the proof of Proposition 4.1.

$\square$

Proof of Theorem 1.4:
The proof of Theorem 1.4 is a consequence of the proof of Proposition 4.1. Indeed, just consider the characteristic equation

$$\begin{cases} \dfrac{\partial X(x,t)}{\partial t} = v(X(x,t),t), \\ X(x,0) = x. \end{cases} \tag{4.25}$$

Then, as in the proof of Proposition 4.1, we use the commutator estimates proved in Lemma 3.1 (1), to show the following estimate:

$$\|g(\cdot,t)\|_{C^r\cap L^p} \le \|g_0\|_{C^r\cap L^p}\, exp\left(C\int_0^t \|v(\cdot,s)\|_{r,p}\, ds\right), \tag{4.26}$$

which proves the result.

$\square$

4.2 Proof of Theorem 1.1

The proof starts with the construction of a successive approximation sequence $\{\rho^n = (\rho^{+,n}, \rho^{-,n})\}_{n\ge 1}$ satisfying

$$\begin{cases} \rho^1 = (\rho_0^+, \rho_0^-) = \rho_0, \\ \dfrac{\partial \rho^{\pm,n+1}}{\partial t} \pm u^n \dfrac{\partial \rho^{\pm,n+1}}{\partial x_1} = 0, \quad \text{on} \quad \mathbb{R}^2 \times (0,T) \\ u^n = R_1^2 R_2^2 \left(\rho^{+,n} - \rho^{-,n}\right), \\ \rho^{\pm,n+1}(x,0) = \rho_0^{\pm}. \end{cases} \tag{4.27}$$

First of all, according to Proposition 2.4, $\rho^1 \in Y_{r,p}$ implies that $u^1 \in C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. Thus, applying Proposition 4.1, we can prove that, for all $T > 0$, there exists a unique solution $\rho^2 \in L^\infty([0,T); Y_{r,p})$ for (4.27) with $n = 2$. Arguing in a similar manner we can show that this approached problem (4.27) has a unique solution ρ^n for all $n \geq 1$.

The rest of the proof can be divided into two major steps. The first step establishes the existence of $T_1 > 0$ such that $\{\rho^n = (\rho^{+,n}, \rho^{-,n})\}_{n\geq 1}$ is uniformly bounded in $Y_{r,p}$ for any $t \in [0, T_1]$. The second step shows that for some $T_2 \in [0, T_1]$, we have $\{\rho^n = (\rho^{+,n}, \rho^{-,n})\}_{n\geq 1}$ is a Cauchy sequence in $C([0, T_2], Y_{r-1,p})$.

Step 1 (A uniform bound): Using similar arguments as in the proof of Proposition 4.1, estimate (4.24) yields, by Proposition 2.4, the following bound on $\{\rho^n = (\rho^{+,n}, \rho^{-,n})\}_{n\geq 1}$:

$$\begin{aligned} \|\rho^{n+1}(\cdot,t)\|_{r,p} \leq \quad & \|\rho_0\|_{r,p} \, exp\left(C_0 \int_0^t \|u(\cdot,s)\|_{C^r \cap L^p} \, ds\right), \\ & \|\rho_0\|_{r,p} \, exp\left(C_0 \int_0^t \|\rho^n(\cdot,s)\|_{r,p} \, ds\right), \end{aligned}$$

where $r > 1$, $p \in (1, +\infty)$ and $C_0 = C_0(r,p)$. Choose T_1 and M satisfying

$$M = 2\,\|\rho_0\|_{r,p} \quad \text{and} \quad \left(exp(C_0 T_1 M) \leq 2 \quad \text{or} \quad T_1 = \frac{\ln(2)}{2C_0\,\|\rho_0\|_{r,p}} \right).$$

Then $\|\rho^n(\cdot,t)\|_{r,p} \leq M$ for all $n \geq 1$ and $t \in [0, T_1]$. Since,

$$\left\|\rho^1\right\|_{r,p} \leq \|\rho_0\|_{r,p} < M$$

and $\left\|\rho^k(\cdot,t)\right\|_{r,p} < M$, we obtain

$$\left\|\rho^{n+1}(\cdot,t)\right\|_{r,p} \leq \|\rho_0\|_{r,p} \, exp(C_0 T_1 M) \leq M. \tag{4.28}$$

Furthermore, since $r > 1$, we use (2.10) and Proposition 2.4 to get:

$$
\begin{aligned}
\left\|\frac{\partial \rho^{\pm,n}}{\partial t}\right\|_{C^{r-1}} &\leq \left\|u^n \frac{\partial \rho^{\pm,n+1}}{\partial x_1}\right\|_{C^{r-1}} \\
&\leq \|u^n\|_{C^{r-1}} \left\|\frac{\partial \rho^{\pm,n+1}}{\partial x_1}\right\|_{L^\infty} + \|u^n\|_{L^\infty} \left\|\frac{\partial \rho^{\pm,n+1}}{\partial x_1}\right\|_{C^{r-1}} \\
&\leq C \|u^n\|_{C^{r-1}} \|\rho^{\pm,n+1}\|_{C^r} \\
&\leq CM^2,
\end{aligned}
$$

where $C = C(r)$. We can also check that the following L^p estimate on $\rho^{\pm,n}$ is valid:

$$
\begin{aligned}
\left\|\frac{\partial \rho^{\pm,n}}{\partial t}\right\|_{L^p} &\leq \left\|u^n \frac{\partial \rho^{\pm,n+1}}{\partial x_1}\right\|_{L^p} \leq \|u^n\|_{L^p} \left\|\frac{\partial \rho^{\pm,n+1}}{\partial x_1}\right\|_{L^\infty} \\
&\leq C \|u^n\|_{L^p} \|\rho^{\pm,n+1}\|_{C^r} \leq CM^2,
\end{aligned}
$$

where we have used Hlder inequality in the first line, then (2.8) and Proposition 2.4 in the second line. Adding the two previous inequalities, we deduce that

$$
\left\|\frac{\partial \rho^{\pm,n}}{\partial t}\right\|_{C^{r-1}} + \left\|\frac{\partial \rho^{\pm,n}}{\partial t}\right\|_{L^p} \leq CM^2, \tag{4.29}
$$

where $C = C(r)$. Thus, by (4.28)-(4.29), we obtain that

$$
\rho^n \in L^\infty([0,T_1]; Y_{r,p}) \cap Lip([0,T_1]; Y_{r-1,p})
$$

is uniformly bounded.

Step 2 (Cauchy sequence): To show that $\{\rho^n = (\rho^{+,n}, \rho^{-,n})\}_{n\geq 1}$ is a Cauchy sequence in $Y_{r-1,q}$, we consider the difference $\eta^{\pm,n} = \rho^{\pm,n} - \rho^{\pm,n-1}$. Rigorously speaking, we should consider the more general difference $\eta^{\pm,m,n} = \rho^{\pm,m} - \rho^{\pm,n}$, but the analysis for $\eta^{m,n} = (\eta^{+,m,n}, \eta^{-,m,n})$ is parallel to what we shall present for $\eta^n = (\eta^{+,n}, \eta^{-,n})$ and thus we consider η^n for the sake of a concise presentation. It follows from (4.27) that $\{\eta^n = (\eta^{+,n}, \eta^{-,n})\}_{n\geq 1}$ satisfies

$$
\begin{cases}
\eta^{\pm,1} = \rho_0^{\pm}, \\
\dfrac{\partial \eta^{\pm,n+1}}{\partial t} \pm u^n \dfrac{\partial \eta^{\pm,n+1}}{\partial x_1} = \mp w^n \dfrac{\partial \rho^{\pm,n}}{\partial x_1}, \\
w^n = R_1^2 R_2^2 (\eta^{+,n} - \eta^{-,n}), \\
\eta^{\pm,n+1}(x,0) = \eta_0^{\pm,n+1}(x) = 0.
\end{cases} \tag{4.30}
$$

Proceeding as in the proof of Proposition 4.1, we obtain for any integer $j \geq -1$,

$$\begin{aligned}
\|\Delta_j \eta^{\pm,n+1}(\cdot,t)\|_{L^\infty} \;&\leq \underbrace{\int_0^t \left\| \left[u^n \frac{\partial}{\partial x_1}, \Delta_j \right] \eta^{\pm,n+1}(\cdot,s) \right\|_{L^\infty} ds}_{K_1} \\
&+ \underbrace{\int_0^t \left\| \Delta_j \left(w^n \frac{\partial \rho^{\pm,n}}{\partial x_1}(\cdot,s) \right) \right\|_{L^\infty} ds.}_{K_2}
\end{aligned}$$

Estimating K_1 by Lemma 3.1 (2), and K_2 by (2.10), we get:

$$\begin{aligned}
\|\eta^{\pm,n+1}(\cdot,t)\|_{C^{r-1}} \;&\leq C \int_0^t \left(\|\nabla u^n(\cdot,s)\|_{L^\infty} \left\|\eta^{\pm,n+1}(\cdot,s)\right\|_{C^{r-1}} + \|u^n(\cdot,s)\|_{C^r} \left\|\eta^{\pm,n+1}(\cdot,s)\right\|_{L^\infty} \right) ds \\
&+ C \int_0^t \left(\|w^n(\cdot,s)\|_{L^\infty} \left\| \frac{\partial \rho^{\pm,n}}{\partial x_1}(\cdot,s) \right\|_{C^{r-1}} + \|w^n(\cdot,s)\|_{C^{r-1}} \left\| \frac{\partial \rho^{\pm,n}}{\partial x_1}(\cdot,s) \right\|_{L^\infty} \right) ds.
\end{aligned}$$

Since $r > 1$, Proposition 2.2 implies,

$$\|\nabla u^n\|_{L^\infty} \leq C\|u^n\|_{C^r}, \qquad \|\eta^{\pm,n+1}\|_{L^\infty} \leq C\|\eta^{\pm,n+1}\|_{C^{r-1}},$$

$$\left\| \frac{\partial \rho^{\pm,n}}{\partial x_1} \right\|_{L^\infty} \leq C\|\rho^{\pm,n}\|_{C^r} \quad \text{and} \quad \|w^n\|_{L^\infty} \leq C\|w^n\|_{C^{r-1}}.$$

Therefore, for a constant C depending only on r,

$$\begin{aligned}
\|\eta^{\pm,n+1}(\cdot,t)\|_{C^{r-1}} \;&\leq C \int_0^t \|u^n(\cdot,s)\|_{C^r} \left\|\eta^{\pm,n+1}(\cdot,s)\right\|_{C^{r-1}} ds \\
&+ C \int_0^t \|w^n(\cdot,s)\|_{C^{r-1}} \left\|\rho^{\pm,n}(\cdot,s)\right\|_{C^r} ds.
\end{aligned}$$

However, it follows from a basic L^p estimate that

$$\begin{aligned}
\|\eta^{\pm,n+1}(\cdot,t)\|_{L^p} \;&\leq C \int_0^t \|\nabla u^n(\cdot,s)\|_{L^\infty} \left\|\eta^{\pm,n+1}(\cdot,s)\right\|_{L^p} ds \\
&+ C \int_0^t \|w^n(\cdot,s)\|_{L^p} \left\| \frac{\partial \rho^{\pm,n}}{\partial x_1}(\cdot,s) \right\|_{L^\infty} ds \\
&\leq C \int_0^t \|u^n(\cdot,s)\|_{C^r} \left\|\eta^{\pm,n+1}(\cdot,s)\right\|_{L^p} ds \\
&+ C \int_0^t \|w^n(\cdot,s)\|_{L^p} \left\|\rho^{\pm,n}(\cdot,s)\right\|_{C^r} ds.
\end{aligned}$$

Adding the last two inequalities, yields

$$\begin{aligned}\|\eta^{n+1}(\cdot,t)\|_{r-1,p} &\leq C\int_0^t \|u^n(\cdot,s)\|_{C^r\cap L^p}\left\|\eta^{n+1}(\cdot,s)\right\|_{r-1,p}ds\\ &\quad +C\int_0^t \|w^n(\cdot,s)\|_{C^{r-1}\cap L^p}\|\rho^n(\cdot,s)\|_{r,p}\,ds.\end{aligned}$$

The components of w^n are the Riesz transforms of η^n and thus, according to Proposition 2.4:

$$\|w^n\|_{C^{r-1}\cap L^p} \leq C\|\eta^n\|_{r-1,p}.$$

We thus have reached an iterative relationship between $\|\eta^n\|_{r-1,p}$ and $\|\eta^{n+1}\|_{r-1,p}$:

$$\begin{aligned}\|\eta^{n+1}(\cdot,t)\|_{r-1,p} &\leq C_1\int_0^t \|\rho^n(\cdot,s)\|_{r,p}\left\|\eta^{n+1}(\cdot,s)\right\|_{r-1,p}ds\\ &\quad +C_1\int_0^t \|\eta^n(\cdot,s)\|_{r-1,p}\|\rho^n(\cdot,s)\|_{r,p}\,ds,\end{aligned} \tag{4.31}$$

where the constants are labeled as C_1 for the purpose of defining T_2. It has been shown in Step 1 that for $t \leq T_1$,

$$\|\rho^n\|_{r,p} \leq M.$$

Now, choose $T_2 > 0$ satisfying

$$T_2 \leq T_1, \qquad C_1MT_2 \leq \frac{1}{4},$$

we can show that $\{\rho^n(\cdot,t)\}_{n\geq 1}$ is a Cauchy sequence in $Y_{r-1,p}$ for $t \leq T_2$. Indeed, for any given $\varepsilon > 0$, if $\|\eta^n\|_{r-1,p} \leq \varepsilon$ for $t \leq T_2$, then (4.31) implies that:

$$\|\eta^{n+1}\|_{r-1,p} \leq C_1\varepsilon MT_2 + C_1M\int_0^t \|\eta^{n+1}(\cdot,s)\|_{r-1,p}ds,$$

is valid for any $t \leq T_2$. It then follows from Gronwall's inequality that

$$\|\eta^{n+1}\|_{r-1,p} \leq \varepsilon,$$

for any $t \leq T_2$ which completes Step 2.

We conclude from Steps 1 and 2 that there exists $\rho = (\rho^+, \rho^-)$ satisfying

$$\rho \in L^\infty([0,T_2];Y_{r,p}) \cap Lip([0,T_2];Y_{r-1,p})$$

such that ρ^n converges to ρ in $C([0,T_2];Y_{r-1,p})$.
The proof of uniqueness follows directly from Step 2. This completes the proof of Theorem 1.1. □

4.3 Proof of Theorem 1.2

It is worth mentioning that the ideas of the proof of Theorem 1.2 are already contained in the proof of Theorem 1.1.

First of all, we note that for all $L \in \mathbb{R}$, if $\rho^{\pm}$ are solutions of (1.1) then

$$\bar{\rho}^{\pm}(x_1, x_2, t) = \rho^{\pm}(x_1, x_2, t) - Lx_1$$

solves the following system:

$$\begin{cases} \dfrac{\partial \bar{\rho}^{\pm}}{\partial t}(x,t) \pm u \dfrac{\partial \bar{\rho}^{\pm}}{\partial x_1}(x,t) = \mp Lu \quad \text{on } \mathbb{R}^2 \times (0,T), \\ \\ u = R_1^2 R_2^2 (\bar{\rho}^+ - \bar{\rho}^-), \end{cases} \tag{4.32}$$

and respects the following initial data:

$$\bar{\rho}_0^{\pm}(x_1, x_2, t) = \rho_0^{\pm}(x_1, x_2) - Lx_1.$$

Now, to prove Theorem 1.2, it suffices to show that, for all initial data $\bar{\rho}_0^{\pm} \in Y_{r,p}$, the system (4.32) has a unique local solution $\bar{\rho}^{\pm} \in L^{\infty}([0,T); Y_{r,p})$ for $r > 1$ and $p \in (1, +\infty)$.

In order to do this, we proceed as in the proof of Theorem 1.1. We consider the following approached system:

$$\begin{cases} \bar{\rho}^1 = (\bar{\rho_0}^+, \bar{\rho_0}^-) = \bar{\rho}_0, \\ \\ \dfrac{\partial \bar{\rho}^{\pm,n+1}}{\partial t} \pm u^n \dfrac{\partial \bar{\rho}^{\pm,n+1}}{\partial x_1} = \mp Lu^n, \quad \text{on} \quad \mathbb{R}^2 \times (0,T) \\ \\ u^n = R_1^2 R_2^2 \left(\bar{\rho}^{+,n} - \bar{\rho}^{-,n}\right), \\ \\ \bar{\rho}^{\pm,n+1}(x,0) = \bar{\rho_0}^{\pm}. \end{cases} \tag{4.33}$$

We remark that, the only change that appears here, compared to the approached system (4.27) is the right-hand side Lu^n of the second equation of (4.33). However, by Proposition 2.4, we know that this term remains bounded in $L^{\infty}([0,T); C^r(\mathbb{R}^2) \cap L^p(\mathbb{R}^2))$ for $r > 1$ and $p \in (1, +\infty)$. Which permits us to easily follow the same steps of the proof of Theorem 1.1. This finally proves that, for some small $T > 0$, we have on the one hand: the sequence $\bar{\rho}^n = (\bar{\rho}^{+,n}, \bar{\rho}^{-,n})$ is uniformly bounded in $L^{\infty}([0,T); Y_{r,p})$, and on the other hand, this sequence is a Cauchy sequence in $L^{\infty}([0,T); Y_{r-1,p})$. This terminate the proof.

$\square$

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