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MULTIVARIATE AFFINE FRACTAL INTERPOLATION

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ABSTRACT. Fractal interpolation functions capture the irregularity of some data very effectively in comparison with the classical interpolants. They yield a new technique for fitting experimental data sampled from real world signals, which are usually difficult to represent using the classical approaches. The affine fractal interpolants constitute a generalization of the broken line interpolation, which appears as a particular case of the linear self-affine functions for specific values of the scale parameters. We study the $\mathcal{L}^p$ convergence of this type of interpolants for $1 \leq p < \infty$ extending in this way the results available in the literature. In the second part, the affine approximants are defined in higher dimensions via product of interpolation spaces, considering rectangular grids in the product intervals. The associate operator of projection is considered. Some properties of the new functions are established and the aforementioned operator on the space of continuous functions defined on a multidimensional compact rectangle is studied.

Keywords Iterated Function System. Fractals. Fractal Interpolation Functions. Smooth Fractal Function. Fractal Operator.
MSC 28A80. 26C15. 41A20. 41A05. 46B15

1. INTRODUCTION

The features of many real phenomena such as financial series, the distribution of galaxies, the spread of bacterial colonies, real time image synthesis, turbulence of fluids, climate data, bioelectric recordings, snowflake, coastlines, Brownian motion, the surface shapes of mountains, topographies, rocks, clouds and fractures, etc can not be apprehended effectively with the help of the classical interpolant methods because they may not provide an interpolant/approximant with a desired precision. The method of iterated function systems (IFSs) supports the understanding and processing of complex sets with the help of Collage Theorem [1]. To analyze self-referential sets and deal with highly irregular data, Barnsley [1] in 1986 first put forward the concept of fractal functions as the fixed point of the Read-Bajraktarević operator defined on a suitable space of functions. It has become a very powerful tool in areas such as signal processing, multiwavelets, computer graphics, electrochemistry, financial series, acoustics, sociology, and weather forecasting. Due to its potential application in various fields, it has attracted numerously authors, see for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and references quoted therein. A fractal interpolation function [1] is very different from the traditional functions considered so far. It may interpolate a specified data set and has non-integer Hausdorff and Minkowski dimensions. The power of fractal interpolation allows us to generalize

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any other interpolant, both smooth and non-smooth [12, 13]. Barnsley [1] introduced affine fractal interpolation functions (AFIFs) which are obtained as attractors of affine transformations in the plane. The AFIFs have non-integral dimension and can be computed easily. The graphs of these AFIFs can be effectively utilized to approximate image components such as the tops of clouds, the profiles of mountain ranges, horizons over forests, etc. Therefore, it is not surprising that the AFIFs are receiving an increasing intensity of investigation even after three decades of its first pronouncement (see for instance [8, 9, 11, 14, 15, 16, 17, 18]; and references therein). Recent applications of this theory include modeling of discrete sequences as in [19], modeling of speech signals as in [20] and compression of static images as in [21]. In many problems arising in day to day life as in engineering and science, we require approximation methods to reproduce physical reality as close as possible. Fractal functions are not well explored in the field of shape preserving interpolation/approximation. Motivated by theoretical and practical needs, the authors have initiated the study of shape preserving interpolation and approximation using fractal functions [22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

In this paper, we study the $\mathcal{L}^p$ convergence of affine fractal interpolants for $1 \leq p < \infty$ extending in this way the results available in the literature. We also find the $\mathcal{L}^p$-estimates of the affine interpolation error. We define bi-affine fractal functions and the operator $\mathcal{D}^{\alpha\beta}$ of bi-affine fractal interpolation. We prove that $\mathcal{D}^{\alpha\beta}(f) \in Lip1$ with some restrictions on scale vectors. We also prove that the operator $\mathcal{D}^{\alpha\beta}$ is linear, bounded, projection, compact, and has a closed range. We also define the adjoint operator of $\mathcal{D}^{\alpha\beta}$ as $(\mathcal{D}^{\alpha\beta})^*$ and claim that it is also compact and has closed range. The possibility is given to construct tensor product of affine fractal interpolants for the approximation of functions with several variables $f(x_1, x_2, \ldots, x_d)$.

In Section 2.2, we review briefly the notion of FIFs including some basic results and notation of the IFSs. In Section 3, we extend the results of the reference [16] and we study the $\mathcal{L}^p$-convergence of affine fractal interpolants for $1 \leq p < +\infty$. We also provide $\mathcal{L}^p$-estimates of the affine interpolation error. We define two-dimensional fractal interpolants (bi-affine fractal functions) in Section 4 and find some analytical properties of the operator of bi-affine fractal interpolation. We leave the possibility to extend and construct tensor product of affine fractal interpolants for the approximation of functions with several variables $f(x_1, x_2, \ldots, x_d)$ in Section 5.

We have used the following notation throughout the article. Denote by $\mathbb{N}$ the set of natural numbers. For any $r \in \mathbb{N}$, let $\mathbb{N}_r$ denote the segment of $\mathbb{N}$ containing the first r numbers and $\mathbb{N}_r^0 := \mathbb{N}_r \cup \{0\}$. Let us denote $\forall\ f \in \mathcal{C}(I)$, $\|f\|_\infty = sup\{|f(t)| : t \in I\}$ and let $\|\cdot\|$ be the norm of an operator with respect to $\|\cdot\|_\infty$ in $\mathcal{C}(I)$. Let us define $|\alpha|_\infty = \max\{|\alpha_n| : n \in \mathbb{N}_N\}$. For $1 \leq p < +\infty$, let us define the $\mathcal{L}^p$-norm: $\|f\|_p = \left(\int_a^b |f(t)|^p dt\right)^{1/p}$.

2. Background and preliminaries

2.1. Iterated Function Systems. An IFS supplies a framework for generating self-referential sets such as fractals which are seen as the attractors of specific IFSs.

Definition 2.1. *Let (X, d) be a complete metric space with metric d. If $w_n : X \to X$, $n \in \mathbb{N}_N$ are continuous functions, then $\mathcal{I} = \{X; w_n : n \in \mathbb{N}_N\}$ is called an*

Iterated Function System or IFS for short. If, in addition, each $w_n, n \in \mathbb{N}_N$ in $\mathcal{I}$ is a contraction map then the IFS $\mathcal{I}$ is called contractive or hyperbolic.

Let $\mathcal{H}(X) = \{A : A \neq \emptyset, A \text{ is compact in } X\}$ be the family of all nonempty compact subsets of X. It is well known from [1] that $\mathcal{H}(X)$ is complete with respect the Hausdorff metric $d_{\mathcal{H}} : \mathcal{H}(X) \times \mathcal{H}(X) \to [0, \infty)$ defined by

$$d_{\mathcal{H}}(B, C) = \max\big\{\max_{b\in B}\min_{c\in C} d(b, c), \max_{c\in C}\min_{b\in B} d(b, c)\big\}.$$

Define $W : \mathcal{H}(X) \to \mathcal{H}(X)$ by $W(B) = \bigcup_{n=1}^{N} w_n(B)\ \forall\ B \in \mathcal{H}(X)$. For all $k \in \mathbb{N}$, let W^k be the k-fold autocomposition of W and let W^0 be the identity map.

Definition 2.2. *The set $A \in \mathcal{H}(X)$ is said to be an attractor or a deterministic fractal of the IFS $\mathcal{I}$ if $\lim_{k\to\infty} W^k(B) = A$ for each $B \in \mathcal{H}(X)$, where the limit is taken with respect to the Hausdorff metric, i.e., $\lim_{k\to\infty} W^k(B) = A \Leftrightarrow \lim_{k\to\infty} d_{\mathcal{H}}\big(A, W^k(B)\big) = 0$. The fixed point A is also sometimes called invariant set or self-referential set as $A = W(A) = \bigcup_{n=1}^{N} w_n(A)$.*

We quote a basic result in the theory of IFS as follows.

Theorem 2.3. *(Barnsley [1]). If the IFS $\mathcal{I}$ is contractive (hyperbolic), then $\mathcal{I}$ has a unique attractor A satisfying $W(A) = A$.*

2.2. **Fractal Interpolation Functions.** Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N]$ be the closed interval that contains them. Let a set of data points $D = \{(t_n, x_n) \in I \times \mathbb{R} :\ n \in \mathbb{N}_N^0\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n$, $n \in \mathbb{N}_N$ be contractive homeomorphisms such that

$$L_n(t_0) = t_{n-1},\ L_n(t_N) = t_n, \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \leq l\ |c_1 - c_2| \quad \forall\ c_1, c_2\ \in I \tag{2}$$

for some $0 \leq l < 1$. Define $F = I \times K$, where K is a suitable compact set in $\mathbb{R}$ and N continuous mappings, $F_n : F \to \mathbb{R}$ be defined such that:

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \quad n \in \mathbb{N}_N, \tag{3}$$

$$|F_n(t, x) - F_n(t, y)| \leq r|x - y|, \quad t \in I, \quad x, y \in \mathbb{R}, \quad 0 \leq r < 1. \tag{4}$$

Now define functions $w_n(t, x) = (L_n(t), F_n(t, x)),\ \forall\ n \in \mathbb{N}_N$.

Theorem 2.4. *([1]): The IFS $\{F;\ w_n : n \in \mathbb{N}_N\}$ defined above admits a unique attractor G. G is the graph of a continuous function $g : I \to \mathbb{R}$ which obeys $g(t_n) = x_n$ for $n \in \mathbb{N}_N^0$.*

The previous function is called a FIF corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^{N}$ and it is unique satisfying the functional equation in [1] as

$$g(t) = F_n(L_n^{-1}(t), g \circ L_n^{-1}(t)), \tag{5}$$

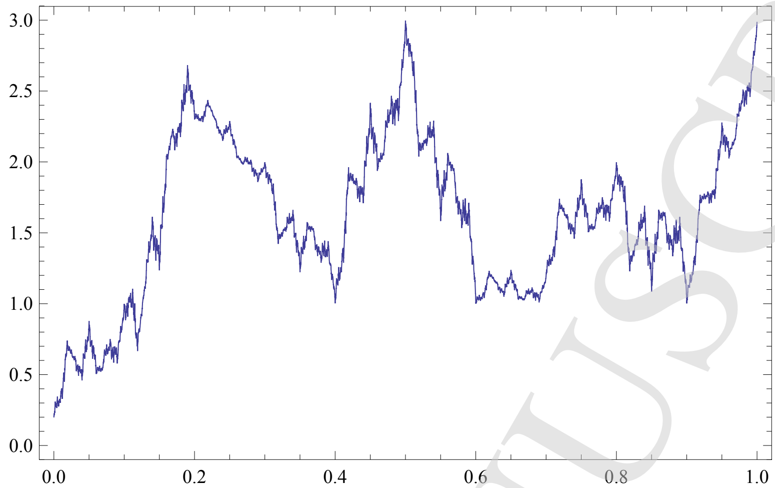

FIGURE 1. Graph of an affine fractal interpolation function.

for $n \in \mathbb{N}_N$ and $t \in I_n = [t_{n-1}, t_n]$. The most widely studied FIFs in theory and applications so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n, \\ F_n(t,x) = \alpha_n x + q_n(t), \end{cases} \tag{6}$$

where $-1 < \alpha_n < 1$ and the coefficients a_n and b_n of the affine maps L_n are determined through the conditions given in (1) as

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0} \quad \text{and} \quad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0}. \tag{7}$$

α_n is called the vertical scaling factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector of the IFS. If $q_n(t)$ is a line, the FIF is termed affine fractal interpolation function (AFIF). In this case, by equation (3), $q_n(t) = c_n t + d_n$, where

$$c_n = \frac{x_n - x_{n-1}}{t_N - t_0} - \alpha_n \frac{x_N - x_0}{t_N - t_0}, \tag{8}$$

$$d_n = \frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0} - \alpha_n \frac{t_N x_0 - t_0 x_N}{t_N - t_0}. \tag{9}$$

This type of functions are non-smooth in general. The Figure 1 represents an AFIF defined on $I = [0,1]$, $N = 10$, with respect to the data: $D = \{(0, 0.2), (0.1, 1), (0.2, 2.3), (0.3, 2), (0.4, 1), (0.5, 3), (0.6, 1), (0.7, 1.2), (0.8, 2), (0.9, 1), (1, 3)\}$ and the scale vector $\alpha = (0.2, -0.3, 0.1, -0.2, 0.3, -0.3, 0.1, 0.2, -0.3, 0.2)$. According to (5), an affine fractal interpolant satisfies the functional equation:

$$g(t) = \overline{g}(t) + \alpha_n (g - r) \circ L_n^{-1}(t) \tag{10}$$

for $t \in I_n = [t_{n-1}, t_n]$, where $\overline{g}$ is the polygonal whose vertices are the data (t_n, x_n) and r is the line passing through (t_0, x_0) and (t_N, x_N). The details can be read in ([17], Lemma 3.2). This type of functions have also been discussed in [16]. In [17] several ways of obtaining the scaling factors from the data were presented.

Remark 2.5. *If $\alpha_n = 0 \; \forall \; n \in \mathbb{N}_N$ in (10) then we get $g(t) = \overline{g}(t)$ and g is piecewise linear (polygonal or broken line interpolant) with vertices (t_n, x_n).*

From now on we denote g^α an AFIF with scale vector α, in order to display the dependence with respect to the vectorial parameter. Let us consider a partition of $I = [a, b]$, $\Delta : a = t_0 < t_1 < \ldots < t_N = b$, and a scale vector $\alpha = (\alpha_1, \ldots, \alpha_N)$.

Definition 2.6. *Let us consider the data $D_k = \{(t_n, \delta_{kn})\}_{n=0}^N$, where $\delta_{kn} = 1$ if $k = n$ and 0 otherwise. The k-th AFIF f_k^α with respect to the scale vector $\alpha \in \mathbb{R}^N$ and the partition Δ is defined by the equalities*

$$f_k^\alpha(t_n) = \delta_{kn} \quad \forall \; n \in \mathbb{N}_N^0, \tag{11}$$

that is to say, f_k^α is the AFIF corresponding to D_k.

In the reference [16], Theorem 3.5, it is proved that the functions $\{f_k^\alpha\}_{k=0}^N$ constitute a nodal basis of the AFIFs so that any other AFIF g^α (with respect to the scale vector $\alpha \in \mathbb{R}^N$ and the partition Δ) can be expressed as

$$g^\alpha = \sum_{k=0}^N x_k f_k^\alpha, \tag{12}$$

where $x_k = f^\alpha(t_k)$. The family $\{f_k^\alpha\}_{k=0}^N$ is an orthonormal system with respect to the form

$$< p, q >= \sum_{k=0}^N p(t_k) q(t_k). \tag{13}$$

Let $\mathcal{B}_\Delta^\alpha$ be the set of AFIFs associated to the partition Δ with scale vector $\alpha \in \mathbb{R}^N$. The system $\{f_k^\alpha\}_{k=0}^N$ is a basis of the linear space $\mathcal{B}_\Delta^\alpha$.

Remark 2.7. *$\mathcal{B}_\Delta^\alpha$ is a space of finite dimension. As a consequence, if $\mathcal{B}_\Delta^\alpha$ is considered as subspace of $\mathcal{C}(I)$ with the uniform norm, $\mathcal{B}_\Delta^\alpha$ is a closed complete subset and so a Banach space.*

In the same way, we define the operator of affine fractal interpolant (AFI) $\mathcal{A}^\alpha$ associated to Δ and α, $\mathcal{A}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$ defined for $f \in \mathcal{C}(I)$ and $t \in I$ as

$$\mathcal{A}^\alpha(f)(t) = \sum_{k=0}^N f(t_k) f_k^\alpha(t). \tag{14}$$

The main properties of $\mathcal{A}^\alpha$ can be found in ([16]) as:
(i) $\mathcal{A}^\alpha$ is linear, bounded and

$$\|\mathcal{A}^\alpha\| \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty}. \tag{15}$$

(ii) $\mathcal{A}^\alpha$ is a projection ($\mathcal{A}^\alpha = \mathcal{A}^\alpha \circ \mathcal{A}^\alpha$).

3. $\mathcal{L}^p$-convergence of affine interpolants

In this section, we extend the results of the reference [16] and we study the $\mathcal{L}^p$-convergence of affine fractal interpolants for $1 \leq p < +\infty$. Let us consider $f \in \mathcal{C}(I)$ and let $\overline{f}$ be the piecewise linear and continuous (polygonal) function whose vertices are $(t_n, f(t_n))$, where $n \in \mathbb{N}_N^0$. Let us define the modulus of continuity of g, $\omega_g(\delta)$, as

$$\omega_g(\delta) = \sup_{|\epsilon|\leq\delta} |g(t+\epsilon) - g(t)|, \tag{16}$$

where $t, t+\epsilon$ are in the domain of g.

Lemma 3.1. *If h is the diameter of the partition $\Delta : a = t_0 < t_1 < \ldots < t_N = b$, that is to say, $h = \max\{t_n - t_{n-1}\}$, for $1 \leq p \leq +\infty$,*

$$\|f - \overline{f}\|_p \leq \omega_f(h)(\mu(I))^{1/p} \tag{17}$$

where $\mu(I) = b - a$.

Proof. It is well known that $\forall\, t \in I$,

$$|f(t) - \overline{f}(t)| \leq \omega_f(h), \tag{18}$$

obtaining the result for $p = +\infty$. For $1 \leq p < +\infty$, using (18),

$$\|f - \overline{f}\|_p^p = \int_I |f(t) - \overline{f}(t)|^p dt \leq (\omega_f(h))^p \mu(I) \tag{19}$$

The statement of Lemma 3.1 is an immediate consequence of this inequality. □

Lemma 3.2. *Let $\mathcal{A}^\alpha(f)$ be the affine interpolant of f with respect to Δ and the scale vector α as defined in (14) and let r be the line passing through $(t_0, f(t_0))$ and $(t_N, f(t_N))$. For $1 \leq p \leq +\infty$,*

$$\|\mathcal{A}^\alpha(f) - \overline{f}\|_p \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|\overline{f} - r\|_p. \tag{20}$$

Proof. According to the equation (10) $\forall\, t \in I_n$,

$$\mathcal{A}^\alpha(f)(t) - \overline{f}(t) = \alpha_n\ (\mathcal{A}^\alpha(f) - r) \circ L_n^{-1}(t),$$

$$\|\mathcal{A}^\alpha(f) - \overline{f}\|_p^p = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} |\alpha_n|^p |(\mathcal{A}^\alpha(f) - r) \circ L_n^{-1}(t)|^p dt.$$

The change of variable $\tilde{t} = L_n^{-1}(t)$ provides

$$\|\mathcal{A}^\alpha(f) - \overline{f}\|_p^p = \sum_{n=1}^N a_n |\alpha_n|^p \int_a^b |(\mathcal{A}^\alpha(f) - r)(\tilde{t})|^p d\tilde{t},$$

$$\|\mathcal{A}^\alpha(f) - \overline{f}\|_p^p = \sum_{n=1}^N a_n |\alpha_n|^p \|\mathcal{A}^\alpha(f) - r\|_p^p. \tag{21}$$

Again,

$$\|\mathcal{A}^\alpha(f) - \overline{f}\|_p^p \leq \sum_{n=1}^N a_n |\alpha|_\infty^p \|\mathcal{A}^\alpha(f) - r\|_p^p.$$

Since according to (7),

$$\sum_{n=1}^N a_n = \sum_{n=1}^N \frac{t_n - t_{n-1}}{t_N - t_0} = 1,$$

one has

$$\|\mathcal{A}^\alpha(f) - \overline{f}\|_p \leq |\alpha|_\infty\ \|\mathcal{A}^\alpha(f) - r\|_p, \tag{22}$$

$$\|\mathcal{A}^\alpha(f) - \overline{f}\|_p \leq |\alpha|_\infty \left(\|\mathcal{A}^\alpha(f) - \overline{f}\|_p + \|\overline{f} - r\|_p\right),$$

and the result follows for $p < +\infty$. The case $p = +\infty$ is proved in Lemma 4.1 of [17]. □

With the help of the former lemmas, the next theorem provides $\mathcal{L}^p$-estimates of the affine interpolation error.

Theorem 3.3. *Let $\mathcal{A}^\alpha(f)$ be the affine interpolant of f with respect to Δ and the scale vector α, let r be the line passing through $(t_0, f(t_0))$ and $(t_N, f(t_N))$ and $\overline{f}$ be the polygonal whose vertices are $(t_n, f(t_n))$ where $n \in \mathbb{N}_N^0$. Then, for $1 \leq p \leq +\infty$,*

$$\|\mathcal{A}^\alpha(f) - f\|_p \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|\overline{f} - r\|_p + \omega_f(h)(\mu(I))^{1/p} \tag{23}$$

Proof. It is a consequence of the triangular inequality

$$\|\mathcal{A}^\alpha(f) - f\|_p \leq \|\mathcal{A}^\alpha(f) - \overline{f}\|_p + \|\overline{f} - f\|_p \tag{24}$$

and Lemmas 3.1-3.2. □

Corollary 3.4. *If $f \in \mathcal{C}(I)$, then*

$$\|\mathcal{A}^\alpha(f) - f\|_\infty \leq \ \omega_f(h) + \frac{2|\alpha|_\infty}{1 - |\alpha|_\infty}\|f\|_\infty. \tag{25}$$

where h is the diameter of the partition Δ.

Proof. It is a consequence of the former theorem in the case $p = +\infty$, considering that

$$\|\overline{f} - r\|_\infty \leq \|\overline{f}\|_\infty + \|r\|_\infty \leq 2\|f\|_\infty.$$

□

Theorem 3.5. *(Sufficient condition of convergence). If the scale factors are chosen so that $|\alpha|_\infty = \mathcal{O}(h^q)$, where $q > 0$ (or any other infinitesimal of h), then the affine fractal interpolant of f tends to f in the $\mathcal{L}^p$-norm as h tends to zero.*

Proof. It is a consequence of the Theorem 3.3. As f is uniformly continuous, $w_f(h) \to 0$ as h tends to zero ([33]). At the same time $\overline{f}$ tends to f ([33]) and the first term goes to zero with α. □

4. Bi-affine fractal interpolation functions

Given a partition $\Delta_1 : a = x_0 < x_1 < \ldots < x_M = b$ of the interval $I = [a, b]$, and a partition $\Delta_2 : c = y_0 < y_1 < \ldots < y_N = d$ of the interval $J = [c, d]$, let us consider the grid $\Delta = \Delta_1 \times \Delta_2$ of $I \times J$ and the data $\{(x_i, y_j, z_{ij}) : i \in \mathbb{N}_M^0; j \in \mathbb{N}_N^0\}$. We seek an interpolant $\hat{f} : I \times J \to \mathbb{R}$ such that $\hat{f}(x_i, y_j) = z_{ij}$ for all i, j. We define a two-dimensional fractal interpolant in the following way. Let $\alpha \in (-1, 1)^M$ and $\beta \in (-1, 1)^N$ be scale vectors for Δ_1 and Δ_2 respectively and let us consider the spaces of affine interpolation $\mathcal{B}^\alpha_{\Delta_1}$ and $\mathcal{B}^\beta_{\Delta_2}$ with respect to I and J, with nodal bases $\{\phi_i^\alpha\}_{i=0}^M$ and $\{\psi_j^\beta\}_{j=0}^N$ such that

$$\phi_i^\alpha(x_k) = \delta_{ik} \quad \text{and} \quad \psi_j^\alpha(y_l) = \delta_{jl}, \tag{26}$$

where δ_{ik} is the delta of Kronecker, $\delta_{ik} = 1$ if $i = k$ and $\delta_{ik} = 0$ otherwise (Definition 2.6).

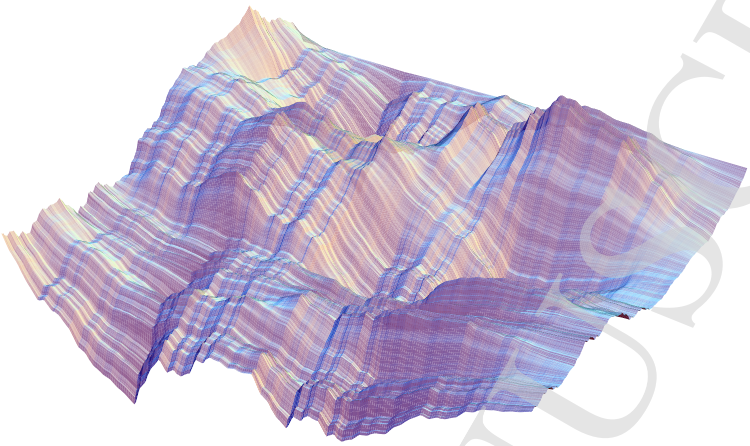

FIGURE 2. A bi-affine fractal interpolation surface.

Definition 4.1. *The space of bi-affine fractal interpolation functions with respect to the grid* Δ *and the scale vectors* α *and* β *is the tensor product of* $\mathcal{B}^{\alpha}_{\Delta_1}$ *and* $\mathcal{B}^{\beta}_{\Delta_2}$*,*

$$\mathcal{T}^{\alpha\beta}_{\Delta} = \mathcal{B}^{\alpha}_{\Delta_1} \otimes \mathcal{B}^{\beta}_{\Delta_2} = span\{\phi^{\alpha}_i(x)\psi^{\beta}_j(y) : i \in \mathbb{N}^0_M; j \in \mathbb{N}^0_N\}$$

.

Remark 4.2. $\mathcal{T}^{\alpha\beta}_{\Delta}$ *is a linear subspace of* $\mathcal{C}(I \times J)$ *whose basis is* $\{\phi^{\alpha}_i(x)\psi^{\beta}_j(y)\}$*.*

Remark 4.3. *In the particular case* $\alpha = \beta = 0$*, the basic functions* ϕ^0_i *and* ψ^0_j *are polygonal (Remark 2.5) and we obtain the space of piecewise bilinear interpolants.*

Definition 4.4. *The operator of bi-affine fractal interpolation is defined as* $\mathcal{D}^{\alpha\beta} : \mathcal{C}(I \times J) \to \mathcal{T}^{\alpha\beta}_{\Delta}$ *expressed by*

$$\mathcal{D}^{\alpha\beta}(f)(x,y) = \sum_{i=0}^{M}\sum_{j=0}^{N} f(x_i, y_j)\phi^{\alpha}_i(x)\psi^{\beta}_j(y) \tag{27}$$

Definition 4.5. *The graph of* $\mathcal{D}^{\alpha\beta}(f)$ *for any* $f \in \mathcal{C}(I \times J)$ *is a bi-affine fractal interpolation surface.*

The Figure 2 represents a bi-affine fractal interpolation surface on $[0,1] \times [0,1]$ for $M = 5$, $N = 4$, $\alpha = (0.3, -0.3, 0.2, -0.1, 0.2)$ and $\beta = (0.2, -0.3, 0.3, -0.1)$.

Definition 4.6. *A function* $f \in \mathcal{C}(I \times J)$ *is Hölder or Lipschitz continuous with exponent* q *(*$f \in Lipq$*) if* $\exists\, K \geq 0$ *such that* $\forall\, (x,y), (x',y') \in I \times J$

$$|f(x,y) - f(x',y')| \leq K\|(x,y) - (x',y')\|^q$$

The set $Lipq$ is a linear subspace of $\mathcal{C}(I \times J)$. For a single variable function the definition is analogous substituting $\|\cdot\|$ by an absolute value.

Lemma 4.7. *If $f_1 : I \to \mathbb{R} \in Lipq$ and $f_2 : J \to \mathbb{R} \in Lipq$, then $f_1(x)f_2(y) \in Lipq$.*

Proof. It is an easy exercise, bearing in mind that f_1, f_2 are continuous on compact intervals and thus bounded. Moreover,

$$|f_1(x)f_2(y)-f_1(x')f_2(y')| \leq |f_1(x)f_2(y)-f_1(x)f_2(y')|+|f_1(x)f_2(y')-f_1(x')f_2(y')|.$$

□

Let us consider the modulus of continuity for functions of several variables:

Definition 4.8. *The modulus of continuity of a function $f \in \mathcal{C}(I \times J)$, along the direction of $e \in R^2$ (where $\|e\| = 1$), with respect to the uniform norm, is defined as*

$$\overline{\omega}_f(\delta; e) = \sup\{|f(t + he) - f(t)| : |h| \leq \delta\}, \tag{28}$$

and, independently of the direction,

$$\overline{\omega}_f(\delta) = \sup\{\overline{\omega}_f(\delta; e); \|e\| = 1\}, \tag{29}$$

that is to say,

$$\overline{\omega}_f(\delta) = \sup\{|f(x, y) - f(x', y')| : \|(x, y) - (x', y')\| \leq \delta\}. \tag{30}$$

Remark 4.9. *We reserve the notation ω for the modulus of functions with one single variable. In the former definitions we assume, of course, that $t, t+he \in I\times J$.*

We note some properties of the modulus from ([34]) as (i) $\overline{\omega}_f(\delta) \leq \overline{\omega}_f(\delta')$ if $\delta \leq \delta'$ (ii) $f \in Lipq \Leftrightarrow \overline{\omega}_f(\delta) \leq K\delta^q$. These items are true for functions with single variable as well.

Lemma 4.10. [35] *Let g be a FIF defined by (6) where q_n are arbitrary but satisfying $q_n(t) \in Lip\ \delta_n$, $0 < \delta_n \leq 1$. Let $\delta = \min\{\delta_n : n \in \mathbb{N}_N\}$ then if $|\alpha|_\infty < h^\delta$, $g \in Lip\ \delta$.*

Lemma 4.11. *If the diameter of the partition Δ is h, and g is an affine fractal interpolant with respect to Δ and a scale vector such that $|\alpha|_\infty < h$, then $g \in Lip1$.*

Proof. In our case the maps q_n are linear, thus $q_n \in Lip1$. Let us consider Lemma 4.10 for $\delta_n = 1$. If $|\alpha|_\infty < h$, then $g \in Lip1$. □

Let h_x, h_y be defined as $h_x = \max\{x_i - x_{i-1} : i \in \mathbb{N}_M\}, h_y = \max\{y_j - y_{j-1} : j \in \mathbb{N}_N\}$.

Proposition 4.12. *If $|\alpha|_\infty < h_x$ and $|\beta|_\infty < h_y$ then $\mathcal{D}^{\alpha\beta}(f) \in Lip1$.*

Proof. According to the former Lemma, ϕ_i^α and ψ_j^β belong to $Lip1$ and so $\mathcal{D}^{\alpha\beta}(f)$ (Lemma 4.7). □

Let us consider the operator $\mathcal{A}_I^\alpha$ of AFIF corresponding to I, Δ_1 and α, along with its analogue $\mathcal{A}_J^\beta$ with respect to J, Δ_2 and β. According to (14), bearing in mind that $\{\phi_i^\alpha\}$ is a basis of AFIF on I, for $g : I \to \mathbb{R}$, $\mathcal{A}_I^\alpha(g) : I \to \mathbb{R}$ is defined as

$$\mathcal{A}_I^\alpha(g)(x) = \sum_{i=0}^{M} g(x_i)\phi_i^\alpha(x), \tag{31}$$

and likewise, for $g' : J \to \mathbb{R}$, $\mathcal{A}_J^\beta(g') : J \to \mathbb{R}$,

$$\mathcal{A}_J^\beta(g')(y) = \sum_{j=0}^{N} g'(y_j)\psi_j^\beta(y). \tag{32}$$

Let us consider the transformations $\widehat{\mathcal{A}}_I^\alpha$, $\widehat{\mathcal{A}}_J^\beta : \mathcal{C}(I \times J) \to \mathcal{C}(I \times J)$ defined as

$$\widehat{\mathcal{A}}_I^\alpha(f)(x,y) = \mathcal{A}_I^\alpha(f_y)(x),$$
$$\widehat{\mathcal{A}}_J^\beta(f)(x,y) = \mathcal{A}_J^\beta(f_x)(y),$$

where $f_x(y) = f(x,y) = f_y(x)$, that is to say ((31), (32)),

$$\widehat{\mathcal{A}}_I^\alpha(f)(x,y) = \sum_{i=0}^{M} f_y(x_i)\phi_i^\alpha(x) = \sum_{i=0}^{M} f(x_i,y)\phi_i^\alpha(x). \tag{33}$$

$$\widehat{\mathcal{A}}_J^\beta(f)(x,y) = \sum_{j=0}^{N} f_x(y_j)\psi_j^\beta(y) = \sum_{j=0}^{N} f(x,y_j)\psi_j^\beta(y). \tag{34}$$

Lemma 4.13. $\mathcal{D}^{\alpha\beta} = \widehat{\mathcal{A}}_I^\alpha \circ \widehat{\mathcal{A}}_J^\beta = \widehat{\mathcal{A}}_J^\beta \circ \widehat{\mathcal{A}}_I^\alpha$.

Proof. For instance, using (33) and (34),

$$\widehat{\mathcal{A}}_I^\alpha(\widehat{\mathcal{A}}_J^\beta(f))(x,y) = \widehat{\mathcal{A}}_I^\alpha(\sum_{j=0}^{N} f(x,y_j)\psi_j^\beta(y)) =$$

$$\sum_{i=0}^{M}(\sum_{j=0}^{N} f(x_i,y_j)\psi_j^\beta(y))\phi_i^\alpha(x) = \sum_{i=0}^{M}\sum_{j=0}^{N} f(x_i,y_j)\phi_i^\alpha(x)\psi_j^\beta(y).$$

□

Theorem 4.14. *$\mathcal{D}^{\alpha\beta}$ is linear, bounded and $\|\mathcal{D}^{\alpha\beta}\| \leq (\frac{1+|\alpha|_\infty}{1-|\alpha|_\infty})(\frac{1+|\beta|_\infty}{1-|\beta|_\infty})$, where $\|\cdot\|$ represents the norm of the operator with respect to the uniform norm in the spaces of functions.*

Proof. The first property is evident according to the definition (27). Moreover,

$$\begin{aligned}
\|\widehat{\mathcal{A}}_I^\alpha(f)\|_\infty &= \max\{|\widehat{\mathcal{A}}_I^\alpha(f)(x,y)| : (x,y) \in I \times J\},\\
&= \max\{|\mathcal{A}_I^\alpha(f_y)(x)| : (x,y) \in I \times J\},\\
&= \max\{\|\mathcal{A}_I^\alpha(f_y)\|_\infty : y \in J\},\\
&\leq \|\mathcal{A}_I^\alpha\| \max\{\|f_y\|_\infty : y \in J\},\\
&\leq \|\mathcal{A}_I^\alpha\|\|f\|_\infty.
\end{aligned}$$

Then, according to (15),

$$\|\widehat{\mathcal{A}}_I^\alpha\| \leq \|\mathcal{A}_I^\alpha\| \leq \frac{1+|\alpha|_\infty}{1-|\alpha|_\infty}.$$

As a consequence of Lemma 4.13,

$$\|\mathcal{D}^{\alpha\beta}\| \leq \|\widehat{\mathcal{A}}_I^\alpha\|\|\widehat{\mathcal{A}}_J^\beta\| \leq (\frac{1+|\alpha|_\infty}{1-|\alpha|_\infty})(\frac{1+|\beta|_\infty}{1-|\beta|_\infty}).$$

□

Theorem 4.15. *For all* $f, g \in \mathcal{C}(I \times J)$,

$$\|\mathcal{D}^{\alpha\beta}(f) - \mathcal{D}^{\alpha\beta}(g)\|_\infty \leq \Big(\frac{1+|\alpha|_\infty}{1-|\alpha|_\infty}\Big)\Big(\frac{1+|\beta|_\infty}{1-|\beta|_\infty}\Big)\|f-g\|_\infty.$$

Proof. It is a straightforward consequence of the former theorem. □

Lemma 4.16. *If* $f, g \in \mathcal{C}(I \times J)$ *agree at the nodes of the grid* $\Delta = \Delta_1 \times \Delta_2$, *then* $\mathcal{D}^{\alpha\beta}(f) = \mathcal{D}^{\alpha\beta}(g)$.

Proof. It is evident from the definition of $\mathcal{D}^{\alpha\beta}$ as defined in (27). □

Theorem 4.17. *The operator* $\mathcal{D}^{\alpha\beta}$ *has the following properties:*
(i) $\mathcal{D}^{\alpha\beta}$ *is a projection,* $\mathcal{D}^{\alpha\beta} = \mathcal{D}^{\alpha\beta} \circ \mathcal{D}^{\alpha\beta}$,
(ii) $\mathcal{D}^{\alpha\beta}$ *has a closed range,*
(iii) $\mathcal{D}^{\alpha\beta}$ *is compact.*

Proof. $\mathcal{D}^{\alpha\beta}(f)$ and f agree at the nodes of the grid, due to the definition of the nodal bases, and thus $\mathcal{D}^{\alpha\beta}(\mathcal{D}^{\alpha\beta}(f)) = \mathcal{D}^{\alpha\beta}(f)$ according to the former Lemma 4.13. The range of $\mathcal{D}^{\alpha\beta}$ is contained in $\mathcal{T}_\Delta^{\alpha\beta}$ (Definition 4.1) and hence is finite dimensional and so closed. The finite dimensionality of $\mathcal{D}^{\alpha\beta}$ along with the continuity of the operator imply that the transformation is compact ([36], Theorem 6.5.2). □

According to the definition of $\mathcal{D}^{\alpha\beta}$, the adjoint operator of $\mathcal{D}^{\alpha\beta}$ is defined as

$$(\mathcal{D}^{\alpha\beta})^* : (\mathcal{T}_\Delta^{\alpha\beta})^* \to (\mathcal{C}(I \times J))^*$$

where $*$ represents the dual in the spaces of functions. This operator is defined as ([37], Definition 6.5.1) $(\mathcal{D}^{\alpha\beta})^*(g^*) = g^* \circ \mathcal{D}^{\alpha\beta}$ if $g^* \in (\mathcal{T}_\Delta^{\alpha\beta})^*$. $(\mathcal{D}^{\alpha\beta})^*$ is linear and bounded and, in this case, has the following properties.

Proposition 4.18. $(\mathcal{D}^{\alpha\beta})^*$ *has closed range, is compact and*

$$\|(\mathcal{D}^{\alpha\beta})^*\| \leq \Big(\frac{1+|\alpha|_\infty}{1-|\alpha|_\infty}\Big)\Big(\frac{1+|\beta|_\infty}{1-|\beta|_\infty}\Big).$$

Proof. The space $(\mathcal{T}_\Delta^{\alpha\beta})^*$ is finite dimensional and thus its image by $(\mathcal{D}^{\alpha\beta})^*$. As a consequence, the range is closed. The finite dimensionality of $(\mathcal{D}^{\alpha\beta})^*$ along with the continuity of the operator imply that $(\mathcal{T}_\Delta^{\alpha\beta})^*$ is compact ([36], Theorem 6.5.2). In general, for a linear and bounded operator of a Banach space $\|\mathcal{D}^{\alpha\beta}\| = \|(\mathcal{D}^{\alpha\beta})^*\|$ ([37], Theorem 6.5.2) and the Theorem 4.14 implies the inequality proposed. □

5. Multidimensional case

The former results can be extended without difficulty to arbitrary dimensions and it is possible to construct tensor product of affine fractal interpolants for the approximation of functions with several variables $f(x_1, x_2, \ldots, x_d)$. The only requirement is the domain of f to be a hyperinterval $I_1 \times I_2 \times \ldots \times I_d$ of an Euclidean space. In this case, we would consider a d-dimensional grid $\Delta = \Delta_1 \times \Delta_2 \times \ldots \times \Delta_d$ defined by means of partitions on each orthogonal direction. The space of functions would be the tensor product $\mathcal{B}_{\Delta_1}^{\alpha^1} \otimes \ldots \otimes \mathcal{B}_{\Delta_d}^{\alpha^d}$.

References

[1] M. F. Barnsley, Fractal functions and interpolation, *Constr. Approx.* **2**(4) (1986) 303–329.

[2] C. J. G. Evertsz, Fractal Geometry of finacial time series, *Fractals* **3**(3) (1995) 609-616.

[3] J. L. Véhel, K. Daoudi and E. Lutton, Fractal modeling of speech signals, *Fractals* **2**(3) (1994) 379-382.

[4] G. W. Wornell, Signal processing with fractals: A Wavelet based approach, *Prentice Hall* (1995).

[5] L. Xin-Fu and L. Xiao-Fan, Seismic data reconstruction with fractal interpolation, *Chin. J. Geophys.* **51**(4) (2008) 855-861.

[6] L. L. Yong and T. Xin, Fractal fitting research on stock prices, *in 2008 Congress on image and signal processing*, **4** (2008) 49-53.

[7] A. V. Tetenov, Self-similar Jordan arcs and graph-directed systems of similarities, *Sib. Math. J.* **47**(5) (2006) 940-949.

[8] H. Y. Wang and J.S. Yu, Fractal interpolation functions with variable parameters and their analytical properties, *J. Approx. Theory* **175** (2005) 118.

[9] P. R. Massopust, Fractal Functions, Fractal Surfaces, and Wavelets, 2nd edn. *Academic Press, San Diego* (1994).

[10] P. R. Massopust, Interpolation and Approximation with Splines and Fractals, *Oxford University Press, Oxford* (2010).

[11] P. R. Massopust, Local fractal interpolation on unbounded domains, *Proc. Edinb. Math. Soc.* **61**(1) (2018) 151-167.

[12] M. A. Navascués and M. V. Sebastián, Generalization of Hermite functions by fractal interpolation, *J. Approx. Theory* **131**(1) (2004) 19-29.

[13] M. A. Navascués and M. V. Sebastián, Smooth fractal interpolation, *J. Inequal. Appl.* Article ID **78734** (2006) 1-20.

[14] A. J. Kurdila, T. Sun, P. Grama and J. Ko, Affine fractal interpolation functions and wavelet-based finite elements, *Comput. Mech.* **17**(3) (1995) 165-189.

[15] L. Dalla, On the parameter identification problem in the plane and the polar fractal interpolation functions, *J. Approx. Theory* **101** (1999) 289-302.

[16] M. A. Navascués and M. V. Sebastián, Error bounds for affine fractal interpolation, *Math. Ineq. Appl.* **9**(2) (2006) 273-288.

[17] M. A. Navascués and M. V. Sebastián, Construction of affine fractal functions close to classical interpolants, *J. Comp. Anal. Appl.* **9**(3) (2007) 271-285.

[18] X. Chen, Q. Guo, and L. Xi, The range of an affine fractal interpolation function, *Int. J. Nonlinear Sci.* **3**(3) (2007) 181186.

[19] D. S. Mazel and M. H. Hayes, Using iterated function systems to model discrete sequences, *IEEE Trans. Signal Process.* **40** (1992) 1724-1734.

[20] J. L. Véhel, K. Daoudi and E. Lutton, Fractal modeling of speech signals, *Fractals* **2** (1994) 379-382.

[21] M. Ali and T. G. Clarkson, Using linear fractal interpolation functions to compress video images, *Fractals* **2** (1994) 417-421.

[22] A. K. B. Chand, S. K. Katiyar and Saravana Kumar G., A new class of rational fractal function for curve fitting, *Proceeding of Computer Aided Engineering CAE 2013*, ISBN No. 78-93-80689-17-3.

[23] A. K. B. Chand, S. K. Katiyar and P. Viswanathan, Approximation using hidden variable fractal interpolation functions, *J. Fractal Geom.* **2**(1) (2015) 81–114.

[24] A. K. B. Chand, M. A. Navascués, P. Viswanathan and S. K. Katiyar, Fractal trigonometric polynomial for restricted range approximation, *Fractals* **24**(2) (2016) 1–11.

[25] M. A. Navascués, P. Viswanathan, A. K. B. Chand, M. V. Sebastián, and S. K. Katiyar, Fractal bases for Banach spaces of smooth functions, *Bull. Aust. Math. Society* **92** (2015) 405–419.

[26] A. K. B. Chand and S. K. Katiyar, Quintic Hermite fractal interpolation in a strip: preserving copositivity, *Springer Proc. Math. Stat.* **143** (2015) 463–475.

[27] S. K. Katiyar and A. K. B. Chand, Toward a unified methodology for fractal extension of various shape preserving spline interpolants, *Springer Proc. Math. Stat.,* **139** (2015) 223–238.

[28] S. K. Katiyar, A. K. B. Chand and M. A. Navascués, Hidden Variable **A**-Fractal Functions and Their Monotonicity Aspects, *Rev. R. Acad. Cienc. Zaragoza* **71** (2016) 7-30.

[29] S. K. Katiyar and A. K. B. Chand, A new class of monotone/convex rational fractal function, *arXiv:1809.10682*.
[30] S. K. Katiyar, Bicubic partially blended rational quartic surface, *arXiv:1910.09822*.
[31] S. K. Katiyar, A. K. B. Chand and Saravana Kumar G., A new class of rational cubic spline fractal interpolation function and its constrained aspects, *Appl. Math. Comp.* **346** (2019) 319-335.
[32] S. K. Katiyar and A. K. B. Chand, Shape Preserving Rational Quartic Fractal Functions, *Fractals* **27**(8) (2019) 1–15.
[33] E. W. Cheney, Approximation Theory, *AMS Chelsea Publ.* (1966).
[34] P. J. Davis, Interpolation and Approximation, *Dover Publ.* (1963).
[35] G. Chen, The smoothness and dimension of fractal interpolation functions, *Appl. Math-JCU* **11B** (1996) 409-418.
[36] L. P. Lebedev, I. I. Vorovich and G. M. L. Gladwell, Functional Analysis, *Kluwer Academic Publ., 2nd. ed.* (2002).
[37] V. Hutson and J. S. Pym, Applications of Functional Analysis and Operator Theory, *Academic Press* (1980).