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ABSTRACT

The purpose of this research is to investigate a set of multiple criteria, decomposition models for hierarchical organizations. To facilitate this investigation the paper has three objectives. First, it presents a generalized decomposition approach to an organizational resource allocation problem. This approach results in a three-level, decision-making hierarchy applicable to a set of decomposition models. Second, it specifies the basic decisions and coordinative mechanisms used by each organizational model within this decision-making hierarchy. Third, it discusses the relationship between these organizational models and pure mathematical decomposition procedures. Finally, an alternative objective function formulation is proposed in order to overcome difficulties with the models.

I. Introduction

In recent years, a number of researchers have noted the strong similarities between the solution procedures utilized by a family of mathematical programming decomposition algorithms and the information exchanges inherent in the decision-making processes within hierarchical organizations. Within the last decade, several alternative formulations, often referred to as organizational models have evolved. These organizational models focus upon a multiple critera formulation of the resource allocation problem faced by a multi-level organization. Although several of these organizational models utilize classical mathematical decomposition algorithms, their authors have defined a specific organizational decision-making hierarchy without reference to an overall organizational problem. The highest level of this decision-making hierarchy is typically concerned with generating goals for and/or allocating resources to lower level units. The decisions at the intermediate levels are structured as goal programming problems and attempt to minimize deviations from the goals generated at the highest level. These deviations are minimized by selecting alternative proposals that are aimed at fulfilling policy goals. The proposals are obtained from subordinate units at the lowest level of the organization.

The initial effort in this type of organizational modeling was Ruefli's Generalized Goal Decomposition (GGD) model [14, 15]. Subsequently, Freeland [9, 11] and Freeland and Baker [10] developed a similar model based upon the principles of Benders' partitioning procedure [2]. After testing these organizational algorithms, Davis [4] and Davis and Talavage [5] proposed two additional algorithms: the

Centralized Goal Decomposition (CGD) and the Hybrid Goal Decomposition (HGD) models. Unlike the Ruefli and Freeland formulations, which used shadow prices as their principal coordinative mechanism, the CGD and HGD models incorporated goal deviations as a coordinative input. Later Davis [7] developed another algorithm, dubbed the Generalized Hierarchical Model (GHM), which incorporated a goal programming structure at each level of the decision hierarchy and relied upon deviations as the sole coordinative mechanisms.

The purpose of this paper is to compare and contrast these organizational models. To facilitate this comparison, the paper has three objectives. First, it will present a generalized decomposition approach for a specific organizational resource allocation problem. This approach will result in a three-level, decision-making hierarchy which is applicable to these organizational models. Second, it will specify the basic decisions and coordinative mechanisms used by each organizational model within this decision-making hierarchy. Finally, it will discuss the relationship between these organizational models and mathematical decomposition procedures. Pursuant to these objectives Section II will outline an overall resource allocation problem faced by a three-level hierarchical organization and will discuss how this problem can be decomposed. Section III provides a description of how each organizational model implements this decomposition. Section IV contains a discussion of these models as pure mathematical decomposition procedures and proposes an alternative objective function formulation to overcome difficulties in the algorithms. A final section provides a summary of the paper and outlines areas for future research.

II. Development of the Decomposition Approach

This section of the paper focuses upon the resource-allocation problem faced by a three-level hierarchical organization. It will begin with the definition of the organization's overall mathematical programming problem. It will be shown that this problem can be decomposed and solved by the algorithms proposed by Ruefli [14], Freeland and Baker [10], Davis and Talavage [5] and Davis [6, 7]. In addition, another potential algorithm, which is a hybrid of the Freeland-Baker and Davis models, will be discussed. It should be noted that each of these algorithms' author(s) used a different set of varible definitions and, in some cases, slightly different constraints in the original statement of his (their) algorithm. This paper presents an universal formulation for the organization's overall problem. Although this formulation's variables and constraints differ slightly from the originals, they are applicable to all of these organizational algorithms. As will become more apparent, the essential differences among the algorithms are not their original variable/constraint definitions, but rather the basic approach that each algorithm employs to decompose and solve the overall problem.

Before developing the decomposition approach for each algorithm, it is appropriate to investigate this overall problem and focus upon the structural interactions among decision variables and constraints. This study will show that the overall problem possesses a specific structure which can be decomposed into three levels of hierarchial decision-making. In adopting this three-level decomposition strategy, however, certain fundamental problems can be expected to occur in the coordination of the decisions at each level. These problems will be discussed in detail. A second paper by the authors [19] will relate the computational consequences of these problems of coordination in a real world application.

The overall organizational problem is defined by equations (1) through (4).

$$\text{Min } \Sigma_{k=1}^{M} [\Sigma_{i=r_{k-1}+1}^{r_k} C_i X_i + W_k^+ Y_k^+ + w_k^+ y_k^+ + W_k^- Y_k^- + w_k^- y_k^- + C_{G_k} G_k] \qquad (1)$$

s.t.

$$B_i X_i - I_{m_k} Y_k^+ + I_{m_k} Y_k^- - G_k = 0 \qquad (2.k)$$

$$B_i' X_i - I_{m_k'} y_k^+ + I_{m_k'} y_k^- = g_k$$

for k=1, ..., M

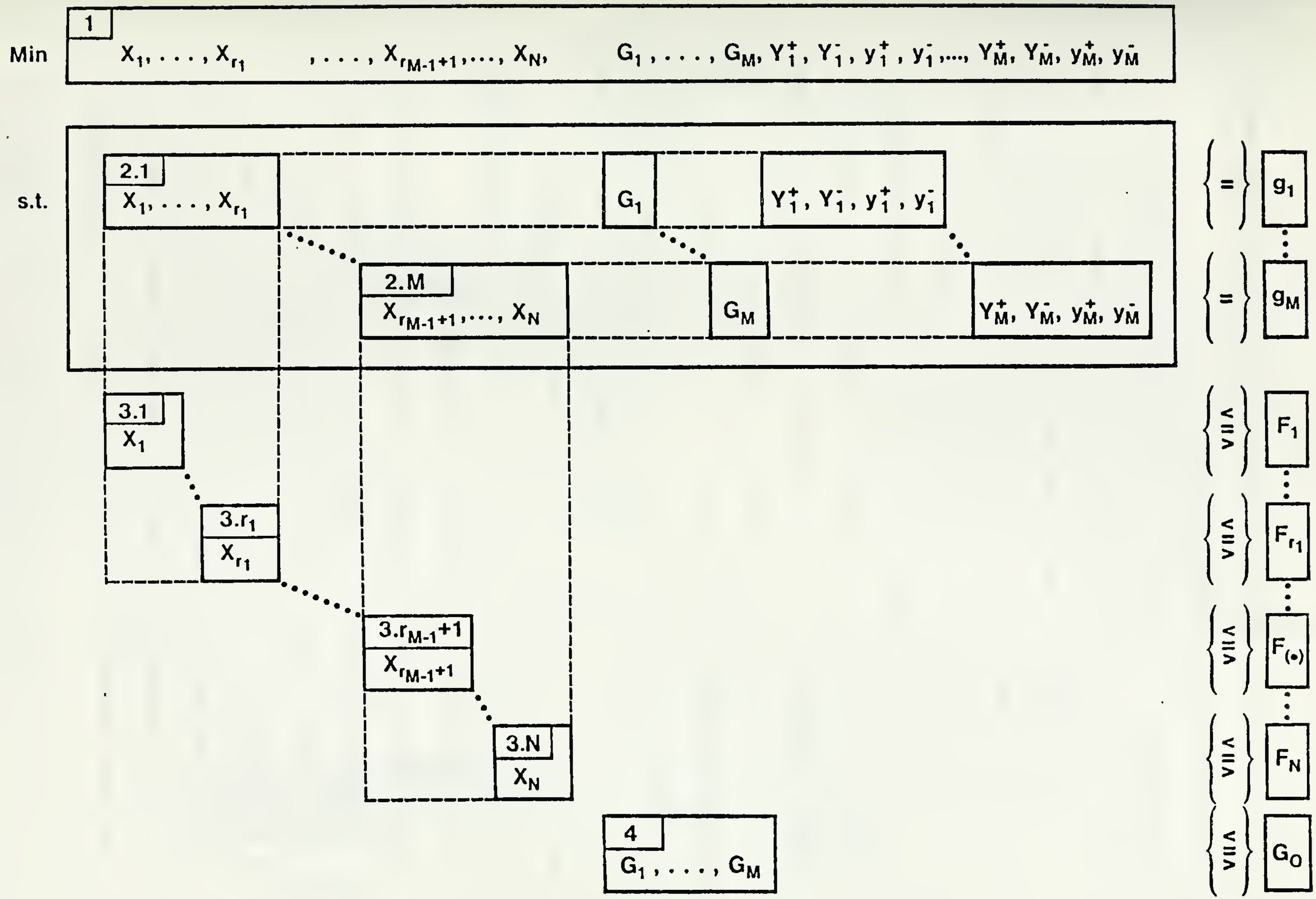
$$D_i X_i \leq F_i \qquad (3.i)$$

for i=1, ..., N

$$\Sigma_{k=1}^{M} P_k G_k \leq G_o \qquad (4)$$

All variables ≥ 0, and I_{m_k} and $I_{m_k'}$ are (m_k x m_k) and (m_k' and m_k') identity matrix, respectively.

Given the complex nature of the equations, the structure of the overall problem is not immediately evident. Figure 1 gives the variable/constraint diagram for this problem. In Figure 1 each row of boxes represents a specific equation of the overall problem; the defining equation is given in the upper left-hand corner of the left-most box. The boxes contain the decision variables for each equation. By grouping equations (2.1) through (2.M), the classic block-angular structure of the overall problem is apparent. Thus a two-level decomposition approach to the problem can easily be applied where equations (2.1) through (2.M) define



VARIABLE/CONSTRAINT DIAGRAM FOR OVERALL ORGANIZATIONAL PROBLEM

Figure 1

the restricted master program and equations (3.1) through (3.N) and (4) define the appropriate column generators.[1]

The structure of this overall problem, however, permits the consideration of a more sophisticated decomposition approach. It is evident that the overall problem is nearly separable into M subproblems; only constraint (4) prevents this separation. Therefore, to permit separation constraint (4) will be initially neglected, and the vectors $(G_1,\ldots,G_M)$ will be assumed to be constant. The first of the M-subproblems would then consist of constraints (2.1) and (3.1) through $(3.r_1)$ with the first term of the summation (k=1) in equation (1) serving as its objective function. Because G_1 is assumed to be a constant, it can be placed in the right-hand-side vector of equation (2.1). The resulting subproblem is shown graphically in Figure 2 as SP.1. In Figure 2, each of the M subproblems, SP.1 through SP.M, possesses a similar block angular structure. For the subproblem SP.k, constraint (2.k) defines the restricted master program, while constraints $(3.r_{k-1} + 1)$ through $(3.r_k)$ define column generators supporting the restricted master.

In the original problem statement, however, these subproblems are coupled through constraint (4). This coupling is illustrated in Figure 2 by the dashed lines. Hence, a mechanism through which constraint (4) can generate the composite vector $(G_1,\ldots,G_M)$ is required. The incorporation of such a mechanism represents the third level of decision-making employed by the decomposition procedure. The development of this essential coordination mechanism is a difficult task. In the next section,

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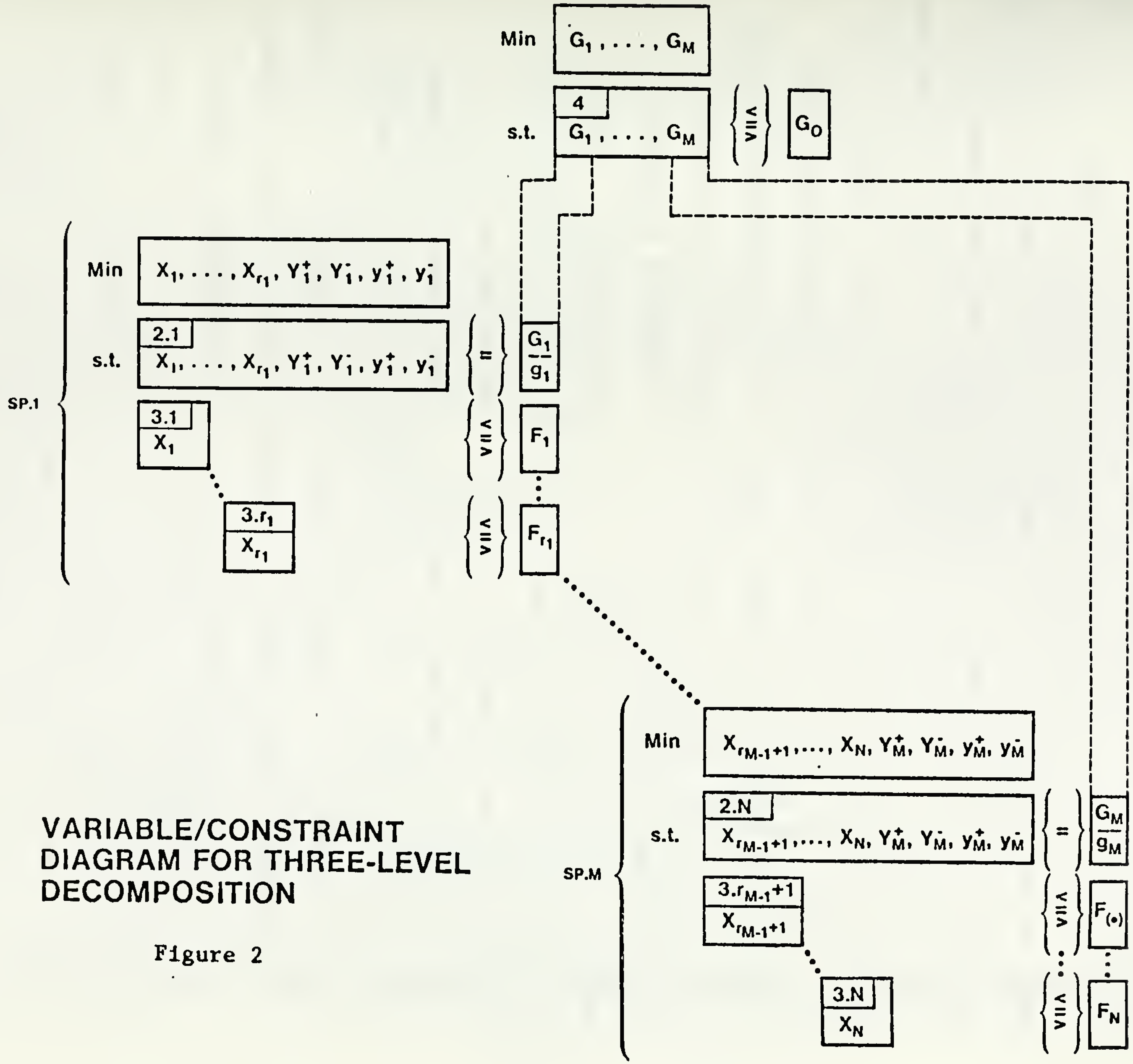
[1]The terms block-angular structure, restricted master program, and column generators are standard terminology in decomposition theory. A brief overview of this theory is given in Appendix 1; in addition, an excellent presentation is given in Lasdon [13].



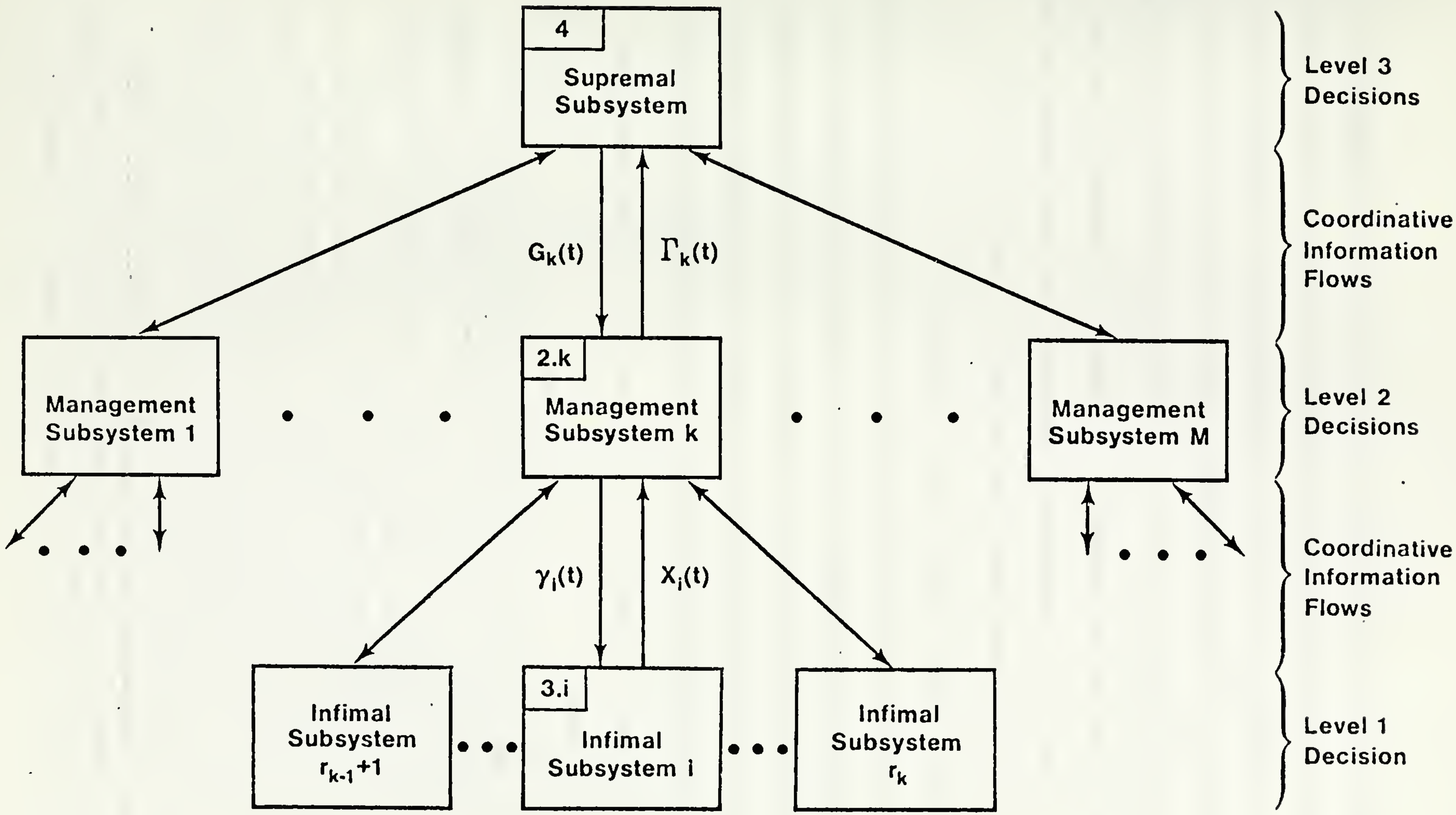
VARIABLE/CONSTRAINT DIAGRAM FOR THREE-LEVEL DECOMPOSITION

Figure 2

three procedures which have been employed in existing three-level (organizational) decomposition models are discussed.

In contrasting the three-level decomposition approach with the two-level approach applicable to block-angular structure of Figure 1, several fundamental differences emerge. First, the single restricted master program resulting from constraints (2.1) through (2.M) in Figure 2 has now been replaced by M restricted master programs defined for each constraint (2.k) (k=1,...,M). In Figure 1, constraint (4) defines an appropriate column generator for the restricted master program. For the three-level approach, it must be used to develop a coordination mechanism for the M separate restricted master programs. Perhaps the most fundamental difference, however, is that under the proposed decomposition procedure, no single decision-making subsystem has been assigned the task of optimizing the organization's overall objective function. That is, each of the M restricted master programs will consider only the k-th term of the summation given in equation (1). Both the two-level and three-level decomposition procedures will, however, use equations (3.1) through (3.N) to define column generators that support their respective restricted master program.

An organizational hierarchy based upon this three-level decomposition approach is depicted in Figure 3. The two lower levels of the organization result from the application of a given decomposition procedure to subproblems SP.1 through SP.M. The restricted master program for each subproblem, SP.k (k=1,...,M), will be called Management Subsystem or Manager k. Manager k will coordinate the decision-making of the Infimal Subsystem or Operating Unit i ($i=r_{k+1}+1,\dots,r_k$). Each subordinate Infimal i will iteratively generate a proposal vector, $X_i(t)$, for



DECISION-MAKING HIERARCHY FOR ORGANIZATIONAL MODELS

Figure 3

its Managment Subsystem at iteration t. To coordinate the generation of this proposal vector, $X_i(t)$, Manager k must generate coordinative input vector, $\gamma_i(t)$ $(i=r_{k-1}+1,\ldots,r_k)$. Each Infimal will incorporate this $\gamma_i(t)$ vector into its decision-making process. Furthermore, the i-th Infimal is responsible for assuring that the proposal vector, $X_i(t)$, satisfies constraint (3.i). In the generation of a composite proposal vector for the i-th Infimal, $X_i^*(t)$, Management Subsystem k must also insure that constraint (2.k) is satisfied while simultaneously attempting to minimize the k-th term of the summation given in equation (1).

The detailed nature of this interaction can not be completely specified until a specific decomposition procedure is applied to subproblem SP.k. Nevertheless, these basic flows of information must occur irrespective of the selected decomposition procedure. Further, because there are M subproblems to be decomposed, there must be M corresponding interactions between Management Subsystem k and its subordinate Infimal Subsystems or column generators.

Finally, to coordinate the simultaneous solution of the M subproblems, SP.1 through SP.M, a third level of decision-making is introduced as the Supremal Subsystem or Central Unit. The Supremal Subsystem will interact with each Management Subsystem k (k=1,...,M). This interaction again will be iterative in nature. To coordinate Management Subsystem k's decision, the Supremal will generate the external goal vector, $G_k(t+1)$. In the generation of $G_k(t)$, the Supremal Subsystem has two primary considerations. First, constraint (4) must be satisfied. Second, the degree of success that Management Subsystem k has experienced in meeting its goal vector, $G_k(t)$, must be considered. To expedite the latter consideration, each Management Subsystem must generate a feedback vector $\Gamma_k(t)$ (k=1,...,M), that the Supremal Subsystem can incorporate into its decision process at the next iteration. Once the Supremal Subsystem generates the vector $G_k(t)$, Management Subsystem k will incorporate it into the right-hand-side of its constraint (2.k). The specific nature of the interaction between the Supremal and the subordinate Management Subsystems will be defined in Section 3.

This section has developed the three levels of decision-making resulting from the basic decomposition approach to the overall organizational problem. In addition the basic flows of coordinative information which allow individual decision-making subsystems to interact have been specified. The next section will describe the specific nature of the interactions among the decision-making subsystems. Through this process the specific organizational models mentioned previously will emerge.

III. Definition of the Organizational Models

In the previous section a general decomposition approach was outlined for an organization's resource allocation problem. This approach resulted in three hierarchial levels of decision-making. In this section specific decisions will be defined for each level of the decision-making hierarchy. Included in this definition will be a short discussion of the decision strategy being employed by the decision-making subsystem. For the Supremal and the Infimal Subsystems, more than one type of decision will be presented. By selecting the alternative decision processes for each level of the hierarchy, specific organizational models will result. In this manner, the decision strategies of several organizational models can be contrasted. Section IV will then return to basic decomposition approach and discuss in greater detail some of the fundamental deficiencies of the organizational models. This discussion will focus upon the model's ability to solve the overall organizational problem. Subsequently, the decisions for each subsystem will be defined.

For the Supremal Subsystem, three basic decisions processes are stated in equations (5) though (15). For Management Subsystem k, a single decision type is given by equation (16) through (21). Finally, for the Infimal Subsystems, two basic decision processes are specified by equations (22) through (29). Because the Management Subsystems serve as the primary coordinators between the Supremal and the Infimal Subsystems, their decisions will be discussed first.

On iteration t, each Mangement Subsystem ($k=1,\ldots,M$) has an external goal vector, $G_k(t)$, which has been generated by the Supremal, and an internal goal vector, g_k, which is assumed to be constant throughout the iterative solution process. For each of its subordinate Infimal Subsystems, ($i=r_{k-1}+1,\ldots,r_k$), Management Subsystem k has a set of vectors, $(X_i(1),\ldots,X_i(t))$, which Infimal Subsystem i has generated during the preceding iterations. These vectors may be interpreted as a series of operating proposals submitted by Infimal i. Using equations (19) and (20), Manager k generates a composite proposal vector, $X_i^*(t)$, for each of its subordinate Infimals as a convex combination of the previous vectors or operating proposals. The B_i and B_i' matrices linearly relate these composite proposal vectors, $X_i^*(t)$, to the external goal vector, $G_k(t)$, and

Supremal Subsystem

Type I

$$\text{Min } \Sigma_{k=1}^{M}[C_{G_k} - \Pi_k(t)]G_k(t+1) \qquad (5)$$

$$\text{s.t. } \Sigma_{k=1}^{M} P_k G_k(t+1) \leq G_0 \qquad (6)$$

$$G_k(t+1) \geq 0 \qquad (7)$$

$$\text{for } k=1,\ldots,M$$

Feedback Information: $\Gamma_k(t+1) = \Pi_k(t)$

Type II

$$\text{Min } \Sigma_{k=1}^{M} \zeta_k(t+1) \qquad (8)$$

$$\text{s.t. } \zeta_k(t+1) + [\Pi_k(s) - C_{G_k}]G_k(t+1) \geq$$

$$z_k^*(s) + [\Pi_k(s) - C_{G_s}]G_k(s) \qquad (9)$$

$$\text{for } k=1,\ldots,M;\ s=1,\ldots,t$$

$$\Sigma_{k=1}^{M} P_k G_k(t+1) \leq G_0 \qquad (10)$$

$$G_k(t+1) \geq 0 \qquad (11)$$

$$\text{for } k=1,\ldots,M$$

Feedback Information: $\Gamma_k(t+1) = \Pi_k(t)$

In equation (9) $z_k^*(s)$ is the optimal value of Manager k's objective function, i.e. equation (16), for iteration s.

<u>Supremal Subsystem</u> (cont'd)

Type III

$$\text{Min } \Sigma_{k=1}^{M}[C_{G_k} G_k(t+1) + W_k^+ S_k^+(t+1) + W_k^- S_k^-(t+1)] \tag{12}$$

$$\text{s.t.} \quad G_k(t+1) + I_{m_k} S_k^+(t+1) - I_{m_k} S_k^-(t+1) = G_k(t) + \Gamma_k(t+1) \tag{13}$$

for k=1,...,M

$$\Sigma_{k=1}^{M} P_k G_k(t+1) \leq G_0 \tag{14}$$

$$G_k(t+1),\ S_k^+(t+1),\ S_k^-(t+1) \geq 0 \tag{15}$$

for k=1,...,M

Feedback Information: $\Gamma_k(t+1) = Y_k^+(t) - Y_k^-(t)$

Managing Subsystem k

$$\text{Min } \sum_{i=r_{k-1}+1}^{r_k} C_i X_i^*(t) + W_k^+ Y_k^+(t) + w_k^+ y_k^+(t) + W_k^- Y_k^-(t) + w_k^- y_k^-(t) \tag{16}$$

$$\text{s.t. } \sum_{i=r_{k-1}+1}^{r_k} B_i X_i^*(t) - I_{m_k} Y_k^+(t) + I_{m_k} Y_k^-(t) = G_k(t) \tag{17}$$

$$\sum_{i=r_{k-1}+1}^{r_k} B_i' X_i^*(t) - I_{m_k'} y_k^+(t) + I_{m_k'} y_k^-(t) = g_k \tag{18}$$

$$X_i^*(t) = \sum_{j=1}^{t} \lambda_i(j) X_i(j) \tag{19}$$

$$\text{for } i=r_{k-1}+1,\ldots,r_k$$

$$\sum_{j=1}^{t} \lambda_i(j) = 1 \tag{20}$$

$$\text{for } i=r_{k-1}+1,\ldots,r_k$$

$$\left.\begin{array}{l} \lambda_i(j) \geq 0 \text{ for } i=r_{k-1}+1,\ldots r_k \text{ and } j=1,\ldots,t \\ Y_k^+(t),\ Y_k^-(t),\ y_k^+(t),\ y_k^-(t) \geq 0 \end{array}\right\} \tag{21}$$

Infimal Subsystem i

Type I

$$\text{Min } [C_i - \Pi_k(t)B_i - \Pi'_k(t)B'_i]X_i(t+1) \quad (22)$$

$$\text{s.t.} \quad D_iX_i(t+1) \leq F_i \quad (23)$$

$$X_i(t+1) \geq 0 \quad (24)$$

$$\text{Coordinative Input:} \quad \gamma_i(t+1) = \begin{bmatrix} \Pi_k(t) \\ \hline \Pi'_k(t) \end{bmatrix}$$

Type II

$$\text{Min } C_iX_i(t+1) + W_k^+\Psi_k^+(t+1) + w_k^+\psi_i^+(t+1) + W_k^-\Psi_i^-(t+1) + w_k^-\psi_i^-(t+1) \quad (25)$$

$$\text{s.t.} \quad \begin{bmatrix} B_i \\ \hline B'_i \end{bmatrix} X_i(t+1) - I_{m_k+m'_k} \begin{bmatrix} \Psi_i^+(t+1) \\ \hline \psi_i^+(t+1) \end{bmatrix} + I_{m_k+m'_k} \begin{bmatrix} \Psi_i^-(t+1) \\ \hline \psi_i^-(t+1) \end{bmatrix} = \gamma_i(t+1) \quad (26)$$

$$D_iX_i(t+1) \leq F_i \quad (27)$$

$$X_i(t+1),\ \Psi_i^+(t+1),\ \psi_i^+(t+1),\ \Psi_i^-(t+1),\ \psi_i^-(t+1) \geq 0 \quad (28)$$

$$\text{Coordinative Input:} \quad \gamma_{i(t+1)} = \begin{bmatrix} B_i \\ \hline B'_i \end{bmatrix} X_i^*(t) - \begin{bmatrix} Y_k^+(t) \\ \hline y_k^+(t) \end{bmatrix} + \begin{bmatrix} Y_k^-(t) \\ \hline y_k^-(t) \end{bmatrix} \quad (29)$$

the internal goal vector, g_k, through equations (17) and (18), respectively. In these constraints, the deviation vectors, $Y_k^+(t)$, $Y_k^-(t)$, $y_k^+(t)$ and $y_k^-(t)$, are computed. The objective function of Manager k minimizes the weighted sum of these deviation vectors in conjunction with the actual cost of the composite proposal vectors.[2] This objective function corresponds to the k-th term of the summation comprising equation (1). Therefore, in solving its decision on iteration t, Management Subsystem k generates the optimal set of composite Infimal proposal vectors $X_i^*(t)$ for $i=r_{k-1}+1,\ldots,r_k$ and an optimal set of deviation vectors $Y_k^+(t)$, $Y_k^-(t)$, $y_k^+(t)$ and $y_k^-(t)$. Associated with this solution are two simplex multipler vectors, $\pi_k(t)$ and $\pi_k'(t)$, for equations (17) and (18), respectively. From this solution, Management Subsystem k generates the essential coordinative inputs for the decisions at the other hierarchial levels. These coordinative inputs include the feedback vector, $\Gamma_k(t)$, for the Supremal Subsystem and the coordinative input vector, $\gamma_i(t)$, for each Infimal Subsystem. The specific formulation of these coordinative inputs are unique to the three decision structures specified for the Supremal in equations (5) through (15) and two decision types for the Infimal specified in equations (22) through (29).

The Supremal Subsystem's role is to coordinate the Management Subsystems. This is achieved through the generation of the set of external goal vectors, $(G_1(t+1),\ldots,G_M(t+1))$, which will be used by the Management Subsystems on the next iteration. Three basic decision types

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[2]Omission of all cost vectors, C, from the mathematical statement of each decision type creates a pure goal programming structure. In certain situations, non-zero cost vectors are desirable; see Davis [6].

have been given for the Supremal Subsystem. In all three decision structures, equation (4) of the overall problem is considered during the generation of the external goal vectors. However, the basic strategy employed by the Supremal Subsystem to generate the goal vectors differs for each decision type. In the Type I Decision, the Supremal simply minimizes the reduced cost of the external goals with respect to Management Subsystems' optimal solutions for the previous iteration. In this manner, the Supremal Subsystem behaves as a column generator for equation (4) in Figure 1. That is it is acting as if the Dantzig-Wolfe decomposition procedure [3] were applied to the block-angular structure displayed in the Figure 1.

The second decision type for the Supremal Subsystem uses the computational approach of Benders' decomposition procedure [2] to generate partioning constraints upon the feasible goal space given by equation (4). Contrasted with Supremal Decision Type I, this approach allows the Supremal to generate any goal vector, $(G_1(t+1),\ldots,G_M(t+1))$, contained in the feasible region defined by equation (4). With Supremal Decision Type I, the Supremal Subsystem can only generate extreme points of this feasible region as potential goals. Like the Supremal Decision Type I, the Supremal Subsystem uses the simplex multiplier associated with equation (17) as the primary feedback information from the Management Subsystem k.

The Supremal Decision Type III differs from the two previous Supremal decisions in that it uses the deviation vectors, $Y_k^+(t)$ and $Y_k^-(t)$, resulting from Managing Subsystem k's decision on the previous iteration as the source of feedback information. Using this information and the goal vector which the Supremal Subsystem generated on the previous iteration for Management Subsystem k, $G_k(t)$, the Supremal Subsystem generates the effective goal vector which Management Subsystem k's current decision $(X^*_{r_{k-1}+1}(t),\ldots,X^*_{r_k}(t))$ would satisfy as an equality. This effective goal vector is given as the right-hand-side to equation (13). The deviation vectors $S^+_k(t+1)$ and $S^-_k(t+1)$ are then introduced to the left-hand-side of equation (13) in order to compute the deviations of $G_k(t+1)$ from this effective goal vector. Like the Supremal Decision Type II, equation (13) allows the Supremal Subsystem to generate any composite goal vector $(G_1(t+1),\ldots,G_M(t+1))$ satisfying equation (4) for consideration by the Management Subsystems on the next iteration. Through this procedure, the Supremal attempts to adjust the composite goal vector so that the combination of the cost of the goals and the weighted sum of the resulting deviations from the effective goal vectors, $G_k(t) + \Gamma_k(t)$ $(k=1,\ldots,M)$, are minimized.

For the Infimal Subsystems, two basic decisions have been presented. The first is simply the basic column generator for equation (3.1) derived from the application of the Dantzig-Wolfe decomposition procedure [3] to the overall organizational problem. In this approach the i-th Infimal Subsystem attempts to minimize the relative cost of its proposal, $X_i(t+1)$, with respect to Management Subsystem k's current solution while simultaneously insuring the feasibility of $X_i(t+1)$ with respect to equation (3.1). The coordinative inputs to the Infimal Subsystem are the simplex multipliers associated with its superordinate Management Subsystem's equations (17) and (18).

The formulation of Infimal Decision Type II is similar to the Supremal's Decision Type III. However, the formulations differ in that the Supremal Subsystem must be concerned with the current solutions for all the Management Subsystems, while the i-th Infimal Subsystem is concerned only with the current solution of its superordinate Management Subsystem. The coordinative input for Infimal Subsystem i's decison is its goal vector, $\gamma_i(t+1)$. Management Subsystem k generates this prospective goal vector, $\gamma_i(t+1)$, using equation (29). If the i-th Infimal Subsystem could generate a proposal vector, $X_i(t+1)$, that fulfills each of the goals contained in $\gamma_i(t+1)$, then the Management Subsystem could completely satisfy its current goals, $G_k(t)$ and g_k. Equation (26) allows Infimal Subsystem i to estimate the deviations from $G_k(t)$ and g_k that will result from its proposal, $X_i(t+1)$. The feasibility of $X_i(t+1)$ with respect to equation (3.1) is also insured. The selection of the optimum $X_i(t+1)$ is determined by the minimization of the cost of the proposal vector and the penalty costs of the deviations.

The discussion of the basic decision types for each of decision-making subsystems is admittedly sketchy in detail. For a more detailed discussion of Supremal Decision Types I and III, Management Decision Type I and Infimal Decision Type I, the reader is referred to Davis [6, 7], Davis and Talavage [5] or Whitford [17, 18]. For a discussion of the Supremal Decision Type II, the reader is referred to Benders [2], Freeland [11] or Freeland and Baker [10]. The remainder of this section will focus upon the application of these basic decision types in the three-level organizational models listed in Table 1. To this end, one decision type will be selected for each decision-making level in the organizational hierarchy depicted in Figure 3.

For this study, five models have been identified in the literature. These models are:

Organizational Model	Subsystem Decision Type		
	Supremal	Managing	Infimal
Ruefli GGD	I	I	I
Freeland & Baker	II	I	I
Davis & Talavage CGD	III	I	I
Davis & Talavage HGD	I / III	I	I
Davis GHM	III	I	II
Study Model F-B/D	II	I	II

Decision Types Employed by Organizational Models

Table 1

1. The Generalized Goal Decomposition Model defined in Ruefli [14, 15]
2. Freeland and Baker's model defined in Freeland [9 , 11] and its two-level version defined in Freeland and Baker [10]

3 and 4. The Centralized and Hybrid Goal Decomposition Models defined in Davis [4] and Davis and Talavage [5]

5. The Generalized Hierarchial Model defined in Davis [7] and Whitford [17].

Table 1 outlines the decision type utilized by each model for each level of the decision-making hierarchy. With the exception of the Hybrid Goal Decomposition (HGD) model, a single decision type has been assigned for each level. The HGD model uses a two-stage decision-making process at the Supremal level. First a tentative goal, $G_k(t+1)$, is generated using the Supremal Decision Type I. The $G_k(t+1)$ vectors are subsequentially modified via Supremal Decision Type III. This model was originally designed to test the advantages of using both simplex multipliers and deviation vectors resulting from the Management Subsystems' decisions as feedback information to the Supremal Subsystem.

One should note that the five models described previously do not encompass all potential combinations of decision types at the appropriate decision-making levels. For example, an alternative model could use Supremal Decision Type II in conjunction with Infimal Decision Type II. This model, referred to as a study or Freeland-Baker/Davis (F-B/D) model in Table 1, represents a hybrid of the Freeland and Baker and Davis Generalized Hierarchial models and decision types. The following section will describe the decomposition procedures of each of these six organizational models.

IV. Organizational Models as Decomposition Procedures

Before discussing the efficacy of the organizational models in Table 1 as decomposition procedures, several points are worth noting. Recently, research has defined organizational models as composition approaches. This terminology implies that the modeler defines the type of decisions used at each level of the hierarchy, independently of any characterization of the overall problem for the organization. Although Sweeney, et al. [16] recently focused upon the overall problem for these composition models, they maintained a distinction between the composition and decomposition approaches. They proposed the existence of two overall problems. The first or "Ideal Organizational Problem" was defined as the organization's actual problem. The second, called the "Design Problem," represented a separate formulation. The optimum solution of the Design problem and the optimum solution generated by a particular organizational algorithm are identical. This necessarily implies that the application of different organizational algorithms could result in different Design Problem formulations. Unfortunately Sweeney et al. provided no mathematical definition of either the Design or Ideal Organizational Problems.

Several difficulties and questions are created by the existence of two organizational problems. First the interrelationship of one problem to the other must be defined. Do the two problems have the same optimal solution? If not, are the set of feasible solutions for both the problems the same? If not, what leads to the differences? Does the composition approach lead to additional constraints necessitated by the introduction of an organizational structure? If so, can the nature of these constraints be defined or explained? The list of questions continues.

The most serious implication created by the existence of two problems infers that a complete specification of either <u>overall</u> organizational problem and, in particular, the Design Problem is a difficult, if not impossible task.

It has also been conjectured that composition models can incorporate the hierarchical structure of a organization. Further it has been suggested that a change in the structure of an organization will result in a change in the <u>composition</u> model's optimal solution. This implies that the Design Problem changes as modifications are made in an organization's structure. These assertions have <u>not</u> been computationally tested.

This paper has treated organizational models as decomposition approaches. Section II defined this approach, while Section III specified the specific decomposition procedure used by each model. Computational experience [19] has shown that these models <u>do</u> converge to a limiting solution quickly. However, in computational testing, the potential for a nonoptimal limiting solution to the "overall problem" given by equations (1) through (4) has been demonstrated [5].

This nonoptimality does <u>not</u> justify the existence of a Design Problem. Instead it points to the ineffectiveness of an individual algorithm's ability to solve the overall problem. There are several potential sources for each algorithm's shortcomings. First, no true master program for the decomposition of the overall problem has been defined. Instead each of the models utilizes a partial master program, in which each algorithm considers only a portion of the overall problem's constraints. Although these partial masters are coordinated through the Supremal Subsystem, the effectiveness of this coordination has not been demonstrated. Further, no individual decisison maker is concerned with the organization's overall objective.

Second, the partial master programs tend to be highly degenerate. This implies the existence of multiple optimal solutions to their dual problems. Because these multiple yet distinct optimal dual solutions or simplex multipliers are used as coordinative mechanisms in certain algorithms, the efficacy of these models is at best marginal. To test the effectiveness of the simplex multiplier as a coordination mechanism, a Dantzig-Wolfe decomposition procedure was applied to the two-level subproblem, SP.i, described by equations (2.k), (3.i) ($i=r_{k-1}+1,...,r_k$), a fixed goal vector, G_k, and the k-th term of the summation in equation (1). This Dantzig-Wolfe formulation converged to a nonoptimal solution. Clearly the existence of multiple dual solutions could have played a role in creating this nonoptimality. However, another and more important factor must be considered. This primary factor is the linear formulation of the objective function. This structure implies a constant marginal penalty cost. In contrast, a quadratic objective function would create a increasing marginal penalty cost as a deviation from an individual goal increases. A quadratic penalty function has been incorporated into the GHM [1, 8], and this quadratic GHM converges to the optimum solution. Further, it appears that insertion of quadratic penalty functions will achieve similar results for the other organizational models. If this is the case, then each of the organizational models in Table 1, is a true decomposition procedure. Accordingly, the existence of both an Ideal and Design problem as advocated by Sweeny *et al.* is suspect.

V. Summary and Conclusions

The purpose of this research has been to investigate a series of decomposition models that focus upon a multicriteria approach to resource allocation in hierarchical organizations. A description of an overall resource allocation problem faced by an organization was given. Next description of the strategies utilized by each algorithm to decompose and solve this overall problem was presented. Difficulties created by the use of shadow prices as coordination mechanisms were shown to limit the efficacy of the Ruefli and Freeland models. Further the inability of all multicriteria algorithms to converge to <u>the</u> optimal solution of the overall problems was discussed. In order to overcome problems of non-optimality, use of a quadratic objective function was proposed.

Several implications follow from the findings of this research. First, Ruefli [15] has noted that a change in the bureaucratic structure of an organization should lead to a change in the allocation of the organization's resources. As long as effective coordinating mechanisms are in place, our findings indicate that this may not be the case, unless of course, a bureaucratic change explicitly modifies the constraints of the organization's overall problem. On the other hand, bureaucratic reshuffling could easily precipitate a shift in penalty weightings associated with goal deviations. These shifts in priorities could impact upon the eventual allocation of resources. However, a priority shift would of course change the overall problem's objective function.

Most organizational models utilize shadow prices as coordinating mechanisms. Clearly information exchanges in most if not <u>all</u> organizations are more detailed that the mere passing of dual variables.

Computational testing [5, 18, 19] indicates that the inclusion of performance targets or goals and deviations from those goals provides a more realistic and efficient mechanism for coordinating hierarchical decision-making systems.

Finally, the linear penalty functions associated with organizational models represents a rather naive or simplistic view of decision making. A quadratic penalty function is clearly more realistic and captures the economic and behavioral concepts of diminishing marginal utility and/or increasing marginal disutility. At the theoretical level, incorporation of a quadratic penalty function solves many if not all of the problems of organizational models as pure mathematical, decomposition procedures. At the intuitive level, a quadratic penalty function injects common sense into the mathematical modeling of organizational behavior.

Computational testing of these conclusions is underway, and partial results are reported in a companion paper [19]. Clearly the need for future research that will confirm, refute and/or extend these findings for actual administrative problems is needed.

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Appendix 1

An Overview of Mathematical Decomposition

A decomposition algorithm can basically be described as a procedure which breaks down a large linear programming problem into a collection of smaller subproblems whose solutions in a prescribed manner will generate the solution to the original linear programming problem. To facilitate the application of a decomposition procedure, the original problem should possess what is known as a block-angular structure. This structure is defined by assuming initially that the original vector of decision variables, X, can be partitioned into a set of decision vectors, X_i, i=1,...,N. This partitioning is effected so that the majority of the constraints can be expressed in the form:

$$D_i X_i \{\leq, =, \geq\} F_1 \text{ for } (i=1,\ldots,N),$$

where D_i is a matrix relating the decision X_i to the right-hand-side vector F_i. The remaining constraints, however, involve a coupling of the components of the X_i vectors. These constraints are of the form:

$$B_1 X_1 + \ldots + B_N X_N \{\leq, =, \geq\} G,$$

where B_i is a matrix is a matrix relating X_i to the right-hand-side vector G. Defining the objective function as

$$\text{Min(Max) } C_1 X_1 + \ldots + C_N X_N,$$

the block-angular structure becomes apparent when the original linear programming problem is summarized in equations (A.1) through (A.4).

$$\text{Min (Max)} \quad C_1X_1 + \ldots + C_NX_N \tag{A.1}$$

$$\text{s.t.} \quad B_1X_1 + \ldots + B_NX_N \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} G \tag{A.2}$$

$$\begin{array}{lll} D_1X_1 & & \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} F_1 \\ & \ddots & \vdots \\ & & D_NX_N \begin{Bmatrix} \leq \\ = \\ \geq \end{Bmatrix} F_N \end{array} \tag{A.3}$$

$$X_i \geq 0 \quad (i=1,\ldots,N) \tag{A.4}$$

The decomposition algorithm consists of three primary components: the restricted master program, the set of N column generators (i=1,...,N) and a coordinative informational flow. The restricted master program has three functions:

1) It attempts to optimize equation (A.1).

2) It furnishes the coordinative information flow to each column generator. The coordinative exchange assists each column generator in the selection of $X_i(t)$ at iteration t of the solution process.

3) It generates a composite decision vector, $\{X_1^*(t),\ldots,X_N^*(t)\}$, from the set of decision vectors from iterations 1 through t, $\{X_i(1),\ldots,X_i(t)\}$, for each column generator (i=1,..,N). The composite decision vector is generated such that it explicitly satisfies equation (A.2) and, through the generation process, implicitly satisfies equations (A.3) and (A.4).

The column generator responds to the flow of coordinative information and iteratively generates $X_i(t)$ such that:

$$D_i X_i(t) \{\substack{\leq \\ = \\ \geq}\} F_i \quad \text{and} \tag{A.5}$$

$$X_i(t) \geq 0. \tag{A.6}$$

The coordinative information directs this iterative process toward the eventual solution of the original linear programming problem.