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Monte Carlo methods for barrier options

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Alla mia famiglia, A mio padre

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Contents

Introduction

This thesis develops efficient statistical procedures to price a barrier option using the Monte Carlo (MC) approach.

European options are very popular standard derivatives. An investor has the right to buy or to sell a risky asset, named *underlying asset*, at a future date, called *maturity*: in the first case, the option is called *call*, in the second case *put*. This right can be exercised by paying (for a call) or receiving (for a put) an amount, called *strike price*.

Barrier options are European options whose existence depends on several predefined prices, named *barriers*. More precisely, we can distinguish two types of barrier options: *knock-out* and *knock-in* (for more details on barrier options we refer to [60, 59, 92]).

A knock-out is active at the beginning of its life, but becomes inactive if the underlying asset reaches the barriers. The payoff of a knock-out is equal to the vanilla payoff if the underlying does not reach the barriers, otherwise it is null.

On the contrary, a knock-in is inactive at the beginning of its life, but it is activated if the underlying reaches the barriers. The payoff of a knock-in is equal to the vanilla payoff if the underlying reaches the barriers, otherwise it is null.

A barrier option can be classified on the basis of the number of monitoring instants (continuous or discrete) and the number of barriers (single or double).

- *Continuous*. The underlying is monitored at every instant.
- *Discrete*. The underlying is monitored at a finite set of instants.
- *Single*. The barrier level is unique.
- *Double*. The barrier level is double.

In this thesis we deal with the problem of pricing a barrier option. A very common approach to price a barrier option is the so-called *martingale approach*: the price of a barrier option can be expressed as the actualized expected option payoff under a particular probability measure, named *risk-neutral measure* (RNM). We study this problem firstly under the assumption of the constant volatility, i.e. *Black-Scholes model* (BSM), and, successively, under the assumption of the stochastic volatility (SV), i.e. *stochastic volatility model* (SVM).

Under the BSM [15], the barrier option pricing problem is well-posed because the completeness of the market ensures the existence and uniqueness of the RNM [29]. From a numerical point of view, the option price can be computed using three types of techniques: *closed formulas*, *MC methods*, *numerical schemes*.

Under a SVM [16, 103], the barrier option pricing problem is not well-posed because the incompleteness of the market implies that more RNMs exist. In this case we proceed as suggested by Back [6]: the RNM is chosen on the basis of the investors' preferences and endowments and on the basis of the production possibilities. From a numerical point of view, since no closed formulas exist, the option price can be computed using *MC methods* or *numerical schemes*.

In this thesis we consider only MC methods. We start from the basic MC methods, known as *standard Monte Carlo* (MCst) *methods*. A MCst method works as follows: firstly, it generates a set of underlying paths over a discrete time set and, successively, it computes the standard average of the actualized option payoff realizations over the underlying paths, called *MCst estimator*. It results evident that the crucial point of a MCst procedure is the *generation of the underlying paths*.

The goodness of a MC estimator is measured by three variables: the precision, the accuracy and the bias.

- *Precision*. The inverse of the variance.
- *Accuracy*. The inverse of the mean squared error.
- *Bias*. The difference between the expected value of the estimator and the real price value.

They are related as

$$\frac{1}{Accuracy} = Bias^2 + \frac{1}{Precision},$$

which implies that, in order to evaluate the performance of a MC estimator, we can analyze only the bias and the precision. In order to achieve a good level of performance, a large number of paths is required because the value of the precision and of the accuracy is inversely proportional to the number of the underlying paths.

The main problems of a MCst estimator for barrier options are: the *high bias* and the *low precision*.

The problem of the bias is mainly related to the case of a continuous barrier option [39]. In order to simulate the payoff correctly, we should check if the underlying reaches the barriers at every instant, but a MCst estimator takes in account the underlying values only over a discrete time set: this loss of information introduces a bias in a MCst estimator.

The problem of the low precision is related to the possibility that the underlying paths could be rejected: this phenomenon is very common for a knock-out with initial underlying value approaching the barriers [96]. This implies that the variance of the estimator is high and, consequently, the precision is low.

These issues can be overcome using a class of MC methods, known as *Bayesian MC* (BMC) *methods*. The goal of these methods is to construct an estimator of the actualized *conditional expected payoff* given a set of variables correlated with the underlying, named *observations*. It is summarized as follows: firstly, at every time step, a sample of underlying values are generated from the conditional density of the underlying value given the observations, then we evaluate the standard average of the actualized option payoff realizations over the underlying paths. In short, a BMC procedure is a MCst procedure to evaluate

the actualized conditional expected payoff and, consequently, the crucial point is the *sampling step*.

In this thesis we propose some sampling schemes: the *Metropolis-Hastings algorithm* (MHA), the *Kalman filter* (KF), *Sequential Monte Carlo methods* (MCse) and the *bootstrap filter* (BF).

The MHA [81] addresses the problem of simulating a Markov process using, at every time step, the past underlying values as observations. It can be summarized as follows: firstly, at every time step, a set of the stochastic process values are sampled from a prefixed density, called *proposal distribution*, and, successively, they are accepted or rejected according to a fixed probability distribution, named *acceptance probability*.

The KF [66] addresses the problem of simulating a linear Markov stochastic process by choosing the observations on the basis of the problem to solve. It works as follows: firstly, it finds a *prior estimate* of the process using the past values of the process and, successively, using the information inferring from the observations, it constructs a new estimator, depending on the prior estimate, in order to minimize the loss function. The great advantage of the KF is that, by exploiting the linearity of the Markov process, we obtain a linear estimate of the process.

A MCse method [64] is more general than the previous algorithms because no too strong assumptions about the state variable are required. As in the KF, the choice of the observations depends on the particular problem to solve. The idea underlying a MCse is that the distribution of the process to simulate is approximated by a family of densities, named *importance densities*, depending on the observations and selected in order to decrease the variance of a MC estimator. A MCse procedure works as follows: firstly, the stochastic process is simulated using the importance densities and, successively, if the variance is not sufficiently low, the simulations are subject to a resampling procedure.

The BF [45] is a particular MCse method, where, at every step, the importance density is given by the conditional density of the stochastic process to simulate given its historical values, called *transition density*, and the observations depend on the particular problem to solve.

In this thesis we firstly present the main BMC methods under the BSM: the *conditional Monte Carlo* (CMC) *method*, the *exponential twisting* (ET) *method* and the *weighting functions* (WF) *method*.

The CMC method estimates the price of a discrete barrier option, whose observations are represented by the underlying values until the first instant when the barriers are crossed. Using this approach, the barrier option is written as the expected value of a vanilla option price computed at the first instant of crossing the barriers. In short, the CMC method generates the paths using the Black-Scholes formula to draw a sample of vanilla price values at the first instant when the barriers are reached and, successively, it finds the standard average of these values.

The ET addresses the problem of pricing a discrete barrier option using a MCse sampling technique: at every time step, the observations are given by the past underlying values, and the importance densities belong to the class of the *exponential family densities*.

The WF approach estimates the price of a continuous barrier option using a MCse method: at every time step, the observations are represented by the past underlying values, and the importance density is the transition density.

As discussed before, in literature the BMC approach has been developed only under the BSM. In the last part of the thesis we describe our contributes to the option pricing problem: *construction of BMC estimators for continuous barrier options to solve the problem of high bias and low precision of a MCst estimator under a SVM*. More precisely, we propose the following procedures: the *extended weighting functions* (EWF) *algorithm* and the *bootstrap BMC* (BBMC) *method*.

The EWF approach generalizes the WF approach, illustrated before, under a SVM: at every time step, the observations are the past underlying and volatility values, and the importance density is the conditional density of the current underlying value given these observations.

The BBMC method addresses the problem of pricing a continuous barrier option using the BF technique, where at every time step, the observations are given by the past underlying values and by the past and current volatility values.

In order to test the validity of the last two schemes, we have applied them to price three continuous barrier options under a particular SVM, the *SABR model* [49]. This model is very common in financial sector because it provides a numerical procedure to estimate a barrier option price, which is described in the following: firstly the implied volatility is evaluated using Hagan formula [49], then the price of the option, whose volatility is given by the implied volatility and all the other parameter are not changed, is calculated by applying the Kunimoto-Ikeda formula. The consequence of replacing the real value with this approximation is that *the bias of both the EWF and BBMC estimators is not null*. In our numerical experiments we have studied the *effect of the distance between the initial underlying value and the barriers on the bias and precision of MC estimators*: firstly we have priced three continuous knock-out put options with single constant barrier by applying the MCst, the EWF and the BBBC estimators with different numbers of time steps, successively the results have been compared in terms of bias and precision. The options differ among them only for the barrier values: in the first one, the initial underlying value is distant from the barriers, in the second one we decrease this distance and, finally, in the third one the initial underlying value approaches the barriers.

The results of the experiments are summarized in the following.

Firstly, while in the first case the three MC estimators provide good results, the picture changes in the other two cases because the MCst estimator has a *high bias* and a *low precision*. As mentioned before, this phenomenon is related to the distance between the initial underlying value and the barriers: the probability that the underlying crosses the barriers is inversely proportional to the distance between the barriers and the initial underlying value.

Secondly, the BBMC method provides better results than the EWF approach. The reason is that in the first one the underlying simulations are obtained using a greater information set than the second one: at every time step, the BBMC uses not only the past underlying and volatility values, as in the EWF approach, but also the current volatility values.

Thesis Organization

This thesis is structured as follows. Chapter 1 formulates the barrier option price problem and reviews the main results about MCst estimators for barrier options. Chapter 2 explores the main BMC methods under the hypothesis of

constant volatility. Chapter 3 discusses the BMC methods under the hypothesis of SV and provides the numerical test. Finally, we draw the conclusions.

Chapter 1

Barrier option pricing problem

This chapter is divided into two parts. In the first part we formulate the barrier option pricing problem: using the martingale approach, the price of a barrier option is expressed as the actualized expected option payoff under a particular measure, named *risk-neutral measure* (RNM). In the last part of the chapter we detail a class of statistical procedures to compute the option price, named *standard Monte Carlo* (MCst) methods, and we investigate their issues: the *bias* and the *low precision* and *accuracy*.

Section 1.1 reviews the basic probabilistic and statistical notions. Section 1.2 is devoted to the financial background. Section 1.3 formulates the martingale approach for barrier options. Section 1.4 treats MCst methods.

1.1 Mathematical Background

This section recalls the basic definitions of probability (Subsection 1.1.1), stochastic calculus (Subsection 1.1.2) and the basic statistical notions (Subsection 1.1.3).

1.1.1 Basic concepts

We start with the definition of probability space and random variable and, successively, we introduce some notions linked to them, as probability density function, expected value, cumulative distribution function. For more details see [36, 14]).

Definition 1. *[Probability space] A* probability space $\mathcal{S}$ *is a triplet* $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P})$, *where:*

(DF1): Ω *is a nonempty set, named* sample space*;*

(DF2): $\mathcal{F}$*, named* σ-algebra*, is a family of subsets of* Ω*, named* events*, closed respect to the union and the intersection;*

(DF3): $\Omega \in \mathcal{F}$*;*

(DF4): the function $\mathbb{P} : \mathcal{F} \to [0;1]$*, named* probability*, satisfies*

$$\mathbb{P}\left(\bigcup_{i=1}^{I} A_i\right) = \sum_{i=1}^{I} \mathbb{P}(A_i),$$

with $\{A_i\}_{i=1}^{I}$ *a finite or infinite family of disjoint events.*

Property (DF4) is known as Kolmogorov axioms*.*

In finance, a σ-algebra is the *set of the available information about a system variable.*

Definition 2. *Let* $\mathcal{S}$ *be a a probability space (1) and* $I \in \mathbb{N}$*. An* I-dimensional random variable X defined in $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P})$ *is a* $\mathcal{F}-$*measurable function* $X : \Omega \to \mathbb{R}^I$*, i.e.*

$$(X \in A) := \{\omega \in \Omega / X(\omega) \in A\} \in \mathcal{F}, \tag{1.1}$$

for every $A \subset \mathbb{R}^I$*.*

The values that a random variable can assume are called *outcomes*. A random variable X is *continuous* if the set of the outcomes is uncountable. A random variable X is *discrete* if the set of outcomes is finite. Given $M \in \mathbb{N}$, a discrete random variable X with M outcomes can be represented as

$$X = \begin{pmatrix} x_1 & \cdots & x_M \\ p_1 & \cdots & p_M \end{pmatrix}, \tag{1.2}$$

or equivalently $X = (x_m; p_m)_{m=1}^{M}$. The values $x_m \in \mathbb{R}^I$ are the *outcomes* of X, the values $p_m = \mathbb{P}(X = x_m)$ are called *probabilistic weights.*

The smallest σ-algebra containing the sets (1.1) is called *σ-algebra generated by* X, and it is indicated with the symbol $<X>$.

Remark 1.1.1. *In the following, given a a probability space* $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P})$ *(1), an* I*-dimensional random variable* $X : \Omega \to R^I$ *(2) will be simply called* random variable*. In the case* $I = 1$*,* X *will be also called* real random variable*.*

Definition 3. *Let* X *be a random variable (2). The* probability density function *(pdf) of* X *is the function* $\pi(\cdot) : \mathbb{R}^I \to [0;1]$ *defined as:*

$$\pi(x) := \mathbb{P}(X = x), \tag{1.3}$$

for every $x \in \mathbb{R}^I$*.*

If a random variable X is continuous, its density function $\pi(\cdot)$ has the following properties:

(P1): $\pi(x) \geq 0, \quad \forall x \in \mathbb{R}^I$;

(P2): $\int \pi(x)\, dx = 1$.

If a random variable X is discrete, the pdf coincides with the probability weights (1.2, which satisfy the following conditions:

(P1): $p_m \geq 0 \quad m = 1, ..., M$;

(P2): $\sum_{m=1}^{M} p_m = 1$.

The notion of pdf lets us define the expected value and variance of a random variable.

Definition 4 (Expected value)**.** *Let X be a random variable with pdf $\pi(\cdot)$ and $f : \mathbb{R}^I \to \mathbb{R}$ an integrable real function, whose domain contains $X(\Omega)$.*
If X is continuous, the expected value *of $f(X)$ is defined as:*

$$\mathbb{E}[f(X)] := \int f(x)\pi(x)\, dx. \tag{1.4}$$

In particular, if X is real and f is the identical function in (1.4), one has the expected value *of X:*

$$\mathbb{E}[X] := \int x\pi(x)\, dx. \tag{1.5}$$

If X is discrete (1.2), the expected value *of $f(X)$ is defined as:*

$$\mathbb{E}[f(X)] := \sum_{m=1}^{n} p_m f(x_m), \tag{1.6}$$

In particular, if X is real and f is the identical function in (1.4), one has the expected value *of X:*

$$\mathbb{E}[X] := \sum_{m=1}^{M} p_m x_m, \tag{1.7}$$

with $M \in \mathbb{N}$.

Definition 5 (Variance and standard deviation)**.** *Let X be a random variable and $f : \mathbb{R}^I \to \mathbb{R}$ an integrable real function, whose domain contains $X(\Omega)$.*
If X is continuous, the variance *of $f(X)$ is defined as:*

$$\mathbb{V}[f(X)] := \mathbb{E}[f(X)^2] - \mathbb{E}^2[f(X)], \tag{1.8}$$

If X is real and f is the identical function in (1.8), one has the variance *of X:*

$$\mathbb{V}[X] := \mathbb{E}[X^2] - \mathbb{E}^2[X], \tag{1.9}$$

The square root $\sqrt{V[X]}$ is the standard deviation *of X.*

In economics the variance and the standard deviation are the most common risk measures.

Definition 6 (Cumulative distribution function)**.** *The* cumulative distribution function *(cdf) $F(\cdot)$ of a real random variable X is the function $F : \mathbb{R} \to [0; 1]$ defined as*

$$F(x) := \mathbb{P}(X \leq x), \tag{1.10}$$

for every $x \in \mathbb{R}$.

If X is continuous, the cdf of X takes the form:

$$F(x) := \int_{-\infty}^{x} \pi(y)\, dy, \quad \forall x \in \mathbb{R}. \tag{1.11}$$

If X is discrete, the cdf of X takes the form:

$$F(x) := \sum_{x_m \leq x} \pi_m x_m, \quad m = 1, ..., M. \tag{1.12}$$

Two common examples of continuous random variables are the Normal and the Uniform.

Definition 7. *[Normal random variable] Given $\mu \in \mathbb{R}$, $\sigma \in]0; +\infty[$, a continuous real random variable X is a* Normal with mean μ and standard deviation σ, *in symbols $X \sim N(\mu; \sigma)$, if its pdf $\pi(\cdot)$ (1.3) is:*

$$\pi(x) = \frac{1}{\sigma\sqrt{2\pi}} \int e^{-\frac{(x-\mu)^2}{\sigma^2}} \, dx, \quad \forall x \in \mathbb{R}. \tag{1.13}$$

If $\mu = 0$ and $\sigma^2 = 1$, X is named standard.

Definition 8. *Given $a, b \in \mathbb{R}$, with $a < b$, a continuous real random variable X is a* Uniform in $[a; b]$, *in symbols $X \sim U([a; b])$, if its pdf $\pi(\cdot)$ is:*

$$\pi(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{othewise,} \end{cases} \tag{1.14}$$

with $a, b \in \mathbb{R}$.

Let $N, M \in \mathbb{N}$. We consider N random variables $X_1, ..., X_N$, defined in the same probability space $\mathcal{S}$ (1). For every $n = 1, ..., N$, we set

(N1): x_n is an outcome of X_n, named *particle of* X_n;

(N2): $X_{1:n} := (X_1, ..., X_n)$;

(N3): $x_{1:n} := (x_1, ..., x_n)$ is a *path* of $X_{1:N}$;

(N4): if X_n is discrete, its M outcomes are denoted with the symbol x_{nm}, $m = 1, ..., M$.

In the following we define the joint density, marginal density and conditional density; successively, we provide the notions of covariance matrix and uncorrelated random variables.

Definition 9. *The* joint density of $X_{1:N}$ *is its pdf:*

$$\pi(x_{1:N}) := \mathbb{P}(X_1 = x_1, ..., X_N = x_N). \tag{1.15}$$

For every $n = 1, ..., N$, the marginal density *of X_n is the pdf of X_n:*

$$\pi(x_n) := \mathbb{P}(X_n = x_n). \tag{1.16}$$

If $\pi(x_{1:n-1}, x_{n+1:N}) \neq 0$, the conditional density of X_n given $X_{1:n-1}, X_{n+1:N}$ *is*

$$\pi(x_n | x_{1:n-1}, x_{n+1:N}) := \frac{\pi(x_{1;N})}{\pi(x_{1:n-1}, x_{n+1:N})}. \tag{1.17}$$

The random variables $X_{1:N}$ are said independent *if:*

$$\pi(x_n | x_{1:n-1}, x_{n+1:N}) = \pi(x_n), \quad n = 1, ..., N.$$

A vector of independent random variables with the same density function is denoted with the symbol i.i.d *(independent and identically distributed).*

In the following, a particle x of X_n is also indicated with the symbol $x \sim \pi(x_n)$, a path x of $X_{1:N}$ is also indicated with the symbol $x \sim \pi(x_{1:N})$.

From an economic point of view, the conditional density $\pi(x_n|x_{1:n-1})$ is *the density of* X_n *using information relative to* $X_{1:n-1}$: this implies that, *if we increase the number of conditioning random variable, we increase the information set* and, consequently, *we improve our estimation.*

Using recursively (1.17), we obtain:

$$\pi(x_{1:n}) = \prod_{n=1}^{N} \pi(x_n|x_{1:n-1}). \tag{1.18}$$

Definition 10 (Covariance matrix)**.** *The* covariance matrix *of a random vector* $X_{1:N}$*, with expected values* $\mu_n = \mathbb{E}[X_n]$ *and variances* $\sigma_n^2 = \mathbb{V}[X_n]$*,* $n = 1, ..., N$*, is the matrix* $C = (c_{i,j})_{i,j=1}^{N} \in \mathbb{R}^N \times \mathbb{R}^N$ *defined as*

$$c_{ij} = \begin{cases} \sigma_i^2 & \text{if } i = j \\ \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)] & \text{if } i \neq j, \end{cases} \tag{1.19}$$

for every $i, j = 1, ..., N$*.*

Definition 11 (Uncorrelation)**.** *For every* $i, j = 1, ..., N$ *with* $i \neq j$*, the random variables* X_i *and* X_j *are* uncorrelated *if:*

$$\mathbb{E}[X_i^T X_j] = \mathbb{E}[X_j^T X_i] = \mathbb{E}[X_i]\mathbb{E}[X_j], \tag{1.20}$$

where A^T *denotes the transpose of a matrix* A *.*

The conditional and joint densities of a random vector are linked by the *marginalization rule* (MR).

Theorem 12 (MR)**.** *If* X_n *is a discrete then:*

$$\pi(x_n|x_{1:n-1}, x_{n+1:N}) = \sum_{m=1}^{M} \pi(x_{1:n-1}, x_{nm}, x_{n+1:N}). \tag{1.21}$$

If X_n *is continuous, then:*

$$\pi(x_n|x_{1:n-1}, x_{n+1:N}) = \int \pi(x_{1:N})\, dx_n, \tag{1.22}$$

for every $n = 1, ..., N$*.*

The marginal densities and the conditional density of a random vector are related by the *law of total probability* (LTP).

Theorem 13 (LTP)**.** *If,* X_n *is discrete, then:*

$$\pi(x_n) = \sum_{m=1}^{M} \pi(x_n|x_{1:n-1,m}, x_{n+1:N,m})\pi(x_{1:n-1,m}, x_{n+1:N;m}) \quad n = 1, ..., N, \tag{1.23}$$

if X_n *are continuous, then:*

$$\pi(x_n) = \int \pi(x_n|x_{1:n-1}, x_{n+1:N})\pi(x_{1:n-1}, x_{n+1:N})\, dx_{1:n-1} dx_{n+1:N}, \tag{1.24}$$

for every $n = 1, ..., N$*.*

The conditional density (1.17) lets us define the conditional expected value and conditional variance of a random variable.

Definition 14 (Conditional expected value and variance)**.** *Let $f : \mathbb{R}^I \to \mathbb{R}$ be an integrable real function whose domain contains $X_n(\Omega)$ for every n. If X_n is continuous, the* conditional expected value of $f(X_n)$ given $X_{1:n-1}, X_{n+1:N}$ *is:*

$$\mathbb{E}[f(X_n)|X_{1:n-1}, X_{n+1:N}] := \int f(x_n)\pi(x_n|x_{1:n-1}, x_{n+1:N})\, dx_n. \tag{1.25}$$

If X_n is discrete, the conditional expected value of $f(X_n)$ given $X_{1:n-1}, X_{n+1:N}$ *is*

$$\mathbb{E}[f(X_n)|X_{1:n-1}, X_{n+1:N}] := \sum_{m=1}^{M} f(x_m)\pi(x_{nm}|x_{1:n-1}, x_{n+1:N}). \tag{1.26}$$

The conditional variance of $f(X_n)$ given $X_{1:n-1}, X_{n+1:N}$ *is:*

$$\begin{aligned} \mathbb{V}[f(X_n)|X_{1:n-1}, X_{n+1:N}] &= \mathbb{E}[f(X_n)^2|X_{1:n-1}, X_{n+1:N}] \\ &- \mathbb{E}^2[f(X_n)|X_{1:n-1}, X_{n+1:N}]. \end{aligned} \tag{1.27}$$

If X_n is real and f is the identical function in (1.25), one has the conditional expected value *and the* conditional variance.

If X_n is continuous, the conditional expected value takes the form:

$$\mathbb{E}[X_n|X_{1:n-1}, X_{n+1:N}] = \int x_n\pi(x_n|x_{1:n-1}, x_{n+1:N})\, dx_n, \tag{1.28}$$

if X_n is discrete, then:

$$\mathbb{E}[X_n|X_{1:n-1}, X_{n+1:N}] := \sum_{m=1}^{M} x_{nm}\pi(x_{nm}|x_{1:n-1}, x_{n+1:N}). \tag{1.29}$$

The conditional variance *is equal to:*

$$\begin{aligned} \mathbb{V}[X_n|X_{1:n-1}, X_{n+1:N}] &= \mathbb{E}[X_n^2|X_{1:n-1}, X_{n+1:N}] + \\ &- \mathbb{E}^2[X_n|X_{1:n-1}, X_{n+1:N}], \end{aligned} \tag{1.30}$$

for every $n = 1, ..., N$.

The variance of a random variable can be expressed in terms of conditional expected value and conditional variance by means of the *law of total variance* (LTV)

Theorem 15 (LTV)**.** *The variance $\mathbb{V}[X_n]$ of X_n can be expressed as:*

$$\begin{aligned} \Omega_n &= \mathbb{E}[X_n|X_{1:n-1}, X_{n+1:N}] \\ \Delta_n &= \mathbb{V}[X_n|X_{1:n-1}, X_{n+1:N}] \\ \mathbb{V}[X_n] &= \mathbb{V}[\Omega_n] + \mathbb{E}[\Delta_n], \end{aligned} \tag{1.31}$$

for every $n = 1, ..., N$.

A theoretical result involving the concepts of expected value and conditional expected value of a random variable is the *law of total expectations* (LTE):

Theorem 16 (LTE). *The expected value $\mathbb{E}[X_n]$ can be expressed as:*

$$\Omega_n = \mathbb{E}[X_n|X_{1:n-1}, X_{n+1:N}] \quad (1.32)$$
$$\mathbb{E}[X_n] = \mathbb{E}[\Omega_n], \quad (1.33)$$

for every $n = 1, ..., N$.

This subsection concludes with the Bayes theorem [10] and the notion of multivariate Normal random variable.

Theorem 17 (Bayes theorem). *Let $X_{1:N}$ and $Y_{1:N}$ be random vectors defined in the same probability space and, for every n, $x_{1:n}$ and $y_{1:n}$ respectively generic outcomes of $X_{1:n}$ and $Y_{1:n}$. If $\pi(y_{1:n}) \neq 0$, the conditional densities $\pi(x_{1:n}|y_{1:n})$ and $\pi(y_{1:n}|x_{1:n})$ (1.17) are related as:*

$$\pi(x_{1:n}|y_{1:n}) = \frac{\pi(y_{1:n}|x_{1:n})}{\pi(y_{1:n})}\pi(x_{1:n}), \quad (1.34)$$

for every $n = 1, ..., N$.

A common example of random vector is the multivariate Normal random variable.

Definition 18 (Multivariate Normal random variable). *Let $\mu_{1:N}$ be a N-dimensional real vector and $C = (c_{ij})_{i,j=1}^N$ a real matrix. A random vector $X_{1:N}$ is a* multivariate Normal random variable with mean $\mu_{1:N}$ *and covariance matrix C (1.19), in symbols $X_{1:N} \sim N(\mu_{1:N}, C)$, if $X_n \sim N(\mu_n; c_{nn})$ for every n (1.13). If $X_n \sim N(0; 1)$, $n = 1, ..., N$, the random vector $X_{1:N}$ is said* standard.

Next section is devoted to basic results of stochastic calculus.

1.1.2 Stochastic calculus

We begin with the definition of filtered probability space and stochastic process (for more references see [97, 86]).

Definition 19 (Filtered Probability Space). *Let T be a positive real number. A filtered probability space $\mathcal{S}_T$ is a triplet $\mathcal{S}_T = (\Omega, (\mathcal{F}_t)_{t\in[0;T]}, \mathbb{P})$, where*

(DF1): Ω is the sample space;

(DF2): $(\mathcal{F}_t)_{t\in[0;T])}$ is a family of continuous σ-algebras increasing respect to the inclusion, called filtration*;*

(DF3): $\mathbb{P}$ is a probability measure.

In finance a filtration is the set of available information in a time interval $[0; T]$.

Definition 20 (Stochastic process). *Let $\mathcal{S}_T$ be a filtered probability space (19), and t_n, $n = 1, ..., N$, a set of N elements of $[0; T]$.*
An uncountably family of random variables $(X_t)_{t\in[0;T]}$ (2) is a continuous stochastic process *if X_t is $\mathcal{F}_t-$measurable for every $t \in [0; T]$.*
A discrete family of N random variables $X_{1:N} := (X_n)_{n=1}^N$ is a discrete stochastic process *if X_n is $\mathcal{F}_{t_n}-$measurable for every $n = 1, ..., N$.*

For the rest of this section all the stochastic process are supposed to be defined in a fixed filtered probability space $\mathcal{S}_T$.

A discrete stochastic process can be seen as a *random vector whose random variables are defined in different probability spaces*: this lets us extend the concepts relative to random vectors (joint density, marginal density and conditional density) to discrete stochastic processes.

For every $n = 1, ..., N$, all these notions have the following financial interpretations:

(I1): a discrete stochastic process $X_{1:n}$ models a time-varying variable of a dynamic system over the discrete time set $t_{1:n}$;

(I2): X_n is the state of the system variable at t_n;

(I3): the marginal density $\pi(x_n)$ represents the probability that the system variable stays at the state n;

(I4): the conditional density $\pi(x_n|x_{n-1})$, named *transition density*, is the probability that the system variables moves from the state $n-1$ to n.

Given an integrable function $f : \mathbb{R}^{NI} \to \mathbb{R}$ and a discrete stochastic process $X_{1:N}$ of continuous random variables, we define the expected value of $f(X_{1:n})$ as

$$\mathbb{E}[f(X_{1:n})] := \int f(x_{1:n})\pi(x_{1:n})\, dx_{1:n}, \quad n = 1, ..., N. \tag{1.35}$$

In the following we consider some very common stochastic processes in finance: martingales, Brownian motion, Brownian bridge motion, Markov processes.

Definition 21. *A discrete stochastic process $X_{1:N}$ satisfying the following conditions:*

(DF1): $\mathbb{E}[|X_n|] < +\infty$ *(1.7),*

(DF2): $\mathbb{E}[X_n|X_{1:n-1}]$ *for every n (1.29),*

is said discrete martingale. *A continuous stochastic process $(X_t)_{t\in[0:T]}$ satisfying the following conditions:*

(DF1): $\mathbb{E}[|X_s|] < +\infty$ *for all $s \in [0;T]$ (1.4);*

(DF2): $\mathbb{E}[X_t|X_{t:s}]$ *for all $t, s \in I$, with $0 \leq t < s$ (1.25).*

is said continuous martingale.

Definition 22 (Brownian motion)**.** *A continuous stochastic process $(W_t)_{t\in[0;T]}$ is a Brownian motion if it has the following properties:*

(DF1): $W_0 = 0$*;*

(DF2): W_t *is $\mathcal{F}_t$-measurable for every $t \in [0;T]$ (2);*

(DF1): the random variable $W_t - W_s$ is a Normal random variable for every $s, t \in [0;T]$ (7);

(DF2): the random variable $W_t - W_s$ is independent of $W_s - W_z$ for every t, s, z with $z < s < t$ (1.17).

Given a Brownian motion $(W_t)_{t\in[0;T]}$ *and* $\mu \in \mathbb{R}$*, the process* $(\widetilde{W}_t)_{t\in[0;T]}$*, defined as*

$$\widetilde{W}_t := \mu_t + W_t \quad \forall t \in [0;T], \tag{1.36}$$

is said Brownian motion with drift μ_t.

Definition 23 (Brownian bridge motion)**.** *Let* $(\widetilde{W}_t)_{t\in[0;T]}$ *be a Brownian motion with drift* μ *(1.36), with* $a, b \in \mathbb{R}$ *and* $W_0 = a$ *and* $W_T = b$*,* $\widetilde{w}_t$ *an outcome of* $\widetilde{W}_t$ *for every* $t \in [0;T]$.

A continuous stochastic process $(B_t)_{t\in[0;T]}$ *is a* Brownian bridge motion *if, given an outcome* b_t *of* B_t*, its marginal density* $\pi(b_t)$ *(9) is the conditional density of* W_t *respect to* a *and* b.

$$\pi(b_t) := \pi(\widetilde{w}_t | W_0 = a, W_T = b), \tag{1.37}$$

for every $t \in [0;T]$.

Definition 24 (Markov process)**.** *A discrete stochastic process is a Markov process if the following condition holds:*

$$\pi(x_n | x_{1:n-1}, x_{n+1:N}) = \pi(x_n | x_{n-1}), \quad n = 1, ..., N. \tag{1.38}$$

The transition density $\pi(x_n|x_{n-1})$ *is also called the* transitional kernel *from* X_{n-1} *to* X_n.

Let $X_{1:N}$ be a Markov process. If X_n is discrete with outcomes x_{nm}, Formula (1.23) and Definition (1.38) imply that:

$$\pi(x_n) = \sum_{m=1}^{M} \pi(x_n | x_{n-1,m}) \pi(x_{n-1,m}), \quad n = 1, ..., N. \tag{1.39}$$

If X_n is continuous, Formula (1.24) and Definition (1.38) imply that:

$$\pi(x_n) = \int_{\mathbb{R}^N} \pi(x_n | x_{n-1}) \pi(x_{n-1}) \, dx_{n-1}, \quad n = 1, ..., N. \tag{1.40}$$

An important result about a Markov process is the Chapman-Kolmogorov law (CKL).

Theorem 25 (CKL)**.** *Let* $X_{1:N}$ *be a Markov process with* X_n *continuous random variables. Then:*

$$\pi(x_i | x_j) = \int \pi(x_i | x_{j+1:i-1}) \pi(x_{j+1:i-1} | x_j) \, dx_{j+1:i-1}, \tag{1.41}$$

for every $i, j = 1, ..., N$*, with* $i > j$.

This subsection concludes with some basic results of stochastic integral and stochastic differential equation (SDE). We set:

(N1): $[0;T]$ is a real interval.

(N2): $0 = t_0 < t_1 < ... < t_n = T$ indicates a partition of $[0;T]$.

(N3): $(W_t)_{t\in[0;T]}$ denotes a Brownian motion.

(N4): $a_i : \Omega \to \mathbb{R}$ are $\mathcal{F}_{t_i}-$measurable function with $\mathbb{E}[a_i^2] < +\infty$ for every $i = 1, ..., n$.

(N5): $(X_t)_{t\in[0;T]}$ indicates a stochastic process with

$$\int_0^T \mathbb{E}[X_t^2]\, dt < +\infty.$$

(N6): $I_i =]t_{i-1}; t_i]$, $i = 1, ..., n$.

(N7): $1_{I_i}(\omega)$ is the index function of I_i, $i = 1, ..., n$.

Definition 26 (Stochastic integral)**.** *The* stochastic integral *of X_t is defined as follows [62, 61]:*

(D1): if $X_t = \sum_{i=1}^n a_i(\omega)1_{I_i}(\omega)$, one has

$$\int_0^T X_t\, dW_t := \sum_{i=1}^n a_i \Delta W_i, \quad \Delta W_i = W_{i+1} - W_i.$$

(D2): if X_t is a generic process, at every $t \in [0; T]$ it can be approximated by a sequence of simple processes X_{nt} such that

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T |X_{nt} - X_t|^2\, dt\right] = 0.$$

The stochastic integral $\int_0^T X_t\, dW_t$ is

$$\int_0^t X_t\, dW_t := \lim_{n\to+\infty} X_{nt},$$

for every $t \in [0; T]$. The process X_t is said Ito integrable.

Definition 27 (Stochastic differential equations)**.** *Let $\mu_t(\omega), \sigma_t(\omega) : [0; T] \times \mathbb{R}^N \to \mathbb{R}$ be real functions such that μ_t is Riemann-integrable and the process $\sigma_t(X_t)$ is Ito-integrable. The process X_t is said an* Ito process *if the following relation holds:*

$$X_t = X_0 + \int_0^t \mu_s\,(X_s)\ ds + \int_0^t \sigma_s\,(X_s)\, dW_s, \quad \forall t \in [0; T], \tag{1.42}$$

which can be rewritten, in terms of a stochastic differential equation (SDE), as:

$$dX_t = \mu_t\,(X_t)\, dt + \sigma_t\,(X_t)\, dW_t, \tag{1.43}$$

for every $t \in [0; T]$.

For the rest of this thesis we make the following assumptions about a SDE [85]:

(A1): For every $n \in \mathbb{N}$, exists a constant K_n such that

$$|\mu_t(x) - \mu_t(y)|^2 + |\sigma_t(x) - \sigma_t(y)|^2 \le K_n|x - y|^2,$$

with $|x|, |y| \le n$ and $t \in [0; T]$.

(A2): exists a $C \in \mathbb{R}$ such that:

$$|\mu_t(x)|^2 + |\sigma_t(x)|^2 \leq C(1 + x^2),$$

for every $x \in \mathbb{R}^N$ and $t \in [0; T]$.

Next subsection explores the notions of accuracy, precision and bias of an estimator (for more details we refer to [23, 57]).

1.1.3 Accuracy, precision, bias

In order to define the notions of accuracy, precision and bias, we introduce the following notations:

(N1): X is a random variable;

(N2): $\theta \in \mathbb{R}$ indicates a real parameter associated to X;

(N3): $\pi(\cdot, \theta)$ denotes the density function of X;

(N4): $x_{1:M}$ is a set of M i.i.d. particles $x_m \sim \pi(\cdot, \theta)$, $m = 1, ..., M$;

(N5): $\widetilde{\theta}(\cdot) : \mathbb{R}^M \to \mathbb{R}$ is a random variable depending on $x_{1:M}$.

The random variable $\widetilde{\theta}$ is named *estimator* and lets us *estimate the value of* θ. Given $D \in \mathbb{N}$, for every $d = 1, ..., D$, we set:

(N1): $x_{d,1:M} := (x_{d1}, ..., x_{dM})$;

(N2): $\theta_d := \theta(X_d)$;

(N3): $\widetilde{\theta}_d := \widetilde{\theta}(X_d)$.

The goodness of an estimator is measured by the *precision*, *accuracy* and *bias*, which are defined in the following.

Definition 28 (Precision, Accuracy and Bias)**.** *The accuracy of* $\widetilde{\theta}$*, indicated with the symbol* $ACC(\theta, \widetilde{\theta})$*, is given by:*

$$ACC(\theta, \widetilde{\theta}) := \frac{D}{\sum_{d=1}^{D}[(\theta_d - \widetilde{\theta}_d)^2]}. \tag{1.44}$$

The precision of $\widetilde{\theta}$*, indicated with the symbol* $PREC(\theta, \widetilde{\theta})$*, is given by:*

$$\mu_d = \frac{1}{D}\sum_{n=1}^{D}\widetilde{\theta}_d \tag{1.45}$$

$$PREC(\theta, \widetilde{\theta}) = \frac{D}{\sum_{d=1}^{D}[(\widetilde{\theta}_d - \mu_d)^2]}. \tag{1.46}$$

Remark 1.1.2. *From (1.7–1.9) it follows that the precision* $PREC(\theta, \widetilde{\theta})$ *is the inverse of the variance of the following discrete random variable (1.2)*

$$\begin{pmatrix} \widetilde{\theta}_1 & \cdots & \widetilde{\theta}_D \\ \frac{1}{D} & \cdots & \frac{1}{D}. \end{pmatrix}$$

Consequently, we can apply all the theoretical results about the variance to $PREC(\theta, \widetilde{\theta})$.

The bias of $\widetilde{\theta}$, indicated with the symbol $BIAS(\theta, \widetilde{\theta})$, is:

$$BIAS(\theta, \widetilde{\theta}) = \frac{1}{D} \sum_{d=1}^{D} \left(\widetilde{\theta}_d - \theta_d \right). \tag{1.47}$$

If $BIAS(\theta, \widetilde{\theta}) = 0$, the estimator $\widetilde{\theta}$ is said unbiased.

The notions of MSE, bias and variance of an estimator $\widetilde{\boldsymbol{\theta}}$ are related as:

$$\frac{1}{ACC(\theta, \widetilde{\theta})} = BIAS^2(\theta, \widetilde{\theta}) + \frac{1}{PREC(\theta, \widetilde{\theta})}. \tag{1.48}$$

Thanks to (1.48), the *analysis of the goodness of an estimator is reduced to the analysis of its bias and of its precision.*

Given a continuous real random variable X, a possible parameter is represented by its expected value $\mathbb{E}[X]$. A very common estimator of $\mathbb{E}[X]$ is the *sample average* μ^M of a set of independent particles $x^{(m)}$, $m = 1, ..., M$, defined as:

$$\mu^M := \frac{1}{M} \sum_{m=1}^{M} x^{(m)}. \tag{1.49}$$

The notion of sample average can be generalized to the case of a discrete stochastic process $X_{1:N}$ (20), with X_n continuous for every n. Given an integrable function $f : \mathbb{R}^{NI} \to \mathbb{R}$, an estimator of the expected value $\mathbb{E}[f(X_{1:N})]$ is the *sample average μ_N^M of the realizations of f over a set of M independent paths* $x_{1:N}^{(m)}$, $m = 1, ..., M$:

$$\mu_N^M := \frac{1}{M} \sum_{m=1}^{M} f(x_{1:N}^{(m)}), \tag{1.50}$$

for every $M \in \mathbb{N}$.

This subsection concludes with the definition of index function of a set.

Definition 29 (Index function)**.** *Let A be a nonempty subset of $\mathbb{R}^N$, with $N \in \mathbb{N}$. The index function $1_A(\cdot)$ is the function $1 : \mathbb{R}^N \to \{0, 1\}$ defined as*

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases} \tag{1.51}$$

for every $x \in \mathbb{R}^N$.

Let $(A_n)_{n=1}^N$ be a sequence of N non-empty sets. Then:

$$\begin{aligned} A_N &:= \bigcap_{n=1}^{N} A_n \\ \prod_{n=1}^{N} 1_{A_n}(x) &:= 1_{A_N}(x). \end{aligned} \tag{1.52}$$

In the case of a uncountable family of non-empty sets $(A_t)_{t \in [0;T]}$, we set:

$$\begin{aligned} A_T &:= \bigcap_{t \in [0;T]} A_t \\ (1_{A_t}(x))_{t \in [0;T]} &:= 1_{A_T}(x), \end{aligned} \tag{1.53}$$

for every $T \in]0; +\infty[$.

1.2 Financial Background

This section provides some basic financial concepts (for more details we refer to [85]). We begin with the definition of security and financial market.

Definition 30 (Securities and financial markets)**.** *A* security *is every tradable asset. The set $\mathcal{M}$ of securities which are traded in the same place is named* financial market [1] .

We make the following assumptions.

(A1): All the the securities belong to the same market.

(A2): The market is composed of a risk-free asset and a risky asset.

(A3): Every trader can invest an amount of money at a constant risk-free interest rate.

(A4): The value function of the risky asset is described by a continuous stochastic process.

In the following we will make use of the following notations.

(N1): $\mathcal{M}$ denotes the market.

(N2): r indicates the constant interest rate, $r \in [0;1]$.

(N3): $[0;T]$ denotes the time interval, $T \in]0;+\infty[$.

(N4): $\mathcal{S}_T$ indicates the fixed filtered probability space (19).

(N5): σ_t is the volatility, $\sigma_t \in]0;1[$ for every $t \in [0;T]$ [2] .

(N6): $(S_t)_{t\in[0;T]}$ indicates the value function of the risky asset.

(N7): $(W_t^s)_{t\in[0;T]}$ and $(W_t^\sigma)_{t\in[0;T]}$ are two Brownian motions (22).

(N8): $S_0 \in \mathbb{R}^0$ and σ_0 denote respectively the values of S_t and σ_t at 0.

(N9): $\alpha_t(\cdot) : [0;T] \times \mathbb{R}^2 \to \mathbb{R}$, $\beta_t(\cdot), \gamma_t(\cdot) : [0;T] \times \mathbb{R} \to \mathbb{R}$ are real continuous functions.

(N10): $0 = t_0 < t_1 < ... < t_N = T$ indicate a discretization of $[0;T]$, with $N \in \mathbb{N}$.

(N11): S_n and σ_n denote respectively the values of S_t and σ_t at t_n.

(N12): $\alpha_n, \beta_n, \gamma_n$ indicate respectively the value of $\alpha_t, \beta_t, \gamma_t$ at t_n.

In option pricing, S_t and σ_t are generally defined as the solution of a particular system of SDE, called *financial security model*. This notion is formalized in Subsection 1.2.1.

[1] In a more general context, the term market indicates the set of all the assets, the place where they added and all the laws to which transactions are subjects (for more details see [47]).

[2] The volatility of a risky asset is the degree of variation of the asset value of the security. It is measured by the standard deviation of logarithmic returns of the security [33].

1.2.1 Financial security models

We begin with the definition of financial security model (FSM).

Definition 31 (FSM). *Let $\alpha_t, \beta_t, \sigma_t$ be functions satisfying (1.1.2). A FSM is a system of SDEs taking the form:*

$$\begin{aligned} dS_t &= rS_t dt + \alpha_t(S_t, \sigma_t) dW_t^s \quad (1.54) \\ d\sigma_t &= \beta_t(\sigma_t) dt + \gamma_t(\sigma_t) dW_t^\sigma, \quad (1.55) \end{aligned}$$

for every $t \in [0; T]$.

In the following we examine two kinds of FSMs: the *Black-Scholes model* (BSM) and *stochastic volatility models* (SVMs).

The BSM [15] is characterized, for every $t \in [0; T]$, by:

(P1): $\alpha_t(S_t, \sigma) = \sigma S_t$.

(P2): $\beta_t(\sigma_t) = 0$.

(P3): $\gamma_t(\sigma_t) = 0$.

System (1.54−1.55) becomes:

$$\begin{aligned} dS_t &= rS_t dt + \sigma_t S_t dW_t^s \quad (1.56) \\ d\sigma_t &= \sigma. \end{aligned}$$

The solution S_t of (1.56) is obtained using *Ito lemma* [62, 61]:

$$S_t = S_0 exp\left[\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^s\right], \quad \forall t \in [0; T], \quad (1.57)$$

which is called *Geometric Brownian motion* (GBM). Since S_t in (1.57) is positive, we can define the process $X_t = \ln S_t$:

$$\begin{aligned} \mu_t &= \ln S_0 + (r - \frac{1}{2}\sigma^2)t \\ X_t &= \mu_t t + \sigma W_t^s, \quad \forall t \in [0; T]. \quad (1.58) \end{aligned}$$

The process X_t is a Brownian motion with time-dependent drift μ_t. Given $0 = t_0 < t_1 < ... < t_N = T$ with constant time step $h = T/N$, the value S_n is related to S_{n-1} by

$$\begin{aligned} X_n &:= X_{n-1} + \left(r - \frac{\sigma^2}{2}\right)h + \sqrt{h}\sigma Z_n \quad Z_n \sim N(0; 1) \quad (1.59) \\ S_n &= exp\left[X_n\right], \quad n = 1, ..., N, \quad (1.60) \end{aligned}$$

and $X_n \sim N(\mu_n; h\sigma^2)$, with $\mu_n = X_{n-1} + \left(r - \frac{\sigma^2}{2}\right)h$. The discrete process $S_{1:N}$ resulting from (1.60) is positive and a Markov process (1.38)

A SVM [16, 103, 55, 49] takes the form (1.54−1.55), with $\alpha_t, \gamma_t \neq 0$. Since a SVM cannot be solved analytically, given $t_n = nh$, $n = 1, ..., N$, we are only able to determine an approximation S_n of S_t at t_n using the Euler scheme [70]:

$$\begin{aligned} S_n &= (1 + rh)S_{n-1} + \sqrt{h}\sigma_{n-1}\alpha_{n-1}Z_n^s, \quad Z_n^s \sim N(0;1) && (1.61)\\ \sigma_n &= \sigma_{n-1} + \beta_{n-1}h + \sqrt{h}\gamma_{n-1}Z_n^\sigma, \quad Z_n^\sigma \sim N(0;1) && (1.62)\\ n &= 1, ..., N. \end{aligned}$$

As under the BSM, the discrete processes ($S_{1:N}$ and $\sigma_{1:N}$) are Markov process. The error of approximation of S_t with S_n, $n = 1, ..., N$, is of order of $\sqrt{h}$, as estimated in [8]−[43]:

$$\sup_{1 \leq n \leq N} |S_n - S_{t_n}| = O(\sqrt{h}).$$

This subsection concludes with two common examples of SVMs: the *Heston model* (HM) and the *SABR model*.

The HM [55] is characterized by:

(P1): $\alpha_t(S_t, \sigma) = \sigma_t S_t$.

(P2): $\beta_t(\sigma_t) = k(\theta - \sigma_t^2)$.

(P3): $\gamma_t(\sigma_t) = \varepsilon\sigma_t$.

The HM takes the form:

$$\begin{aligned} dS_t &= rS_t dt + \sigma_t S_t dW_t^s && (1.63)\\ d\sigma_t^2 &= k(\theta - \sigma_t^2)dt + \varepsilon\sigma_t dW_t^\sigma. && (1.64) \end{aligned}$$

In (1.64) θ is the expected value of μ_t when t tends to infinite, k is the mean reversion rate towards θ, which controls the speed of the volatility going back to its mean, and ε is the volatility of the volatility, also called *vol-vol*. Here all these parameters have been supposed to be constant, but an extended HM assumes that the parameters are time-dependent [82, 12].

The SABR model, introduced by Hagan in [49] and successively developed in [48, 2], is characterized, for every $t \in [0; T]$, by:

(P1): $\alpha_t(S_t, \sigma) = \sigma_t S_t^\beta$, $\gamma \in [0; 1]$.

(P2): $\beta_t(\sigma_t) = 0$.

(P3): $\gamma_t(\sigma_t) = \alpha\sigma_t$.

The SABR model takes the following form:

$$\begin{aligned} dS_t &= rS_t dt + \sigma_t S_t^\beta dW_t^s && (1.65)\\ d\sigma_t &= \alpha\sigma_t dW_t^\sigma. && (1.66) \end{aligned}$$

In (1.65−1.66) β determines the shape of the risky asset, α, named *vol-vol*, is the volatility of σ_t.

Equation (1.65) models the underlying as a CEV model [26], Equation (1.66) models the volatility as a GBM (1.57):

$$\sigma_t = \sigma_0 exp\left(-\frac{\alpha^2}{2}t + \alpha W_t^\sigma\right), \quad t \in [0;T]. \tag{1.67}$$

Subsection 2.3 introduces the concept of (RNM) and its link with the properties of no-arbitrage (NA) and a complete market (CM).

1.2.2 Arbitrage and completeness

Following the martingale approach, a barrier option price is expressed as the *actualized expected option payoff under the RNM*. For this reason, the existence and uniqueness of this particular measure, linked to the notions of NA and CM, are central issues in option pricing.

We start with the definitions of portfolio, arbitrage and completeness.

Definition 32 (Portfolio definition)**.** *A* portfolio *is every linear combination of assets of* $\mathcal{M}$.

Definition 33 (Arbitrage definition)**.** *An* arbitrage portfolio *(or simply called* arbitrage*) is a portfolio whose value function* $(S_t)_{t\in[0;T]}$ *satisfies the following properties:*

(P1): $\mathbb{P}(S_0 = 0) = 1$.

(P2): $\mathbb{P}(\exists t \in]0;T] : S_t \neq 0) = 1$.

An arbitrage is a *free-cost portfolio which ensures a positive future payoff*. If the market has no-arbitrage (NA) portfolios, it is said that in the market the *no-arbitrage principle* (NAP) *holds*.

Definition 34 (Completness definition)**.** *A market* $\mathcal{M}$ *is complete if and only if all the tradable assets belong to the market.*

The NAP and CM are related to the notion of RNM thanks to the *First Fundamental Theorem of Asset Pricing* (FFTAP) and *Second Fundamental Theorem of Asset Pricing* (SFTAP) (for more details see [29, 52, 53]).

Theorem 35 (FFTAP)**.** *The NAP holds if and only if exists a density function* $q(\cdot)$*, such that the process* $(e^{-rt}S_t)_{t\in[0;T]}$ *results a martingale (21):*

$$e^{rs}S_s = \mathbb{E}[e^{rt}S_t|S_u,\ s \leq u \leq t], \tag{1.68}$$

for every $s, t \in [0;T]$ *with* $s < t$.

Theorem 36 (SFTAP)**.** *A market M is complete if and only if a RNM exists and is unique.*

As it will be discussed in Section 1.3, the notion of the RNM lets us define the NA price of a barrier option. Under the BSM, the SFTAP holds: this implies that the barrier option price problem is well-posed. Under a SVM, where the SFTAP does not hold, many densities functions satisfying (1.68) exist: in this

case the barrier option price problem is not well-posed. In order to overcome this issue, we proceed as in Back [6]: firstly, we assume for the rest of this thesis that the NAP holds, then we select the RNM among all the possible candidates on the basis of the investors' preferences and endowments and production possibilities.

Formula (1.68) can be generalized to the case of a derivative. A derivative is a contract whose payoff depends on the value function of a risky asset S_t, in symbols $F(S_t)$. From (1.68) it follows that also the process $F(S_t)$ is a martingale:

$$F(S_s) = e^{r(t-s)}\mathbb{E}[F(S_t)], \quad \forall s, t \in [0;T],\ s < t, \tag{1.69}$$

Section 3 formalizes the notion of the NA price.

1.3 Barrier option pricing problem

We preliminarily introduce the concepts of vanilla and barrier option (for more details see [60, 59, 47] and for barrier options also [92]).

Definition 37. *A vanilla option is a contract giving the right to buy (call) or to sell (put) a quantity of a risky asset (underlying) at a future date (maturity). In order to exercise this right, an investor pays (in the case of a call) or receives (in the case of a put) an amount (strike price).*

Definition 38. *A barrier option is a vanilla option which is activated (knock-in) or extinguished (knock-out) if the underlying reaches some specified levels (barriers).*

In Introduction barrier options have been classified on the basis of the number of monitoring instants sets (continuous or discrete) and of the number of the barriers (single barrier or double barriers).

The payoff of a knock-in coincides with the one of the vanilla option with the same financial variables (underlying, maturity, strike price) if the underlying crosses the barriers, otherwise it is null.

The payoff of a knock-out coincides with the one of the vanilla option with the same financial variables (underlying, maturity, strike price) if the underlying does not hit the barrier, otherwise it is null.

Now we formulate the barrier pricing problem using the notion of the RNM. We introduce the following notations:

(N1): $L_t : [0;T] \to \mathbb{R}$ is a decreasing function, named *lower barrier*.

(N2): $U_t : [0;T] \to \mathbb{R}$ is an increasing function, named *upper barrier*, with $L_t < U_t$ for every $t \in [0;T]$.

(N3): I_t indicates the *barrier interval*, defined as

$$I_t = \begin{cases}]L_t; U_t[& \text{if knock-out} \\ (]L_t; U_t[)^c & \text{if knock-in } ^3 \end{cases}, \tag{1.70}$$

for every $t \in [0;T]$.

(N4): K indicates the strike price;

(N5): $h(\cdot)$ denotes the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$h(x) = \begin{cases} \max\{x - K; 0\} & \text{if Call} \\ \max\{K - x; 0\} & \text{if Put,} \end{cases} \tag{1.71}$$

which is the payoff of a vanilla option;

(N6): $\mathbb{E}[\cdot]$ denotes the expected value under the RNM q.

(N7): $0 < t_1 < ... < t_N = T$ is a set of N instants of $[0; T]$.

(N8): L_n and U_n indicate respectively the values of L_t and U_t at t_n, $n = 1, ..., N$;

(N9): I_n denote the value of I_t at t_n, $n = 1, ..., N$;

(N10): Ψ_t^c is the value of a continuous barrier option at $t \in [0; T]$;

(N11): Ψ_t^d is the value at $t \in [0; T]$ of a discrete barrier option with monitoring instants t_n, $n = 1, ..., N$;

The payoff Ψ_T^c of a continuous option is equal to

$$\Psi_T^c = h(S_T) \left(1_{I_t}(S_t)\right)_{t \in [0;T]}. \tag{1.72}$$

Using (1.69), the price Ψ_0^c of a barrier option can be expressed as

$$\Psi_0^c = \mathbb{E}[e^{-rT} \Psi_T^c]. \tag{1.73}$$

The payoff Ψ_T^d of a discrete barrier option is equal to

$$\Psi_T^d = h(S_N) \left(\prod_{n=1}^{N} 1_n(S_n) \right). \tag{1.74}$$

The price Ψ_0^d of a discrete barrier option can be expressed as

$$\Psi_0^d = \mathbb{E}[e^{-rT} \Psi_T^d]. \tag{1.75}$$

Pricing closed formulas for continuous barrier options have been derived under the strong assumption of the flat structure of market parameters ([79, 56, 74, 91]). In general, a continuous barrier option can be priced only via simulation because the integrals in (1.73) are analytically intractable or the density function of the underlying is unknown. The most common simulation techniques are: *quadrature formulas* [5, 101, 28], *finite difference schemes* [30], *lattice rules* ([58, 68, 9]), *Monte Carlo* (MC) *methods* [42, 63].

Discrete barrier option can be priced only using the same simulation techniques [72, 100, 84].

In the following we will treat a particular family of MC methods, MCst methods (for general results about MC methods we refer to the following textbooks [87, 73, 93, 50]).

1.4 Standard Monte Carlo

MC methods, introduced by Metropolis in a very celebrated paper [80], is a very large and heterogeneous family of statistical procedures to evaluate the expected value of a discrete stochastic process. They find application in many financial fields, as portfolio management [21], capital allocation [102] and, in partiular, in option pricing [19, 18]. In option pricing, they are a good tool to price a discrete barrier option (1.75) and a continuous barrier option (1.73) because, as illustrated in Section 1.3, the barrier option price can be expressed as the actualized expected value of the payoff under the RNM.

The most common MC procedures, i.e. MCst methods, address the problem of computing the expected value $\mathbb{E}[f(X_{1:N})]$ (1.35). A MCst scheme is summarized in the following: firstly, M independent paths $x_{1:N}^{(m)}$ of $X_{1:N}$, $m = 1, ..., M$, are simulated and, successively, it computes the sample average of $f(x_{1:N}^{(m)})$, called *standard MCst estimator*:

$$\Delta_N^M = \frac{1}{M} \sum_{m=1}^{M} f(x_{1:N}^{(m)}). \tag{1.76}$$

The crucial point of a MCst is the simulation step. This can be performed by sampling from the joint density $\pi(x_{1:N})$ or, as in the case of the pricing of barrier option, by applying a numerical scheme if $X_{1:N}$ is the solution of a SDE. The variance of Δ_N^M is equal to

$$\mathbb{V}[\Delta_N^M] = \frac{\mathbb{V}[f(x_{1:N}^1)]}{M}. \tag{1.77}$$

Equation (1.77) implies that, *in order to achieve low levels of variance, a large number of paths is required.*

In the following we enunciate the *Law of large numbers* (LLN), which claims that Δ_N^M is an estimator of $\mathbb{E}[f(X_{1:N})]$.

Theorem 39 (LLN)**.** *Let $(X_m)_{m\in\mathbb{N}}$ be an infinite sequence of i.i.d random variables, with the same expected value $\mu \in \mathbb{R}$, and, for every $M \in \mathbb{N}$, $\overline{X}_M$ the sample average of $X_{1:M}$:*

$$\overline{X}_M = \frac{1}{M} \sum_{m=1}^{M} X_m.$$

The sequence $\overline{X}_M$ converges in probability *to μ, i.e.*

$$\mathbb{P}\left(\lim_{M \to +\infty} (\overline{X}_M - \mu) \right) = 1,$$

i.e. Δ_N^M is an estimator of μ.

Now we develop the MCst approach for barrier option, which can be formalized as follows:

(A1): The process $X_{1:N}$ is represented by $S_{1:N}$.

(A2): The function f is the actualized payoff $e^{-rT}\Psi_T^d$ (1.74).

(A3): The price Ψ_0^d is $\mathbb{E}[f(S_{1:N})]$.

(A4): The independent paths $s_{1:N}^{(m)}$, $m = 1, ..., M$, are generated using (1.57) (under the BSM) or (1.61−1.62) (under the SVM).

The MCst estimator Δ_N^M of a barrier option price (1.75−1.73) is:

$$\Delta_N^M = \frac{1}{M} \sum_{m=1}^{M} \left(\prod_{n=1}^{N} 1_{I_n}(s_n^{(m)}) h(s_N^{(m)}) \right). \tag{1.78}$$

As illustrated in Subsection 1.1.3, for (1.48), the goodness of a MCst estimator is measured by the bias and the precision. Differently to other kinds of options, MCst estimators for barrier options are *biased* and have a *low precision*.

The problem of the bias is related to a continuous barrier option [39]. In (1.73), the payoff depends on the underlying for every instant. But a MCst estimator considers only a finite set of underlying values: this implies that a MCst procedure is not able to simulate correctly the option payoff. This loss of information introduces a bias in the MCst estimator.

The problem of the precision is linked to the number of survival paths. For (1.77), a high value of M is required in order to achieve good levels of precision. In the case of a barrier option, the number of paths could be low because many paths generally cross the barriers and they are rejected: this phenomenon is very common when S_0 approaches the barriers, as observed in [96, 64, 94, 84].

In order to overcome these issues, in the following chapters we discuss a particular MC approach, named *Bayesian MC approach*, under the BSM (Chapter 2) and a SVM (Chapter 3).

Chapter 2

Bayesian MC methods under the BSM

In this chapter we present the Bayesian MC methods under the BSM, a class of MC methods to solve the problems of the bias and the low precision of MCst methods, investigated in Chapter 1. A Bayesian MC method estimates the actualized conditional expected option payoff given a set of variables correlated with the underlying, named *observations*. It is summarized in the following: firstly a set of underlying paths are generated using a family of sampling techniques, named *Bayesian MC sampling techniques*, then the standard average of the actualized payoff realizations over the underlying paths is computed.

Section 2.1 recalls the main Bayesian sampling techniques for the simulation of the paths. Section 2.2 describes the general Bayesian MC framework. Section 2.3 proposes the main Bayesian MC schemes for barrier options.

2.1 Bayesian sampling techniques

Bayesian methods sampling techniques simulate a stochastic process, named *state variable*, using information coming from a set of random variable correlated with the state variable, named *observations*. For the rest of this section the state variable and the observations will be indicated respectively with $X_{1:N}$ and $Y_{1:N}$, both defined in the same filtered probability space $\mathcal{S}_T$ (20). The symbols $x_{1:n}$ and $y_{1:n}$ denote respectively a generic outcome of $X_{1:N}$ and $Y_{1:n}$, $n = 1, ..., N$.

We assume that:

(A1): X_n is correlated with $X_{1:n-1}$ and $Y_{1:n}$, $n = 1, ..., N$;

(A2): X_n is independent of $Y_{n+1:N}$, $n = 1, ..., N$;

Our goal is to sample from $\pi(x_{1:N}|y_{1:N})$. This can be factorized, using (1.18), as:

$$\pi(x_{1:N}|y_{1:N}) = \prod_{n=1}^{N} \pi(x_n|x_{1:n-1}, y_{1:n}). \tag{2.1}$$

In particular, if $X_{1;N}$ is a Markov process (1.38), Expression (2.1) is simplified

as

$$\pi(x_{1:N}|y_{1:N}) = \prod_{n=1}^{N} \pi(x_n|x_{n-1}, y_{1:n}). \tag{2.2}$$

Thanks to (2.1−2.2), in the following *we address the problem of sampling from the densities* $\pi(x_n|x_{1:n-1}, y_{1:n})$, $n = 1, ..., N$.

This section examines the following Bayesian sampling methods: *Metropolis-Hastings algorithm* (MHA) (Subsection 2.1.1), *Kalman filter* (KF) (Subsection 2.1.2), *Sequential Monte Carlo* (MCse) *methods* (Subsection 2.1.3) and the *Bootstrap filter* (BF) (Subsection 2.1.4)

2.1.1 Metropolis-Hastings algorithm

The MHA, introduced by Metropolis [81] and successively redeveloped by Hastings [54], was originally applied in chemistry to simulate a liquid in equilibrium with its gas phase. It is an *acceptance-rejection method for the simulation of a Markov process* (for a general treatment of the MHA see [25, 40, 83]).

The MHA is based on the following assumptions:

(A1): $Y_n := 0$, $n = 1, ..., N$.

(A2): $X_{1:N}$ is a Markov process (1.38);

(A3): the transition densities are independent of time, i.e. $\pi(x_n|x_{n-1}) = K(x_{n-1}, x_n)$, $n = 1, ..., N$ (*homogeneity*);

(A4): the marginal densities are independent of time (*invariance*) and of the initial density (*ergodicity*).

Expression (2.2) takes the form:

$$\pi(x_{1:N}|y_{1:N}) = \prod_{n=2}^{N} K(x_{n-1}, x_n). \tag{2.3}$$

The marginal densities are named *target densities*, the function $K(\cdot, \cdot)$ is named *invariant kernel* and it satisfies the following condition, known as the *detailed balance equation* [24, 3].

$$\pi(x_{n-1})K(x_{n-1}, x_n) = \pi(x_n)K(x_n, x_{n-1}), \quad n = 2, ..., N. \tag{2.4}$$

Every invariant kernel is obtained by solving the previous equation.

Given an input density $q(\cdot)$, named *proposal density*, the MHA is divided into two steps:

(S1): Construction of the following kernel:

$$K(x, y) := \alpha(x, y)q(x, y) \tag{2.5}$$

$$\alpha(x, y) := \begin{cases} \min\left\{1; \frac{\pi(y)q(y,x)}{\pi(x)q(x,y)}\right\} & \text{if } \pi(x)q(x, y) > 0 \\ 1 & \text{otherwise,} \end{cases} \tag{2.6}$$

with $x, y \in \mathbb{R}^N$. The proof that the kernel $K(\cdot, \cdot)$ satisfies (2.4) can be found in [24, 3].

(S2): Sampling from $K(\cdot,\cdot)$.

The MHA is summarized in Algorithm MH: in input it receives the initial value x_1 and N, q, in output it produces a simulation $x_{1:N}$ of $X_{1:N}$.

Algorithm 1 MH

begin
  for $n = 2$ *to* N **do**
    1) Generate a vector y from a proposal density $q(x_{n-1}, \cdot)$.
    2) Compute the acceptance ratio $\alpha := \alpha(x_{n-1}, y)$ (2.6).
    3) Draw a uniform number $u \sim U([0;1])$.
    if $\alpha > u$ **then**
      4) Set $x_n := y$.
      else
      5) Set $x_n := x_{n-1}$**.**
    end
  end
  return $x_{1:N} := (x_1, ..., x_N)$.
end

In short, the MHA *approximates the unknown target density with a prefixed proposal density*. For every n, the value of X_n is accepted if α is higher than a uniform threshold, otherwise it is rejected: for this reason, the function α is also named *acceptance-rejection probability*.

2.1.2 Kalman Filter

The Kalman filter (KF) was discovered by the mathematician Rudolf Emil Kalman during the *Apollo program* and published in [66, 65]. This algorithm addresses the problem of *simulating a linear Markov process using a set of linear observations*.

The KF relies on the following assumptions:

(A1): The Markov processes $X_{1:N}$ and $Y_{1:N}$ are governed by the following linear system:

$$X_n = \Psi X_{n-1} + W_n \tag{2.7}$$
$$Y_n = H X_n + V_n, \quad n = 1, ..., N, \tag{2.8}$$

with $\Psi, H \in \mathbb{R}^N \times \mathbb{R}^N$. Both $X_{1:N}$ and $Y_{1:N}$ are Markov processes.

(A2): $W_{1:N}$ and $V_{1:N}$ are Normal processes (18):

$$Q = \mathbb{E}[W_n W_n^T] \tag{2.9}$$
$$R = \mathbb{E}[V_n V_n^T], \tag{2.10}$$
$$W_{1:N} \sim N(0, Q),\ V_{1:N} \sim N(0, R).$$

The covariance matrices are time-independent.

(A3): V_n and W_n are uncorrelated.

The goal of the KF is to derive an estimator $\widehat{X}_n$ of X_n which minimizes the *error covariance matrix*:

$$\begin{aligned} e_n &:= X_n - \widehat{X}_n. && (2.11)\\ P_n &:= \mathbb{E}[e_n e_n^T] && (2.12)\\ n &= 2, ..., N. \end{aligned}$$

Given in input the value x_1 of X_1, the KF involves the following steps: the *prediction* and the *update*.

- *Prediction.* A *prior estimate* $\widehat{X}_n^{'}$ is obtained using the value X_{n-1} in (2.7):

$$\widehat{X}_n^{'} = \Psi \widehat{X}_{n-1}^{'} + \widehat{W}_n. \tag{2.13}$$

  We calculate the *error of the prior estimate* $e_i^{'}$:

$$e_n^{'} = X_n - \widehat{X}_n^{'}, \tag{2.14}$$

  the *prior estimate* $P_i^{'}$ *of* P (2.11−2.12):

$$P_n^{'} = \mathbb{E}[e_n^{'}(e_n^{'})^T], \tag{2.15}$$

  and the *prior estimate of the observations* $\widehat{z}_n^{'}$:

$$\widehat{Z}_n^{'} = H\widehat{X}_n^{'}, \tag{2.16}$$

  for every $n = 1, ..., N$.

- *Update.* The KF constructs the final estimate $\widehat{X}_n$ taking the the form:

$$\widehat{X}_n = \widehat{X}_n^{'} + K_n\left(Z_n - \widehat{Z}_n^{'}\right). \tag{2.17}$$

  The matrix $K_n \in \mathbb{R}^N \times \mathbb{R}^N$, named *Kalman gain*, is chosen in order to minimize P_n (2.11−2.12).

  Theorem (40) provides the expression of K_n and P_n (the proof can be found in [105].).

  Theorem 40 (Matrix Covariance Expression)**.** *Let H be the transaction matrix of $X_{1:N}$ (2.7), $P_n^{'}$ the matrix given in (2.15), R the covariance matrix of $V_{1:n}$ (2.8). Then, for every $n = 1, ..., N$, the matrix $K_n \in \mathbb{R}^N \times \mathbb{R}^N$ minimizing P_n (2.11−2.12) is:*

$$K_n = P_n^{'}H^T(HP_n^{'}H + R)^{-1}, \tag{2.18}$$

  and the corresponding value of P_n is

$$P_n = (I - K_nH)P_n^{'}, \tag{2.19}$$

  where I indicates the N-dimensional identical matrix.

In short, by exploiting the linearity of the state variable and of the observations, the KF *provides a linear and exact estimate of the state variable.*

2.1.3 Sequential Importance Sampling

Sequential important sampling (SIS) methods were introduced in [51, 71] and successively detailed in [88, 50, 99]. Differently from the MHA and the KF, a SIS method does not rely on too strong assumptions about $X_{1:N}$ and $Y_{1:N}$.

We define a new family of N density functions $q_n(\cdot)$, called *importance densities*, selected in order to decrease the variance of MC estimators (MC estimators will be treated in Section 2.2), and a set of variables $w_n(\cdot)$, named *importance weights*, as

$$w_n(x_n) := \frac{\pi(x_n|x_{1:n-1}, y_{1:n})}{q(x_n|x_{1:n-1}, y_{1:n})}. \tag{2.20}$$

Using (2.20), the density $\pi(x_n|x_{1:n-1}, y_{1:n})$ can be rewritten as

$$\pi(x_n|x_{1:n-1}, y_{1:n}) = w_n(x_n)q(x_n|x_{1:n-1}, y_{1:n}). \tag{2.21}$$

A SIS scheme addresses the *problem of evaluating Formula* (2.21) *by performing the following steps* for every $n = 1, ..., N$:

(S1): sampling $x_n^{(m)}$ from $q(x_n|x_{1:n-1}, y_{1:n})$;

(S2): evaluation of $w_n^{(m)} := W_n(x_n^{(m)})$;

(S3): resampling of $x_n^{(m)}$ according to the normalized weights $W_n^{(m)}$:

$$W_n^{(m)} = \frac{w_n^{(m)}}{\sum_{m=1}^{M} w_n^{(m)}}. \tag{2.22}$$

The last point is justified as follows. The steps (S1)−(S2) generate a set of not equally distributed particles $x_n^{(m)}$ with weights $w_n^{(m)}$. As discussed in [35, 95, 78], after a few iterations most importance weights $w_n^{(m)}$ become null: the consequence is that the variance of the importance weights, given by [77, 76]

$$\widehat{N}_n = \frac{1}{\sum_{m=1}^{M} W_n^{(m)}}, \tag{2.23}$$

increases (this phenomenon is known as *sample impoverishment*). In order to overcome this issue, *we resample the particles* $x_n^{(m)}$ from the *following discrete random variable* $\widehat{X}_n$ (1.2):

$$\widehat{X}_n = \begin{pmatrix} x_n^{(1)} & \cdots & x_n^{(M)} \\ W_n^{(1)} & \cdots & W_n^{(M)}. \end{pmatrix} \tag{2.24}$$

A SIS algorithm is summarized in Algorithm SIS: in input it receives N, M, q_n, x_1, in output it produces M independent paths $x_{1:N}^{(m)}$, $m = 1, ..., M$.

Algorithm 2 SIS

begin
  for $n = 2$ *to* N **do**
    //Generation of the particles.
    for $m = 1$ *to* M **do**
      1) Generate $x_n^{(m)}$ from the density $q(x_{1:n}|x_{1:n-1}, y_{1:n})$.
      2) Compute the normalized weights $W_n^{(m)}$ (2.22).
    end
    //Resampling from a discrete random variable.
    3) Calculate $\widehat{N}_n$ (2.23).
    if $\widehat{N}_n > N_T$ **then**
      4) Set a real variable $cdf = 0$.
      5) Set $j = 1$ and $k = 1$.
      6) Draw a uniform number $u \sim [0; 1]$.
      while $cdf < u$ *and* $j \leq M$ **do**
        7) Calculate $cdf = cdf + W_n^{(j)}$.
        8) Set $x_n^{(j)} := x_n^{(k)}$ and $W_n^{(j)} := W_n^{(k)}$.
        9) Set $j = j + 1$
      end
      10) Set $k = k + 1$.
    end
  end
  return $\widehat{X}_n := (x_n^{(m)}, W_n^{(m)})_{m=1}^M$.
end

As pointed out before, the *crucial point of a SIS algorithm is the resampling from a discrete variable*: this is implemented only if the variance of the importance weights is too high, i.e. if $\widehat{N}_n$ is higher than a threshold N_T.

The procedure works as follows: for every $n = 1, ..., N$, firstly we determine the outcome $x_n^{(k)}$ of X_n for which the cdf (6) is higher than a uniform threshold u, then we replace all the values $x_n^{(j)}$ and $W_n^{(j)}$, $1 \leq j \leq k$, respectively with $x_n^{(k)}$ and $W_n^{(k)}$, and, finally, we start from $k + 1$; we repeat the previous steps until $k = M$.

In conclusion, a SIS method is identified by *the choice of the importance density* and of the *initial value*.

2.1.4 Bootstrap Filter

The Bootstrap filter (BF) has been introduced by Gordon [45, 46] and successively developed by different authors [4, 22, 31, 32]. The BF is a *SIS method* which addresses the *problem of simulating a nonlinear Markov process*. For more details about the BF we refer to [20, 27].

The BF is based on the following assumptions.

(A1): The Markov processes $X_{1:N}$ and the observations $Y_{1:N}$ are governed by

the following nonlinear system:

$$\begin{aligned} X_n &= F(X_{n-1}) + W_n && (2.25)\\ Y_n &= G(X_n) + V_n, \quad n = 1, ..., N, && (2.26) \end{aligned}$$

with $F, G : \mathbb{R}^N \to \mathbb{R}^N$ two integrable functions.

(A2): $W_{1:N}$ and $V_{1:N}$ are Normal processes (18) with null mean and time-independent covariance matrices (1.19), i.e. $W_{1:N} \sim N(0, Q)$ and $V_{1:N} \sim N(0, R)$, where

$$\begin{aligned} Q &= \mathbb{E}[W_n W_n^T] && (2.27)\\ R &= \mathbb{E}[V_n V_n^T], && (2.28)\\ W_{1:N} &\sim N(0, Q),\ V_{1:N} \sim N(0, R). \end{aligned}$$

for every $n = 1, ..., N$;

(A3): V_n and W_n are uncorrelated.

In the BF, we select the transition densities $\pi(x_n|x_{n-1})$ as importance densities:

$$q_n(x_n|x_{1:n-1}, y_{1:n}) := \pi(x_n|x_{n-1}), \quad n = 1, ..., N, \tag{2.29}$$

which is, for every n, the density of the Normal process $N(F(X_{n-1}), Q)$ (2.25).

The importance weights $w_n(x_n)$ (2.20) are given by

$$\begin{aligned} w_n(x_n) &:= \frac{\pi(x_n|x_{n-1}, y_{1:n})}{q(x_n|x_{n-1}, y_{1:n})} && (2.30)\\ &= \frac{\pi(y_n|x_n)}{\int_{\mathbb{R}^N} \pi(y_n|x_n)\pi(x_n)\, dx_n}. && (2.31) \end{aligned}$$

For (2.26), $\pi(y_n|x_n)$ is the density of the Normal process $N(G(X_n), R)$.

Formula (2.31) follows from the choice of q_n and from the following theorem, which provides a factorization for $\pi(x_n|x_{n-1}, y_n)$.

Theorem 41 (Expression of $\pi(x_n|x_{n-1}, y_n)$)**.** *The density $\pi(x_n|x_{n-1}, y_n)$ can be expressed as*

$$\pi(x_n|x_{n-1}, y_n) = \frac{\pi(y_n|x_n)\pi(x_n|x_{n-1})}{\int_{\mathbb{R}^N} \pi(y_n|x_n)\pi(x_n)\, dx_n}, \tag{2.32}$$

for every $n = 1, ..., N$.

Proof. We apply the Bayes theorem (1.34) to $\pi(x_n|x_{n-1}, y_n)$:

$$\pi(x_n|x_{n-1}, y_n) = \frac{\pi(y_n|x_n, x_{n-1})\pi(x_n|x_{n-1})}{\pi(y_n|x_{n-1})}, \quad n = 1, ..., N,$$

which can be simplified, using (2.25−2.26), as

$$\pi(x_n|x_{n-1}, y_n) = \frac{\pi(y_n|x_n)\pi(x_n|x_{n-1})}{\pi(y_n)}. \tag{2.33}$$

Since $Y_{1:M}$ is a Markov process, the marginal density $\pi(y_n)$ can be expressed, using the CKL (25), as

$$\pi(y_n) = \int_{\mathbb{R}^N} \pi(y_n|x_n)\pi(x_n)\, dx_n.$$

Substituting this expression into (2.33), the assertion follows. □

The BF algorithm involves the following steps:

(S1): sampling $x_n^{(m)}$ from $\pi(x_n|x_{n-1})$;

(S2): evaluation of the denominator in (2.31) using the MCst approach (1.76):

$$\widehat{\pi}_n = \frac{1}{M}\sum_{m=1}^{M} \pi(y_n|x_n);$$

(S3): calculation of $w_n^{(m)} := w_n(x_n^{(m)})$

$$w_n^{(m)} := \frac{\pi(y_n|x_n^{(m)})}{\sum_{m=1}^{M} \pi(y_n|x_n^{(m)})}; \tag{2.34}$$

(S3): resampling of $x_n^{(m)}$ according to $w_n^{(m)}$.

We observe that no normalization is required as in a general SIS (2.22) because the weights $w_n^{(m)}$ are already normalized.

The BF is described in Algorithm BF (for more detailes see [75]): in input it receives N, M, x_1, F, G, Q, R, in output it produces M independent paths $x_{1:N}^{(m)}$, $m = 1, ..., M$.

Algorithm 3 BF

begin
  for $n = 2$ *to* N **do**
    //Generation of the particles.
    for $m = 1$ *to* M **do**
      1) Generate $x_n^{(m)} \sim N(F(X_{n-1}), Q)$ (2.25).
      2) Compute the normalized weights $w_n^{(m)}$ (2.34).
      3) Calculate $\widehat{N}_n$ (2.23).
    end
    //Sampling from a discrete random variable.
    if $\widehat{N}_n > N_T$ **then**
      4) Set a real variable $cdf = 0$.
      5) Set $j = 1$ and $k = 1$.
      6) Draw a uniform number $u \sim [0; 1]$.
      while $cdf < u$ *and* $j \leq M$ **do**
        7) Calculate $cdf = cdf + w_n^{(j)}$.
        8) Set $x_n^{(j)} := x_n^{(k)}$ and $w_n^{(j)} := w_n^{(k)}$.
        9) Set $j = j + 1$
      end
      10) Set $k = k + 1$.
    end
  end
  return $\widehat{X}_n := (x_n^{(m)}, w_n^{(m)})_{m=1}^{M}$.
end

In short, the BF Algorithm is derived from the *SIS Algorithm with importance densities* (2.29) and *importance weights* (2.34).

2.2 Bayesian MC approach

As discussed in Section 1.4, MCst estimators have a low precision and, in the case of a continuous barrier option, are biased. In order to solve them, in this section we propose a MC approach, named *Bayesian MC* (BMC) *approach.*

Firstly we firstly describe the BMC approach from a general point of view (for more detailes we refer to [13, 38, 17, 37]) and successively we use it to price a barrier option.

Given a state variable $X_{1:N}$, $n = 1, ..., N$ and an integrable function $f : \mathbb{R}^{nI} \to \mathbb{R}$, a BMC procedure estimates the following *conditional expected value*:

$$\Psi_n := \int f(x_{1:n})\pi(x_{1:n}|y_{1:n})\, dx_{1:n}, \quad n = 1, ..., N. \tag{2.35}$$

Now we formulate the BMC for discrete and continuous barrier options under the BSM.

In the case of a discrete barrier option, the BMC can be summarized as follows:

(P1): The state variable is represented by S_n, $n = 1, ..., N$, resulting from (1.60).

(P2): The functions f in (2.1) is the option payoff Ψ_T^d (1.74) with $n = N$.

Using (1.18) and the Markov property of S_n, Expression (2.35) becomes:

$$\Psi_N^d = \int_{\mathbb{R}^N} \Psi_T^d \left(\prod_{n=1}^{N} \pi(s_n|s_{n-1}, y_{1:n}) \right) ds_{1:N}. \tag{2.36}$$

A BMC procedure for discrete barrier options works as follows: firstly, a set of M independent paths $s_{1:N}^{(m)}$ of $S_{1:N}$, $m = 1, ..., M$, are generated using a Bayesian sampling technique (Section 2.1), then we compute the MCst estimator of (2.36) over the paths $s_{1:N}^{(m)}$:

$$\Theta_N^M = \frac{e^{-rT}}{M} \sum_{m=1}^{M} f(s_{1:N}^{(m)}). \tag{2.37}$$

In order to extend the BMC approach to the case of continuous barrier options, we preliminarily introduce the notion of exit probability.

Definition 42. *Let $s, t \in [0; T]$ and $x, y \in [0; +\infty]$ respectively the values of S_t at s and z, and I_t the value of the barrier interval at $t \in [0; T]$ (1.70). The exit probability is the function $p(\cdot) : \mathbb{R}^2 \to [0; 1]$, defined as*

$$p(x, y) := \mathbb{P}(\exists t \in [s; z] : S_t \notin I_t | S_s = x, S_z = y), \tag{2.38}$$

for every $s, z \in [0; T]$.

The exit probability is the*probability that the underlying hits the barriers once at least.* Theorem (43) provides the formulas of $p(x, y)$ in the case of a single barrier [1, 11] and double barriers [86] under the BSM.

Theorem 43. *The exit probability $p(x, y)$ (2.38) of a single barrier option, i.e. $L_t = 0$ and $U_t = B_t \in]0; +\infty[$ or $L_t = B_t \in]0; +\infty[$ and $U_t = +\infty$ for every $t \in [0; T]$, is equal to:*

$$p(x, y) = \begin{cases} 1 & \text{if } x \notin I_x \text{ or } y \notin I_y \\ exp\left(-2\frac{(\ln x - \ln B_y)(\ln y - \ln B_y)}{(z-s)\sigma^2}\right) & \text{otherwise,} \end{cases} \tag{2.39}$$

for every $s, z \in [0; T]$, with $s < z$.

In the case of a double barrier option, i.e. $L_t, U_t \in]0; +\infty[$ and $L_t \neq U_t$ for every $t \in [0; T]$, with $x \in]L_x; U_x[$, the exit probability $p(x, y)$ (2.38) is equal to:

$$p(x, y) = 1,$$

if $y \notin]L;U[$, *and by*

$$\begin{aligned} p(x,y) &= \sum_{m=1}^{+\infty} [R(\alpha m - \gamma, \delta)] + R_n(-\alpha m + \beta, \delta)] + \\ &+ \sum_{m=1}^{+\infty} [R(\alpha m, \delta)] + R(-\alpha m, \delta)], \end{aligned} \tag{2.40}$$

where

$$\delta = ln\frac{y}{x}, \quad \alpha = 2\ln\frac{U_z}{L_z}, \quad \beta = 2\ln\frac{U_z}{x}\gamma = 2\ln\frac{x}{L_z}, \quad R(z,x) = exp\left(-\frac{z(z-2x)}{2(t-s)\sigma^2}\right),$$

if $y \in]L_y;U_y[$, *for every* $s, z \in [0;T]$, *with* $s < z$.

Formulas (2.39–2.40) are derived from the laws of the maximum and minimum of a Brownian motion (for more details see [67]).

Now we introduce the following weights $g(x,y)$:

$$g(x,y) = 1_{]s;z[}(y)p(x,y).$$

If the barriers have been crossed, i.e. $g(S_{n-1}, S_n) = 0$, we are informed if the the barriers have been crossed. If the barriers have not been crossed, we measure the probability of the underlying not to hit the barriers.

The starting point of the BMC approach for continuous barrier options is the following theorem, which provides a new expression of Ψ_0^c [44].

Theorem 44. *The price of a continuous barrier option* Ψ_0^c *can be rewritten as*

$$\Psi_0^N = h(s_N)\left(\prod_{n=1}^{N} g(s_{n-1}, s_n)\right) \tag{2.41}$$

$$\Psi_0^c = e^{-rT}\int \Psi_0^N \pi(s_{1:N})\, ds_{1:N}, \tag{2.42}$$

with $N \in \mathbb{N}$.

The main consequences of Theorem (1.73) are that we can *redefine the price* Ψ_0^c as (2.41) and *every MCst estimator of* (2.41)*is unbiased.*

Now we are able to develop the BMC approach for continuous barrier options, which is characterized by the following points:

(P1): $Y_n = 0$ for every $n = 1, ..., N$.

(P2): The state variable is represented by S_n, $n = 1, ..., N$, obtained by applying (1.60).

(P3): The functions f in (2.1) is Ψ_0^N (2.41).

Bayesian problem (2.35) takes the form (see [96]):

$$\begin{aligned} \Psi^N &= e^{-rT}\int_{\mathbb{R}} h(s_N)g(s_{N-1}, s_N)\pi(s_N|s_{N-1})\, ds_N \cdots \\ &\cdots \int_{\mathbb{R}} g(s_0, s_1)\pi(s_1|s_0)\, ds_1. \end{aligned} \tag{2.43}$$

For every $n = 1, ..., n$, one proceeds as follows: firstly we draw $s_n^{(m)} \sim \pi(s_n|s_{n-1})$, $m = 1, ..., M$, secondly we introduce the following variables $h_n^{(m)}$, $m = 1, ..., M$:

$$h_n^{(m)} = \begin{cases} h(s_N^{(m)})g(s_{N-1}^{(m)}, s_N^{(m)}) & \text{if } n = N \\ \\ g(s_{n-1}^{(m)}, s_n^{(m)}) & \text{otherwise,} \end{cases} \tag{2.44}$$

thirdly every integral is evaluated using a MCst estimator (1.76) over $s_n^{(m)}$ and, finally, we construct an estimator Θ_N^M as the *product of the MCst estimates found at the previous step*:

$$\Theta_N^M = \frac{e^{-rT}}{M^N} \prod_{n=1}^{N} \left(\sum_{m=1}^{M} g_n^{(m)} \right). \tag{2.45}$$

Theorem (45) states that Θ_n^M (2.37−2.45) *is unbiased* and *has a higher precision than the MCst* Δ_n^M (1.76).

Theorem 45. *The estimator* Θ_N^M *defined in (2.37−2.45) is unbiased and has a lower variance than the MCst* Δ_n^M *(1.76). In symbols:*

$$\mu_n = \mathbb{E}[\Theta_n^M] \tag{2.46}$$

$$\mathbb{V}[\Theta_n^M] \leq \mathbb{V}[\Delta_n^M], \tag{2.47}$$

for every $n = 1, ..., N$.

Proof. Equation (2.46) follows from the application of the LTE (1.32−1.33) to μ_n:

$$\begin{aligned} \mu_n &= \mathbb{E}[\Omega_n] \\ \Omega_n &= \mathbb{E}[f(X_{1:n})|Y_{1:n}] \\ n &= 1, ..., N. \end{aligned}$$

Now we prove (2.47). For every $n = 1, ..., N$, the variance $\mathbb{V}[\Delta_n^M]$ of Δ_n^M can be expressed, using (15), as:

$$\begin{aligned} \mathbb{V}[\Delta_n^M] &= \mathbb{V}[\Omega_n^s] + \mathbb{E}[\Delta_n^s] \\ \Omega_n^s &= \mathbb{E}[\Delta_n^M|Y_{1:n}] \\ \Delta_n &= \mathbb{V}[\Delta_n^M|Y_{1:n}] \end{aligned} \tag{2.48}$$

The positiveness of f implies that:

$$\mathbb{V}[\Delta_n^M] \geq \mathbb{V}[\Omega_n^s]. \tag{2.49}$$

The variable Ω_n^s 2.48 can be rearranged as:

$$\Omega_i^s = \frac{1}{M} \sum_{m=1}^{M} \mathbb{E}[f(x_{1:i}^{(m)}] = \tag{2.50}$$

$$= \mathbb{E}[f(x_{1:i}^{(1)}]. \tag{2.51}$$

Equation (2.50) derives from the linearity of the conditional expected value. Equation (2.51) is a consequence of the property of i.i.d of the particles.

Finally, the assertion follows from (2.49−2.51) and the definition of Θ_i^M. □

In short, the main features of a BMC method are: the *choice of the observations* and *the Bayesian sampling technique* (Section 2.1).

2.3 Bayesian MC methods

This section is devoted to some BMC methods to price a barrier option under the BSM.

Subsection 2.2.1 examines the *conditional MC* technique. Subsection 2.2.2 describes the *exponential twisting* method. Subsection 2.2.3 explores the *weighting functions* approach.

2.3.1 Conditional MC methods

The conditional MC method (CMCM), introduced by [58, 19] and successively proposed in [84, 89], is a BMC method to price a discrete barrier option (1.75)

We preliminarily provide some basic definitions. The first instant in which the barriers are crossed is denoted with τ, we define the following variables:

$$\begin{aligned} \Psi(x) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \\ d1 &:= \frac{\ln(s/k) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \\ d2 &:= d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

$$BSM(s,k,r,T,t) := \begin{cases} s\Psi(d1) - k\Psi(d2) & \text{if Call} \\ ke^{-r(T-t)}\Psi(-d2) - s\Psi(-d1) & \text{if Put,} \end{cases} \tag{2.52}$$

As shown in [15], the functions h and BSM are related by

$$BSM(S_0, K, r, T, 0) = \mathbb{E}[h(S_T)|S_{1:\tau}]. \tag{2.53}$$

Formula (2.53) is the so unknown *Black-Scholes formula* [15]. Using (2.53), Bayesian problem (2.36) becomes:

$$\Psi_T^d = \mathbb{E}\left[BSM(S_\tau, K, r, T, \tau)\right], \tag{2.54}$$

A CMCM estimator of Ψ_T^d is given by

$$\widehat{\Psi}_T^d = \frac{e^{-r\tau}}{M} \sum_{m=1}^{M} BSM(S_\tau^{(m)}, K, r, T, \tau), \quad s_\tau^{(m)} \sim \pi(S_\tau). \tag{2.55}$$

In short, the CMC is the *MCst estimator of Formula* (2.54).

2.3.2 Exponential twisting

The exponential twisting (ET), introduced in [19, 98] and successively proposed in [41, 90, 84], is a *BMC procedure to price a discrete barrier option with a constant single barrier.* In our case the barrier will be indicated with the symbol B, the barrier interval, if a knock-out, takes the form:

$$I_t = \begin{cases}]B; +\infty[& \text{if } S_0 > B \\ ]0; B[& \text{if } S_0 < B, \end{cases} \tag{2.56}$$

if the option is a knock-in, then

$$I_t = \begin{cases}]B;+\infty[& \text{if } S_0 < B \\ ]0;B[& \text{if } S_0 > B, \end{cases} \tag{2.57}$$

We preliminarily introduce the following variables:

$$\begin{aligned} b &= \ln\left(\frac{S_0}{B}\right); \quad c = \frac{K}{S_0} \quad a = \frac{1}{2} - \frac{r}{\sigma^2}; \quad d = \frac{2b+c}{N\sigma^2+h} \\ t^- &:= a - d; \quad t^+ = a + d, \end{aligned}$$

τ is the first instant when the B is hit, and the following function $M(\cdot) : [0;T] \to]0;+\infty]$ as

$$M(t) := exp\left[t\left(r - \frac{\sigma^2}{2}\right)h + \frac{1}{2}\sigma^2 t\right]. \tag{2.58}$$

Bayesian problem (2.35) is simplified as

$$\Psi_0^d = \int_{\mathbb{R}^N} \Psi_T^d \left(\prod_{n=1}^{N} \pi(s_n|s_{n-1})\right) ds_{1:N}. \tag{2.59}$$

The ET approach is described in the following: firstly, we sample from $\pi(s_n|s_{n-1})$ using a SIS scheme (**Algorithm SIS**), with importance densities:

$$q_n(s_n|s_{n-1}) := \begin{cases} \frac{e^{t^-}\pi(s_n|s_{s_{n-1}})}{M(t^-)} & \text{if } t \le \tau \\ \frac{e^{t^+}\pi(s_n|s_{s_{n-1}})}{M(t^+)} & \text{otherwise,} \end{cases} \tag{2.60}$$

and importance weights $w(s_n)$ (2.20):

$$w(s_n) \begin{cases} \frac{M(t^-)}{exp[t^- s_n]} & \text{if } t \le \tau \\ \frac{M(t^+)}{exp[t^+ s_n]} & \text{otherwise,} \end{cases} \tag{2.61}$$

successively we evaluate (2.2) using the following MCst estimator (1.76) over the paths $s_{1:N}^{(m)}$:

$$\Delta = \frac{e^{-rT}}{M} \sum_{m=1}^{M} \left[h(s_N^{(m)}) \left(\prod_{n=1}^{N} 1_{I_n}(s_{1:N}^{(m)})\right)\right]. \tag{2.62}$$

The ET scheme is summarized in **Algorithm ET**: in input it receives $T, S_0, r, \sigma, K, N, M$. In output it produces the value Ψ_0^d.

Algorithm 4 ET

begin
  for $n = 2$ *to* N **do**
    //Generation of the particles.
    for $m = 1$ *to* M **do**
      1) Generate $s_n^{(m)} \sim q(s_n|s_{n-1})$ (2.60).
      2) Compute $w_n^{(m)} := w(s_n^{(m)}, s_{n-1}^{(m)})$ (2.61).
      2) Evaluate the normalized weights $W_n^{(m)}$ (2.22).
    end
    //Sampling from a discrete random variable.
    3) Set a real variable $cdf = 0$.
    4) Set $j = 1$ and $k = 1$.
    5) Draw a uniform number $u \sim [0; 1]$.
    while $cdf < u$ *and* $j \leq M$ **do**
      5) Calculate $cdf = cdf + W_n^{(j)}$.
      6) Set $s_n^{(j)} := s_n^{(k)}$.
      7) Set $j = j + 1$
    end
    8) Set $k = k + 1$.
  end
  return Θ_N^M (2.62).
end

In short, for every $= 1, ..., N$, the ET is based on a *SIS procedure, with observations represented by the past value of the underlying.* An extension of ET method is treated in [89], where the authors combine the CMCM and the ET.

2.3.3 Weighting functions method

The weighting functions (WF) method, developed in [64, 96, 94], is a *BMC method to price a continuous barrier option under the BSM* (1.56).

The WF proceeds as follows. Firstly, for every $n = 1, ..., N$, we simulate a set of particles $x_n^{(m)}$ using a SIS technique (**Algorithm SIS**), with importance densities q_n given by the transition densities:

$$\begin{aligned} x_n &:= \ln S_n \\ q_n(x_n|x_{n-1}) &:= \pi(\ln s_n | \ln s_{n-1}) \\ n &= 1, ..., N, \end{aligned} \tag{2.63}$$

which is the density of a Normal $N(\mu_n, \widetilde{\sigma})$ with

$$\mu_n := x_{n-1} + (r - \frac{\sigma^2}{2})h, \quad \widetilde{\sigma} := h\sigma.$$

Secondly, we find the underlying particles

$$s_n^{(m)} = exp(x_n^{(m)}),$$

and we determine the importance weights $w_n(s_n)$ (2.20), given by:

$$w_n(s_n) = g(s_{n-1}, s_n). \tag{2.64}$$

Finally, we evaluate the estimator Θ_N^M (2.45).

The WF scheme is described in **Algorithm WFM**: in input it receives $T, S_0, r, \sigma, K, N, M, \widehat{N}_n$, in output it produces the value of Θ_N^M.

Algorithm 5 WFM

begin
  for $n = 2$ *to* N **do**
    //Generation of the particles.
    for $m = 1$ *to* M **do**
      1) Draw $x_n^{(m)} \sim N(\mu_n, \widetilde{\sigma})$.
      2) Compute the unnormalized weights $w_n^{(m)}$ (2.64).
      3) Normalization $W_n^{(m)}$ (2.22).
    end
    //Resampling from a discrete random variable.
    4) Calculate $\widehat{N}_n$ (2.23).
    if $\widehat{N}_n > N_T$ **then**
      5) Set a real variable $cdf = 0$.
      6) Set $j = 1$ and $k = 1$.
      7) Draw a uniform number $u \sim [0; 1]$.
      while $cdf < u$ *and* $j \leq M$ **do**
        8) Calculate $cdf = cdf + W_n^{(j)}$.
        9) Set $x_n^{(j)} := x_n^{(k)}$ and $W_n^{(j)} := W_n^{(k)}$.
        10) Set $j = j + 1$
      end
      10) Set $k = k + 1$.
    end
    11) Set $s_n^{(m)} = exp[x_n^{(m)}]$.
  end
  return Θ_N^M.
end

In short, in the WF approach the sampling procedure is a *SIS with transition densities as importance densities and the observations are represented*, for every n, by *the past values of the underlying.*

Chapter 3

Bayesian MC methods under a SVM

This section discusses two new BMC approaches to price a continuous barrier option under a SVM, whose goal is, as highlighted in the previous chapters, to construct unbiased estimators with high precision. The first one extends the WF approach (Subsection 2.3.3), the second one is based on the BF technique (Subsection 2.1.4). In the last part of the chapter we provide some numerical tests, which confirm the applied contribute of the algorithms examined in this chapter.

Section 3.1 presents the MC methods under a SVM. Section 3.2 provides the numerical tests.

3.1 Bayesian MC methods

As explored in Section 2.2, the goal of a BMC approach is to evaluate the following conditional expected value [13, 38, 17, 37].

$$\Psi_n := \int f(x_{1:n})\pi(x_{1:n}|y_{1:n})\, dx_{1:n}, \quad n = 1, ..., N.$$

Theorem (45) claims that every MC estimator of the previous expression is an unbiased estimator with high precision.

In order to develop a BMC approach to price a continuous barrier option under a SVM (1.54−1.55), we generalize the notion of exit probability given in Section 2.2.

Definition 46. *Let $s, t \in [0; T]$, $x, y, \sigma \in \mathbb{R}$ with $S_s = x$, $S_t = y$ and $\sigma_s = \sigma$. The exit probability is the function $p(\cdot) : \mathbb{R}^3 \to [0; 1]$ defined as*

$$p(x, y, \sigma) := \mathbb{P}(\exists t \in [s; z] : S_t \notin I_t | S_s = x, S_z = y, \sigma_s = \sigma). \tag{3.1}$$

for every $s, z \in [0; T]$.

Theorem (47) provides the expression of the trigger probability under a SV model (for the proof see [7]).

Theorem 47. *The exit probability $p(x,y,\sigma)$ (2.38) of a single barrier option is equal to:*

$$p(x,y,\sigma) = \begin{cases} 1 & \text{if } x \notin I_x \text{ or } y \notin I_y \\ exp\left(-2\frac{(x-B_s)(y-B_z)}{(z-s)\sigma^2}\right) & \text{otherwise,} \end{cases} \tag{3.2}$$

for every $s,z \in [0;T]$, with $s<z$.

In the case of a double barrier option, the exit probability $p(x,y,\sigma)$ (2.38) is equal to:

$$p(x,y,\sigma) = 1,$$

if $y \notin]L;U[$, and by

$$\begin{aligned} p(x,y,\sigma) &= \sum_{m=1}^{+\infty} [R(\alpha m - \gamma, \delta)] + R_n(-\alpha m + \beta, \delta)] + \\ &+ \sum_{m=1}^{+\infty} [R(\alpha m, \delta)] + R(-\alpha m, \delta)], \end{aligned} \tag{3.3}$$

where

$$\delta = y-x, \quad \alpha = 2(U_z - L_z), \quad \beta = 2(U_z - x), \quad \gamma = 2(x - L_z), \quad R(z,x) = exp\left(-\frac{z(z-2x)}{2(z-s)\sigma^2}\right),$$

if $y \in]L_y;U_y[$, for every $s,z \in [0;T]$, with $s<z$.

Now we introduce the following weights $g(x,y,\sigma)$:

$$g(x,y,\sigma) = 1_{]s;z[}(y)p(x,y,\sigma). \tag{3.4}$$

The starting point of our BMC is the following theorem, which provides a new expression of Ψ_0^c (for more details see [7]).

Theorem 48. *The price Ψ_0^c can be rearranged as*

$$\Psi_T^N = e^{-rT} h(S_N) \left(\prod_{n=1}^{N} g(s_{n-1}, s_n, \sigma_{n-1}) \right) \tag{3.5}$$

$$\Psi_0^c = \int \Psi_T^N \pi(s_{1:N})\, ds_{1:N}, \tag{3.6}$$

where s_{n-1} and s_n are respectively the outcomes of S_{n-1} and S_n.

The BMC approach for continuous barrier options under a SVM relies on the following points:

(P1): the state variable is represented by S_n;

(P2): the observations are $S_{1:n-1}$ and $\sigma_{1:n}$;

(P3): The function f in (2.1) is the variable Ψ_T^N.

Problem (2.35) becomes for $n = N$:

$$\Psi_N = e^{-rT} \int_{\mathbb{R}^N} h(s_n) \left(\prod_{n=1}^{N} g(s_{n-1}, s_n, \sigma_{n-1}) \right) \pi(s_{1:N}|\sigma_{1:N})\, ds_{1:N}. \tag{3.7}$$

Theorem (45) guarantees that *every estimator of* (3.7) *is unbiased* and *has a higher precision than a MCst estimator of* (3.6).

Using (1.18–24) and the reduction formulas of the multiple integrals, Formula (3.7) can be rewritten as

$$\begin{aligned} \Psi^N &= e^{-rT} \int_{\mathbb{R}} h(s_N) g(s_{N-1}, s_N, \sigma_{N-1}) \pi(s_N|s_{N-1}, \sigma_{N-1})\, ds_N \cdots \\ &\cdots \int_{\mathbb{R}} g(s_0, S_1, \sigma_0) \pi(s_1|s_0, \sigma_0)\, ds_1. \end{aligned} \tag{3.8}$$

A BMC algorithm proceeds as follows.

(S1): Using (1.61–1.62), simulate M independent paths $s_{1:N}^{(m)}$ and $\sigma_{1:N}^{(m)}$, $m = 1, ..., M$.

(S2): Every integral in (3.8) is estimated using a MCst approach (1.76).

(S3): The products of the quantities found at (S2) is calculated.

In short, given the following quantities $h_n^{(m)}$:

$$h_n^{(m)} = \begin{cases} h(s_N^{(m)}) g(s_{N-1}^{(m)}, s_N^{(m)}, \sigma_{N-1}^{(m)}) & \text{if } n = N \\ g(s_{n-1}^{(m)}, s_n^{(m)}, \sigma_{n-1}^{(m)}) & \text{otherwise,} \end{cases} \tag{3.9}$$

a BMC estimator Π_N^M of Ψ_0^c (3.6) is:

$$\Pi_N^M = \frac{e^{-rT}}{M^N} \prod_{n=1}^{N} \left(\sum_{m=1}^{M} h_n^{(m)} \right). \tag{3.10}$$

As pointed out in Chapter 2, a BMC method is identified uniquely by the *observations* and the *sampling procedure*.

Subsection 3.1.1 details the extended WF method (EWF). Subsection 3.1.2 proposes the bootstrap BMC (BBMC).

3.1.1 Extended weighting functions method

As mentioned at the beginning of this chapter, the EWF *generalizes the WF approach examined in Subsection* 2.3.3 *under a SVM* (1.54–1.55).

The WF procedure works as follows. Firstly, for every $n = 1, ..., N$, we generate M particles $x_n^{(m)}$ using a SIS technique (**Algorithm SIS**), with importance densities q_n given by the transition densities:

$$q_n(s_n|s_{1:n-1}, \sigma_{1:n}) := \pi(s_n|s_{n-1}, \sigma_{n-1}) \quad n = 1, ..., N, \tag{3.11}$$

and importance weights $w_n(s_n)$:

$$w_n(s_n) = g(s_{n-1}, s_n, \sigma_{n-1}). \tag{3.12}$$

For (1.61−1.62), q_n is the density of a Normal $N(\mu_n, \widetilde{\sigma}_n)$ with

$$\mu_n := (1+r)s_{n-1}, \quad \widetilde{\sigma}_n = h(\sigma_{n-1}\alpha_{n-1})^2.$$

Finally, we evaluate estimator Π_N^M (3.10).

The EWF procedure is described in **Algorithm SVM-WF**: in input it receives $T, S_0, r, \sigma_0, K, N, M, \widehat{N}_n$ and the functions $\alpha_t, \beta_t, \gamma$, in output it produces the value of the estimator Π_N^M.

Algorithm 6 SVM-WFM

begin
  for $n = 2$ *to* N **do**
    //Generation of the particles.
    for $m = 1$ *to* M **do**
      1) Determine $(s_n^{(m)}, \sigma_n)$ using (1.61−1.62).
      2) Compute the importance weights $w_n^{(m)} := w_n(s_n^{(m)})$ (3.12).
      3) Normalization $W_n^{(m)}$ (2.22).
    end
    //Resampling from a discrete random variable.
    3) Calculate $\widehat{N}_n$ (2.23).
    if $\widehat{N}_n > N_T$ **then**
      4) Set a real variable $cdf = 0$.
      5) Set $j = 1$ and $k = 1$.
      6) Draw a uniform number $u \sim [0; 1]$.
      while $cdf < u$ *and* $j \leq M$ **do**
        7) Calculate $cdf = cdf + W_n^{(j)}$.
        8) Set $s_n^{(j)} := s_n^{(k)}$ and $W_n^{(j)} := W_n^{(k)}$.
        9) Set $j = j + 1$
      end
      10) Set $k = k + 1$.
    end
  end
  return Π_N^M.
end

In short, at every time step $n = 1, ..., N$, the EWF samples $s_n^{(m)} \sim \pi(s_n|s_{n-1}, \sigma_{n-1})$, $m = 1, ..., M$, and then it uses them to find the value of the BMC estimator.

3.1.2 Bootstrap Filter Approach

In this section we illustrate the BBMC method to price a continuous barrier option under a SVM (1.54−1.55). As suggested by the name, this algorithm,

developed originally by Kim et al. [69] to estimate the volatility of a SVM, simulates the independent underlying paths *by applying the BF* (**Algorithm BF**).

The starting point of the BBMC approach is the following theorem, which reformulates a discrete SVM (1.61−1.62) as proposed by Taylor [104]

Theorem 49 (Taylor, 1994)**.** *The discrete model (1.54−1.55) can be rewritten as*

$$\begin{aligned} \sigma_n &= \sigma_{n-1} + \beta_{n-1}h + \sqrt{h}\gamma_{n-1}Z_n^\sigma, \quad Z_n^\sigma \sim N(0;1) && (3.13)\\ R_n &= \sigma_n Z_n^s, \quad Z_n^s \sim N(0;1), && (3.14) \end{aligned}$$

where R_n is the asset return of S_n at t_n:

$$R_n = \frac{S_n - S_{n-1}}{S_{n-1}}. \tag{3.15}$$

Model (3.13−3.14) satisfies Assumptions (2.1.4) by setting:

$$\begin{aligned} X_n &:= \sigma_n, \quad F(x) := x + \beta_{n-1}h, \quad W_n := \sqrt{h}\gamma_{n-1}Z_n^\sigma \\ Y_n &:= R_n, \quad G(x) := 0, \quad V_n := \sigma_n z_n^s, \end{aligned}$$

for every $n = 1, ..., N$.

Model (3.13−3.14) is obtained by exploiting the definition of volatility as the *standard deviation of the asset returns* [34].

The BMC is summarized as follows: firstly, at every time step $n = 1, ..., N$, we simulate $(\sigma^{(m)}, r_n^{(m)})$ by applying the BF technique to (3.13−3.14), and we compute:

$$s_n^{(m)} := (1 + r_n^{(m)})s_{n-1}^{(m)}, \tag{3.16}$$

finally we evaluate the estimator Π_N^M (3.10) using $s_n^{(m)}$, $n = 1, ..., N$.

The BBMC method is described in **Algorithm BBMC**: in input it receives $T, S_0, r, \sigma_0, K, N, M, \widehat{N}_n$ and the functions $\alpha_t, \beta_t, \gamma$, in output it produces the value of the estimator Π_N^M.

Algorithm 7 BBMC

begin
  for $n = 2$ *to* N **do**
    //Generation of the particles.
    for $m = 1$ *to* M **do**
      1) Compute $(\sigma_n^{(m)}, r_n^{(m)})$ using (3.13−3.14).
      2) Compute the normalized weights $w_n^{(m)}$ (2.34).
    end
    //Sampling from a discrete random variable.
    3) Calculate $\widehat{N}_n$ (2.23).
    if $\widehat{N}_n > N_T$ **then**
      4) Set a real variable $cdf = 0$.
      5) Set $j = 1$ and $k = 1$.
      6) Draw a uniform number $u \sim [0; 1]$.
      while $cdf < u$ *and* $j \leq M$ **do**
        7) Calculate $cdf = cdf + w_n^{(j)}$.
        8) Set $\sigma_n^{(j)} := \sigma_n^{(k)}$ and $r_n^{(j)} := r_n^{(k)}$.
        9) Set $j = j + 1$
      end
      10) Set $k = k + 1$.
    end
  end
  11) Compute $s_n^{(m)}$ (3.16).
  return Π_N^M.
end

In short, for every $n = 1, ..., N$, the BBMC generates a set of i.i.d. particles $s^{(m)} \sim \pi(s_n|s_{n-1}, \sigma_n)$, $m = 1, ..., M$, and it uses them to determine the value of the BMC estimator. Since for (1.62) the variables σ_n are generated using σ_{n-1}, one has:

$$\pi(s_n|s_{n-1}, \sigma_n) = \pi(s_n|s_{n-1}, \sigma_{n-1}, \sigma_n).$$

The previous expression means that the *BBMC conditions on more random variables than the EWF*, where the importance density is given by $\pi(s_n|s_{n-1}, \sigma_{n-1})$: as pointed out in Subsection 1.1.1, this implies that *the variance of the BBMC is lower than the variance of the EWF*, and, consequently, has a *higher precision*.

3.2 Numerical experiments

In this chapter we have developed two BMC methods to solve the issues of MCst estimators under a SVM: this section provides some numerical experiments in order to test the validity of this approach. More precisely, we have analyzed the *consequences of the distance between the initial underlying value and the barriers on the bias and precision of MC estimators*. In the following we will report our results. We have priced three euro continuous knock-out puts

with single constant barrier, with $T = 4$ years, $K = 0.4$ €, $S_0 = 0.4$ €, $\sigma_0 = 0.3$, $r = 0.02$ and barrier B given by:

(B1): the barrier of the first option is $B = 10$ €;

(B2): the barrier of the second option is $B = 6$ €;

(B3): the barrier of the third option is $B = 3$ €;

using a MCst scheme, the EWF approach and the BBBC method with different numbers of time steps $N = 10, 20, 30, 40, 50, 60, 80, 90$ and $M = 1000$, and we have compared the results in terms of bias and precision with a number of samples $D = 1000000$ in (1.47−1.45−1.46). We have assumed that the underlying S_t and the volatility σ_t follow a SABR model (1.65−1.66) with $\alpha = 0.4$, $\beta = 0.5$, $r = 0.02$.

The SABR model takes the form:

$$\begin{aligned} dS_t &= 0.02 S_t dt + \sigma_t S_t^{0.5} dW_t^s \\ d\sigma_t &= 0.4 \sigma_t dW_t^\sigma. \end{aligned}$$

The calculation of the bias requires the knowledge of the real price of the option (1.47) but, as investigated in Chapter 1, no closed formulas exist to price a barrier option under a SVM: for this reason, in our case we have found only an *approximation of this price* by proceeding as follows: firstly we have estimated the implied volatility using Hagan formula [49], then we have priced the option, whose volatility is given by the implied volatility and all the other parameter are not changed, by applying the Kunimoto-Ikeda formula. The consequence of replacing the real value with this approximation is that *the bias of the EWF and BBMC estimators is not null.*

In our case, the implied volatility is about 0.56, and:

(P1): the price of (B1) is 8.23 €;

(P2): the price of (B2) is 8.07 €;

(P3): the price of (B3) is 7.02 €.

In all the figures the MCst is indicated with the term *Standard*, the EWF estimator with *Sequential*, the BBMC estimator with *Bootstrap*. To simplify the notations, we omitt the symbol € in both the bias and precision.

Figures (3.1−3.2) show the precision and the bias of the three estimators for the first option.

Firstly, we observe that the three estimators give good results both in terms of the bias and precision. The explanation is that not many paths are rejected because the initial underlying value is sufficiently distant from the barrier.

Secondly, we observe that the number of time steps has a significant effect only in the case of the Standard. The reason is that, as described in [96], the number of survival path in the MCst procedure is inversely proportional to the number of time steps, while, in the case of both Sequential and Bootstrap, the resampling procedure on which the two algorithms are based lets us replace the rejected paths with the survival ones.

Finally we observe that the Bootstrap provides better performances than Sequential. The reason is that, as discussed in Section 3.1, at every time step the Bootstrap is obtained using more information than the Sequential one: in fact, as discussed in Section 3.1, at every time step in the Bootstrap we condition on the past underlying values and past and current volatility values, in the Sequential we condition on the past underlying and volatility values.

Figures (3.3−3.4) show the precision and the bias of the three estimators for the second option.

Firstly, we firstly observe that the decrease of the distance between the initial underlying value and the barrier implies that the three estimators provide worse performance than to the previous case.

Secondly, we observe that, as in the previous case, thanks to the resampling procedure, the Sequential and the Bootstrap estimators have a lower bias and a higher precision than the Standard and the number of time steps impacts the bias and the precision only of the Standard.

Finally, we observe that the Bootstrap still provides better results than the Sequential (for the justification see the previous case).

Figures (3.5−3.6) show the precision and the bias of the three estimators for the third option.

Firstly, we observe that, since the initial underlying value approaches the barrier, the performance of the three estimators changes respect to the previous cases.

Finally we observe that, as in the previous cases, the number of time steps has a low impact only on the Sequential and the Bootstrap and that the Bootstrap has the lowest bias and the highest precision among the three estimators (for the justification see the previous points).

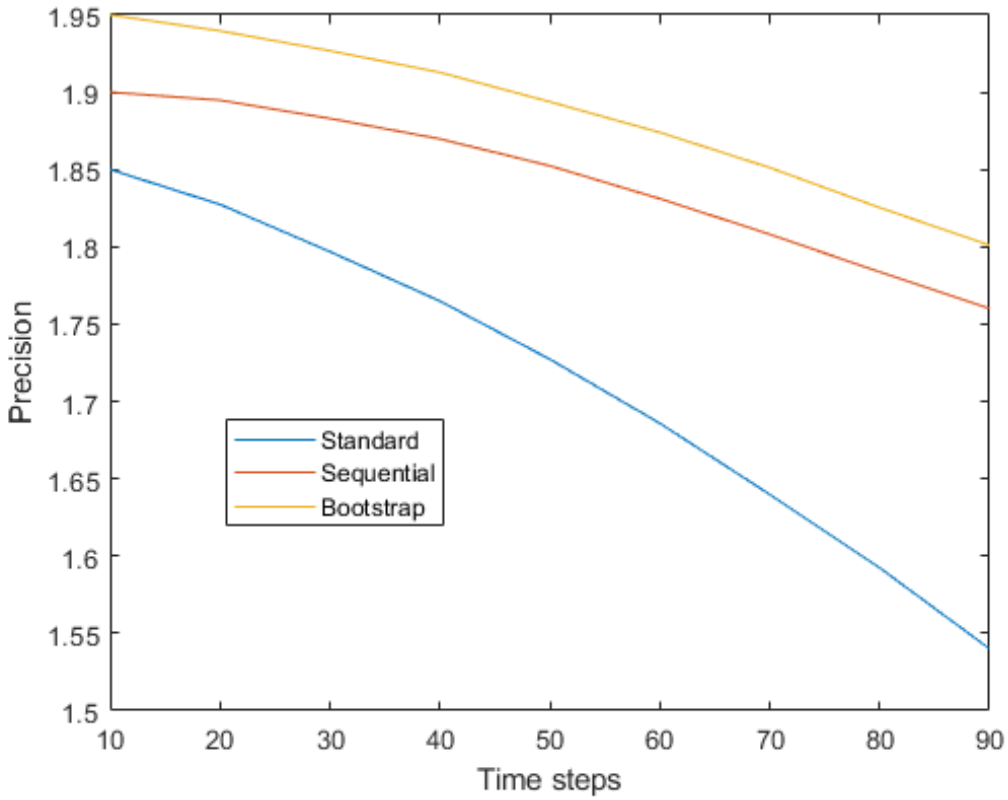

Figure 3.1: **Precision of MC estimators for $B = 10$.**

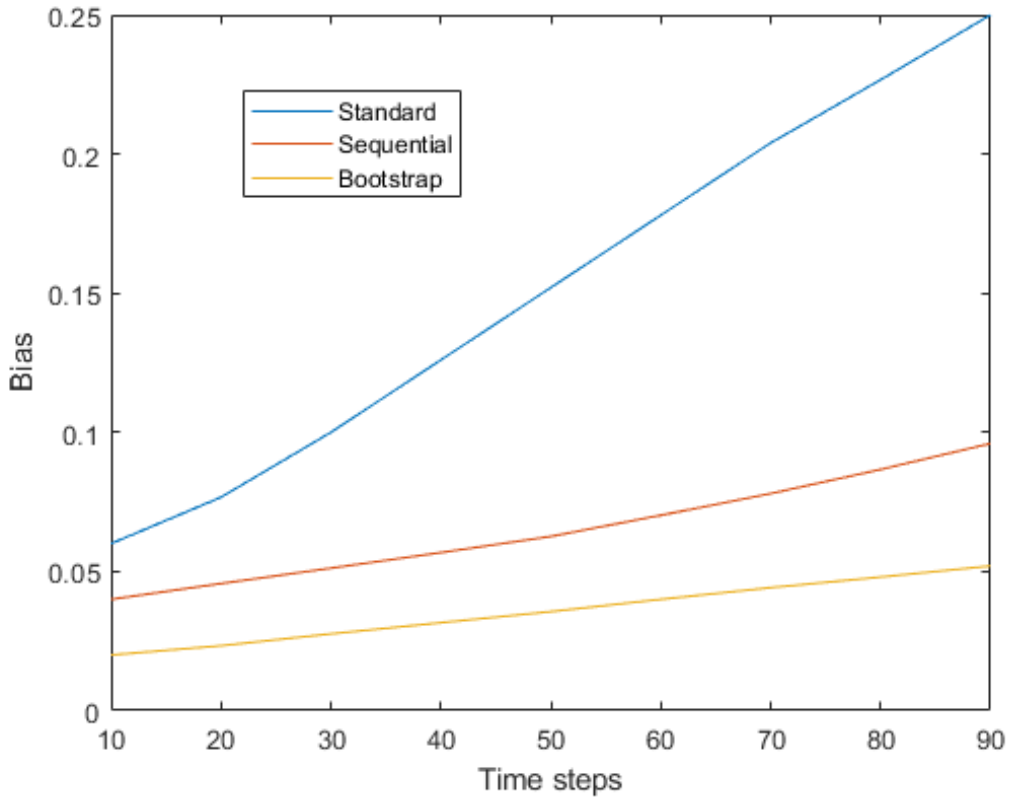

Figure 3.2: **Bias of MC estimators for $B = 10$.**

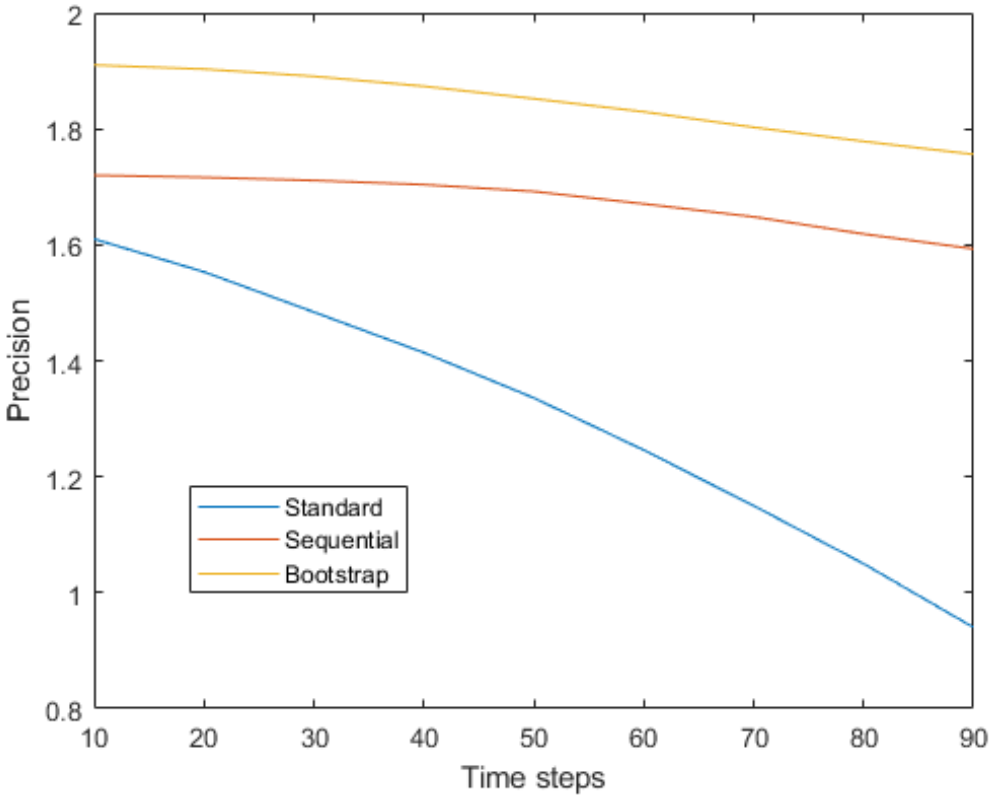

Figure 3.3: **Precisions of MC estimators for $B = 6$.**

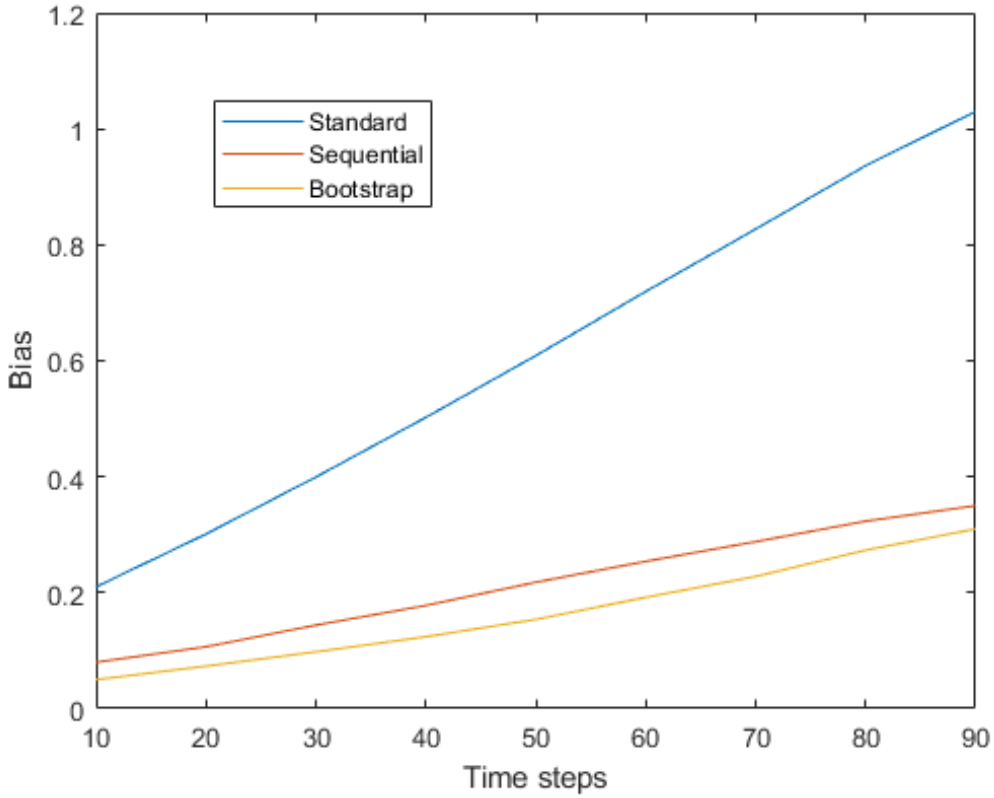

Figure 3.4: **Bias of MC estimators for $B = 6$.**

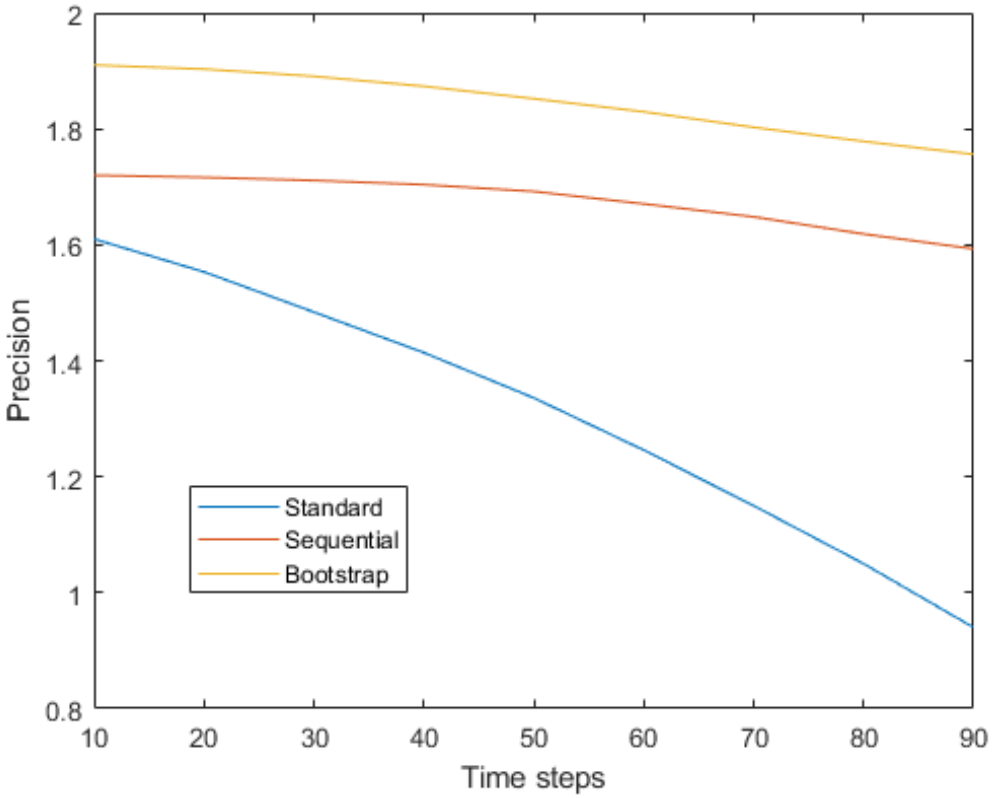

Figure 3.5: **Precision of MC estimators for $B = 3$.**

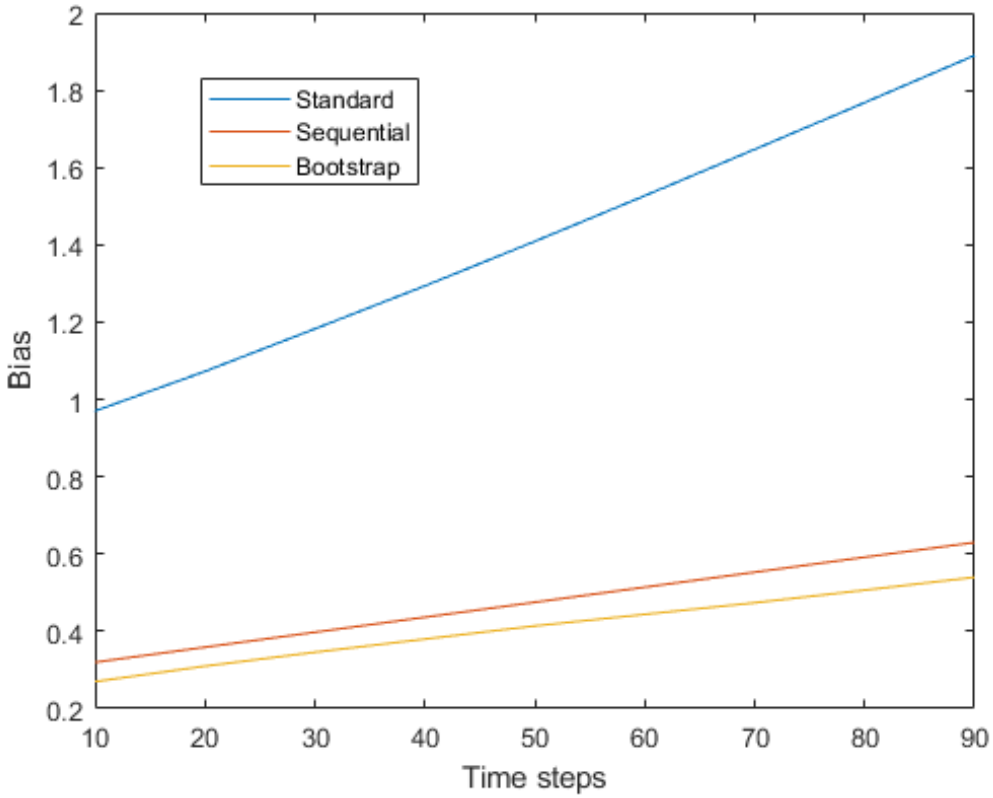

Figure 3.6: **Bias of MC estimators for $B = 3$.**

Conclusions

In this thesis we have examined MC methods to price a barrier option. Following the classical martingale approach, the barrier option price is written as the expectation of the actualized payoff under the RNM. Firstly we have considered the MCst methods and we have highlighted their main issues: the *bias* and the *low precision*. In order to overcome these ones, we have discussed a class of MC methods, named *BMC methods*. A BMC method produces in output the following option pricing estimator, named *BMC estimator*: the standard average of the actualized payoff realizations over a set of underlying paths, simulated using the *conditional density of the underlying given a set of observations*.

In the first part of the thesis we have described the following BMC techniques under the hypothesis of the constant volatility: the *CMC method*, the *ET method* and the *WF method*.

In the last part of the thesis we have discussed our contributes to the option pricing problem: *development of two BMC procedures for continuous barrier options, the EWF and the BBMC, to derive a MC estimator characterized by a lower bias and higher precision than a MCst estimator*. Both the schemes can be summarized as follows: firstly, the underlying paths are generated from a fixed family of density functions, named *importance densities*, depending on the observations, successively they are subject to a resampling procedure if the variance of the BMC estimator is too high, finally we evaluate the BMC estimator. In short, the two schemes differ for the *choice of the observations* and, consequently, the *form of the importance densities*: in the EWF, the observations are represented, at every time instant, by the past underlying and volatility values, in the BBMC, the observations are represented, at every time instant, by the past underlying values and by the past and current volatility values.

The validity of these two methods has been tested by some numerical experiments, where we have studied the *impact of the distance between the initial underlying value and the barriers on the bias and precision of a MC estimator*. More precisely, we have proceeded as follows: firstly, we have estimated the price of three continuous knock-out options under a SABR model using the MCst, the EWF and the BBMC estimators for different time steps, and, successively, we have compared the results in terms of bias and precision. The choice of the SABR model lets us determine an approximation of the price, which is necessary to evaluate the precision of MC estimators. The procedure can be summarized as follows: firstly, we have computed the implied volatility by applying the Hagan formula, then we have determined the price of the option, whose volatility is given by the implied volatility and all the other parameters are not changed, using the Kunimoto-Ikeda formula. The consequence of replacing the real value with this approximation is that *the bias of the EWF and BBMC estimators is*

not null.

The numerical results can be summarized as follows.

(R1): The three estimators achieve similar levels of bias and precision when the underlying initial value is far from the barriers.

(R2): The MCst estimator is biased and has a lower precision level than the EWF and the BBMC when the initial underlying value approaches the barriers.

(R3): The BBMC provides better results than the EWF.

The justification of (R1)−(R2) is that the probability of crossing the barrier is inversely proportional to the distance between the initial underlying value and the barriers. The justification of R3 is that in the BBMC uses a larger set of observations than the EWF.

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