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A topological analysis of competitive economies

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Abstract

This thesis deals with some central issues in the theory of competitive economies and with the intrinsic topological nature that they exhibit. The topics discussed focus on economic models with a multitude of agents and many commodities and are divided in an extended introduction and four chapters.

In Chapter 2 we prove a convexity result for the range of finitely-additive vector measures and correspondences with values in a locally convex space. These results are based on a new topological reformulation of the so-called saturation property which, in some form, generalizes the notion of non-atomicity of a measure space and plays an important role in many economic problems.

Chapter 3 investigates what topological assumptions on the commodity space are essential to formulate and study the problem of existence of competitive equilibrium in a Walrasian competitive economy with many commodities. It includes a general theorem on the existence of equilibrium prices in abstract markets with infinite dimensional commodity spaces that is inspired by Nikaidô (1959).

In Chapter 4 we study a coalitional model of an exchange economy in which coalitions are represented as the elements of a topological Boolean ring, allocations are finitely additive vector measures and the commodity space is an ordered locally convex space. In this general framework we will use the results of Chapter 2 to provide a topological condition for a perfectly competitive economy and use it to prove two extensions of classical theorems on the veto power of coalitions (Schmeidler (1972) and Vind (1972)). This will allow us to conclude that the economic power of coalitions is an essentially topological property.

Finally, Chapter 5 studies the competitive objection mechanism in a saturated economy with a separable Banach space of commodities whose positive cone has non-empty interior. By doing so we will provide a new characterization of Mas-Colell's bargaining set in the infinite dimensional setting. We will then introduce stronger notions of bargaining set that allow to extend our analysis to the case of markets with imperfections.

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Contents

Chapter 1

Introduction

A competitive market is one in which individual agents act as if their personal choices had no influence on the price formation and, more generally, on the overall outcome of the trades. For this to happen, every trader must conform passively to the given price system that, even if determined by the totality of individuals' choices, remains for him an uncontrollable data.

When agents are aware of how prices are generated, this idea of competition is closely related to that of economic negligibility of individuals and it is conceivable only if large multitudes of economic agents participate in the economy. In fact, in a market with only finitely many traders the determination of the prices can easily be manipulated by a single agent altering his own supplies and demands for goods. In other words, a competitive economy is to be thought as an idealized construct that can only be achieved in the presence of infinitely many, negligible economic agents.

To give a proper formalization of a competitive economy with a finite number of commodities, Aumann proposed in (1964) a model in which agents are represented as the points of a non-atomic measure space and the aggregation of individuals' choices is obtained through a suitable integration procedure. From a mathematical point of view, the effectiveness of this representation is twofold: while it provides a concrete, measure-theoretical expression of the informal notion of economic negligibility of individuals, it also shows how the aggregation over a large number of economic agents has a convexifying effect on preferences, demands and consumption sets. The first of these two points is clear: if every agent is identified with a point of a non-atomic measure space, and hence with a set of measure zero, any change in his demands and supplies will not affect the global outcome of the trades (that depends on the integral of all individual demands and supplies). Rational traders will then be expected to behave as price-takers even when they are perfectly informed of the market mechanism.

On the other hand, the mentioned convexifying effect is a somehow more subtle point that should be discussed in details. What Aumann observed is that if one assigns to each agent a set of commodity bundles and then averages these sets over a significant share of agents (i.e. a non-negligible coalition) then the resulting set must be convex. This allowed him to conclude that in a non-atomic economy with finitely many commodities the aggregate demands and supplies of any significant coalition can be modelled as convex-valued correspondences even when individual preferences are not convex. In the years, many key properties of non-atomic economies have been based on this convexifying effect and contributed to the popularity of Aumann's model. A non-exhaustive list of classical examples includes the Core-Walras equivalence, as proved in Aumann (1964), Vind (1964), Cornwall (1969), the existence of competitive equilibria, as in Aumann (1966), Hildenbrand (1970) and the characterizations of value allocations in Aumann (1975), of fair allocations in Varian (1976) and of competitive notions of bargaining set in Mas-Colell (1989). The impact of this effect has indeed transcended the study of competitive economies with many successful applications to problems dealing with cooperative and non-cooperative games as well as abstract economies. We mention, as examples, the classical contributions published in Aumann and Shapley (1974), Schmeidler (1973) and Khan and Vohra (1984).

Unfortunately, the non-atomic model becomes much less effective in many important situations that emerge in the economic analysis and that require the use of infinite dimensional commodity spaces. This convexifying effect on agents' preferences, in fact, relies on the finite number of commodities considered by Aumann and it cannot be achieved, in general, in a model presenting an infinite dimensional commodity space. This failure represents a major obstacle to the extension of many important results to non-atomic economies with many commodities.

It becomes therefore necessary to replace the assumption of non-atomicity of the measure space of agents with a stronger property that ensures a new convexifying effect even when an infinite number of commodities is taken into account.

1.1 General outline of the thesis

In this work we discuss different aspects of competitive economies that become significant when we want to study markets with many commodities and we cannot rely on Aumann's non-atomic model. Our primary interest will be to understand the fundamental requirements needed to replicate the convexyifing effect mentioned above and to give a proper formalization of the intuition that 'aggregation eliminates nonconvexity' even in more general settings. Clearly, this intention can be pursued only together with a focus on what features of competitive economies we want to capture in the new mathematical representation of the market.

To facilitate its exposition, the analysis we present is divided in four chapters devoted to four different issues: (i) the mathematical problem of generalizing the notion of non-atomic measure space, (ii) the need of infinite dimensional commodity spaces in Walrasian economies, (iii) the representation of the economic power of coalitions intended as their capacity to affect the trading activities and (iv) agents' reaction to non-competitive allocations in a fully decentralized economy. We briefly discuss these points in the following paragraphs.

The role of Lyapunov's Theorem and the saturation property

From a technical point of view, the convexifying effect in Aumann's model is a consequence of Lyapunov's Theorem (1940). The Theorem states that the range of any $\mathbb{R}^\ell$-valued measure defined on a non-atomic measure space is convex and compact[1]. As it is known, the validity of this statement depends directly on the finite dimension of $\mathbb{R}^\ell$ and, in general, it does not hold if we consider measures with values in infinite dimensional spaces (see (Diestel and Uhl, 1977, Chapter IX) for a classical exposition of this issue). The problem of generalizing Aumann's model is therefore strongly related to finding suitable extensions of Lyapunov's Theorem to the infinite dimensional settings.

The centrality of this problem in the theory of competitive economies is witnessed by the many articles on infinite dimensional versions of Lyapunov's Theorem that have been published in the economic literature over the last three decades. Two major contributions in this direction were given first by Gretsky and Ostroy (1985) and then by Rustichini and Yannelis (1991) with the introduction of new economic models that contemplate a Banach space of commodities and, in some form, generalize Aumann's idea of a non-atomic economy. Thanks to their new definitions, they were able to restore the convexifying effect on preferences with the aid of two generalizations of Lyapunov's Theorem proved respectively by Kingman and Robertson (1968) and by Knowles (1975) (see Tourky and Yannelis (2001) for a broad analysis of Rustichini and Yannelis's model).

Since the late 1990's, significant improvements were obtained by moving the attention from non-atomic measure spaces to Loeb probability spaces, introduced by Loeb in (1975), and then to saturated measure spaces, as defined in Hoover and Keisler (1984). By considering Loeb spaces, in fact, important techniques from the theory of non-standard analysis could be formulated in a measure-theoretical register and used to prove different convexity results for Banach space valued

[1] Aumann explicitly refers to this formulation of Lyapunov's Theorem in Aumann (1965). However it was Vind who first proved in (1964) the connections between the convexifying effect and Lyapunov's Theorem. See also (Hildenbrand, 1974, Theorem 3, page 62) for a short proof and further comments. Classical proofs of Lyapunov's Theorem are given in Halmos (1948) and Lindenstrauss (1966).

correspondences, as in Sun (1996, 1997). The latter were then successfully applied to a variety of economic problems dealing with multitudes of negligible economic agents. We mention, among the many, the results in Khan and Sun (1996, 1997); Sun and Zhang (2015) and the surveys Khan and Sun (2002), Anderson (2008).

It is, however, the property of saturation to be the most interesting in our perspective. Saturated measure spaces are in fact a special class of non-atomic spaces for which it is possible to reproduce many key properties of Loeb spaces including those that we mentioned above (see Sun and Yannelis (2008) and Loeb and Sun (2009)). Together with this, it is proved in Podczeck (2008) and Keisler and Sun (2009) that the property of saturation is not only sufficient to ensure many convexity results for Banach space valued correspondences, but it is also necessary. This makes saturated measure spaces the natural candidate for any generalization of the non-atomic spaces used by Aumann in his finite dimensional economic model.

In 2013, profound extensions of Lyapunov's Theorem were finally obtained in Khan and Sagara (2013) with a reformulation of the saturation property that is based on Maharam's Theorem on classification of measure algebras (1942). In their main theorem, which still relays on Knowles (1975), the authors considered a Banach space E and proved that any E-valued measure defined on a homogeneous measure algebra (Σ, λ) has a convex and weakly compact range provided that the Maharam-type of Σ is strictly greater than the dimension of E. These results were then sharpened in Greinecker and Podczeck (2013) and applied to prove the core-Walras equivalence in a special class of economies with a Banach space of commodities. Further extensions were then obtained in Khan and Sagara (2015, 2016) and Urbinati (2019).

The analysis in Chapter 2, which was partially presented in Urbinati (2019), takes off from Khan and Sagara's extension of Lyapunov's Theorem. Using the so-called Frechét-Nikodym approach, in which a measure is studied through the topology it induces on its domain, we will be able to give a topological reformulation of the saturation property using the notion of *degree of saturation* of a measure, introduced in Urbinati (2019). With this approach, we will extend the main results in Khan and Sagara (2013, 2016) and in Greinecker and Podczeck (2013) and prove a convexity theorem for the range of finitely additive vector measures with values in a locally convex space. Thanks to a decomposition theorem that was first presented in Urbinati and Weber (2017), it will also be possible to base the proof on a simpler version of Knowles' Theorem than the one needed in Khan and Sagara (2016).

All of these results will then be applied to the study of finitely additive correspondences with convex range and values that is presented here for the first time.

What space is the right commodity space for Walrasian competitive economies?

In the late 1950's, Hukakane Nikaidô extended some of the results obtained in (1956a) and gave what is one of the first studies of equilibrium existence for Walrasian economies with many commodities. His work, which remains surprisingly little known, precedes most of the literature on the topic by over a decade and anticipates some of its most characterizing concerns. Beside its historical relevance, the approach presented in Nikaidô (1957, 1959) stands out for the very elementary description of the commodity space in which commodity bundles are described as the formal object of agents' choices deprived from any physical attribute. In Chapter 3, that was developed with Professor M. Ali Khan, we follow Nikaidô's early contributions to reflect on the mathematical description of infinite dimensional commodity spaces needed in the Walrasian program.

As it is known, while there is essentially only one way to choose a topological and order structure on the Euclidean space $\mathbb{R}^\ell$, on a general linear space there are many different possibilities to do so. This means that when we consider economies with infinitely many commodities it is no longer possible to refer to a *natural* structure of the commodity space and the mathematical representation of commodities and prices has to depend on the nature of the economic phenomena one wants to study. Since the early 1970's, the literature on competitive economies with many commodities has extended in many directions and a variety of different topological, order and algebraic properties have been considered, mostly to allow precise applications of the theory. Being interested specifically on the working of a competitive economy, a natural question is whether or not any of these properties is essential in the Walrasian analysis. In other words, we ask what mathematical assumptions on the commodity space are necessary to study the existence of a Walrasian equilibrium in a competitive economy.

As a matter of fact, the choice of a specific infinite dimensional commodity space in modelling competitive economies is usually related to the particular kind of allocations one wants to consider. In a rough simplification, we could say that in the classical Malinvaud (1953), Debreu (1954), Peleg and Yaari (1970) and Bewley (1972) the authors were primarily concerned with studying allocation over time or different states of the nature, in Mas-Colell (1975) and Jones (1984, 1983) with the analysis of commodity differentiation and in Hart (1975) with particular allocations that arise in financial economics (see also Duffie (2010) for more references and examples). In each of these cases, one is first brought to ask "how do we represent a specific kind of allocation?" and only secondarily "how do we model a market that involves such allocations?".

In Chapter 3 we will try to move away from this approach and to shift the attention from the physical description of allocations to the decision problem faced by the agents, in line with Nikaidô's view. Formally, we will present an equilibrium existence theorem in which the commodity and price spaces are described in a purely algebraic form (i.e. they will not be endowed with any intrinsic topological or order structure) and the only topological and order considerations will follow directly from the linearity of prices and the characteristics of the agents. This will allow us to argue that the study of the equilibrium existence, which is essentially a fixed-point problem, requires the commodity space to be endowed with a structure that is much weaker than the one necessary for other solution concepts (see, for example, the study in Aliprantis and Burkinshaw (1991) of the necessary assumptions to prove the Core-Walras equivalence).

The economic weight of coalitions in competitive economies

A central notion in the study of large economies is represented by the so-called *economic weight* of coalitions, intended as the capacity of a group of agents to take part and influence the trades. As we approach competitive economies, where the idea of economic negligibility of individual traders plays a crucial role, the problem of understanding how the actors in the economy and their economic weight should be represented is an old and significant issue that we face in Chapter 4.

Following Aumann's non-atomic model, it is common to address this problem by representing the agents of the economy as the points of a measure space (T, Σ, λ) where the elements of the σ-algebra Σ represent all the possible coalitions that can be formed in the economy and λ can be thought as a measure of the economic weight of coalitions.

A different but effective way to study the trading process in large economies is to move the attention from the set of agents to the family of coalitions through the so called *coalitional representation*, introduced in Vind (1964) and then generalized in Cornwall (1969) and Armstrong and Richter (1984, 1986). The main idea of this approach is to ignore all individual agents without economic weight and take the coalitions in the economy as the primitive entities of the model. Coalitions will therefore be seen as elements of an abstract Boolean ring $\mathcal{R}$ and allocations as vector measures assigning to every coalition in $\mathcal{R}$ the correspondent (aggregated) commodity bundle. Since it is always possible to derive a coalitional representation of an individual model of the economy (via a suitable integration process) we can take the coalitional approach as a more general environment to study large economies (see the discussions in Debreu (1967) and in (Armstrong and Richter, 1984, pp. 117–118, 141)).

In Chapter 4 we consider a very general coalitional model of an exchange economy with infinitely many commodities and focus on the problem of representing mathematically the economic weight of coalitions. Specifically, we will show how in every exchange economy the set of all coalitions can be represented as a topological Boolean ring so that coalitions with "small" economic power correspond to "topologically small" elements of the ring, an idea presented for the first time in Urbinati (2019). What will emerge is that the topological approach we propose here is not only a natural consequence of the commodity-price duality, but also a necessary tool to explore the study of economies with a locally convex space of commodities without imposing significant (and apparently unjustified) restrictions on the model.

Concretely, the economic model considered in Chapter 4 will be a generalization of Armstrong and Richter (1984) and Cheng (1991) in which the commodity space is an ordered locally convex space and allocations may be only finitely-additive. In this framework, the topological extension of Lyapunov's Theorem proved in Chapter 2 will allow us to formulate a condition for competitive markets which, in some sense, replicates the 'many more agents than commodities' condition presented in Rustichini and Yannelis (1991). Finally, we will use these results to prove extensions of classical theorems on the veto power of coalitions originally proved in Schmeidler (1972) and Vind (1972) and more recently extended to the infinite dimensional settings in Hervés-Beloso et al. (2000), Evren and Hüsseinov (2008) and Bhowmik and Graziano (2015) among the many.

The competitive objections mechanism

In a way, our original interest for competitive economies was motivated by the possibility of studying optimal allocations of resources as the outcome of a fully decentralized interaction of individual agents. When this happens, an equilibrium state is defined as a balance-of-power situation where no group of agents is effective and so there is no incentive for individuals to cooperate. To put it in another way, in a properly formulated competitive economy it should always be possible that a significant share of agents can improve upon a non-equilibrium allocation without recurring to any centralized organization. Therefore, not only we expect to find a coalition able to object any non-competitive allocation but we also require that such an objection can be reached through a fully decentralized process.

This intuition can be formalized by means of the *competitive objection mechanism* that is described as follows. Imagine that the agents of an exchange economy are asked to choose between accepting the bundles assigned to them by a feasible allocation f or to trade their initial resources at a given price system p. If f is not competitive, some agents will certainly reject f in favor of other consumption plans in their budget set that they find more profitable. A competitive objection to f is reached if the transactions associated to these plans are performable, which is to say that the total demands of all the deviating agents do not exceed their total supplies.

The notion of competitive objection was introduced in Mas-Colell (1989) to study the bargaining set of an atomless economy with finitely many commodities. Under assumptions that are close to those needed in Aumann's existence and equivalence results (1964; 1966), he was able to show that every non-competitive allocation could be objected "competitively" and that every competitive objection is justified, in the sense that it cannot in turn be counter-objected. This allowed him to conclude that, in atomless economies with finitely many commodities, the core, the bargaining set and the competitive allocations are all equivalent solution concepts.

From a normative point of view, the interest in studying the competitive objection mechanism is therefore twofold: if on the one hand it provides a fully decentralized mechanism to find objections against non-competitive allocations, on the other it helps studying the bargaining set of large economies.

In Chapter 5 we present a first study of the competitive objection mechanism and Mas-Colell's bargaining set in the infinite dimensional settings. Formally, we will consider an economy with a saturated measure space of agents and a separable Banach space of commodities and prove that, in this framework, every feasible but non-competitive allocation can be objected competitively. This will prove that, in the economic model described, every allocation in the bargaining set is competitive.

We will then weaken the assumptions on the measure space of agents and study some alternative notions of bargaining set in the presence of oligopolies and market imperfections. This will also give us some infinite dimensional extensions of recent results obtained in Hervés-Beloso et al. (2018).

1.2 What do we mean by *topological* analysis?

We have decided to base our analysis of competitive economies on a topological approach. Accordingly, to facilitate the focus on purely topological considerations, many of the arguments are presented in highly abstract settings (e.g. topological Boolean rings, general topological linear spaces etc.) that may need a little elucidation. In this paragraph we briefly discuss this choice and the method used to construct our main research question.

The idea that many aspects of large economies should be studied with a topological perspective goes back to the dawn of economic theory. In a famous passage of his *Mathematical principle of the Theory of Wealth* (1838), for example, Cournot discusses the relation between the continuity of the market demand function and the necessity of having a large number of agents interacting in the economy, see (Cournot, 1938, pp. 49–50). With the development in both the fields, topology and economic theory, an increasing number of topological techniques have been successfully applied in several economic problems and have now become standard arguments.

In a purely instrumental view of mathematics, the role of topology in the economic analysis may be reduced to providing the technical tools that allow to bypass or simplify some aspects of a given economic model. Any topological consideration is therefore justified as a necessary, mechanical expedient that is imported from an extraneous discipline and applied to the specific economic problem.

The arguments in this thesis, however, are based on a completely different standpoint that is very well framed in Samuelson's famous opening statement of his *Foundation of Economic Analysis*: "Mathematics *is* language"[2]. In the approach we want to give here, in fact, the focus on a topological register is to be thought as a precise declaration of the language we want to use not only in the description, but also in the formulation and construction of the economic question. The adjective *topological* in the title is therefore used to indicate a precise way to *think* of the economic problem rather than a tool for solving it.

On a more concrete level, this thesis is an attempt to put the emphasis on the topological implications underlying many models of perfectly competitive economies. By doing so, it will be shown how these considerations allow to prove extensions of known results as well as to discuss the minimal requirements to formalize some central concepts in the equilibrium analysis. As a way of illustration, we could say that in Chapter 2 it is shown how the saturation property of measure spaces, on which the results in Chapter 4 and 5 are based, can be translated into topological property of Boolean rings, Chapter 3 focuses on the topological register needed for Walrasian models of competitive economies while Chapter 4 exposes how the problem of studying the economic power of coalitions is essentially topological.

[2]Samuelson quoted Gibbs' motto "Mathematics is a language" in the first edition of *Foundation of Economic Analysis* in (1947) only to shorten it to "Mathematics *is* language" in the second edition of 1952. This second formulation was reissued as Article 26 in Stiglitz' collection (1966). A broad exposition of Samuelsons' approach is given in Dixit (2005).

Chapter 2

The saturation property and the range of vector measures

The aim of this chapter is to provide conditions under which an additive correspondence with values in an infinite dimensional space has a convex and weakly compact range. More formally, we will be dealing with a Boolean ring $\mathcal{R}$, a locally convex space E and a correspondence $\Phi\colon \mathcal{R} \twoheadrightarrow E$ that is additive in the sense that $\Phi(0) = \{0\}$ and $\Phi(a \vee b) = \Phi(a) + \Phi(b)$ whenever $a, b \in \mathcal{R}$ are disjoint. In these settings, the range of Φ is defined as the union $R(\Phi) := \bigcup\{\Phi(a) : a \in \mathcal{R}\}$. With these definitions, our main concern can be seen as an attempt to find extensions to the classical Theorem of Lyapunov on the range of vector measures that allow to consider multiplicity of values and finite additivity in an infinite dimensional framework.

Among the many possible lines of investigations, we will be primarily interested in the case of an additive correspondence Φ that can be entirely studied through the family $\mathcal{S}_\Phi$ of its *selections*, which is the collection of every (finitely additive) measure $\mu\colon \mathcal{R} \to E$ with the property that $\mu(a) \in \Phi(a)$ when $a \in \mathcal{R}$. For correspondences of this type, that are called *rich in selections*, many properties of the range can be reduced to the study of *spliceable families of measures*[1], a notion that will be formalized and analyzed at page 16. The problem of characterizing additive correspondences with a convex range can then be reduced to the following question:

> Let $\mathcal{M}$ be a *spliceable* set of E-valued measures defined on $\mathcal{R}$. When is it true that the set $\mathcal{M}(a) := \{\mu(a) : \mu \in \mathcal{M}\}$ is convex for every $a \in \mathcal{R}$?

Our main research question is therefore in line with that explored in Artstein (1972); Costé and Pallu de la Barrière (1979) or Basile (1998) among the many.

[1] A set $\mathcal{M}$ of E-valued measures on $\mathcal{R}$ is said to be *spliceable* if for every $\alpha, \beta \in \mathcal{M}$ and $a \in \mathcal{R}$ the function $\eta\colon x \mapsto \alpha(x \wedge a) + \beta(x \setminus a)$, for $x \in \mathcal{R}$, is still a measure in $\mathcal{M}$. See Definition 2.1.10.

To allow the use of important tools of integration theory, in the literature it is common to address the question above under the additional assumption that all the measures in $\mathcal{M}$ are absolutely continuous with respect to some probability measures $\lambda:\mathcal{R} \to [0,1]$. By doing so, one can move the attention from the set $\mathcal{M}$ to the measure λ and find conditions on the latter that are sufficient to ensure the convexity of the range of each $\mu \in \mathcal{M}$.

To allow a more general view-point, however, we will answer the question above using the so-called Fréchet-Nikodym approach, in which a class of measures $\mathcal{M}$ is studied via the topological structure it induces on the Boolean ring where it is defined.

The mathematical setting

We start by introducing some of the notation that will be used throughout this chapter.

- Given a set X we will write $\mathcal{P}(X)$ for its power set and $\mathcal{P}_0(X)$ for the collection of all non-empty subsets of X. When $A, B \subset X$ are not empty $A+B$ will be the usual Minkowski sum $\{x + y : \ x \in A, \ y \in B\}$. A *correspondence* between two sets X and Y is a function $\Phi: X \to \mathcal{P}(Y)$. In this case we will write $\Phi: X \twoheadrightarrow Y$.

- E will be a complete, Hausdorff locally convex topological vector space with continuous dual E^*. For $x^* \in E^*$, $x \in E$ we will also write $\langle x^*, x\rangle$ instead of $x^*(x)$. A subset $\mathcal{F}$ of E^* is said to *separate the points* of a non-empty $Y \subset E$ if for every $x, y \in Y$ there is a $x^* \in \mathcal{F}$ such that $\langle x^*, x\rangle \neq \langle x^*, y\rangle$.

- We agree to denote by $\mathcal{R}$ a Boolean ring and to write $\triangle, \wedge, \vee, \setminus$ and $\leq$ respectively for the symmetric difference (sum), infimum (multiplication), supremum, difference and the natural ordering. When $\mathcal{R}$ is a Boolean algebra, i.e. a Boolean ring with unit, we shall write e for the unit of $\mathcal{R}$ and write a^c for the complement $e \setminus a$ of a whenever $a \in \mathcal{R}$. The principal ideal generated by a $a \in \mathcal{R}$ will be the set $\mathcal{R}_a := \mathcal{R} \wedge a := \{b \in \mathcal{R} : \ b \leq a\}$. For algebras of sets, i.e. sub-algebras of the power sets of a non-empty set, we will also use the standard set notation.

- By *measure* we will always mean a finitely additive function defined on a Boolean ring. When μ is a measure on $\mathcal{R}$ we will refer to the set $N(\mu) := \{a \in \mathcal{R} : \ \mu(b) = 0 \ \forall b \leq a\}$ as the ideal of *μ-null elements* and denote by $\tilde{\mathcal{R}}_\mu$ the quotient algebra $\mathcal{R}/N(\mu)$ so that the elements of $\tilde{\mathcal{R}}_\mu$ are the classes of equivalence determined by the relation $a \sim_\mu b \iff a \triangle b \in N(\mu)$ for $a, b \in \mathcal{R}$.

Other notation conventions will be introduced throughout the chapter. As our main references, we cite Bhaskara Rao and Bhaskara Rao (1983) and Weber (2002) for the theory of finitely additive measures (charges) and topological Boolean rings, Diestel and Uhl (1977), Aliprantis and Border (2006) and Fabian et al. (2011) for elements of vector measures, integration and functional analysis. For what concerns additive correspondences we will mainly refer to Drewnowski (1976a) and Basile (1994), other classical references are Schmeidler (1971), Artstein (1972) and Costé and Pallu de la Barrière (1979).

2.1 Definitions and preliminary results

We start this section by recalling some of the notions needed to formalize the idea of an "infinite sum" in a topological linear space. Most of these results are known in the literature but we recall them here as they are usually presented under different notation and perspectives. We will mainly follow the approach given in (Urbinati and Weber, 2017, Section 3) but still, we mention (Lindenstrauss and Tzafiri, 1977, Chapter 1) for an introduction on sums and series in Banach spaces, Bourbaki (2013) for a more general view in the context of topological groups and (Drewnowski, 1976a, Section 1) for the analysis of infinite sums of sets.

Given an index set $\mathcal{I}$ let $\mathcal{F}(\mathcal{I})$ denote the collection of all its finite subsets and endow $\mathcal{F}(\mathcal{I})$ with the inclusion order.

Definition 2.1.1. *A family $(x_i)_{i\in\mathcal{I}}$ of elements of E is said to be* summable *if the net of finite partial sums $(\sum_{i\in F} x_i)_{F\in\mathcal{F}(\mathcal{I})}$ converges to some $x \in E$. In this case we write $s((x_i)_{i\in\mathcal{I}}) \coloneqq \sum_{i\in\mathcal{I}} x_i \coloneqq x$.*

The set of all summable families in E with index set $\mathcal{I}$ is denoted by $\ell^1(\mathcal{I}, E)$.

With the definition given the set $\ell^1(\mathcal{I}, E)$ can be thought as a linear sub-space of the product space $E^{\mathcal{I}}$ so that the function $s\colon \ell^1(\mathcal{I}, E) \to E$ is a linear map.

Definition 2.1.2. *We say that a subset $A \subset E^{\mathcal{I}}$ is* uniformly summable *if $A \subseteq \ell^1(\mathcal{I}, E)$ and for every 0-neighborhood U in E there exists a $F_0 \in \mathcal{F}(\mathcal{I})$ such that whenever $x = (x_i)_{i\in\mathcal{I}} \in A$ and $F \in \mathcal{F}(\mathcal{I})$ is such that $F_0 \subseteq F$ one has:*

$$s(x) - \sum_{i\in\mathcal{I}} x_i \in U.$$

As a consequence of the fact that E has a 0-neighborhood basis consisting of closed sets, from the definition given we derive the following characterization of uniformly summable families that is sometimes called *Cauchy's criterion* (see (Bourbaki, 2013, Section 5.2) for a proof).

Proposition 2.1.3 (Cauchy's Criterion). *A family $A \subseteq E^{\mathcal{I}}$ is uniformly summable if and only if for any 0-neighborhood U in E there exists $F_U \in \mathcal{F}(\mathcal{I})$ such that $\sum_{i\in G} x_i \in U$ whenever $G \in \mathcal{F}(\mathcal{I})$ is disjoint from F_U and $(x_i)_{i\in\mathcal{I}} \in A$.*

We stress that in Proposition 2.1.3 the completeness of the space E plays a crucial role and cannot be dropped. The following theorem shows how the continuity of the sum operations can be partially extended to infinite sums.

Theorem 2.1.4. *Consider the topology τ_p induced on $\ell^1(\mathcal{I}, E)$ by the product topology on $E^{\mathcal{I}}$ and let $A \subseteq \ell^1(\mathcal{I}, E)$ be a uniformly summable set. Then the restriction of s to A is a uniformly continuous function with respect to τ_p.*

Proof. Let $U \subseteq E$ be a 0-neighborhood, then take a closed, symmetric 0-neighborhood $U_0 \subset E$ such that $U_0 + U_0 + U_0 \subset U$. By the Cauchy Criterion there must be a $F_0 \in \mathcal{F}(\mathcal{I})$ such that $\sum_{i\in G} x_i \in U_0$ whenever $G \in \mathcal{F}(\mathcal{I})$ is disjoint from F_0 and $(x_i)_{i\in\mathcal{I}} \in A$. Being F_0 finite, there must be a 0-neighborhood $V \subset E$ such that $\sum_{i\in F_0} V \subset U_0$.

Consider now the 0-neighborhood in $(\ell^1(\mathcal{I}, E), \tau_p)$ defined as the set $W := \{(x_i)_{i\in\mathcal{I}} \in \ell^1(\mathcal{I}, E) : \ x_i \in V \ \ \forall i \in F_0\}$. If we choose any two $x = (x_i)_{i\in\mathcal{I}}$ and $y = (y_i)_{i\in\mathcal{I}}$ in A such that $x - y \in W$ we will have that $\sum_{i\in F_0}(x_i - y_i) \in \sum_{i\in F_0} V \subset U_0$ while $\sum_{i\in G} x_i \in U_0$, $\sum_{i\in G} y_i \in U_0$ for every $G \in \mathcal{F}(\mathcal{I})$ disjoint from F_0. But then, by the closedness of the set U_0, we will have that:

$$s(x) - s(y) = \sum_{i\in F_0}(x_i - y_i) + \sum_{i\notin F_0} x_i - \sum_{i\notin F_0} y_i \in U_0 + U_0 + U_0 \subset U.$$

which proves that the restriction of s to A is a uniformly continuous function. □

Corollary 2.1.5. *Let $(A_i)_{i\in\mathcal{I}}$ be a family of non-empty, convex and weakly compact subsets of E such that $A := \prod_{i\in\mathcal{I}} A_i$ is uniformly summable. Then $\sum_{i\in\mathcal{I}} A_i$ is convex and weakly compact too.*

Proof. Let us denote by w the sub-space topology induced on $\ell^1(\mathcal{I}, E)$ by the product topology on $(E, \sigma(E, E^*))$. Being each of the A_i's convex and weakly compact, by Tychonov's Theorem the product $A = \prod_{i\in\mathcal{I}} A_i$ must be convex and compact with respect to the topology w.

Observe now that A is uniformly summable even in the weak topology and so, by Theorem 2.1.4, the restriction of s to A, beside being linear, is uniformly continuous with respect to w and the weak topology on E. It follows that $\sum_{i\in\mathcal{I}} A_i$, which is the image of A under the function s, is convex and weakly compact too. □

2.1.A Measures and additive correspondences

In what follows we talk about an *additive correspondence* $\Phi:\mathcal{R} \twoheadrightarrow E$ meaning a map that assigns to every $a \in \mathcal{R}$ a non-empty subset of $\Phi(a) \subseteq E$ and that is such that $\Phi(0) = \{0\}$ and $\Phi(a \vee b) = \Phi(a) + \Phi(b)$ whenever $a, b \in \mathcal{R}$ are disjoint. Clearly, a measure can be seen as a special case of an additive correspondence whose values are singleton.

Together with the definition given, that is purely algebraic, it will be often needed to make the following topological considerations on additive correspondences.

Definition 2.1.6. *An additive correspondence $\Phi:\mathcal{R} \twoheadrightarrow E$ is said to be:*

- Exhaustive *if for every every sequence $(a_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{R}$ and 0-neighborhood U in E there is a $m_U \in \mathbb{N}$ such that $\Phi(a_n) \subseteq U$ for every $n \geq m_U$.*
- Countably additive *if for every sequence $(a_n)_{n\in\mathbb{N}}$ of pairwise disjoint elements of $\mathcal{R}$ the system $(\Phi(a_n))_{n\in\mathbb{N}}$ is uniformly summable and $\sum_{n\in\mathbb{N}} \Phi(a_n) = \Phi(\sup_n a_n)$ whenever $\sup_n a_n \in \mathcal{R}$.*
- Completely additive *if for every net $(a_i)_{i\in\mathcal{I}}$ of pairwise disjoint elements of $\mathcal{R}$ the system $(\Phi(a_i))_{i\in\mathcal{I}}$ is uniformly summable and $\sum_{i\in\mathcal{I}} \Phi(a_i) = \Phi(\sup_i a_i)$ whenever $\sup_i a_i \in \mathcal{R}$.*

Clearly, being every measure an additive correspondence, we can say that a measure is *exhaustive*, *σ-additive* or *completely additive* measure whenever it is exhaustive, countably or completely additive as a correspondence. In particular, a measure μ will be *exhaustive* if $\mu(a_n) \to 0$ whenever $(a_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint elements of $\mathcal{R}$.

It follows from the definitions that every completely additive correspondence is countably additive and every countably additive correspondence is exhaustive (see Drewnowski (1976a)). The converse is, in general, not true but we still have the following characterization of exhaustive correspondences.

Theorem 2.1.7. *The following conditions are equivalent:*

1. *Φ is an exhaustive correspondence.*
2. *For every net $(a_i)_{i\in\mathcal{I}}$ of pairwise disjoint elements of $\mathcal{R}$ the system $(\Phi(a_i))_{i\in\mathcal{I}}$ is uniformly summable.*

Proof. To prove that $(1 \Rightarrow 2)$ we assume by contradiction that there is a net $(a_i)_{i\in\mathcal{I}}$ of pairwise disjoint elements of $\mathcal{R}$ such that the family $(\Phi(a_i))_{i\in\mathcal{I}} \subset E^{\mathcal{I}}$

violates the Cauchy Criterion for summability (see 2.1.3). This is to say that for some 0-neighborhood U in E it is possible to associate to every $F \in \mathcal{F}(\mathcal{I})$ a finite $G_F \subset \mathcal{I} \setminus F$ such that $\sum_{i \in G_F} \Phi(a_i) \notin U$. With an inductive procedure we can therefore find a sequence $(F_n)_{n \in \mathbb{N}}$ of pairwise disjoint subsets of $\mathcal{I}$ such that $\sum_{i \in F_n} \Phi(a_i) \notin U$ for every $n \in \mathbb{N}$. But then, if we call $b_n := \sup_{i \in F_n} a_i$ for every $n \in \mathbb{N}$, we have that $(b_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint elements of $\mathcal{R}$ such that $\Phi(b_n) = \sum_{i \in F_n} \Phi(a_i) \notin U$ for every $n \in \mathbb{N}$, in contradiction with the assumption that Φ is exhaustive.

Suppose now that (2) holds and let $(a_n)_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint elements of $\mathcal{R}$. By the Cauchy Criterion on summability, for every 0-neighborhood U in E there is a finite set $F \subset \mathbb{N}$ such that $\sum_{i \in G} \Phi(a_i) \subset U$ whenever $G \subset \mathcal{I} \setminus F$ is finite. But then it must be the case that $\Phi(a_i) \subseteq U$ for every $i \notin F$ and Φ is therefore exhaustive. □

We shall now talk about additive selections of correspondences. Formally, if R is a rule between two sets X and Y, i.e. a subset of the Cartesian product $X \times Y$, we usually call a *selection of R* any function $f\colon X \to Y$ such that $(x, f(x)) \in R$ for every $x \in X$. In the framework of additive correspondences a special importance will be given to selections that are measures.

Definition 2.1.8. *Let $\Phi\colon \mathcal{R} \twoheadrightarrow E$ be a correspondence. An* additive selection *of Φ is a measure $\mu\colon \mathcal{R} \to E$ such that $\mu(a) \in \Phi(a)$ for every $a \in \mathcal{R}$. The set of additive selections of Φ is denoted by $\mathcal{S}_\Phi$.*

The correspondence Φ is rich in additive selections *if $\Phi(a) = \{\mu(a) : \mu \in \mathcal{S}_\Phi\}$ for every $a \in \mathcal{R}$.*

In the following we may say that an additive correspondence Φ is *rich in (exhaustive, completely additive) selections* if for every $a \in \Phi$ and $x \in \Phi(a)$ there is an (exhaustive, completely additive) $\mu \in \mathcal{S}_\Phi$ such that $\mu(a) = x$.

It is clear that the study of an additive correspondence Φ that is rich in selections can be reduced to that of the set $\mathcal{S}_\Phi$. On the other hand, we might ask when for a given set $\mathcal{M}$ of E-valued measures on $\mathcal{R}$ the correspondence $\Phi_\mathcal{M}$ that maps each $x \in \mathcal{R}$ in the set $\{\mu(x) : \mu \in \mathcal{M}\}$ is additive.

Proposition 2.1.9. *Let $\mathcal{M}$ be a set of measures defined on $\mathcal{R}$ with values in E. Then the following are equivalent:*

(i) The correspondence $\Phi_\mathcal{M}\colon x \mapsto \{\mu(x) : \mu \in \mathcal{M}\}$, for $x \in \mathcal{R}$, is additive.

(ii) For every $a \in \mathcal{R}$, $\mu_1, \mu_2 \in \mathcal{M}$ the function $\eta\colon x \mapsto \mu_1(x \wedge a) + \mu_2(x \setminus a)$, $x \in \mathcal{R}$, is an additive selection of $\Phi_\mathcal{M}$.

Proof. Suppose that (i) holds and select $\mu_1, \mu_2 \in \mathcal{M}$, $a \in \mathcal{R}$. Define the measure $\eta: \mathcal{R} \to E$ as in point (ii) and observe that, by construction, $\eta(x) \in \Phi(x \wedge a) + \Phi(x \setminus a)$ for every $x \in \mathcal{R}$. But then, by the additivity of $\Phi_{\mathcal{S}}$, $\eta(x) \in \Phi(x)$ for every $x \in \mathcal{R}$ and hence η is a selection of $\Phi_{\mathcal{M}}$.

Assume now that $\mathcal{M}$ satisfies condition (ii). To show (i) let $a_1, a_2 \in \mathcal{R}$ be disjoint and take $x_1, x_2 \in E$ such that $x_i \in \Phi_{\mathcal{M}}(a_i)$ for $i = 1, 2$. Take $\mu_1, \mu_2 \in \mathcal{M}$ such that $\mu_i(a_i) = x_i$ for $i = 1, 2$ (such μ_i exists by the definition of $\Phi_{\mathcal{M}}$) and define the measure η by setting $\eta(x) = \mu_1(x \wedge a_1) + \mu_2(x \setminus a_1)$ for every $x \in \mathcal{R}$. But then η is a selection of $\Phi_{\mathcal{M}}$ (by point (ii)) and by construction it is such that $\eta(a_1 \vee a_2) = \mu_1(a_1) + \mu_2(a_2) = x_1 + x_2$. It follows that $x_1 + x_2 \in \Phi_{\mathcal{M}}(a_1 \vee a_2)$ and thus the additivity $\Phi_{\mathcal{M}}$.

□

We stress that point (ii) does not imply that the measure η is itself in $\mathcal{M}$, it is however clear that, if interested in the study of the correspondence $\Phi_{\mathcal{M}}$, one gives a special importance to all the measures that can be built from $\mathcal{M}$ through the *splicing* operation described in point (ii) of Proposition 2.1.9.

Definition 2.1.10. *A set $\mathcal{M}$ of E-valued measures on $\mathcal{R}$ is* spliceable*, or* closed under splicing*, if for every $\mu_1, \mu_2 \in \mathcal{M}$ and $a \in \mathcal{R}$ the function $\eta: x \mapsto \mu_1(x \wedge a) + \mu_2(x \setminus a)$, for $x \in \mathcal{R}$, is a measure in $\mathcal{M}$.*

A direct consequence of Proposition 2.1.9 shows that a correspondence Φ rich in additive selections is additive if and only if $\mathcal{S}_\Phi$ is a spliceable set. In the spirit of this observation one may also think that a similar result holds for exhaustive correspondences, i.e. that a correspondence rich in exhaustive selections is exhaustive if and only if the set of its exhaustive selections is spliceable. The following example shows that this is not the case and that given a spliceable set $\mathcal{M}$ of exhaustive measures it is possible that the correspondence $\Phi_{\mathcal{M}}$ is not exhaustive.

Example 2.1.11. Consider the set $\mathcal{M}$ consisting of all E-valued measures defined on the algebra $\mathcal{P}(\mathbb{N})$ that have a finite support, i.e. all the measures $\mu: \mathcal{P}(\mathbb{N}) \to E$ such that $\{n : \mu(\{n\}) > 0\}$ is finite. It is clear that $\mathcal{M}$ is spliceable and that each $\mu \in \mathcal{M}$ is exhaustive. At the same time, since $\Phi_{\mathcal{M}}(F) = E$ for every non-empty $F \subseteq \mathbb{N}$, the correspondence $\Phi_{\mathcal{M}}$ cannot be exhaustive.

Last we introduce one of the central notions of this chapter: the *range* of an additive correspondence. This will simply be defined as the union of its values.

Definition 2.1.12. *The* range *of the additive correspondence $\Phi: \mathcal{R} \twoheadrightarrow E$ is the set: $R(\Phi) := \bigcup_{a \in \mathcal{R}} \Phi(a)$. In the case of a measure μ, we also write $\mu(\mathcal{R})$ to denote its range $R(\mu)$.*

Example 2.1.13. A relevant class of additive correspondences is formed by those that are naturally obtained from a single measure. Given a measure $\mu : \mathcal{R} \to E$ we can define Φ_μ as the correspondence that assigns to each $a \in \mathcal{R}$ the set $\mu(\mathcal{R}_a)$ or, equivalently, the correspondence generated by the spliceable set of measures $\{\mu_b : b \in \mathcal{R}_a\}$ (where μ_b maps $x \in \mathcal{R}$ into $\mu(b \wedge x)$). In this case, one sees that Φ_μ is exhaustive if and only if μ is so.

A large share of literature on convex-ranged measures is devoted to study the cases in which, given a measure μ on $\mathcal{R}$, each of the sets $\mu(\mathcal{R}_a)$ are convex and compact (see (Diestel and Uhl, 1977, Chapter IX) and its references for a survey of classical examples). In our framework, this is equivalent with studying conditions under which Φ_μ has convex and compact values.

2.1.B Topological Boolean rings

When λ is a positive, bounded, scalar measure on $\mathcal{R}$ the function $d_\lambda : (x, y) \mapsto \lambda(x \vartriangle y)$, for $x, y \in \mathcal{R}$, defines an invariant pseudo-metric[2] on $(\mathcal{R}, \vartriangle)$ and hence a group-topology $\tau(\lambda)$. Such a topology, whose 0-neighborhood system is generated by the sets $(\{x : \lambda(x) \leq 2^{-n}\})_{n \in \mathbb{N}}$, is called the λ-*topology* and it is the coarsest one making λ a (uniformly) continuous function. By moving the attention from the measure λ to the uniform structure it induces on $\mathcal{R}$, i.e. the topology $\tau(\lambda)$, one can easily translate in a topological register many important features of the measure space $(\mathcal{R}, \lambda)$ and facilitate many considerations that would be hard to formulate otherwise. As a way of illustration, let us consider any measure $\mu : \mathcal{R} \to \mathbb{R}^N$ and observe that μ is absolutely continuous with respect to λ [3] if and only if it is continuous with respect to the topology $\tau(\lambda)$. Furthermore, the topology $\tau(\lambda)$ benefits from the following interesting properties:

- Its 0-neighborhood system is generated by a family of solid sets, where a subset X of $\mathcal{R}$ is *solid* if $y \in X$ whenever $y \leq x$ for some $x \in X$.

- The operations $\vartriangle$ and $\wedge$ are both uniformly continuous.

On a broader level, a group-topology on $(\mathcal{R}, \vartriangle)$ that satisfies any of the two properties listed above (which are in fact equivalent by (Weber, 2002, Proposition 1.6)) belongs to the family of the so-called *Fréchet-Nikodym topologies* (or simply FN-topologies). We will give a special importance to these topologies and, even though FN-topologies do not represent all the possible topologies that can be defined on $\mathcal{R}$, we will agree with the following definition.

[2] We say that a pseudo-metric d on a group $(G, +)$ is *invariant* if $d(x + z, y + z) = d(x, y)$ for every $x, y, z \in G$.

[3] We refer here to the $\epsilon - \delta$ notion of absolute continuity as defined in (Bhaskara Rao and Bhaskara Rao, 1983, Definition 6.1.1): i.e. μ is absolutely continuous with respect to λ if and only if for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|\mu(y)| \leq \epsilon$ whenever $\lambda(x) \leq \delta$ for all $y \leq x \in \mathcal{R}$.

Definition 2.1.14. *A* topological Boolean ring *is a pair* $(\mathcal{R}, u)$ *such that* $\mathcal{R}$ *is a Boolean ring and* u *is a group-topology on* $\mathcal{R}$ *whose* 0*-neighborhood system is generated by a family of solid sets.*

We will say that a topological Boolean ring $(\mathcal{R}, u)$ is *exhaustive* if every sequence of pairwise disjoint elements of $\mathcal{R}$ converges to 0 or, equivalently, if every monotone net in $\mathcal{R}$ is Cauchy (Weber, 2002, Proposition 3.4). On the other hand, if every monotone net in $\mathcal{R}$ order converging to some $x \in \mathcal{R}$ is also topologically convergent to x we will call u an *order continuous* topology. These two classes of measures, which link the algebraic and the uniform nature of topological Boolean rings, are related by the following property.

Proposition 2.1.15 (Proposition 4.2 in Weber (2002))**.** *Let* $(\mathcal{R}, u)$ *be an exhaustive, Hausdorff topological Boolean ring that is complete (as a uniform space). Then* $\mathcal{R}$ *is a complete Boolean algebra and* u *is order continuous.*

Following the same approach used for a scalar measure λ we may want to study any given measure $\mu: \mathcal{R} \to E$ through the uniform structure it induces on $\mathcal{R}$. This will be given by the group-topology $\tau(\mu)$ on $\mathcal{R}$ whose 0-neighborhood system is generated by the sets $\{x: \ y \in U \text{ for all } y \leq x\}$ with U ranging over the 0-neighborhoods in E. Just like in the case of scalar measures, $\tau(\mu)$ will result to be the coarsest FN-topology on $\mathcal{R}$ making μ a (uniformly) continuous function, see (Weber, 2002, Proposition 1.10).

Definition 2.1.16. *Let* $\mu: \mathcal{R} \to E$ *be a measure. We call* μ-topology*, and denote by* $\tau(\mu)$*, the coarsest* FN*-topology on* $\mathcal{R}$ *making* μ *a uniformly continuous function.*

It is clear that definition we just gave generalizes that of λ-topology we considered for positive, scalar measures. Furthermore, one sees that a measure μ on $\mathcal{R}$ is exhaustive if and only if $(\mathcal{R}, \tau(\mu))$ is an exhaustive Boolean ring. Even the notion of *absolute continuity* of measures, whose most general formalization is quite involved, can be now expressed by topological means.

Definition 2.1.17. *Given two measures* μ *and* ν *over* $\mathcal{R}$*, we say that* ν *is* absolutely continuous with respect to μ*, and write* $\nu \ll \mu$*, if* $\tau(\nu)$ *is coarser than* $\tau(\mu)$*.*

In view of Proposition 2.1.15, we give a special importance to those measures inducing a complete FN-topology on $\mathcal{R}$, namely the *closed measures*.

Definition 2.1.18. *A measure* μ *on* $\mathcal{R}$ *is* closed *if the topological Boolean ring* $(\mathcal{R}, \tau(\mu))$ *is complete as a uniform space.*

Let us call $N(u)$ the closure of $\{0\}$ in $(\mathcal{R}, u)$. Since $N(u)$ is a closed ideal in $(\mathcal{R}, u)$ (that coincides with $N(\mu)$ if u is the FN-topology induced by a measure μ) the quotient $(\hat{\mathcal{R}}, \hat{u}) \coloneqq (\mathcal{R}, u)/N(u)$ will be a Hausdorff topological Boolean ring which is exhaustive or (uniformly) complete whenever u is so. Furthermore, if μ is a u-continuous measure on $\mathcal{R}$ then $\hat{\mu}: \hat{x} \mapsto \mu(x)$, for $x \in \hat{x} \in \hat{\mathcal{R}}$, defines on $\hat{\mathcal{R}}$ a $\hat{u}$-continuous measure that has the same range as μ. This, together with Proposition 2.1.15, gives us the following key result.

Proposition 2.1.19. *Let $(\mathcal{R}, u)$ be a complete and exhaustive topological Boolean ring and let $(\hat{\mathcal{R}}, \hat{u})$ be the quotient $(\mathcal{R}, u)/N(u)$. Then $\hat{\mathcal{R}}$ is a complete Boolean algebra and $\hat{u}$ is order continuous.*

In addition, if $\mu: \mathcal{R} \to E$ is a u-continuous measure and $\hat{\mu}: \hat{\mathcal{R}} \to E$ is the function defined by $\hat{\mu}(\hat{x}) = \mu(x)$ for $x \in \hat{x} \in \hat{\mathcal{R}}$ then $\hat{\mu}$ is a completely additive measure and $\mu(\mathcal{R}) = \hat{\mu}(\hat{\mathcal{R}})$.

It follows from the Proposition above that for closed and exhaustive measures μ and ν over a complete ring, $\nu \ll \mu$ if and only if $N(\mu) \subseteq N(\nu)$. This has to do with the fact that for any two order continuous topologies u and v over a complete Boolean ring $\mathcal{R}$, $u \subseteq v$ if and only if $N(v) \subseteq N(u)$ (Weber, 2002, Theorem 4.8).

Remark 2.1.20. In the study of scalar valued measure (or even Banach space valued measures) it is often much easier to work with σ-additive measures defined on σ-algebras rather than finitely-additive ones. To make an example, if $\lambda, \eta: \mathcal{R} \to \mathbb{R}$ are σ-additive measures on a σ-algebra then it is well known that $\lambda \ll \eta$ if and only if $N(\eta) \subset N(\lambda)$ and that there is a partition of $\mathcal{R}$ based on the Lebesgue-decomposition of λ with respect to η. None of these properties hold, in general, if λ, η are only finitely additive.

Everything changes when we consider measures with values in a space that is non-metrizable. In this case, most of the nice properties that made σ-additive measures more appealing than finitely-additive ones (including those mentioned above) fail for reasons that are similar to those that emerge in the scalar-valued, finitely-additive settings. To restore the same properties in this more general framework one needs to impose that the topologies induced by the measures are complete (see for example the results in (Kluvánek and Knowles, 1976, Chapter V) where an equivalent definition of closed measure is considered).

The topological approach we exposed here gives a nice and elegant explanation for this phenomena: σ-additive measures defined on σ-algebras are in fact automatically closed when they take values in a metrizable space but they may not be so in the general case. In other words, it is the combination of *exhaustivity* and *completeness* of the topology induced by a measure, rather than its σ-additivity and the σ-completeness of its domain, the fundamental condition needed in many situations. All this shows us the importance of Proposition 2.1.19.

2.1.C The degree of saturation

Given a measure μ on $\mathcal{R}$, by *μ-atom* is meant any element $a \in \mathcal{R} \setminus N(\mu)$ such that, for every $x \leq a$, either $x \in N(\mu)$ or $a \setminus x \in N(\mu)$. It is therefore clear that if we denote by $\pi: \mathcal{R} \to \mathcal{R}/N(\mu)$ the quotient map, $a \in \mathcal{R} \setminus N(\mu)$ is a μ-atom if and only if $\pi(\mathcal{R} \wedge a) = \{0, \pi(a)\}$. With an inductive argument we can conclude that the measure μ is *non-atomic*, i.e. that there are no μ-atoms in $\mathcal{R}$, if and only if:

$$\inf\{|\pi(\mathcal{R} \wedge x)| : x \in \mathcal{R} \setminus N(\mu)\} = \infty. \tag{2.1}$$

In this section we want to extend the notion of non-atomicity of a measure μ by introducing a cardinal function on topological Boolean rings that gives us an estimation of the minumum 'size' of its non-trivial principal ideals. We will call this function *degree of saturation*.

Recall that the *density* of a topological group G, denoted by $dens(G)$, is the least among the cardinalities of all dense subsets of G. It is straightforward to see that, if H is the closure of the identity in G, then the Hausdorff quotient G/H has the same density as G. In general, when H is a subset of G, it is not necessarily true that $dens(H) = dens(G)$. Consequently, for a given topological Boolean ring $(\mathcal{R}, u)$ and $x \in \mathcal{R}$ we could have that the sub-space $\mathcal{R} \wedge x$, considered with the topology induced by u, has density strictly smaller than $dens(\mathcal{R})$. This observation brings us to the following definition.

Definition 2.1.21. *The* degree of saturation *of a topological Boolean ring* $(\mathcal{R}, u)$*, denoted by* $sat(u)$*, is the least among the densities of all* $\mathcal{R} \wedge x$*, with* $x \in \mathcal{R} \setminus N(u)$*, each one considered as a topological sub-space of* $(\mathcal{R}, u)$*.*

If μ *is a measure over* $\mathcal{R}$*, we also write* $sat(\mu)$ *to denote* $sat(\tau(\mu))$ *and call it* degree of saturation *of the measure* μ*.*

Just like the density character, we note that the degree of saturation of a topological Boolean ring $(\mathcal{R}, u)$ is the same as the one of the correspondent Hausdorff quotient. To see this is enough to observe that if $\pi: (\mathcal{R}, u) \to (\mathcal{R}, u)/N(u)$ denotes the quotient map then $dens(\mathcal{R} \wedge x) = dens(\pi(\mathcal{R} \wedge x))$ for every $x \in \mathcal{R}$. In other words we can alternatively define the degree of saturation of $(\mathcal{R}, u)$ as the cardinal:

$$sat(u) = \inf\{dens(\pi(\mathcal{R} \wedge x)) : x \in \mathcal{R} \setminus N(u)\}$$

where, as usual, each principal ideal $\mathcal{R} \wedge x$ is considered endowed with the sub-space topology induced by $(\mathcal{R}, u)$. The connections between the definition above and equation 2.1 are then clear and we can see that a measure μ is non-atomic if and only if $sat(\mu)$ is infinite. This explains why we can take the notion of degree of saturation as a generalization of that of non-atomicity.

Proposition 2.1.22. *Let $\mathcal{R}$ be a complete Boolean algebra and u, v two order continuous FN-topologies on $\mathcal{R}$ such that $v \subseteq u$. Then $sat(u) \leq sat(v)$.*

Proof. We prove that for every $x \in \mathcal{R} \setminus N(v)$ there is a continuous function $f: (\mathcal{R}, v) \to (\mathcal{R}, u)$ such that $f(\mathcal{R} \wedge x)$ is of the form $\mathcal{R} \wedge y$ for some $y \in \mathcal{R} \setminus N(u)$. This way, for every v-dense subset D of $\mathcal{R} \wedge x$, $f(D)$ is a u-dense subset of $\mathcal{R} \wedge y$ with cardinality smaller or equal than $|D|$. We do it only for $x := e$, as the proof strategy remains the same for a generic $x \in \mathcal{R} \setminus N(v)$.

Since the ideal $N(v)$ can be seen as a monotone net in $\mathcal{R}$, it must converge to $a := \sup N(v) \in \mathcal{R}$ (which exists by the completeness of $\mathcal{R}$) by the order continuity assumption. This, being $N(v)$ closed, implies that $N(v)$ can be written as the principal ideal $\mathcal{R} \wedge a$.

Let $b := a^c$ and call u_b and v_b the sub-space topologies induced on $\mathcal{R} \wedge b$ by u and v respectively. v_b and u_b are order-continuous topologies defined on a complete Boolean algebra and, moreover, by the choice of b, $N(v_b) = N(v) \wedge b = \{0\} = N(u) \wedge b = N(u_b)$. But then, a glance at (Weber, 2002, Theorem 4.8) gives us $v_b = u_b$. Let $f: \mathcal{R} \to \mathcal{R} \wedge b$ be the function that assigns $x \wedge b$ to each $x \in \mathcal{R}$. Of course, f is surjective and continuous with respect to v and v_b. Since $v_b = u_b$, f is the desired function. $\square$

Corollary 2.1.23. *Let $(\mathcal{R}, u)$ be a complete and exhaustive topological Boolean ring and μ a u-continuous measure on $\mathcal{R}$. Then $sat(u) \leq sat(\mu)$.*

Proof. Let $(\hat{\mathcal{R}}, \hat{u})$, $\hat{\mu}$ be as in Proposition 2.1.19 so that $\hat{\mathcal{R}}$ is a complete Boolean algebra and $\hat{u}$ and $\tau(\hat{\mu})$ are order-continuous topologies on $\mathcal{R}$. Since $\hat{u}$ and $\tau(\hat{\mu})$ satisfy the assumptions of Proposition 2.1.22, $sat(\hat{u}) \leq sat(\tau(\hat{\mu})) = sat(\hat{\mu})$. The thesis follows from the fact that $sat(u) = sat(\hat{u})$ and $sat(\mu) = sat(\hat{\mu})$. $\square$

Remark 2.1.24. The notion of *saturation* of a measure space has been widely employed in different applications of measure and probability theory in the last decades (see Fajardo and Keisler (2004); Keisler and Sun (2009) and their references for a survey). However, it is in Khan and Sagara (2013) that we find this notion adapted to topological Boolean algebras with the following definition: a measure μ on an algebra $\mathcal{R}$ is *saturated* if there is no $x \in \mathcal{R} \setminus N(\mu)$ such that $\mathcal{R} \wedge x$, endowed with the topology $\tau(\mu)$, is separable. The definition of *degree of saturation* we gave in 2.1.21 can be seen as a natural extension of this concept: in fact a measure μ is saturated in the sense given by Khan and Sagara if and only if $sat(\mu)$ is uncountable.

2.2 The Lyapunov property

The classical Theorem of Lyapunov states that any $\mathbb{R}^N$-valued, σ-additive and non-atomic measure defined on a σ-algebra has a convex and compact range. As a direct consequence of this, if $\mathcal{R}$ is a σ-algebra and $\lambda: \mathcal{R} \to [0, +\infty[$ a σ-additive, non-atomic measure, then the range of every $\mathbb{R}^N$-valued measure absolutely continuous with respect to λ is convex and compact. With an abuse of the terminology introduced in Khan and Sagara (2013) we may therefore say that the measure space $(\mathcal{R}, \lambda)$ has the *Lyapunov property* with respect to any finite dimensional space. This, in our general context of topological Boolean rings, brings us to the following definition.

Definition 2.2.1. *We say that a topological Boolean ring* $(\mathcal{R}, u)$ *has the* Lyapunov property *with respect to the space* E *if every* u*-continuous measure* $\nu: \mathcal{R} \to E$ *has a convex and weakly compact range.*

We will say that a measure μ *on* $\mathcal{R}$ *has the* Lyapunov property *with respect to* E *if* $\tau(\mu)$ *has the Lyapunov property with respect to* E*.*

In other words, a measure $\mu: \mathcal{R} \to E$ has the Lyapunov property with respect to E if every E-valued measure absolutely continuous with respect to μ has a convex and weakly compact range. Our main problem can then be written in the following way:

> **Problem:** Given the locally convex space E, which topological Boolean rings have the Lyapunov property with respect to E?

We divide our analysis in two steps: first we consider only topological Boolean rings whose uniform structure is induced by a scalar measure, then we tackle the problem in the general case.

2.2.A The range of measures admitting a control

We say that a vector measure $\mu: \mathcal{R} \to E$ has a *control measure* $\lambda: \mathcal{R} \to [0, +\infty[$ if λ is bounded and $\mu \ll \lambda$, i.e. if $\lim_n \mu(x_n) = 0$ whenever $(x_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{R}$ such that $\lim_n \lambda(x_n) = 0$. In this case, μ is continuous with respect to the λ-topology and therefore it is exhaustive and bounded and it is σ-additive when λ is σ-additive. Moreover, both μ and λ will be closed whenever $\mathcal{R}$ is a σ-algebra and λ is σ-additive (see (Weber, 2002, Corollary 3.7) or (Aliprantis and Border, 2006, Lemma 13.13)).

In general not all vector measures are controlled. However, a slight generalization of Bartle-Dunford-Schwartz's Theorem ensures that if μ is an exhaustive measure with values in a metrizable, locally convex space then it admits a control measure λ which can be taken σ-additive if μ is so (Weber, 2002, Corollary 7.5).

A useful advantage of working with measures that are controlled is that it is possible to use functional analytical tools that would not be available otherwise. In the following, we are going to use a special integration procedure to assign to every controlled measure μ an integral operator T_μ (a more profound analysis of this procedure can be found, for example, in Kluvánek and Knowles (1976)). Let $\mathcal{A}$ be a σ-algebra of subsets of a non-empty Ω and let $\lambda : \mathcal{A} \to [0, +\infty[$ be a σ-additive measure. If we identify the functions that are equal λ-almost everywhere we can associate every λ-continuous measure $\mu : \mathcal{A} \to E$ with the unique continuous operator $T_\mu : L^\infty(\lambda) \to E$ with the property that $T_\mu(\chi_F) = \mu(F)$ for every $F \in \mathcal{A}$, where χ_F denotes the characteristic function of F. This operator T_μ, that we call the *integral operator associated to* μ, will result to be continuous with respect to the weak* topology on $L^\infty(\lambda)$ and the weak topology on E (Diestel and Uhl, 1977, IX.1.4). In the following, we shall also write $\int f\, d\mu$ for $T_\mu(f)$ and observe that, by a continuity argument, for all $x^* \in E^*$ and $f \in L^\infty(\lambda)$, $x^* \circ T_\mu(f) = x^*\left(\int f\, d\mu\right) = \int f\, d(x^* \circ \mu) = T_{x^* \circ \mu}(f)$.

The following Theorem shows the relation between the non-injectiveness of the operator T_μ and the convexity of the range of μ. Its proof can be found in (Diestel and Uhl, 1977, IX.1.4) for Banach-space valued measures and in (Urbinati and Weber, 2017, Proposition 2.3) for the locally convex case.

Proposition 2.2.2. *Let $\mu : \mathcal{A} \to E$ be a vector measure over a σ-algebra of sets and $\lambda : \mathcal{A} \to [0, +\infty[$ a σ-additive control for μ. For every $A \in \mathcal{A} \setminus N(\lambda)$, assume that the restriction of the operator T_μ to the space $L^\infty(\lambda_A)$, consisting of functions in $L^\infty(\lambda)$ vanishing off A, is non-injective. Then $\mu(\mathcal{A} \cap A)$ is weakly compact and convex for all $A \in \mathcal{A}$.*

In view of the above, our next aim is to find conditions on the measures μ and λ ensuring that each of the operators $T_\mu : L^\infty(\lambda_A) \to E$, $A \in \mathcal{A} \setminus N(\lambda)$, is non-injective. For example, we could ask that $dim\ (L^\infty(\lambda_A)) > dim\ E$[4] for every $A \in \mathcal{A} \setminus N(\lambda)$, a condition studied in Rustichini and Yannelis (1991) and again in Tourky and Yannelis (2001). The approach below closely follows the line of Greinecker and Podczeck (2013) and it is included here for the sake of completeness. First we will need the following lemma.

Lemma 2.2.3. *Let $\lambda : \mathcal{A} \to [0, +\infty[$ be a σ-additive measure over a σ-algebra of sets. Then $dens(\mathcal{A}, \tau(\lambda)) = dens(L^1(\lambda), \|\cdot\|_1)$ whenever $dens(\mathcal{A}, \tau(\lambda))$ is infinite.*

Proof. We first prove the inequality $dens(\mathcal{A}, \tau(\lambda)) \leq dens(L^1(\lambda), \|\cdot\|_1)$. Take a dense set $\mathcal{F} \subset L^1(\lambda)$ and for $f \in \mathcal{F}$ define $B_f := \{x : |1 - f(x)| \leq \frac{1}{2}\} \in \mathcal{A}$. Our goal is to show that for any $A \in \mathcal{A}$ and $\epsilon > 0$ we can take $f \in \mathcal{F}$ such that $\lambda(A \triangle B_f) < 2\epsilon$. This way $\{B_f : f \in \mathcal{F}\}$ is dense in $(\mathcal{A}, \tau(\lambda))$ and so the claim will follow from the generality of $\mathcal{F}$.

[4] Here $dim\ E$ stands for the algebraic dimension of the linear space E.

Choose $A \in \mathcal{A}$, $\epsilon > 0$ and take $f \in \mathcal{F}$ such that $\|\chi_A - f\|_1 < \epsilon$. We have that:

$$\begin{aligned}\epsilon > \|\chi_A - f\|_1 = &\int |\chi_A - f(x)|\, d\lambda(x) \geq \\ \geq &\int_{A \setminus B_f} |1 - f(x)|\, d\lambda(x) + \int_{B_f \setminus A} |f(x)|\, d\lambda(x).\end{aligned}$$

By construction, $|f(x)| \geq \frac{1}{2}$ for $x \in B_f$ while $|1 - f(x)| \geq \frac{1}{2}$ for $x \notin B_f$ so from the previous equation follows that:

$$\begin{aligned}\epsilon > \|\chi_A - f\|_1 \geq &\int_{A \setminus B_f} \frac{1}{2}\, d\lambda + \int_{B_f \setminus A} \frac{1}{2}\, d\lambda = \\ = &\frac{1}{2}\lambda(A \setminus B_f) + \frac{1}{2}\lambda(B_f \setminus A) = \frac{1}{2}\lambda(A \triangle B_f)\end{aligned}$$

as claimed.

We now prove that the inequality $dens(\mathcal{A}, \tau(\lambda)) \geq dens(L^1(\lambda), \|\cdot|_1)$ holds when $dens(\mathcal{A}, \tau(\lambda))$ is infinite. For a dense subset $\mathcal{B} \subset \mathcal{A}$, let $\mathcal{F}$ be the collection of all (finite) linear combinations of elements of $\{\chi_B : B \in \mathcal{B}\}$ with rational coefficient. Since $\mathcal{F}$ and $\mathcal{B}$ have the same cardinality and the space of simple functions[5] $S(\mathcal{A})$ is dense in $L^1(\lambda)$, it will be enough to show that $\mathcal{F}$ is dense in $S(\mathcal{A})$ to prove that $dens(L^1(\lambda), \|\cdot\|_1) \leq |\mathcal{B}|$.

Let $f := \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ be any simple function in $S(\mathcal{A})$ and $\epsilon > 0$. For every $i \leq n$ choose $\beta_i \in \mathbb{Q}$ and $B_i \in \mathcal{B}$ so that $|\beta_i - \alpha_i| \leq \epsilon/n$ and $\lambda(B_i \triangle A_i) \leq \epsilon/n\alpha$, where $\alpha := \sup_{i \leq n} |\alpha_i|$. This way $g := \sum_{i=1}^{n} \beta_i \chi_{B_i}$ is a function in $\mathcal{F}$ such that:

$$\begin{aligned}|g - f| \leq &\sum_{i=1}^{n} |\beta_i \chi_{B_i} - \alpha_i \chi_{A_i}| \leq \\ \leq &\sum_{i=1}^{n} |\beta_i - \alpha_i| \chi_{B_i} + |\alpha_i| \cdot |\chi_{B_i} - \chi_{A_i}| = \\ = &\sum_{i=1}^{n} |\beta_i - \alpha_i| \chi_{B_i} + |\alpha_i| \chi_{B_i \triangle A_i}.\end{aligned}$$

But then:

$$\begin{aligned}\int |g - f|\, d\lambda \leq &\sum_{i=1}^{n} |\beta_i - \alpha_i| \lambda(B_i) + |\alpha_i| \lambda(B_i \triangle A_i) \leq \\ \leq &\frac{\epsilon}{n} \sum_{i=1}^{n} \lambda(B_i) + \alpha \sum_{i=1}^{n} \lambda(B_i \triangle A_i) \leq \epsilon + \epsilon\end{aligned}$$

proving that $\mathcal{F}$ is dense in $S(\mathcal{A})$. □

[5] i.e. all linear combinations of characteristic functions of all $A \in \mathcal{A}$.

In Greinecker and Podczeck (2013), the authors consider a σ-algebra of sets $\mathcal{A}$ and for every infinite cardinal number κ they define a class of κ-*atomless* measures. The latter consists of σ-additive measure $\lambda: \mathcal{A} \to [0,1]$ such that $dens L^1(\lambda_A) \geq \kappa$ for every $A \in \mathcal{A} \setminus N(\lambda)$ (by (Greinecker and Podczeck, 2013, Fact 1) this is equivalent to the original definition of κ-atomless measures). By doing so, they were able to prove in (Greinecker and Podczeck, 2013, Section 3) that if $\lambda: \mathcal{A} \to [0,1]$ is a σ-additive, κ-atomless measure and E is a Banach space separated by a family $\mathcal{F} \subset E^*$ with $|\mathcal{F}| < \kappa$ then every measure $\mu: \mathcal{A} \to E$ absolutely continuous with respect to λ has a convex and weakly compact range.

Next Theorem can be seen as an extension of Greinecker and Podczeck's main result to the case of measures with values in a locally convex space.

Theorem 2.2.4. *Let $\mu: \mathcal{A} \to E$ be a measure on a σ-algebra of sets and let $\lambda: \mathcal{A} \to [0, +\infty[$ be a σ-additive control measure for μ with infinite degree of saturation. Assume that there exists a family $\mathcal{F} \subset E^*$ that separates the points of $\overline{span}\mu(\mathcal{A})$ with $|\mathcal{F}| < sat(\lambda)$. Then $\mu(\mathcal{A})$ is convex and weakly compact.*

Proof. We identify functions which are λ-almost everywhere equal.

By Proposition 2.2.2 it will be sufficient to prove that for any $A \in \mathcal{A} \setminus N(\lambda)$, the restriction of the operator $T_\mu: f \mapsto \int f \, d\mu$ to $L^\infty(\lambda_A)$ is non-injective. We will do this for $A = \Omega$, since the proof remains the same for the general case.

If $\mathcal{F}$ is finite then $\overline{span}\mu(\mathcal{A})$ must be finite dimensional. At the same time, being $sat(\lambda)$ infinite, the space $L^\infty(\lambda)$ has infinite dimension and so the operator $T_\mu: L^\infty(\lambda) \to E$ cannot be injective. Therefore we can assume that $\mathcal{F}$ is infinite.

By the Radon-Nikodym Theorem, to every $x^* \in \mathcal{F}$ we can associate a function $g_{x^*} \in L^1(\lambda)$ so that the measure $x^* \circ \mu$ is described by the relation $A \mapsto \int_A g_{x^*} \, d\lambda$ for $A \in \mathcal{A}$. Put $Y := \overline{span}\{g_{x^*} : x^* \in \mathcal{F}\}$. Since the set of finite linear combinations of the g_{x^*}'s with rational coefficients is a dense subset of Y with cardinality $|\mathcal{F}|$ and the latter is strictly smaller than $sat(\lambda)$ by hypothesis, we have that $dens(Y) \leq |\mathcal{F}| < dens(\mathcal{A}, \tau(\lambda))$. Consequently, Y cannot be the whole $L^1(\lambda)$, since $L^1(\lambda)$ has density greater or equal to $(\mathcal{A}, \tau(\lambda))$ by Lemma 2.2.3.

Now, because Y is a closed proper sub-space of $L^1(\lambda)$ and $L^1(\lambda)^* = L^\infty(\lambda)$, as a consequence of the Hahn-Banach Theorem there must be a $f \in L^\infty(\lambda) \setminus \{0\}$ such that for all $x^* \in \mathcal{F}$, $\int f \, d(x^* \circ \mu) = 0$ and so, by a continuity argument, $x^* \circ T_\mu(f) = 0$. But $T_\mu(f)$ belongs to $\overline{span}\mu(\mathcal{A})$, so $x^* \circ T_\mu(f) = 0$ for all $x^* \in \mathcal{F}$ implies that $T_\mu(f) = 0$ and hence that T_μ is non-injective as claimed. $\square$

In the assumptions of Theorem 2.2.4, λ is a σ-additive real valued measure defined on a σ-algebra and as such it is closed. The corollary that follows shows that this property alone is enough to guarantee the validity of the results.

Corollary 2.2.5. *Let $\lambda: \mathcal{R} \to [0, +\infty[$ be a closed control measure for $\mu: \mathcal{R} \to E$ with infinite degree of saturation and assume that there is a family $\mathcal{F} \subset E^*$ separating* *the points of* $\overline{span}\mu(\mathcal{R})$ *such that* $|\mathcal{F}| < sat(\lambda)$. *Then* $\mu(\mathcal{R})$ *is convex and weakly compact.*

Proof. Let $(\hat{\mathcal{R}}_\lambda, \hat{u})$ be the quotient $(\mathcal{R}, \tau(\lambda))/N(\lambda)$ and $\hat{\mu}: \hat{\mathcal{R}}_\lambda \to E$, $\hat{\lambda}: \hat{\mathcal{R}}_\lambda \to [0, +\infty[$ be the measures defined by $\hat{\mu}(\hat{x}) = \mu(x)$, $\hat{\lambda}(\hat{x}) = \lambda(x)$ for $x \in \hat{x} \in \hat{\mathcal{R}}_\lambda$. By Proposition 2.1.19, being $(\mathcal{R}, u)$ complete and exhaustive, $\hat{\mathcal{R}}_\lambda$ is complete and $\hat{\lambda}$ is a completely additive control measure for $\hat{\mu}$.

By the Loomis-Sikorski representation Theorem (Sikorski, 1960, 29.1), there exists a σ-algebra of sets $\mathcal{A}$ and a surjective homomorphism $\pi: \mathcal{A} \to \hat{\mathcal{R}}_\lambda$ such that $Ker\pi$ is a σ-ideal in $\mathcal{A}$ thus $\mathcal{A}/Ker\pi$ is isomorphic to $\hat{\mathcal{R}}_\lambda$. Let us define the measures $\lambda_\pi := \hat{\lambda} \circ \pi: \mathcal{A} \to [0, +\infty[$ and $\mu_\pi := \hat{\mu} \circ \pi: \mathcal{A} \to E$. By construction, λ_π is a σ-additive control measure for μ_π, $sat(\lambda_\pi) = sat(\hat{\lambda}) = sat(\lambda)$ and $\mu_\pi(\mathcal{A}) = \mu(\mathcal{R})$. In other words, $\overline{span}\mu_\pi(\mathcal{A})$ is separated by the family $\mathcal{F} \subset E^*$ with $|\mathcal{F}| < sat(\lambda_\pi)$ and so, being satisfied the condition of Theorem 2.2.4, $\mu_\pi(\mathcal{A})$, and therefore $\mu(\mathcal{R})$, is convex and weakly compact as claimed. □

2.2.B The range of general measures

In the absence of a control measure for $\mu: \mathcal{R} \to E$, it is much harder to obtain a result close to Theorem 2.2.4 with a similar approach. This is mainly due to the difficulties that can arise in generalizing some of the functional analytic tools used throughout the proofs of Proposition 2.2.2, Lemma 2.2.3 and Theorem 2.2.4, in which the properties of $L^\infty(\lambda)$ and $L^1(\lambda)$ were intensively employed.

Our main goal in this Section is to prove the following.

> Let $\mu: \mathcal{R} \to E$ be a closed and exhaustive measure with infinite degree of saturation and suppose that there is a family $\mathcal{F} \subset E^*$ that separates the points of $\overline{span}\mu(\mathcal{R})$ with $|\mathcal{F}| < sat(\mu)$. Then $\mu(\mathcal{R})$ is convex and weakly compact.

The idea behind the proof is simple and it follows what has been done in Urbinati and Weber (2017). It consists in decomposing μ as a sum $\mu = \sum_{i \in \mathcal{I}} \mu_i$ in which each one of the μ_i's is a measure satisfying the hypothesis of Corollary 2.2.5 and then showing that $\mu(\mathcal{R}) = \sum_i \mu_i(\mathcal{R})$. However, if on one hand the writing $\sum_{i \in \mathcal{I}} \mu(\mathcal{R})$ has a clear meaning when $\mathcal{I}$ is finite, in the infinite case things must be handled much more carefully and a little more notation is needed. We refer to the exposition at the beginning of the chapter (pages 13 to 15) for a short survey on infinite sums and uniform summability.

The following Theorem is going to be necessary.

Theorem 2.2.6 (Theorem 4.5 in Urbinati and Weber (2017)). *Let* $\mu: \mathcal{R} \to E$ *be a closed and exhaustive measure. Then there is a system* $a_i \in \mathcal{R}$ *of almost* *disjoint*[6] *elements and* $x_i^* \in E^*$, $i \in \mathcal{I}$, *such that the measures* $\mu_i: \mathcal{R} \to E$ *defined by* $\mu_i(x) = \mu(x \wedge a_i)$, $x \in \mathcal{R}$, *satisfy the following conditions:*

1. *for each* $i \in \mathcal{I}$ *the measure* μ_i *is absolutely continuous with respect to* $|x_i^* \circ \mu|$, *the variation of* $x_i^* \circ \mu$;

2. *for all* $x \in \mathcal{R}$, $(\mu_i(x))_{i \in \mathcal{I}}$ *is summable and* $\mu(x) = \sum_{i \in \mathcal{I}} \mu_i(x)$;

3. $\prod_{i \in \mathcal{I}} \mu_i(\mathcal{R})$ *is uniformly summable and* $\sum_{i \in \mathcal{I}} \mu_i(\mathcal{R}) = \mu(\mathcal{R})$.

What Theorem 2.2.6 ensures is that whenever we have a closed and exhaustive measure $\mu: \mathcal{R} \to E$, we can always write it as the infinite sum of some controlled measures μ_i's that can be chosen so that $(\mu_i(\mathcal{R}))_{i \in \mathcal{I}}$ is uniformly summable. We stress in particular that point (3) in the Theorem follows from (2) and from Theorem 2.1.7: the correspondence $\Phi_\mu: \mathcal{R} \twoheadrightarrow E$ that maps each $x \in \mathcal{R}$ in $\mu(\mathcal{R}_x)$ is in fact exhaustive (as noticed in Example 2.1.13) and, by construction, it is such that $\Phi(a_i) = \mu_i(\mathcal{R})$ for every $i \in \mathcal{I}$. It is therefore sufficient to apply the implication $(1 \Rightarrow 2)$ in Theorem 2.1.7 to the family $(a_i)_{i \in \mathcal{I}}$ to conclude that the system $(\Phi(a_i))_{i \in \mathcal{I}}$, and hence $(\mu_i(\mathcal{R}))_{i \in \mathcal{I}}$, is uniformly summable.

We now have all the ingredients to prove our main theorem.

Theorem 2.2.7. *Let* $\mu: \mathcal{R} \to E$ *be a closed and exhaustive measure with infinite degree of saturation and suppose that there is a family* $\mathcal{F} \subset E^*$ *that separates the points of* $\overline{span}\mu(\mathcal{R})$ *with* $|\mathcal{F}| < sat(\mu)$. *Then* $\mu(\mathcal{R})$ *is convex and weakly compact.*

Proof. Let $x_i^* \in E^*$, $a_i \in \mathcal{R}$ and $\mu_i: \mathcal{R} \to E$, $i \in \mathcal{I}$, be as in Proposition 2.2.6 so that $\prod_{i \in \mathcal{I}} \mu_i(\mathcal{R})$ is uniformly summable and $\mu(\mathcal{R}) = \sum_{i \in \mathcal{I}} \mu_i(\mathcal{R})$. If we prove that each of the μ_i's satisfies the assumptions of Corollary 2.2.5, so that the $\mu_i(\mathcal{R})$'s are all convex and weakly compact subsets of E, the thesis will follow from Corollary 2.1.5.

Fix a $i \in \mathcal{I}$ and call λ the measure $|x_i^* \circ \mu|: \mathcal{R} \to [0, +\infty[$, which is a control measure for μ_i by point (1) in 2.2.6. By construction, λ is absolutely continuous with respect to μ and therefore, beside being closed, it has a degree of saturation greater or equal than $sat(\mu)$ so that $|\mathcal{F}| < sat(\lambda)$.

Moreover, since $\mu_i(\mathcal{R}) \subseteq \mu(\mathcal{R})$, the family $\mathcal{F}$ separates the points of $\overline{span}\mu_i(\mathcal{R})$ too. But then all the assumptions on Corollary 2.2.5 are satisfied and $\mu_i(\mathcal{R})$ is convex and weakly compact as claimed. □

Corollary 2.2.8. *Let* $\mu: \mathcal{R} \to E$ *be a closed and exhaustive measure with infinite degree of saturation and suppose that there is a family* $\mathcal{F} \subset E^*$ *that separates the points of* E *with* $|\mathcal{F}| < sat(\mu)$. *Then* μ *has the Lyapunov property with respect to* E.

[6] i.e. such that $a_i \wedge a_j \in N(\mu)$ for all $i, j \in \mathcal{I}$ distinct.

Proof. Let $\nu\colon\mathcal{R}\to E$ be a measure absolutely continuous with respect to μ. Then ν is closed, exhaustive and has degree of saturation greater or equal to $sat(\mu)$, where the latter is strictly greater than $|\mathcal{F}|$ by assumption. Since $\mathcal{F}$ separates the points of E, and consequently of $\overline{span}\,\nu(\mathcal{R})$, ν satisfies all the assumptions of Theorem 2.2.7 and as such it has a convex and weakly compact range. □

Corollary 2.2.9. *Let $(\mathcal{R},u)$ be a complete and exhaustive topological Boolean ring such that $sat(u)$ is infinite. Furthermore, assume that there is a family $\mathcal{F}\subset E^*$ that separates the points of E with $|\mathcal{F}|<sat(u)$. Then $(\mathcal{R},u)$ has the Lyapunov property with respect to E.*

Proof. It follows directly from Corollary 2.1.23 that every u-continuous measure $\mu\colon\mathcal{R}\to E$ satisfies the assumptions of Corollary 2.2.8 and therefore it has a convex and weakly compact range. □

Remark 2.2.10 (Remarks on the main theorem)**.** As mentioned before, what makes it quite easier to work with a vector measure $\mu\colon\mathcal{R}\to E$ admitting a control $\lambda\colon\mathcal{R}\to[0,+\infty[$ is the possibility of employing many fine properties of the spaces $L^1(\lambda)$ and $L^\infty(\lambda)$. When such a λ does not exist, it is necessary to study other function spaces in order to replace $L^1(\lambda)$ and $L^\infty(\lambda)$. This is done, for example, in Kluvánek and Knowles (1976) where a generalization of Proposition 2.2.2 is given. Following this line of investigation, in (2016) Khan and Sagara proved that a closed, σ-additive measure over a σ-algebra $\mu\colon\mathcal{A}\to E$ has convex and weakly compact range if it is homogeneous of type strictly greater than the topological dimension of E, generalizing a previous result they had proved in (2013). The problem with this approach is mainly due to the very deep analytical tools employed which seem to be a very high price to be payed in this framework.

To prove Theorem 2.2.7, which can be seen as a general case of the above mentioned result of Khan and Sagara, we decided to follow a completely different path inspired by Urbinati and Weber (2017). Theorem 2.2.7 improves previous results in two respects: the less restrictive hypothesis, in which neither σ-additiveness of the measures nor the σ-completeness of the algebra are required, and the proof strategy itself which seems to be more flexible to further developments.

2.2.C The range of additive correspondences

We now go back to the study of the range of an additive correspondence $\Phi\colon\mathcal{R}\twoheadrightarrow E$ that is rich in additive selections. Our first result is a standard argument that allows to deduce the convexity of the values of Φ from the convexity of the ranges of the measures belonging to a sufficiently rich subset of $\mathcal{S}_\Phi$. We will need the following Lemma.

Lemma 2.2.11. *Let $(\mathcal{R}, u)$ and E be as in Corollary 2.2.9 and let $\mu_1, \mu_2 : \mathcal{R} \to E$ be two u-continuous measures. Then, for every $a \in \mathcal{R}$ and every $t \in [0,1]$ there is a $b \le a$ such that $t\mu_1(a) = \mu_1(b)$ and $(1-t)\mu_2(a) = \mu_2(a \setminus b)$.*

Proof. The first thing we observe is that, under these assumptions, it is possible to find a family $\mathcal{F} \subseteq (E \times E)^*$ that separates the points of $E \times E$ and that is such that $|\mathcal{F}| < sat(u)$. Therefore, by Corollary 2.2.9, every u-continuous measure with values in $E \times E$ will necessarily have a convex range.

Let us choose $a \in \mathcal{R}$ and $t \in [0,1]$. By setting $\eta(x) = (\mu_1(x \wedge a), \mu_2(x \wedge a))$ for every $x \in \mathcal{R}$ we can define a u-continuous measure $\eta : \mathcal{R} \to E \times E$ that has a convex range by the argument above. This will mean in particular that both 0 and $\eta(a)$ belong to $\eta(\mathcal{R})$ and so there must be a $y \in \mathcal{R}$ such that $\eta(y) = t\eta(a)$. Call $b := y \wedge a$, then observe that, by construction, we have that $\mu_i(b) = \mu_i(y \wedge a) = t\mu_i(a)$ for $i = 1,2$. But then $\mu_1(b) = t\mu_1(a)$ while $(1-t)\mu_2(a) = \mu_2(a) - t\mu_2(a) = \mu_2(a) - \mu_2(b) = \mu_2(a \setminus b)$ as claimed. □

Theorem 2.2.12. *Let $(\mathcal{R}, u)$ and E be as in Corollary 2.2.9, $\mathcal{M}$ be a spliceable family of u-continuous measures with values in E and call $\Phi_{\mathcal{M}}$ the correspondence that maps $x \in \mathcal{R}$ into $\{\mu(x) : \mu \in \mathcal{M}\}$. Then:*

1. *$\Phi_{\mathcal{M}}(a)$ is convex for every $a \in \mathcal{R}$.*
2. *$R(\Phi) = \bigcup_{x \in \mathcal{R}} \Phi(x)$ is convex.*

Proof. To prove point (1) let us take $a \in \mathcal{R}$, $v_1, v_2 \in \Phi_{\mathcal{M}}(a)$ and $t \in [0,1]$. We need to prove that $v := tv_1 + (1-t)v_2$ belongs to $\Phi_{\mathcal{M}}(a)$ or, equivalently, that there is a $\nu \in \mathcal{M}$ such that $\nu(a) = v$. Let us choose, for $i = 1,2$, a measure $\mu_i \in \mathcal{M}$ such that $\mu_i(a) = v_i$. Being each μ_i a u-continuous measure, we can apply Lemma 2.2.11 and find a $b \le a$ such that $\mu_1(b) = t\mu_1(a)$ and $\mu_2(a \setminus b) = (1-t)\mu_2(a)$. In particular we will have that $\mu_1(b) + \mu_2(a \setminus b) = tv_1 + (1-t)v_2 = v$. The claim is proved once we observe that the function ν that assign to each $x \in \mathcal{R}$ the vector $\nu(x) = \mu_1(x \wedge b) + \mu_2(x \setminus b)$ is a measure in $\mathcal{M}$ (because $\mathcal{M}$ is spliceable) and it is such that $\nu(a) = \mu_1(b) + \mu_2(a \setminus b) = v$ as desired.

We focus now on point (2). To prove that $R(\Phi_{\mathcal{M}})$ is convex let us choose $v_1, v_2 \in R(\Phi_{\mathcal{M}})$, $t \in [0,1]$ and claim that $v := tv_1 + (1-t)v_2 \in R(\Phi_{\mathcal{M}})$, which is to say that $v = \nu(d)$ for some $\nu \in \mathcal{M}$ and $d \in \mathcal{R}$. Let us take $a_i \in \mathcal{R}$ and $\mu_i \in \mathcal{M}$ be such that $\mu_i(a_i) = v_i$ for every $i = 1,2$, then define $b_0 := a_1 \wedge a_2$, $b_1 := a_1 \setminus a_2$ and $b_2 := a_2 \setminus a_1$. As a consequence of Theorem 2.3.1, the range of each μ_i is a convex set and so we can take $c_1 \le b_1$ and $c_2 \le b_2$ such that $t\mu_1(b_1) = \mu_1(c_1)$ and $(1-t)\mu_2(b_2) = \mu_2(c_2)$. At the same time, by point (1) there must be a $\nu \in \mathcal{M}$ such that $\mu_0(b_0) = t\mu_1(b_0) + (1-t)\mu_2(b_0)$. To conclude the proof let us define ν as the measure that assigns $\mu_1(x \wedge b_1) + \mu_0(x \wedge b_0) + \mu_2(x \setminus a_1)$ to each $x \in \mathcal{R}$ and call

$d := c_1 \vee b_0 \vee c_2$. This way the fact that $\mathcal{M}$ is closed under splicing ensures that $\nu \in \mathcal{M}$ while the identity $\nu(d) = v$ follows from the following series of equations:

$$\begin{aligned}\nu(d) =&\mu_1(c_1) + \mu_0(b_0) + \mu_2(c_2) = \\ =&t\mu_1(b_1) + (t\mu_1(b_0) + (1-t)\mu_2(b_0)) + (1-t)\mu_2(b_2) = \\ =&t\mu_1(a_1) + (1-t)\mu_2(a_2) = tv_1 + (1-t)v_2 = v.\end{aligned}$$

□

Thanks to Proposition 2.1.9 we can rephrase the Theorem above in the following formulation that results more suitable to certain applications.

Corollary 2.2.13. *Let $(\mathcal{R}, u)$ be a closed and exhaustive topological Boolean ring with an infinite degree of saturation and suppose that there is a $\mathcal{F} \subseteq E^*$ that separates the points of E and that is such that $|\mathcal{F}| < sat(u)$. Then every additive correspondence $\Phi: \mathcal{R} \twoheadrightarrow E$ that is rich in u-continuous selections has convex values and a convex range.*

Remark 2.2.14. On the set $\mathcal{P}_0(E)$ we can define a topology, sometimes called the Hausdorff topology, defined by the semi-metrics:

$$d_p(X, Y) := \sup_{x \in X} \inf_{y \in Y} p(x - y)$$

with p ranging over continuous semi-norms on E. Thanks to this topological structure on $\mathcal{P}_0(E)$, one could define the *FN-topology induced by Φ on $\mathcal{R}$* as the coarsest FN-topology on $\mathcal{R}$ making Φ a continuous function and denote it by $\tau(\Phi)$. This idea is followed, for example in Drewnowski (1976a), Avallone and Basile (1993) and Basile (1994, 1998).

In the approach we have proposed here we have studied an additive correspondence $\Phi: \mathcal{R} \twoheadrightarrow E$ through the weakest FN-topology u_Φ on $\mathcal{R}$ that makes every selection in $\mathcal{S}_\Phi$ a uniformly continuous function. When Φ is rich in selections this is equivalent with saying that u_Φ is the weakest FN-topology making Φ an upper hemicontinuous correspondence. This u_Φ is usually coarser than the topology $\tau(\Phi)$ described above. In the example 2.1.11 it is clear that u_Φ is exhaustive while $\tau(\Phi)$ is not.

2.3 Refinements

As it was first proved in (Tweddle, 1968, Theorem 3), a finitely additive measure taking values in a locally convex space has a relatively weakly compact range if and only if it is exhaustive (see also or (Diestel and Uhl, 1977, Corollary 18.1.I) for the case of Banach-space valued measures). This implies that whenever $\mu: \mathcal{R} \to E$ is exhaustive, the space $\overline{span}\mu(\mathcal{R})$ belongs to the class of weakly compactly generated spaces, where a linear sub-space Y of E is *weakly compactly generated* if it is the closed linear span of a weakly compact subset of E.

In the light of this remark, we might agree with saying that most of the results on the range of E-valued exhaustive measures can be reformulated in terms of weakly compactly generated subsets of E. The following Theorem is a way to do this.

Theorem 2.3.1. *Let $(\mathcal{R}, u)$ be a complete and exhaustive topological Boolean ring with infinite degree of saturation and assume that for every weakly compactly generated sub-space Y of E there is a family $\mathcal{F} \subset E^*$ separating the points of Y such that $|\mathcal{F}| < sat(\mu)$. Then $(\mathcal{R}, u)$ has the Lyapunov property with respect to E.*

Proof. Let $\nu: \mathcal{R} \to E$ be a u-continuous measure and call $Y := \overline{span}\nu(\mathcal{R})$. Our goal is to prove that ν has a convex and weakly compact range by showing that is satisfies the assumptions of Theorem 2.2.7.

Being ν exhaustive, Y is a weakly compactly generated sub-space of E and so, by hypothesis, its points are separated by a family $\mathcal{F} \subset E^*$ with $|\mathcal{F}| < sat(\mu)$. The fact that ν is closed, and so $sat(\mu) \leq sat(\nu)$ by Corollary 2.1.23, concludes the proof. □

We stress that Theorem 2.3.1 is a significant improvement of Corollary 2.2.8 as it allows us to consider a much wider class of measures. As a way of illustration, in the following example, we describe a locally convex, infinite dimensional space whose weakly compactly generated sub-spaces are finite dimensional.

Example 2.3.2. Consider the infinite dimensional space $X = c_{00}$ consisting of all real sequences with finite support (i.e. sequences $(x_n)_n \subset \mathbb{R}$ such that $x_n = 0$ for all but a finite number of indexes $n \in \mathbb{N}$). On E we take the topology τ_B generated by the base:

$$\left\{ E \cap \left(\prod_{n \in \mathbb{N}} U_n \right) : \; U_n \text{ is an open set of } \mathbb{R} \text{ for all } n \in \mathbb{N} \right\}.$$

Such τ_B is commonly known as *box topology* and, by (Jarchow, 1981, section 6.6), it makes (X, τ_B) a complete locally convex space. Furthermore, one observes that bounded sets in E must lie in finite dimensional sub-spaces of X (see for example (Zabeti, 2016, Theorem 4)).

Consider now the algebra $\mathcal{B}$ of measurable subsets of the real unit interval $[0,1]$ with the Lebesgue measure λ. Since the range of every measure $\mu: \mathcal{B} \to X$ absolutely continuous with respect to λ lies in a finite dimensional sub-space of X, by the classical Lyapunov's Theorem $\mu(\mathcal{B})$ must be compact and convex. This implies that $(\mathcal{B}, \tau(\lambda))$ has the Lyapunov's property with respect to X even though there is no family of functionals $\mathcal{F} \subset X^*$ that separates the points of X with $|\mathcal{F}| < sat(\lambda)$.

Similarly with what is done in Khan and Sagara (2013, 2015) and (Greinecker and Podczeck, 2013, Corollary 1), one might want to find a relation between the density of the space E and the degree of saturation of a E-valued measure with the Lyapunov property with respect to E. In order to do this, we will recall the following preliminary result, due to Amir and Lindenstrauss (1968), whose proof can be found in (Fabian et al., 2011, Theorem 13.3) for Banach spaces and in (Cascales and Orihuela, 1987, Theorem 13) for a general class of spaces that includes locally convex metrizable spaces.

Proposition 2.3.3. *If E is a metrizable weakly compactly generated locally convex space then $dens(E) = dens(E^*)$ where E^* is considered with the weak* topology.*

Proposition 2.3.3 allows us reformulate the conditions in 2.3.1 in terms of the density of weakly compactly generated sub-spaces of E.

Proposition 2.3.4. *Let $(\mathcal{R}, u)$ be a complete and exhaustive topological Boolean ring and assume that every weakly compactly generated sub-space of E is linearly homeomorphic to some metrizable space with density strictly smaller than $sat(u)$. Then μ has the Lyapunov property with respect to E.*

Proof. Let $\nu: \mathcal{R} \to E$ be a u-continuous measure. We need to prove that $\nu(\mathcal{R})$ is convex and weakly compact. Since ν is closed and exhaustive, it is sufficient to show that the points of $Y := \overline{span}\,\nu(\mathcal{R})$ are separated by an infinite family $\mathcal{F} \subset E^*$ with $|\mathcal{F}| < sat(\nu)$, then apply Theorem 2.2.7.

Being ν exhaustive, Y is a weakly compactly generated sub-space of E and as such it is metrizable by assumption. Thus, by applying Proposition 2.3.3, we can take a family $\mathcal{F} \subset Y^*$ with cardinality $dens(Y)$ that is dense in Y^* with respect to the weak* topology. The family $\mathcal{F}$ has therefore cardinality strictly smaller than $sat(u)$ by assumption and it separates the points of Y as a consequence of the Hahn-Banach Theorem (Fabian et al., 2011, Proposition 3.39).

This, together with the fact that $sat(u) \leq sat(\nu)$ (Proposition 2.1.23), implies that $|\mathcal{F}| < sat(\nu)$ as desired. □

Remark 2.3.5. In the setting of Proposition 2.3.4, the measure μ takes values in a sub-space of E whose topology can be induced by a metric. Thus, by the Theorem of Bartle-Dunford-Schwartz (as formulated in (Weber, 2002, Corollary 7.5)) the measure μ is equivalent with respect to a scalar measure $\lambda: \mathcal{R} \to [0, +\infty[$ (i.e. $\tau(\mu) = \tau(\lambda)$).

This makes possible to prove Proposition 2.3.4 via Corollary 2.2.5 without the need of employing Theorem 2.2.7.

2.3.A A characterization of the Lyapunov property

It is known that Lyapunov's Theorem also characterizes finite dimensional spaces. In fact, if E is a F-space[7] such that every E-valued, non-atomic and σ-additive measure on σ-algebras has a compact or convex range, E cannot have infinite dimension (see (Diestel and Uhl, 1977, Corollary 6 on pg 265) for the case E is a Banach space, for the general result see Wnuk (1980)). In the terminology we have introduced, this is equivalent with saying that an F-space E has a finite dimension if and only if every σ-additive, non-atomic measure $\lambda: \mathcal{R} \to [0, +\infty[$ defined on a σ-algebra has the Lyapunov property with respect to E.

We wonder whether a similar statement can be generalized to spaces with higher dimension, proving that those conditions that in Theorem 2.2.7 were shown to be sufficient for the convexity result are also necessary. In other words, we ask if the following question can be answered positively:

> **Question:** Let $\mu: \mathcal{R} \to E$ be a closed and exhaustive measure with the Lyapunov property with respect to E. Is it true that $sat(\mu)$ must be strictly greater then the cardinality of the minimum family $\mathcal{F} \subset E^*$ that separates the points of $\overline{span}\, \mu(\mathcal{R})$?

Following an idea of Wnuk (1980), we use the existence of a topologically independent sequence in E to provide a partial answer to the previous question.

Recall that $(v_n)_{n \in \mathbb{N}} \subset E$ is a *topologically linearly independent sequence* in E if for every $f \in \ell^\infty(\mathbb{N})$ (i.e. the space of bounded functions $f: \mathbb{N} \to \mathbb{R}$) $\sum_{n \in \mathbb{N}} f(n) v_n = 0$ implies $f = 0$. In Drewnowski et al. (1981) it is proved that every infinite dimensional metrizable vector space (X, τ) contains a topologically linearly independent sequence $(v_n)_{n \in \mathbb{N}}$.

Proposition 2.3.6. *Suppose that E is metrizable and infinite dimensional. Let $\mu: \mathcal{R} \to E$ be an exhaustive measure such that every measure $\nu: \mathcal{R} \to E$ absolutely continuous with respect to μ has a convex range. Then $sat(\mu)$ is uncountable.*

Proof. Since E is metrizable and μ exhaustive, by Bartle-Dunford-Schwartz's Theorem (Weber, 2002, Corollary 7.5) there is a measure $\lambda: \mathcal{R} \to [0, +\infty[$ which is equivalent to μ, i.e. such that $\tau(\lambda) = \tau(\mu)$. Clearly, if $sat(\lambda)$ is uncountable there is nothing left to prove so we assume by contradiction that $sat(\lambda) \le |\mathbb{N}|$. This means that there exists a $a \in \mathcal{R} \setminus N(\lambda)$ such that $\mathcal{R} \wedge a$ is a separable topological subspace of $(\mathcal{R}, \tau(\lambda))$. Without loss of generality we can assume that $\mathcal{R}$ is a Boolean algebra, that $a = e$ and take a sequence b_n, $n \in \mathbb{N}$ dense in $(\mathcal{R}, \tau(\lambda))$.

On $\mathcal{R}$ we define the family of scalar measures $\lambda_n: x \mapsto \lambda(x \wedge b_n)$, $n \in \mathbb{N}$, and observe that $x \triangle y \notin N(\lambda)$ implies that $\lambda_n(x) \ne \lambda_n(y)$ for at least one $n \in \mathbb{N}$.

[7] i.e. a complete metrizable topological vector space.

Since E is metrizable, we can select a topologically linearly independent sequence v_n, $n \in \mathbb{N}$, in E and choose a sequence of non-zero $t_n \in \mathbb{R}$, $n \in \mathbb{N}$, so that $(t_n v_n)_{n\in\mathbb{N}}$ is summable in E. Then, we can define the measure $\nu: \mathcal{R} \to E$ by setting $\nu(x) := \sum_{n\in\mathbb{N}} \lambda_n(x) t_n v_n$ for $x \in \mathcal{R}$. Since ν is absolutely continuous with respect to μ, $\nu(\mathcal{R})$ must be a convex subset of E, meaning that there is a $d \in \mathcal{R}$ such that $\nu(d) = \nu(d^c) = \nu(e)/2$. But then:

$$0 = \nu(d) - \nu(d^c) = \sum_{n\in\mathbb{N}} t_n \left(\lambda_n(d) - \lambda_n(d^c)\right) v_n$$

and so, having taken v_n, $n \in \mathbb{N}$, topologically linearly independent and t_n non-zero, it must be $\lambda_n(d) = \lambda_n(d^c)$ for each $n \in \mathbb{N}$.

The contradiction follows from the fact that, by construction, $\lambda(d \triangle d^c) > 0$ and so there must be a $n \in \mathbb{N}$ with $\lambda_n(d) \neq \lambda_n(d^c)$. □

We stress that in the settings of Proposition 2.3.6 the assumption of metrizability of the space E cannot be directly dropped. As seen in example 2.3.2, if E is infinite dimensional but not metrizable it is possible to find a measure $\mu: \mathcal{R} \to E$ with the Lyapunov property with respect to E such that $sat(\mu)$ is countable.

Corollary 2.3.7. *Let E be separable, infinite dimensional and metrizable and let $\mu: \mathcal{R} \to E$ be a closed and exhaustive measure. Then the following are equivalent:*

1. *μ has the Lyapunov property with respect to E.*
2. *every $\nu: \mathcal{R} \to E$ with $\nu \ll \mu$ has convex range.*
3. *$sat(\mu)$ is uncountable.*

Proof. The implication $(1 \Rightarrow 2)$ is obvious while $(2 \Rightarrow 3)$ is a consequence of Proposition 2.3.6. Finally, $(1 \Rightarrow 2)$ can be seen as a special case of Theorem 2.3.4. □

Remark 2.3.8. Corollary 2.3.7 still holds if we replace the hypothesis on the metrizability and separability of E with the less restrictive hypothesis that every weakly compactly generated subset of E is linearly homeomorphic to a separable and metrizable space. In fact, under this milder assumptions, the proofs remains identical.

Remark 2.3.9. Using (Diestel and Uhl, 1977, Corollary 6, pg. 265), Khan and Sagara proved in (2013, Section 4.2) that if E is an infinite dimensional separable Banach space and $\mu: \mathcal{A} \to E$ is a σ-additive homogeneous measure over a σ-algebra then μ is saturated[8] if and only if every $\nu: \mathcal{A} \to E$ absolutely continuous with respect to μ has a convex and weakly compact range.

[8] i.e. $sat(\mu)$ is uncountable.

Their result is extended in this section via Proposition 2.3.6 and Corollary 2.3.7 to include finitely additive measures that are not necessarily homogeneous and that can take values in locally convex metrizable spaces. Moreover, the proof is significantly simplified by avoiding the necessity of recurring to Maharam's Theorem of classification of homogeneous measure algebras.

Chapter 3

The choice of the commodity space in Walrasian competitive economies

In this chapter we focus on the problem of choosing the right mathematical representation of bundles and price systems in a competitive economy with many commodities. As it is known, there have been many studies exploring the existence of Walrasian equilibrium with an infinity of commodities and a variety of infinite dimensional commodity spaces have been considered. Depending on specific applications of the theory, different order or topological properties on the commodity spaces have been used to allow a better representation of specific kinds of allocations. A natural question, that we explore in this chapter, is whether there is a common subtext underlying all of these representations. To put it in a different form, we ask the following.

> What mathematical structure on the commodity space is *essential* to describe and study the working of a Walrasian competitive economy?

The chapter is organized as follows. In the Section 3.1 we investigate the algebraic aspects of the duality between commodities and prices. We first focus on the linear representation of commodity bundles, seen as the formal object of agents' choices rather than lists of quantities of goods, then we discuss the nature of price systems and the different possible ways of representing them.

In Section 3.2.A we analyze the working of a Walrasian economy with infinitely many commodities and define a very elementary economic model which is entirely

[0]The results in this chapter were developed together with Professor M. Ali Khan from the Johns Hopkins University of Baltimore, USA. Still, I take full responsibility of any error or imprecision in this exposition.

determined by means of the excess of supply correspondence and the set of bundles that can be disposed freely by the economic agents. We will call this essential model an *abstract market*, rephrasing the definition given by McKenzie in (1959).

We will then adapt the approach developed by Nikaidô (1957, 1959) to prove the existence of a competitive price for a class of abstract markets. This result will be based on an infinite dimensional extension of the Gale-Debreu-Nikaidô Lemma, that will be proved in 3.3.1, which stands out for the weakness of the topological assumptions on the commodity and price spaces. In this case, in fact, the excess of supply correspondence and the set of freely disposable goods will induce respectively a very weak locally convex topology and a pseudo-order on the commodity space while no topological structure will be required for the space of prices.

We will conclude the chapter with a discussion on how the Theorem 3.3.1 shows the difficulties of giving an axiomatic definition of infinite dimensional commodity spaces in Walrasian competitive economies (Sections 3.3.B and 3.4).

The mathematical setting

For the sake of the exposition we summarize here the definitions and the notation conventions that will be used throughout this chapter.

- (E, F) will be a dual pair of linear spaces. For every non-empty $X \subset F$, $\sigma(E, X)$ will denote the weakest linear topology on E making every function in X continuous. If not otherwise specified, E and F will be considered endowed with the weak topologies that follow from the duality between them.

- By *wedge* we will mean a non-empty subset W of a linear space satisfying $W + W \subseteq W$ and $\lambda W \subseteq W$ for any $\lambda > 0$. It follows that any wedge W is automatically convex and $W \cap -W$ is a linear sub-space. A wedge is called a *cone* if it contains only the trivial sub-spaces, i.e. if $W \cap -W = \{0\}$[1]. A wedge W is *proper* if $W \neq E$ and *non-trivial* if it is not a linear subspace of E.

  If $X \subseteq E$ is non-empty, the set $X^* := \{p \in F : \ p \cdot x \geq 0 \text{ for every } x \in X\}$ is a convex and closed wedge in F. We call X^* the *dual wedge* of X and write $\langle X^* \rangle$ to denote $X^* \setminus \{0\}$. If $W \subset E$ is a wedge with non-empty interior, then its dual W^* is a cone (See (Aliprantis and Tourky, 2007, Chapter 2.1)).

- We will write $\phi: X \twoheadrightarrow Y$ to say that ϕ is a correspondence between X and Y. The correspondence $\phi: X \twoheadrightarrow Y$ between subsets of topological linear spaces

[1]These definitions have been taken from Aliprantis and Tourky (2007) and are not universal. Sometimes, a non-empty set C is called a *cone* if it is closed under multiplication by a positive scalar and it is said to be *pointed* if $C \cap -C = \{0\}$.

is *upper hemicontinuous* if $\phi^{-1}(U) := \{x : \phi(x) \subset U\}$ is open whenever U is an open set in Y. For any $A \subseteq X$, we will say that $\phi(A) := \cup\{\phi(x) : x \in A\}$ is the *image* of A under ϕ and call $\phi(X)$ the *range* of ϕ. If X is convex, then ϕ is a *Kakutani map* if it is upper hemicontinuous and has non-empty, compact and convex values[2].

Our main reference for the general analysis of the working of a Walrasian economy will be the classical Debreu (1959), Nikaidô (1968) and Arrow and Hahn (1971) for the finite dimensional setting. For the case of economies with infinitely many commodities we refer to Aliprantis et al. (1990) and Chapter 5 in Florenzano (2003). From a mathematical perspective, we mainly refer to Aliprantis and Tourky (2007) for the study of Wedges, cones and duality.

3.1 The duality between commodities and prices

In the economies we will consider, commodity bundles and all possible price systems will be given as primitive notions. In the following sections we will base our model on a dual pair (E, F) of linear spaces representing, respectively, the set of commodity bundles and the set of price systems. The bilinear map $\langle \cdot, \cdot \rangle : F \times E \to \mathbb{R}$ will be thought as the price evaluation function that assigns to every $x \in E$ and $p \in F$ the value $p \cdot x$ of the bundle x under the price system p. Here, the assumption that F separates the points of E is required to ensure that for any two distinct bundles $x, y \in E$ we can always find a price system making x more expensive than y.

However elementary, the settings described above are based on profound observations and choices that is worth discussing in details. We do this in the following paragraphs addressing separately the representation of commodity bundles and price systems.

3.1.A Linear representation of commodity bundles

Bundles of commodities are the central entity of the Walrasian model and the object of consumers' and producers' activity. By moving the attention from the physical attributes of commodities to the constraint decision problems faced by economic agents, it becomes clear that, rather than lists of quantities of goods, bundles are to be thought as formal objects that can be exchanged, aggregated or reproduced in certain quantities. Therefore, we may define algebraically a commodity space as a set E, whose elements are the bundles, that is closed under some

[2] This definition is not standard and it was taken from (Dugundji and Granas, 1982, Definition 8.1, pg. 166).

properly defined operations that capture the intuitive notions of aggregation, exchange and reproduction of bundles. In this perspective, there is a clear advantage in endowing E with a linear structure: for any two bundles $x, y \in E$, in fact, one can define $x + y$ as their aggregate, $x - y$ as the 'net trade' that allows to obtain y from x and αx, for $\alpha > 0$, as the bundles one obtains by reproducing x α times. In other words, we can justify the linear space structure on the commodity space E by the necessity of formalizing algebraically some operations between bundles.

In this general framework one observes that the assumption that E has a finite dimension becomes very restrictive and the absence of a 'natural basis' for E makes it impossible to refer directly to commodities but only to bundles. To overcome these difficulties one can decide to take a set $\mathcal{I}$ of commodities as the primitive entity of the model and to define a bundle x as a specification of how much of each commodity is contained in x. For a given commodity space E, this idea can be preserved only if E admits some form of Schauder basis $(e_i)_{i\in\mathcal{I}}$ so that for every $x \in E$ there is a unique way to write x as a linear combination $\sum_{i\in\mathcal{I}} \alpha_i e_i$. By associating each $i \in \mathcal{I}$ to a single commodity, the representation $x = \sum_{i\in\mathcal{I}} \alpha_i e_i$ gives a precise meaning to the sentence "the bundle x contains α_i units of the commodity i". It is crucial to point out that this approach can only be pursued in very specific cases and when the commodity space E is already endowed with a topological structure. Without such a topology, not only it is impossible to define a Schauder basis, but also the expression $\sum_{i\in\mathcal{I}} \alpha_i e_i$ has no mathematical meaning.

By choosing to work with a very generic linear space E there will be no real advantage in fixing a preferred basis for E, for even if we take a generic coordinate system $(e_i)_{i\in\mathcal{I}}$ of E, a bundle $x \in E$ could admit two different representations $\sum_{i\in\mathcal{I}} \alpha_i e_i$ and $\sum_{i\in\mathcal{I}} \beta_i e_i$. One could still associate each $i \in \mathcal{I}$ to a different commodity but in this case the the question "how many units of commodity i are present in x?" will have multiple valid answers. This will mean that the formalization of any economic concept will be given in an abstract way, without referring to a specific coordinate system. We can call this kind of approach a *coordinate-free* representation.

Remark 3.1.1. We stress that the coordinate-free approach to the representation of commodity spaces can be useful even in the case one decides to work with finite dimensional commodity spaces. In his work of (1956), for example, Debreu discusses the importance of considering agents whose consumption sets and preferences may be described independently from any fixed base on the commodity space even when the latter has a finite dimension.

Remark 3.1.2. It should be stressed that the linear structure on E is only one of the many that one can consider. Depending on the economic question that one wants to explore, it may be convenient to define operations on E in a different way according to the specific contingencies of the problem. As a way of illustration, we mention a recent stream of research in which a set of bundle E is endowed with an independent mathematical structure, called *abstract convex structure*, that allows to formalize the intuitions described above without recurring to any linear structure of E. An exposition of the main features of this approach can be found in Martínez-Legaz (2005) and the monograph Urai (2010). See also Singer (1997) for a classical exposition of abstract notions of convexity and duality in optimization problems. Other interesting questions on the algebraic structure of the commodity space are widely influenced by the study of abstract economies and more general notions of competition, see for example the work in Richter and Rubinstein (2015, 2018).

3.1.B Price systems

It should be stressed that, while commodity bundles are the natural object of any economic analysis, the assumption that prices should be given exogenously is a strong one that is proper to the Walrasian approach. In fact, while the Paretian and Edgeworthian notions of equilibrium depend exclusively on the way agents rank consumption bundles, the definition of Walrasian equilibrium is *price-dependent* and can only be formalized by means of agents' best response to independent prices. This centrality of price systems can be traced down to the importance of obtaining full decentralization of allocations, which is on of the main focuses of the Walrasian program.

A *price system* is essentially a specification of the exchange rates at which commodities can be traded. In this perspective, the duality between prices and commodities is the rule that describes what exchanges of bundles are made possible when a given price system emerges.

We can formalize the ideas above through the following abstraction of the duality between prices and commodities. If E is the collection of all commodity bundles and F that of price systems we can define a correspondence $\pi: F \times E \twoheadrightarrow E$ that assigns to each price system $p \in F$ and commodity bundle $x \in E$ the set $\pi(p, x) \subseteq E$ of all the alternatives to x that are available under the price system p.

Definition 3.1.3. *An* abstract commodity-price duality *is a triple* (E, F, π) *where* E *is a* set of commodity bundles, F *is the* set of price systems *and* $\pi: F \times E \twoheadrightarrow E$ *is called* price evaluation correspondence.

In our specific framework, where we have decided to endow E with a linear structure, we will define F as a set of functionals on E and consider the price evaluation correspondence π that assigns to every $p \in F$ and $x \in E$ the half-space $\pi(p, x) = \{y : p \cdot y \leq p \cdot x\}$. This way, the duality will be in some form consistent with the operations defined on E and the intuitive notion of *exchange rate* is preserved by means of linear transformations as in the finite dimensional case.

We stress that, with these definitions, the 0 functional in F corresponds to a degenerate price system.

Remark 3.1.4. This definition of π is of course a choice and as such can be criticized. As a way of illustration, one could want to allow exchanges that are symmetric in the sense that, under any price system p, the bundle x can be obtained from y if and only if y can be obtained from x. In this case $\pi(p, x)$ should be defined as the hyper-plane $\{y : p \cdot y = p \cdot x\}$. Other more sophisticated examples for the choice of π can be found in the literature on incomplete markets.

Remark 3.1.5. Let us go back to the map $\pi: (p, x) \mapsto \{y : p \cdot y \leq p \cdot x\}$ assigning any $x \in E$ and $p \in F$ to all the bundles that can be obtained from x at price p. Once F is endowed with a linear structure it is clear that π is homogeneous of degree zero which is saying that any multiplication of a vector $p \in F$ by a positive scalar will not affect the exchanges that are possible under the price system p.

In a more formal way, we should define *price systems* as the equivalence classes of the type $[p]$, for $p \in F$, formed by all the $q \in F$ such that $p = \lambda q$ for some $\lambda > 0$. With a little more freedom of language we will continue talk about a price system meaning the single functional $p: E \to \mathbb{R}$ instead of the entire equivalence class.

Remark 3.1.6. We have given a description of the commodity price duality that is purely algebraic. This approach differs from most of the literature on infinite dimensional competitive economies where it is common to introduce a commodity space E as a linear space endowed with a topological (or lattice) structure and to define a price system as any linear functional on E which is continuous (or positive). This way, the notion of *prices* is derived from that of *commodity bundles* and from the structure that is defined on the commodity space. In a purely Walrasian framework, however, the central role played by price systems that are exogenously generated suggests that prices should be defined together with commodity bundles as if it was not possible to talk about the latter without a full description of the former. In other words, it should be the choice of the topology (and order) on the commodity space to depend on how prices are defined and not vice versa.

3.2 Equilibrium of excess of supply

A fundamental feature of Walrasian competitive economies is that agents' behaviour can be described by means of supply and demand. The notion of competitive equilibrium itself, which is the main interest of this theory, is determined by the possibility of agents to make individual consumption and production choices that are optimal under an exogenously given set of prices. In other words, even in Walrasian models that are preference-based, the analysis of equilibrium existence is directly related to the possibility of defining the excess of supply for some price systems.

Following the argument above, we will give a very abstract and synthetic description of the economy whose primitive elements will be the (total) excess of supply correspondence and the set of all the bundles that can be freely disposed by the agents. Intuitively, the excess of supply correspondence ζ assigns to every $p \in F$ the residuals of all the optimal transactions at price p. This is to say that any bundle $z \in \zeta(p)$ can be seen as the difference of a virtual aggregated production plan y and an aggregated consumption plan x both of which are optimal under the price p[3]. If z can be disposed at no cost, which is to say that $z \in Y$, then the transaction associated to z is performable and p is an *equilibrium price*.

Definition 3.2.1. *An* abstract market *is a pair* (Y, ζ) *where* Y *is a closed wedge in* E*, whose elements we call the* freely disposable bundles*, and* $\zeta\colon \langle Y^* \rangle \twoheadrightarrow E$ *is a correspondence, called* excess of supply*.*

A vector $\overline{p} \in \langle Y^* \rangle$ *is called* equilibrium price *for the abstract market* (Y, ζ) *if* $\zeta(\overline{p}) \cap Y \neq \varnothing$*.*

We observe that our notion of freely disposable bundles, i.e. the vectors in Y, is not related to any pre-existent property of the commodities themselves and it may depend exclusively on the characteristic of the agents. Vectors in Y could be, for example, the bundles that can absorbed by the production sector or consumed by some agent at the equilibrium price. At the same time, every wedge Y defines on E a *pseudo-order* $\geq_Y$ defined by $x \geq_Y z \iff x - y \in Y$ (see (Aliprantis and Tourky, 2007, Chapter 1) for definitions and basic properties of pseudo-ordered linear spaces).

Our original concern can now be formulated as follows: if there is a competitive price in every abstract market (Y, ζ) whose commodity-price duality is given by the dual pair (E, F), is there a natural topological and order structure on E? In what follows we will give a negative answer to this question.

Remark 3.2.2. In the literature, a special relevance is given to the case in which the cone Y has non-empty interior, as it was in Nikaidô (1956b). In Bewley (1972), for example, the set Y coincides with the negative orthant $-L^\infty(\mu)_+$, in Toussaint (1984) it is the production technology set and in Mas-Colell (1986) it is derived from the uniform properness of preferences. However related to different interpretations, in all of these examples the study of the geometric properties of the cone Y is essentially the same, see Kajii (1988) for a discussion on this issue.

[3] The adjective *virtual* is borrowed from the register used in physics where it indicates a potential displacement that may not meet the constraint of the model. In this case it is used to stress the idea that the production and consumption plans y and x may not be actuated since the correspondent transaction may not be feasible.

3.2.A Gale-Debreu-Nikaidô Lemma

The main result of this section (Theorem 3.3.1) is an elaboration of (Nikaidô, 1957, Main Theorem) and can be seen as an extension of the so-called Gale-Debreu-Nikaidô Lemma (3.2.3) to general vector spaces. Given the dual pair (E, F), this result provides conditions on a wedge Y and a correspondence ζ that are sufficient to guarantee the existence of equilibrium prices for the abstract market (Y, ζ). To introduce some key observations, we first present it in the simplest case in which both E and F are the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$ and Y coincides with the positive orthant $\mathbb{R}^\ell_+$. These are the classic assumptions used, among the others, by Gale (1955), Nikaidô (1956a), Kuhn (1956) and Debreu (1956), (1959, pg 82).

Lemma 3.2.3 (Gale-Debreu-Nikaidô)**.** *Let $K \subset \mathbb{R}^\ell_+$ be the unit simplex and suppose that the correspondence $\phi: K \twoheadrightarrow \mathbb{R}^\ell$ is such that:*

(*i*) *ϕ is a Kakutani map.*

(*ii*) *The Walras' law prevails, i.e. $p \cdot x \geq 0$ for every $p \in K$ and $x \in \phi(x)$.*

Then there is a $\overline{p} \in K$ such that $\zeta(\overline{p}) \cap \mathbb{R}^\ell_+ \neq \varnothing$.

The idea of the proof is to model the interaction between the whole society and a price-adjusting mechanism so that the equilibrium prices can be obtained as the fixed point of a correspondence opportunely constructed. We sketch the proof here and break it in different steps to facilitate the successive discussion.

Proof. The first thing we observe is that, being K compact and ζ upper hemicontinuous, the range $\zeta(K)$ must be a compact set too (see Lemma 3.3.3).

1. Fix a $z \in \zeta(K)$. Since the map $p \mapsto p \cdot z$ is linear and continuous on the compact and convex set K, $\theta(z) := \{p \in K : \ p \cdot z \leq q \cdot z \text{ for all } q \in K\}$ must be a non-empty, convex and compact set too. Let us define the correspondence $\theta: \zeta(K) \twoheadrightarrow K$ by the relation $x \mapsto \theta(x)$ and observe that θ is upper hemicontinuous by Berge's Theorem (for a proof see (Aliprantis and Border, 2006, Theorem 17.31)).

2. Define now $\psi: K \times \zeta(K) \to K \times \zeta(K)$ as the product $(p, z) \mapsto \theta(z) \times \zeta(p)$. By construction, ψ will be upper hemicontinuous and have non-empty, compact and convex values (it is a Kakutani map) so that all the assumptions of Kakutani's fixed point Theorem are met (see (Florenzano, 2003, Theorem 1.1.2) for a proof). This is saying that that there must be a $(\overline{p}, \overline{z}) \in K \times \zeta(K)$ such that $\overline{p} \in \theta(\overline{z})$ and $\overline{z} \in \zeta(\overline{p})$.

3. We claim that $\overline{z} \in \mathbb{R}^\ell_+$ so that $\overline{p}$ is the desired vector in K. To see this we observe that $\overline{p} \cdot \overline{z} \geq 0$ by the Walras' law (condition (ii)) and for every $q \in K$ we have $\overline{p} \cdot \overline{z} \leq q \cdot \overline{z}$ by the definition of θ. This means that $0 \leq q \cdot \overline{z}$ for every $q \in K$, proving that $\overline{z} \in \mathbb{R}^\ell_+$.

□

In the proof of Lemma 3.2.3, the finite dimension of the space $\mathbb{R}^\ell$ plays a twofold role: it ensures that K, the unit simplex, is a compact set and it allows to apply Kakutani fixed point Theorem in Step 2. But is it really necessary? Can it be dispensed by assuming that K is a compact set?

Imagine that (E, F) is any dual pair, that $K \subset F$ is a convex and compact set and $\zeta: K \twoheadrightarrow E$ is a Kakutani map satisfying the Walras' law (as in condition (ii) of the Lemma 3.2.3). The first thing one observes is that the Step 1 of the proof can be reproduced identically in this new context. Step 2 can also be easily adapted to this infinite dimensional setting by applying the so-called Kakutani-Fan-Glicksberg Theorem (proved in Fan (1952) and Glicksberg (1952). An expository proof can be found in (Florenzano, 2003, Corollary 1.1.2)) instead of the classical Kakutani fixed point Theorem to the correspondence ψ. Finally, following the procedure of Step 3, one can prove the existence of a $\overline{p} \in K$ and a $\overline{z} \in \zeta(\overline{p})$ such that $q \cdot \overline{z} \geq 0$ for every $q \in K$. In other words, we have just adapted the proof of 3.2.3 to this specific infinite dimensional extension of Gale-Debreu-Nikaidô Lemma.

Proposition 3.2.4. *Suppose that $K \subset F$ is a convex and compact set and that the correspondence $\zeta: K \twoheadrightarrow E$ is such that:*

(i) *ζ is a Kakutani map.*

(ii) *The Walras' law prevails, i.e. $p \cdot x \geq 0$ for every $p \in K$ and $x \in \phi(x)$.*

Then there are a $\overline{p} \in K$ and a $\overline{z} \in \zeta(\overline{p})$ such that $q \cdot \overline{z} \geq 0$ for every $q \in K$.

Reading some of the references in Nikaidô (1957), we have reason to believe that a similar version of Proposition 3.2.4 was presented by Nikaidô in a technical report of the university of Stanford in 1956 with the title “On the existence of competitive equilibrium with infinitely many commodities”, (1956b). We did not manage to find a copy of the mentioned technical report, whose results are anyway extended in Nikaidô (1957) and again in (1959). It shall be stressed that the approach we chose to state 3.3.1 is independent of Proposition 3.2.4 as we will weaken the assumption that the excess of supply correspondence ζ is upper hemicontinuous.

3.2.B A note on the Walras' law

From an economic perspective, the assumption that the excess of supply correspondence ζ satisfies the so-called Walras' law (condition (ii) in Lemma 3.2.3) is a direct consequence of the idea that the consumers' choices are subject to budget constraints. This is to say that, at any price, the value of the total demand cannot exceed that of the total supply.

From a more technical point of view, the Walras' law plays an important role in the proof of Gale-Debreu-Nikaidô Lemma as it ensures that the values of the excess of supply correspondence ζ computed on $p \in F$ are all contained in the half-space $\{x : p \cdot x \geq 0\}$. In the literature (see Yannelis (1985), Mehta and Tarafdar (1987), for a broader exposition see (Florenzano, 2003, Chapter 2)) it is common to give a direct proof of the Lemma 3.2.3 replacing the assumption (ii) with the following, weaker condition.

$$\forall p \in K \text{ there is a } x \in \phi(p) \text{ such that } p \cdot x \geq 0. \qquad (ii')$$

This simplification is easily explained if one observes that, under assumption (ii'), the correspondence $\phi' : p \mapsto \{x \in \phi(p) : \ p \cdot x \geq 0\}$ is a Kakutani map satisfying all the assumptions of Lemma 3.2.3. By applying the Lemma to ϕ', and observing that $\phi'(p) \subseteq \phi(p)$ for every p, we obtain that $\phi(K) \cap \mathbb{R}^\ell_+ \neq \varnothing$. With this idea one can replicate the proof of Lemma 3.2.3 and obtain the following.

Corollary 3.2.5. *Suppose that $K \subset \mathbb{R}^\ell$ is a compact and convex set and the correspondence $\phi : K \twoheadrightarrow \mathbb{R}^\ell$ is such that $p \mapsto \{x \in \phi(p) : \ p \cdot x \geq 0\}$ is a Kakutani map. Then there is a $\overline{p} \in K$ such that $\phi(\overline{p}) \cap \mathbb{R}^\ell_+ \neq \varnothing$.*

In the next section the same line of Corollary 3.2.5 could be used to prove Theorem 3.3.1 replacing the Walras' law with the weaker (ii'). However, for the sake of the exposition, we will not pursue this idea.

3.3 Nikaidô's extension to the infinite dimensional case

We are now ready to face the main Theorem of this section in its generality. With minor exceptions, the proof will follow that of Nikaidô (1957) and (1959, Theorem 5). Let us say that a wedge $Y \subset E$ is *non-trivial* if it is not a linear sub-space of E.

Theorem 3.3.1. *Suppose that Y is a non-trivial, closed wedge in E and the correspondence $\zeta : \langle Y^* \rangle \twoheadrightarrow E$ satisfies the following assumptions:*

1. *For every finite dimensional sub-space L of F, the restriction of ζ to $L \cap \langle Y \rangle$ is a Kakutani map when E is endowed with the topology $\sigma(E, L)$.*

2. *The range of ζ is a compact set.*

3. *The Walras' law prevails, i.e. $p \cdot x \geq 0$ whenever $p \in \langle Y^* \rangle$ and $x \in \zeta(p)$.*

Then there is a $p \in \langle Y^* \rangle$ such that $\zeta(p) \cap Y \neq \varnothing$.

Among the many approaches to infinite-dimensional extensions of the Gale-Debreu-Nikaidô Lemma, Theorem 3.3.1 stands out for the weakness of the continuity assumptions stated in point (1). Assuming that the correspondence ζ is not globally continuous, but it is well-behaving when restricted to finite dimensional sub-spaces of its domain, has two major consequences. In the interest of our research, it allows us to prove the existence of equilibrium prices without recurring to any topology on the space of prices: in fact, the only topological considerations needed on F are related to its finite-dimensional sub-spaces which are naturally endowed with a topological structure (the fact that F does not need to be a topological linear space is already observed in the assumptions of (Nikaidô, 1959, Theorem 4)). On the other hand, allowing some discontinuity of the excess of supply when the commodity space is infinite dimensional has important microeconomic justifications, as seen in Florenzano (1983).

3.3.A Proof of the Theorem 3.3.1

As for most infinite dimensional extensions of the Gale-Debreu-Nikaidô Lemma, the main idea in the proof of Theorem 3.3.1 is to apply the Lemma 3.2.3 to a class of properly defined abstract markets, precisely we will approximate the wedge $\langle Y^* \rangle$ through some of its finite dimensional sub-sets. As we shall see, differently from Proposition 3.2.4, this Theorem cannot be proved directly with an approach involving elementary extensions of Kakutani fixed point Theorem.

For the sake of the exposition we divide the proof in several steps and agree to write P instead of $\langle Y^* \rangle$. Our claim will be proved if we show that the intersection between $\zeta(P)$ and Y is not empty. The strategy of the proof goes as follows.

1. It is shown that Y is the intersection of a family of cones Y_i, $i \in \mathcal{I}$, such that each $\langle Y_i^* \rangle$ is a finite dimensional subset of P.

2. Lemma 3.2.3 is used to show that for each $i \in \mathcal{I}$ the set $\zeta(P) \cap Y_i$ is closed and non-empty.

3. It is proved that the family $\zeta(P) \cap Y_i$, with $i \in \mathcal{I}$, has the local intersection property. By the compactness of the range of ζ it will follow that:
$$(\zeta(P) \cap Y) = \bigcap_{i \in \mathcal{I}} (\zeta(P) \cap Y_i)$$

is non-empty, and so there exists at least an equilibrium price for (Y, ζ).

We stress that, in this case, the approximation is made on the space of prices F and not directly on E. Still, the compactness of the range of ζ will be crucial in concluding the proof.

We first prove some auxiliary lemmas.

Lemma 3.3.2. *Let $W \subset E$ be a proper closed wedge with non-empty interior and W^* the relative dual wedge. Then $\langle W^* \rangle$ is convex and non-empty.*

Proof. Since W is closed, W can be seen as the dual wedge of W^* by a standard separation argument (see for example (Aliprantis and Tourky, 2007, Theorem 2.13 (3))). Therefore, $W^* = \{0\}$ would imply that $W = E$ in contradiction with the fact that $E \neq W$.

To see that $\langle W^* \rangle$ is convex let us choose $p, q \in \langle W^* \rangle$, $t \in (0,1)$ and claim that $tp + (1-t)q \in \langle W^* \rangle$. Since W^* is automatically convex, it is enough to show that $tp + (1-t)q \neq 0$. By assumption W has an interior point u and so we know that $p \cdot u > 0$ and $q \cdot u > 0$ (Aliprantis and Tourky, 2007, Lemma 2.17). But then $(tp + (1-t)q) \cdot u > 0$ proves the claim. $\square$

Lemma 3.3.3. *Let K be a compact set and $\phi \colon K \twoheadrightarrow E$ be an upper hemicontinuous correspondence. Then $\phi(K)$ is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $\zeta(K)$ and claim that it has a finite sub-cover. By construction, $\mathcal{V} := \{\phi^{-1}(U) : U \in \mathcal{U}\}$ is a cover of the compact set K whose elements must be open by the upper hemi-continuity of ϕ. We can therefore select $U_1, \ldots, U_k$ in $\mathcal{U}$ such that $\{\phi^{-1}(U_1), \ldots, \phi^{-1}(U_k)\}$ covers K. But then $U_1, \ldots, U_k$ form a finite sub-cover of $\zeta(K)$ and the proof is concluded. $\square$

Step 1 (Lemma 9 in Nikaidô (1959))**.** *There is a family $(Y_i)_{i \in \mathcal{I}} \subset E$ of proper closed wedges with non-empty interior that is such that $Y = \bigcap_i Y_i$.*

Proof. The first thing we observe is that, being Y non-trivial, there must be a $u \in Y$ that does not belong to $-Y$. This, together with the closedness of Y, allows us to claim the existence of a 0-neighborhood U in E such that $(-u + U) \cap Y = \varnothing$. Let $\mathcal{V} := (V_i)_{i \in \mathcal{I}}$ be the filter of 0-neighborhoods in E that are convex, symmetric and contained in U. The idea is to build each Y_i by 'inflating' Y through the open set V_i. To do this we consider the cone Q_i generated by $u + V_i$ and call Y_i the closure of $Y + Q_i$. Please observe that every element of Y_i can be written as a sum $x + t(u + v) + z$ for some properly chosen $x \in Y$, $t \geq 0$ and $v + z \in V_i$.

We show that the $(Y_i)_i$ is the claimed family of cones. First we fix $i \in \mathcal{I}$ and observe that Y_i is closed (by definition), it has non-empty interior (since $u + V_i \subseteq Y_i$) and it does not coincide with E. To show this latter point let us assume by contradiction that $-u \in Y_i$: by the argument above, this means that we can find $x \in Y$, $t \geq 0$ and $v, z \in V_i$ such that $-u = x + t(u+v) + z$ and hence:

$$-u = \frac{1}{t+1}x + \frac{t}{t+1}v + \frac{1}{t+1}z \in Y + \frac{t}{t+1}V_i + \frac{1}{t+1}V_i \subseteq Y + V_i$$

Where the last inclusion follows from the convexity of V_i. But then since V_i is symmetric and $V_i \subseteq U$ it must be that $-u + U \cap Y \neq \varnothing$, in contradiction with the choice of u and U.

There is only left to show that $\bigcap_i Y_i \subseteq \overline{Y}$. This, together with the fact that Y is closed and contained in each of the Y_i's, will conclude our proof. Take $y \in \bigcap_i Y_i$ and let $V \subseteq U$ be a symmetric 0-neighborhood. We want to prove that there is a $y_V \in Y$ such that $y - y_V \in V$.

By construction, for every $i \in \mathcal{I}$ there are $x_i \in Y$, $t_i \geq 0$ and $v_i, z_i \in V_i$ such that $y = x_i + t_i(u + v_i) + z_i$. If $\lim_i t_i = 0$ then there must be a $j \in \mathcal{I}$ such that $t_j(u + v_j) + z_j \in V$. In this case we can put $y_V = x_j$. If $(t_i)_{i \in \mathcal{I}}$ does not converge to 0 nor to ∞ then, for a $j \in \mathcal{I}$ we will have that $t_j v_j + z_j \in V$. But then it will be enough to put $y_V := x_j + t_j u$. Last we observe that, if $\lim_i t_i = \infty$ then for a $j \in \mathcal{I}$ we would have that $-y/t_j + v_j + z_j/t_j \in U$. But then we would have that $-u = x_j/t_j + (-y/t_j + v_j + z_j/t_j) \in Y + U$ which is in contradiction with the choice of u. □

Step 2. *For every index i and every finite subset A of $\langle Y_i^* \rangle$ there are a $p \in co(A)$ and a $z \in \zeta(p)$ such that $q \cdot z \geq 0$ for every $q \in A$.*

Proof. Fix an index i and observe that $\langle Y_i^* \rangle$ is non-empty and convex as Y_i has an interior point (see Lemma 3.3.2). Furthermore, from the inclusion $Y \subset Y_i$ we derive that $\langle Y_i^* \rangle \subset P$ so that, whenever $A \subset \langle Y_i^* \rangle$ is finite, the correspondence ζ is defined on the whole $co(A)$. This proves that the statement above is well formulated.

Suppose now that $A \subset \langle Y_i^* \rangle$ is finite and non-empty. Without loss of generality we can assume that all vectors in A are linearly independent so that the convex hull $co(A)$ is linearly and topologically isomorphic to the unit simplex K in Euclidean space $\mathbb{R}^A$. Call L the linear sub-space of F spanned by $co(A)$, let $\alpha: \mathbb{R}^A \to L$ be the linear isomorphism that maps K into $co(A)$ and define the function $\beta: E \to \mathbb{R}^A$ by $\beta_q(x) = q \cdot x$ for every $q \in A$, $x \in E$. Then consider the composition

$$\tilde{\zeta} := \beta \circ \zeta \circ \alpha : K \twoheadrightarrow \mathbb{R}^A.$$

Using the linearity and continuity of α and β together with the upper hemi-continuity of the restriction of ζ to the linear sub-space L (condition (1)) we derive that $\tilde{\zeta}$ is a Kakutani map that satisfies the Walras' law. This, by Lemma 3.2.3, implies that there must be a $\overline{p} \in K$ such that $\tilde{\zeta}(\overline{p}) \cap \mathbb{R}^A_+ \neq \varnothing$.

Take now any $x \in E$ such that $\beta(x) \in \tilde{\zeta}(\overline{p}) \cap \mathbb{R}^A_+$ and observe that, by construction, $x \in \zeta(\alpha(\overline{p}))$ and $q \cdot x \geq 0$ for every $q \in co(A)$. This proves the claim. □

It is worth emphasizing that another way to write the result in Step 2 is the following: for every $i \in \mathcal{I}$ and every finite subset A of $\langle Y_i^* \rangle$ the set $\zeta(P) \cap A^*$ is non-empty. Here by A^* we mean the dual wedge of A, i.e. the set of all $x \in E$ such that $q \cdot x \geq 0$ for all $q \in A$.

Step 3. *The set $\zeta(P)$ intersects Y.*

Proof. Let us first fix some simplifying notation. For any $i \in \mathcal{I}$ let $\mathcal{F}_i$ be the collection of all finite, non-empty subsets of $\langle Y_i^* \rangle$. It is clear that Y_i will be the intersection of all the A^* with $A \in \mathcal{F}_i$.

Consider the family $\mathcal{G} := \{A^* \cap \zeta(P) : \ A \in \mathcal{F}_i \text{ for some } i \in \mathcal{I}\}$. A little computation, together with the Step 1, shows that:

$$\bigcap \mathcal{G} = \bigcap_i \left(\bigcap_{A \in \mathcal{F}_i} (A^* \cap \zeta(P)) \right) = \bigcap_i (Y_i \cap \zeta(P)) = Y \cap \zeta(P).$$

To show that this intersection is non-empty, and hence prove the claim, it is therefore sufficient to show that the sets in $\mathcal{G}$, which are subsets of the compact set $\zeta(P)$, have the finite intersection property.

Let us choose two elements $A^* \cap \zeta(P)$ and $B^* \cap \zeta(P)$ in $\mathcal{G}$ and prove that $A^* \cap B^* \cap \zeta(P)$ is a non-empty set in $\mathcal{G}$. By the way we defined $\mathcal{G}$ there must be indexes $k, j \in \mathcal{I}$ such that $A \in \mathcal{F}_k$ and $B \in \mathcal{F}_j$.

We first observe that $(A \cup B) \in \mathcal{F}_h$ for some $h \in \mathcal{I}$. In fact, from Step 1 we know that $Y = \cap_i Y_i$ and so there must be a $h \in \mathcal{I}$ such that $Y_h \subseteq Y_k \cap Y_j$. But this means that $Y_h^* \supseteq Y_k^* \cup Y_j^*$ and so $\langle Y_h^* \rangle \supset (A \cup B)$ (remember that A and B are finite subsets of $\langle Y_k^* \rangle$ and $\langle Y_j^* \rangle$ respectively). We then apply the result in Step 2 to $(A \cup B)$ and Y_h to obtain that there must be a $p \in co(A \cup B)$ and a $z \in \zeta(p)$ such that $q \cdot z \geq 0$ for every $q \in (A \cup B)$. This shows that $\zeta(p)$ intersects the set $(A \cup B)^*$ which is equal to $A^* \cap B^*$. □

3.3.B Comments on the Theorem

Let us make a parallel with the finite dimensional version of the Gale-Debreu-Nikaidô as exposed in Lemma 3.2.3. If E is the Euclidean space $\mathbb{R}^\ell$ and Y its positive orthant, the assumptions (1) and (3) are exactly identical to conditions (i) and (ii) in Lemma 3.2.3. On the other hand, in the finite dimensional setting condition (2) is superfluous if ζ is assumed to be homogeneous of degree 0[4]. To see this it is sufficient to observe that the range of ζ coincides with $\zeta(K)$, where K is the unit simplex of $\mathbb{R}^\ell$, and that the image of a compact set under an upper hemicontinuous correspondence is a compact set (see Lemma 3.3.3). Furthermore,

[4]Recall that ζ is *homogeneous of degree* 0 if $\lambda\zeta(p) = \zeta(p)$ for every $\lambda > 0$ and $p \in F$ such that $\zeta(p) \neq \varnothing$.

condition (1) is in fact the more general assumption that can generalizes (i) in 3.2.3 and can hardly be weakened. At the same time, James' Theorem allows to rewrite condition (2) in the algebraic requirement that every $p \in Y$ must attain its maximum on the range of ζ, a condition that is always satisfied in the finite dimensional case.

Among the assumptions needed to prove Theorem 3.3.1, the fact that the excess of supply correspondence must have non-empty value for every $p \in Y$ represents a strong limitation to the class of economic phenomena that can be studied with this approach. In fact, if we think of ζ as the difference between a supply correspondence η and a demand correspondence ξ, this assumption coincides with asking that $\xi(p)$ is non-empty whenever $\eta(p)$ is so. As it is now very well known, in infinite dimensional economies this is rarely the case (see Aliprantis and Burkinshaw (2006) for an exposition of this issue). Interesting developments in this direction are obtained by assuming that the domain of the correspondence ζ is a dense subset of $\langle Y^* \rangle$ and ζ satisfies some *boundary conditions*. This idea is pursued, for example, in Aliprantis and Brown (1983), Florenzano (1983) and Mehta and Tarafdar (1987).

Remark 3.3.4. As mentioned, we proved Theorem 3.3.1 to show how the study of existence of equilibria can be carried out with very weak topological requirements on the commodity and price spaces. In fact, what emerges in Theorem 3.3.1 is that no topology is needed on the space of prices while the only topological considerations on the commodity space are related to the weak topology induced by the duality with prices. This is a result that can hardly be improved.

Remark 3.3.5. The proof of Theorem 3.3.1 depends on that of 3.2.3 which is known to be equivalent to Kakutani fixed-point Theorem (see Uzawa (1962b)). As such, condition (1) should not come as a surprise, just like condition (i) in 3.2.3. Still it is important to stress that a similar result can be obtained if other fixed-point Theorems are used instead of Kakutani's. In Nikaidô (1959), for example, the whole argument is based on Montgomery's fixed-point Theorem and the excess of supply correspondence is asked to be acyclic instead of convex-valued. We have decided to present Theorem 3.3.1 in this form mainly because the shift to Montgomery-like assumptions on the excess of demand correspondence did not seem to bring significant contributions to our study of the commodity spaces. In fact, having chosen to base the commodity-price duality on a dual pair of linear spaces, we automatically gave a special importance to locally convex topologies and hence to convex sets and Kakutani maps.

Remark 3.3.6. It seems that Theorem 3.3.1 can be easily extended by considering some other, more general, fixed-point Theorems. Apart from the Montgomery's Fixed point Theorem, one could for example weaken the point (1) in Theorem

3.3.1 and assume that each restriction of the ζ have the *local direction property*, as in Park Park (2004), the *generalized transferable open-lower sections property*, as in Scalzo (2015), the *continuous inclusion property*, as in He and Yannelis (2016) or, in general, can be majorized as in Prokopovych (2016). Other interesting extensions could be obtained by thinking of Theorem 3.3.1 as a purely variational inclusion problem and studied using the techniques presented in Kristály and Varga (2003).

However interesting, it seems that none of these possible extensions of Theorem 3.3.1 can really contribute to our research focus on the topology needed on the commodity and price spaces (as discussed in the Remark 3.3.4). For this reason, we decided to present this more elementary version of the Theorem.

3.4 Conclusions

We have based our discussion on the description of the minimal mathematical structure needed to represent an infinite dimensional commodity space in a Walrasian economy. A similar question had been raised by Aliprantis and Burkinshaw in (1991) to understand what properties of commodities and prices were necessary to ensure the equivalence of Core and competitive allocations in every "well-behaving"economy based on the same sets of commodities and prices. What we did in 3.3.1 is showing the great independence of the equilibrium existence problem from any pre-existent topological assumption on E as the topological and order properties that are used in the hypothesis and the proof are exactly those that naturally follow from the (algebraic) duality established between E and F (i.e. locally convex topologies and linear pre-orders). This contribution shows the impossibility of giving an axiomatic definition of commodity space in Walrasian economies as long as the linearity of prices is not dropped (see the discussion in the paragraph 3.1.B).

Another observation we can derive from Theorem 3.3.1 is that almost all the relevant properties of the commodity space are to be derived from the characteristics of the abstract market we wish to study. Intuitively, this means that it is pointless to describe the commodities and price systems without specifying the actors that animate the economy. As an example, it is pointless to talk about a "natural order"on the commodity (and price) space unless we know that all positive bundles are desirable by all the agents. In the same way, there is no reason to endow E with a specific topology as long as we do not know for which price systems the total demand and supply are well defined . Even the notion of free disposal equilibrium should depend on agents' reactions to the increase of some specific commodities. By pushing these observations to their extremes, one could even argue that the dimension of the commodity space itself is dependent on the agents' behaviour as it makes no sense to differentiate among bundles if agents' preferences are not sophisticated enough to distinguish among them. In an extreme exemplification, it would be useless to distinguish among a large class of physical goods in a market where agents are indifferent between all of them.

Chapter 4

Competitive economies in coalitional form

A fundamental feature of competitive economies is that individual agents have no incentives to form alliances in order to affect the outcome of the trades. Every large coalition is therefore unstable and can be easily broken down into many, equally powerful subgroups. In other words, in a competitive economy we expect small coalitions to be equally effective in opposing or objecting any unbalanced distribution of resources.

In the light of the above, an important issue in modelling competitive economies is to give a precise representation of the so-called economic weight, which is a measure of the influence that each coalition can exert on the trades. We study this problem in this chapter. To facilitate this analysis, we will consider an economic model in which every actor unable to influence the economic activity is ignored and the attention moves from individual agents to coalitions. This idea, that was introduced in Vind (1964), is the foundation of what is often called the *coalitional representation* of competitive economies.

Concretely, our model will extend the one presented in Urbinati (2019) and will be articulated around these four main ingredients:

1. The *infinite number of commodities*: we will consider an infinite dimensional, linear space E as space of commodities and endow it with the locally convex topology that naturally follows from the commodity-price duality.

2. The *finitely additive coalitional representation* of the economy: coalitions will be defined as the elements of an abstract Boolean ring $\mathcal{R}$ while allocations will be represented by finitely additive vector measures $\alpha\colon \mathcal{R} \to E_+$. Individual agents will therefore disappear from the model.

3. The *topological description of the economic weight*: we will endow $\mathcal{R}$ with a

ring-topology u which will be the smallest one making the initial endowment a continuous function. Since u "regulates" the economic interactions within coalitions, u will be taken as a non-numerical description of how the economic weight is distributed in $\mathcal{R}$.

4. The *topological extension of Lyapunov's Theorem*: using the results proved in Chapter 2, we will introduce a condition on $(\mathcal{R}, u)$ that ensures the convexity of the range of every allocation. This property, which mimics the convexifying effect in Aumann's non-atomic model, will allow us to formulate a condition for competitive markets which, in some sense, replicates the 'many more agents than commodities' condition formulated in Rustichini and Yannelis (1991).

The chapter is organized as follows. In Section 4.1 we describe the economic model, in 4.2 we introduce our topological description of the economic weight and use the results in Chapter 2 to formulate a condition for perfect competition.

Last in 4.3 we prove that, when the conditions for perfect competition are met, every allocation outside the core can be blocked by arbitrarily small and arbitrarily large coalitions, as in the celebrated Theorems of Schmeidler and Vind (respectively in (1972) and (1972)). By using the topology u to describe 'small' and 'large' coalitions we will argue that these properties of competitive economies are essentially topological.

4.1 The economic model

This section is devoted to the description of an exchange economy $\mathcal{E}$ with infinitely many agents and commodities. We mainly adapt the abstract coalitional approach introduced in Armstrong and Richter (1984) to obtain a finitely additive economy with an infinite dimensional locally convex space of commodities. At this stage we will try not to make any restrictive assumption on the model so as to allow comparisons with other popular representations of exchange economies.

We will represent the commodity-price duality via a dual pair (E, E^*) of infinite dimensional ordered linear spaces, where the positive orthant of E, denoted by E_+, stands for the spaces of commodity bundles while $E^*_+ \setminus \{0\}$ for the set of prices. As usual, for $v \in E_+$ and $p \in E^*_+$ we write $\langle p, v \rangle$ or $p(v)$ to denote the value of the bundle v at price p. Moreover, we let E be endowed with any linear topology consistent with the duality and reserve the writing $\sigma(E, E^*)$ (respectively $\sigma(E^*, E)$) to denote the *weak topology* induced by E^* on E (the *weak* topology* on E^* induced by E).

An *exchange economy in coalitional form* with commodity space E, or simply *coalitional exchange economy*, will be a description of all of the coalitions that are able to take part in the trading activity, the set of possible ways to allocate the totality of resources and coalitions' preferences toward the different allocations. The goal of the rest of this section is to formalize and study all of the notions mentioned above and introduce some of the assumptions that will be needed for further applications.

4.1.A Coalitions, allocations and consumption

As mentioned, we will consider a model in which coalitions, instead of individual agents, are taken as the primitive decision makers of the economy. The natural way to do so is to describe coalitions as the elements of an abstract Boolean ring $\mathcal{R}$ where the 0 element represents the empty coalition, $\leq$ the natural order on $\mathcal{R}$ and the usual operations of symmetric difference (sum), infimum (multiplication), supremum and set difference are respectively denoted by $\triangle$, $\wedge$, $\vee$ and $\smallsetminus$. When $\mathcal{R}$ is a Boolean algebra we will denote its unit by e and refer to it as the *grand coalition*. Even if $\mathcal{R}$ is a purely abstract object, by the Stone representation Theorem we can identify $\mathcal{R}$ with a ring of subsets of some set of agents $T \neq \varnothing$ so that each coalition can be thought as a group of agents taking a joint decision. The following definitions are standard.

Definition 4.1.1. *We will call* coalitions of the economy *the elements of Boolean ring $\mathcal{R}$. For any $a \in \mathcal{R}$, all the elements of $\mathcal{R}_a := \{b \in \mathcal{R} : \ b \leq a\}$ will be called* sub-coalitions *of a.*

Intuitively, the agents in each coalition will rank by unanimity the different possible redistributions of resources. Therefore we shall consider allocations, rather than the single consumption bundles, as the formal objects of coalitions' choices and include in the description of the model a set $\mathcal{H}$ as a specification of all the consumption plans that can be chosen by coalitions. Formally, a *consumption plan*, or *allocation*, is represented as a vector measure $\mu: \mathcal{R} \to E_+$ with the intuition that $\mu(a)$ is the bundle assigned to the coalition a through μ.

We will assume that all of the resources that are exchanged in the economic activity are initially owned by the coalitions. Therefore, we will give a special importance to a specific allocation $\nu \in \mathcal{H}$ which we call *initial endowment*. Since we want the total amount of resources available in the economy to be bounded, we require that the initial endowment ν has a totally bounded range which, in our framework, is equivalent with asking that ν is an exhaustive measure[1]. An allocation $\mu \in \mathcal{H}$ is said to be *attainable* by the coalition $a \in \mathcal{R}$ if all the resources initially owned by a are redistributed through μ among a and its sub-coalitions, i.e. if $\mu(a) = \nu(a)$. When $\mathcal{R}$ is a Boolean algebra we will call *feasible* an allocation that is attainable by the grand coalition.

[1] i.e. $\nu(a_n) \to 0$ for all sequences $(a_n)_n \subset \mathcal{R}$ of pairwise disjoint coalitions (see (Diestel and Uhl, 1977, Corollary 18.1.I)).

For any allocation μ and coalition a the measure $\mu_a \colon x \mapsto \mu(a \wedge x)$, for $x \in \mathcal{R}$, is to be thought as the way in which the agents in a reallocate among themselves the resources that are listed in $\mu(a)$. In this perspective, the set $\mathcal{H}_a \coloneqq \{\mu_a : \mu \in \mathcal{H}\}$ is the set of allocations on which the coalition a has some influence and represents the consumption plans that depend on a alone. We can therefore say that a coalition a is *null* if $\mathcal{H}_a$ is the zero measure, which means that a has no influence at all.

Following Armstrong and Richter's axiomatic approach, we will consider only sets of allocations that are *spliceable cones*.

Definition 4.1.2. *A* spliceable cone of allocations *is a set $\mathcal{H}$ of measures on $\mathcal{R}$ with values in E_+ such that:*

$H1:$ *$\mathcal{H}$ is a cone, i.e. $\alpha + \beta \in \mathcal{H}$ and $t\alpha \in \mathcal{H}$ for every $\alpha, \beta \in \mathcal{H}$ and $t > 0$,*

$H2:$ *$\mathcal{H}$ is closed under splicing, i.e. for every $a \in \mathcal{R}$ and $\alpha, \beta \in \mathcal{H}$ the measure defined by $\eta(x) \coloneqq \alpha(x \wedge a) + \beta(x \smallsetminus a)$, for $x \in \mathcal{R}$, is an allocation in $\mathcal{H}$,*

$H3:$ *for every $\alpha \in \mathcal{H}$, $v \in E_+$ and every $p \in E_+^*$ the measure defined by $\eta(x) \coloneqq \alpha(x) + v\langle p, \nu(x)\rangle$, for $x \in \mathcal{R}$, belongs to $\mathcal{H}$.*

We will say that a coalition a is null *if $\mathcal{H}_a = \{\alpha_a : \alpha \in \mathcal{H}\}$ is the singleton $\{0\}$.*

Condition $H2$ is a way to ensure that coalitions can act autonomously: if α and β are two allocations in $\mathcal{H}$, any coalition a can conceive both the distributions α_a and β_a independently from how resources are allocated outside a. This is a fundamental feature of competitive models.

On the space $a(\mathcal{R}, E)$ of all E-valued measures defined on $\mathcal{R}$ it is possible to define a locally convex topology that follows directly from the commodity-price duality. In a market regulated by a price system $p \in E_+^*$, the total worth of an allocation μ is given by the p-variation of μ, i.e. by:

$$|\mu|_p \coloneqq \sup_{\pi \in \Pi_{\mathcal{R}}} \sum_{a \in \pi} |\langle p, \mu(a)\rangle|$$

where $\Pi_{\mathcal{R}}$ is the family of all finite subsets of pairwise disjoint elements of $\mathcal{R}$.[2] It is clear that the higher $|\mu|_p$ is, the more significant the allocation μ will be in our market. We follow this intuition and endow $a(\mathcal{R}, E)$ with the topology ρ induced by all of the pseudo-norms $|\cdot|_p$ with p ranging in E_+^*. This will be a locally convex topology consistent with the pointwise-ordering and finer than the topology of the pointwise weak-convergence on $a(\mathcal{R}, E)$. With an abuse of notation, we will write ρ also for the subspace topology induced on $\mathcal{H}$ and call $(\mathcal{H}, \rho)$ the *space of allocations* of our economy.

[2] In the specific case where μ takes values in E_+ and $p \in E_+^*$, $|\mu|_p = \sup\{\langle p, \mu(a)\rangle : a \in \mathcal{R}\}$.

Remark 4.1.3. An alternative method to describe the possible ways to allocate resources in the economy is through a *consumption set correspondence* $X: \mathcal{R} \twoheadrightarrow E_+$ that maps every $a \in \mathcal{R}$ into the non-empty set $X(a) \subset E_+$ of the bundles that she can own or consume on an aggregate level. Intuitively, if a and b are disjoint coalitions and $v, w \in E_+$ are bundles that are owned by a and b respectively, we would expect $v+w$ to be consumable by the joint coalition $a \vee b$, i.e. $v+w \in X(a \vee b)$. Thus we shall assume that X is non-empty-valued, maps the empty coalition in the set $\{0\}$ and it is finitely additive in the sense that $X(a) + X(b) = X(a \vee b)$ whenever a and b are disjoint coalitions.

In the presence of a consumption set correspondence X it is always possible to define the set of allocations as the family of all additive selections of X, i.e. the measures $\mu: \mathcal{R} \to E$ that are such that $\mu(a) \in X(a)$ for every $a \in \mathcal{R}$. On the other hand, given a set of allocations $\mathcal{M}$, the correspondence $X: \mathcal{R} \twoheadrightarrow E_+$ that assigns the set $X(a) := \{\mu(a) : \ \mu \in \mathcal{M}\}$ to each $a \in \mathcal{R}$ is a consumption plan correspondence provided that $\mathcal{M}$ is closed under splicing, i.e. if for every $a \in \mathcal{R}$ and $\alpha, \beta \in \mathcal{M}$ the measure $\eta: \mathcal{R} \to E$ defined by $\eta(x) := \alpha_a(x) + \beta(x \setminus a)$, for $x \in \mathcal{R}$, is an allocation in $\mathcal{M}$.

This approach was introduced and studied in Cornwall (1969) in a σ-additive setting but it can easily be adapted to our model, as it has been done in Basile (1993) and Donnini and Graziano (2009).

Example 4.1.4. With the axiomatic definition of a spliceable cone of allocations given in 4.1.2 we can include a large variety of cases considered in the literature. We leave to (Armstrong and Richter, 1986, pg. 137) and (Cheng, 1991, Section 5) for classical examples both in the finite and the infinite dimensional settings.

In most of the situations, however, we usually work with a spliceable cone of allocations that is formed by exhaustive measures alone. This is always the case in finite-dimensional coalitional models (where order-boundedness and exhaustivity are equivalent concepts) and it is often assumed, mostly implicity, in the infinite dimensional settings. We list here a few examples:

- When every allocation is assumed to be absolutely continuous with respect to a σ-additive, or bounded, measure $\lambda: \mathcal{R} \to [0,1]$. This is the case of Zame (1986), Greinecker and Podczeck (2013), where the measures are asked to be σ-additive, but also of Cheng (1991), Donnini and Graziano (2009) in the finitely additive framework.

- When $\mathcal{R}$ is a Boolean algebra and E is a convex-solid Riesz space. In this case every order-bounded set is topologically bounded and therefore, since the range of every E_+-valued measure defined on $\mathcal{R}$ is contained in an order-interval, every allocation has a topologically bounded range (see (Aliprantis and Burkinshaw, 2006, Section 3.3)). But then any of the following conditions will imply that all the possible allocations are exhaustive:

- E is a separable space and $\mathcal{R}$ is σ-complete (see Labuda's Theorem in Labuda (1975)).
- There are no sub-spaces of E topologically isomorphic to ℓ^∞ and $\mathcal{R}$ is σ-complete.
- There are no sub-spaces of E topologically isomorphic to c_0.

The last two conditions are known with the name of Diestel-Faires Theorems and can be found, for the Banach space-valued case, in Diestel and Faires (1974) and in Drewnowski (1975, 1976b) for the extension to locally convex spaces (see also the discussion in (Diestel and Uhl, 1977, pg. 34)).

4.1.B Preferences

If $\mathcal{R}$ is a Boolean ring of coalitions and $\mathcal{H}$ a spliceable cone of allocations, a *coalitional preference profile* on $(\mathcal{R},\mathcal{H})$ is a system $\succ := (\succ_a)_{a\in\mathcal{R}}$ of irreflexive binary relations on $\mathcal{H}$ that are not necessarily transitive or complete. When $a \in \mathcal{R}$ and $\alpha,\beta \in \mathcal{H}$, $\alpha \succ_a \beta$ reads that almost every agent in a prefers the bundle she receives through the allocation α rather than β. Once $\succ$ is defined we can introduce weak preference relations $(\succcurlyeq_a)_{a\in\mathcal{R}}$ on $\mathcal{H}$ by setting $\alpha \succcurlyeq_a \beta$ if and only if there is no non-null coalition $a' \leq a$ such that $\beta \succ_{a'} \alpha$.

The following definition is standard in coalitional models of exchange economies.

Definition 4.1.5. *A* coalitional preference profile *on $(\mathcal{R},\mathcal{H})$ is a system $\succ := (\succ_a)_{a\in\mathcal{R}}$ of irreflexive binary relations on $\mathcal{H}$ that are such that, for every $\alpha,\beta \in \mathcal{H}$ and $a \in \mathcal{R}$:*

$P1$: the set $\{x \in \mathcal{R} : \alpha \succ_x \beta\}$ is an ideal in $\mathcal{R}$,

$P2$: $\alpha \succ_a \beta$ if and only if $\alpha_a \succ_a \beta$ if and only if $\alpha \succ_a \beta_a$,

Condition $P1$, sometimes called *ideal*, is necessary to ensure that the formation of coalitions is totally voluntary as it guarantees that a coalition a considers an allocation α more profitable than β if and only if all of its sub-coalitions agree. Condition $P2$, called *selfish*, states that the preferences of coalitions depend exclusively on their own consumptions and it is needed to rule out the presence of externalities on consumption.

If $\succ := (\succ_a)_{a\in\mathcal{R}}$ is a coalitional preference profile, for every $a \in \mathcal{R}$ and $\alpha \in \mathcal{H}$ it is well defined the *upper contour set*:

$$P_\alpha(a) := \{\beta(a) : \beta \in \mathcal{H}, \beta \succ_a \alpha\} \subseteq E_+. \tag{4.1}$$

The following proposition is a direct consequences of conditions $P1$ and $P2$ in the definition of coalitional preference profile.

Proposition 4.1.6. *Let* $\succ := (\succ_a)_{a\in\mathcal{R}}$ *be a coalitional preference profile on* $(\mathcal{R},\mathcal{H})$*. Then, for every* $\alpha \in \mathcal{H}$ *the upper countour set correspondence* $x \mapsto P_\alpha(x)$ *is a finitely additive correspondence on* $\mathcal{R}$ *that is rich in selections in* $\mathcal{H}$*.*

Proof. Let us fix $\alpha \in \mathcal{H}$ then take two disjoint $a_1, a_2 \in \mathcal{R}$ and $v_1, v_2 \in E_+$ so that $v_i \in P_\alpha(a_i)$ for $i = 1, 2$. We need to show that $v := v_1 + v_2$ can be written as $\gamma(a)$, where $a := a_1 \vee a_2$ and γ is an allocation in $\mathcal{H}$ such that $\gamma \succ_a \alpha$.

For $i = 1, 2$ let $\beta_i \in \mathcal{H}$ be such that $\beta_i \succ_{a_i} \alpha$ and $\beta_i(a_i) = v_i$, then call γ the measure that maps each $x \in \mathcal{R}$ into $\beta_1(x \wedge a) + \beta_2(x \setminus a)$. This way, being $\mathcal{H}$ closed under the splicing operation, γ belongs to $\mathcal{H}$ and it is such that $\gamma(a) = \beta_1(a_1) + \beta_2(a_2) = v$. To show that $\gamma \succ_a \alpha$ it is enough to observe that, for $i = 1, 2$, $\gamma(x) = \beta_i(x)$ for each $x \le a_i$ and so, by Condition $P2$, $\gamma \succ_{a_i} \beta_i$. But then $\gamma \succ_a \alpha$ follows from condition $P1$. □

We stress that no assumptions on completeness nor transitivity of preferences are necessary. We might, however, need to consider the case in which preferences are somehow consistent with the topological structure on the commodity space E and, consequently, with the commodity-price duality. Following Armstrong and Richter (1984) we will introduce the following definition to express the idea that small changes in the bundle assigned to a coalition do not affect significantly its choices.

Definition 4.1.7. *A coalitional preference profile* $\succ := (\succ_a)_{a\in\mathcal{R}}$ *is* lower semi-continuous *if, for every* $a \in \mathcal{R}$ *and* $\alpha \in \mathcal{H}$*, the upper contour set* $P_\alpha(a)$ *is relatively open in* E_+

In the definition above we are simply requiring that if a coalition $a \in \mathcal{R}$ prefers an allocation β to α and $v \in E_+$ is sufficiently similar to $\beta(a)$ then there is a way to allocate the resources listed in v among the sub-coalitions of a that is still preferred to α by a. On this line, we might need to express the idea that a coalition can always be made better off with small changes in its consumption. This notion is formalized with the following definition.

Definition 4.1.8. *A coalitional preference profile* $\succ := (\succ_a)_{a\in\mathcal{R}}$ *is* weakly locally non-satiated *if, for every non-null* $a \in \mathcal{R}$ *and* $\alpha \in \mathcal{H}$*, the bundle* $\alpha(a)$ *belongs to the weak closure of* $P_\alpha(a)$*.*

The topological considerations behind the definitions 4.1.7 and 4.1.8 are relative to the space of commodities E, not to the space of allocations. With a different approach one could call the coalitional preference profile $\succ$ *continuous* if the set

$\{(\beta,\alpha): \beta \succcurlyeq_a \alpha\}$ is closed for every $a \in \mathcal{R}$ (see for, for example, Vind (1964), Cornwall (1969) and Cheng (1991)). Similarly, one could call $\succ$ *coalitionwise locally non-satiated* if every α belongs to the closure of $\{\beta \in \mathcal{H}: \beta \succ_a \alpha\}$ for every non-null $a \in \mathcal{R}$. It is clear that if a coalitional preference profile $\succ$ is coalitionwise locally non-satiated then it is weakly locally non-satiated.

Last we analyse the notion of monotonicity of preferences. As discussed in Cornwall (1969), given $\alpha, \beta \in \mathcal{H}$ and a non-null $a \in \mathcal{R}$ we say that $\alpha \geq_a^* \beta$ if and only if $\alpha(x) > \beta(x)$ for every non-null $x \leq a$.

Definition 4.1.9. *A coalitional preference profile $\succ := (\succ_a)_{a\in\mathcal{R}}$ is* monotone *if for every non-null $a \in \mathcal{R}$, $\alpha \geq_a^* \beta$ implies $\alpha \succ_a \beta$ for every $\alpha, \beta \in \mathcal{H}$.*

There are of course other ways in which a coalitional preference profile may depend on the order relation on E and, indirectly, on $\mathcal{H}$. We shall discuss some of them relative to Assumption 4 in Section 4.3.

4.1.C The general model

In the light of all the notions introduced above we define an *exchange economy in coalitional form* on the commodity space E, a list:

$$\mathcal{E} := \langle \mathcal{R}, \mathcal{H}, \nu, \succ \rangle$$

where:

- $\mathcal{R}$ is the Boolean ring representing all the coalitions in the economy,
- $\mathcal{H}$ is a spliceable cone of allocations,
- $\nu \in \mathcal{H}$ is an exhaustive measure representing the *initial endowment*,
- $\succ := (\succ_a)_{a\in\mathcal{R}}$ is a coalitional preference profile on $(\mathcal{R}, \mathcal{H})$.

In what follows, we shall also refer to $\mathcal{E}$ as a *coalitional exchange economy*.

We remark that the model we have described stands out for two main reasons: the finitely additive environment in which the sets of coalitions and assignments are described and the generality of the commodity space on whose topological structure we made no restrictions. A third peculiarity of this model is that we did not impose that the initial endowment of resources ν had to be absolutely continuous with respect to a scalar measure $\lambda: \mathcal{R} \to \mathbb{R}_+$, a condition that is often required in coalitional economies (as in Vind (1964); Armstrong and Richter (1984) or Cheng (1991), where infinitely many commodities are considered). We will discuss the details (and the consequences) of this choice in the paragraph 4.2.B.

Remark 4.1.10. A significant contribution towards a more abstract formulation of coalitional economies was given in the research memorandum by Cornwall (1968) where he proposed to represent coalitions as the elements of a measure algebra instead of the measurable sets of a probability space[3] Cornwall's approach can therefore be seen as an attempt to consider the economy that one would obtain by quotiening an economy "a' la Vind" by the ideal of economically negligible coalitions, when such an operation was well defined. See also (Cornwall, 1969, Remark at pg. 355).

4.2 The space of coalitions

As mentioned in the introduction, by *economic weight* we mean the capacity of a coalition to take part and influence the trades. Loosely speaking, in a pure exchange framework coalitions with "better" initial endowment will more likely play a significant role in the economic activity and therefore have a larger economic weight. Following this intuition, we should measure how powerful a coalition $a \in \mathcal{R}$ is in terms of what she will be able to attain if she decides to deviate from the rest of the economy and act independently. In other words, we need to focus on the set:

$$\nu(\mathcal{R}_a) := \{\nu(x) : \ x \in \mathcal{R}, \ x \leq a\}$$

which is the collection of all bundles initially owned by a and its sub-coalitions. In the light of this, the smaller the set $\nu(\mathcal{R}_a)$ is, the "weaker" we expect the coalition a to be.

With the observations above we focus on the structure induced on $\mathcal{R}$ by the correspondence $a \mapsto \nu(\mathcal{R}_a)$. Precisely, if we consider $\mathcal{R}$ as a ring, there is a natural topology on $\mathcal{R}$ whose 0-neighborhood system is generated by the sets of the form $\{x \in \mathcal{R} : \ \nu(\mathcal{R}_x) \subset U\}$ where U ranges over a 0-neighborhood base of (E, ρ). This topology, which is the coarsest ring topology on $\mathcal{R}$ making ν a continuous function, is often called ν-topology and belongs to the class of *Frechét-Nikodym topologies*.

Definition 4.2.1. *We call* distribution of the economic weight *the ν-topology on $\mathcal{R}$ and denote it by the letter u.*

With the definition given, we can always refer to the *space of coalitions* in the economy $\mathcal{E}$ as the topological Boolean ring $(\mathcal{R}, u)$. Each allocation can therefore be seen as a E-valued function on the topological space $(\mathcal{R}, u)$ that may, or may not, be continuous with respect to u. For this reason, for the rest of these notes we will make the following assumption on the model.

[3]Cornwall does not explicitly mention measure algebras, however he assumes that coalitions form a Boolean σ-algebra Σ that supports a strictly positive σ-additive measure, i.e. a σ-additive probability measure $\mu: \Sigma \to [0,1]$ that is such that $\mu(F) = 0$ if and only if $F = 0$.

Assumption 1. *Each $\alpha \in \mathcal{H}$ is absolutely continuous*[4] *with respect to ν.*

With the Assumption above we are restricting the class of allocations to those that are "topologically dominated" by the initial endowment so that $\mathcal{H}$ and $\mathcal{H}_u$ coincide and every allocation is automatically exhaustive. Similar Assumptions are explicitly made in Vind (1964), Cheng (1991), Basile (1993) and Donnini and Graziano (2009) and implicitly in Cornwall (1969) (with conditions $(X.3)$ and $(Y.3)$).

Remark 4.2.2. As it has been mentioned in the Example 4.1.4, the requirement that every allocation is exhaustive is not very restrictive in most of the economic models usually employed in the literature. As a further way of illustration, let us consider the case of an economy $\mathcal{E}$ where E is a symmetric Riesz space and $\mathcal{R}$ is a Boolean algebra: in this case the range of every feasible allocation α is relatively weakly compact as it must lie in the interval $[0, \sup \nu(\mathcal{R})]$ (which is a weakly compact set by (Aliprantis and Border, 2006, Theorem 8.60)). This means that α is exhaustive by (Diestel and Uhl, 1977, Corollary 18.1.I).

Remark 4.2.3. Under mild assumptions it can be shown that, even in models in which Assumption 1 is violated and not every allocation is absolutely continuous with respect to ν, most of the economic analysis can be reduced to the study of the allocations in $\mathcal{H}_u$ of all u-continuous allocations in $\mathcal{H}$. This is the approach used, for example, in Richter (1971) and Armstrong and Richter (1984). In these articles, where E is assumed to have a finite dimension and ν to be non-atomic, it is proved that for monotone preference profiles every core allocation belongs to $\mathcal{H}_u$ and if $\alpha \in \mathcal{H}_u$ is blocked by some $\beta \in \mathcal{H}$ it is also blocked by an allocation in $\mathcal{H}_u$ (see for example (Richter, 1971, Propositions 1, 3) or (Armstrong and Richter, 1984, Proposition 1)). In other words, the cores of the economies $\mathcal{E} = (\mathcal{R}, \mathcal{H}, \nu, \succ)$ and $\mathcal{E}' = (\mathcal{R}, \mathcal{H}_u, \nu, \succ)$ coincide.

In our framework it is possible to pursue the same strategy used in Richter (1971) by assuming that $(\mathcal{R}, u)$ is uniformly complete and that every $\alpha \in \mathcal{H}$ is exhaustive. In fact, under these assumptions, the Lebesgue decomposition Theorem used in (Richter, 1971, Proposition 1) can be replaced by (Weber, 2002, Theorem 8.2).

Remark 4.2.4. The idea that the economic weight of coalitions can be studied as a topology on $\mathcal{R}$, so that coalitions with a small influence are the 'topologically small' ones, was introduced in (Urbinati, 2019, Section 4). It is however not the first time that considerations on the influence of coalitions are expressed in a purely topological form: in Basile (1995), for example, a FN-topology is defined on the

[4]Recall that for two $\alpha, \beta \in a(\mathcal{R}, E)$ we say that α is *absolutely continuous* with respect to β if $\alpha(x_i) \to 0$ for every net $(x_i)_{i \in \mathcal{I}}$ in $\mathcal{R}$ such that $\beta(x_i) \to 0$.

coalitions of a voting game and used to study the political influence of each group of voters.

In a way, the notion of economic weight we defined here is not too distant from that of *diameter* introduced in Grodal (1972). See the Remark 4.2.7 for a broader discussion.

4.2.A An equivalent topology on coalitions

A possible alternative way of defining the economic weight of coalitions, more closely related to the commodity-price duality, is to measure the economic potential of each coalition under all the possible price systems that can emerge. Formally, one could associate to every price system $p \in E^*_+ \setminus \{0\}$ the positive measure:

$$\nu_p : a \mapsto \sup\{\langle p, \nu(x)\rangle : \ x \in \mathcal{R}_a\} \tag{4.2}$$

which assign to every coalition $a \in \mathcal{R}$ the maximum possible income she can attain at price p if she deviates from the rest of the economy. With this idea, we are brought to say that the economic power of a coalition $a \in \mathcal{R}$ should be "small" whenever $\nu_p(a)$ is small for some $p \in E^*_+ \setminus \{0\}$. Quite surprisingly, this approach can be shown to be equivalent with the one showed above thanks to the following proposition.

Proposition 4.2.5 (Corollary 7.3 in Weber (2002))**.** *For any net of coalitions x_i, $i \in \mathcal{I}$, and x_0 in the Boolean ring $\mathcal{R}$ it is equivalent to say that:*

1. *the net x_i converges to x_0 in $(\mathcal{R}, u)$,*

2. *for all $p \in E^*_+$, the net $\nu_p(x_i)$ converges to $\nu_p(x_0)$.*

In other words, the topology induced on $\mathcal{R}$ by the collection ν_p, $p \in E^*_+$, coincides with the ν-topology and hence with the *distribution of economic weight* as defined in definition 4.2.1. Our notion of economic weight can therefore be directly derived from the commodity-price duality.

There are a few useful consequences of Proposition 4.2.5: the first is that the topology u on $\mathcal{R}$ does not really depend on the specific topology chosen on E but only on the duality (E, E^*), the second is that u is fully described by the class of u-valued probability measures that can be defined on $\mathcal{R}$.

Corollary 4.2.6. *Let U be a 0-neighborhood in $(\mathcal{R}, u)$. Then there are a finite set $\lambda_i : \mathcal{R} \to [0,1]$, $i = 1, \dots, n$, of u-continuous probability measures and an $\varepsilon > 0$ such that:*

$$\{x \in \mathcal{R} : \ \lambda_i(x) \leq \varepsilon \ \forall i \leq n\} \subseteq U.$$

Proof. By Proposition 4.2.5 the topology u is fully determined by the semi-metrics $\nu_p\colon \mathcal{R} \to \mathbb{R}_+$, with p ranging over E^*_+. Since U is a 0-neighborhood in $(\mathcal{R}, u)$, there must be a finite set $p_1, \dots, p_n \in E^*_+$ and $\delta > 0$ such that:

$$\{x \in \mathcal{R} :\ \nu_{p_i}(x) \le \delta,\ \forall i \le n\} \subseteq U.$$

We stress that, for every $i \le n$, being ν_{p_i} exhaustive we have $\sup \nu_{p_i} < \infty$. Without loss of generality we can therefore assume that $\sup \nu_{p_i} > 0$ for every $i \le n$.

To conclude the proof it is enough to define for each $i \le n$ the probability measure λ_i that maps each $x \in \mathcal{R}$ into $\nu_{p_i}(x)/\sup \nu_{p_i}$ and put $\varepsilon := \sup\{\delta/\sup \nu_{p_i} : i \le n\}$. □

Remark 4.2.7. When the space of commodities E is equipped with a norm $\|\cdot\|$, it is common to define the *diameter* the function $a \mapsto |a|_\nu := \sup\{\|x\| :\ x \in \mathcal{R}_a\}$, $a \in \mathcal{R}$ (see for example Grodal (1972) or Hervés-Beloso et al. (2000); Evren and Hüsseinov (2008) for the case in which the dimension of E is infinite). Since every $p \in E^*_+$ defines on E the semi-norm $x \mapsto |\langle p, x\rangle|$, we could see the function ν_p as a special case of a diameter function.

It is also worth stressing that, when E is normed, the diameter function $|\cdot|_\nu$ provides a sort of numerical description of the economic weight in the sense that the sets $U_\varepsilon := \{x \in \mathcal{R} :\ |x|_\nu \le \varepsilon\}$, with $\varepsilon > 0$, form a 0-neighborhood base of $(\mathcal{R}, u)$. All this implies that a coalition has "small" diameter if and only if it belongs to a "small" open set in the space of coalitions $(\mathcal{R}, u)$.

4.2.B Numerical evaluations of the economic weight

A significant situation is when the initial endowment ν is a *controlled measure* i.e. it is absolutely continuous with respect to a certain scalar measure $\lambda\colon \mathcal{R} \to \mathbb{R}_+$ (then we call λ a *control measure* for ν). If this is the case, the topology of the space of coalitions $(\mathcal{R}, u)$ is fully determined by the semi-metric $d_\lambda\colon (x, y) \mapsto \lambda(x \bigtriangleup y)$, for $x, y \in \mathcal{R}$, and $\lambda(x)$ indicates the "distance" of coalition x from the empty coalition. It seems therefore reasonable to think of λ as a measure of how powerful each coalition is, justifying the following definition.

Definition 4.2.8. *We call* numerical evaluation of the economic weight *any scalar measure $\lambda\colon \mathcal{R} \to \mathbb{R}_+$ such that ν is absolutely continuous with respect to λ.*

In general, not all vector measures with values in a locally convex space are controlled, thus a priori we cannot tell whether or not we can find a numerical evaluation of the economic weight in the economy we are considering. However, when the space of commodities E is a Banach space, or more generally a metrizable space, a slight generalization of a theorem by Bartle, Dunford and Schwartz guarantees the existence of a control measure for ν (Weber, 2002, Corollary 7.5). This means that in every economy with a metrizable space of commodities we can always describe the space of coalitions via the pair $(\mathcal{R}, \lambda)$, where λ is a control measure for ν.

Remark 4.2.9. In most of the literature on coalitional economies it is common to describe the space of coalitions as a measurable space $(\mathcal{R}, \lambda)$, where $\mathcal{R}$ is the ring of coalitions and $\lambda: \mathcal{R} \to \mathbb{R}_+$ is a control measure of the initial endowment. For the reasons explained in this section, such a measure λ is often called *measure of the economic power* of coalitions. However, as we have pointed out earlier, this approach, while fully justified for economies with a Banach space of commodities, puts a strong limitation on the set of allocations and on the possible choices of initial endowment. A limitation which does not seem to have a real economical justification.

It is to avoid this kind of eventuality that we decided to follow a complete different line and to define the space of coalitions as a topological Boolean ring. See also the discussion in Sections 2.2.A and 2.2.B in Chapter 2.

Remark 4.2.10. In the specific case in which E is a Banach space, Rybakov's Theorem ensures the existence of a $p \in E_+^*$ such that ν is absolutely continuous with respect to the measure $p \circ \nu$, meaning that $p \circ \nu$ is a numerical evaluation of the economic weight (see (Diestel and Uhl, 1977, Theorem IX.2.2) and (Weber, 2002, Corollary 7.5) for the finitely additive approach). This shows how, in this particular framework, the condition $H3$ given in the Definition 4.1.2 implies the conditions $(S2)$ in Armstrong and Richter (1984) and $H.1$ in Cheng (1991).

The observation above implies, in particular, that if there is a $p \in E_+^*$ such that $\nu \ll p \circ \nu$ (i.e. a Rybakov functional for ν) it is always possible to distribute a bundle $v \in E_+ \setminus \{0\}$ among all the non-null coalitions in the economy. In fact, by condition $H3$ in 4.1.2 the relation $x \mapsto v(\langle p, \nu(x) \rangle)$ defines an allocation in $\mathcal{H}$ that assigns a non-zero bundle to each non-null $x \in \mathcal{R}$.

In our general framework, when E is a locally convex space and ν does not have any control measure this property might not hold. It could in fact be impossible to allocate a positive amount of resources among all the non-null coalitions in the economy. We will see in Section 4.3 how this affects the definitions of the blocking mechanism.

4.2.C A condition for perfect competition

When the commodity space E has a finite dimension, Aumann's notion of perfect competitiveness of the market is stated in terms of non-atomicity of the initial endowment and, consequently, of all allocations (see Aumann (1964)). We follow his idea to extend this notion to the case of infinite dimensional spaces by requiring that ν satisfies the following conditions.

Assumption 2. *$sat(\nu)$ is infinite and there is a family $\mathcal{F} \subset E^*$ separating the points of E such that $|\mathcal{F}| < sat(\nu)$.*

Once again we stress that in the finite dimensional settings Assumption 2 is equivalent to the condition of non-atomicity of allocations and therefore to Aumann's notion of perfect competitive market. Also, in line with Aumann, as a direct consequence of Theorem 2.2.7 we have that, under Assumption 2, every closed allocation has convex and weakly compact range. It is in view of this that throughout we will also assume the following:

Assumption 3. *$(\mathcal{R}, u)$ is a uniformly complete topological ring.*

Assumptions 2, 3 give us important results on convexity of preferences. Precisely, if we fix a coalition $a \in \mathcal{R}$ and an allocation $\alpha \in \mathcal{H}$ we can represent the set of bundles preferred to $\alpha(a)$ by means of the set $P_\alpha(a)$ which we claim to be convex.

Theorem 4.2.11. *Under assumptions 2 and 3 the correspondence $x \mapsto P_\alpha(x)$, for $x \in \mathcal{R}$, has convex values for every $\alpha \in \mathcal{H}$.*

Proof. Let $\alpha \in \mathcal{H}$. By Proposition 4.1.6, P_α is a finitely additive correspondence rich in selections that belongs to $\mathcal{H}$. But then, under Assumptions 2 and 3, P_α satisfies all of the requirements of Theorem 2.2.12 and, as such, it has convex values. □

Theorem 4.2.11 reflects the idea that, when the structure of the space of coalitions is much richer than that of the space of commodities there is a *convexifying effect* on preferences.

Remark 4.2.12. The conjecture that in a regime of perfect competition the aggregation among agents could have a convexifying effect on their total demand (even when individual preferences are not convex) was proved in Aumann (1964). Even before that, Uzawa in (1962a) discussed a similar convexifying effect on production functions as a consequence of the aggregation of infinitely many negligible producers, a result which he attributes to Farrell (who proved it in (1959) in a simple case) and to Hurwicz and Uzawa (who had presented it in the unpublished manuscript Hurwicz and Uzawa (shed) and then in (1977)).

It was Vind, however, who first showed in (1964) how the convexifying effect on preferences proved by Aumann is a direct consequence of Lyapunov's Theorem on the range of vector measures. In this perspective, Theorem 4.2.11 can be seen as the natural extension of Vind's intuition to the infinite dimensional setting in which Theorem 2.2.7 was proved.

Remark 4.2.13. It shall be stressed that the Theorem 4.2.11 does not imply that the coalitional preference profile $\succ$ is convex, i.e. that $\{\beta \in \mathcal{H} : \ \beta \succ_a \alpha\}$ is a convex subset of $\mathcal{H}$ for every $\alpha, \beta \in \mathcal{H}$ and $a \in \mathcal{R}$. In general, the convexity of a coalitional preference profile is a much stronger property than the convexity of all the relative upper contour sets we have defined in Equation 4.1. Under very common assumptions it can even be shown that if a coalitional preference profile $\succ$ is derived from a system of individual preferences on E then $\succ$ is convex if and only if almost every agent's preference relation is convex. This property was conjectured by Armstrong in 1984 at the Institute for Mathematics and Its Applications at the University of Minnesota and proved Mauldin (1986) for the case of economies with finitely many commodities.

What follows from Theorem 4.2.11, however, is that for every $\alpha \in \mathcal{H}$ and non null $a \in \mathcal{R}$ one has $\alpha \notin co\left(\{\beta \in \mathcal{H} : \ \beta \succ_a \alpha\}\right)$, a condition similar to those introduced by Shafer and Sonnenschein in (1975, Theorem 1) to replace the convexity of preferences in the context of abstract economies.

4.3 On the core of exchange economies

We now move our attention to the problem of determining when a coalition is capable of improving upon a given allocation. Intuitively, a coalition a can improve upon a given allocation α if it finds a more profitable way to re-allocate its own resources among its sub-coalitions. This intuition can be formalized in different ways, depending on whether we consider strict or weak preferences. For the sake of completeness, we include here some definitions that are standard in the literature.

Definition 4.3.1. *Let $\alpha, \beta \in \mathcal{H}$. We say that α is* dominated *by β if there exists a non-null coalition $a \in \mathcal{R}$ such that $\beta(a) = \nu(a)$ and $\beta \succ_a \alpha$. If this is the case we also say that a* blocks, *or* improves upon, *the allocation α via β.*

The core *is defined as the set $\mathcal{C}(\mathcal{E})$ of all feasible allocations that are not dominated.*

Loosely speaking, a coalition a will block the allocation α via β if almost every agent in a finds β more profitable than α. This condition can of course be weakened by requiring that only a significant share of agents in a prefers α to β while the remaining agents are indifferent between the two allocations. We formalize this intuition by defining a relaxed veto mechanism based on the notion of $\star$-core, as introduced in (Shitovitz, 1973, pg. 479).

Definition 4.3.2. *Let $\alpha, \beta \in \mathcal{H}$ and $a \in \mathcal{R}$. We say that a* $\star$-blocks *α via β if $\beta(a) = \nu(a)$ and there is a non-null $a' \leq a$ such that $\beta \succ_{a'} \alpha$ and $\beta(x) = \alpha(x)$ for every $x \leq a \setminus a'$.*

The $\star$-core *of the economy* $\mathcal{E}$ *is the set* $\mathcal{C}^\star(\mathcal{E})$ *of all feasible allocations that cannot be* $\star$*-blocked by any coalition.*

With the definition given, an allocation is $\star$-blocked whenever it is blocked in the ordinary sense and so the $\star$-core of the economy is a subset of $\mathcal{C}(\mathcal{E})$. The converse inclusion, in general, is not true in our environment and $\mathcal{C}(\mathcal{E})$ will usually be strictly larger than $\mathcal{C}^\star(\mathcal{E})$. It is however possible to reconcile the two notions of core under some additional assumptions.

Assumption 4. *For every non-zero* $v \in E_+$ *and every* $\alpha \in \mathcal{H}$*,* $a \in \mathcal{R}$ *we have* $\alpha(a) + v \in P_\alpha(a)$*.*

Condition 4 can be thought as a coalitional form of desiderability of positive bundles and it is a much stronger requirement than the simple monotonicity of preferences. It is equivalent to asking that, for every allocation $\alpha \in \mathcal{H}$, every non-null $a \in \mathcal{R}$ and every non-zero $v \in E_+$ there exists a $\zeta \in \mathcal{H}$ such that $\zeta \succ_a \alpha$ and $\zeta(a) = \alpha(a) + v$.

We can now prove the following.

Proposition 4.3.3. *Suppose that* $\succ$ *is lower semi-continuous, that Assumption 4 is met and let* α *be an allocation that is* $\star$*-blocked by a coalition* $a \in \mathcal{R}$*. Then* a *blocks* α *in the ordinary sense. In particular,* $\mathcal{C}^\star(\mathcal{E}) = \mathcal{C}(\mathcal{E})$*.*

Proof. Since a $\star$-blocks α we can take a $\beta \in \mathcal{H}$ and a non-null $a' \le a$ such that $\beta(a) = \nu(a)$, $\beta(x) = \alpha(x)$ for every $x \le a \setminus a'$ and $\beta \succ_{a'} \alpha$.

By definition, the bundle $\beta(a')$ must belong to the set $P_\alpha(a')$, that is open by the lower semi-continuity of preferences. We can therefore pick a $t \in (0,1)$ such that $t\beta(a')$ is still in $P_\alpha(a')$ and call $v \coloneqq (1-t)\beta(a')$. Clearly $v \in E_+$ and so, by Assumption 4, $\alpha(a \setminus a') + v \in P_\alpha(a \setminus a')$. This means in particular that there will be a $\gamma \in \mathcal{H}$ such that $\alpha(a \setminus a') + v = \gamma(a \setminus a')$ and $\gamma \succ_{a \setminus a'} \alpha$.

Let us call $\mu\colon \mathcal{R} \to E_+$ the measure that assigns to every $x \in \mathcal{R}$ the bundle:

$$\mu(x) \coloneqq \gamma(x \setminus a') + t\beta(x \wedge a').$$

The first thing we observe is that the allocation μ is attainable by coalition a. In fact:

$$\begin{aligned}\mu(a) =& \gamma(a \setminus a') + t\beta(a') = \alpha(a \setminus a') + v + t\beta(a') = \\ =& \beta(a \setminus a') + (1-t)\beta(a') + t\beta(a') = \beta(a) = \nu(a).\end{aligned}$$

Furthermore, it follows directly from the selfish property of $\succ$ (condition $P2$ in the Definition 4.1.5) that that $\mu \succ_{a \setminus a'} \alpha$ and $\mu \succ_{a'} \alpha$. But then, from the ideal property of $\succ$ (condition $P1$ in 4.1.5), $\mu \succ_a \alpha$. We conclude that α is blocked by a in the ordinary sense.

The equivalence $\mathcal{C}(\mathcal{E}) = \mathcal{C}^\star(\mathcal{E})$ is proved by observing that, by the argument above, every feasible allocation outside $\mathcal{C}^\star(\mathcal{E})$ is blocked in the ordinary sense and so the inclusion $\mathcal{C}(\mathcal{E}) \subseteq \mathcal{C}^\star(\mathcal{E})$ holds. □

Remark 4.3.4. Please observe that, in particular, the condition presented in 4 is always met when E is a Banach space and $\succ$ is lower semi-continuous and monotone. In fact, in this case, given a non-null $a \in \mathcal{R}$ and a $v \in E_+ \setminus \{0\}$, by the observations made in the Remark 4.2.10 it is always possible to find a $\beta \in \mathcal{H}$ such that $\beta(a) = v$ and $\beta(x) \neq 0$ for every non-null $x \leq a$. But then, for any $\alpha \in \mathcal{H}$, $\zeta := \alpha + \beta$ is an allocation such that $\zeta(x) > \alpha(x)$ for every non-null $x \leq a$ and so $\zeta \succ_a \alpha$ by the monotonicity of preferences. Following this argument, one could prove the claims of Proposition 4.3.3 hold whenever E is a Banach space and $\succ$ is monotone and lower semi-continuous.

It is worth mentioning that equivalence results similar to those shown in Proposition 4.3.3 were proved in (Shitovitz, 1973, Lemma 4) and (Hüsseinov, 1998, Lemma at pg. 133) in an individualistic framework and under different conditions on preferences. In particular, Husseinov was able to prove the equivalence $\mathcal{C}(\mathcal{E}) = \mathcal{C}^\star(\mathcal{E})$ by assuming only measurability, continuity and local non-satiation of preferences.

4.3.A On the veto power of small coalitions

In (1972) Schmeidler proved that in a competitive economy with a finite number of commodities any allocation that does not belong to the core can be blocked by arbitrarily small coalitions. In our framework, where it may be impossible to refer to the exact "size" of a coalition, we can still formulate Schmeidler's idea using the topological structure on the space of coalitions. Formally we shall say that a property P holds for *arbitrarily small* coalitions if and only if for every 0-neighborhood U in $(\mathcal{R}, u)$ there is a coalition $a \in U$ for which property P holds or, equivalently, if every null-coalition is a limit point of the set $\{a \in \mathcal{R} : P \text{ holds for } a\}$.

Schmeidler's Theorem can therefore be formulated as follows.

Theorem 4.3.5. *Under the Assumptions 2 and 3 let α be an allocation that is blocked by a coalition $a \in \mathcal{R}$ via a given $\beta \in \mathcal{H}$ and let U be a 0-neighborhood in $(\mathcal{R}, u)$. Then there is a coalition $b \leq a$ in U that blocks α via β.*

Using the terminology we have introduced, we can rephrase the Theorem above and claim that, under Assumptions 2 and 3, any allocation outside the core can be blocked by arbitrarily small coalitions.

Proof. If U is a 0-neighborhood in $(\mathcal{R}, u)$, by Corollary 4.2.6 we can find a finite number of u-continuous measures $\lambda_i: \mathcal{R} \to [0,1]$, $i = 1, \dots, n$, and a $\varepsilon \in (0,1)$ such that $x \in U$ whenever $\lambda_i(x) \leq \varepsilon$ for every $i \leq n$.

Let us define the function $\eta: \mathcal{R} \to E \times E \times \mathbb{R}^n$ that assigns to every $x \in \mathcal{R}$ the vector:

$$\eta(x) \coloneqq (\nu(x), \beta(x), \lambda_1(x), \dots, \lambda_n(x))$$

and observe that η is a u-continuous measure that takes values in a space that is separated by a family of functionals $\mathcal{F}$ such that $|\mathcal{F}| < sat(u)$. The set $\eta(\mathcal{R} \wedge a)$ is then convex by Theorem 2.2.7 and it contains both the 0 vector and $\eta(a)$. Let $b \leq a$ be such that $\eta(b) = \varepsilon\eta(a)$. We claim that b is the desired coalition.

First of all we observe that b is non-null since $\nu(b) = \varepsilon\nu(a) \neq 0$. On the other hand, being $\lambda_i(b) \leq \varepsilon$ for all $i \leq n$, $b \in U$. To see that b blocks α via β observe that β is an allocation attainable by b, because $\nu(b) = \varepsilon\nu(a) = \varepsilon\beta(a) = \beta(b)$, and that $b \leq a$ and $\beta \succ_a \alpha$ imply that $\beta \succ_b \alpha$ because of the ideal property of preferences (Condition $P1$ in 4.1.5). □

In general, the larger is the economy, the harder it is to check whether a given distribution of resources is a core allocation or not. Here we have shown that the area in which we have to look for blocking coalitions can be narrowed from the whole $\mathcal{R}$ to any neighborhood of 0 in $(\mathcal{R}, u)$. Alternatively, we might also say that under the Assumptions of Theorem 4.3.5 any non-core allocation can be blocked by re-allocating an arbitrarily small amount of resources. This last intuition is formalized in the following corollary.

Corollary 4.3.6. *Under the Assumptions 2 and 3 let α be a non-core allocation and let V be a 0-neighborhood in $(\mathcal{H}, \rho)$. Then there is an allocation in V that dominates α.*

Proof. Let $\beta \in \mathcal{H}$ be an allocation that dominates α. By Theorem 4.3.5 we can find a net $(a_i)_{i \in \mathcal{I}}$ of coalitions that converges to 0 in $(\mathcal{R}, u)$ and that is such that each a_i blocks α via β. In particular, this means that whenever we choose $(x_i)_{i \in \mathcal{I}} \subset \mathcal{R}$ so that $x_i \leq a_i$ for $i \in \mathcal{I}$ we will have that $\nu(x_i) \to 0$ in E.

Recall now that, by assumption, $\beta \in \mathcal{H}$ implies that $\beta \ll \nu$ and therefore $\sup\{\langle p, \beta(x)\rangle : x \leq a_i\}$ converges to 0 for every non-zero $p \in E^*_+$.

For every $i \in \mathcal{I}$ call $\beta_i \coloneqq \beta_{a_i}$ and observe that a_i blocks α via β_i. By construction we will have that $|\beta_i|_p = \sup\{\langle p, \beta(x)\rangle : x \leq a_i\}$ converges to 0 for every $p \in E^*_+ \setminus \{0\}$. But then, for a sufficiently large $j \in \mathcal{I}$, β_j is an allocation in V that dominates α. □

4.3.B On the veto power of large coalitions

In the spirit of the work of Schmeidler, Vind presented in (1972) some sufficient conditions under which any feasible allocation outside the core can be blocked by an arbitrarily large coalition. This result, however close to that of Schmeidler, is obtained under stronger assumptions on preferences and requires a more careful analysis. To overcome some of the issues that arise in our settings, in this paragraph we present an infinite dimensional extension of Vind's Theorem in which we consider a relaxed veto mechanism based on the notion of $\star$-core.

Using the blocking mechanism that defines the $\star$-core we can prove the following version of Vind's Theorem on the veto power on large coalitions (Vind (1972)) in which, like in the case of Theorem 4.3.5, the notion of *arbitrarily large* coalitions is formalized using the topological structure on $(\mathcal{R}, u)$. Once again, we shall say that a property P holds for *arbitrarily large coalitions* if $\mathcal{R}$ is a Boolean algebra and for every neighborhood U of e in $(\mathcal{R}, u)$ the property P holds a coalition $a \in U$.

Theorem 4.3.7. *Suppose that $\mathcal{R}$ is a Boolean algebra, that $\succ$ is lower semi-continuous and weakly locally non-satiated and that Assumptions 2 and 3 hold. Let α be a feasible allocation that can be $\star$-blocked and let U be a neighborhood of e in $(\mathcal{R}, u)$. Then there is a coalition in U that $\star$-blocks α.*

In other words, we are claiming that, under the assumptions above, every feasible allocation outside the $\star$-core can be $\star$-blocked by arbitrarily large coalitions.

Proof. If U is a neighborhood of e in $(\mathcal{R}, u)$, by Corollary 4.2.6 we can select a finite number of u-continuous measures $\lambda_i : \mathcal{R} \to [0,1]$ and a $\varepsilon \in (0,1)$ that are such that:

$$\{x \in \mathcal{R} : \lambda_i(e \triangle x) \le 1 - \varepsilon,\ \forall i \le n\} \subseteq U.$$

To prove our statement it will therefore be sufficient to find a coalition d that $\star$-blocks α and that is such that $\lambda_i(d) \ge \varepsilon \lambda_i(e)$ for all $i \le n$ (this way $1 - \varepsilon \ge \lambda(e) - \lambda(d) = \lambda_i(e \triangle d)$).

Suppose that $a \in \mathcal{R}$ is a coalition that $\star$-blocks α. This means that there exist $\beta \in \mathcal{H}$ and a non-null $a' \le a$ such that: (i) $\beta(a) = \nu(a)$, (ii) $\beta \succ_{a'} \alpha$ and (iii) $\beta(x) = \alpha(x)$ for every $x \le a \setminus a'$. Call $\eta : \mathcal{R} \to E \times E \times E \times \mathbb{R}^n$ the function that assigns to each $x \in \mathcal{R}$ the vector:

$$\eta(x) := (\nu(x), \alpha(x), \beta(x), \lambda_1(x), \ldots, \lambda_n(x))$$

and observe that η is a u-continuous measure that satisfies all of the assumptions of Theorem 2.2.7. We can therefore find $b' \le a'$ and $b'' \le a \setminus a'$ such that $\eta(b') = \varepsilon \eta(a')$ and $\eta(b'') = \varepsilon \eta(a \setminus a')$. Similarly, we can define $b := b' \vee b''$ and find a $c \le b^c$ such that $\eta(c) = \varepsilon \eta(b^c)$.

We claim that the coalition $d := b \vee c$ is the desired one. First of all observe that, for every $i \leq n$, $\lambda_i(d) = \lambda_i(b) + \lambda_i(c) = \varepsilon\lambda_i(a) + \varepsilon\lambda_i(b^c) \geq \varepsilon\lambda_i(e)$ which means that $d \in U$. So we only need to prove that d $\star$-blocks α.

Let us focus on b. The first observation we make is that b $\star$-blocks α via β. In fact: (i) $\nu(b) = \varepsilon\nu(a) = \varepsilon\beta(a) = \beta(b)$, (ii) b' is a non-null sub-coalition of a' and so $\beta \succ_{b'} \alpha$ (by selfishness of preferences) and (iii) $b \smallsetminus b' = b'' \leq a \smallsetminus a'$, meaning that $\alpha(x) = \beta(x)$ for all $x \leq b \smallsetminus b'$. Furthermore, being $\succ_{b'}$ weakly locally non-satiated, $\alpha(b')$ belongs to the closure of the set $P_\alpha(b')$ which, in turn, is convex by Theorem 4.2.11. This means that $v = \varepsilon\alpha(b') + (1-\varepsilon)\beta(b') \in P_\alpha(b')$ and hence $v = \gamma(b')$ for some allocation γ such that $\gamma \succ_{b'} \alpha$.

Let $\zeta: \mathcal{R} \to E$ be the allocation that assigns to each $x \in \mathcal{R}$ the bundle:

$$\zeta(x) := \alpha(x \smallsetminus b') + \varepsilon\alpha(x \wedge b') + (1-\varepsilon)\beta(x \wedge b').$$

First of all, we observe that b' is a non-null subcoalition of d that is such that $\zeta \succ_{b'} \alpha$ and $\zeta(x) = \alpha(x)$ for every $x \leq d \smallsetminus b'$. To prove that d $\star$-blocks α via ζ we only have to show that $\zeta(d) = \nu(d)$. By construction, $\zeta(b) = \varepsilon\alpha(b) + (1-\varepsilon)\beta(b)$ and $\zeta(c) = \alpha(c)$ and so we can write:

$$\zeta(d) = \zeta(b) + \zeta(c) = (1-\varepsilon)\beta(b) + \varepsilon\alpha(b) + \alpha(c).$$

Furthermore, we know that $\beta(b) = \nu(b)$, that α is feasible and, by the choice of c, that $\alpha(c) = \varepsilon\alpha(b^c)$. The equation above can then be rewritten in the form:

$$\zeta(d) = (1-\varepsilon)\nu(b) + \varepsilon[\alpha(b) + \alpha(b^c)] = (1-\varepsilon)\nu(b) + \varepsilon\nu(e).$$

But then:

$$\zeta(d) = \nu(b) + \varepsilon[\nu(e) - \nu(b)] = \nu(b) + \varepsilon\nu(b^c) = \nu(b) + \nu(c) = \nu(d)$$

as claimed. □

As it has been mentioned, the core and the $\star$-core of an economy may not coincide and it is possible that an allocation is $\star$-blocked even if there are no coalitions that block it in the standard sense. Therefore, Theorem 4.3.7 alone is not sufficient to guarantee that any feasible, non-core allocation is blocked (in the standard sense) by arbitrarily large coalitions. However, thanks to Proposition 4.3.3, we can prove the following corollary.

Corollary 4.3.8. *Under the Assumptions of Theorem 4.3.7 suppose that Assumption 4 holds. Let α be a feasible allocation that can be $\star$-blocked and let U be a neighborhood of e in $(\mathcal{R}, u)$. Then there is a coalition in U that blocks α in the ordinary sense.*

Proof. By Theorem 4.3.7 we can take a $a \in U$ that $\star$-blocks α. It follows from Proposition 4.3.3 that a blocks α also in the ordinary sense. □

Remark 4.3.9. In a simpler framework, the idea of a topological description of the economic weight was introduced and studied in (Urbinati, 2019, Section 4). This allowed to prove a preliminary version of Theorem 4.3.5 in (Urbinati, 2019, Theorem 4.12). The Theorem 4.3.7, however, is a much stronger result than those presented in Urbinati (2019) where the only results dealing with the veto power of large coalitions were proved in (2019, Theorem 4.15) and can be seen as a simplified version of Corollary 4.3.8.

Remark 4.3.10. It is thanks to the assumptions 2, 3 on the economic weight of coalitions that we could prove Theorems 4.3.5 and 4.3.7 using a classical approach similar to what is done in the finite dimensional cases. Without such a restriction on u, some additional assumptions on the commodity space are needed: in Hervés-Beloso et al. (2000), for example, the commodity space is taken as the sequence space ℓ^∞ while in Evren and Hüsseinov (2008) E is required to be a Banach lattice whose positive cone has an interior point. Another approach, used in Bhowmik and Cao (2012), consists in restricting the attention to allocations with the equal treatment property. Last, in Bhowmik and Graziano (2015) the authors impose the existence of a cone of arbitrage which allows them extend their analysis to economies with atoms.

Remark 4.3.11. As mentioned in remark 4.2.7, when E is a Banach space endowed with a norm $\|\cdot\|$ it is common to call *diameter* the function $x \mapsto \sup\{\|\nu(b)\| : b \in \mathcal{R} \wedge x\}$, for $x \in \mathcal{R}$, as in the equation 4.2. What we have is that for all $\varepsilon > 0$ the set of coalitions with diameter smaller than ε forms a 0-neighborhood in $(\mathcal{R}, \tau(\nu))$ showing that our definition of *distribution of economic power* is very closely related to the idea of diameter.

In this view, when E is a Banach space it follows from Theorem 4.3.5 that every non-core allocation can be blocked by coalitions with arbitrarily small diameter. This result mimics the infinite dimensional reformulation of Grodal's Theorem (1972) that was given in (Hervés-Beloso et al., 2000, Theorem 1).

Chapter 5

Competitive objection mechanism in economies with many commodities

In this chapter we study the competitive objection mechanism and the characterization of competitive allocations through the notion of bargaining set in economies with many commodities. At the best of my knowledge, this is the first contribution of this type.

The main concern will be to provide conditions under which it is always possible that some agents can raise a competitive objection against a feasible, non-competitive allocation. This analysis will be carried out in two different scenarios and will consider different notions of bargaining set in which both standard and Aubin coalitions (i.e. fuzzy sets of agents) are considered in the objection and counter-objection process.

We will mainly focus on the problem of existence of competitive objections in competitive economies with a separable Banach space of commodities and on the crucial role played by the "convexifying effect on preferences" that in the finite dimensional settings is guaranteed by Lyapunov's convexity Theorem. More precisely, we will study the problem of existence of competitive objections in exchange economies in which: (i) the commodity space is a separable Banach space whose positive cone has non-empty interior and (ii) both standard and fuzzy coalitions are considered in the objection and counter-objection mechanisms.

The chapter is organized as follows. In Section 5.1 we describe a very standard model of an exchange economy with a measure space of agents and with a separable Banach space of commodities whose positive cone has non-empty interior. The assumptions of the model are going to be almost identical to those described in Khan and Yannelis (1991a) if not for the weaker impositions we make on the choice of the initial endowment (see Remark 5.1.1 for a more detailed comparison with the literature). We will assume that the measure space of agents is 'rich' enough to ensure some form of convexifying effect as a consequence of the saturation property and the 'many agent of every type' assumption introduced in Podczeck (1997).

In Section 5.2 we redefine Mas-Colell's bargaining set and the notion of competitive objections in this new setting. In 5.2.3 we study the coalitions that can raise competitive objections at a given price while in Proposition 5.2.5 we prove that the existence of competitive objections against feasible allocations depends on the existence of equilibrium prices of a properly defined correspondence. These results are then used in Theorem 5.2.7 to show that, in our framework, every feasible but non-competitive allocation can be objected competitively. This will also allow us to conclude that every allocation in the bargaining set is competitive.

In Section 5.3 we weaken the assumptions on the measure space of agents to allow the presence of market imperfections and oligopolists. In paragraph 5.3.A we develop new definitions of bargaining sets in which we allow Aubin coalitions to take part in the objection and counter-objection processes (as in Hervés-Beloso et al. (2018)). Thanks to a new convexity result, which is proved in Lemma 5.3.6, we will be able to adapt what had been done in Section 5.2 and prove new characterizations of competitive allocations by means of the different notions of bargaining set considered.

The mathematical setting

We shall start by recalling some of the notation that will be used throughout this chapter.

- Let (T, Σ, m) be a complete probability space. For every $A \in \Sigma$, the *principal ideal generated by* A will be the set $\Sigma_A := \{F \in \Sigma : \; F \subseteq A\}$. We will write m_A for the measure $m_A : F \mapsto m(A \cap F)$, for $F \in \Sigma$, and identify it with the restriction of m to the algebra Σ_A. As usual, we shall say that $A \in \Sigma$ is *m-null* if $m(F) = 0$ for every $F \in \Sigma_A$ and write $N(m)$ for the ideal of m-null elements in Σ.

- E will be an ordered Banach space. As usual, we will write $\leq$ for the order relation on E and call $E_+ := \{x \in E : \; x \geq 0\}$ the *positive cone of* E. E^* denotes the topological dual of E, i.e. the space of all continuous linear functions from E to $\mathbb{R}$. For any $x \in E$ and $p \in E^*$ we will write both $\langle p, x\rangle$ and $p \cdot x$ for the value $p(x)$. The duality between E and E^* will induce on E^* a structure of Banach lattice for which the positive dual cone is the set $E^*_+ := \{p \in E^* : \; p(x) \geq 0 \; \forall x \in E_+\}$.

- For any subset $A \subset E$, the *interior*, *closure* and *convex hull* of A will be respectively denoted by $int(A)$, $\overline{A}$ and $co(A)$. The *closed convex hull* of A

will in turn be denoted by $\overline{co}(A)$. To avoid the risk of ambiguity we shall explicitly indicate what is the topology we are considering at each time and write $\overline{A}^w$ when the closure is considered with respect to the weak topology $\sigma(E, E^*)$ on E.

- Given a correspondence $\varphi: T \twoheadrightarrow E$, by *selection* of φ we will mean a function $f: T \to E$ such that $f(t) \in \varphi(t)$ for almost every $t \in T$. The set of all selections of φ that are integrable will be denoted by $\mathcal{L}(\varphi)$. We will talk about the *integral* of a correspondence φ always referring to the Aumann-integral defined by:

$$\int_A \varphi \, dt := \left\{ \int_A f(t) \, dt : \ f \in \mathcal{L}(\varphi) \right\}, \quad A \in \Sigma.$$

The correspondence φ is said to be *integrably bounded* if there is a real-valued, integrable function $h: T \to \mathbb{R}_+$ such that $\sup\{\|y\| : y \in \varphi(t)\} \leq h(t)$ for almost every $t \in T$. φ will be *measurable* when its graph $Gr(\varphi) := \{(t, x) : \ x \in \varphi(t)\}$ belongs to the product algebra $\Sigma \otimes \mathcal{B}(E)$, where $\mathcal{B}(E)$ stands for the Borel σ-algebra on E.

We will primarily refer to (Khan and Yannelis, 1991b, Part I) and (Aliprantis and Burkinshaw, 2006, Chapters 17, 18) as references for the general theory of integration of correspondences and Podczeck (2008), Sun and Yannelis (2008) for the specific case of integration over a saturated measure space. For what concerns the study of the bargaining set and the competitive objection mechanism we refer to Mas-Colell (1989) and the survey in (Grodal, 2009, Sections 4,5).

5.0.A Preliminaries on saturated measure spaces

Let (T, Σ, m) be a complete probability space. As usual, we will say that $A \in \Sigma$ is a *m-atom* if $m(F) \in \{0, m(A)\}$ for every $F \subseteq A$. The measure space (T, Σ, m) is said to be *non-atomic*, or *atomless* if there are no m-atoms in Σ.

We will say that (T, Σ, m) is *separable* if there is a countable family $(F_n)_n \subseteq \Sigma$ such that $\inf_n m(A \cap F_n) = 0$ for every $A \in \Sigma$. This is equivalent to asking that the quotient algebra $\tilde{\Sigma}_m := \Sigma / N(m)$ is generated by a countable family of elements. We will say that (T, Σ, m) is *saturated* if for every non-null $A \in \Sigma$ the measure space (A, Σ_A, m_A) is not separable. It is worth observing that, with the given definition, every saturated measure space is non-atomic while the converse may not hold.

The proof of the following result can be found in (Podczeck, 2008, Theorem 1 and 2) in terms of *super-atomless* measure spaces. The equivalence between super-atomless and saturated measure spaces is proved in (Podczeck, 2008, Fact at page 839).

Lemma 5.0.1. *Let (T, Σ, m) be a saturated measure space and E an infinite dimensional Banach space. Then the following conditions are equivalent.*

1. *(T, Σ, m) is saturated.*

2. *$\int_S \varphi(t)\, dt$ is convex for every correspondence $\varphi : T \twoheadrightarrow E$.*

3. *$\int \varphi(t)\, dt$ is weakly compact for every correspondence $\varphi : T \twoheadrightarrow E$ that is integrably bounded and has weakly compact values.*

Another significant property will be given by the following result proved in (Sun and Yannelis, 2008, Proposition 1.6).

Lemma 5.0.2. *Let (T, Σ, m) be a saturated measure space, E a separable Banach space and Y a metric space. Suppose that $\varphi : Y \times T \twoheadrightarrow E$ is a weakly compact valued correspondence for which all the following assumptions are met:*

1. *$t \mapsto \varphi(y, t)$, $t \in T$, is measurable for every $y \in Y$.*

2. *$y \mapsto \varphi(y, t)$, $y \in Y$, is upper hemicontinuous with respect to the weak topology on E for almost every $t \in T$.*

3. *$y \mapsto \varphi(y, t)$, $y \in Y$, is dominated[1] by an integrably bounded correspondence $K : Y \twoheadrightarrow E$.*

Then $y \mapsto \int_T \varphi(y, t)\, dt$, $y \in Y$, is upper hemicontinuous with respect to the weak topology in E.

5.1 The economic model

The commodity space is an ordered, separable Banach space E whose positive cone E_+ is closed and has non-empty interior. We will refer to the vectors in E_+ as the *commodity bundles* and call *price systems* the non-zero elements of E_+^*. Throughout, we will consider an economy $\mathcal{E} := [(T, \Sigma, m), (X(t), e(t), \succcurlyeq_t)_{t \in T}]$ with the following specifications:

- individual agents will be represented as the points of a complete probability space (T, Σ, m). As usual, the non-null elements of Σ will represent all the possible coalitions that can be formed in the economy while $m : \Sigma \to [0, 1]$ is a numerical evaluation of the economic power of each coalition. We will refer to (T, Σ, m) as the *space of agents*;

[1] There is a correspondence $\tilde{\varphi} : T \twoheadrightarrow E$ such that $\varphi(y, t) \subseteq \tilde{\varphi}(y, t)$ for almost every $T \in T$.

- $X(t) \subset E_+$ is the *consumption set* of agent t. The map $X: T \twoheadrightarrow E_+$ is called the *consumption set correspondence*;
- $e: T \to E_+$ is an integrable selection of the consumption set correspondence X assigning to each individual agent her initial endowments;
- $\succcurlyeq_t$ is a reflexive, transitive and complete binary relation on $X(t)$ representing agent t's preferences over her consumption bundles. The symmetric and asymmetric components of $\succcurlyeq_t$ are respectively called *indifference* and *strict preference relation* and denoted by $\sim_t$ and $\succ_t$.

Choose a $u \in int(E_+)$ and let $\Delta := \{p \in E_+^* : p \cdot u = 1\}$ be a price space equipped with the weak* topology $\sigma(E^*, E)$. This way Δ is a convex set, it is weak* compact by Alaoglu's Theorem (see for example (Aliprantis and Border, 2006, Theorem 5.105)) and, being E separable, it is metrizable (see (Aliprantis and Border, 2006, Theorem 6.30)). Given a price system $p \in \Delta$, for every $t \in T$ we can define agent t's *budget set* at p as:

$$\beta(p, t) := \{x \in X(t) : \ p \cdot x \leq p \cdot e(t)\}.$$

The *demand set* of agent t at the price system p is the set $\xi(p, t)$ of those consumption bundles in $\beta(p, t)$ that are maximal with respect to $\succcurlyeq_t$. In symbols:

$$\xi(p, t) := \{x \in \beta(p, t) : \ x \succcurlyeq_t \beta(p, t)\}^{2}.$$

An *allocation*, or *consumption plan*, is a function in $\mathcal{L}(X)$, the family of all integrable selections of the consumption set correspondence $X: T \twoheadrightarrow E_+$. We will say that an allocation f is *feasible* if $\int_T f(t)\, dt \leq \int_T e(t)\, dt$.

A *competitive equilibrium* for $\mathcal{E}$ is a price-consumption pair (p, f) where $p \in \Delta$ and f is a feasible allocation such that $f(t) \in \xi(p, t)$ for almost every $t \in T$. In this case we call f a *competitive allocation* and write $f \in W(\mathcal{E})$.

5.1.A Further assumptions

Throughout the rest of this chapter we will assume that the economy $\mathcal{E}$ satisfies some additional properties listed below. Precisely, we will need to make restrictions on the consumption set correspondence $X: T \twoheadrightarrow E_+$ and the preferences relations that can be considered in our analysis.

Assumption 5. *The consumption set correspondence $X: T \twoheadrightarrow E$ satisfies all of the following conditions:*

[2] For any $x \in X(t)$ and subset $A \subset X(t)$, the expression $x \succcurlyeq_t A$ will be used meaning that $x \succcurlyeq_t y$ for every $y \in A$. Similar notation will be adopted for $\succ_t$ and $\sim_t$.

(C1) *for every* $t \in T$, $X(t)$ *is non-empty, weakly compact and convex,*

(C2) $X : T \twoheadrightarrow E$ *is integrably bounded and has a measurable graph,*

(C3) *for every* $t \in T$ *there is a* $y_t \in X(t)$ *such that* $e(t) - y_t \in int(E_+)$.

The Assumption $(C3)$ imposes that the initial amount of resources of agent t is not "minimal" in her consumption set $X(t)$. The following assumptions on preferences are standard continuity and measurability assumptions.

Assumption 6. *The preference relations* $(\succcurlyeq_t)_{t \in T}$ *satisfy all of the following conditions.*

(P1) *For every* $t \in T$, *the set* $\{(x, y) \in X(t) \times X(t) : x \succcurlyeq_t y\}$ *is closed in the product topology.*

(P2) *For every* $t \in T$, *the set* $\{(t, x, y) \in T \times X(t) \times X(t) : x \succcurlyeq_t y\}$ *belongs to* $\Sigma \otimes \mathcal{B}(E) \otimes \mathcal{B}(E)$.

The last set of assumptions in this section will be related to the notion of satiability of agents' preferences. More formally, for every $t \in T$ and $x \in X(t)$ we will call $P_t(x) := \{y \in X(t) : y \succ_t x\}$ the *better than set* of agent t with respect to x. We will then say that x is a *satiation point* for t if $P_t(x)$ is empty.

Assumption 7. *For every* $t \in T$ *and* $x \in X(t)$ *all of the following conditions are satisfied.*

(P3) *The set* $\{y \in X(t) : y \succcurlyeq_t x\}$ *is closed in the weak topology.*

(P4) *If* x *is a satiation point for* t *then* $x \geq e(t)$.

(P5) *If* x *is not a satiation point for* t *then it belongs to the closure of* $P_t(x)$ *in the weak topology.*

Assumption $(P4)$ rules out the possibility that an agent will accept to trade her initial endowments for a strictly smaller bundle. Assumption $(P5)$, on the other hand, can be seen as a weak local non-satiation rule on non-satiation points.

Remark 5.1.1. The Assumptions made here are all quite standard in economic models involving a measure space of agents and an infinite dimensional commodity space. In particular, in the models considered in Khan and Yannelis (1991a), Rustichini and Yannelis (1991) and Podczeck (1997) all of the three assumptions 5, 6 and 7 are satisfied. A discussion on all of these assumptions with reference to the literature can be found in Tourky and Yannelis (2001) and Martins-da Rocha (2003).

As it is known, the evaluation map $(p,x) \mapsto p \cdot x$, for $p \in E^*$ and $x \in E$, is no longer jointly continuous when we endow E^* and E with the weak* and the weak topology respectively (see (Aliprantis and Border, 2006, pp. 241–242)). As a consequence, we will usually have that the standard demand correspondence ξ may fail to be upper hemicontinuous and it is therefore much harder to use it in fixed point arguments. In order to overcome these difficulties we can follow the approach given in Podczeck (1997) and introduce a new correspondence $\tilde{\xi}: \Delta \times T \twoheadrightarrow E$ that assigns to every price system p and agent t the set:

$$\tilde{\xi}(p,t) := \{x \in X(t) : \ x \succcurlyeq_t \beta(p,t)\}.$$

We call $\tilde{\xi}$ the *extended demand* correspondence. Differently from ξ, under the Assumptions above this new correspondence benefits from many nice properties which we recall here for the sake of completeness. We refer to (Podczeck, 1997, pp.415–416) for the proofs of these results.

Lemma 5.1.2. *For every $p \in \Delta$ and almost every $t \in T$ one has that:*

(*i*) *$\tilde{\xi}(p,t)$ is non-empty and weakly compact.*

(*ii*) *$p \cdot x \geq p \cdot e(t)$ for every $x \in \tilde{\xi}(p,t)$.*

(*iii*) $\overline{co}\,\tilde{\xi}(p,t) \cap \{x \in E : \ p \cdot x = p \cdot e(t)\} = \overline{co}\,\xi(p,t)$.

(*iv*) *For every $t \in T$ the correspondence $\tilde{\xi}(\cdot,t): p \mapsto \tilde{\xi}(p,t)$, for $p \in \Delta$, is upper hemicontinuous with respect to the weak topology on E.*

(*v*) *For every $p \in \Delta$ the correspondence $\tilde{\xi}(p,\cdot): t \mapsto \tilde{\xi}(p,t)$, for $t \in T$, has a measurable graph.*

We shall see in Paragraph 5.2.9 that it is possible to study the existence of competitive objections even with an alternative approach that does not require the introduction of the extended demand correspondence.

5.1.B The space of agents

Since we did not require m to be non-atomic, it is allowed in Σ the presence of a family $\mathcal{A}$ of m-atoms, i.e. any non-null $A \in \Sigma$ such that $\mu(B) = 0$ or $\mu(A \setminus B) = 0$ for every $B \in \Sigma_A$. Given that $\mathcal{A}$ is at most countable, the union $T_1 := \cup\mathcal{A}$ is a measurable set in Σ which we call *atomic component* of (T,Σ,m). Let us call $T_0 := T \setminus T_1$ the *atomless component* of (T,Σ,m) and write Σ_0 for $\Sigma \cap T_0$. We will refer to agents in T_1 and T_0 as the *large* and *small* traders respectively.

It will often be necessary to impose the following condition.

Assumption 8. *For all $t \in T_1$ the preference relation $\succcurlyeq_t$ is convex, i.e. the set $\{y \in X(t) : \; y \succcurlyeq_t x\}$ is convex whenever $x \in X(t)$.*

Let us focus on the atomless component T_0. We can partition T_0 as the union of two disjoint, measurable sets T_0^{sep} and T_0^{sat} such that, for every $B \in \Sigma$, it must be:

- $B \subseteq T_0^{sep}$ if and only if (B, Σ_B, m_B) is separable,
- $B \subseteq T_0^{sat}$ if and only if (B, Σ_B, m_B) is saturated.

We will refer to T_0^{sat} as the *saturated component* and to T_0^{sep} as the *separable component* of (T, Σ, m).

We say that two agents $s, t \in T$ are *of the same type*, or *equivalent*, if $X(s) = X(t)$, $e(s) = e(t)$ and $\succcurlyeq_s = \succcurlyeq_t$. The relation $t \sim s \iff$ "t is equivalent to s" is then an equivalence relation on T that is measurable with respect to Σ, in the sense that $\{s \in T : \; s \sim t\} \in \Sigma$ for every $t \in T$. An allocation $f \in \mathcal{L}(X)$ is said to have the *equal treatment property* if every agent $t \in T$ will be indifferent between the bundles that are assigned by f to the agents in T that are equivalent to t.

Observe that, by the measurability of X, e and $\succcurlyeq_t$, each equivalence class in $\Sigma / \sim$ is therefore the projection of some $A \in \Sigma$ representing a *type* of consumers. Let us call $C := T/\sim$ and $\pi_C : T \to C$ the corresponding quotient map. On the quotient algebra $\Sigma_C := \Sigma/\sim$ we can define a new measure ν by putting $\nu(F) = m \circ \pi_C^{-1}(F)$ for every $F \in \Sigma_C$. We will call (C, Σ_C, ν) the *space of agent types*.

Following Podczeck (1997) we formalize here the idea that there are many agents of every type in a given coalition.

Definition 5.1.3. *We say that in a coalition F there are* many agents of every type *if there exists a family $(m_c)_{c \in C}$ of measures on Σ such that:*

1. *$m_c^*(\pi_C^{-1}(\{c\})) = 1$ for all $c \in C$, where m_c^* denotes the inner measure induced by m_c,*
2. *for every $A \in \Sigma$ such that $A \subseteq F$, the mapping $c \mapsto m_c(A)$, $c \in C$, is measurable and $\int m_c(A)\, d\nu(c) = m(A)$,*
3. *the restriction of m_c to $\Sigma \cap F$ is atomless for ν-almost every $c \in C$.*

If in $F \in \Sigma$ there are many agents of every type then each of the measures m_c gives us an estimate of how many agents of type c belong to any given allocation. Thus, using $(m_c)_{c \in C}$, we can say that two coalitions A, B are *equivalent* if $m_c(A) = m_c(B)$ for ν-almost every $c \in C$. Observe that condition (2) ensures that the measure of a coalition $A \subseteq F$ can be obtained knowing, for every type of agent $c \in C$, the share of agents of c that belong to B.

Assumption 9. *In T_0^{sep} there are many agents of every type.*

The following is Theorems 3.1 and 3.3 in Podczeck (1997) and Proposition 5.0.1.

Theorem 5.1.4. *Suppose that Assumptions 8 and 9 hold and let $\varphi: T \twoheadrightarrow E$ be an integrably bounded, non-empty and weakly compact valued correspondence with measurable graph and that is such that $\varphi(t) = \varphi(s)$ whenever $t, s \in T_0^{sep}$ are of the same type. Then:*

$$\int_T \overline{co}\, \varphi(t)\, dt = \int_T \varphi(t)\, dt.$$

In particular, $\int_{T_0} \varphi(t)\, dt$ is convex and weakly compact.

More comments and observations on Definition 5.1.3, Assumption 9 and Theorem 5.1.4 can be found in Podczeck (1997) together with examples of economies in which all the three conditions in 5.1.3 are satisfied. Compare also with Assumption (A) in Martins-da Rocha (2003).

5.2 Mas-Colell's bargaining set

The notion of *bargaining set* for atomless economies, as introduced by Mas-Colell in (1989), can be naturally adapted to the economic framework we have described so far. Let us start by considering an allocation f that is not necessarily feasible. A coalition S can *block*, or *object*, f via an allocation g if:

- $\int_S g(t)\, dt \leq \int_S e(t)\, dt$,
- $g(t) \succcurlyeq_t f(t)$ for almost every $t \in S$,
- $m(\{t \in S : \; g(t) \succ_t f(t)\}) > 0$.

In this case we say that (S, g) is an *objection* to f and write $(S, g) \in Ob(f)$. If f is a feasible allocation that cannot be blocked by any coalition, then f is said to belong to the *core* of the economy, denoted by $C(\mathcal{E})$.

When (S, g) is an objection to f, a coalition Q is said to *counter-object* (S, g) if there is an allocation h such that:

- $\int_Q h(t)\, dt \leq \int_Q e(t)\, dt$,
- $h(t) \succ_t g(t)$ for all $t \in Q \cap S$,
- $h(t) \succ_t f(t)$ for every $t \in Q \setminus S$.

In this case we call (Q,h) a *counter-objection* to (S,g) and write $(Q,h) \in COb_f(S,g)$. The objection (S,g) is said to be *justified* if it cannot be counter-objected, i.e. if $COb_f(S,g) = \varnothing$.

Definition 5.2.1. *The* bargaining set *of the economy $\mathcal{E}$ is the collection $BS(\mathcal{E})$ of all feasible allocations against which it is not possible to raise any justified objection.*

In other words, a feasible allocation f will belong to the bargaining set of the economy if whenever (S,g) is an objection to f we have $COb_f(S,g) \neq \varnothing$. Clearly every allocation in the core, and hence every competitive allocation, belongs to the bargaining set as it cannot be objected at all.

5.2.A Competitive objections

As mentioned at the beginning of the chapter, the idea of *competitive objections* is that of imposing a price system p and letting each individual agent decide whether he want to keep the bundle assigned to him by the allocation f or to trade his initial resources in a market regulated by the price system p. This notion is formalized as follows.

Definition 5.2.2. *Let f be a feasible allocation. An objection (S,g) to f is* competitive *at a price system p if for almost every $t \in T$ and every $x \in X(t)$ one has:*

- $p \cdot x \geq p \cdot e(t)$ *whenever* $t \in S$ *and* $x \succcurlyeq_t g(t)$,
- $p \cdot x \geq p \cdot e(t)$ *whenever* $t \notin S$ *and* $x \succcurlyeq_t f(t)$.

A first observation that can be made is that an objection (S,g) to f will be competitive if and only if there is a price system p such that $g(t) \in \xi(p,t)$ for almost every $t \in S$ while $f(t) \succcurlyeq_t \beta(p,t)$ whenever $t \notin S$.

In order to fully understand this mechanism, let us introduce some additional notation that will be necessary for the rest of the chapter. Given a feasible allocation f and a price system p we can define the sets:

$$C_f(p) := \{t \in T : \ \xi(p,t) \succ_t f(t)\}, \ D_f(p) := \{t \in T : \ \xi(p,t) \succcurlyeq_t f(t)\}.$$

Intuitively, an agent t will belong to $C_f(p)$ (resp. $D_f(p)$) if she strictly (resp. weakly) prefers what she obtains by trading $e(t)$ at price p over the bundle $f(t)$. Please observe that a feasible allocation f will be competitive if and only if there is a price system p such that $m(C_f(p)) = 0$.

A first result on the characterization of competitive objections is the following.

Proposition 5.2.3. *Let f be a feasible allocation that is not competitive. A pair (S, g) is a competitive objection against f if and only if there is a price system p such that:*

1. *$g(t) \in \xi(p, t)$ for almost every $t \in S$,*
2. *$C_f(p) \subseteq S \subseteq D_f(p)$ almost everywhere[3],*
3. *$\int_S (g(t) - e(t))\ dt \leq 0$.*

Proof. Let us assume that (S, g) is competitive and call p the relative price system. By definition, for almost every $t \in S$ $g(t)$ is a maximal element in $\beta(p, t)$ for $\succcurlyeq_t$, which in turn is equivalent to saying that $g(t) \in \xi(p, t)$ as claimed in (1). If point (2) is violated then either $m(C_f(p) \setminus S) > 0$ or $m(S \setminus D_f(p)) > 0$. In the first case let $t \in C_f(p) \setminus S$ and take $x \in \xi(p, t)$: by definition we will have $x \succ_t f(t)$ and $p \cdot x \leq p \cdot e(t)$ and so (S, g) is not competitive. On the other hand, for $t \in S \setminus D_f(p)$ then $f(t) \succ_t g(t)$ contradicts the fact that (S, g) is an objection against f. Last, point (3) follows from the fact that (S, g) is an objection to f.

Suppose now that (S, g) and p satisfy conditions (1), (2) and (3). We first need to prove that $(S, g) \in Ob(f)$. The requirement that $\int_S \gamma(t) g(t)\ dt \leq \int_S \gamma(t) e(t)\ dt$ follows from point (3). For almost every $t \in S$ we will have that $t \in D_f(p)$ (point (2)) and $g(t) \in \xi(p, t)$ (point (1)) meaning that $g(t) \succcurlyeq_t f(t)$. Furthermore, from (1) and (2) we also derive that $\{t \in S :\ g(t) \succ_t f(t)\} = C_f(p)$ and has non-zero measure (recall that, being f non competitive, $m(C_f(q)) > 0$ for every price system q). To prove that (S, g) is competitive let us we pick $x \in E_+$ and observe that if $x \succcurlyeq_t g(t)$ for some $t \in S$ then it must be that $x \succcurlyeq_t \xi(p, t)$ and so $p \cdot x \geq p \cdot e(t)$. On the other hand, if $t \notin S$ then $t \notin C_f(p)$ and so, by the completeness of preferences, $f(t) \succcurlyeq_t \xi(p, t)$. But then $x \succcurlyeq_t f(t)$ implies that $x \succcurlyeq_t \xi(p, t)$ and again $p \cdot x \geq p \cdot e(t)$. □

The last Theorem of this section, which is the equivalent formulation of in (Mas-Colell, 1989, Theorem 1), will play a crucial role in characterizing objections that are justified and hence the bargaining set.

Theorem 5.2.4. *Every competitive objection to a feasible allocation is justified.*

Proof. Let f be a feasible allocation and (S, g) an objection to f that is competitive at a price p. If, by contradiction, (S, g) was not justified there would be a counter-objection (Q, h) to (S, g). This would mean that $h(t) \succ_t g(t)$ for every $t \in Q \cap S$ and $h(t) \succ_t f(t)$ for every other $t \in Q$. By points (1) and (2) in Proposition 5.2.3, this would imply that for every $t \in Q$ we have $h(t) \succ_t \xi(p, t)$ and hence $h(t) \notin \beta(p, t)$. But then $\int \delta(t) h(t)\, dt \nleq \int \delta(t) e(t)\, dt$. □

[3] $A \subseteq B$ almost everywhere if and only if $m(B \setminus A) = 0$.

5.2.B Existence of competitive objections

We will devote this section to the study of conditions under which every feasible but non-competitive allocation can be objected competitively. The main idea of our approach, that does not differ substantially from that used by Mas-Colell in (1989) for the finite dimensional case, is to characterize competitive objections as the equilibrium prices of a specific correspondence and then use some variation of the Gale-Debreu-Nikaidô Lemma to prove the existence of such prices. In this perspective, we shall give a special importance to the characterization of competitive objections that was given in Proposition 5.2.3.

Proposition 5.2.5. *Let f be a feasible but non-competitive allocation. Then there is a competitive objection to f at a price system p if and only if the negative cone $-E_+$ intersects the set:*

$$\Psi_f(p) := \left\{ \int_S \left(\tilde{\xi}(p,t) - e(t)\right) dt : \ C_f(p) \subseteq S \subseteq D_f(p) \right\}.$$

Proof. Let us first assume that $(S,g) \in Ob(f)$ is competitive at the price system p and prove that $x := \int_S (g(t) - e(t))\, dt \in \Psi_f(p)$. By Condition (2) in 5.2.3, $C_f(p) \subseteq S \subseteq D_f(p)$ while Condition (3) ensures that $x := \int_S (g(t) - e(t))\, dt \in -E_+$. Furthermore, for almost every $t \in S$ $g(t)$ belongs to $\xi(p,t)$ (and hence to $\tilde{\xi}(p,t)$) by point (1). It follows that x belongs to $\Psi_f(p)$ as claimed.

Suppose now that there are $p \in \Delta$ and $x \in \Psi_f(p)$ such that $x \in -E_+$. By the definition of Ψ_f there must be an allocation g and a coalition S such that: (i) $g(t) \in \tilde{\xi}(p,t)$ for every $t \in S$, (ii) $C_f(p) \subseteq S \subseteq D_f(p)$ and (iii) $\int_S (g(t) - e(t))\, dt = x$. To show that (S,g) is a competitive objection to f at price p it will be sufficient to show that $g(t) \in \xi(p,t)$ for almost every $t \in S$ and then apply Proposition 5.2.3. For every $t \in S$, being $g(t) \in \tilde{\xi}(p,t)$ it must be that $p \cdot (g(t) - e(t)) \geq 0$ (see Lemma 5.1.2). At the same time, from point (iii) we know that $\int_S p \cdot (g(t) - e(t))\, dt = p \cdot x \leq 0$ and so it must be that $p \cdot g(t) = p \cdot e(t)$ for almost every $t \in S$. We conclude that $g(t)$ belongs to $\beta(p,t)$, and hence to $\xi(p,t)$, for almost every $t \in S$. □

The Proposition 5.2.5 will play a crucial role in proving the existence of competitive objections to a given a feasible but non-competitive allocation f. The main idea will be to find conditions under which the correspondence $\Psi_f \colon \Delta \twoheadrightarrow E$ satisfies the following infinite dimensional variation of the Gale-Debreu-Nikaidô Lemma.

Theorem 5.2.6 (Yannelis (1985))**.** *Let $\varphi \colon \Delta \twoheadrightarrow E$ be a correspondence that satisfies the following assumptions:*

1. *$\varphi(p)$ is non-empty, convex and weakly compact for every $p \in \Delta$,*

2. *φ is upper-hemicontinuous with respect to the weak*-topology on Δ and the weak topology on E,*

3. *for all $p \in \Delta$ there exists a $z \in \varphi(p)$ such that $p \cdot z \le 0$.*

Then there exists a $\overline{p} \in \Delta$ such that $\varphi(\overline{p}) \cap (-E_+) \neq \varnothing$.

For the rest of this section, our concern will be to find conditions under which Ψ_f satisfies all of the three requirements in Theorem 5.2.6. As it will be shown, the main difficulty will be understanding when $\Psi_f(p)$ is convex and weakly compact for every $p \in \Delta$.

Theorem 5.2.7. *Suppose that (T, Σ, m) is a saturated measure space and f is a feasible but non-competitive allocation. Then there is a competitive objection to f.*

Proof. Let us start by defining the modified extended demand correspondence $\psi_f \colon \Delta \times T \twoheadrightarrow E$ as the one that assigns to every $p \in \Delta$ and $t \in T$ the set:

$$\psi_f(p,t) := \begin{cases} \tilde{\xi}(p,t) - \{e(t)\}, & \text{if } t \in C_f(p), \\ \tilde{\xi}(p,t) - \{e(t)\} \cup \{0\}, & \text{if } t \in D_f(p) \setminus C_f(p), \\ \{0\}, & \text{otherwise.} \end{cases}$$

Let us observe that for the correspondence ψ_f all of the following conditions are met (see Lemma 5.1.2): (i) $\psi_f(p,t)$ is weakly compact for every $p \in \Delta$ and $t \in T$; (ii) for every $t \in T$, the correspondence $p \mapsto \psi_f(p,t)$, for $p \in \Delta$, is integrably bounded and upper hemicontinuous with respect to the weak topology on E; (iii) for every $p \in \Delta$, the correspondence $t \mapsto \psi_f(p,t)$, for $t \in T$, is measurable. Moreover, by construction, for every $p \in \Delta$ we have that:

$$\Psi_f(p) = \int_T \psi_f(p,t)\, dt.$$

We only need to show that Ψ_f satisfies all of the three conditions in Theorem 5.2.6 and then apply Proposition 5.2.5. Let us start by observing that, being (T, Σ, m) a saturated measure space, point (i) and Lemma 5.0.1 imply that $\int_T \psi_f(p,t)\, dt$ (and hence $\Psi_f(p)$) is a convex and weakly compact set for every $p \in \Delta$. At the same time, by point (ii), (iii) and Lemma 5.0.2 it follows that $p \mapsto \Psi_f(p)$, for $p \in \Delta$, is upper hemicontinuous with respect to the weak topology on E. Last, for every $p \in \Delta$, by the measurability of $t \mapsto \xi(p,t)$, for $t \in T$, we can find an allocation g^p such that $g^p(t) \in \xi(p,t) \subseteq \tilde{\xi}(p,t)$ almost everywhere. But then $p \cdot (g^p(t) - e(t)) \le 0$ for almost every $t \in T$ and so $z := \int_{D_f(p)} (g^p(t) - e(t))\, dt$ is a point in $\Psi_f(p)$ such that $p \cdot z \le 0$. Since all the assumptions of Theorem 5.2.6 are met, there is a $\overline{p} \in \Delta$ with $\Psi_f(\overline{p}) \cap -E_+ \neq \varnothing$ and so we can apply Proposition 5.2.5 to obtain our claim. $\square$

Corollary 5.2.8. *Suppose that* (T, Σ, m) *is a saturated measure space. Then* $W(\mathcal{E})$ *is non-empty and coincides with* $BS(\mathcal{E})$*.*

Proof. Let us assume that the initial endowment e is non-competitive. By Theorem 5.2.7 there must be an objection (S, g) to e that is competitive for some price system p. This means that $g(t) \in \xi(p,t)$ for almost every $t \in S$ and $e(t) \succcurlyeq_t \beta(p,t)$ for almost every $t \notin S$ which in turn implies that $e(t) \in \xi(p,t)$. But then the allocation $\tilde{g} := g\chi_S + e\chi_{S^c}$ is feasible because $\int_T (\tilde{g}(t) - e(t))\, dt = \int_S (g(t) - e(t))\, dt$ and it is such that $g(t) \in \xi(p,t)$ for almost every $t \in T$. This proves that g is competitive.

To prove the equivalence $W(\mathcal{E}) = BS(\mathcal{E})$ observe that every $f \in W(\mathcal{E})$ is a feasible allocation that cannot be blocked and therefore it belongs to $BS(\mathcal{E})$. To show that the reverse inclusion holds, suppose that $f \notin W(\mathcal{E})$ is feasible and apply Theorem 5.2.7 to find a competitive objection to f. Since every competitive objection is justified by Theorem 5.2.4, f does not belong to $BS(\mathcal{E})$ either. □

Remark 5.2.9. From a technical point of view, one of the hustles in proving Theorem 5.2.7 was the lack of global continuity of the demand correspondence. For this reason, we had to follow the intuition given by Podczeck (1997), move the attention to the extended demand correspondence and then apply Yannelis' generalized version of Gale-Debreu-Nikaidô Lemma 5.2.6. A different way to skirt this continuity issue would be to use an approach similar to that of Nikaidô (1959) and Florenzano (1983) and presented in Chapter 3. By focusing on finite dimensional restrictions of the demand functions, in fact, it would be possible to use a form of Theorem 3.3.1, instead of Theorem 5.2.6, without recurring to the extended demand.

More formally, if one proves that the restriction of the standard demand correspondence ξ to each finitely generated simplex in Δ[4] is upper-hemicontinuous, it should be possible to prove Theorem 5.2.7 as follows. First of all, one can define the correspondence Ψ'_f as the one that assigns to each $p \in \Delta$ the set:

$$\Psi'_f(p) := \left\{ \int_S (\xi(p,t) - e(t))\, dt : \; C_f(p) \subseteq S \subseteq D_f(p) \right\}.$$

Now, even if the correspondence Ψ'_f may not be globally upper-hemicontinuous, and so it may be impossible to apply Theorem 5.2.6, it should be possible to show that the restrictions of Ψ'_f to each finite dimensional subspace of Δ is upper-hemicontinuous. If this is the case, following the same path used in the proof of 5.2.7 one can show that Ψ'_f satisfies all the assumptions of Theorem 3.3.1. This would prove the existence of an equilibrium price for Ψ'_f and hence the existence of a competitive objection to f by Proposition 5.2.5.

[4]By *finitely generated simplex* in Δ we mean the convex hull of a finite subset of Δ.

5.3 Objection mechanism in imperfect markets

In Theorem 5.2.7 we have proved the existence of competitive objections assuming that the space of agents was saturated. In general, when this is not the case, it is possible that a feasible but non-competitive allocation cannot be blocked by "standard" coalitions and therefore the core is strictly greater than $W(\mathcal{E})$. It is also possible that $W(\mathcal{E})$ coincides with the core but is still strictly contained in $BS(\mathcal{E})$. In this section we study different notions of bargaining set in which we relax the class of coalitions that can raise objections and counter-objections, then study the relations between this new notion of bargaining set and competitive allocations.

Recall that by *coalition* we mean an element of Σ of positive measure. In the following, we will often refer to them as *standard*, or *crisp*, coalitions. For the rest of this section we will identify each $A \in \Sigma$ with the corresponding characteristic function χ_A, which is the one that assigns 1 to each $t \in A$ and 0 to every other agent. This way we can think of Σ as a subset of:

$$\mathcal{B} := \{\gamma : T \to [0,1] : \ \gamma \text{ is measurable}\}$$

An element $\gamma \in \mathcal{B}$ is an *Aubin coalition* if its support S_γ, which is the set $\{t \in T : \gamma(t) > 0\}$, is a non-null set. Let us write $m(\gamma)$ instead of $\int \gamma(t)\,dt$ and observe that γ is an Aubin coalition if and only if $m(\gamma) > 0$. Intuitively, for an Aubin coalition γ the value $\gamma(t)$ represents agent t's share of resources employed in the formation of the coalition γ while $m(\gamma)$ can be thought as a numerical evaluation of the economic weight of γ (for a standard coalition it simply coincides with its measure).

5.3.A Aubin-bargaining sets

The notion of *objections*, *counter-objections* and *bargaining set*, as introduced in Mas-Colell (1989), can be extended to include also Aubin coalitions. Let us fix an allocation f. An Aubin coalition γ with support S can *object* or *block* f if there is an allocation g such that:

(i) $\int \gamma(t)g(t)\,dt \le \int \gamma(t)e(t)\,dt$,

(ii) $g(t) \succcurlyeq_t f(t)$ for almost every $t \in S$,

(iii) $m(\{t \in S : \ g(t) \succ_t f(t)\}) > 0$.

In this case, we call (γ, g) an *Aubin-objection* to f and write $(\gamma, g) \in Ob_A(f)$. We stress that a standard coalition $S \in \Sigma$ can object f via some allocation g if and only if (χ_S, g) is an Aubin-objection to f. In the light of this, we call *standard* *objection* any Aubin-objection that can be obtained from a standard coalition, i.e. any pair $(\gamma, g) \in Ob_A(f)$ that is such that $\gamma = \chi_A$ for some $A \in \Sigma$.

Let us now suppose that (γ, g) is an Aubin-objection to f. An Aubin coalition δ with support Q *counter-objects* (γ, g) if there is an allocation h such that:

(i) $\int \delta(t)h(t)\,dt \leq \int \delta(t)h(t)\,dt$,

(ii) $h(t) \succ_t g(t)$ for every $t \in Q$ such that $\delta(t) + \gamma(t) > 1$,

(iii) $h(t) \succ_t f(t)$ for every $t \in Q$ such that $\delta(t) + \gamma(t) \leq 1$.

In this case we say that (δ, h) is an *Aubin-counter-objection* to (γ, g) and write $(\delta, h) \in Cob_A(\gamma, g)$. Similarly to what we did for objections, we call $(\delta, h) \in Cob_A(\gamma, g)$ a *standard counter-objection* to (γ, g) if $\delta = \chi_Q$ for some $Q \in \Sigma$.

A *bargaining set* is the collection of all feasible allocations against which it is impossible to raise an objection that is not counter-objected itself. Different notions of bargaining sets can therefore be obtained by specifying which classes of objections and counter-objections are allowed at each time.

With the definitions given above, we can introduce four different types of bargaining sets depending on whether or not Aubin or standard objections and counter-objections are considered. Formally, for a feasible allocation f we will say that:

- $f \in BS_{ss}$ if all standard objections to f have a standard counter-objection.
- $f \in BS_{as}$ if all Aubin-objections to f have a standard counter-objection.
- $f \in BS_{sa}$ if all standard objections to f have an Aubin-counter-objection.
- $f \in BS_{aa}$ if all Aubin-objections to f have an Aubin-counter-objection.

One of the main interests in this chapter is to determine the relations between these four notions of bargaining sets and the set of competitive allocations. Clearly, every $f \in W(\mathcal{E})$, is a feasible allocation that cannot be objected at all and as such it belongs to each of the bargaining sets we have defined. Another observation we can make is that a bargaining set will shrink whenever we allow a larger set of objections or a smaller set of counter-objections. This can be used to prove the following result.

Proposition 5.3.1. *The following inclusions always hold.*

- $W(\mathcal{E}) \subseteq BS_{sa} \subseteq BS_{ss} \subseteq BS_{as}$.
- $W(\mathcal{E}) \subseteq BS_{sa} \subseteq BS_{aa} \subseteq BS_{as}$.

In general, without making any further assumption on preferences, all the inclusions in the Proposition above are not strict.

Remark 5.3.2. Extensions of the notion of bargaining set to Aubin coalitions have been already introduced, among the others, in Hervés-Estévez and Moreno-García (2018a,b) and again in Hervés-Beloso et al. (2018) to study finite economies and their replicas. In the upcoming work by Graziano et al. (2019a), similar notions of bargaining sets are used to study competitive allocations in mixed markets.

We shall point out here that our notion of *Aubin-counter-objection* differs from that given in the mentioned articles where, instead of condition (ii) is asked that:

$$h(t) \succ_t g(t) \text{ for all } t \in Q \cap S_\gamma. \tag{$(ii)'$}$$

It is clear that $(ii)'$ implies the condition (ii) we have chosen here. This will make it harder to find Aubin-objections against which it is not possible to raise Aubin-counter-objections in our settings.

Other notions of bargaining set for mixed markets are described in the forthcoming by Graziano et al. (2019b).

Last, we adapt to this new framework the notion of *competitive objection*.

Definition 5.3.3. *Let f be a feasible allocation. An Aubin-objection (γ, g) to f is* competitive *at a price system p if for almost every $t \in T$ and every $x \in X(T)$ one has:*

- $p \cdot x \geq p \cdot e(t)$ *whenever* $\gamma(t) = 1$ *and* $x \succcurlyeq_t g(t)$*,*
- $p \cdot x \geq p \cdot e(t)$ *whenever* $\gamma(t) < 1$ *and* $x \succcurlyeq_t f(t)$*.*

Once again we point out that if (S, g) is a competitive (standard) objection to f if and only if (χ_S, g) is a competitive Aubin-objection to f. Another observation we can make is that $(\gamma, g) \in Ob_A(f)$ will be competitive if and only if there is a price system p such that $f(t) \succcurlyeq_t \beta(p, t)$ for every t such that $\gamma(t) < 1$ and $g(t) \succcurlyeq_t \beta(p, t)$ for every $t \in S_\gamma$. To see this notice that, for almost every $t \in S_\gamma$, either $\gamma(t) = 1$, and so $g(t) \succcurlyeq_t \beta(p, t)$ by definition, or $\gamma(t) < 1$ and so $g(t) \succcurlyeq_t f(t)$ which in turn is weakly preferred to every $x \in \beta(p, t)$.

5.3.B Existence of competitive Aubin-objections

We now show how, under suitable assumptions, it is possible to find a competitive Aubin-objection to a given feasible but non-competitive allocation. More formally we want to prove the following variation on the Theorem 5.2.7.

Theorem 5.3.4. *Suppose that Assumptions 8 and 9 are satisfied and let f be a feasible but non-competitive allocation with the equal treatment property. Then there is a competitive Aubin-objection to f.*

Once again, the ideas of the proof will be standard and are somehow similar to those given in section 5.2.B. More precisely, we will first characterize the competitive Aubin-objections by means of the equilibrium prices of the correspondence $\overline{co}\Psi_f$ (where Ψ_f is the one defined in Proposition 5.2.5) and then show that such a correspondence satisfies all of the conditions of Theorem 5.2.6.

Proposition 5.3.5. *If the assumptions of Theorem 5.3.4 are met then there is a competitive Aubin-objection to f at a price system p if the negative cone $-E_+$ intersects the set:*

$$K(p) := \left\{ \int \gamma(t)(g(t) - e(t))\,dt : \; C_f(p) \le \gamma \le D_f(p), \; g \in \mathcal{L}\left(\tilde{\xi}(p, \cdot)\right) \right\}^{5}.$$

Proof. Suppose that there are a $p \in \Delta$ and a $x \in K(p)$ such that $x \in -E_+$. By definition, there must be a selection g of $\tilde{\xi}(p, \cdot)$ and a $\gamma \in \mathcal{B}$ such that $C_f(p) \subseteq \gamma \subseteq D_f(p)$ and $x = \int \gamma(t)(g(t) - e(t))\,dt$. Let S denote the support of γ and observe that, being $C_f(p)$ a subset of S of positive measure, S is a coalition.

We will first show that (γ, g) is an Aubin-objection to f. Being g a selection of $\tilde{\xi}(p, \cdot)$ we know that $g(t) \succcurlyeq_t f(t)$ for every $t \in D_p(f)$ with strict preference when $t \in C_f(p)$. But then $g(t) \succcurlyeq_t f(t)$ for almost every $t \in S$ and $m(\{t \in S : \; g(t) \succ_t f(t)\}) = m(C_f(p)) > 0$. Last, the condition for which $\int \gamma(t)g(t)\,dt \le \int \gamma(t)e(t)\,dt \le 0$ is satisfied because $\int \gamma(t)(g(t) - e(t))\,dt = x$ and $x \in -E_+$.

We now prove that (γ, g) is competitive at the price p. Pick any $t \in T$ and let $x \in X(t)$. If $\gamma(t) = 1$, then $t \in S$ and so $g(t) \in \tilde{\xi}(p, t)$. Therefore, $p \cdot x \ge p \cdot e(t)$ whenever $x \succcurlyeq_t g(t)$. If $\gamma(t) < 1$, then $t \notin C_f(p)$ and so $f(t) \succcurlyeq_t \beta(p, t)$. But then $x \succcurlyeq_t f(t)$ implies that $p \cdot x \ge p \cdot e(t)$. □

Lemma 5.3.6. *Under the Assumptions of Theorem 5.3.4, for every $p \in \Delta$, the set $K(p)$ is weakly compact and coincides with $\overline{co}\Psi_f(p)$.*

Proof. Fix a $p \in \Delta$. We first prove that $K(p)$ is a convex and weakly compact subset of E. Let us define $\mathcal{B}_p := \{\gamma \in \mathcal{B} : \; C_f(p) \le \gamma \le D_f(p)\}$ and call $\mathcal{S}$ the set of integrable selections of $\tilde{\xi}(p, \cdot) - e(\cdot)$. As a consequence of Assumption 8 and Theorem 5.1.3, we can rewrite the set $K(p)$ in the following way:

$$\left\{ \int \gamma(t)g(t)\,dt : \; \gamma \in \mathcal{B}_p, \; g \in \overline{co}\mathcal{S} \right\}.$$

[5] Since we are identifying sets in Σ with functions in $\mathcal{B}$, for every $A \in \Sigma$ and $\gamma \in \mathcal{B}$ we write $A \le \gamma$ if and only if $\chi_A(t) \le \gamma(t)$ almost everywhere.

In other words, the set $K(p)$ is the image of $\mathcal{B}_p \times \overline{co}\mathcal{S}$ under the operator $F\colon L^\infty(m) \times L^1(m,E) \to E$ that assigns to every $\gamma \in L^\infty(m)$ and $g \in L^1(m,E)$ the integral $\int \gamma(t)g(t)\,dt$.

Observe now that $\mathcal{B}_p$ is a bounded, closed subset of $L^\infty(m)$ and hence it is weakly* compact by the Alaoglou's Theorem (see (Aliprantis and Border, 2006, Theorem 5.105)). At the same time $\mathcal{S}$ is a relatively weakly compact subset of $L^1(m,E)$ by Diestel's Theorem (Diestel (1977)). This means that the set $\mathcal{B}_p \times \overline{co}\mathcal{S}$ is a convex subset in $L^\infty(m) \times L^1(m,E)$ that is compact in the product topology τ obtained from the weak* topology on $L^\infty(m)$ and the weak topology on $L^1(m,E)$. But then, being the restriction of F to $\mathcal{B}_p \times \overline{co}\mathcal{S}$ continuous with respect to the topology τ and the weak topology on E, $K(p) = F(\mathcal{B}_p \times \overline{co}\mathcal{S})$ is convex and weakly compact too.

To prove that $\overline{co}\Psi_f(p) = K(p)$ let us first observe that $\Psi_f(p) \subseteq K(p)$ holds by construction. To show the reverse inclusion consider a simple function $\gamma \in \mathcal{B}_p$, a $g \in \mathcal{S}$ and claim that $x := \int \gamma(t)g(t)\,dt$ belongs to $co\Psi_f(p)$. This will prove that a weakly dense subset in $K(p)$ is contained in $\overline{co}\Psi_f(p)$ and so $K(p) \subseteq \Psi_f(p)$. Since γ is simple there will be $A_1,\dots,A_n \in \Sigma$ and $\theta_1,\dots,\theta_n \in [0,1]$ such that $\gamma = \sum_{i=1}^n \theta_i \chi_{A_i}$ and $1 = \sup_t |\gamma(t)| = \sum_{i=1}^n \theta_i$. Furthermore, since $C_f(p) \le \gamma \le D_f(p)$, we can choose the A_i's so that $C_f(p) \subseteq A_i \subseteq D_f(p)$ for every $i \le n$. We can conclude that:

$$x = \int \sum_{i=1}^n \theta_i \chi_{A_i}(t) g(t)\,dt = \sum_{i=1}^n \theta_i \int_{A_i} g(t)\,dt.$$

By the choice of the A_i's, we have that $x_i := \int_{A_i} g(t)\,dt \in \Psi_f(p)$ for every $i \le n$ which concludes the proof. □

We now have all of the ingredients to prove the main theorem of this section.

Proof of Theorem 5.3.4. We shall prove that the correspondence $\overline{co}\Psi_f\colon \Delta \twoheadrightarrow E$ satisfies all of the requirements in Theorem 5.2.6. This way there would be a price system $\overline{p} \in \Delta$ such that $-E_+ \cap K(\overline{p}) \neq \varnothing$ and so, by Proposition 5.3.5, there would be a competitive Aubin-objection to f.

It has been proved in Theorem 5.2.7 that: (i) the correspondence Ψ_f is upper hemicontinuous with respect to the weak topology on E, (ii) for every $p \in \Delta$ there is a $z \in \Psi_f(p)$ such that $p \cdot z \le 0$. Furthermore, by Lemma 5.3.6 we know that (iii) $\overline{co}\Psi_f(p)$ is convex and weakly compact for every $p \in \Delta$. By Theorem (Aliprantis and Border, 2006, 17.35), points (i) and (ii) are enough to ensure that the correspondence $\overline{co}\Psi_f$ is upper-hemicontinuous with respect to the weak topology on E as well. Since all of the conditions in Theorem 5.2.6 are satisfied, there must be a $\overline{p} \in \Delta$ and a $z \in \overline{co}\Psi_f(\overline{p}) = K(\overline{p})$ that is such that $z \le 0$. This concludes the proof. □

Corollary 5.3.7. *Suppose that Assumptions 8 and 9 hold. Then $W(\mathcal{E})$ is non-empty and coincides with:*

$$\{f \in BS_{as} : \ f \textit{ has the equal treatment property}\}.$$

Proof. Let us assume that the initial endowment e is non-competitive. By Theorem 5.3.4 there must be an Aubin-objection (γ, g) to e that is competitive for some price system p. This means that, for almost every $t \in S_\gamma$, $g(t) \in \tilde{\xi}(p,t)$ and so $p \cdot (g(t) \geq p - e(t))$ by Lemma 5.1.2. But then, since $0 \geq p \cdot x = \int \gamma(t) p \cdot (g(t) - e(t))\, dt$, it must be that $p \cdot g(t) = p \cdot e(t)$. This means that $g(t) \in \xi(p,t)$ for almost every $t \in S_\gamma$. At the same time we know that $e(t) \succsim_t \beta(p,t)$ for almost every $t \notin S_\gamma$ which in turn implies that $e(t) \in \xi(p,t)$. But then the allocation $h := g\chi_{S_\gamma} + e\chi_{S_\gamma^c}$ is feasible because $\int_T (\tilde{g}(t) - e(t))\, dt = \int_S (g(t) - e(t))\, dt$ and it is such that $h(t) \in \xi(p,t)$ for almost every $t \in T$. This proves that h is competitive.

To prove the equivalence observe that every $f \in W(\mathcal{E})$ is an allocation in BSf_{as} with the equal treatment property that cannot be blocked. To show that the reverse inclusion holds suppose that $f \notin W(\mathcal{E})$ is feasible and has the equal treatment property and apply Theorem 5.3.4 to find a competitive Aubin-objection to f. Since every competitive objection is Aubin-justified by Theorem 5.2.4, f does not belong to BS_{as} either. $\square$

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