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The $O(D)$-equivariant fuzzy hyperspheres

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Chapter 0

Introduction

Heisenberg's uncertainty principle bounds the precisions with which one can measure the position and the momentum of a particle simultaneously, but puts no bounds on each of them separately. A fundamental bound on the precision of position measurements is expected to arise in any consistent quantum theory of gravitation, whatever it will be; this bound is of the order of 10^{-33}cm, the socalled *Planck length*. As first suggested by Heisenberg[1], a fundamental length might also play a role as a parameter regularizing the divergences arising in quantum field theory (even on Minkowski spacetime); and a more commonly adopted regularization parameter is an energy cutoff. Such lengths may also help to unify fundamental interactions (see e.g. [3, 4, 5, 6, 7]) and they might naturally arise from small but non-vanishing commutators between different coordinates.

These are some physical motivations of *Non-Commutative Geometry* [8, 9], whose program is to develop the analog of differential geometry after replacing the commutative algebra of functions on a manifold by a noncommutative one, e.g. generated by a set of non-commutative coordinates (the quantization of the phase space of a particle in nonrelativistic quantum mechanics can be seen as the first example of non-commutative geometry).

Often one deals with a one (or more) parameter family of noncommutative geometries which become commutative when the parameter(s) go to some limit (e.g. $\hbar \to 0$ in the previous example).

Fuzzy Spaces are particular examples parametrized by a positive integer n, such that the *noncommutative* algebra $\mathcal{A}_n$ (playing the role of 'algebra of functions') has a finite dimension increasing and diverging with n and $\mathcal{A}_n \overset{n\to\infty}{\longrightarrow} \mathcal{A}$, which is the algebra of regular functions on an ordinary manifold.

The first and seminal fuzzy space is the Fuzzy 2-Sphere (FS) of Madore and

[1]Heisenberg proposed it in a letter to Peierls [1] to solve the problem of divergent integrals in relativistic quantum field theory. The idea propagated via Pauli to Oppenheimer. In 1947 Snyder, a student of Oppenheimer, published the first proposal of a quantum theory built on a noncommutative space [2].

Hoppe [10, 11], it is a sequence of $SO(3)$-equivariant[2] [$SO(D)$ is the rotation group in D dimensions], finite noncommutative algebras $\mathcal{A}_n$ isomorphic to $\mathcal{M}_n$ (the algebra of $n \times n$ matrices); each matrix represents the expansion in spherical harmonics of an element of $C(S^2)$ truncated at level n. The algebra $\mathcal{A}_n$ is generated by coordinate operators $\{x_i\}_{i=1}^3$ fulfilling

$$[x_i, x_j] = \frac{2i}{\sqrt{n^2-1}} \varepsilon^{ijk} x_k \qquad \text{and} \qquad \sum_{i=1}^{3} x_i x_i \equiv 1, \tag{1}$$

with $n \in \mathbb{N} \setminus \{1\}$. In fact, these operators are obtained by the rescaling

$$x_i = \frac{2L_i}{\sqrt{n^2-1}}, \quad i = 1, 2, 3 \tag{2}$$

of the elements L_i of the standard basis of $\boldsymbol{so}(3)$ in the irrep $(\pi_l, V_{l,3})$ characterized by $\boldsymbol{L}^2 := \sum_{i=1}^3 L_i L_i \equiv l(l+1)I$, or equivalently the one of dimension $n = 2l+1$.

In this thesis we propose and study a new class of fuzzy spaces: for every dimension $D \geq 2$ we propose an $O(D)$-equivariant [$O(D)$ is the orthogonal group in D dimensions] fuzzy hypersphere S^d_Λ, where $d := D - 1$. The relations (1) are covariant under $SO(3)$, but not under the whole $O(3)$, in particular not under parity $x_i \mapsto -x_i$; this is in contrast with the $O(3)$-covariance of both the ordinary sphere S^2 [where the right-hand side of $(1)_1$ is zero] and the new $O(3)$-equivariant fuzzy 2-sphere S^2_Λ [where the right-hand side of $(1)_1$ depends on the angular momentum components, as in Snyder [2] commutation relations]; the coordinate operators $\{\overline{x}_i\}_{i=1}^D$ of these new fuzzy spaces generate the whole algebra of observables $\mathcal{A}_{\Lambda,D}$, as for the FS. Moreover, while the Hilbert space $V_{l,3}$ of the FS carries an irreducible representation of $SO(3)$, that $\mathcal{L}^2(S^d)$ of a quantum particle on S^d decomposes as the direct sum of *all* the vector irreducible representations of $SO(D)$:

$$\mathcal{L}^2(S^d) = \bigoplus_{l=0}^{\infty} V_{l,D}. \tag{3}$$

It turns out that the one $\mathcal{H}_{\Lambda,D}$ of S^d_Λ decomposes as the direct sum $\mathcal{H}_{\Lambda,D} = \bigoplus_{l=0}^{\Lambda} V_{l,D}$, and therefore also in this aspect S^d_Λ better approximates the configuration space S^d in the limit $\Lambda \to \infty$, so the Madore FS algebra $\mathcal{A}_n$ should be seen simply as the spin phase space algebra, not as a fuzzyfication of the algebra of configuration space observables on S^2.

According to this, we believe that this framework is an improvement of Madore fuzzy approximation of the sphere because

[2] The concept of equivariance (or covariance) is very relevant in Physics: given a map f from an algebra $\mathcal{M}$ to an algebra $\mathcal{N}$ (not necessarily distinct from $\mathcal{M}$) one says that it is equivariant with respect to an abstract group G (which acts on $\mathcal{M}$ and $\mathcal{N}$) if applying a G transformation and then computing the function produces the same result as computing the function and then applying the G transformation.

1. We obtain $O(D)$-equivariant algebraic relations between the operators, while the Madore's ones are covariant with respect to a smaller group.

2. In the commutative limit, our Hilbert space of admitted states coincides with $\mathcal{L}^2(S^d)$, and this is not true for the FS.

3. We are able to fuzzy approximate, in every dimension $D \geq 2$, all the coordinate operators and also all the quantum angular momentum operator components of $\mathbb{R}^D$; and this is not true for the FS.

The commutation relations among our coordinates x^i are similar to the ones among the coordinates in Snyder's quantized spacetime algebra [2]. The latter is generated by 4 coordinate operators $\{\hat{x}^\mu\}_{\mu=0,1,2,3}$, and 4 momentum operators $\{\hat{p}_\mu\}_{\mu=0,1,2,3}$; they fulfill (here α is a suitable constant)

$$[\hat{p}_\mu, \hat{p}_\nu] = 0, \qquad [\hat{x}^\mu, \hat{p}_\nu] = i\hbar(\delta^\mu_\nu - \alpha\hat{p}^\mu\hat{p}_\nu), \qquad [\hat{x}^\mu, \hat{x}^\nu] = -i\hbar\alpha L^{\mu\nu}, \tag{4}$$

where we have set $L^{\mu\nu} = \hat{x}^\mu\hat{p}^\nu - \hat{x}^\nu\hat{p}^\mu$ and we raise and lower indices by the Minkowski metric $\eta = \mathrm{diag}(1, -1, -1, -1) = \eta^{-1}$: $v^\mu = \eta^{\mu\nu}v_\nu$, $v_\mu = \eta_{\mu\nu}v^\nu$. These relations are invariant under inversion of the axes, in particular under parity. The $L^{\mu\nu}$ span the Lorentz Lie algebra, and their commutation relations with the 4-vectors $\hat{p}^\mu, \hat{x}^\mu$ are as on the Minkowski space.

This thesis can be divided in four parts. The readers that are familiar with Lie groups, Lie algebras representations, root vectors, weights and noncommutative spaces can skip the first part (chapter 1), which is based on [12] and contains a short introduction to the above topics, and to noncommutative geometries; group representations are applied in the second part (which is partially based on an unpublished article) to construct new noncommutative-geometry toy-models S^d_Λ (chapter 2), which may be useful in some modern applications to condensed matter physics and quantum field theory; the readers that want to give a first look to the simplest elements of our new class of fuzzy spaces, i.e. the fuzzy low-dimensional spheres S^1_Λ, S^2_Λ, S^3_Λ and S^4_Λ, can firstly read chapter 3. Finally, in the third (chapter 4) and fourth (chapter 5) parts we study further aspects of these new noncommutative spaces.

First of all, in section 1.1 the notion of group representation is introduced, that is a homomorphism between the group and a group of operators over a vector space. Then reducible and irreducible representations are introduced and characterized; these characterizations are necessary since one cannot always recognize if a representation is irreducible only by the use of a matrix realization.

Then we do the proof of the Schur's lemma, which is very useful in group representations, for example when there are operators of a Lie algebra which commute with all the others; section 1.2 contains a summary about semisimple Lie algebras, root vectors, weights and weight spaces. Sections 1.3 and 1.4, which are about the algebra $\boldsymbol{so}(D)$.

In this thesis we construct a fuzzy approximation of the unit d-sphere S^d obtained as the hypersurface of $\mathbb{R}^D$ $S^d = \left\{x \in \mathbb{R}^D : \|x\|^2 = 1\right\}$. An alternative way to obtain S^d would be as the coset space $S^d = SO(D)/SO(d)$. For completeness, in section 1.5 we give a short review on coset space geometry and systematic harmonic analysis on coset spaces.

A simple mechanism to modify a quantum mechanical model with commuting coordinates into one with non-commuting coordinates is illustrated by the well-known Landau model, which describes a charged quantum particle interacting only with an uniform magnetic field (in the z direction) $\boldsymbol{B}$. The separation between the levels of energy is $\frac{\|e\boldsymbol{B}\|}{mc}$; if $B := \|\boldsymbol{B}\|$ is strong (or, equivalently, m is small) and the energy is constrained to be below a cutoff $\overline{E}$, then the Hilbert space of states is projected to the subspace of lowest energy, and $\frac{e}{c}Bx, y$ become canonically conjugates, i.e. have a non-zero (but constant) commutator.

Inspired by the Landau model, in chapter 2 a quantum particle is considered in every dimension $D \geq 2$, where the Hamiltonians consist of the standard kinetic terms and rotation invariant potential energies $V(r)$ with a very deep minimum (well) on the d-dimensional sphere of unit radius. The imposition of an energy cutoff makes only a finite-dimensional Hilbert subspace $\mathcal{H}'$ accessible and the coordinates become noncommutative on $\mathcal{H}'$; they also generate the whole algebra of observables of $\mathcal{H}'$. On $\mathcal{H}'$ the distance from the origin is not strictly 1, but its spectrum collapses to 1 (with the exception of highest square angular momentum eigenvalue) in the limit of an infinitely narrow and deep well; the latter can be considered as a quantum version of the constraint $r = 1$.

In other words, in that chapter, the procedure used in [13, 14] is applied to the generic D-dimensional case; in this way the fuzzy spheres constructed are equivariant not only under $SO(D)$, but under the full orthogonal group $O(D)$, obtaining then an $O(D)-$equivariant fuzzy sphere, for every dimension D. Furthermore, the algebra of observables $\mathcal{A}_{\Lambda,D}$ is realized through a suitable irreducible vector representation of $U\boldsymbol{so}\,(D+1)$ (the universal enveloping algebra of $\boldsymbol{so}\,(D+1)$; see definition 1.1.11), and then we do the proof of the convergence (in a certain sense) of this new fuzzy sphere to ordinary quantum mechanics on the sphere S^d.

The aforementioned procedure does not strictly depend on the dimension of the carrier space, but one has to replace all the 2-dimensional and 3-dimensional objects by the corresponding D-dimensional ones; for instance, the D-dimensional spherical harmonics are needed, together with the action on them of the D-dimensional angular momentum operator components.

For this reason, let

$$L_{h,j} := \frac{1}{i}\left(x_h \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_h}\right) \quad \text{with} \quad h, j \in \{1, 2, \cdots, D\} \tag{5}$$

be a component of the quantum angular momentum in $\mathbb{R}^D$, and

$$C_p := \sum_{1 \le h < j \le p} L_{h,j}^2 \quad \text{with } p \in \{2, 3, \cdots, D\} \tag{6}$$

be the realization of the quadratic Casimir of $U\boldsymbol{so}(p)$; in particular, $C_D \equiv \boldsymbol{L}^2$ is the opposite of the Laplace-Beltrami operator Δ_{S^d} on the sphere S^d. This and the fact that the action of $C_{\widetilde{D}}$ in $S^{\widetilde{D}-1}$ coincides with the one in S^d (see section 7.0.1) imply that C_p is the opposite of the Laplace-Beltrami operator $\Delta_{S^{p-1}}$ on the sphere S^{p-1} in every dimension $D \ge p \ge 2$, and its eigenvalues (see [15], p. 169, theorem 22.1) are

$$l_{p-1}(l_{p-1} + p - 2), \quad \text{with} \quad l_1 \in \mathbb{Z} \quad \text{and} \quad l_{p-1} \in \mathbb{N}_0 \quad \forall p > 2. \tag{7}$$

Following [13, 14], start with a quantum particle in $\mathbb{R}^D$ subject to a confining potential well $V(r)$, which has a very deep minimum in $r = 1$ [$\Rightarrow V'(1) = 0$, $V'(1) =: 4k_D$, with $k_D \gg 0$]; assume that, when $r \approx 1$, it can be approximated with the potential of a one-dimensional harmonic oscillator, in symbols $V(r) \simeq V_0 + 2k_D(r-1)^2$, where $V_0 := V(1)$ and k_D plays the role of a confining parameter.

This choice of $V(r)$ ensures that, in the limit $k_D \to +\infty$, the quantum particle is forced to stay on the unit d-dimensional sphere S^d, and this leads to prove also the convergence of this new fuzzy space to ordinary quantum mechanics on the sphere, in that limit.

Once introduced this $V(r)$, one has to study the eigenvalue equation

$$H\boldsymbol{\psi} = \left[-\frac{1}{2}\Delta + V(r)\right]\boldsymbol{\psi} = E\boldsymbol{\psi}, \tag{8}$$

which is a PDE in the unknowns $\boldsymbol{\psi}, E$ and its resolution provides a basis of the Hilbert space of quantum states $\mathcal{H}_D$; in addition, from

$$\begin{gathered} [H, L_{i,j}] = 0 \quad \forall 1 \le i < j \le D, \\ [L_{1,2}, C_{p_2}] = [C_{p_1}, C_{p_2}] = 0, \quad \forall p_1, p_2 \in \{2, 3, \cdots, D\} \end{gathered} \tag{9}$$

it follows that H, $L_{1,2}$ and all these C_p operators can be simultaneously diagonalized in the resolution of (8).

In order to do this, let's look for an eigenfunction $\boldsymbol{\psi}$ in the form

$$\boldsymbol{\psi} = f(r)Y(\theta_d, \theta_{d-1}, \cdots, \theta_1), \tag{10}$$

where Y is a common eigenfunction of the CSCO (Complete set of commuting observables, i.e. a set of commuting operators whose set of eigenvalues completely specify the state of a system) $L_{1,2}$, $C_2, \cdots, C_d$ and $\boldsymbol{L}^2$; while $r, \theta_d, \theta_{d-1}, \cdots, \theta_1$ are polar coordinates. It is obvious that, in order to have $\boldsymbol{\psi} \in \mathcal{L}^2\left(\mathbb{R}^D\right)$, it is necessary that $r^d f \in \mathcal{L}^2(\mathbb{R}_+)$ and $Y \in \mathcal{L}^2\left(S^d\right)$.

The Ansatz (10) transforms the PDE $H\boldsymbol{\psi} = E\boldsymbol{\psi}$ into an ODE in the unknown f, which is solved in section 2.1.1; while in section 2.1.2 an orthonormal basis of $\mathcal{L}^2\left(S^d\right)$ of eigenfunctions of $\boldsymbol{L}^2$ is determined, in particular we prove that every basis-function Y is uniquely determined by a collection of d indices $\boldsymbol{l} := (l_d, \cdots, l_2, l_1)$, fulfilling

$$C_p Y_{\boldsymbol{l}} = l_{p-1}(l_{p-1} + p - 2)Y_{\boldsymbol{l}} \quad , \quad l_d \geq \cdots \geq l_2 \geq |l_1| \quad \text{and} \quad l_i \in \mathbb{Z} \quad \forall i.$$

Then, it turns out that an orthonormal basis $\mathcal{B}_D$ of the space of quantum states $\mathcal{H}_D$ is (here and later on $l := l_d$)

$$\mathcal{B}_D = \left\{ f_{n,l,D}(r) Y_{\boldsymbol{l}}(\theta_d, \theta_{d-1}, \cdots, \theta_1) | n \in \mathbb{N}_0, l \geq l_{d-1} \geq \cdots \geq l_2 \geq |l_1|, l_i \in \mathbb{Z} \forall i \right\}.$$

Furthermore, the consequence of the imposition of a sufficiently low energy cutoff $E \leq \overline{E}$ (see section 2.2) is that the Hilbert space of 'admitted' states $\mathcal{H}_{\overline{E},D} \subset \mathcal{H}_D$ becomes finite-dimensional and spanned by all the H-eigenstates having eigenvalues $E \leq \overline{E}$. We also replace every observable A by the corresponding projected one $\overline{A} := P_{\overline{E},D} A P_{\overline{E},D}$ (here and later on $P_{\overline{E},D}$ is the projection on $\mathcal{H}_{\overline{E},D}$) and we give to $\overline{A}$ the same physical interpretation; in this way we have only states and operators that are 'physical'.

The condition $\overline{E} < 2\sqrt{2k_D}$ implies that the Hamiltonian operator H can be seen, in a first approximation, as the square angular momentum operator $\boldsymbol{L}^2$ (in other words radial excitations are 'frozen'), while two crucial steps, necessary to obtain a fuzzy space, are the choice of a Λ-dependent energy cutoff $\overline{E} := \overline{E}(\Lambda)$ so that $\overline{E}(\Lambda)$ diverges with $\Lambda \in \mathbb{N}$, and the assumption that also k_D depends on Λ in a way such that $\overline{E}(\Lambda) < 2\sqrt{2k_D(\Lambda)}$. This implies that the Hilbert space of admitted states can be definitively re-labeled as $\mathcal{H}_{\Lambda,D}$, and the corresponding algebra of observables $End(\mathcal{H}_{\Lambda,D})$ as $\mathcal{A}_{\Lambda,D}$; then the sequence $\{\mathcal{A}_{\Lambda,D}\}_{\Lambda\in\mathbb{N}}$ is made of finite-dimensional algebras, which become infinite dimensional in the limit $\Lambda \to +\infty$.

In order to calculate the algebraic relations between the generators of $\mathcal{A}_{\Lambda,D}$ we need to determine the action of every $\overline{L}_{h,j} \equiv L_{h,j}$ and $\overline{x}_h$ on a basis of $\mathcal{H}_{\Lambda,D}$. Because of (10), it is possible to use the knowledge of the action of $L_{h,j}$ on the spherical harmonics $Y_{\boldsymbol{l}}$ obtained from the above CSCO, to recover the one on $\boldsymbol{\psi}_{\boldsymbol{l},D}$; since we have not found the action of this in the literature when $D > 3$, we have explicitely calculated it in 2.3.1, while in section 2.3.2 we compute the action of coordinate operators $\overline{x}_h$.

As in [13, 14], in section 2.3.3 it is shown that $\overline{x}_h, \overline{x}_j$ fulfill *Snyder* commutation relations, in other words their commutator is proportional to the component $L_{h,j}$ of the D-dimensional angular momentum operator, up to a scalar operator depending on $\boldsymbol{L}^2$. Then there is a list of all the relations involving the projectors of $\mathcal{A}_{\Lambda,D}$ which show that the $\overline{x}$s generate the whole algebra of observables, for instance every component of the angular momentum operator can be written as a ordered polynomial in the $\overline{x}_h$. The square distance from the origin operator

$\boldsymbol{x}^2 := \sum_h \overline{x}_h \overline{x}_h$ is not identically 1, but a function of $\boldsymbol{L}^2$ such that nevertheless its spectrum is very close to 1 and collapses to 1 in the $k_D \to \infty$ limit.

Furthermore, in section 2.4 some tools of Lie algebra theory are used in order to realize the algebra of observables $\mathcal{A}_{\Lambda,D}$ through a suitable irreducible vector representation $\pi_{\Lambda,D+1}$ of $U\boldsymbol{so}(D+1)$; this is suggested by the fact that the dimension of $\mathcal{H}_{\Lambda,D}$ coincides with the one of the representation space $V_{\Lambda,D+1}$ of $\pi_{\Lambda,D+1}$; then (up to isomorphisms)

$$\mathcal{H}_{\Lambda,D} = \bigoplus_{l=0}^{\Lambda} V_{l,D} = V_{\Lambda,D+1}.$$

That realization is $O(D)$-equivariant and the algebra isomorphism Φ fulfills

$$[\Phi(A)]^\dagger = \Phi\left(A^\dagger\right). \tag{11}$$

The proof of the aforementioned convergence is a sort of certification of the goodness of this approximation of quantum mechanics on the sphere S^d, and this job is done in section 2.5; this is inspired by the behavior of the potential $V(r)$ in the limit $k_D \to +\infty$, where it forces the quantum particle to stay on the unit d-dimensional sphere S^d. The 'projected' spherical harmonics are firstly identified as a basis of a space of all spherical harmonics, $\mathcal{A}_{\Lambda,D}$ as a subalgebra of $\mathcal{B}\left(S^d\right)$, the algebra of bounded functions on S^d [or $\mathcal{C}\left(S^d\right)$, the algebra of polynomial functions on S^d], and then we prove the convergence (in a certain sense) of the operators in $\mathcal{A}_{\Lambda,D}$ to the corresponding ones in $\mathcal{B}\left(S^d\right)$ [$\mathcal{C}\left(S^d\right)$, respectively]; furthermore, we use a $k_D(\Lambda)$ growing faster with $\Lambda \in \mathbb{N}$ to prove this result.

It is possible to see the explicit constructions of S^d_Λ for $d = 1, 2, 3, 4$ in chapter 3, while in Appendix A (chapter 7) there are lengthy computations and proofs of that chapter.

For a coordinate operator x_i (from now later on we identify $x_i \equiv \overline{x}_i$) to approximate well and in an $O(D)$-equivariant way the corresponding coordinate of a quantum particle forced to stay on the unit sphere S^d, its spectrum Σ_{x_i} should fulfill at least the following properties, which are fulfilled also by the Madore FS:

1. The spectrum Σ_{x_i} of each x_i, for all choices of the orthogonal axes, is the same.

2. If α is an eigenvalue of x_i, then also $-\alpha$ is.

3. In the commutative limit the spectrum Σ_{x_i} becomes uniformly dense in $[-1, 1]$, in particular the maximal and the minimal eigenvalues converge to 1 and -1, respectively.

Then, in chapter 4 we present the study of the x_i-eigenvalue equation on S^d_Λ, based on [16], in particular to show that Σ_{x_i} fulfills these and other properties.

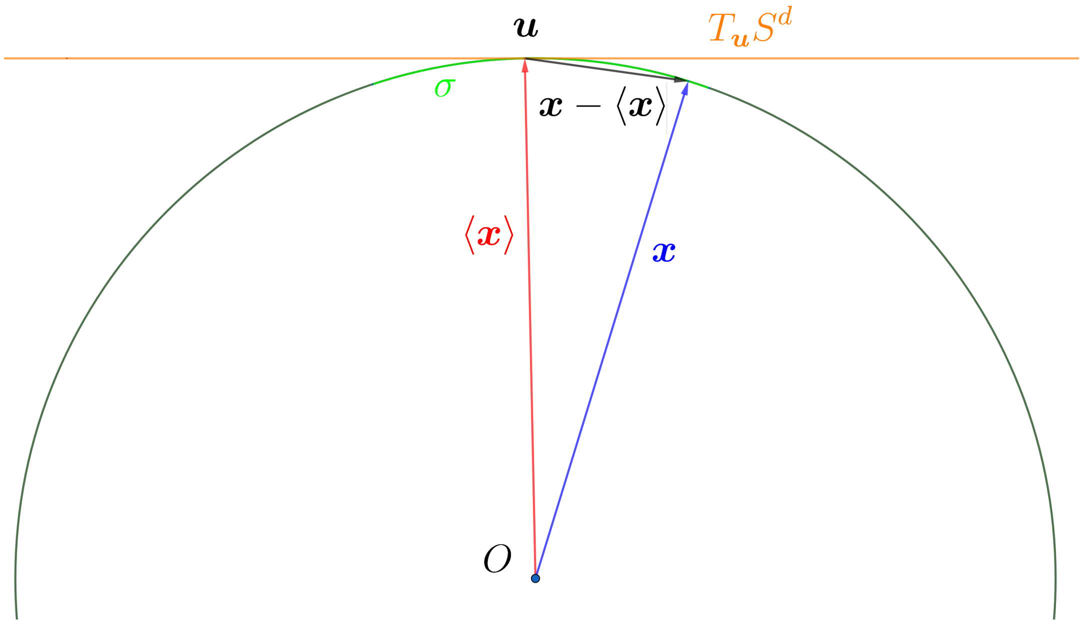

Figure 1: The vectors $\boldsymbol{x}$, $\langle \boldsymbol{x} \rangle$, $\boldsymbol{x} - \langle \boldsymbol{x} \rangle$, the region σ and the tangent plane $T_{\boldsymbol{u}}S^d$ at $\boldsymbol{u}$.

Among the latter one, not shared by the FS, justifies why (see chapter 6) the S^2_Λ can be interpreted as a fuzzy *configuration space*, while the FS should be interpreted only as a fuzzy *spin phase space*: namely that the eigenstate of x_3 with maximal eigenvalue (this is very localized around the North pole of S^2) is an eigenstate of L_3 with zero eigenvalue. In section 4.1 we do a summary about the diagonalization of a Toeplitz tridiagonal matrix; the x_i-eigenvalue equation on S^1_Λ, S^2_Λ and S^d_Λ when $d > 2$ is studied in sections 4.2, 4.3 and 4.4, respectively; then in chapter 6 we do a comparison between the results on S^2_Λ and FS, while in Appendix B (chapter 8) there are lengthy computations and proofs of that chapter.

The x_i-eigenvalue problem is strictly linked to the one of finding the most localized (and therefore closest to 'classical') states of this new fuzzy spaces: first of all, as a measure of localization of a state in configuration space its *spacial dispersion* is here adopted, i.e. the expectation value

$$(\Delta \boldsymbol{x})^2 := \sum_{i=1}^{D} (\Delta x_i)^2 \equiv \left\langle (\boldsymbol{x} - \langle \boldsymbol{x} \rangle)^2 \right\rangle = \left\langle \boldsymbol{x}^2 \right\rangle - \langle \boldsymbol{x} \rangle^2 \tag{12}$$

on the state; here $\boldsymbol{x} \equiv (x_1, ..., x_n)$, $\langle \boldsymbol{x} \rangle \equiv (\langle x_1 \rangle, ..., \langle x_n \rangle)$ pinpoints the average position of the particle in the ambient Euclidean space $\mathbb{R}^D$, the scalar observable $\boldsymbol{x}^2 := \sum_{i=1}^{D} x_i x_i$ measures the square distance from the origin, the vector observable $\boldsymbol{x} - \langle \boldsymbol{x} \rangle$ measures the displacement from the average position, and expression (12) is the average of the square of the latter. This choice is motivated by the fact that it is manifestly $O(D)$-invariant and that if the state is localized in a small region $\sigma \subset S^d$ around a point $\boldsymbol{u} \equiv \langle \boldsymbol{x} \rangle \in S^d$ then $(\Delta \boldsymbol{x})^2$ essentially reduces to the average square displacement in the tangent plane at $\boldsymbol{u}$, and the metric on the sphere is induced by the one in the ambient Euclidean space, as wished.

The above $O(D)$-symmetry means $(\Delta \boldsymbol{x})^2_{\boldsymbol{\psi}} = (\Delta R\boldsymbol{x})^2_{\boldsymbol{\psi}}$ for every state $\boldsymbol{\psi} \in \mathcal{H}_\Lambda$ and $O(D)$-transformation R. This implies that one can equivalently try to minimize $(\Delta \boldsymbol{x})^2 = \langle \boldsymbol{x}^2 \rangle - \langle x_i \rangle^2$ with a fixed $i \in \{1, \cdots, D\}$. On the other hand, since $\boldsymbol{x}^2 \simeq 1$ on the new fuzzy spheres, the most localized states are obtained once one determines the x_i-eigenstates corresponding to the maximal eigenvalues. The knowledge of the x_i-eigenvectors and most localized states will be essential for investigating the quantum metric aspects of these new fuzzy spheres, in particular for studying the 'distance' (either the spectral distance of Connes [8, 17, 18], or alternative ones, see e.g. [19, 20]) between two such pure states. Moreover, most localized states, especially when arranged in systems of coherent states [21, 22, 23, 24], are an extremely useful tool for a number of purposes (see e.g. [25, 26]), notably for studying path integrals (partition and correlation functions) in quantum field theory (QFT) both with a finite and with an infinite number of degrees of freedom. In particular, they may decisively simplify the computation of path integrals representing propagators, correlation functions and their generating functionals; this has applications in nuclear, atomic, condensed matter physics, as well as in QFT and elementary particle physics (see e.g. [27, 28, 29]). From a more foundation-minded viewpoint, the Berezin quantization procedure itself [30] on Kähler manifolds is based on the existence of sets of coherent states (see e.g. chpt. 16 in [31]). The 'cutoff' n (Λ in these new models) works as a regularizing parameter of ultraviolet divergences on all fuzzy spaces, so that integration over fields amounts to integration over matrices (see e.g. [4, 32] for the first QFT on the FS, and [33, 34, 35, 36] for examples of QFT on fuzzy spheres of higher dimensions); it has been recently proposed [5] that it may also parametrize the large (but finite) amount of information hidden in a black hole; finally, if spacetime M is enlarged to a higher-dimensional one $M' = M \times S_n$ - where S_n is a fuzzy space, instead of a compact manifold S - it reduces the number of massive Kaluza-Klein modes of a field theory on M' to a finite value [7]. According to this, the main aim of chapter 5, which is based on [37], is to introduce on S^d_Λ $(d = 1, 2)$ various systems of coherent states (SCS) and study their different localization properties in configuration as well as (angular) momentum space, which are respectively expressed in terms of the uncertainties $\Delta x_i, \Delta L_{ij}$; for equivariance reasons it is convenient to adopt $O(D)$ or - when this is redundant - $S^d = SO(D)/SO(d)$ as a label space parametrizing the elements of the SCS. Then, the SCS are considered both in the *strong sense*, i.e. providing a resolution of the identity, and in the *weak sense*, i.e. making up an (over)complete set in $\mathcal{H}_\Lambda$. $\Delta x_i, \Delta L_{ij}$ must fulfill a number of uncertainty relations and other inequalities following from the algebraic relations (commutation, etc.) among the x_i, L_{ij}. Neither on the commutative nor on the fuzzy spheres is it possible to saturate all of these inequalities (and their consequences, a fortiori), and for this reason in chapter 5 there is a preliminary discussion about the saturation of suitable $O(D)$-equivariant inequalities first on S^d, then on S^d_Λ; we privilege the latter because they have a physical meaning independent of the particular chosen reference frame, and because a state saturating them is automatically mapped into another one by the unitary transformation $U(g)$ corresponding to any orthogonal transformation $g \in O(D)$ [by definition $g_{ij}x_j = U^{-1}(g)\, x_i\, U(g)$, etc.]. Eq. (12) can be seen as a generalization of the square dispersion $(\Delta\boldsymbol{L})^2$ of the spin $\boldsymbol{L}$ as introduced by Perelomov [31], to which it reduces upon replacing $\boldsymbol{x}$ by $\boldsymbol{L}$. In fact, $(\Delta\boldsymbol{L})^2$ is adopted as a measure of localization of the state in (angular) momentum space. Given a state, consider an orthogonal transformation $g\in O(D)$ such that $g\langle\boldsymbol{x}\rangle = (|\langle\boldsymbol{x}\rangle|,0,...,0)$; then the state is mapped by $U(g)$ into a new one with the same $\langle\boldsymbol{x}^2\rangle$, $\langle x_1\rangle = |\langle\boldsymbol{x}\rangle|$, $\langle x_i\rangle = 0$ for $i>1$ (of course one obtains the same result replacing x_1 by any other x_i, or by the L_i). If $\boldsymbol{x}^2$ is central in the algebra of observables and the representation of the latter is irreducible, then $\langle\boldsymbol{x}^2\rangle$ is state-independent, and (12) is minimal on the state(s) that are eigenvectors of x_1 with the highest (in absolute value) eigenvalue. In particular, in Madore's FS it is $x_i \propto L_i$, $\boldsymbol{x}^2 \equiv 1$, and the spacial uncertainty (12) coincides up to a factor with the aforementioned $(\Delta\boldsymbol{L})^2$; hence on the representation space V_l it is minimized by the same SCS, on which it amounts to

$$(\Delta\boldsymbol{x})^2_{min} = \frac{2}{n+1} = \frac{1}{l+1}. \tag{13}$$

Using the results of chapter 4, it is possible to show that on the new fuzzy spheres S^d_Λ

$$(\Delta\boldsymbol{x})^2_{min} < \frac{C_d}{(\Lambda+1)^2}, \quad \text{where } C_d = \begin{cases} 3.5 & \text{if } d=1, \\ 11 & \text{if } d=2, \end{cases} \tag{14}$$

and that the states minimizing $(\Delta\boldsymbol{x})^2$ make up a weak SCS. Its elements can be considered as the closest [31] states to pure classical states - i.e. points - of S^d , because they are in one-to-one correspondence with points of S^d, are optimally localized around the latter and are mapped into each other by the symmetry group $O(D)$. In the case $d=2$ the right-hand side goes to zero as $\Lambda\to\infty$ much faster than the uncertainty (13) for all irreducible components appearing in the decomposition $\mathcal{H}_\Lambda = \bigoplus_{l=0}^{\Lambda} V_l$, including the one $(\Delta\boldsymbol{x})^2_{min} = 1/(\Lambda+1)$ corresponding to the highest l. In this sense the optimally localized states on the new S^2_Λ have a sharper spacial localization than the CS on Madore FS[3]. It is also possible to determine various strong SCS, in particular one with $(\Delta\boldsymbol{x})^2 < 1/(\Lambda+1)$; the elements of the latter SCS are eigenvectors of a suitable component of the angular momentum, so that the corresponding states (rays or equivalently 1-dim projections) are in one-to-one correspondence with points of S^d, and the resolution

[3]Of course a future, more precise determination of $(\Delta\boldsymbol{x})^2_{min}$ will indicate an even sharper localization.

of the identity holds also integrating just over the coset space S^d; furthermore, in Appendix C (chapter 9) there are lengthy computations and proofs of that chapter. In view of potential physical applications to a quantum particle moving within a very thin domain with the shape of a sphere (like e.g. an electron in fullerene), the three-dimensional model is more useful than the FS because in this new model the restriction to the unit sphere is obtained 'a posteriori' from the dynamics. Beside their theoretical interest as toy-models of fuzzy geometries in quantum gravity, these models may thus have some application to one- and two-dimensional quantum systems, which are a very 'hot' topic of research in condensed matter physics (quantum waveguides or nanotubes; fullerene, graphene[4], quantum Hall-effect, etc.). In all cases there are very thin layers of matter where electrons are confined by potential energies with very deep minima there and steep gradients in the normal direction.

[4]Graphene is an allotrope of carbon in the form of a two-dimensional layer of carbon atoms and it has the resistance of diamond and the flexibility of plastic.

Chapter 1

Theoretical framework

1.1 Lie Group and Lie algebra representations

1.1.1 Basics notions about Lie groups and Lie algebras

1.1.1.1 Definition of a Matrix Lie Group

We begin with a very important class of groups, the general linear groups.

Definition 1.1.1 *The general linear group over the real numbers, denoted $GL(n;\mathbb{R})$ is the group of all $n \times n$ invertible matrices with real entries. The general linear group over the complex numbers, denoted $GL(n;\mathbb{C})$, is the group of all $n \times n$ invertible matrices with complex entries.*

The general linear groups are indeed groups under the operation of matrix multiplication: The product of two invertible matrices is invertible, the identity matrix is an identity for the group, an invertible matrix has (by definition) an inverse, and matrix multiplication is associative.

Definition 1.1.2 *Let $\mathcal{M}_n(\mathbb{C})$ denote the space of all $n \times n$ matrices with complex entries.*

Definition 1.1.3 *Let A_m be a sequence of complex matrices in $\mathcal{M}_n(\mathbb{C})$. We say that A_m converges to a matrix A if each entry of A_m converges (as $m \to +\infty$) to the corresponding entry of A [i.e., if $(A_m)_{kl}$ converges to A_{kl} for all $1 \leq k, n \leq n$].*

Definition 1.1.4 *A matrix Lie group is any subgroup G of $GL(n;\mathbb{C})$ with the following property: If A_m is any sequence of matrices in G, and A_m converges to some matrix A then either $A \in G$, or A is not invertible.*

The condition on G amounts to saying that G is a closed subset of $GL(n;\mathbb{C})$ [this does not necessarily mean that G is closed in $\mathcal{M}_n(\mathbb{C})$]. Thus, definition 1.1.4 is equivalent to saying that a matrix Lie group is a closed subgroup of $GL(n;\mathbb{C})$.

The condition that G be a closed subgroup, as opposed to merely a subgroup, should be regarded as a technicality, in that most of the interesting subgroups of $GL(n;\mathbb{C})$ have this property. Most of the matrix Lie groups G we will consider have the stronger property that if A_m is any sequence of matrices in G, and A_m converges to some matrix A, then $A \in G$ [i.e., that G is closed in $\mathcal{M}_n(\mathbb{C})$].

1.1.1.2 The orthogonal and special orthogonal groups, $O(n)$ and $SO(n)$

A $n \times n$ real matrix A is said to be orthogonal if the column vectors that make up A are orthonormal, that is, if

$$\sum_{l=1}^{n} A_{lj} A_{lk} = \delta_{jk}, \quad 1 \leq j, k \leq n,$$

where δ_{jk} is the Kronecker delta, equal to 1 if $j = k$ and equal to zero if $j \neq k$. Equivalently, A is orthogonal if it preserves the inner product, namely if $\langle x, y\rangle = \langle Ax, Ay\rangle$ for all vectors $x, y \in \mathbb{R}^n$ (angled brackets denote the usual inner product on $\mathbb{R}^n$, $\langle x, y\rangle = \sum_k x_k y_k$. Still another equivalent definition is that A is orthogonal if $A^T A = I$, i.e. if $A^T = A^{-1}$ (here, A^T is the transpose of A, $(A^T)_{kl} = A_{lk}$).

Since $\det A^T = \det A$, we see that if A is orthogonal, then $\det(A^T A) = (\det A)^2 = \det I = 1$. Hence, $\det A = \pm 1$, for all orthogonal matrices A.

This formula tells us in particular that every orthogonal matrix must be invertible. However, if A is an orthogonal matrix, then

$$\left\langle A^{-1}x, A^{-1}y\right\rangle = \left\langle A\left(A^{-1}x\right), A\left(A^{-1}y\right)\right\rangle = \langle x, y\rangle.$$

Thus, the inverse of an orthogonal matrix is orthogonal. Furthermore, the product of two orthogonal matrices is orthogonal, since if A and B both preserve inner products, then so does AB. Thus, the set of orthogonal matrices forms a group.

The set of all $n \times n$ real orthogonal matrices is the orthogonal group $O(n)$, and it is a subgroup of $GL(n;\mathbb{C})$. The limit of a sequence of orthogonal matrices is orthogonal, because the relation $A^T A = I$ is preserved under taking limits. Thus, $O(n)$ is a matrix Lie group. The set of $n \times n$ orthogonal matrices with determinant one is the special orthogonal group $SO(n)$. Clearly, this is a subgroup of $O(n)$, and hence of $GL(n;\mathbb{C})$. Moreover, both orthogonality and the property of having determinant one are preserved under limits, and so $SO(n)$ is a matrix Lie group. Since elements of $O(n)$ already have determinant ± 1, $SO(n)$ is 'half' of $O(n)$.

Geometrically, elements of $O(n)$ are either rotations or combinations of rotations and reflections. The elements of $SO(n)$ are just the rotations.

1.1.1.3 Homomorphisms and Isomorphisms

Definition 1.1.5 *Let G and H be matrix Lie groups. A map $\mathbf{\Phi}$ from G to H is called a Lie group homomorphism if*

1. *$\mathbf{\Phi}$ is a group homomorphism,*

2. *$\mathbf{\Phi}$ is continuous.*

If, in addition, $\mathbf{\Phi}$ is one-to-one and onto and the inverse map $\mathbf{\Phi}^{-1}$ is continuous, then $\mathbf{\Phi}$ is called a Lie group isomorphism.

Note that the inverse of a Lie group isomorphism is continuous (by definition) and a group homomorphism (by elementary group theory), and thus a Lie group isomorphism. If G and H are matrix Lie groups and there exists a Lie group isomorphism from G to H, then G and H are said to be isomorphic, and we write $G \simeq H$. Two matrix Lie groups which are isomorphic should be thought of as being essentially the same group.

The simplest interesting example of a Lie group homomorphism is the determinant, which is a homomorphism of $GL(n;\mathbb{C})$ into $\mathbb{C}$. Another simple example is the map $\mathbf{\Phi} : \mathbb{R} \to SO(2)$ given by

$$\mathbf{\Phi}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

1.1.1.4 Lie Groups

A Lie group is something that is simultaneously a smooth manifold and a group. As the terminology suggests, every matrix Lie group is a Lie group. The reverse is not true: Not every Lie group is isomorphic to a matrix Lie group.

Definition 1.1.6 *A Lie group is a differentiable manifold G which is also a group and such that the group product*

$$G \times G \to G$$

and the inverse map $g \to g^{-1}$ are differentiable.

Now let us think about the question of whether every matrix Lie group is a Lie group. This is certainly not obvious, since nothing in our definition of a matrix Lie group says anything about its being a manifold (indeed, the whole point of considering matrix Lie groups is that one can define and study them without having to go through manifold theory first!. Nevertheless, it is true that every matrix Lie group is a Lie group, and it would be a particularly misleading choice of terminology if this were not so.

Theorem 1.1.1 *Every matrix Lie group is a smooth embedded submanifold of $\mathcal{M}_n(\mathbb{C})$ and is thus a Lie group.*

It is customary to call a map $\mathbf{\Phi}$ between two Lie groups a Lie group homomorphism if $\mathbf{\Phi}$ is a group homomorphism and $\mathbf{\Phi}$ is smooth, whereas we have (in definition 1.1.5) required only that $\mathbf{\Phi}$ be continuous. However, the following proposition shows that our definition is equivalent to the more standard one.

Proposition 1.1.1 *Let G and H be Lie groups and let $\mathbf{\Phi}$ be a group homomorphism from G to H. If $\mathbf{\Phi}$ is continuous, it is also smooth.*

1.1.1.5 The Matrix Exponential

The exponential of a matrix plays a crucial role in the theory of Lie groups.

The exponential enters into the definition of the Lie algebra of a matrix Lie group and is the mechanism for passing information from the Lie algebra to the Lie group. Since many computations are done much more easily at the level of the Lie algebra, the exponential is indispensable in studying (matrix) Lie groups.

Let X be an $n \times n$ real or complex matrix. We wish to define the exponential of X, denoted e^X or $\exp X$, by the usual power series

$$e^X = \sum_{m=0}^{+\infty} \frac{X^m}{m!}. \tag{1.1}$$

We will follow the convention of using letters such as X and Y for the variable in the matrix exponential.

Proposition 1.1.2 *For any $n \times n$ real or complex matrix X, the series (1.1) converges. The matrix exponential e^X is a continuous function of X.*

1.1.1.6 The Lie Algebra of a Matrix Lie Group

The Lie algebra is an indispensable tool in studying matrix Lie groups. On the one hand, Lie algebras are simpler than matrix Lie groups, because (as we will see) the Lie algebra is a linear space. Thus, we can understand much about Lie algebras just by doing linear algebra. On the other hand, the Lie algebra of a matrix Lie group contains much information about that group. Thus, many questions about matrix Lie groups can be answered by considering a similar but easier problem for the Lie algebra.

Definition 1.1.7 *Let G be a matrix Lie group. The Lie algebra of G, denoted $\boldsymbol{g}$, is the set of all matrices X such that e^{tX} is in G for all real numbers t.*

This means that X is in $\boldsymbol{g}$ if and only if the one-parameter subgroup generated by X lies in G. Note that even though G is a subgroup of $GL(n;\mathbb{C})$ [and not necessarily of $GL(n;\mathbb{R})$], we do not require that e^{tX} be in G for all complex numbers t, but only for all real numbers t. Also, it is definitely not enough to have just e^X in G. That is, it is easy to give an example of an X and a G such that $e^X \in G$ but such that $e^{tX} \notin G$ for some real values of t. Such an X is not in the Lie algebra of G.

It is customary to use lowercase characters such as $\boldsymbol{g}$ to refer to Lie algebras.

It is possible to show that every matrix Lie group is an embedded submanifold of $GL(n;\mathbb{C})$, and then that $\boldsymbol{g}$ is the tangent space to G at the identity. This means that $\boldsymbol{g}$ can alternatively be defined as the set of all derivatives of smooth curves through the identity in G.

The following ones are basic properties of the Lie algebra of a matrix Lie group.

Proposition 1.1.3 *Let G be a matrix Lie group, with Lie algebra $\boldsymbol{g}$. Let X be an element of $\boldsymbol{g}$, and A an element of G. Then, AXA^{-1} is in $\boldsymbol{g}$.*

Theorem 1.1.2 *Let G be a matrix Lie group, $\boldsymbol{g}$ its Lie algebra, and X and Y elements of $\boldsymbol{g}$. Then*

1. *$sX \in \boldsymbol{g}$ for all numbers s,*
2. *$X + Y \in \boldsymbol{g}$,*
3. *$XY - YX \in \boldsymbol{g}$.*

Definition 1.1.8 *Given two $n\times n$ matrices A and B, the bracket (or commutator) of A and B, denoted $[A, B]$, is defined to be*

$$[A, B] = AB - BA.$$

According to last theorem, the Lie algebra of any matrix Lie group is closed under brackets.

We return now to the setting of general, not necessarily complex, matrix Lie groups. The following very important theorem tells us that a Lie group homomorphism between two Lie groups gives rise in a natural way to a map between the corresponding Lie algebras. In particular, this will tell us that two isomorphic Lie groups have 'the same' Lie algebras (i.e., the Lie algebras are isomorphic).

Theorem 1.1.3 *Let G and H be matrix Lie groups, with Lie algebras $\boldsymbol{g}$ and $\boldsymbol{h}$, respectively. Suppose that $\boldsymbol{\Phi} : G \to H$ is a Lie group homomorphism. Then, there exists a unique real linear map $\phi : \boldsymbol{g} \to \boldsymbol{h}$ such that*

$$\boldsymbol{\Phi}\left(e^{X}\right) = e^{\phi(X)}$$

for all $X \in \boldsymbol{g}$. The map ϕ has following additional properties:

1. *$\phi\left(AXA^{-1}\right) = \boldsymbol{\Phi}(A)\,\phi(X)\,\boldsymbol{\Phi}(A)^{-1}$, for all $X \in \boldsymbol{g}$, $A \in G$,*
2. *$\phi([X, Y]) = [\phi(X), \phi(Y)]$, for all $X, Y \in \boldsymbol{g}$,*
3. *$\phi(X) = \frac{d}{dt}\boldsymbol{\Phi}\left(e^{tX}\right)|_{t=0}$, for all $X \in \boldsymbol{g}$.*

Suppose that G, H, and K are matrix Lie groups and $\boldsymbol{\Phi} : H \to K$ and $\boldsymbol{\Psi} : G \to H$ are Lie group homomorphisms. Let $\boldsymbol{\Lambda} : G \to K$ be the composition of $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, $\boldsymbol{\Lambda}(A) = \boldsymbol{\Phi}(\boldsymbol{\Psi}(A))$. Let ϕ, ψ, and λ be the associated Lie algebra maps. Then,

$$\lambda(X) = \phi(\psi(X)).$$

In practice, given a Lie group homomorphism $\mathbf{\Phi}$, the way one goes about computing ϕ is by using Property 3. Of course, since ϕ is (real) linear, it suffices to compute ϕ on a basis for $\boldsymbol{g}$. In the language of differentiable manifolds, Property 3 says that ϕ is the derivative (or differential) of $\mathbf{\Phi}$ at the identity, which is the standard definition of ϕ.

A linear map with Property 2 is called a Lie algebra homomorphism. This theorem says that every Lie group homomorphism gives rise to a Lie algebra homomorphism. The converse is true under certain circumstances. Specifically, suppose that G and H are Lie groups and that $\phi : \boldsymbol{g} \to \boldsymbol{h}$ is a Lie algebra homomorphism. If G is simply connected, then there exists a unique Lie group homomorphism $\mathbf{\Phi} : G \to H$ such that $\mathbf{\Phi}$ and ϕ are related as in the last theorem.

1.1.1.7 Lie Algebras

We now consider the abstract notion of a Lie algebra, not necessarily given to us as the Lie algebra of a matrix Lie group.

Definition 1.1.9 *A finite-dimensional real or complex Lie algebra is a finite-dimensional real or complex vector space $\boldsymbol{g}$, together with a map $[\cdot,\cdot]$ from $\boldsymbol{g} \times \boldsymbol{g}$ into $\boldsymbol{g}$, with the following properties:*

1. $[\cdot,\cdot]$ *is bilinear.*
2. $[X, Y] = -[Y, X]$ *for all* $X, Y \in \boldsymbol{g}$.
3. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ *for all* $X, Y, Z \in \boldsymbol{g}$.

Condition 2 is called ‘skew symmetry’. Condition 3 is called the Jacobi identity. Note also that Condition 2 implies that $[X, X] = 0$ for all $X \in \boldsymbol{g}$.

We will deal only with finite-dimensional Lie algebras and will from now on interpret ‘Lie algebra’ as ‘finite-dimensional Lie algebra’.

It should be emphasized here that $\boldsymbol{g}$ can be any vector space (not necessarily a space of matrices) and that the ‘bracket’ operation $[\cdot,\cdot]$ can be any bilinear, skew-symmetric map that satisfies the Jacobi identity. In particular, $[X, Y]$ is not necessarily equal to $XY - YX$; indeed, the expression $XY - YX$ does not even make sense in general, since $\boldsymbol{g}$ does not necessarily have a product operation defined on it.

Although the bracket operation in a Lie algebra does not have to be given to us as $[X, Y] = XY - YX$, it is possible to construct Lie algebras in this way.

That is to say, if $\mathcal{A}$ is an associative algebra and we define $[\cdot,\cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ by $[X, Y] = XY - YX$, then this operation does, indeed, make $\mathcal{A}$ into a Lie algebra. This operation is clearly bilinear and skew-symmetric, and it is a simple computation to check, using the associativity of $\mathcal{A}$, the Jacobi identity. For any Lie algebra, the Jacobi identity means that the bracket operation behaves as if it were $XY - YX$, even if it is not actually defined this way. Indeed, it can be shown that every Lie algebra $\boldsymbol{g}$ can be embedded into some associative algebra $\mathcal{A}$ in such a way that the bracket on $\boldsymbol{g}$ corresponds to the operation $XY - YX$ in $\mathcal{A}$.

If $\boldsymbol{g}$ is a Lie algebra, we can think of the bracket operation as making $\boldsymbol{g}$ into an algebra in the general sense. This algebra, however, is not associative. The Jacobi identity is to be thought of as a substitute for associativity.

Proposition 1.1.4 *The space $\mathcal{M}_n(\mathbb{R})$ of all $n \times n$ real matrices is a real Lie algebra with respect to the bracket operation $[A, B] = AB - BA$. The space $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ complex matrices is a complex Lie algebra with respect to the same bracket operation.*

Let V be a finite-dimensional real or complex vector space, and let $\boldsymbol{gl}(V)$ denote the space of linear maps of V into itself. Then, $\boldsymbol{gl}(V)$ becomes a real or complex Lie algebra with the bracket operation $[A, B] = AB - BA$.

The last proposition shows that the Lie algebra of a matrix Lie group is indeed a Lie algebra in the abstract sense.

Definition 1.1.10 *A subalgebra of a real or complex Lie algebra $\boldsymbol{g}$ is a subspace $\boldsymbol{h}$ of $\boldsymbol{g}$ such that $[H_1, H_2] \in \boldsymbol{h}$ for all H_1 and $H_2 \in \boldsymbol{h}$. If $\boldsymbol{g}$ is a complex Lie algebra and $\boldsymbol{h}$ is a real subspace of $\boldsymbol{g}$ which is closed under brackets, then $\boldsymbol{h}$ is said to be a real subalgebra of $\boldsymbol{g}$.*

If $\boldsymbol{g}$ and $\boldsymbol{h}$ are Lie algebras, then a linear map $\phi : \boldsymbol{g} \to \boldsymbol{h}$ is called a Lie algebra homomorphism if $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \boldsymbol{g}$.

If, in addition, ϕ is one-to-one and onto, then ϕ is called a Lie algebra isomorphism. A Lie algebra isomorphism of a Lie algebra with itself is called a Lie algebra automorphism.

A subalgebra of a Lie algebra is, again, a Lie algebra. The inverse of a Lie algebra isomorphism is, again, a Lie algebra isomorphism.

Proposition 1.1.5 *The Lie algebra $\boldsymbol{g}$ of a matrix Lie group G is a real Lie algebra.*

Proof. By Theorem 1.1.2, $\boldsymbol{g}$ is a real subalgebra of the space $\mathcal{M}_n(\mathbb{C})$ of all complex matrices and is, thus, a real Lie algebra. □

We end this section with the following

Definition 1.1.11 (Universal enveloping algebra)
Let $\boldsymbol{g}$ be a Lie algebra over a field $\mathbb{K}$. The universal enveloping algebra of $\boldsymbol{g}$ is a pair $(U\boldsymbol{g}; i)$, satisfying the following:

1. *$U\boldsymbol{g}$ is an associative algebra with unit over $\mathbb{K}$.*

2. *$i : \boldsymbol{g} \to U\boldsymbol{g}$ is linear and $i(X)i(Y) - i(Y)i(X) = i([X, Y])$, for all $X, Y \in \boldsymbol{g}$.*

3. *For any associative algebra A with unit over $\mathbb{K}$ and for any linear map $j : \boldsymbol{g} \to A$ satisfying $j(X)j(Y) - j(Y)j(X) = j([X,Y])$ for each $X, Y \in \boldsymbol{g}$, there exists a unique homomorphism of algebras $\phi : U\boldsymbol{g} \to A$ such that $\phi \circ i = j$.*

1.1.1.8 Structure constants

Let $\boldsymbol{g}$ be a finite-dimensional real or complex Lie algebra, and let $X_1, \cdots, X_n$ be a basis for $\boldsymbol{g}$ (as a vector space). Then, for each i and j, $[X_i, X_j]$ can be written uniquely in the form

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ijk} X_k$$

The constants c_{ijk} are called the structure constants of $\boldsymbol{g}$ (with respect to the chosen basis). Clearly, the structure constants determine the bracket operation on $\boldsymbol{g}$. The structure constants satisfy the following two conditions:

$$c_{ijk} + c_{jik} = 0,$$

$$\sum_m (c_{ijm} c_{mkl} + c_{jkm} c_{mil} + c_{kim} c_{mjl}) = 0$$

for all i, j, k, l. The first of these conditions comes from the skew symmetry of the bracket, and the second comes from the Jacobi identity.

1.1.2 The Complexification of a Real Lie Algebra

Definition 1.1.12 *If V is a finite-dimensional real vector space, then the complexification of V, denoted $V_{\mathbb{C}}$, is the space of formal linear combinations*

$$v_1 + iv_2$$

with $v_1, v_2 \in V$. This becomes a real vector space in the obvious way and becomes a complex vector space if we define

$$i(v_1 + iv_2) = -v_2 + iv_1.$$

We will regard V as a real subspace of $V_{\mathbb{C}}$ in the obvious way.

Proposition 1.1.6 *Let $\boldsymbol{g}$ be a finite-dimensional real Lie algebra and $\boldsymbol{g}_{\mathbb{C}}$ its complexification (as a real vector space). Then, the bracket operation on $\boldsymbol{g}$ has a unique extension to $\boldsymbol{g}_{\mathbb{C}}$ which makes $\boldsymbol{g}_{\mathbb{C}}$ into a complex Lie algebra. The complex Lie algebra $\boldsymbol{g}_{\mathbb{C}}$ is called the complexification of the real Lie algebra $\boldsymbol{g}$.*

1.1.3 Representations

Definition 1.1.13 *Let G be a matrix Lie group. Then, a finite-dimensional complex representation of G is a Lie group homomorphism*

$$\Pi : G \to GL(n; \mathbb{C})$$

$(n \geq 1)$ or, more generally, a Lie group homomorphism

$$\Pi : G \to GL(V)$$

where V is a finite-dimensional complex vector space (with $\dim V \geq 1$). A finite-dimensional real representation of G is a Lie group homomorphism Π of G into $GL(n; \mathbb{R})$ or into $GL(V)$, where V is a finite-dimensional real vector space. If $\boldsymbol{g}$ is a real or complex Lie algebra, then a finite-dimensional complex representation of $\boldsymbol{g}$ is a Lie algebra homomorphism π of $\boldsymbol{g}$ into $\boldsymbol{gl}(n; \mathbb{C})$ or into $\boldsymbol{gl}(V)$, where V is a finite-dimensional complex vector space. If $\boldsymbol{g}$ is a real Lie algebra, then a finite-dimensional real representation of $\boldsymbol{g}$ is a Lie algebra homomorphism π of $\boldsymbol{g}$ into $\boldsymbol{gl}(n; \mathbb{R})$ or into $\boldsymbol{gl}(V)$.

If Π or π is a one-to-one homomorphism, then the representation is called faithful.

One should think of a representation as a linear action of a group or Lie algebra on a vector space [since, say, to every $g \in G$, there is associated an operator $\Pi(g)$, which acts on the vector space V]. In fact, we will use terminology such as 'Let Π be a representation of G acting on the space V'.

Even if $\boldsymbol{g}$ is a real Lie algebra, we will consider mainly complex representations of $\boldsymbol{g}$.

Definition 1.1.14 *Let Π be a finite-dimensional real or complex representation of a matrix Lie group G, acting on a space V. A subspace W of V is called invariant if $\Pi(A)w \in W$ for all $w \in W$ and all $A \in G$. An invariant subspace W is called nontrivial if $W \neq \{0\}$ and $W \neq V$. A representation with no nontrivial invariant subspaces is called irreducible. The terms invariant, nontrivial, and irreducible are defined analogously for representations of Lie algebras.*

Definition 1.1.15 *Let G be a matrix Lie group, let Π be a representation of G acting on the space V, and let Σ be a representation of G acting on the space W. A linear map $\phi : V \to W$ is called an intertwining map of representations if*

$$\phi\left(\Pi(A)v\right) = \Sigma(A)\phi(v)$$

for all $A \in G$ and all $v \in V$. The analogous property defines intertwining maps of representations of a Lie algebra.

If ϕ is an intertwining map of representations and, in addition, ϕ is invertible, then ϕ is said to be an equivalence of representations. If there exists an isomorphism between V and W, then the representations are said to be equivalent.

Two equivalent representations should be regarded as being 'the same' representation. A typical problem in representation theory is to determine, up to equivalence, all of the irreducible representations of a particular group or Lie algebra.

Proposition 1.1.7 *Let G be a matrix Lie group with Lie algebra $\boldsymbol{g}$ and let Π be a (finite-dimensional real or complex) representation of G acting on the space V. Then, there is a unique representation π of $\boldsymbol{g}$ acting on the same space such that*

$$\Pi(e^X) = e^{\pi(X)}$$

for all $X \in \boldsymbol{g}$. The representation π can be computed as

$$\pi(X) = \frac{d}{dt}\Pi\left(e^{tX}\right)|_{t=0}$$

and satisfies

$$\pi\left(AXA^{-1}\right) = \Pi(A)\pi(X)\Pi(A)^{-1}$$

for all $X \in \boldsymbol{g}$ and $A \in G$.

Proposition 1.1.8

1. *Let G be a connected matrix Lie group with Lie algebra $\boldsymbol{g}$. Let Π be a representation of G and π the associated representation of $\boldsymbol{g}$. Then, Π is irreducible if and only if π is irreducible.*

2. *Let G be a connected matrix group, let H_1 and H_2 be representations of G, and let π_1 and π_2 be the associated Lie algebra representations. Then, π_1 and π_2 are equivalent if and only if Π_1 and Π_2 are equivalent.*

Definition 1.1.16 *Let G be a matrix Lie group, let $\mathcal{H}$ be a Hilbert space, and let $U(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$. Then, a homomorphism $\Pi : G \to U(\mathcal{H})$ is called a unitary representation of G if Π satisfies the following continuity condition: If $A_n, A \in G$ and $A_n \to A$, then*

$$\Pi(A_n)v \to \Pi(A)v$$

for all $v \in \mathcal{H}$. A unitary representation with no nontrivial closed invariant subspaces and such that $\Pi(A^{-1}) = [\Pi(A)]^\dagger$ is called irreducible.

1.1.3.1 Schur's Lemma

Let Π and Σ be representations of a matrix Lie group G, acting on spaces V and W. Schur's Lemma is an extremely important result which tells us about intertwining maps of irreducible representations. Part of Schur's Lemma applies to both real and complex representations, but part of it applies only to complex representations.

It is desirable to be able to state Schur's Lemma simultaneously for groups and Lie algebras. In order to do so, we need to indulge in a common abuse of notation. If, say, Π is a representation of G acting on a space V, we will refer to V as the representation, without explicit reference to Π.

Lemma 1.1.1 (Schur's Lemma) *1. Let V and W be irreducible real or complex representations of a group or Lie algebra and let $\phi : V \to W$ be an intertwining map. Then, either $\phi = 0$ or ϕ is an isomorphism.*

2. Let V be an irreducible complex representation of a group or Lie algebra and let $\phi : V \to V$ be an intertwining map of V with itself. Then, $\phi = \lambda I$, for some $\lambda \in \mathbb{C}$.

3. Let V and W be irreducible complex representations of a group or Lie algebra and let $\phi_1, \phi_2 : V \to W$ be nonzero intertwining maps. Then, $\phi_1 = \lambda \phi_2$ for some $\lambda \in \mathbb{C}$.

1.1.3.2 Adjoint representation of an algebra

Let A be an algebra and $X \in A$, one can consider the linear transformation

$$ad(X) : Z \in A \to [Z, X] \in A$$

If $Y, Z, K \in A$, then (according to the Jacobi identity)

$$\begin{aligned}[ad(Y), ad(Z)]K &= ad(Y)ad(Z)K - ad(Z)ad(Y)K = ad(Y)[Z, K] - ad(Z)[Y, K] \\ &= [Y, [Z, K]] - [Z, [Y, K]] = [[Y, Z], K] = ad\left([Y, Z]\right) K.\end{aligned}$$

So the map ad provides a representation of the algebra, which is called 'adjoint representation'.

1.2 Semisimple Lie Algebras

In this section, we will consider a class of Lie algebras (the complex semisimple ones) that their representations can be described by a 'theorem of the highest weight'.

Definition 1.2.1 *If $\boldsymbol{g}$ is a complex Lie algebra, then an ideal in $\boldsymbol{g}$ is a complex subalgebra $\boldsymbol{h}$ of $\boldsymbol{g}$ with the property that for all X in $\boldsymbol{g}$ and H in $\boldsymbol{h}$, we have $[X, H]$ in $\boldsymbol{h}$.*

Note that the definition of an ideal is stronger than that of a subalgebra. For a subalgebra, we require only that the bracket of two elements of the subalgebra remain in the subalgebra. For an ideal, we require that the bracket of an element of the ideal with any element of $\boldsymbol{g}$ be, again, in the ideal. Any Lie algebra $\boldsymbol{g}$ has two 'trivial' examples of ideals: $\boldsymbol{g}$ itself and the zero ideal $\boldsymbol{h} = \{0\}$.

Definition 1.2.2 *A complex Lie algebra $\boldsymbol{g}$ is called indecomposable if the only ideals in $\boldsymbol{g}$ are $\boldsymbol{g}$ and $\{0\}$. A complex Lie algebra $\boldsymbol{g}$ is called simple if $\boldsymbol{g}$ is indecomposable and* $\dim \boldsymbol{g} > 2$.

Definition 1.2.3 *A complex Lie algebra is called reductive if it is isomorphic to a direct sum of indecomposable Lie algebras. A complex Lie algebra is called semisimple if it isomorphic to a direct sum of simple Lie algebras.*

Definition 1.2.4 *If $\boldsymbol{g}$ is a complex semisimple Lie algebra, then a compact real form of $\boldsymbol{g}$ is a real subalgebra $\boldsymbol{p}$ of $\boldsymbol{g}$ with the property that every X in $\boldsymbol{g}$ can be written uniquely as $X = X_1 + iX_2$ with X_1 and X_2 in $\boldsymbol{p}$ and such that there is a compact simply-connected matrix Lie group P_1 such that the Lie algebra $\boldsymbol{p}_1$ of P_1 is isomorphic to $\boldsymbol{p}$.*

One can prove that every complex semisimple Lie algebra has a compact real form. The compact real form is not unique, but it is 'unique up to conjugation'.

1.2.1 Cartan Subalgebras

We now begin to develop the structure that we will use in describing the representations of complex semisimple Lie algebras. These same structures are used to give a classification of semisimple Lie algebras.

Definition 1.2.5 *If $\boldsymbol{g}$ is a complex semisimple Lie algebra, then a Cartan subalgebra of $\boldsymbol{g}$ is a complex subspace $\boldsymbol{h}$ of $\boldsymbol{g}$ with the following properties:*

1. *For all H_1 and H_2 in $\boldsymbol{h}$, $[H_1, H_2] = 0$.*
2. *For all X in $\boldsymbol{g}$, if $[H, X] = 0$ for all H in $\boldsymbol{h}$, then X is in $\boldsymbol{h}$.*
3. *For all H in $\boldsymbol{h}$, ad_H is diagonalizable, where $ad_H(Y) = [H, Y]$, for all Y in $\boldsymbol{g}$.*

Condition 1 says that $\boldsymbol{h}$ is a commutative subalgebra of $\boldsymbol{g}$. Condition 2 says that $\boldsymbol{h}$ is a maximal commutative subalgebra (i.e., not contained in any larger commutative subalgebra). Condition 3 says that each ad_H is diagonalizable. Since the Hs in $\boldsymbol{h}$ commute, the ad_H's also commute, and thus they are simultaneously diagonalizable.

Of course, the definition of a Cartan subalgebra makes sense in any Lie algebra, semisimple or not. However, if $\boldsymbol{g}$ is not semisimple, then $\boldsymbol{g}$ may not have any Cartan subalgebras. In fact, one can prove that every semisimple Lie algebra has a Cartan subalgebra, and all Cartan subalgebras of a given complex semisimple Lie algebra have the same dimension. In light of this result, the following definition makes sense.

Definition 1.2.6 *If $\boldsymbol{g}$ is a complex semisimple Lie algebra, then the rank of $\boldsymbol{g}$ is the dimension of any its Cartan subalgebra.*

1.2.2 Roots and Root Spaces

From now on we assume that we have chosen (and one can show that these choices are always possible if $\boldsymbol{g}$ is a complex semisimple Lie algebra) a compact real form $\boldsymbol{p}$ of $\boldsymbol{g}$ and a maximal commutative subalgebra $\boldsymbol{t}$ of $\boldsymbol{p}$, the Cartan subalgebra $\boldsymbol{h} = \boldsymbol{t} + i\boldsymbol{t}$ and an inner product on $\boldsymbol{g}$ that is invariant under the adjoint action of P (P is the subgroup of G whose Lie algebra is $\boldsymbol{p}$) and that takes real values on $\boldsymbol{p}$.

Definition 1.2.7 *A root of $\boldsymbol{g}$ (relative to the Cartan subalgebra $\boldsymbol{h}$) is a nonzero linear functional α on $\boldsymbol{h}$ such that there exists a nonzero element X of $\boldsymbol{g}$ with*

$$[H, X] = \alpha(H)X$$

for all H in $\boldsymbol{h}$.

The set of all roots is denoted R.

The condition on X says that X is an eigenvector for each ad_H, with eigenvalue $\alpha(H)$. Note that if X is actually an eigenvector for each ad_H with H in $\boldsymbol{h}$, then the eigenvalues must depend linearly on H. That is why we insist that α be a linear functional on $\boldsymbol{h}$. So, a root is just a (nonzero) collection of simultaneous eigenvalues for the ad_H's. Note that any element of $\boldsymbol{h}$ is a simultaneous eigenvector for all the ad_H's, with all eigenvalues equal to zero, but we only call α a root if α is nonzero. Of course, for any root α, some of the $\alpha(H)$'s may be equal to zero; we just require that not all of them be zero.

Definition 1.2.8 *If α is a root, then the root space $\boldsymbol{g}_\alpha$ is the space of all X in $\boldsymbol{g}$ for which $[H, X] = \alpha(H)X$ for all H in $\boldsymbol{h}$. An element of $\boldsymbol{g}_\alpha$ is called a root vector (for the root α).*

More generally, if α is any element $\boldsymbol{h}^*$, the space of real-valued linear functionals on $\boldsymbol{h}$, we define $\boldsymbol{g}_\alpha$ to be the space of all X in $\boldsymbol{g}$ for which $[H, X] = \alpha(H)X$ for all H in $\boldsymbol{h}$ (but we do not call $\boldsymbol{g}_\alpha$ a root space unless α is actually a root).

Taking $\alpha = 0$, we see that $\boldsymbol{g}_0$ is the set of all elements of $\boldsymbol{g}$ that commute with every element of $\boldsymbol{h}$. Since $\boldsymbol{h}$ is a maximal commutative subalgebra, we conclude that $\boldsymbol{g}_0 = \boldsymbol{h}$. If α is not zero and not a root, then $\boldsymbol{g}_\alpha = \{0\}$.

Now, since $\boldsymbol{h}$ is commutative, the operators ad_H, $H \in \boldsymbol{h}$, all commute. Furthermore, by the definition of a Cartan subalgebra, each $ad_H, H \in \boldsymbol{h}$, is diagonalizable. It follows that the ad_H's, $H \in H$, are simultaneously diagonalizable. As a result, $\boldsymbol{g}$ can be decomposed as the direct sum of $\boldsymbol{h}$ and the root spaces $\boldsymbol{g}_\alpha$.

Thus, we have established the following.

Proposition 1.2.1 *The Lie algebra $\boldsymbol{g}$ can be decomposed as a direct sum as follows:*

$$\boldsymbol{g} = \boldsymbol{h} \oplus \bigoplus_{\alpha \in R} \boldsymbol{g}_\alpha$$

This means that every element of $\boldsymbol{g}$ can be written uniquely as a sum of an element of $\boldsymbol{h}$ and one element from each root space $\boldsymbol{g}_\alpha$.

In addition,

Proposition 1.2.2 *1. If $\alpha \in \boldsymbol{h}^*$ is a root, then so is $-\alpha$.*

2. The roots span $\boldsymbol{h}^*$.

And

Theorem 1.2.1 *1. If α is a root, then the only multiples of α that are roots are α and $-\alpha$.*

2. If α is a root, then the root space $\boldsymbol{g}_\alpha$ is one dimensional.

3. For each root α, we can find nonzero elements X_α in $\boldsymbol{g}_\alpha$, Y_α in $\boldsymbol{g}_{-\alpha}$ and H_α in $\boldsymbol{h}$ such that

$$\begin{aligned} [H_\alpha, X_\alpha] &= 2X_\alpha, \\ [H_\alpha, Y_\alpha] &= -2Y_\alpha, \\ [X_\alpha, Y_\alpha] &= H_\alpha. \end{aligned}$$

The element H_α is unique (i.e., independent of the choice of X_α and Y_α).

Point 3 of the theorem tells us that X_α, Y_α, and H_α span a subalgebra of $\boldsymbol{g}$. The elements H_α of $\boldsymbol{h}$ given in Point 3 of the theorem are called the co-roots. Their properties are closely related to the properties of the roots themselves.

Theorem 1.2.2 *The roots form a finite set of nonzero elements of a real inner-product space E and have the following properties (we denote the inner product by $\langle \cdot, \cdot \rangle$):*

1. The roots span E.

2. If α is a root, then $-\alpha$ is a root and the only multiples of α that are roots are α and $-\alpha$.

3. *If α is a root, let w_α denote the linear transformation of E given by*

$$w_\alpha[\beta] = \beta - 2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha.$$

   Then, for all roots α and β, $w_\alpha[\beta]$ is also a root.

4. *If α and β are roots, then the quantity*

$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}$$

   is an integer.

Any collection of vectors in a finite-dimensional real inner-product space having these properties is called a root system.

1.2.3 Positive Roots

What we need now is simply some consistent notion of higher and lower that will allow us to divide the root vectors X_α into 'raising operators' and 'lowering operators'. This should be done in such a way that the commutator of two raising operators is, again, a raising operator and not a lowering operator. This means that we want to divide the roots into two groups, one of which will be called 'positive' and the other 'negative'. This should be done is such a way that if the sum of positive roots is again a root, that root should be positive. There is no unique way to make the division into positive and negative; any consistent division will do.

Definition 1.2.9 *Suppose that E is a finite-dimensional real inner-product space and that $R \subset E$ is a root system. Then, a base for R is a subset $\Delta = \{\alpha_1, \cdots, \alpha_r\}$ of R such that Δ forms a basis for E as a vector space and such that for each $\alpha \in R$, we have*

$$\alpha = n_1\alpha_1 + n_2\alpha_2 + \cdots + n_r\alpha_r,$$

where the n_j's are integers and either all greater than or equal to zero or all less than or equal to zero.

Once a base Δ has been chosen, the α's for which $n_j \geq 0$ are called the positive roots (with respect to the given choice of Δ) and the α's with $n_j \leq 0$ are called the negative roots. The elements of Δ are called the positive simple roots.

Therefore, to be a base (in the sense of root systems), $\Delta \subset R$ must, in particular, be a basis for E in the vector space sense. In addition, the expansion of any $\alpha \in R$ in terms of the elements of Δ must have integer coefficients and all of the nonzero coefficients (for a given α) must be of the same sign.

Theorem 1.2.3 *For any root system, a base exists.*

Definition 1.2.10 *An element $\mu \in \boldsymbol{h}$ is called a dominant integral element if $\langle \mu, H_\alpha \rangle$ is a non-negative integer for each positive simple root α. Equivalently μ is a dominant integral element if*

$$2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}$$

is a non-negative integer for each positive simple root α.

If μ is dominant integral, then $\langle \mu, H_\alpha \rangle$ will automatically be a non-negative integer for each positive root α, not just the positive simple ones.

1.2.4 The Theorem of the Highest Weight

We begin with elementary properties of the representations of $\boldsymbol{g}$.

Definition 1.2.11 *Suppose π is a finite-dimensional representation of $\boldsymbol{g}$ on a vector space V. Then, $\mu \in \boldsymbol{h}$ is called a weight for π if there exists a nonzero vector v in V such that*

$$\pi(H)v = \langle \mu, H \rangle v \tag{1.2}$$

for all $H \in \boldsymbol{h}$. A nonzero vector v satisfying (1.2) is called a weight vector for the weight μ, and the set of all vectors satisfying (1.2) (zero or nonzero) is called the weight space with weight μ. The dimension of the weight space is called the multiplicity of the weight.

To understand this definition, suppose that $v \in V$ is a simultaneous eigenvector for each $\pi(H)$, $H \in \boldsymbol{h}$. This means that for each $H \in \boldsymbol{h}$, there is a number λ_H such that $\pi(H)v = \lambda_H v$. Since the representation $\pi(H)$ is linear in H, the λ_H's must depend linearly on H as well; that is, the map $H \to \lambda_H$ is a linear functional on $\boldsymbol{h}$. Then, there is a unique element μ of $\boldsymbol{h}$ such that $\lambda_H = \langle \mu, H \rangle$. Thus, a weight vector is nothing but a simultaneous eigenvector for all the $\pi(H)$'s and the vector μ is simply a convenient way of encoding the eigenvalues.

Definition 1.2.12 *Let μ_1 and μ_2 be two elements of $\boldsymbol{h}$. Then, μ_1 is higher than μ_2 (or, equivalently, μ_2 is lower than μ_1) if there exist non-negative real numbers $a_1, \cdots, a_r$ ar such that*

$$\mu_1 - \mu_2 = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_r\alpha_r$$

where $\{\alpha_1, \cdots, \alpha_r\} = \Delta$ is the set of positive simple roots. This relationship is written as $\mu_1 \succeq \mu_2$ or $\mu_2 \preceq \mu_1$.

If π is a representation of $\boldsymbol{g}$, then a weight μ_0 for π is said to be a highest weight if for all weights μ of π, $\mu \preceq \mu_0$.

Theorem 1.2.4 (Theorem of the Highest Weight)

1. *Every irreducible representation has a highest weight.*

2. *Two irreducible representations with the same highest weight are equivalent.*

3. *The highest weight of every irreducible representation is a dominant integral element.*

4. *Every dominant integral element occurs as the highest weight of an irreducible representation.*

1.3 The orthogonal algebras $\boldsymbol{so}(2n,\mathbb{C})$

The root system for $\boldsymbol{so}(2n,\mathbb{C})$ is denoted D_n. We consider $\boldsymbol{so}(2n,\mathbb{C})$, the space of $2n \times 2n$ skew-symmetric complex matrices, with compact real form $\boldsymbol{so}(2n)$, the space of $2n \times 2n$ skew-symmetric real matrices. We consider in $\boldsymbol{so}(2n)$ the maximal commutative subalgebra $\boldsymbol{t}$ consisting of 2×2 block-diagonal matrices in which the k^{th} diagonal block is of the form

$$\begin{pmatrix} 0 & a_k \\ -a_k & 0 \end{pmatrix} \tag{1.3}$$

for some $a_k \in \mathbb{R}$. We then consider the Cartan subalgebra $\boldsymbol{h} = \boldsymbol{t}+i\boldsymbol{t}$ of $\boldsymbol{so}(2n,\mathbb{C})$, which consists of 2×2 block-diagonal matrices in which the k^{th} diagonal block is of the form (1.3) with $a_k \in \mathbb{C}$ [The calculations in the next two paragraphs show that $\boldsymbol{so}(2n,\mathbb{C})$ decomposes as a direct sum of $\boldsymbol{h}$ and root spaces $\boldsymbol{g}_\alpha$ corresponding to (nonzero) elements $\alpha \in \boldsymbol{h}^*$. It follows from this that $\boldsymbol{t}$ is actually a maximal commutative subalgebra of $\boldsymbol{so}(2n)$, which is not obvious from the definition of $\boldsymbol{t}$]. The root vectors are now 2×2 block matrices having a 2×2 matrix C in the (k,l) block $(k < l)$, the matrix $-C^T$ in the (l,k) block, and zero in all other blocks, where C is one of the four matrices

$$C_1 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad C_4 = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

A little calculation shows that these are, indeed, root vectors and that the corresponding roots are the linear functionals on $\boldsymbol{h}$ given by $i(a_k + a_l)$, $-i(a_k + a_l)$, $i(a_k - a_l)$, and $-i(a_k - a_l)$, respectively.

We may consider the inner product $\langle X, Y\rangle := trace(X^*Y)$ on $\boldsymbol{so}(2n,\mathbb{C})$ which is invariant under the adjoint action of $SO(2n)$. If we use this inner product to identify $\boldsymbol{h}^*$ with $\boldsymbol{h}$, then the roots are thought of as elements of $\boldsymbol{h}$ instead of $\boldsymbol{h}^*$. Let Θ_k denote the 2×2 block-diagonal matrix whose k^{th} diagonal block is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and whose other diagonal blocks are zero. The roots (as elements of $\boldsymbol{h}$) are then the matrices

$$\frac{i}{2}\left(\pm\Theta_k \pm \Theta_l\right),$$

with $1 \leq k < l \leq n$. Each of the roots has length 1 with respect to the given inner product. The inner product of $\frac{i}{2}\left(\pm\Theta_k \pm \Theta_l\right)$ with $\frac{i}{2}\left(\pm\Theta_{k'} \pm \Theta_{l'}\right)$ is zero if the set $\{k, l\}$ is disjoint from $\{k', l'\}$, and the inner product is $\pm\frac{1}{2}$ if the intersection of $\{k, l\}$ and $\{k', l'\}$ has one element. The root $\frac{i}{2}\left(\Theta_k - \Theta_l\right)$ is orthogonal to the root $\frac{i}{2}\left(\Theta_k + \Theta_l\right)$.

As a base, we may take the $n-1$ roots

$$\frac{i}{2}\left(\Theta_1 - \Theta_2\right), \frac{i}{2}\left(\Theta_2 - \Theta_3\right), \cdots, \frac{i}{2}\left(\Theta_{n-2} - \Theta_{n-1}\right), \frac{i}{2}\left(\Theta_{n-1} - \Theta_n\right) \tag{1.4}$$

together with the one additional root,

$$\frac{i}{2}\left(\Theta_{n-1} + \Theta_n\right) \tag{1.5}$$

Note that for $1 \leq k < l \leq n$, we have the following formulas:

$$\Theta_k - \Theta_l = \left(\Theta_k - \Theta_{k+1}\right) + \left(\Theta_{k+1} - \Theta_{k+2}\right) + \cdots + \left(\Theta_{l-1} - \Theta_l\right),$$

$$\Theta_k + \Theta_n = \left(\Theta_k - \Theta_{n-1}\right) + \left(\Theta_{n-1} + \Theta_n\right),$$
$$\Theta_k + \Theta_l \left(\Theta_k + \Theta_n\right) + \left(\Theta_l - \Theta_n\right).$$

This shows that every root of the form $\frac{i}{2}\left(\Theta_k - \Theta_l\right)$ or $\frac{i}{2}\left(\Theta_k + \Theta_l\right)$ can be written as a linear combination of the base in (1.4) and (1.5) with non-negative integer coefficients. The roots of this form are then positive and the remaining roots are negative.

1.4 The orthogonal algebras $\boldsymbol{so}\,(2n+1, \mathbb{C})$

The root system for $\boldsymbol{so}\,(2n+1, \mathbb{C})$ is denoted B_n. We consider $\boldsymbol{so}\,(2n+1, \mathbb{C})$, the space of $(2n+1) \times (2n+1)$ skew-symmetric complex matrices, with compact real form $\boldsymbol{so}\,(2n+1)$, the space of $(2n+1) \times (2n+1)$ skew-symmetric real matrices. We consider in $\boldsymbol{so}\,(2n+1)$ the maximal commutative subalgebra $\boldsymbol{t}$ consisting of block diagonal matrices with n blocks of size 2×2 followed by one block of size 1×1. We take the 2×2 blocks to be of the same form as in $\boldsymbol{so}\,(2n)$ and we take the 1×1 block to be zero. The associated Cartan subalgebra $\boldsymbol{h}$ of $\boldsymbol{so}\,(2n+1, \mathbb{C})$ is then matrices of the same form as in $\boldsymbol{t}$ except that the off-diagonal elements of the 2×2 blocks are permitted to be complex.

The Cartan subalgebra in $\boldsymbol{so}\,(2n+1, \mathbb{C})$ is identifiable in an obvious way with the Cartan subalgebra in $\boldsymbol{so}\,(2n, \mathbb{C})$. In particular, both $\boldsymbol{so}\,(2n, \mathbb{C})$ and $\boldsymbol{so}\,(2n+1, \mathbb{C})$ have rank n. With this identification of the Cartan subalgebras, every root for $\boldsymbol{so}(2n, \mathbb{C})$ is also a root for $\boldsymbol{so}(2n+1, \mathbb{C})$. There are $2n$ additional roots for $\boldsymbol{so}(2n+1, \mathbb{C})$. The root vectors for these additional roots are as follows. First, the matrices having

$$B_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

in entries $(2k, 2n+1)$ and $(2k+1, 2n+1)$ and having $-B_1^T$ in entries $(2n+1, 2k)$ and $(2n+1, 2k+1)$. Second, the matrices having

$$B_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

in entries $(2k, 2n+1)$ and $(2k+1, 2n+1)$ and having $-B_2^T$ in entries $(2n+1, 2k)$ and $(2n+1, 2k+1)$. The corresponding roots, viewed as elements of $\boldsymbol{h}^*$, are given by ia_k and $-ia_k$.

Let Θ_k have the same meaning as in the previous subsection, except that now Θ_k is a $(2n+1)\times(2n+1)$ matrix. We use the inner product $\langle X, Y\rangle = trace(X^*Y)$, which is invariant under the adjoint action of $SO(2n+1)$, to identify $\boldsymbol{h}^*$ with $\boldsymbol{h}$. In that case, the additional roots for the $\boldsymbol{so}(2n+1, \mathbb{C})$ case are given by

$$\pm\frac{i}{2}\Theta_k.$$

These additional roots have length $\frac{1}{\sqrt{2}}$ with respect to the given inner product, whereas the roots that are the same as for $\boldsymbol{so}(2n, \mathbb{C})$ have length 1.

As a base for our root system, we may take the $n-1$ roots

$$\frac{i}{2}(\Theta_1 - \Theta_2), \frac{i}{2}(\Theta_2 - \Theta_3), \cdots, \frac{i}{2}(\Theta_{n-2} - \Theta_{n-1}), \frac{i}{2}(\Theta_{n-1} - \Theta_n) \tag{1.6}$$

[exactly as in the $\boldsymbol{so}(2n, \mathbb{C})$ case] together with the one additional root,

$$\frac{i}{2}\Theta_n.$$

The positive roots are those of the form $\frac{i}{2}(\Theta_k - \Theta_l)$ or $\frac{i}{2}(\Theta_k + \Theta_l)$ $(k < l)$ and those of the form $\frac{i}{2}\Theta_k$ $(1 \leq k \leq n)$.

1.5 Coset spaces

1.5.1 Coset spaces geometry

We give now a short review, based on [38] (pag. 190-195), of coset space geometry, beginning with a few definitions.

First of all

Definition 1.5.1 (Transitive action of a group on a metric space)
Let G be a group, M be a metric space and $\varphi : G \times M \to M$ be an action of G on M. The action φ is said transitive if any two points of the space are connected through the group action.

Definition 1.5.2 (Homogeneous space)
Let G be a group, M be a metric space and $\varphi : G \times M \to M$ be a transitive action of G on M. The metric space G is said to be homogeneous if φ is an isometry.

In addition, the subgroup H of G which leaves a point X fixed is called the isotropy subgroup, so any other point $X' = gX(g \in G, g \notin H)$ is invariant under a subgroup gHg^{-1} of G isomorphic to H.

Example 1.5.1 *The unit sphere S^2 in $\mathbb{R}^3$ is isometric under the transitive action of $SO(3)$, and any point remains fixed under $SO(2)$ rotations around the axis passing through that point, so that $SO(2)$ is the isotropy subgroup.*

The points X of a homogeneous space will be labeled, in the next lines, using the parameters which identify the G-group element which transform a fixed X_0 (the origin) into X. These parameters are redundant and there are infinitely many group elements g such that $X = gX_0$, because of H-isotropy. According to this, it is natural to characterize the points of a homogeneous space by the cosets gH, and a further action of another $g' \in G$ on the coset gH is $g'gH$.

A homogeneous space is then a coset space G/H and, according to example 1.5.1, the two-sphere S^2 can be considered as the coset space $SO(3)/SO(2)$; one can show that, in general, for a d-sphere $S^d = SO(D)/SO(d)$.

In the case of a Lie group G, one obtains coset manifolds, endowed with a Riemannian structure. The Lie algebra of G can be split as:

$$\boldsymbol{g} = \boldsymbol{h} \oplus \boldsymbol{k} \tag{1.7}$$

where $\boldsymbol{h}$ is the Lie algebra of H and $\boldsymbol{k}$ contains the remaining generators, called 'coset generators', and the commutation relations

$$\begin{aligned}
&[H_i, H_j] = c^k_{ij} H_k \quad H_i \in \boldsymbol{h} \\
&[H_i, K_a] = c^j_{ia} H_j + c^b_{ia} K_b \quad K_a \in \boldsymbol{k} \\
&[K_a, K_b] = c^j_{ab} H_j + c^c_{ab} K_c
\end{aligned} \tag{1.8}$$

define the structure constants of G.

One can show that, if H is compact or semisimple, it is always possible to determine a set of K_a such that all the $c^j_{ia} = 0$, and the c^b_{ia} are antisymmetric in a, b.

In a generic exponential

$$g = exp(y^a K_a) exp(x^i H_i). \tag{1.9}$$

the G coordinates are y_a, x_i, and it is easy to see that is clear that the $\dim G - \dim H$ parameters y_a corresponding to the K_a generators characterize the cosets gH.

So, each coset (which is labeled by the y parameters) can be mapped into an element $L(y)$ of G, the coset representative. For instance

$$L(y) = exp(y^a K_a), \tag{1.10}$$

and this means that the whole geometry of G/H can be constructed in terms of coset representatives.

A left multiplication by a generic element g of G, sends $L(y)$ to $L(y') = L(y')h$

$$gL(y) = L(y')h, \quad h \in H, \tag{1.11}$$

and in general it belongs to another equivalence class, while y' and h depend on y and g, and on the way of choosing representatives.

1.5.2 H-Analysis on Coset Spaces

In this section we want to give a short review on harmonic analysis on G/H, based on [39] (pag. 1175 - 1182).

It is well known how to write the Fourier expansion of a sufficiently regular function on S^1, and in the next lines we generalize this to arbitrary G/H manifolds, where G is a compact group manifold. In this case a complete functional basis is given by the matrix elements D of the unitary irreps of G. So, if ϕ is a function on G, it can be expanded as

$$\phi(g) = \sum_{(\mu)} \sum_{m,n=1}^{\dim(\mu)} c_{mn}^{(\mu)} D_{mn}^{(\mu)}(g) \tag{1.12}$$

m, n being indices in the unitary irrep labeled by (μ), and $\dim(\mu)$ the dimension of the (μ)-irrep. In addition, the matrix elements fulfill the following

$$\begin{gathered}
\int_G D_{mn}^{(\mu)}(g) D_{sr}^{(\nu)}\left(g^{-1}\right) dg = \frac{vol(G)}{\dim(\mu)} \delta_{mr} \delta_{ns} \delta^{\mu\nu}, \\
\sum_{(\mu)} D_{mn}^{(\mu)}(g) D_{nm}^{(\nu)}\left(g'^{-1}\right) \dim(\mu) = \delta\left(g - g'\right) vol(G).
\end{gathered} \tag{1.13}$$

where the G-invariant measure dg fulfills

$$\int_G dg = vol(G),$$

so, from (1.12) (1.13), it follows.

$$c_{mn}^{(\mu)} = \frac{\dim(\mu)}{vol(G)} \int_G D_{nm}^{(\mu)}\left(g^{-1}\right) \phi(g) dg.$$

In addition, if the given function $\phi(g)$ transforms under right G-multiplication as

$$\phi_m^{(\mu)}(gg') = \phi_n^{(\mu)}(g)D_{nm}^{(\mu)}(g')\,, \tag{1.14}$$

then it turns out that

$$\phi_m^{(\mu)}(g) = c_n^{(\mu)}D_{nm}^{(\mu)}(g) \quad [\text{no sum on } (\mu)] \tag{1.15}$$

with

$$c_n^{(\mu)} = \frac{1}{vol(G)}\int_G D_{mn}^{(\mu)}\left(g^{-1}\right)\phi_m^{(\mu)}(g)dg \quad [\text{no sum on } (\mu)]. \tag{1.16}$$

Condition (1.15) substantially means that

$$D_{nm}^{(\mu)} \quad \mu, n \text{ fixed} \tag{1.17}$$

is a complete basis for functions $\phi_m^{(\mu)}(g)$ fulfilling (1.14), and it is the subset of the complete functional basis which transforms as in (1.14).

The expansion (1.15) always exists because choosing $g = e$ (the identity in G) in (1.14) yields (1.15) with $c_n^{(\mu)} = \phi_n^{(\mu)}(e)$; while (1.16) follows from $D_{mn}^{(\mu)}\left(g^{-1}\right)\phi^{(\mu)}(g) = \phi^{(\mu)}(e)$.

If $\phi(y)$ is a function on G/H, then the matrix elements $D_{mn}^{(\mu)}\,|L(y)|$ are still a complete functional basis: they can be seen as vectors and the dimension of their vector is $vol(G/H)$.

All the above formulas continue to hold, but one has to replace $vol(G)$ with $vol(G/H) = vol(G)/vol(H)$ and also dg with $d\mu(y) =$ invariant measure on G/H .

Assume that a function $\phi(L(y))$ [here $\phi(L(y))$ is considered as a function of $L(y) \in G$ rather than of the coset coordinate y] in an irrep (α) of the subgroup H, fulfills

$$\phi_i^{(\alpha)}(L(y)h) = D_{ji}^{(\alpha)}(h)\phi_j^{(\alpha)}(L(y)); \tag{1.18}$$

then ϕ^α is not constant over the points of a coset gH but varies linearly under right action of H. In [39] it is shown that

- Every $D_b^a(h)$ is generated by the structure constants c_{ib}^a, i.e.

$$D_b^a(h) = [exp(c_i)]_b^a,$$

- In the expansion (1.15) one has to include all the G-irreps containing (α) under reduction to H, this means

$$\phi_i^{(\alpha)}(L(y)) = \sum_{(\nu)}\sum_n c_n^{(\nu)}D_{ni}^{(\nu)}(L(y)) \tag{1.19}$$

  where the sum on (ν) is restricted because of

$$(\nu) \xrightarrow{H} \cdots + (\alpha) + \cdots$$

- The
$$D_{ni}^{(\nu)} \quad i \text{ fixed}, \quad (\nu) \text{ contains } (\alpha) \tag{1.20}$$
form a complete basis for coset functions $\phi^{(\alpha)}(y)$ fulfilling (1.18).

Indeed, out of the complete set $D_{mn}^{(\nu)}$, the particular subset (1.20) transforms as in (1.18).

In order to invert (1.19), one needs to calculate

$$\int D_{mk}^{(\mu)}(L(y)) D_{kr}^{(\nu)}\left(L(y)^{-1}\right) d\mu \qquad \begin{array}{lll} m:\ (\mu) & - & \text{irrep index} \\ r:\ (\nu) & - & \text{irrep index} \\ k:\ (\alpha) & - & \text{irrep index contained in both} \\ & & (\mu) \text{ and } (\nu) \text{ irreps} \end{array}$$

and, from $(1.13)_1$, it follows

$$\begin{aligned} \frac{vol(G)}{\dim(\mu)}\delta_{mr}\delta_{ij}\delta^{\mu\nu} &= \int_G D_{mi}^{(\mu)}(g) D_{jr}^{(\nu)}\left(g^{-1}\right) dg \\ &= \int_H dh \int_{G/H} D_{mi}^{(\mu)}(L(g)h) D_{jr}^{(\nu)}\left(h^{-1}L^{-1}(g)\right) d\mu \\ &= \int_H D_{ki}^{(\alpha)}(h) D_{jl}^{(\alpha)}\left(h^{-1}\right) dh \int_{G/H} D_{mk}^{(\mu)}(L(y)) D_{lr}^{(\nu)}\left(L^{-1}(y)\right) d\mu \\ &= \frac{vol(H)}{\dim(\alpha)}\delta_{ij} \int_{G/H} D_{mk}^{(\mu)}(L(y)) D_{kr}^{(\nu)}\left(L^{-1}(y)\right) d\mu, \end{aligned}$$

and therefore

$$\int_{G/H} D_{mk}^{(\mu)}(L(y)) D_{kr}^{(\nu)}\left(L^{-1}(y)\right) d\mu = \frac{\dim(\alpha)}{\dim(\mu)} vol\left(\frac{G}{H}\right)\delta_{mr}\delta^{\mu\nu}.$$

So, the $c_n^{(\nu)}$ in (1.19) are

$$c_n^{(\nu)} = \frac{1}{vol(G/H)}\frac{\dim(\mu)}{\dim(\alpha)} \int_{G/H} D_{in}^{(\nu)}(L(y))\phi_i\left(L(y)^{-1}\right).$$

The coset functions $D_{in}^{(\nu)}(L(y))$ are called the H-harmonics on G/H.

Using the action of the covariant Lie derivative L_K on $L(y)$ (we use the simplified notation $L_A \equiv L_{K_A}$),

$$L_A L(y) = K_A L(y) - L(y) T_i W_A^i(y) = T_A L(y),$$

one can prove the following

Theorem 1.5.1 *The H-harmonics are eigenfunctions of the covariant Laplace-Beltrami operators:*

$$g^{A_1 A_1'} g^{A_2 A_2'} \cdots g^{A_n A_n'} Tr[c_{A_1} c_{A_2} \cdots c_{A_n}] L_{A_1'} L_{A_2'} \cdots L_{A_n'},$$

with

$$g^{AB} = G\text{-group metric}, \; c_{AB}^{C} = G\text{-structure constants}.$$

We end this section with some useful results on Laplace-Beltrami operators

1. If $\Box$ is a differential operator on G/H and
$$[\Box, L_A] = 0 \quad \text{for every } A \tag{1.21}$$
$\Box$ is called an invariant operator on G/H, and is diagonal on the harmonics. The Laplace-Beltrami operators are a complete set of invariant operators: every $\Box$ satisfying (1.21) is a function of them.

2. There are $r = rank(G/H)$ independent Laplace-Beltrami operators. The rank of G/H can be defined to be the maximal number of mutually commuting generators in the 'coset algebra' $\boldsymbol{k}$.

3. In a coset space of rank r, the first r Laplace-Beltrami operators can be chosen as a complete basis of invariant operators. The remaining higher-order Laplace-Beltrami operators are functionally dependent on the first r Laplace-Beltrami operators.

Points 1. and 2. generalize the well known fact that a group of rank r has r independent Casimir operators.

1.6 Non commutative geometry and Fuzzy Spaces

1.6.1 Introduction to non-commutative geometries

In a broad sense, the first example of a non-commutative geometry is the non-commutative version of phase space at the basis of quantum mechanics; in fact, according to this definition

Definition 1.6.1 (C^*-algebra)
A C^*-algebra $\mathcal{A}$ is

1. *a linear associative algebra over the field $\mathbb{C}$ of complex numbers, i.e. a vector space over $\mathbb{C}$ with an associative product linear in both factors,*

2. *a normed space, i.e. a norm $\|\|$ is defined on $\mathcal{A}$:*
$$\|A\| \geq 0, \quad \|A\| = 0 \Leftrightarrow A = 0, \quad \forall A \in \mathcal{A},$$
$$\|\lambda A\| = |\lambda| \|A\|, \quad \forall \lambda \in \mathbb{C}, \forall A \in \mathcal{A}$$
$$\|A + B\| \leq \|A\| + \|B\|, \quad \forall A, B \in \mathcal{A},$$
with respect to which the product is continuous:
$$\|AB\| \leq \|A\| \|B\|,$$
and $\mathcal{A}$ is a complete space with respect to the topology defined by the norm (thus $\mathcal{A}$ is a Banach algebra),

3. *a $*$-(Banach) algebra, i.e. there is an involution $* : \mathcal{A} \to \mathcal{A}$,*

$$(A+B)^* = A^* + B^*, \quad (\lambda A)^* = \overline{\lambda} A^*, \quad (AB)^* = B^* A^*, \quad (A^*)^* = A,$$

4. *with the property (C^*-condition)*

$$\|A^* A\| = \|A\|^2.$$

it turns out that the algebra of observables of a classical system is an abelian C^*-algebra, but this is no longer true when one deals with atomic or other microscopic systems.

C^*-algebras are now an important tool in the theory of unitary representations of locally compact groups, and are also used in algebraic formulations of quantum mechanics; moreover, the C^*-algebraic formulation of Quantum Mechanics, which has unquestionable advantages for logic and conceptual economy, especially for a mathematically oriented audience, and has played a crucial role for the recent non-commutative extensions of Calculus, Geometry, Probability etc. , has not yet become standard in quantum mechanics textbooks.

They were first considered primarily for their use in quantum mechanics to model algebras of physical observables. This line of research began with Werner Heisenberg's matrix mechanics and in a more mathematically developed form with Pascual Jordan around 1933. Subsequently, John von Neumann attempted to establish a general framework for these algebras which culminated in a series of papers on rings of operators. These papers considered a special class of C^*-algebras which are now known as von Neumann algebras.

Around 1943, the work of Israel Gelfand and Mark Naimark yielded an abstract characterisation of C^*-algebras making no reference to operators on a Hilbert space.

The measurement of the position of a particle requires an experimental apparatus which distinguishes points at very small scales and in macroscopic systems, it is enough to identify the position with a precision of a few orders of magnitude smaller than the size of the body, than for the realizability of the measurements one needs a control of the physics at scales which are still macroscopic. The situation changes if one wants to localize the position of an atomic particle of size $10^{-8}cm$ or of a nucleus of size $10^{-13}cm$, in fact there are intrinsic limitations. Heisenberg showed that any attempt to localize an atomic particle with sharp precision will produce a large disturbance on the microscopic system, with the result that the mean square deviation of the measurements of the momentum becomes larger and larger. For example, a precise localization of the particle can be obtained by taking a photograph, which requires sending light on the particle; the picture is the result of a reflection of light by the particle and, since light rays carry energy and momentum, the reflection of light changes the momentum of the particle. The result of these analysis led Heisenberg to the conclusion that for any state there is an intrinsic limitation in the relative precision by which x (the position) and p (the momentum) can be measured, independently of the state. The Heisenberg bound indicates that for all the states, if x_j and p_j denote the cartesian coordinates and the components of the momentum of the the particle, then

$$\Delta x_j \Delta p_j \geq \frac{h}{4\pi}, \text{ where } h \text{ is the Planck's constant.}$$

The above relations, called the Heisenberg uncertainty relations, should be regarded as unavoidable limitations for the preparation of states with sharper and sharper values of position or momentum. Clearly, since h is very small, the above inequality is relevant only for microscopic systems and this is the crucial point where atomic physics departs from classical physics. Heisenberg's idea is that the uncertainty relations arise as direct consequences of the following Heisenberg commutation relations

$$[x_j, p_k] = i\hbar\delta_{jk}\mathcal{I}, \qquad \text{where} \quad \hbar = \frac{h}{2\pi}.$$

Thus, the position and momentum of an atomic particle cannot be described by a commutative algebra, and, in a broad sense, the phase space of quantum mechanics is an example of non-commutative geometry. Planck's constant h plays the role of a continuous *deformation parameter*, i.e. a parameter that controls the noncommutativity, and $h \to 0$ is the *commutative limit*, i.e. the limit in which noncommutativity disappears [Technically speaking, x_j, p_j do not belong to a C^*-algebra, but using their exponentials $e^{i(a^j x_j + b^j p_j)}$ one can construct a C^*-algebra containing all the observables]. In a stricter sense, in noncommutative geometry also the subalgebra generated by the space(time) coordinates alone is noncommutative. By the Gelfand-Naimark Theorem, every C^*-algebra is isomorphic to an algebra of bounded operators in a Hilbert space, the vectors of which describe a full set of states; such a general Hilbert space description is equivalent to a representation in terms of continuous functions and probability measures only if the algebra of observables is abelian. The Gelfand-Naimark Theorem is therefore very important for the mathematical description of a physical system, because it settles the basic difference between classical and quantum physics.

Theorem 1.6.1 (Gelfand-Naimark) *Every C^*-algebra $\mathcal{A}$ is isomorphic to an algebra of (bounded) operators on a Hilbert space.*

One can get the Gelfand-Naimark characterization of abelian C^*-algebras from the above theorem; in fact it is possible to show that the irreducible representations π_ω are defined by pure states, which for abelian C^*-algebras are multiplicative, so that the corresponding representation are one-dimensional $\pi_\omega(A) = \omega(A)\mathcal{I}$. Then the family $\mathcal{F}$ of all inequivalent irreducible representations coincides with the Gelfand spectrum and the faithful representation $\pi(A) = \oplus_{\omega\in\mathcal{F}}\pi_\omega(A)$

is given by the collection $\{\omega(A), \omega \in \mathcal{F}\}$, that is by the function $\tilde{A}(\omega) = \omega(A)$. Furthermore,

$$\|\tilde{A}\|_\infty = \sup_\omega |\tilde{A}(\omega)| = \sum_\omega |\omega(A)| = \|A\|.$$

With the weak$*$ topology one can show that $\mathcal{F}$ is a compact Hausdorff topological space and the functions $\tilde{A}$ are continuous; this approach shows one basic difference between the abelian and the non-abelian case. In the first case, the set of pure states defines a 'classical' space; in the second case, the set of pure states defines a 'quantum' or 'non-commutative' space, whose points are rays in Hilbert spaces.

Another example is the Tannaka-Krein Theorem which is a generalization of the Gelfand-Naimark Theorem for compact groups.

1.6.2 The Fuzzy Sphere of Madore

This is a noncommutative model of a curved 2-dimensional space, more precisely of a sphere S^2; it is based on the algebra of $n \times n$ complex matrices which replaces the one of functions on S^2. The former looks like the latter above the length scale $k \approx \frac{r}{n}$, where r is the radius of the sphere; n plays the role of a *discrete* deformation parameter, and $n \to \infty$ the role of classical limit. In general, fuzzy spaces are noncommutative geometries based on a sequence of finite-dimensional algebras which become infinite-dimensional and commutative in the limit $n \to \infty$. The fuzzy sphere was proposed by Madore also to construct on it a toy-model of a quantum field theory on a (Wick-rotated) spacetime and investigate whether the ultraviolet divergences due to local field interactions could be regularized by a finite n. In fact, in the classical formulation of the quantum fields theory on Minkowski spacetime there are ultraviolet divergences which are corrections coming from perturbative methods applied to point-like field interactions. It was an idea of Heisenberg [1] to avoid this problem replacing the notion of points by some alternative structure which makes the infinitely precise measurements of position impossible. For example one can suppose that with length less than k the coordinates of a point are non-commuting operators and the position of the particle does not have an exact meaning. For instance, one can choose k less than the Compton wavelength, that is $\lambda_c = \frac{h}{m_0 c}$ with m_0 the mass of the particle; then the internal structure gives an uncertainty to the point less than the quantum uncertainty of the position of the particle.

The geometry of a manifold can be described using the algebra of functions defined on it, the coordinates are the generators of the algebra and the vector fields are the derivations; but one can describe the differential geometry using the operators over an algebra of functions and it is natural try to develop a new non-commutative version of the differential geometry replacing the algebra of functions $\mathcal{C}$ with a non-commutative one $\mathcal{A}$. It is possible to use a lattice structure, which eliminates the ultraviolet divergences since the associated algebra of functions is finite-dimensional. The finiteness of the algebra is linked to its non-commutativity and the algebra of matrices which will be used recalls finite versions of the algebra of observables on a phase space. The matricial geometries recall classical phase spaces where they have a symplectic form and they recall a quantum phase space since their algebra is not commutative.

Consider $\mathbb{R}^3$ with coordinates $\widetilde{x}_a$, $1 \leq a \leq 3$, the Euclidean metric $g_{a,b} = \delta_{a,b}$, the sphere

$$g_{a,b}\widetilde{x}_a\widetilde{x}_b = r^2, \tag{1.22}$$

and then the algebra $\mathcal{C}(S^2)$ of complex-valued polynomial functions $f(\widetilde{x}_a)$ on S^2

$$f(\widetilde{x}_a) = f_0 + f_a\widetilde{x}_a + \frac{1}{2}f_{a,b}\widetilde{x}_a\widetilde{x}_b + \cdots \tag{1.23}$$

This is an algebra which separates points and it is dense in the algebra of smooth functions. Madore constructs a sequence of non-commutative approximations of $\mathcal{C}(S^2)$.

A truncation of all functions to the constant term implies that the algebra $\mathcal{C}(S^2)$ is reduced to $\mathcal{A}_1 = \mathbb{C}$ of complex numbers and the geometry of S^2 is reduced to that of a point.

Keeping the term linear in the $\widetilde{x}_a$, the output is a four-dimensional vector space $\mathcal{A}_2$. It is possible to define a new product in the $\widetilde{x}_a$, so that $\mathcal{A}_2$ becomes an algebra.

If we require that the radical of $\mathcal{A}_2$ is equal to zero then there are two ways. We can define the product so that $\mathcal{A}_2$ becomes equal to the direct sum of four copies of $\mathbb{C}$, then the resulting algebra is commutative, the sphere looks like a set of four points and this would be a lattice approximation. The second possibility is to define the product so that $\mathcal{A}_2$ becomes equal to the algebra $\mathcal{M}_2$, of complex 2×2 matrices. That is, we replace $\widetilde{x}_a$ with $x_a = \kappa\sigma^a$, where σ^a are Pauli's matrix and κ is such that $r^2 = 3\kappa^2$. The sphere is not well described and one can distinguish only two points, because all x_a admit only two, opposite eigenvalues; the eigenvectors of e.g. x_3 can be identified with the north and the south pole.

Suppose next that we keep the term quadratic in the $\widetilde{x}_a$, then the resulting vector space $\mathcal{A}_3$ has dimension 9 and one can introduce in it a product such that it becomes equal to the algebra $\mathcal{M}_3$ of complex and square matrices of order 3; the $\widetilde{x}_a$ can be replaced with $x_a = \kappa J^a$, where the J^a form the three-dimensional and irreducible representation of $SU(2)$, that is $[J^a, J^b] = 2i\varepsilon_{abc}J^c$, with κ such that $r^2 = 8\kappa^2$. The sphere is now less fuzzy and one can distinguish the equator and the poles, corresponding to the three eigenvalues of x_3.

In general suppose that we suppress the terms of degree larger than n in the $\widetilde{x}_a$. The resulting set is a vector space $\mathcal{A}_n$. Let $N_l = \binom{2+l}{2}$ be the number of components of a completely symmetric tensor $f_{a_1,\cdots,a_l}$. Because of the constraint (1.22) for $l \geq 2$, N_{l-2} of these components would not contribute to the expansion (1.23). Therefore there are $N_l - N_{l-2} = 2l + 1$ independent monomials of degree l and $\sum_{l=0}^{n-1}(2l+1) = n^2$ components in all. So $\mathcal{A}_n$ is of dimension n^2 and we can introduce a new product in the $\widetilde{x}_a$ which will make it into the algebra $\mathcal{M}_n$ of complex $n \times n$ matrices. That is, one makes the replacement

$$\widetilde{x}_a \mapsto x_a = \kappa J^a \tag{1.24}$$

but where the J^a form the n-dimensional irreducible representation of the Lie algebra of $SU(2)$, and the parameter n is related to r by the equation $r^2 = (n^2-1)\kappa$. For large n we have

$$\kappa \simeq \frac{r}{n}$$

and so $\kappa \to 0$ as $n \to +\infty$. Introduce the constant

$$k = 4\pi\kappa r$$

It has the dimension of (length)^2 and plays here a role analogous to that played by Planck's constant in quantum mechanics. The commutative limit is given by $k \to 0$. It is convenient also to define $\bar{k} := \frac{k}{2\pi} = 2\kappa r$.

The generators x_a of the algebra $\mathcal{M}_n$, satisfy the commutation relations

$$[x_a, x_b] = i\bar{k} C^c_{ab} x_c \quad C_{abc} = r^{-1}\epsilon_{abc}.$$

So in the limit they commute and all of the points of the sphere can be distinguished.

We shall be more interested in the mapping ϕ_n, of $\mathcal{M}_n$ into $\mathcal{C}(S^2)$ given by the inverse of (1.24) on the generators x_a. Every element $f \in \mathcal{M}_n$ has a unique expansion

$$f = \sum_{l=0}^{n-1} \frac{1}{l!} f_{a_1 \cdots a_l} x_{a_1} \cdots x_{a_l}$$

where $f_{a_1 \cdots a_l}$ is a symmetric trace-free tensor. Let $\widetilde{f}$ be the element of $\mathcal{C}(S^2)$ obtained from f by replacing x_a by $\widetilde{x}_a$ in this expansion. Then $f \mapsto \widetilde{f}$ defines a linear mapping ϕ_n of $\mathcal{M}_n$ into $\mathcal{C}(S^2)$. The range of ϕ_n is the subspace of functions on S^2 which are polynomials in the $\widetilde{x}_a$ of degree up to and including $n-1$. If we consider the vector space W_l of elements in $\mathcal{M}_n$ which possess an expansion of degree at most $l \le n-1$ then we have for $f, g \in W_l$

$$\phi_n(fg) - \phi_n(f)\phi_n(g) \sim o\left(\frac{l}{n}\right) \tag{1.25}$$

To see this consider first the case $l = 1$. Then $f = f_0 + f_a x_a$ and $g = g_0 + g_a x_a$ and

$$\phi_n(fg) - \phi_n(f)\phi_n(g) = \frac{1}{2} i\bar{k} f_a g_b C^{abc} \widetilde{x}_c.$$

If $l > 1$ then each monomial except the first will contribute in general a term containing a factor $\bar{k}$. This yields $l-1$ terms each of which will vanish in the limit $n \to +\infty$. If $l = n-1$ then (1.25) is an empty assertion. In the limit of very large n however ϕ_n can be considered as a morphism between the algebra of polynomials in x_a and the algebra of polynomials in $\widetilde{x}_a$. As the order of the polynomials involved approaches $n-1$, the error involved in considering ϕ_n an algebra morphism becomes more and more important.

One wishes to approximate a commutative algebra $\mathcal{C}\left(S^2\right)$ by a sequence of noncommutative approximations $\mathcal{M}_n$. To see in which sense this can be done one can define a norm on $\mathcal{M}_n$ and show that in the limit $n \to +\infty$ the algebra $\mathcal{C}\left(S^2\right)$ can be considered as the image of the diagonal matrices in $\mathcal{M}_n$. For each element $f \in \mathcal{M}_n$ one sets

$$\|f\|_n^2 = \frac{1}{n} Tr\left(f^* f\right). \tag{1.26}$$

The generators x_a have a norm given by

$$\|x_a\|_n^2 = \frac{1}{3} r^2.$$

This is independent of n. If we define the norm of an element $\widetilde{f} \in \mathcal{C}\left(S^2\right)$ by

$$\left\|\widetilde{f}\right\|^2 = \frac{1}{4\pi r^2} \int \left|\widetilde{f}\right|^2$$

then $\|x_a\|_n = \|\widetilde{x}_a\|$. Let $f \in \mathcal{M}_n$ and set $\widetilde{f} = \phi_n(f)$. Then we have

$$\frac{1}{n} Tr(f) \mapsto \frac{1}{4\pi r^2} \int \widetilde{f}$$

as $n \to +\infty$. Indeed the left-hand side tends to an $SO(3)$-invariant integral over S^2.

The normalization is fixed by considering the case $f = 1$. In particular

$$\|f\|_n \to \left\|\widetilde{f}\right\|.$$

A general element $f \in \mathcal{M}_n$ with entries $O(1)$ will have a norm $\|f\|_n = O(n)$. A diagonal matrix with entries $O(1)$ or a matrix with only a number $O(1)$ of off-diagonal terms will on the other hand have a norm $\|f\|_n = O(1)$. For large n then bounded functions will be the image of near-diagonal matrices, that is of matrices which commute with each other to within order $\bar{k}$.

We cannot speak of the position of a particle because of the absence of localization but the state of a particle on the sphere is described as in quantum mechanics by a state vector ψ, which we shall assume to be normalized. For the matrix algebra $\mathcal{M}_n$ then a particle is described by a vector ψ, which we shall assume to be normalized. For the matrix algebra $\mathcal{M}_n$ then a particle is described by a vector $\psi \in \mathbb{C}^n$ with $\psi^* \psi = 1$. An observable associated to the particle is a Hermitian element of $\mathcal{M}_n$ and the value of an observable f is given by the real number $\psi^* f \psi$. For example, what corresponds to the position of the particle is given by two of the three numbers $\psi^* x_a \psi$. When $n \to +\infty$, these must converge to well defined values of the coordinates $\widetilde{x}_a$ if the particle is to be considered as localized. The state vectors lie in $\mathbb{P}^{n-1}(\mathbb{C})$, a space of complex dimension $n-1$. So the $\psi^* x_a \psi$ do not determine a state. The additional $2n-4$ real numbers needed to fix a point in $\mathbb{P}^{n-1}(\mathbb{C})$ give information about the dispersion of the particle. If we measure one generator, say x_3, then after the measurement ψ becomes its eigenvector and is completely determined. The expectation values of the other two generators then vanish since there is equal probability of a positive and negative value. The most general state vector can be written in terms of the eigenvectors of x_3. The matrix which takes the latter to the former corresponds to what in quantum mechanics would be the Schrödinger wavefunction. As $k \to 0$ it becomes more and more dilficult to distinguish a vector uniquely using x_3. In the limit the eigenvalues of one of the other two generators must be used as well. In the limit the function $\widetilde{\psi}(\widetilde{x}_a)$ gives the (purely classical) probability of finding the particle at the point with coordinates $\widetilde{x}_a$.

The analogue of a general coordinate transformation is a change of generators

$$x_a \mapsto x'_a$$

of the algebra $\mathcal{M}_n$. This mapping does not necessarily respect the relations of the algebra and it does not necessarily possess an extension to an automorphism of $\mathcal{M}_n$.

As an example we shall briefly consider a second set of generators (u, v) of the algebra $\mathcal{M}_n$ which are in no way related to the group $SU(2)$ and which satisfy the relations

$$u^n = 1, \quad v^n = 1, \quad uv = e^{\frac{2\pi i}{n}} vu. \tag{1.27}$$

This describes a particular case of the two-dimensional quantum plane, and the relations (1.27) are invariant under the transformations

$$u \mapsto u' = \exp(2\pi i p/n)\, u, \quad v \mapsto v' = \exp(2\pi i q/n)\, v,$$

for (p, q) in the discrete group $\mathbb{Z} \times \mathbb{Z}$. So in the limit $n \to +\infty$ the sequence of matrices (u, v) tends to generators $(\widetilde{u}, \widetilde{v})$ of an algebra of functions defined on the torus

$$\widetilde{u} = \exp(2\pi i x/r), \quad \widetilde{v} = \exp(2\pi i y/r),$$

where $0 \leq x, y \leq r$. Abbreviating $q := \exp(2\pi i/n)$, a concrete realization of u, v fulfilling (1.27) is provided by the socalled *clock* and *shift* matrices

$$u = \begin{pmatrix} 1 & & & & \\ & q & & & \\ & & q^2 & & \\ & & & ... & \\ & & & & q^{n-1} \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 0 & & ... & 1 \\ 1 & 0 & & ... & \\ 0 & 1 & 0 & ... & \\ & & & ... & \\ & & & ... & 1 & 0 \end{pmatrix}.$$

For any matrix in $\mathcal{M}_n$, in particular for the x_a, there exist matrix polynomials $x_a = x_a(u,v)$ and their inverses $u = u(x_a)$ and $v = v(x_a)$, for each value of n. However if the limit exists when $n \to +\infty$ the corresponding functions would have to be discontinuous since they would otherwise define a homeomorphism of the sphere with the torus. Identify the torus as the region of the real plane (x,y) with $0 \leq x,y \leq r$ defined by an algebra of functions which are periodic at the two boundaries and the sphere as the same region but defined by an algebra of functions which have a constant value around the boundary. Then there is an embedding of the set of continuous functions on the sphere into the continuous functions on the torus. We can consider however the algebras of all functions on the two manifolds to be identical. They are both the limit of the sequence of matrix algebras $\mathcal{M}_n$. The generators of $\mathcal{M}_n$ which we have used above have two different symmetries and these symmetries are to be found in the manifold which is defined in the limit. The $SO(3)$ symmetry defines an S^2 geometry; the $\mathbb{Z} \times \mathbb{Z}$ defines the torus.

A diffeomorphism of S^2 defines and is defined by an automorphism of the algebra of smooth functions on S^2. Let ϕ be a diffeomorphism of S^2. Then ϕ has an extension ϕ^a to $\mathbb{R}^3$ and we can set $\widetilde{x}'_a = \phi^a(\widetilde{x}_b)$. This defines an automorphism of $\mathcal{C}(S^2)$ which is independent of the extension. Conversely such an automorphism ϕ^a restricted to the generators $\widetilde{x}_a$ defines a coordinate transformation of $\mathbb{R}^3$ and by restriction a diffeomorphism of S^2. The non-commutative analogue of a diffeomorphism of S^2 is therefore an automorphism of $\mathcal{M}_n$. Since $\mathcal{M}_n$ is a simple algebra all of its automorphims are of the form $f \mapsto f' = g^{-1}fg$ where g is a fixed arbitrary element of $\mathcal{M}_n$ which has an inverse. We have considered complex-valued functions on S^2 and the algebra $\mathcal{C}(S^2)$ has a $*$-operation $\widetilde{f} \mapsto \widetilde{f}^*$ obtained by taking the complex conjugate of $\widetilde{f}$. A diffeomorphism of S^2 will define an automorphism of $\mathcal{C}(S^2)$ which respects this $*$-operation: $\widetilde{f}'^* = \widetilde{f}^{*\prime}$. We must therefore require the same condition on the automorphisms of $\mathcal{M}_n$:$(g^{-1}fg)^* = g^{-1}f^*g$. This means that $g^* = g^{-1}$ and therefore that

$$x'_a = g^{-1} x_a g, \quad g \in SU(N). \tag{1.28}$$

A different choice of x'_a not related to x_a by this formula would be equivalent to a different choice of differential or topological structure. An element $f \in \mathcal{M}_n$ has an expansion $f = f(x_a)$ in the basis x_a and an expansion $f = f'(x'_a)$ in the basis x'_a. If (1.28) is considered as an automorphism then $f \mapsto g^{-1}fg$. If it is to be regarded as a change of generators then $f \mapsto f$ and $f'(y_a)$ is determined in terms of $f(y_a)$ by the identity $f'(x'_a) = f(x_a)$.

A smooth global vector field on S^2 defines and is defined by a derivation of the algebra $\mathcal{C}(S^2)$. The non-commutative analogue of a global vector field on S^2 is therefore a derivation of the algebra $\mathcal{M}_n$, that is, a linear map X of $\mathcal{M}_n$ into itself which satisfies Leibnitz's rule: $X(fg) = X(f)g + fX(g)$. Since $\mathcal{M}_n$ is a simple algebra all of its derivations are of the form $X = ad(h)$ where h is a fixed arbitrary element of $\mathcal{M}_n$. Since we wish also to have $(X(f))^* = X(f^*)$ we shall require that h be antiHermitian. A diffeomorphism of S^2 leaves the set of smooth global vector fields invariant. The change of generators (1.28) takes X into $X' = ad(g^{-1}hg)$ and so all automorphisms of $\mathcal{M}_n$ are analogues of diffeomorphisms of S^2.

Let $\widetilde{x}'_a$ be the limit of the sequence $\{x'_a\}$ defined above. Then the map $\widetilde{x}_a \mapsto \widetilde{x}'_a$ is a coordinate transformation of $\mathbb{R}^3$. If g is near to the identity we can write

$$x'_a \simeq x_a - [h, x_a], \quad g = 1 + h.$$

An important special case is given by

$$h = \frac{1}{ik} h_a x_a. \tag{1.29}$$

In this case $x'_a \simeq x_a + C^a_{bc} h^b x_c$ and therefore in the limit it corresponds to an infinitesimal rotation about the axis h_a in $\mathbb{R}^3$. The formula (1.28) with h small and given by (1.29) yields the adjoint action of the Lie algebra of $SO(3)$ on $\mathcal{M}_n$. On the algebra $\mathcal{M}_n$, we have a representation of $SO(3)$ which contains exactly once the irreducible representation of dim $2j+1$ for $0 \le j \le n-1$.

1.6.3 Fuzzy spaces through energy cutoff and confining potentials

As known, in classical mechanics a charged particle (e.g. an electron) in the plane $z = 0$ subject to a magnetic field $\boldsymbol{B}$ pointing along the z direction moves with constant speed $\boldsymbol{v}$ along a circle of radius $r = mvc/\|e\boldsymbol{B}\|$ (the cyclotron radius), where m, e are the mass and the charge of the particle and $v := \|\boldsymbol{v}\|$, $B := |\boldsymbol{B}\|$. The Lorentz force causes the electron to spiral around and the centrifugal force must balance the Lorentz force, that is

$$\frac{mv^2}{r} = \frac{|e|}{c} vB;$$

then it is possible to calculate the cyclotron radius

$$r = \frac{mcv}{|e|B}$$

and the angular frequency of the cyclotron motion is

$$\omega = 2\pi \frac{v}{2\pi r} = \frac{|e|B}{mc}.$$

In quantum mechanics the energy levels of this system (the socalled Landau model) are quantized and the x, y coordinates from the center of the circle do not commute.

The literature is very rich of works about this topic; in particular, Peierls [40] firstly studied the one band Hamiltonian for a Bloch electron in a magnetic field, while in [41, 42] R. Jackiw and G. Magro analyze the Landau Hamiltonian and show that the x, y coordinates of the particle themselves become noncommuting if one imposes an energy cut-off, namely projects down all the observables to the subspace of the Hilbert space of states characterized by and energy below a certain quantity (the cut-off). Thus this mechanism can provide an example of non-commutative geometry. The Hamiltonian of this system is

$$H = \frac{1}{2m}\left(\mathbf{p} - e\frac{\mathbf{A}}{c}\right)^2,$$

where $\mathbf{A}$ is the electromagnetic vector potential, that is $\mathbf{B} = \nabla \times \mathbf{A}$. Choosing a suitable gauge, one has $\mathbf{A} = (0, Bx, 0)$ and the Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{1}{2m}\left(p_y - \frac{qBx}{c}\right)^2.$$

The operator y is not into this Hamiltonian, then p_y commutes with H, so this operator can be replaced by its eigenvalue $\hbar k_y$, and using the cyclotron frequency one has

$$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2\left(x - \frac{\hbar k_y}{m\omega}\right)^2.$$

The eigenvalue equation of the Hamiltonian is the same of the quantum harmonic oscillator; then there is a quantization of energies:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \qquad n \in \mathbb{N}_0.$$

These values of the energy increase with n and correspond to the so-called Landau levels, they have infinite degeneration. Let $\mathcal{H}$ be the Hilbert space of all quantum states, the imposition of an energy cut-off $\overline{E}$ means that one assumes to make only low-energy measurements; the consequence is that the space of physical states is reduced from $\mathcal{H}$ to the subspace $\mathcal{H}_{\overline{E}}$ spanned by the eigenfunctions with $E_n \leq \overline{E}$. In this situation (as for the new fuzzy hyperspheres that are introduced in the next chapter) some commutation relations become non trivial after the projection on $\mathcal{H}_{\overline{E}}$. In particular if one confines the particle in a bounded region of the plane imposing some boundary conditions (or an infinite barrier of potential) the flux of $\boldsymbol{B}$ through the surface must be quantized and this value determines the degeneration which is finite; alternatively, one can consider the same problem on a torus and impose that the wavefunction is periodic up to a phase factor, and this yields a finite degeneration. In these situations one has a finite-dimensional subspace of $\mathcal{H}$ and, through the use of a projector, any operator can be transformed into an endomorphism of $\mathcal{H}_{\overline{E}}$.

More generally, if the Hamiltonian has a confining potential, then classically the position of the particle can be only in a compact region, and also p is compact because $E_k \leq \overline{E}$. The quantization ensures that $\mathcal{H}_{\overline{E}}$ has finite dimension (that can be estimated even without solving the eigenvalue problem). In this way one can construct a non-commutative fuzzy geometry if the original coordinates projected on $\mathcal{H}_{\overline{E}}$ do not commute and generate (through non-ordered polynomials[1]) the algebra $\mathcal{A}_{\overline{E}}$ of all endomorphisms of $\mathcal{H}_{\overline{E}}$.

[1]If one wants to use only generators with a pre-determined order some more generators may be necessary.

Chapter 2

The general construction of S_Λ^d with $d \in \mathbb{N}$

2.1 General setting

As mentioned before, consider a quantum particle in $\mathbb{R}^D$, with a Hamiltonian operator

$$H := -\frac{1}{2}\Delta + V(r)$$

such that the potential $V(r)$ has a very sharp minimum at $r = 1$ with a very large $k_D := V'(1)/4 > 0$, and fix $V_0 := V(1)$ so that the ground state has zero energy, i.e. $E_0 = 0$. In addition, impose here that the energy cutoff $\overline{E}$ is chosen so that

$$V(r) \simeq V_0 + 2k_D(r-1)^2 \quad \text{if } r \text{ fulfills} \quad V(r) \leq \overline{E}, \tag{2.1}$$

then one can neglect terms of order higher than 2 in the Taylor expansion of $V(r)$ around $r = 1$ and approximate the potential with a harmonic one in the classical region $b_{\overline{E}} \subset \mathbb{R}^D$ compatible with the energy cutoff $V(r) \leq \overline{E}$. The equality $\boldsymbol{L}^2 Y(\theta_d, \theta_{d-1}, \cdots, \theta_1) = l(l + D - 2) Y(\theta_d, \theta_{d-1}, \cdots, \theta_1)$ and the Ansatz (10) are used to simplify the resolution of the PDE $H\boldsymbol{\psi} = E\boldsymbol{\psi}$, in fact this problem is consequently split in two:

1. Solve the corresponding ODE for $f(r)$;

2. Determine all the eigenfunctions of $\boldsymbol{L}^2$, which will be also square-integrable because S^2 is compact and $\boldsymbol{L}^2$ is regular.

In addition, it is also necessary to verify if H is a self-adjoint operator on the Hilbert space $\mathcal{H}_D$ of pure quantum states.

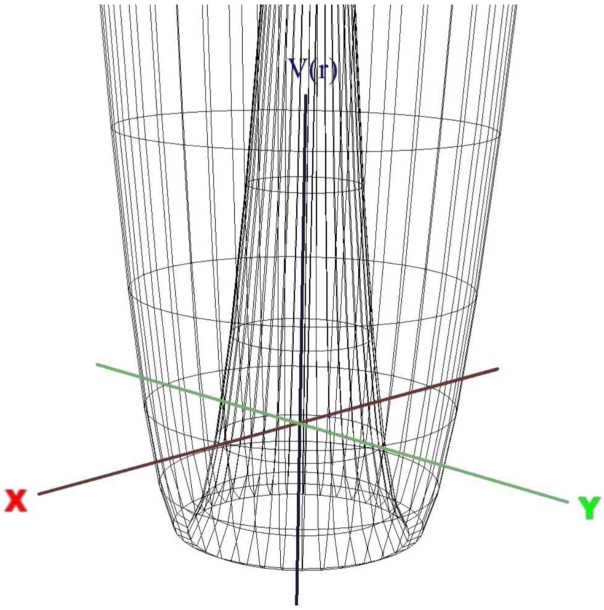

Figure 2.1: Three-dimensional plot of $V(r)$

2.1.1 Resolution of $H\psi = E\psi$ – Step 1

The ODE for $f(r)$ turns out to be equivalent to equation (9) in [13, 14]; this means that one has to solve

$$\left[-\partial_r^2 - (D-1)\frac{1}{r}\partial_r + \frac{1}{r^2}l\,(l+D-2) + V(r)\right] f(r) = Ef(r). \tag{2.2}$$

In section 7.0.2 it is shown that the hypothesis $r^2 V(r) \overset{r\to 0^+}{\longrightarrow} T \in \mathbb{R}^+$ [which is obviously compatible with (2.1)] and the request that $\psi \in D(H) \equiv D(H^*)$ (self-adjointness of H) imply that $f(r)$ is regular at $r = 0$, and then the same applies to the function $g(r) := f(r) r^{\frac{D-1}{2}}$. Consequently, (2.2) becomes

$$-g''(r) + g(r)\frac{[D^2 - 4D + 3 + 4l(l+D-2)]}{4r^2} + V(r)g(r) = Eg(r). \tag{2.3}$$

For the purposes of this thesis, the solution of this last equation is interesting only around $r = 1$; this means that one can use the equalities (at leading order)

$$\frac{1}{r^2} = 1 - 2(r-1) + 3(r-1)^2, \quad V(r) = V_0 + 2k_D(r-1)^2,$$

which lead to this 1-dimensional harmonic oscillator equation

$$-g''(r) + g(r)k_{l,D}\,(r - \widetilde{r}_{l,D})^2 = \widetilde{E}_{l,D}\,g(r), \tag{2.4}$$

where

$$\begin{aligned} b(l,D) &:= \frac{D^2-4D+3+4l(l+D-2)}{4}, \quad k_{l,D} := 3b(l,D)+2k_D, \\ \widetilde{r}_{l,D} &:= \frac{4b(l,D)+2k_D}{3b(l,D)+2k_D}, \quad \widetilde{E}_{l,D} := E - V_0 - \frac{2b(l,D)\,[k_D+b(l,D)]}{3b(l,D)+2k_D}; \end{aligned} \tag{2.5}$$

so at leading order the lowest eigenvalues E are those of the 1-dimensional harmonic oscillator approximation of (2.3).

The (Hermite) square-integrable solutions of (2.4) are ($M_{n,l,D}$ is a suitable normalization constant)

$$g_{n,l,D}(r) = M_{n,l,D}\; e^{-\frac{\sqrt{k_{l,D}}}{2}\left(r-\widetilde{r}_{l,D}\right)^2} \cdot H_n\left(\left(r-\widetilde{r}_{l,D}\right)\sqrt[4]{k_{l,D}}\right) \qquad \text{with } n \in \mathbb{N}_0, \tag{2.6}$$

implying

$$f_{n,l,D}(r) = \frac{M_{n,l,D}}{r^{\frac{D-1}{2}}}\; e^{-\frac{\sqrt{k_{l,D}}}{2}\left(r-\widetilde{r}_{l,D}\right)^2} \cdot H_n\left(\left(r-\widetilde{r}_{l,D}\right)\sqrt[4]{k_{l,D}}\right) \qquad \text{with } n \in \mathbb{N}_0. \tag{2.7}$$

The corresponding 'eigenvalues' in (2.4) are $\widetilde{E}_{n,l,D} = (2n+1)\sqrt{k_{l,D}}$ and this leads to energies

$$E_{n,l,D} = (2n+1)\sqrt{k_{l,D}} + V_0 + \frac{2b(l,D)[k_D+b(l,D)]}{3b(l,D)+2k_D}. \tag{2.8}$$

As mentioned before, V_0 is fixed requiring that the lowest energy level, which corresponds to $n=l=0$, is $E_{0,0,D}=0$; this implies

$$V_0 = -\sqrt{k_{0,D}} - \frac{2b(0,D)\,[k_D+b(0,D)]}{3b(0,D)+2k_D}; \tag{2.9}$$

while the expansions of $\widetilde{r}_{l,D}$ and $E_{n,l,D}$ at leading order in k_D are the following ones:

$$\begin{aligned} \widetilde{r}_{l,D} &= 1 + \frac{b(l,D)}{2k_D} - \frac{3b(l,D)^2}{4k_D^2} + O\left(k_D^{-3}\right), \\ V_0 &= -\sqrt{2k_D} - b(0,D) - \frac{3b(0,D)}{2\sqrt{2k_D}} + \frac{b(0,D)^2}{2k_D} + \frac{9b(0,D)^2}{8\,(2k_D)^{\frac{3}{2}}} - \frac{3b(0,D)^3}{4k_D^2} + O\left(k_D^{-\frac{5}{2}}\right), \\ E_{n,l,D} &= (2n+1)\sqrt{2k_D} + V_0 + b(l,D) + (2n+1)\frac{3b(l,D)}{2\sqrt{2k_D}} \\ &\quad - \frac{b(l,D)^2}{2k_D} - (2n+1)\frac{9b(l,D)^2}{16k\sqrt{2k_D}} + \frac{3b(l,D)^3}{4k_D^2} + O\left(k_D^{-\frac{5}{2}}\right) \\ &= 2n\sqrt{2k_D} + l(l+D-2) + \frac{1}{\sqrt{2k_D}}\left[3nb(l,D) + \frac{3}{2}l(l+D-2)\right] \\ &\quad + \frac{1}{2k_D}\left[-l(l+D-2)\right]\left[\frac{2D^2-8D+6+4l(l+D-2)}{4}\right] + O\left(k_D^{-\frac{3}{2}}\right). \end{aligned} \tag{2.10}$$

2.1.2 Resolution of $H\boldsymbol{\psi} = E\boldsymbol{\psi}$ – Step 2

In section 7.0.3 it is shown that an orthonormal basis of $\mathcal{L}^2(S^d)$, made up of $\boldsymbol{L}^2$-eigenfunctions, is the collection of all the

$$Y = Y_{\boldsymbol{l}}(\theta_d, \cdots, \theta_2, \theta_1) = \frac{e^{il_1\theta_1}}{\sqrt{2\pi}} \left[\prod_{n=2}^{d} {}_n\overline{P}_{l_n}^{l_{n-1}}(\theta_n) \right], \quad \boldsymbol{l} = (l_d, \cdots, l_2, l_1), \tag{2.11}$$

where

$${}_j\overline{P}_L^M(\theta) := \sqrt{\frac{2L+j-1}{2}} \sqrt{\frac{(L+M+j-2)!}{(L-M)!}} \left[\sin\theta\right]^{\frac{2-j}{2}} P_{L+\frac{j-2}{2}}^{-\left(M+\frac{j-2}{2}\right)}(\cos\theta), \tag{2.12}$$

$l_d \geq \cdots \geq l_2 \geq |l_1|$, $l_i \in \mathbb{Z}\ \forall i$ and P_l^m is the associated Legendre function of first kind (see [43] for a summary about these special functions).

They fulfill

$$\begin{gathered} L_{1,2}Y_{\boldsymbol{l}} = l_1 Y_{\boldsymbol{l}} \Rightarrow C_2 Y_{\boldsymbol{l}} = l_1^2 Y_{\boldsymbol{l}}, \quad C_p Y_{\boldsymbol{l}} = l_{p-1}(l_{p-1}+p-2)Y_{\boldsymbol{l}}, \\ \text{and} \qquad \int_{S^d} Y_{\boldsymbol{l}} Y_{\boldsymbol{l'}}^* d\alpha = \delta_{\boldsymbol{l}}^{\boldsymbol{l'}}, \end{gathered} \tag{2.13}$$

where $d\alpha$ is the usual measure on S^d,

$$d\alpha = \left[\sin^{d-1}(\theta_d)\sin^{d-2}(\theta_{d-1}) \cdots \sin(\theta_2)\right] d\theta_1 d\theta_2 \cdots d\theta_d.$$

According to these last equations, every $\bar{l} \in \mathbb{N}_0$ identifies a

$$V_{\bar{l},D} := span\left\{Y_{\bar{\boldsymbol{l}}} : \bar{\boldsymbol{l}} := \left(\bar{l}, l_{d-1}, \cdots, l_2, l_1\right), \bar{l} \geq l_{d-1} \geq \cdots \geq l_2 \geq |l_1|, l_i \in \mathbb{Z}\forall i\right\}, \tag{2.14}$$

which is the representation space of an irrep of $U\boldsymbol{so}(D)$, and $\{L_{1,2}, C_2, \cdots, C_D\}$ is a CSCO of this irrep, where CSCO stands for *complete set of commuting observables*, i.e. a set of commuting operators whose set of eigenvalues completely specify elements of a basis of $\mathcal{H}_{\Lambda,D}$.

In addition, in section 7.0.3.4 it is shown that

$$V_{l,D} \quad \text{is isomorphic to} \quad \bigoplus_{m=0}^{l} V_{m,d} \quad \text{if } D > 3,$$

$$\text{while } V_{l,3} \quad \text{is isomorphic to} \quad \bigoplus_{m=-l}^{l} V_{m,2};$$

this decomposition can be also applied to $\mathcal{H}_{\Lambda,D}$, up to isomorphisms, and this job is done in section 2.4.

So, the pure quantum states (the elements of an orthonormal basis of $\mathcal{H}_D$) are the following ones:

$$\boldsymbol{\psi}_{n,\boldsymbol{l},D}(r, \theta_d, \cdots, \theta_2, \theta_1) := f_{n,\boldsymbol{l},D}(r) Y_{\boldsymbol{l}}(\theta_d, \cdots, \theta_2, \theta_1), \tag{2.15}$$

with $n \in \mathbb{N}_0$, $l \equiv l_d \geq \cdots \geq l_2 \geq |l_1|$, $l_i \in \mathbb{Z}\ \forall i$.

2.2 The imposition of the cutoff

As mentioned before, a low enough energy cutoff $E \leq \overline{E}$ is imposed in a way such that it excludes all the states with $n > 0$; according to this, it must be $\overline{E} < 2\sqrt{2k_D}$, which (from the physical point of view) means that radial oscillations are 'frozen' ($\Rightarrow n = 0$, as wanted), so that all corresponding classical trajectories are circles; the energies E below $\overline{E}$ will therefore depend only on l and D, and are consequently denoted by $E_{l,D}$.

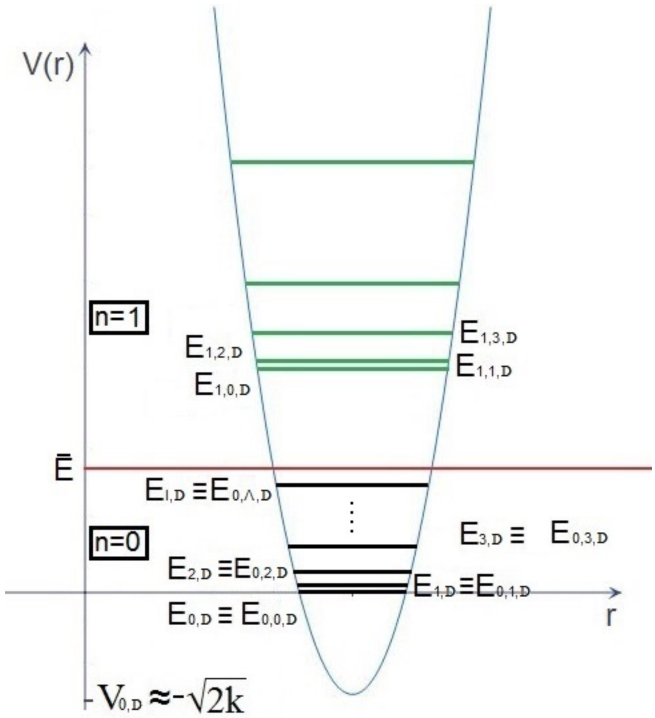

Figure 2.2: Two-dimensional plot of $V(r)$ including the energy-cutoff

The Hilbert space of 'admitted' states is $\mathcal{H}_{\overline{E},D} \subset \mathcal{H}$, it is finite-dimensional and spanned by the states $\boldsymbol{\psi}$ fulfilling the cutoff condition; on the other hand, one has also to replace every observable A by $\overline{A} := P_{\overline{E},D} A P_{\overline{E},D}$, where $P_{\overline{E},D}$ is the projection on $\mathcal{H}_{\overline{E},D}$, and we give to $\overline{A}$ the same physical interpretation.

Then, at leading orders in $1/\sqrt{k_D}$,

$$\begin{gathered} H = E_{l,D} = l(l+D-2) + O\left(\frac{1}{\sqrt{k_D}}\right); \\ \boldsymbol{\psi}_{\boldsymbol{l},D}(r,\theta_{D-1},\cdots,\theta_1) = \frac{M_{l,D}}{r^{\frac{D-1}{2}}} e^{-\frac{\sqrt{k_{l,D}}}{2}\left(r-\widetilde{r}_{l,D}\right)^2} Y_{\boldsymbol{l}}\left(\theta_d,\cdots,\theta_1\right), \end{gathered} \tag{2.16}$$

where the normalization factor $M_{l,D}$ is fixed so that $M_{l,D} > 0$ and all $\boldsymbol{\psi}_{\boldsymbol{l},D}$ have unit norm in $\mathcal{L}^2\left(\mathbb{R}^D\right)$ (see section 7.0.5).

The choice of a Λ-dependent energy cutoff $\overline{E} = \overline{E}(\Lambda) := \Lambda(\Lambda + D - 2)$, implies that the condition $E \leq \overline{E}$ becomes equivalent to the projection of the theory onto the Hilbert subspace $\mathcal{H}_{\Lambda,D} \equiv \mathcal{H}_{\overline{E},D}$ spanned by all the states $\boldsymbol{\psi}_{\boldsymbol{l},D}$ with $l(l+D-2) \leq \Lambda(\Lambda+D-2) \Leftrightarrow l \leq \Lambda$. For consistency it must be

$$\Lambda(\Lambda + D - 2) < 2\sqrt{2k_D}, \tag{2.17}$$

and for instance one can define $k_D(\Lambda) \geq [\Lambda(\Lambda + D - 2)]^2$, while in section 7.0.12 a larger $k_D(\Lambda)$ is used in order to prove the convergence to ordinary quantum mechanics on S^d. According to this first choice of $k_D(\Lambda)$, all $E_{l,D}$ are smaller than the energy levels corresponding to $n > 0$; this is also sufficient to guarantee that $k_{l,D} \gg 1$ for all $l \leq \Lambda$ [by the way, $k_{l,D} > 0$ is a necessary condition for (2.4) to be the eigenvalue equation of a harmonic oscillator]; furthermore, the spectrum of $\overline{H}$ becomes the whole spectrum $\{l(l+D-2)\}_{l\in\mathbb{N}_0}$ of $\boldsymbol{L}^2$ in the commutative limit, i.e. $\Lambda \to \infty$.

2.3 The algebra $\mathcal{A}_{\Lambda,D}$

2.3.1 The action of angular momentum components on $Y_{\boldsymbol{l}}$

In the next lines there are the R coefficients, they are determined in section 7.0.6 and used in the following definition (which is given by induction) of the action of a generic $L_{h,j}$ on a spherical harmonic $Y_{\boldsymbol{l}}$.

Definition 2.3.1 *For $D = 2$ there is only one angular momentum component, $L_{1,2}$, and its action is $L_{1,2}Y_{l_1} = l_1 Y_{l_1}$. For $D > 2$, let*

$$d_{L,l,D} := \sqrt{(L+1)(L+D-3) - l(l+D-4)} = \sqrt{(L-l+1)(L+l+D-3)}$$

and

$$R_{h,D}\left(\boldsymbol{l},\boldsymbol{l'}\right) := \left\langle Y_{\boldsymbol{l'}}, t_h Y_{\boldsymbol{l}}\right\rangle;$$

the action of the angular momentum operators is defined in this way:

$$L_{\nu,D}Y_{\boldsymbol{l}} := \frac{1}{i} \sum_{\substack{l'_j : |l_j - l'_j| = 1 \\ \text{for } j = \nu-1,\cdots,d-1}} \left\{ d_{l,l_{d-1},D} R_{\nu,d}\left(\boldsymbol{l}, \widetilde{\boldsymbol{l'}}_\nu\right) Y_{\widetilde{\boldsymbol{l'}}_\nu} - d_{l,l_{d-1}+1,D} R_{\nu,d}\left(\boldsymbol{l}, \widehat{\boldsymbol{l'}}_\nu\right) Y_{\widehat{\boldsymbol{l'}}_\nu} \right\}, \tag{2.18}$$

where

$$\widetilde{\boldsymbol{l'}}_\nu := \left(l, l_{d-1} - 1, l'_{d-2}, \cdots, l'_{\nu-1}, l_{\nu-2}, \cdots, l_1\right),$$
$$\widehat{\boldsymbol{l'}}_\nu := \left(l, l_{d-1} + 1, l'_{d-2}, \cdots, l'_{\nu-1}, l_{\nu-2}, \cdots, l_1\right),$$

for $\nu \in \{1, \cdots, d-2\}$, $l_0 \equiv l_1$ *and*

$$\widetilde{\boldsymbol{l'}}_{d-1} := (l, l_{d-1}-1, l_{d-2}, \cdots, l_1), \quad \widehat{\boldsymbol{l'}}_{d-1} := (l, l_{d-1}+1, l_{d-2}, \cdots, l_1).$$

Furthermore,

$$L_{D,j} := -L_{j,D} \quad , \quad L_{\pm,\nu} := \frac{L_{2,\nu} \mp iL_{1,\nu}}{\sqrt{2}} \quad \forall \nu \geq 3$$

and the action of $L_{h,\widetilde{D}}$ *on a* D*-dimensional spherical harmonic, when* $h < \widetilde{D} < D$, *is defined as the same of* $L_{h,\widetilde{D}}$ *on a* $\widetilde{D}$*-spherical harmonic in* $\mathbb{R}^{\widetilde{D}}$*; then it, when acts in* $\mathbb{R}^D$*, does not 'affect' the indices* $l, l_{d-1}, \cdots l_{\widetilde{D}-1}$.

Summarizing,

- In section 7.0.6 the action in $\mathbb{R}^D$ of the coordinate operators $t_\nu := \frac{x_\nu}{r}$ on the D-dimensional spherical harmonics $Y_{\boldsymbol{l}}$ is calculated, this action essentially defines the aforementioned $R_{\nu,D}$ coefficients;

- This implies that one can easily derive the action in $\mathbb{R}^{D-1}$ of coordinate operators t_h on a generic $(D-1)$-dimensional spherical harmonic $Y_{l_{d-1},\cdots,l_1}$, which consequently uses the $R_{h,d}$ coefficients;

- So, in definition 2.3.1 the action of $L_{\nu,D}$ on $Y_{\boldsymbol{l}}$ is the same, up to the $\frac{d_{l,l_{d-1},D}}{i}$ and $-\frac{d_{l,l_{d-1}+1,D}}{i}$ coefficients, of t_ν on $Y_{l_{d-1},\cdots,l_1}$; this is also in agreement with the Wigner-Eckart theorem, because

$$\langle Y_{\boldsymbol{l'}}, L_{\nu,D} Y_{\boldsymbol{l}} \rangle = \begin{cases} \frac{1}{i} d_{l,l_{d-1},D} R_{\nu,d} \left(\boldsymbol{l}, \widetilde{\boldsymbol{l'}}_\nu\right) & \text{if } l'_{d-1} = l_{d-1} - 1, \\ -\frac{1}{i} d_{l,l_{d-1}+1,D} R_{\nu,d} \left(\boldsymbol{l}, \widehat{\boldsymbol{l'}}_\nu\right) & \text{if } l'_{d-1} = l_{d-1} + 1, \\ 0 \quad \text{otherwise}, \end{cases}$$

  where the first factor depends only on the index l_{d-1}, which identifies the $SO(d)$ irrep, while the second one is a Clebsch-Gordan coefficient.

In sections 7.0.7 and 7.0.8 the following relations are explicitly checked for the reader's convenience:

$$\begin{aligned} \boldsymbol{L}^2 Y_{\boldsymbol{l}} &= \sum_{1 \leq h < j \leq D} L_{h,j}^2 Y_{\boldsymbol{l}} = l\,(l+D-2)\,Y_{\boldsymbol{l}}, \\ [L_{h,j}, L_{p,s}] &= i\left(\delta_{h,p} L_{j,s} + \delta_{j,s} L_{h,p} - \delta_{h,s} L_{j,p} - \delta_{j,p} L_{h,s}\right). \end{aligned} \tag{2.19}$$

2.3.2 The action of 'projected' operators on $\mathcal{H}_{\Lambda,D}$

The Hilbert space of admitted states $\mathcal{H}_{\Lambda,D}$, constructed in section 2.2, is spanned by all the states $\boldsymbol{\psi}_{\boldsymbol{l},D}$ fulfilling $l \leq \Lambda$. In the following lines we do a complete study of the action of the 'projected' angular momentum operators $\overline{L}_{h,j}$ and of the 'projected' coordinate operators $\overline{x}_h$ on the pure quantum states. The definition 2.3.1 implies $\overline{L}_{h,j}\boldsymbol{\psi}_{\boldsymbol{l},D} = L_{h,j}\boldsymbol{\psi}_{\boldsymbol{l},D}$, which is a consequence of the invariance of H (and therefore $P_{\overline{E},D}$) with respect to rotations (i.e. they commute with every $L_{h,j}$); from this and the fact that the action of every $L_{h,j}$ does not 'affect' the index l it follows that the action of $L_{h,j}$ on a $\boldsymbol{\psi}_{\boldsymbol{l},D}$ essentially coincides with the one of $Y_{\boldsymbol{l}}$. Then

$$L_{\nu,D}\boldsymbol{\psi}_{\boldsymbol{l},D} := \frac{1}{i} \sum_{\substack{l'_j:|l_j-l'_j|=1 \\ \text{for } j=\nu-1,\cdots,d-1}} \left\{ d_{l,l_{d-1},D} R_{\nu,d}\left(\boldsymbol{l},\widetilde{\boldsymbol{l}'}_\nu\right)\boldsymbol{\psi}_{\widetilde{\boldsymbol{l}'}_\nu,D} - d_{l,l_{d-1}+1,D} R_{\nu,d}\left(\boldsymbol{l},\widehat{\boldsymbol{l}'}_\nu\right)\boldsymbol{\psi}_{\widehat{\boldsymbol{l}'}_\nu,D}\right\};$$

$$L_{D,j}\boldsymbol{\psi}_{\boldsymbol{l},D} := -L_{j,D}\boldsymbol{\psi}_{\boldsymbol{l},D} \quad , \quad L_{\pm,\nu}\boldsymbol{\psi}_{\boldsymbol{l},D} := \left(L_{2,\nu} \mp iL_{1,\nu}\right)\boldsymbol{\psi}_{\boldsymbol{l},D} \quad \forall \nu \geq 3, \tag{2.20}$$

and the action of $L_{h,\widetilde{D}}$ on a $\boldsymbol{\psi}_{\boldsymbol{l},D}$, when $\widetilde{D} < D$, is essentially the same of $L_{h,\widetilde{D}}$ on a $\widetilde{D}$-spherical harmonic in $\mathbb{R}^{\widetilde{D}}$, as for (2.18) and (2.20).

On the other hand, the action of $\overline{x}_h$ on a state $\boldsymbol{\psi}_{\boldsymbol{l},D}$ can be obtained from the one of the multiplication operator $t_h\cdot$ on a D-dimensional spherical harmonic $Y_{\boldsymbol{l}}$ (see section 7.0.6), while sometimes it is useful to consider the operators

$$\overline{x}_\pm := \overline{x}_1 \pm i\overline{x}_2.$$

It is easy to see that the action of projected coordinate operators 'affect' the index l, for this reason further calculations are needed, because in this case the integral

$$\int_0^{+\infty} r f_{l,D}(r) f_{l',D}(r)dr$$

is not trivial, unlike what happens for the action of $L_{h,j}$.

According to this,

$$\overline{x}_h\psi_{\boldsymbol{l},D} = \sum_{\substack{|l_j-l'_j|=1 \\ j\in\{h-1,\cdots,d-1\}}} \left[c_{l,D}R_{h,D}\left(\boldsymbol{l},\widetilde{\widetilde{\boldsymbol{l}'}}_h\right)\boldsymbol{\psi}_{\widetilde{\widetilde{\boldsymbol{l}'}}_h,D} + c_{l+1,D}R_{h,D}\left(\boldsymbol{l},\widehat{\widehat{\boldsymbol{l}'}}_h\right)\boldsymbol{\psi}_{\widehat{\widehat{\boldsymbol{l}'}}_h,D}\right], \tag{2.21}$$

where

$$\widetilde{\widetilde{\boldsymbol{l}'}}_h := (l-1, l'_{d-1}, \cdots, l'_{h-1}, l_{h-2}\cdots, l_1), \;\; \widehat{\widehat{\boldsymbol{l}'}}_h := (l+1, l'_{d-1}, \cdots, l'_{h-1}, l_{h-2}\cdots, l_1),$$
$$c_{l,D} := \int_0^{+\infty} r f_{l,D}(r) f_{l-1,D}(r)dr, \quad c_{l+1,D} := \int_0^{+\infty} r f_{l,D}(r) f_{l+1,D}(r)dr,$$
$$c_{-\Lambda,2} = c_{\Lambda+1,2} := 0 \quad \text{and} \quad c_{0,D} = c_{\Lambda+1,D} := 0 \quad \forall D \geq 3; \tag{2.22}$$

the explicit values of $c_{l,D}$ are calculated in section 7.0.9 and

$$c_{l,D} \overset{(7.76)}{=} \sqrt{1 + \frac{[b(l,D) + b(l-1,D)]}{2k_D}} \quad \text{up to terms} \quad O\left(\frac{1}{(k_D)^{\frac{3}{2}}}\right). \tag{2.23}$$

2.3.3 The commutation relations and the action of $\boldsymbol{x}^2$

The calculations of section 7.0.8.1 can be used to determine the action of $[\overline{x}_h, \overline{x}_j]$ on $\mathcal{H}_{\Lambda,D}$, this because the action of $\overline{x}_h$ on $\boldsymbol{\psi}_{\boldsymbol{l},D}$ is essentially the same of $L_{h,D+1}$ on $Y_{l_D,\boldsymbol{l}}$; the only difference is the replacement of $-\frac{1}{i}d_{l_D,l+1,D+1}$ with $c_{l+1,D}$ and $\frac{1}{i}d_{l_D,l,D+1}$ with $c_{l,D}$, respectively. These arguments and (7.77) are sufficient to prove that (see section 7.0.10.1 for explicit calculations)

$$[\overline{x}_h, \overline{x}_j] = \left[-\frac{I}{k_D} + \left(\frac{1}{k_D} + \frac{(c_{\Lambda,D})^2}{2\Lambda + D - 2}\right)\widehat{P}_{\Lambda,D}\right]\overline{L}_{h,j}, \tag{2.24}$$

where $\widehat{P}_{\Lambda,D}$ is the projector on the $\Lambda(\Lambda + D - 2)$-eigenspace of $\boldsymbol{L}^2$.

On the other hand, it is obvious that $\widehat{P}_{\Lambda,D} := \widehat{P}_{\overline{E},D}$ commutes with $L_{h,j}$, for all $1 \leq h < j \leq D$; this and

$$\left[L_{h,s}, x^j\right] \overset{(5)}{=} \frac{1}{i}\left(\delta^s_j x^h - \delta^h_j x^s\right) \tag{2.25}$$

imply

$$\begin{aligned}
\left[\overline{L}_{h,s}, \overline{x}_j\right] &= \widehat{P}_{\Lambda,D} L_{h,s} \widehat{P}_{\Lambda,D} \widehat{P}_{\Lambda,D} x_j \widehat{P}_{\Lambda,D} - \widehat{P}_{\Lambda,D} x_h \widehat{P}_{\Lambda,D} \widehat{P}_{\Lambda,D} L_{h,s} \widehat{P}_{\Lambda,D} \\
&= \widehat{P}_{\Lambda,D} L_{h,s} x_j \widehat{P}_{\Lambda,D} - \widehat{P}_{\Lambda,D} x_j L_{h,s} \widehat{P}_{\Lambda,D} \\
&= \widehat{P}_{\Lambda,D} \left[L_{h,s}, x_j\right] \widehat{P}_{\Lambda,D} \\
&= \frac{1}{i}\left(\delta^s_j \overline{x}_h - \delta^h_j \overline{x}_s\right).
\end{aligned}$$

Furthermore, if one defines $\boldsymbol{x}^2 := \sum_h \overline{x}_h \overline{x}_h$, then the calculations of section 7.0.7 can be used to prove that [see section 7.0.10.2 for the explicit calculations, while here the $b(l,D)$ coefficients are the ones defined in (2.5)]

$$\begin{aligned}
\boldsymbol{x}^2 \boldsymbol{\psi}_{\boldsymbol{l},D} = \Bigg\{ & 1 + \frac{b(l,D) + [b(l+1,D)]\frac{l+D-2}{2l+D-2} + [b(l-1,D)]\frac{l}{2l+D-2}}{2k_D(\Lambda)} \\
& - \left[\left(1 + \frac{b(\Lambda,D) + b(\Lambda+1,D)}{2k_D(\Lambda)}\right)\frac{\Lambda + D - 2}{2\Lambda + D - 2}\right]\widehat{P}_{\Lambda,D}\Bigg\}\boldsymbol{\psi}_{\boldsymbol{l},D}.
\end{aligned} \tag{2.26}$$

In addition (here $\widetilde{P}_{h,j}$ is the projector on the eigenspace of C_{D-h} corresponding to $l_{D-h} \equiv j$),

$$\prod_{l=0}^{\Lambda}\left[\boldsymbol{L}^2 - l(l+D-2)I\right] = 0 \quad , \quad \prod_{l_{d-1}=0}^{l}\left[C_{D-1} - l_{d-1}(l_{d-1}+D-3)I\right]\widetilde{P}_{1,l} = 0,$$

$$\cdots \quad , \quad \prod_{l_1=-l_2}^{l_2}\left[L_{1,2} - l_1 I\right]\widetilde{P}_{D-2,l_2} = 0, \quad (\overline{x}_\pm)^{2\Lambda+1} = 0, \text{ and } (L_{\nu,\pm})^{2\Lambda+1} = 0, \forall\nu \geq 3 \tag{2.27}$$

The relations (2.24)-(2.27) imply that the coordinate operators generate the whole algebra of observables $\mathcal{A}_{\Lambda,D}$, in fact every $L_{h,j}$ can be written in terms of $[\overline{x}_h, \overline{x}_j]$ and therefore every projector $\widetilde{P}_{h,j}$ can be written as a non-ordered polynomial in the $\overline{x}_p$.

2.4 Realization of $\mathcal{A}_{\Lambda,D}$ through $U\boldsymbol{so}(D+1)$

Let $\Lambda \in \mathbb{N}$, $\pi_{\Lambda,D+1}$ be the irreducible representation of $U\boldsymbol{so}(D+1)$ having $l_D \equiv \Lambda$ and $V_{\Lambda,D+1}$ be the corresponding representation space [see (2.14)]. First of all, in section 7.0.3.4 it is shown that $\dim\mathcal{H}_{\Lambda,D} = \dim V_{\Lambda,D+1}$, and if one identifies $\boldsymbol{\psi}_{\boldsymbol{l},D} \equiv Y_{\Lambda,\boldsymbol{l}} \in V_{\Lambda,D+1}$, then the operators on $\mathcal{H}_{\Lambda,D}$, in particular $\overline{L}_{h,j}$ and $\overline{x}_h$, are naturally realized in $\pi_{\Lambda,D+1}[U\boldsymbol{so}(D+1)]$.

In fact one has [here the $L_{h,j}$s are seen as basis elements of $\boldsymbol{so}(D+1)$]

$$\begin{gathered}\overline{L}_{h,j} = L_{h,j} \quad \text{if } h<j<D+1 \quad \text{and} \quad \overline{x}_h = p_D^*(\lambda)\, L_{h,D+1} p_D(\lambda), \\ \text{where} \quad \lambda := \frac{2-D+\sqrt{(D-2)^2+4\boldsymbol{L}^2}}{2}.\end{gathered} \tag{2.28}$$

It turns out that the function p_D has to fulfill

$$\begin{aligned} p_D^*(l+1)\, p_D(l) &= \frac{1}{i}\frac{c_{l+1,D}}{d_{\Lambda,l+1,D+1}} = \frac{1}{i}\frac{\sqrt{1+\frac{b(l,D)+b(l+1,D)}{4k_D(\Lambda)}}}{\sqrt{(\Lambda-l)(\Lambda+l+D-1)}}, \\ p_D^*(l-1)\, p_D(l) &= i\frac{c_{l,D}}{d_{\Lambda,l,D+1}} = i\frac{\sqrt{1+\frac{b(l,D)+b(l-1,D)}{4k_D(\Lambda)}}}{\sqrt{(\Lambda-l+1)(\Lambda+l+D-2)}}; \end{aligned} \tag{2.29}$$

it can be determined recursively, starting from $p_D(0) := 1$ and then using the last formulas.

This means that

Theorem 2.4.1 *Formulas (2.28), (2.29) and section define an $O(D)$-equivariant $*$-algebra isomorphism between the algebra $\mathcal{A}_\Lambda = End(\mathcal{H}_\Lambda)$ of observables (endomorphisms) on $\mathcal{H}_\Lambda$ and the $C_{D+1} = \Lambda\left[\Lambda + (D+1) - 2\right]$ irreducible representation*

$\pi_{\Lambda,D+1}$ *of* $U\boldsymbol{so}(D+1)$*:*

$$\mathcal{A}_\Lambda := End(\mathcal{H}_\Lambda) \simeq M_N(\mathbb{C}) \simeq \boldsymbol{\pi}_\Lambda[U\boldsymbol{so}(D+1)],$$
$$where \quad \dim \mathcal{H}_{\Lambda,D} \equiv N :\overset{(7.20)}{=} \binom{\Lambda+D-2}{\Lambda-1}\frac{2\Lambda+D-1}{\Lambda}. \tag{2.30}$$

As already recalled, the group of $*$-automorphisms of $M_N(\mathbb{C}) \simeq \mathcal{A}_\Lambda$ is inner and isomorphic to $SU(N)$, i.e. of the type

$$a \mapsto g\,a\,g^{-1}, \qquad a \in \mathcal{A}_\Lambda,$$

with g an unitary $N \times N$ matrix with unit determinant. A special role is played by the subgroup $SO(D+1)$ acting in the representation $\boldsymbol{\pi}_\Lambda$, namely $g = \boldsymbol{\pi}_\Lambda\left[e^{i\alpha}\right]$, where $\alpha \in \boldsymbol{so}(D+1)$. In particular, choosing $\alpha = \alpha_{h,j}L_{h,j}$ ($\alpha_{h,j} \in \mathbb{R}$ and $h < j \leq D$) the automorphism amounts to a $SO(D) \subset SO(D+1)$ transformation (a rotation in D-dimensional space). Parity $(L_{h,j}, L_{p,D+1}) \mapsto (L_{h,j}, -L_{p,D+1})$, is an $O(D) \subset SO(D+1)$ transformation with determinant -1 in the $L_{p,D+1}$ space, and therefore also in the $\bar{x}_p$ space. This shows that (2.28) is equivariant under $O(D)$, which plays the role of isometry group of this fuzzy sphere.

2.5 Convergence to $O(D)$-equivariant quantum mechanics on S^d

Here it is explained how this new fuzzy space converges to $O(D)$-equivariant quantum mechanics on the sphere S^d as $\Lambda \to \infty$.

The fuzzy analogs of the vector spaces $B(S^d), C(S^d)$ are defined as [see (7.88) for the explicit definition of $\widehat{Y}_{\boldsymbol{l}}$]

$$\mathcal{C}_{\Lambda,D} := span_{\mathbb{C}}\left\{\widehat{Y}_{\boldsymbol{l}} : 2\Lambda \geq l \equiv l_d \geq \cdots \geq l_2 \geq |l_1|, l_i \in \mathbb{Z}\forall i\right\} \subset \mathcal{A}_{\Lambda,D} \subset B[\mathcal{L}^2(S^d)], \tag{2.31}$$

and here the highest l is 2Λ because $\widehat{Y}_{2\Lambda,2\Lambda,\cdots,2\Lambda}$ is the 'highest' multiplying operator acting nontrivially on $\mathcal{H}_{\Lambda,D}$ (it does not annihilate $\boldsymbol{\psi}_{\Lambda,\Lambda,\cdots,-\Lambda,D}$).

So

$$\mathcal{C}_{\Lambda,D} = \bigoplus_{l=0}^{2\Lambda} V_{l,D} \tag{2.32}$$

is the decomposition of $\mathcal{C}_{\Lambda,D}$ into irreducible components under $O(D)$; furthermore, $V_{l,D}$ is trace-free for all $l > 0$, i.e. its projection on the single component $V_{0,D}$ is zero and it is easy to see that (2.32) becomes the decomposition of $B(S^d), C(S^d)$ in the limit $\Lambda \to \infty$.

In addition, the fuzzy analog of $f \in B(S^d)$ is

$$\hat{f}_\Lambda := \sum_{l=0}^{2\Lambda} \sum_{\substack{l_{d-1}\le l \\ l_{h-1}\le l_h \text{ for } h=d-1,\cdots,3 \\ |l_1|\le l_2}} f_{\boldsymbol{l}} \widehat{Y}_{\boldsymbol{l}} \in \mathcal{A}_{\Lambda,D} \subset B[\mathcal{L}^2(S^2)]; \tag{2.33}$$

while the $\boldsymbol{\psi}_{\boldsymbol{l},D} \in \mathcal{H}_{\Lambda,D}$ are the fuzzy analogs of the spherical harmonics $Y_{\boldsymbol{l}}$ considered just as elements of an orthonormal basis of the Hilbert space $\mathcal{L}^2(S^d)$; for this reason, consider the $O(D)$-covariant embedding $\mathcal{I} : \mathcal{H}_{\Lambda,D} \hookrightarrow \mathcal{L}^2(S^d)$ defined by

$$\mathcal{I}\left(\sum_{l=0}^{\Lambda} \sum_{\substack{l_{d-1}\le l \\ l_{j-1}\le l_j \text{ for } j=d-1,\cdots,3 \\ |l_1|\le l_2}} \phi_{\boldsymbol{l}} \boldsymbol{\psi}_{\boldsymbol{l},D} \right) = \sum_{l=0}^{\Lambda} \sum_{\substack{l_{d-1}\le l \\ l_{j-1}\le l_j \text{ for } j=d-1,\cdots,3 \\ |l_1|\le l_2}} \phi_{\boldsymbol{l}} Y_{\boldsymbol{l}},$$

and below the symbol $\mathcal{I}$ is dropped and then simply identified $\boldsymbol{\psi}_{\boldsymbol{l},D} \equiv Y_{\boldsymbol{l}}$.

The decomposition of $\mathcal{H}_{\Lambda,D}$ into irreducible components under $O(D)$ reads

$$\mathcal{H}_{\Lambda,D} = \bigoplus_{l=0}^{\Lambda} V_l, \quad V_l := \left\{ \sum_{\substack{l_{d-1}\le l \\ l_{j-1}\le l_j \text{ for } j=d-1,\cdots,3 \\ |l_1|\le l_2}} \phi_{\boldsymbol{l}} \boldsymbol{\psi}_{\boldsymbol{l},D} : \phi_{\boldsymbol{l}} \in \mathbb{C} \right\}, \tag{2.34}$$

and $(2.34)_1$ becomes the decomposition of $\mathcal{L}^2(S^d)$ in the limit $\Lambda \to \infty$.

For all $\phi \in \mathcal{L}^2(S^d)$ let

$$\phi_\Lambda := \sum_{l=0}^{\Lambda} \sum_{\substack{l_{d-1}\le l \\ l_{j-1}\le l_j \text{ for } j=d-1,\cdots,3 \\ |l_1|\le l_2}} \phi_{\boldsymbol{l}} \boldsymbol{\psi}_{\boldsymbol{l},D},$$

where $\phi_{\boldsymbol{l}}$ are the coefficients of the decomposition of ϕ in the orthonormal basis of spherical harmonics; clearly $\phi_\Lambda \to \phi$ in the $\mathcal{L}^2(S^d)$-norm $\|\ \|$, and in this sense $\mathcal{H}_{\Lambda,D}$ invades $\mathcal{L}^2(S^d)$ as $\Lambda \to \infty$.

Let $B\left[\mathcal{L}^2\left(S^d\right)\right]$ be the algebra of bounded operators on $\mathcal{L}^2\left(S^d\right)$, the embedding $\mathcal{I}$ induces the one $\mathcal{J} : \mathcal{A}_{\Lambda,D} \hookrightarrow B\left[\mathcal{L}^2\left(S^d\right)\right]$ and by construction $\mathcal{A}_{\Lambda,D}$ annihilates $\mathcal{H}_{\Lambda,D}^{\perp}$; the operators $L_{h,j}, \overline{L}_{h,j}$ coincide on $\mathcal{H}_{\Lambda,D}$, while one can easily check that $\overline{L}_{h,j} \to L_{h,j}$ strongly as $\Lambda \to \infty$ on the domain $D\left(L_{h,j}\right) \subset \mathcal{L}^2\left(S^d\right)$ [1] and, similarly, $f(\overline{L}_{h,j}) \to f(L_{h,j})$ strongly on $D[f(L_{h,j})]$ for all measurable function $f(s)$.

[1]The strict inclusion it follows from the fact that $L_{h,j}$ is unbounded; for example $\phi \in D(L_{1,2})$ only if $\sum_{l=0}^{+\infty} \sum_{l_{d-1}\le l,\cdots} |l_1|^2 |\phi_{\boldsymbol{l}}|^2 < +\infty$.

Bounded (in particular, continuous) functions f on the sphere S^d, acting as multiplication operators $f\cdot : \phi \in \mathcal{L}^2(S^d) \mapsto f\phi \in \mathcal{L}^2(S^d)$, make up a subalgebra $B(S^d)$ [resp. $C(S^d)$] of $B\left[\mathcal{L}^2(S^d)\right]$. An element of $B(S^d)$ is actually an equivalence class $[f]$ of bounded functions differing from f only on a set of zero measure, because this ensures that for any $f_1, f_2 \in [f]$, and $\phi \in \mathcal{L}^2(S^d)$, $f_1\phi$ and $f_2\phi$ differ only on a set of zero measure, and therefore are two equivalent representatives of the same element of $\mathcal{L}^2(S^d)$. Since f belongs also to $\mathcal{L}^2(S^d)$, then

$$f_N(\theta_d, \cdots, \theta_1) := \sum_{l=0}^{N} \sum_{\substack{l_{d-1} \leq l \\ l_{j-1} \leq l_j \text{ for } j=d-1,\cdots,3 \\ |l_1| \leq l_2}} f_{\boldsymbol{l}} Y_{\boldsymbol{l}}(\theta_d, \cdots, \theta_1)$$

converges to $f(\theta_d, \cdots, \theta_1)$ in the $\mathcal{L}^2(S^d)$ norm as $N \to \infty$.

In section 7.0.12 it is shown that every projected coordinate operator $\overline{x}_h$ converges strongly to the corresponding t_h as $\Lambda \to \infty$ if

$$k_D(\Lambda) \geq \Lambda \left[\dim \mathcal{H}_{\Lambda,D}\right]^2 b(\Lambda, D).$$

Again, since for all $\Lambda > 0$ the operator $\overline{x}_h$ annihilates $\mathcal{H}_{\Lambda,D}^{\perp}$, $\overline{x}_h$ *does not* converge to t_h *in operator norm.* It is possible to prove also this more general result:

Theorem 2.5.1 *Choosing*

$$k_D(\Lambda) \geq \Lambda^2 [\dim \mathcal{H}_{2\Lambda,D}]^3 \left[(2\Lambda)!\right]^D 2^{\Lambda D} \left[(2\Lambda+1)!!\right]^{2D} b(\Lambda, D) \sqrt{\dim \mathcal{H}_{\Lambda,D}}, \quad (2.35)$$

then for all $f, g \in B(S^d)$ the following strong limits as $\Lambda \to \infty$ hold: $\hat{f}_\Lambda \to f\cdot$, $\widehat{(fg)}_\Lambda \to fg\cdot$ and $\hat{f}_\Lambda \hat{g}_\Lambda \to fg\cdot$.

In other words, the product in $\mathcal{A}_{\Lambda,D}$ between the approximations $\widehat{f}_\Lambda$ and $\widehat{g}_\Lambda$ goes to the product in $B\left[\mathcal{L}^2(S^d)\right]$ between $f\cdot$ and $g\cdot$ [although $\left(\widehat{fg}\right)_\Lambda \neq \widehat{f}_\Lambda \widehat{g}_\Lambda$].

Chapter 3

The cases $2 \leq D \leq 5$

3.1 S^1_Λ

When $D = 2$ the choices $\overline{E} = \overline{E}(\Lambda) := \Lambda^2$ and $k_2 = k_2(\Lambda) \geq \Lambda^4$ imply that the Hilbert space of admitted states $\mathcal{H}_{\Lambda,2}$ is generated by all the functions (see sections 2.1.1 and 2.1.2)

$$\boldsymbol{\psi}_{l,2} = \boldsymbol{\psi}_{l,2}(r,\theta) := f_{l,1}(r)Y_l(\theta) = f_{l,1}(r)\frac{e^{il\theta}}{\sqrt{2\pi}}, \quad |l| \leq \Lambda$$

hence

$$\dim \mathcal{H}_{\Lambda,2} \overset{(7.20)}{=} \binom{\Lambda}{\Lambda-1}\frac{2\Lambda+1}{\Lambda} = 2\Lambda + 1.$$

The only one angular momentum component is $L := L_{1,2}$ and it acts as follows (see definition 2.3.1 in section 2.3.1):

$$L\boldsymbol{\psi}_l = l\boldsymbol{\psi}_l;$$

with

$$\boldsymbol{L}^2\boldsymbol{\psi}_l = l^2\psi_l.$$

The coordinate operators are $\overline{x}_1, \overline{x}_2, \overline{x}_\pm := \overline{x}_1 \pm i\overline{x}_2$, and they act on $\mathcal{H}_{\Lambda,2}$ as follows (see section 2.3.2):

$$\overline{x}_1\boldsymbol{\psi}_l = \frac{c_{l+1,2}}{2}\boldsymbol{\psi}_{l+1} + \frac{c_{l,2}}{2}\boldsymbol{\psi}_{l-1} \quad , \quad \overline{x}_2\boldsymbol{\psi}_l = \frac{c_{l+1,2}}{2i}\boldsymbol{\psi}_{l+1} - \frac{c_{l,2}}{2i}\boldsymbol{\psi}_{l-1},$$

$$\overline{x}_+\boldsymbol{\psi}_l = c_{l+1,2}\boldsymbol{\psi}_{l+1} \quad , \quad \overline{x}_-\boldsymbol{\psi}_l = c_{l,2}\boldsymbol{\psi}_{l-1},$$

where

$$c_{l,2} \overset{(2.23)}{=} \begin{cases} \sqrt{1 + \frac{l^2 - l + \frac{3}{4}}{k_2}} & \text{if } -\Lambda + 1 \leq l \leq \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

They fulfill (see section 2.3.3)

$$[\overline{x}_1, \overline{x}_2] = \frac{1}{i}\left[-\frac{I}{k_2} + \left(\frac{1}{k_2} + \frac{(c_{-\Lambda+1,2})^2}{2\Lambda}\right)\widehat{P}_{-\Lambda,2} + \left(\frac{1}{k_2} - \frac{(c_{\Lambda,2})^2}{2\Lambda}\right)\widehat{P}_{\Lambda,2}\right]\overline{L}_{1,2},$$
$$[L, \overline{x}_2] = \frac{1}{i}\overline{x}_1, \quad [L, \overline{x}_1] = -\frac{1}{i}\overline{x}_2, \quad [L, \overline{x}_+] = \overline{x}_+, \quad [L, \overline{x}_-] = -\overline{x}_-, \tag{3.1}$$

$$\boldsymbol{x}^2 := \sum_{h=1}^{2} \overline{x}_h \overline{x}_h = \left\{1 + \frac{2\boldsymbol{L}^2+1}{2k_2(\Lambda)} - \left[\frac{1}{2}\left(1 + \frac{4\Lambda^2+4\Lambda+3}{4k_2(\Lambda)}\right)\right]\left(\widehat{P}_{\Lambda,2} + \widehat{P}_{-\Lambda,2}\right)\right\} \tag{3.2}$$

and

$$\prod_{l_1=-l_2}^{l_2}[L_{1,2} - l_1 I] = 0 \quad , \quad (\overline{x}_\pm)^{2\Lambda+1} = 0. \tag{3.3}$$

According to this, the algebra of observables is generated by the coordinate operators, in fact every projector can be written as a ordered polynomial in the $\overline{x}_\nu$.

Furthermore, the $SO(3)$-irrep $\pi_{\Lambda,3}$, the one characterized by $C_3 \equiv \Lambda(\Lambda+1)I$ with representation space

$$V_{\Lambda,3} := span\left\{Y_{\Lambda,l}(\theta_2, \theta_1) : \Lambda \geq |l|; \Lambda, l \in \mathbb{Z}\right\},$$

can be used to identify $\boldsymbol{\psi}_l \equiv Y_{\Lambda,l}$, and also the operators

$$L_{h,j} \equiv L_{h,j} \quad \text{for} \quad 1 \leq h < j \leq 2 \quad \text{and} \quad \overline{x}_s \equiv p_2(\lambda) L_{s,3} p_2(\lambda), \tag{3.4}$$

where

$$\lambda := \sqrt{\boldsymbol{L}^2},$$

while $p_2(\lambda)$ is an analytic function and the values $p_2(l)$, when $l \in \mathbb{N}_0$, can be obtained recursively from (2.29) starting from $p_2(0) := 1$.

Furthermore, in order to prove the convergence of S^1_Λ to ordinary quantum mechanics on S^1, it is convenient to identify $\boldsymbol{\psi}_l \equiv Y_l$ and then to consider their fuzzy counterparts $\widehat{Y}_l$ [see (7.88)], which can be used to approximate a generic $f \in B(S^1)$ or $f \in C(S^1)$; this is possible because the Y_l are an orthonormal basis of $\mathcal{L}^2(S^1)$, and also homogeneous polynomials in the $t_h := x_h/r$ variables. Then,

$$\widehat{f}_\Lambda := \sum_{l=-2\Lambda}^{2\Lambda} f_l \widehat{Y}_l, \quad \text{where} \quad f_l := \langle Y_l, f\rangle,$$

is an approximation of f because of the following two theorems (see section 2.5)

Theorem 3.1.1 *Every projected coordinate operator* $\overline{x}_h$ *converges strongly to the corresponding* t_h *as* $\Lambda \to \infty$ *if*

$$k_2(\Lambda) \geq \Lambda (2\Lambda + 1)^2 \left(\frac{4\Lambda^2 - 1}{4} \right).$$

Theorem 3.1.2 *Choosing* $k_2(\Lambda)$ *fulfilling (2.35) for* $D = 2$*, then for all* $f, g \in B(S^1)$ *the following strong limits as* $\Lambda \to \infty$ *hold:* $\hat{f}_\Lambda \to f\cdot$, $\widehat{(fg)}_\Lambda \to fg\cdot$ *and* $\hat{f}_\Lambda \hat{g}_\Lambda \to fg\cdot$.

3.2 S^2_Λ

When $D = 3$ the choices $\overline{E} = \overline{E}(\Lambda) := \Lambda(\Lambda + 1)$ and

$$k_3 = k_3(\Lambda) \geq [\Lambda(\Lambda + 1)]^2 \tag{3.5}$$

imply that the Hilbert space of admitted states $\mathcal{H}_{\Lambda,3}$ is generated by all the functions (see sections 2.1.1 and 2.1.2)

$$\boldsymbol{\psi}_{l,l_1,3} = \boldsymbol{\psi}_{l,l_1,4}(r, \theta_1, \theta_2) := f_{l,3}(r) Y_{l,l_1}(\theta_1, \theta_2), \quad l \leq \Lambda$$

hence

$$\dim \mathcal{H}_{\Lambda,3} \overset{(7.20)}{=} \binom{\Lambda + 1}{\Lambda - 1} \frac{2\Lambda + 2}{\Lambda} = (\Lambda + 1)^2 .$$

The angular momentum components are $L_{1,2} =: L_0 = L_3 = L_z$, $L_{1,3} =: -L_2 = -L_y$, $L_{2,3} =: L_1 = L_x$, $L_\pm := L_x \pm i L_y$ and they act as follows (see definition 2.3.1 in section 2.3.1):

$$\begin{aligned}
L_z \boldsymbol{\psi}_{l,l_1} &= l_1 \boldsymbol{\psi}_{l,l,l_1}, \\
L_y \boldsymbol{\psi}_{l,l_1} &= \frac{1}{i} \left[-\frac{d_{l,l_1,3}}{2} \boldsymbol{\psi}_{l,l_1-1} + \frac{d_{l,l_1+1,3}}{2} \boldsymbol{\psi}_{l,l_1+1} \right], \\
L_x \boldsymbol{\psi}_{l,l_1} &= \frac{1}{i} \left[-\frac{d_{l,l_1,3}}{2i} \boldsymbol{\psi}_{l,l_1-1} - \frac{d_{l,l_1+1,3}}{2i} \boldsymbol{\psi}_{l,l_1+1} \right], \\
L_+ \boldsymbol{\psi}_{l,l_1} &= d_{l,l_1+1,3} \boldsymbol{\psi}_{l,l_1+1}, \\
L_- \boldsymbol{\psi}_{l,l_1} &= d_{l,l_1,3} \boldsymbol{\psi}_{l,l_1-1},
\end{aligned} \tag{3.6}$$

where $d_{l,l_1,3} = \sqrt{(l - l_1 + 1)(l + l_1)}$.

They fulfill

$$[L_i, L_j] = i\varepsilon^{ijh} L_h, \quad \boldsymbol{L}^2 \boldsymbol{\psi}_{l,l_1} = l(l+1) \boldsymbol{\psi}_{l,l_1} \quad \text{and} \quad C_2 \psi_{l,l_1} = l_1^2 \psi_{l,l_1}.$$

The coordinate operators are $\overline{x}_1, \overline{x}_2, \overline{x}_3 =: \overline{x}_0$, $\overline{x}_\pm = \overline{x}_1 \pm i\overline{x}_2$ and they act on $\mathcal{H}_{\Lambda,3}$ as follows (see section 2.3.2):

$$
\begin{aligned}
\overline{x}_1 \boldsymbol{\psi}_{l,l_1} =& \left[\frac{c_{l,3} A_l^{-,l_1}}{2} \boldsymbol{\psi}_{l-1,l_1-1} + \frac{c_{l,3} A_l^{+,l_1}}{2} \boldsymbol{\psi}_{l-1,l_1+1} \right. \\
& \left. + \frac{c_{l+1,3} B_l^{-,l_1}}{2} \boldsymbol{\psi}_{l+1,l_1-1} + \frac{c_{l+1,3} B_l^{+,l_1}}{2} \boldsymbol{\psi}_{l+1,l_1+1} \right], \\
\overline{x}_2 \boldsymbol{\psi}_{l,l_1} =& \left[-\frac{c_{l,3} A_l^{-,l_1}}{2i} \boldsymbol{\psi}_{l-1,l_1-1} + \frac{c_{l,3} A_l^{+,l_1}}{2i} \boldsymbol{\psi}_{l-1,l_1+1} \right. \\
& \left. - \frac{c_{l+1,3} B_l^{-,l_1}}{2i} \boldsymbol{\psi}_{l+1,l_1-1} + \frac{c_{l+1,3} B_l^{+,l_1}}{2i} \boldsymbol{\psi}_{l+1,l_1+1} \right], \\
\overline{x}_3 \boldsymbol{\psi}_{l,l_1} =& c_{l,3} A_l^{0,l_1} \boldsymbol{\psi}_{l-1,l_1} + c_{l+1,3} B_l^{0,l_1} \boldsymbol{\psi}_{l+1,l_1}.
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
\overline{x}_+ \boldsymbol{\psi}_{l,l_1} &= c_{l,3} A_l^{+,l_1} \boldsymbol{\psi}_{l-1,l_1+1} + c_{l+1,3} B_l^{+,l_1} \boldsymbol{\psi}_{l+1,l_1+1}, \\
\overline{x}_- \boldsymbol{\psi}_{l,l_1} &= c_{l,3} A_l^{-,l_1} \boldsymbol{\psi}_{l-1,l_1-1} + c_{l+1,3} B_l^{-,l_1} \boldsymbol{\psi}_{l+1,l_1-1},
\end{aligned}
$$

where

$$
c_{l,3} \overset{(2.23)}{=} \begin{cases} \sqrt{1 + \frac{l^2}{k_3}} & \text{if } 1 \leq l \leq \Lambda, \\ 0 & \text{otherwise,} \end{cases}
$$

and, according to (7.32),

$$
\begin{aligned}
B_l^{+,l_1} &:= A(l,l_1,2) = \sqrt{\frac{(l+l_1+1)(l+l_1+2)}{(2l+1)(2l+3)}}, \\
A_l^{+,l_1} &:= B(l,l_1,2) = -\sqrt{\frac{(l-l_1-1)(l-l_1)}{(2l+1)(2l-1)}}, \\
B_l^{-,l_1} &:= C(l,l_1,2) = -\sqrt{\frac{(l-l_1+2)(l-l_1+1)}{(2l+1)(2l+3)}}, \\
A_l^{-,l_1} &:= D(l,l_1,2) = \sqrt{\frac{(l+l_1)(l+l_1-1)}{(2l+1)(2l-1)}}, \\
B_l^{0,l_1} &:= F(l,l_1,2) = \sqrt{\frac{(l+l_1+1)(l-l_1+1)}{(2l+1)(2l+3)}}, \\
A_l^{0,l_1} &:= G(l,l_1,2) = \sqrt{\frac{(l-l_1)(l+l_1)}{(2l+1)(2l-1)}}.
\end{aligned}
$$

They fulfill (see section 2.3.3)

$$[\overline{x}_h, \overline{x}_j] = \left[-\frac{I}{k_3} + \left(\frac{1}{k_3} + \frac{(c_{\Lambda,3})^2}{2\Lambda+1}\right)\widehat{P}_{\Lambda,3}\right]\overline{L}_{h,j}, \quad [L_{h,s}, \overline{x}_j] = \frac{1}{i}\left(\delta^s_j \overline{x}_h - \delta^h_j \overline{x}_s\right), \tag{3.8}$$

$$\boldsymbol{x}^2 := \sum_{h=1}^{3} \overline{x}_h \overline{x}_h = \left\{1 + \frac{\boldsymbol{L}^2+1}{k_3(\Lambda)} - \left[\left(1 + \frac{(\Lambda+1)^2}{k_3(\Lambda)}\right)\frac{\Lambda+1}{2\Lambda+1}\right]\widehat{P}_{\Lambda,3}\right\} \tag{3.9}$$

and

$$\begin{gathered}\prod_{l=0}^{\Lambda}\left[\boldsymbol{L}^2 - l(l+1)I\right] = 0 \quad , \quad \prod_{l_1=-l_2}^{l_2}\left[L_{1,2} - l_1 I\right]\widetilde{P}_{1,l} = 0, \\ (\overline{x}_\pm)^{2\Lambda+1} = 0, \text{ and } (L_\pm)^{2\Lambda+1} = 0,\end{gathered} \tag{3.10}$$

where $\widetilde{P}_{h,j}$ is the projector on the eigenspace of C_{3-h} corresponding to $l_{3-h} \equiv j$.

According to this, the algebra of observables is generated by the coordinate operators, in fact every projector can be written as a ordered polynomial in the $\overline{x}_\nu$.

Furthermore, the $SO(4)$-irrep $\pi_{\Lambda,4}$, the one characterized by $C_4 \equiv \Lambda(\Lambda+2)I$ with representation space

$$V_{\Lambda,4} := span\left\{Y_{\Lambda,l,l_1}(\theta_3,\theta_2,\theta_1) : \Lambda \geq l \geq |l_1|, l_i \in \mathbb{Z}\forall i\right\},$$

can be used to identify $\boldsymbol{\psi}_{l,l_1} \equiv Y_{\Lambda,l,l_1}$, and also the operators

$$L_{h,j} \equiv L_{h,j} \quad \text{for} \quad 1 \leq h < j \leq 3 \quad \text{and} \quad \overline{x}_s \equiv p_3(\lambda) L_{s,4} p_3(\lambda), \tag{3.11}$$

where

$$\lambda := \frac{-1+\sqrt{1+4\boldsymbol{L}^2}}{2},$$

while $p_3(\lambda)$ is an analytic function and the values $p_3(l)$, when $l \in \mathbb{N}_0$, can be obtained recursively from (2.29) starting from $p_3(0) := 1$.

Furthermore, in order to prove the convergence of S^2_Λ to ordinary quantum mechanics on S^2, it is convenient to identify $\boldsymbol{\psi}_{l,l_1} \equiv Y_{l,l_1}$ and then to consider their fuzzy counterparts $\widehat{Y}_{l,l_1}$ [see (7.88)], which can be used to approximate a generic $f \in B(S^2)$ or $f \in C(S^2)$; this is possible because the Y_{l,l_1} are an orthonormal basis of $\mathcal{L}^2(S^2)$, and also homogeneous polynomials in the $t_h := x_h/r$ variables. Then,

$$\widehat{f}_\Lambda := \sum_{l=0}^{2\Lambda}\sum_{l_1=-l}^{l} f_{l,l_1}\widehat{Y}_{l,l_1}, \quad \text{where} \quad f_{l,l_1} := \langle Y_{l,l_1}, f\rangle,$$

is an approximation of f because of the following two theorems (see section 2.5)

Theorem 3.2.1 *Every projected coordinate operator $\overline{x}_h$ converges strongly to the corresponding t_h as $\Lambda \to \infty$ if*

$$k_3(\Lambda) \geq \Lambda^2 (\Lambda+1)^5 .$$

Theorem 3.2.2 *Choosing $k_3(\Lambda)$ fulfilling (2.35) for $D=3$, then for all $f,g \in B(S^2)$ the following strong limits as $\Lambda \to \infty$ hold: $\hat{f}_\Lambda \to f\cdot, \widehat{(fg)}_\Lambda \to fg\cdot$ and $\hat{f}_\Lambda \hat{g}_\Lambda \to fg\cdot$.*

3.3 S^3_Λ

When $D=4$ the choices $\overline{E} = \overline{E}(\Lambda) := \Lambda(\Lambda+2)$ and $k_4 = k_4(\Lambda) \geq [\Lambda(\Lambda+2)]^2$ imply that the Hilbert space of admitted states $\mathcal{H}_{\Lambda,4}$ is generated by all the functions (see sections 2.1.1 and 2.1.2)

$$\boldsymbol{\psi}_{l,l_2,l_1,4} = \boldsymbol{\psi}_{l,l_2,l_1,4}(r,\theta_1,\theta_2,\theta_3) := f_{l,4}(r) Y_{l,l_2,l_1}(\theta_1,\theta_2,\theta_3), \quad l \leq \Lambda$$

hence

$$\dim \mathcal{H}_{\Lambda,4} \stackrel{(7.20)}{=} \binom{\Lambda+2}{\Lambda-1} \frac{2\Lambda+3}{\Lambda} = \frac{2\Lambda^3+9\Lambda^2+13\Lambda+6}{6} = \frac{1}{3}(\Lambda+1)(\Lambda+2)\left(\Lambda+\frac{3}{2}\right).$$

The angular momentum components are $\{L_{h,j} : 1 \leq h < j \leq 4\}$, $L_{\pm,4} := L_{2,4} \mp i L_{1,4}$ and they act as follows (see definition 2.3.1 in section 2.3.1):

$$\begin{aligned}
L_{1,2}\boldsymbol{\psi}_{l,l_2,l_1} &= l_1 \boldsymbol{\psi}_{l,l_2,l_1}, \\
L_{1,3}\boldsymbol{\psi}_{l,l_2,l_1} &= \frac{1}{i}\left[\frac{d_{l_2,l_1,3}}{2}\boldsymbol{\psi}_{l,l_2,l_1-1} - \frac{d_{l_2,l_1+1,3}}{2}\boldsymbol{\psi}_{l,l_2,l_1+1}\right], \\
L_{2,3}\boldsymbol{\psi}_{l,l_2,l_1} &= \frac{1}{i}\left[-\frac{d_{l_2,l_1,3}}{2i}\boldsymbol{\psi}_{l,l_2,l_1-1} - \frac{d_{l_2,l_1+1,3}}{2i}\boldsymbol{\psi}_{l,l_2,l_1+1}\right], \\
L_{1,4}\boldsymbol{\psi}_{l,l_2,l_1} &= \frac{1}{i}\left[\frac{d_{l,l_2,4}B(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_2-1,l_1+1} + \frac{d_{l,l_2,4}D(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_2-1,l_1-1}\right. \\
&\qquad \left. -\frac{d_{l,l_2+1,4}A(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_2+1,l_1+1} - \frac{d_{l,l_2+1,4}C(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_2+1,l_1-1}\right], \\
L_{2,4}\boldsymbol{\psi}_{l,l_2,l_1} &= \frac{1}{i}\left[\frac{d_{l,l_2,4}B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_2-1,l_1+1} - \frac{d_{l,l_2,4}D(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_2-1,l_1-1}\right. \\
&\qquad \left. -\frac{d_{l,l_2+1,4}A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_2+1,l_1+1} + \frac{d_{l,l_2+1,4}C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_2+1,l_1-1}\right], \\
L_{3,4}\boldsymbol{\psi}_{l,l_2,l_1} &= \frac{1}{i}\left[d_{l,l_2,4}G(l_2,l_1,2)\boldsymbol{\psi}_{l,l_2-1,l_1} - d_{l,l_2+1,4}F(l_2,l_1,2)\boldsymbol{\psi}_{l,l_2+1,l_1}\right], \\
L_{+,4}\boldsymbol{\psi}_{l,l_2,l_1} &= -\,d_{l,l_2,4}B(l_2,l_1,2)\boldsymbol{\psi}_{l,l_2-1,l_1+1} + d_{l,l_2+1,4}A(l_2,l_1,2)\boldsymbol{\psi}_{l,l_2+1,l_1+1}, \\
L_{-,4}\boldsymbol{\psi}_{l,l_2,l_1} &= d_{l,l_2,4}D(l_2,l_1,2)\boldsymbol{\psi}_{l,l_2-1,l_1-1} - d_{l,l_2+1,4}C(l_2,l_1,2)\boldsymbol{\psi}_{l,l_2+1,l_1-1},
\end{aligned}$$

where $d_{l,l_2,4}=\sqrt{(l-l_2+1)(l+l_2+1)}$.

They fulfill

$$[L_{h,j},L_{p,s}]\,\boldsymbol{\psi}_{l,l_2,l_1}=i\left(\delta_{h,p}L_{j,s}+\delta_{j,s}L_{h,p}-\delta_{h,s}L_{j,p}-\delta_{j,p}L_{h,s}\right)\boldsymbol{\psi}_{l,l_2,l_1},$$

$$\boldsymbol{L}^2\boldsymbol{\psi}_{l,l_2,l_1}=l(l+2)\boldsymbol{\psi}_{l,l_2,l_1},\quad C_3\psi_{l,l_2,l_1}=l_2(l_2+1)\boldsymbol{\psi}_{l,l_2,l_1}\quad\text{and}\quad C_2\psi_{l,l_2,l_1}=l_1^2\psi_{l,l_2,l_1}.$$

The coordinate operators are $\overline{x}_1,\overline{x}_2,\overline{x}_3,\overline{x}_4$ $\overline{x}_\pm:=\overline{x}_1\pm i\overline{x}_2$, and they act on $\mathcal{H}_{\Lambda,4}$ as follows (see section 2.3.2):

$$\begin{aligned}
\overline{x}_1\boldsymbol{\psi}_{l,l_2,l_1}=&\Big[\frac{c_{l,4}D(l,l_2,3)D(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l-1,l_2-1,l_1-1}+\frac{c_{l,4}D(l,l_2,3)B(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l-1,l_2-1,l_1+1}\\
&+\frac{c_{l,4}B(l,l_2,3)C(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l-1,l_2+1,l_1-1}+\frac{c_{l,4}B(l,l_2,3)A(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l-1,l_2+1,l_1+1}\\
&+\frac{c_{l+1,4}C(l,l_2,3)D(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l+1,l_2-1,l_1-1}+\frac{c_{l+1,4}C(l,l_2,3)B(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l+1,l_2-1,l_1+1}\\
&+\frac{c_{l+1,4}A(l,l_2,3)C(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l+1,l_2+1,l_1-1}+\frac{c_{l+1,4}A(l,l_2,3)A(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l+1,l_2+1,l_1+1}\Big],\\
\overline{x}_2\boldsymbol{\psi}_{l,l_2,l_1}=&\Big[\frac{c_{l,4}D(l,l_2,3)D(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_2-1,l_1-1}-\frac{c_{l,4}D(l,l_2,3)B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_2-1,l_1+1}\\
&+\frac{c_{l,4}B(l,l_2,3)C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_2+1,l_1-1}-\frac{c_{l,4}B(l,l_2,3)A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_2+1,l_1+1}\\
&+\frac{c_{l+1,4}C(l,l_2,3)D(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_2-1,l_1-1}-\frac{c_{l+1,4}C(l,l_2,3)B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_2-1,l_1+1}\\
&+\frac{c_{l+1,4}A(l,l_2,3)C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_2+1,l_1-1}-\frac{c_{l+1,4}A(l,l_2,3)A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_2+1,l_1+1}\Big],\\
\overline{x}_3\boldsymbol{\psi}_{l,l_2,l_1}=&\left[c_{l,4}D(l,l_2,3)G(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_2-1,l_1}+c_{l,4}B(l,l_2,3)F(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_2+1,l_1}\right.\\
&\left.+c_{l+1,4}C(l,l_2,3)G(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_2-1,l_1}+c_{l+1,4}A(l,l_2,3)F(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_2+1,l_1}\right],\\
\overline{x}_4\boldsymbol{\psi}_{l,l_2,l_1}=&c_{l,4}G(l,l_2,3)\boldsymbol{\psi}_{l-1,l_2,l_1}+c_{l+1,4}F(l,l_2,3)\boldsymbol{\psi}_{l+1,l_2,l_1},\\
\overline{x}_+\boldsymbol{\psi}_{l,l_2,l_1}=&\left[c_{l,4}D(l,l_2,3)B(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_2-1,l_1+1}+c_{l,4}B(l,l_2,3)A(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_2+1,l_1+1}\right.\\
&\left.+c_{l+1,4}C(l,l_2,3)B(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_2-1,l_1+1}+c_{l+1,4}A(l,l_2,3)A(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_2+1,l_1+1}\right],\\
\overline{x}_-\boldsymbol{\psi}_{l,l_2,l_1}=&\left[c_{l,4}D(l,l_2,3)D(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_2-1,l_1-1}+c_{l,4}B(l,l_2,3)C(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_2+1,l_1-1}\right.\\
&\left.+c_{l+1,4}C(l,l_2,3)D(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_2-1,l_1-1}+c_{l+1,4}A(l,l_2,3)C(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_2+1,l_1-1}\right],
\end{aligned}$$

where

$$c_{l,4}\overset{(2.23)}{=}\begin{cases}\sqrt{1+\frac{l^2+l+\frac14}{k_4}} & \text{if } 1\le l\le\Lambda,\\ 0 & \text{otherwise,}\end{cases}$$

and, according to (7.32),

$$A(l,l_2,3) = \sqrt{\frac{(l+l_2+2)(l+l_2+3)}{(2l+2)(2l+4)}},$$
$$B(l,l_2,3) = -\sqrt{\frac{(l-l_2-1)(l-l_2)}{(2l+2)(2l)}},$$
$$C(l,l_2,3) = -\sqrt{\frac{(l-l_2+2)(l-l_2+1)}{(2l+2)(2l+4)}},$$

$$D(l,l_2,3) = \sqrt{\frac{(l+l_2+1)(l+l_2)}{(2l+2)(2l)}},$$
$$F(l,l_2,3) = \sqrt{\frac{(l+l_2+2)(l-l_2+1)}{(2l+2)(2l+4)}},$$
$$G(l,l_2,3) = \sqrt{\frac{(l-l_2)(l+l_2+1)}{(2l+2)(2l)}}.$$

They fulfill (see section 2.3.3)

$$[\overline{x}_h, \overline{x}_j] = \left[-\frac{I}{k_4} + \left(\frac{1}{k_4} + \frac{(c_{\Lambda,4})^2}{2\Lambda+2}\right)\widehat{P}_{\Lambda,4}\right]\overline{L}_{h,j}, \quad [L_{h,s}, \overline{x}_j] = \frac{1}{i}\left(\delta^s_j \overline{x}_h - \delta^h_j \overline{x}_s\right),$$

$$\boldsymbol{x}^2 := \sum_{h=1}^{4} \overline{x}_h \overline{x}_h = \left\{1 + \frac{4\boldsymbol{L}^2+9}{4k_4(\Lambda)} - \left[\left(1 + \frac{4\Lambda^2+12\Lambda+9}{4k_4(\Lambda)}\right)\frac{\Lambda+2}{2\Lambda+2}\right]\widehat{P}_{\Lambda,4}\right\}$$

and

$$\begin{gathered}\prod_{l=0}^{\Lambda}\left[\boldsymbol{L}^2 - l(l+2)I\right] = 0 \quad , \quad \prod_{l_2=0}^{l}\left[C_3 - l_2(l_2+1)I\right]\widetilde{P}_{1,l} = 0, \\ \prod_{l_1=-l_2}^{l_2}\left[L_{1,2} - l_1 I\right]\widetilde{P}_{2,l_2} = 0, \quad (\overline{x}_\pm)^{2\Lambda+1} = 0, \text{ and } (L_{\nu,\pm})^{2\Lambda+1} = 0, \forall \nu \geq 3,\end{gathered} \tag{3.12}$$

where $\widetilde{P}_{h,j}$ is the projector on the eigenspace of C_{4-h} corresponding to $l_{4-h} \equiv j$.

According to this, the algebra of observables is generated by the coordinate operators, in fact every projector can be written as a ordered polynomial in the $\overline{x}_\nu$.

Furthermore, the $SO(5)$-irrep $\pi_{\Lambda,5}$, the one characterized by $C_5 \equiv \Lambda(\Lambda+3)I$ with representation space

$$V_{\Lambda,5} := span\left\{Y_{\Lambda,l,l_2,l_1}(\theta_4,\theta_3,\theta_2,\theta_1) : \Lambda \geq l \geq l_2 \geq |l_1|, l_i \in \mathbb{Z}\forall i\right\},$$

can be used to identify $\boldsymbol{\psi}_{l,l_2,l_1} \equiv Y_{\Lambda,l,l_2,l_1}$, and also the operators

$$L_{h,j} \equiv L_{h,j} \quad \text{for} \quad 1 \leq h < j \leq 4 \quad \text{and} \quad \overline{x}_s \equiv p_4(\lambda) L_{s,5} p_4(\lambda), \tag{3.13}$$

where

$$\lambda := \frac{-2 + \sqrt{4 + 4\boldsymbol{L}^2}}{2} = \sqrt{1 + \boldsymbol{L}^2} - 1,$$

while $p_4(\lambda)$ is an analytic function and the values $p_4(l)$, when $l \in \mathbb{N}_0$, can be obtained recursively from (2.29) starting from $p_4(0) := 1$.

Furthermore, in order to prove the convergence of S^3_Λ to ordinary quantum mechanics on S^3, it is convenient to identify $\boldsymbol{\psi}_{l,l_2,l_1} \equiv Y_{l,l_2,l_1}$ and then to consider their fuzzy counterparts $\widehat{Y}_{l,l_2,l_1}$ [see (7.88)], which can be used to approximate a generic $f \in B(S^3)$ or $f \in C(S^3)$; this is possible because the Y_{l,l_2,l_1} are an orthonormal basis of $\mathcal{L}^2(S^3)$, and also homogeneous polynomials in the $t_h := x_h/r$ variables. Then,

$$\widehat{f}_\Lambda := \sum_{l=0}^{2\Lambda} \sum_{l_2=0}^{l} \sum_{l_1=-l_2}^{l_2} f_{l,l_2,l_1} \widehat{Y}_{l,l_2,l_1}, \quad \text{where} \quad f_{l,l_2,l_1} := \langle Y_{l,l_2,l_1}, f \rangle,$$

is an approximation of f because of the following two theorems (see section 2.5)

Theorem 3.3.1 *Every projected coordinate operator $\overline{x}_h$ converges strongly to the corresponding t_h as $\Lambda \to \infty$ if*

$$k_4(\Lambda) \geq \Lambda \frac{1}{9} (\Lambda+1)^2 (\Lambda+2)^2 \left(\Lambda + \frac{3}{2}\right)^2 \frac{4\Lambda(\Lambda+2)+3}{4}$$
$$\Lambda \frac{1}{9} (\Lambda+1)^2 (\Lambda+2)^2 \left(\Lambda + \frac{3}{2}\right)^3 \left(\Lambda + \frac{1}{2}\right).$$

Theorem 3.3.2 *Choosing $k_4(\Lambda)$ fulfilling (2.35) for $D = 4$, then for all $f, g \in B(S^3)$ the following strong limits as $\Lambda \to \infty$ hold: $\hat{f}_\Lambda \to f\cdot$, $\widehat{(fg)}_\Lambda \to fg\cdot$ and $\hat{f}_\Lambda \hat{g}_\Lambda \to fg\cdot$.*

3.4 S^4_Λ

When $D = 5$ the choices $\overline{E} = \overline{E}(\Lambda) := \Lambda(\Lambda+3)$ and $k_5 = k_5(\Lambda) \geq [\Lambda(\Lambda+3)]^2$ imply that the Hilbert space of admitted states $\mathcal{H}_{\Lambda,5}$ is generated by all the functions

$$\boldsymbol{\psi}_{l,l_3,l_2,l_1,5} = \boldsymbol{\psi}_{l,l_3,l_2,l_1,4}(r, \theta_1, \theta_2, \theta_3, \theta_4) := f_{l,5}(r) Y_{l,l_3,l_2,l_1}(\theta_1, \theta_2, \theta_3, \theta_4), \quad l \leq \Lambda$$

hence

$$\dim \mathcal{H}_{\Lambda,5} = \binom{\Lambda+3}{\Lambda-1} \frac{2\Lambda+4}{\Lambda} = \frac{1}{12} (\Lambda+1)(\Lambda+2)^2(\Lambda+3).$$

The angular momentum components are $\{L_{h,j} : 1 \le h < j \le 5\}$, $L_{\pm,5} := L_{2,5} \mp iL_{1,5}$ and they act as follows:

$$\begin{aligned}
L_{1,2}\boldsymbol{\psi}_{l,l_3,l_2,l_1} &= l_1\boldsymbol{\psi}_{l,l_3,l_2,l_1},\\
L_{1,3}\boldsymbol{\psi}_{l,l_3,l_2,l_1} &= \frac{1}{i}\left[\frac{d_{l_2,l_1,3}}{2}\boldsymbol{\psi}_{l,l_3,l_2,l_1-1} - \frac{d_{l_2,l_1+1,3}}{2}\boldsymbol{\psi}_{l,l_3,l_2,l_1+1}\right],\\
L_{2,3}\boldsymbol{\psi}_{l,l_3,l_2,l_1} &= \frac{1}{i}\left[-\frac{d_{l_2,l_1,3}}{2i}\boldsymbol{\psi}_{l,l_3,l_2,l_1-1} - \frac{d_{l_2,l_1+1,3}}{2i}\boldsymbol{\psi}_{l,l_3,l_2,l_1+1}\right],\\
L_{1,4}\boldsymbol{\psi}_{l,l_3,l_2,l_1} &= \frac{1}{i}\left[\frac{d_{l_3,l_2,4}B(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3,l_2-1,l_1+1} + \frac{d_{l_3,l_2,4}D(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3,l_2-1,l_1-1}\right.\\
&\qquad\left.-\frac{d_{l_3,l_2+1,4}A(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3,l_2+1,l_1+1} - \frac{d_{l_3,l_2+1,4}C(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3,l_2+1,l_1-1}\right],\\
L_{2,4}\boldsymbol{\psi}_{l,l_3,l_2,l_1} &= \frac{1}{i}\left[\frac{d_{l_3,l_2,4}B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3,l_2-1,l_1+1} - \frac{d_{l_3,l_2,4}D(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3,l_2-1,l_1-1}\right.\\
&\qquad\left.-\frac{d_{l_3,l_2+1,4}A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3,l_2+1,l_1+1} + \frac{d_{l_3,l_2+1,4}C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3,l_2+1,l_1-1}\right],\\
L_{3,4}\boldsymbol{\psi}_{l,l_3,l_2,l_1} &= \frac{1}{i}\left[d_{l_3,l_2,4}G(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3,l_2-1,l_1} - d_{l_3,l_2+1,4}F(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3,l_2+1,l_1}\right],
\end{aligned}$$

$$\begin{aligned}
L_{1,5}\boldsymbol{\psi}_{l,l_3,l_2,l_1} = \frac{1}{i}&\left[\frac{d_{l,l_3,5}D(l_3,l_2,3)D(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3-1,l_2-1,l_1-1}\right.\\
&+\frac{d_{l,l_3,5}D(l_3,l_2,3)B(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3-1,l_2-1,l_1+1}\\
&+\frac{d_{l,l_3,5}B(l_3,l_2,3)C(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3-1,l_2+1,l_1-1}\\
&+\frac{d_{l,l_3,5}B(l_3,l_2,3)A(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3-1,l_2+1,l_1+1}\\
&-\frac{d_{l,l_3+1,5}C(l_3,l_2,3)D(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3+1,l_2-1,l_1-1}\\
&-\frac{d_{l,l_3+1,5}C(l_3,l_2,3)B(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3+1,l_2-1,l_1+1}\\
&-\frac{d_{l,l_3+1,5}A(l_3,l_2,3)C(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3+1,l_2+1,l_1-1}\\
&\left.-\frac{d_{l,l_3+1,5}A(l_3,l_2,3)A(l_2,l_1,2)}{2}\boldsymbol{\psi}_{l,l_3+1,l_2+1,l_1+1}\right],
\end{aligned}$$

$$\begin{aligned}
L_{2,5}\boldsymbol{\psi}_{l,l_3,l_2,l_1} = \frac{1}{i}\Bigg[& \frac{d_{l,l_3,5}D(l_3,l_2,3)D(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3-1,l_2-1,l_1-1} \\
& - \frac{d_{l,l_3,5}D(l_3,l_2,3)B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3-1,l_2-1,l_1+1} \\
& + \frac{d_{l,l_3,5}B(l_3,l_2,3)C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3-1,l_2+1,l_1-1} \\
& - \frac{d_{l,l_3,5}B(l_3,l_2,3)A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3-1,l_2+1,l_1+1} \\
& - \frac{d_{l,l_3+1,5}C(l_3,l_2,3)D(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3+1,l_2-1,l_1-1} \\
& + \frac{d_{l,l_3+1,5}C(l_3,l_2,3)B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3+1,l_2-1,l_1+1} \\
& - \frac{d_{l,l_3+1,5}A(l_3,l_2,3)C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3+1,l_2+1,l_1-1} \\
& + \frac{d_{l,l_3+1,5}A(l_3,l_2,3)A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l,l_3+1,l_2+1,l_1+1}\Bigg],
\end{aligned}$$

$$\begin{aligned}
L_{3,5}\boldsymbol{\psi}_{l,l_3,l_2,l_1} = \frac{1}{i}\big[& d_{l,l_3,5}D(l_3,l_2,3)G(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3-1,l_2-1,l_1} \\
& + d_{l,l_3,5}B(l_3,l_2,3)F(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3-1,l_2+1,l_1} \\
& - d_{l,l_3+1,5}C(l_3,l_2,3)G(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3+1,l_2-1,l_1} \\
& - d_{l,l_3+1,5}B(l_3,l_2,3)A(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3+1,l_2+1,l_1}\big],
\end{aligned}$$

$$L_{4,5}\boldsymbol{\psi}_{l,l_3l_2,l_1} = \frac{1}{i}\left[d_{l,l_3,5}G(l_3,l_2,3)\boldsymbol{\psi}_{l,l_3-1,l_2,l_1} - d_{l,l_3+1,5}F(l_3,l_2,3)\boldsymbol{\psi}_{l,l_3+1,l_2,l_1}\right],$$

$$\begin{aligned}
L_{+,5}\boldsymbol{\psi}_{l,l_3,l_2,l_1} = & d_{l,l_3,5}D(l_3,l_2,3)B(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3-1,l_2-1,l_1+1} \\
& + d_{l,l_3,5}B(l_3,l_2,3)A(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3-1,l_2+1,l_1+1} \\
& - d_{l,l_3+1,5}C(l_3,l_2,3)B(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3+1,l_2-1,l_1+1} \\
& - d_{l,l_3+1,5}A(l_3,l_2,3)A(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3+1,l_2+1,l_1+1},
\end{aligned}$$

$$\begin{aligned}
L_{-,5}\boldsymbol{\psi}_{l,l_3,l_2,l_1} = & - d_{l,l_3,5}D(l_3,l_2,3)D(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3-1,l_2-1,l_1-1} \\
& - d_{l,l_3,5}B(l_3,l_2,3)C(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3-1,l_2+1,l_1-1} \\
& + d_{l,l_3+1,5}C(l_3,l_2,3)D(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3+1,l_2-1,l_1-1} \\
& + d_{l,l_3+1,5}A(l_3,l_2,3)C(l_2,l_1,2)\boldsymbol{\psi}_{l,l_3+1,l_2+1,l_1-1},
\end{aligned}$$

where $d_{l,l_3,5} = \sqrt{(l-l_3+1)(l+l_3+2)}$.

They fulfill

$$[L_{h,j},L_{p,s}]\,\boldsymbol{\psi}_{l,l_3,l_2,l_1} = i\left(\delta_{h,p}L_{j,s} + \delta_{j,s}L_{h,p} - \delta_{h,s}L_{j,p} - \delta_{j,p}L_{h,s}\right)\boldsymbol{\psi}_{l,l_3,l_2,l_1},$$

$$\boldsymbol{L}^2\boldsymbol{\psi}_{l,l_3,l_2,l_1} = l(l+3)\boldsymbol{\psi}_{l,l_3,l_2,l_1}, \quad C_4\psi_{l,l_3,l_2,l_1} = l_3(l_3+2)\boldsymbol{\psi}_{l,l_3,l_2,l_1},$$

$$C_3\psi_{l,l_3,l_2,l_1} = l_2(l_2+1)\boldsymbol{\psi}_{l,l_3,l_2,l_1} \quad \text{and} \quad C_2\psi_{l,l_3,l_2,l_1} = l_1^2\psi_{l,l_3,l_2,l_1}.$$

The coordinate operators are $\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5$, $\overline{x}_\pm := \overline{x}_1 \pm i\overline{x}_2$, and they act on $\mathcal{H}_{\Lambda,5}$ as follows:

$$\begin{aligned}
\overline{x}_1 \boldsymbol{\psi}_{l,l_3,l_2,l_1} =& \frac{c_{l,5} D(l,l_3,4) D(l_3,l_2,3) D(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l-1,l_3-1,l_2-1,l_1-1} \\
&+ \frac{c_{l+1,5} C(l,l_3,4) D(l_3,l_2,3) D(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l+1,l_3-1,l_2-1,l_1-1} \\
&+ \frac{c_{l,5} B(l,l_3,4) C(l_3,l_2,3) D(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l-1,l_3+1,l_2-1,l_1-1} \\
&+ \frac{c_{l+1,5} A(l,l_3,4) C(l_3,l_2,3) D(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l+1,l_3+1,l_2-1,l_1-1} \\
&+ \frac{c_{l,5} D(l,l_3,4) B(l_3,l_2,3) C(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l-1,l_3-1,l_2+1,l_1-1} \\
&+ \frac{c_{l+1,5} C(l,l_3,4) B(l_3,l_2,3) C(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l+1,l_3-1,l_2+1,l_1-1} \\
&+ \frac{c_{l,5} B(l,l_3,4) A(l_3,l_2,3) C(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l-1,l_3+1,l_2+1,l_1-1} \\
&+ \frac{c_{l+1,5} A(l,l_3,4) A(l_3,l_2,3) C(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l+1,l_3+1,l_2+1,l_1-1} \\
&+ \frac{c_{l,5} D(l,l_3,4) D(l_3,l_2,3) B(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l-1,l_3-1,l_2-1,l_1+1} \\
&+ \frac{c_{l+1,5} C(l,l_3,4) D(l_3,l_2,3) B(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l+1,l_3-1,l_2-1,l_1+1} \\
&+ \frac{c_{l,5} B(l,l_3,4) C(l_3,l_2,3) B(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l-1,l_3+1,l_2-1,l_1+1} \\
&+ \frac{c_{l+1,5} A(l,l_3,4) C(l_3,l_2,3) B(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l+1,l_3+1,l_2-1,l_1+1} \\
&+ \frac{c_{l,5} D(l,l_3,4) B(l_3,l_2,3) A(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l-1,l_3-1,l_2+1,l_1+1} \\
&+ \frac{c_{l+1,5} C(l,l_3,4) B(l_3,l_2,3) A(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l+1,l_3-1,l_2+1,l_1+1} \\
&+ \frac{c_{l,5} B(l,l_3,4) A(l_3,l_2,3) A(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l-1,l_3+1,l_2+1,l_1+1} \\
&+ \frac{c_{l+1,5} A(l,l_3,4) A(l_3,l_2,3) A(l_2,l_1,2)}{2} \boldsymbol{\psi}_{l+1,l_3+1,l_2+1,l_1+1}, \\
\overline{x}_2 \boldsymbol{\psi}_{l,l_3,l_2,l_1} =& \frac{c_{l,5} D(l,l_3,4) D(l_3,l_2,3) D(l_2,l_1,2)}{2i} \boldsymbol{\psi}_{l-1,l_3-1,l_2-1,l_1-1} \\
&+ \frac{c_{l+1,5} C(l,l_3,4) D(l_3,l_2,3) D(l_2,l_1,2)}{2i} \boldsymbol{\psi}_{l+1,l_3-1,l_2-1,l_1-1} \\
&+ \frac{c_{l,5} B(l,l_3,4) C(l_3,l_2,3) D(l_2,l_1,2)}{2i} \boldsymbol{\psi}_{l-1,l_3+1,l_2-1,l_1-1} \\
&+ \frac{c_{l+1,5} A(l,l_3,4) C(l_3,l_2,3) D(l_2,l_1,2)}{2i} \boldsymbol{\psi}_{l+1,l_3+1,l_2-1,l_1-1}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{c_{l,5}D(l,l_3,4)B(l_3,l_2,3)C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_3-1,l_2+1,l_1-1}\\
&+ \frac{c_{l+1,5}C(l,l_3,4)B(l_3,l_2,3)C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_3-1,l_2+1,l_1-1}\\
&+ \frac{c_{l,5}B(l,l_3,4)A(l_3,l_2,3)C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_3+1,l_2+1,l_1-1}\\
&+ \frac{c_{l+1,5}A(l,l_3,4)A(l_3,l_2,3)C(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_3+1,l_2+1,l_1-1}\\
&- \frac{c_{l,5}D(l,l_3,4)D(l_3,l_2,3)B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_3-1,l_2-1,l_1+1}\\
&- \frac{c_{l+1,5}C(l,l_3,4)D(l_3,l_2,3)B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_3-1,l_2-1,l_1+1}\\
&- \frac{c_{l,5}B(l,l_3,4)C(l_3,l_2,3)B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_3+1,l_2-1,l_1+1}\\
&- \frac{c_{l+1,5}A(l,l_3,4)C(l_3,l_2,3)B(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_3+1,l_2-1,l_1+1}\\
&- \frac{c_{l,5}D(l,l_3,4)B(l_3,l_2,3)A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_3-1,l_2+1,l_1+1}\\
&- \frac{c_{l+1,5}C(l,l_3,4)B(l_3,l_2,3)A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_3-1,l_2+1,l_1+1}\\
&- \frac{c_{l,5}B(l,l_3,4)A(l_3,l_2,3)A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l-1,l_3+1,l_2+1,l_1+1}\\
&- \frac{c_{l+1,5}A(l,l_3,4)A(l_3,l_2,3)A(l_2,l_1,2)}{2i}\boldsymbol{\psi}_{l+1,l_3+1,l_2+1,l_1+1},
\end{aligned}
$$

$$
\begin{aligned}
\overline{x}_3\boldsymbol{\psi}_{l,l_3,l_2,l_1} =& c_{l,5}D(l,l_3,4)D(l_3,l_2,3)G(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3-1,l_2-1,l_1}\\
&+ c_{l+1,5}C(l,l_3,4)D(l_3,l_2,3)G(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3-1,l_2-1,l_1}\\
&+ c_{l,5}B(l,l_3,4)C(l_3,l_2,3)G(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3+1,l_2-1,l_1}\\
&+ c_{l+1,5}A(l,l_3,4)C(l_3,l_2,3)G(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3+1,l_2-1,l_1}\\
&+ c_{l,5}D(l,l_3,4)B(l_3,l_2,3)F(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3-1,l_2+1,l_1}\\
&+ c_{l+1,5}C(l,l_3,4)B(l_3,l_2,3)F(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3-1,l_2+1,l_1}\\
&+ c_{l,5}B(l,l_3,4)A(l_3,l_2,3)F(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3+1,l_2+1,l_1}\\
&+ c_{l+1,5}A(l,l_3,4)A(l_3,l_2,3)F(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3+1,l_2+1,l_1},
\end{aligned}
$$

$$
\begin{aligned}
\overline{x}_4\boldsymbol{\psi}_{l,l_3,l_2,l_1} =& c_{l,5}D(l,l_3,4)G(l_3,l_2,3)\boldsymbol{\psi}_{l-1,l_3-1,l_2,l_1} + c_{l,5}B(l,l_3,4)F(l_3,l_2,3)\boldsymbol{\psi}_{l-1,l_3+1,l_2,l_1}\\
&+ c_{l+1,5}C(l,l_3,4)G(l_3,l_2,3)\boldsymbol{\psi}_{l+1,l_3-1,l_2,l_1} + c_{l+1,5}A(l,l_3,4)F(l_3,l_2,3)\boldsymbol{\psi}_{l+1,l_3+1,l_2,l_1}\ ,\\
\overline{x}_5\boldsymbol{\psi}_{l,l_3,l_2,l_1} =& c_{l,5}G(l,l_3,4)\boldsymbol{\psi}_{l-1,l_3,l_2,l_1} + c_{l+1,5}F(l,l_3,4)\boldsymbol{\psi}_{l+1,l_3,l_2,l_1},
\end{aligned}
$$

$$\begin{aligned}
\overline{x}_+\boldsymbol{\psi}_{l,l_3,l_2,l_1} =& c_{l,5}D(l,l_3,4)D(l_3,l_2,3)B(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3-1,l_2-1,l_1+1}\\
&+ c_{l+1,5}C(l,l_3,4)D(l_3,l_2,3)B(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3-1,l_2-1,l_1+1}\\
&+ c_{l,5}B(l,l_3,4)C(l_3,l_2,3)B(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3+1,l_2-1,l_1+1}\\
&+ c_{l+1,5}A(l,l_3,4)C(l_3,l_2,3)B(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3+1,l_2-1,l_1+1}\\
&+ c_{l,5}D(l,l_3,4)B(l_3,l_2,3)A(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3-1,l_2+1,l_1+1}\\
&+ c_{l+1,5}C(l,l_3,4)B(l_3,l_2,3)A(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3-1,l_2+1,l_1+1}\\
&+ c_{l,5}B(l,l_3,4)A(l_3,l_2,3)A(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3+1,l_2+1,l_1+1}\\
&+ c_{l+1,5}A(l,l_3,4)A(l_3,l_2,3)A(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3+1,l_2+1,l_1+1},
\end{aligned}$$

$$\begin{aligned}
\overline{x}_-\boldsymbol{\psi}_{l,l_3,l_2,l_1} =& c_{l,5}D(l,l_3,4)D(l_3,l_2,3)D(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3-1,l_2-1,l_1-1}\\
&+ c_{l+1,5}C(l,l_3,4)D(l_3,l_2,3)D(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3-1,l_2-1,l_1-1}\\
&+ c_{l,5}B(l,l_3,4)C(l_3,l_2,3)D(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3+1,l_2-1,l_1-1}\\
&+ c_{l+1,5}A(l,l_3,4)C(l_3,l_2,3)D(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3+1,l_2-1,l_1-1}\\
&+ c_{l,5}D(l,l_3,4)B(l_3,l_2,3)C(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3-1,l_2+1,l_1-1}\\
&+ c_{l+1,5}C(l,l_3,4)B(l_3,l_2,3)C(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3-1,l_2+1,l_1-1}\\
&+ c_{l,5}B(l,l_3,4)A(l_3,l_2,3)C(l_2,l_1,2)\boldsymbol{\psi}_{l-1,l_3+1,l_2+1,l_1-1}\\
&+ c_{l+1,5}A(l,l_3,4)A(l_3,l_2,3)C(l_2,l_1,2)\boldsymbol{\psi}_{l+1,l_3+1,l_2+1,l_1-1},
\end{aligned}$$

where

$$c_{l,5} \overset{(2.23)}{=} \begin{cases} \sqrt{1+\frac{l^2+2l+1}{k_5}} & \text{if } 1 \leq l \leq \Lambda, \\ 0 & \text{otherwise,} \end{cases}$$

and, according to (7.32),

$$\begin{aligned}
A(l,l_3,4) &= \sqrt{\frac{(l+l_3+3)(l+l_3+4)}{(2l+3)(2l+5)}},\\
B(l,l_3,4) &= -\sqrt{\frac{(l-l_3-1)(l-l_3)}{(2l+3)(2l+1)}},\\
C(l,l_3,4) &= -\sqrt{\frac{(l-l_3+2)(l-l_3+1)}{(2l+3)(2l+5)}},\\
D(l,l_3,4) &= \sqrt{\frac{(l+l_3+2)(l+l_3+1)}{(2l+3)(2l+1)}},\\
F(l,l_3,4) &= \sqrt{\frac{(l+l_3+3)(l-l_3+1)}{(2l+3)(2l+5)}},\\
G(l,l_3,4) &= \sqrt{\frac{(l-l_3)(l+l_3+2)}{(2l+3)(2l+1)}}.
\end{aligned}$$

They fulfill (see section 2.3.3)

$$[\overline{x}_h, \overline{x}_j] = \left[-\frac{I}{k_5} + \left(\frac{1}{k_5} + \frac{(c_{\Lambda,5})^2}{2\Lambda+3}\right)\widehat{P}_{\Lambda,5}\right]\overline{L}_{h,j}, \quad [L_{h,s}, \overline{x}_j] = \frac{1}{i}\left(\delta^s_j \overline{x}_h - \delta^h_j \overline{x}_s\right),$$

$$\boldsymbol{x}^2 := \sum_{h=1}^{5} \overline{x}_h \overline{x}_h = \left\{1 + \frac{2\boldsymbol{L}^2+8}{4k_5(\Lambda)} - \left[\left(1 + \frac{2\Lambda^2+8\Lambda+8}{4k_5(\Lambda)}\right)\frac{\Lambda+3}{2\Lambda+3}\right]\widehat{P}_{\Lambda,4}\right\}$$

and

$$\begin{gathered}\prod_{l=0}^{\Lambda}\left[\boldsymbol{L}^2 - l(l+2)I\right] = 0 \quad , \quad \prod_{l_2=0}^{l}\left[C_3 - l_2(l_2+1)I\right]\widetilde{P}_{1,l} = 0, \\ \prod_{l_1=-l_2}^{l_2}\left[L_{1,2} - l_1 I\right]\widetilde{P}_{2,l_2} = 0, \quad (\overline{x}_\pm)^{2\Lambda+1} = 0, \text{ and } (L_{\nu,\pm})^{2\Lambda+1} = 0, \forall \nu \geq 3,\end{gathered} \tag{3.14}$$

where $\widetilde{P}_{h,j}$ is the projector on the eigenspace of C_{5-h} corresponding to $l_{5-h} \equiv j$.

According to this, the algebra of observables is generated by the coordinate operators, in fact every projector can be written as a ordered polynomial in the $\overline{x}_\nu$.

Furthermore, the $SO(6)$-irrep $\pi_{\Lambda,6}$, the one characterized by $C_6 \equiv \Lambda(\Lambda+4)I$ with representation space

$$V_{\Lambda,6} := span\left\{Y_{\Lambda,l,l_3,l_2,l_1}(\theta_5,\theta_4,\theta_3,\theta_2,\theta_1) : \Lambda \geq l \geq l_3 \geq l_2 \geq |l_1|, l_i \in \mathbb{Z}\forall i\right\},$$

can be used to identify $\boldsymbol{\psi}_{l,l_3,l_2,l_1} \equiv Y_{\Lambda,l,l_3,l_2,l_1}$, and also the operators

$$L_{h,j} \equiv L_{h,j} \quad \text{for} \quad 1 \leq h < j \leq 5 \quad \text{and} \quad \overline{x}_s \equiv p_5(\lambda) L_{s,6} p_5(\lambda), \tag{3.15}$$

where

$$\lambda := \frac{-3 + \sqrt{9 + 4\boldsymbol{L}^2}}{2},$$

while $p_5(\lambda)$ is an analytic function and the values $p_5(l)$, when $l \in \mathbb{N}_0$, can be obtained recursively from (2.29) starting from $p_5(0) := 1$.

Furthermore, in order to prove the convergence of S^4_Λ to ordinary quantum mechanics on S^4, it is convenient to identify $\boldsymbol{\psi}_{l,l_3,l_2,l_1} \equiv Y_{l,l_3,l_2,l_1}$ and then to consider their fuzzy counterparts $\widehat{Y}_{l,l_3,l_2,l_1}$, which can be used to approximate a generic $f \in B(S^4)$ or $f \in C(S^4)$; this is possible because the Y_{l,l_3,l_2,l_1} are an orthonormal basis of $\mathcal{L}^2(S^4)$, and also homogeneous polynomials in the $t_h := x_h/r$ variables. Then,

$$\widehat{f}_\Lambda := \sum_{l=0}^{2\Lambda}\sum_{l_3=0}^{l}\sum_{l_2=0}^{l_3}\sum_{l_1=-l_2}^{l_2} f_{l,l_3,l_2,l_1}\widehat{Y}_{l,l_3,l_2,l_1}, \quad \text{where} \quad f_{l,l_3,l_2,l_1} := \langle Y_{l,l_3,l_2,l_1}, f\rangle,$$

is an approximation of f because of the following two theorems (see section 2.5)

Theorem 3.4.1 *Every projected coordinate operator* $\overline{x}_h$ *converges strongly to the corresponding* t_h *as* $\Lambda \to \infty$ *if*

$$k_5(\Lambda) \geq \Lambda \frac{1}{144}(\Lambda+1)^2(\Lambda+2)^4(\Lambda+3)^2 \frac{4\Lambda(\Lambda+3)+8}{4} = \frac{1}{144}(\Lambda+1)^3(\Lambda+2)^5(\Lambda+3)^2.$$

Theorem 3.4.2 *Choosing* $k_5(\Lambda)$ *fulfilling (2.35) for* $D = 5$*, then for all* $f, g \in B(S^4)$ *and* $C(S^4)$ *the following strong limits as* $\Lambda \to \infty$ *hold:* $\hat{f}_\Lambda \to f\cdot$, $\widehat{(fg)}_\Lambda \to fg\cdot$ *and* $\hat{f}_\Lambda \hat{g}_\Lambda \to fg\cdot$.

Chapter 4

The x_i-eigenvalue problem

4.1 Diagonalization of Toeplitz tridiagonal matrices

A real Toeplitz tri-diagonal matrix is a $n \times n$ matrix

$$P_n\left(a,b,c\right) := \begin{pmatrix} a & b & 0 & 0 & 0 & 0 & 0 & 0 \\ c & a & b & 0 & 0 & 0 & 0 & 0 \\ 0 & c & a & b & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a & b & 0 \\ 0 & 0 & 0 & 0 & \cdots & c & a & b \\ 0 & 0 & 0 & 0 & \cdots & 0 & c & a \end{pmatrix} \quad \text{where } a,b,c \in \mathbb{R}. \tag{4.1}$$

Its eigenvalues are (see e.g. [44] p. 2-3)

$$\lambda_h = a + 2\sqrt{bc}\cos\left(\frac{h\pi}{n+1}\right), \quad h = 1, \cdots, n \tag{4.2}$$

and the corresponding eigenvectors χ^h are columns with the following components

$$\chi^{h,k_D} = \left(\frac{c}{b}\right)^{\frac{k_D}{2}} \sin\left(\frac{hk\pi}{n+1}\right), \qquad h, k_D = 1, 2, \cdots, n, \tag{4.3}$$

up to normalization. In the symmetric case ($b = c$) all eigenvalues are real and the highest one is clearly λ_1; the norm of χ^1 is easily computed:

$$\left\|\chi^1\right\|_2 = \sum_{k_D=1}^{n} \sin^2\left(\frac{k_D\pi}{n+1}\right) = \frac{n+1}{2}. \tag{4.4}$$

4.2 Spectrum of x_i in the $O(2)$-equivariant fuzzy circle

In this subsection the spectrum of x_1 is studied. This is not a restriction because the algebraic relations (3.1-3.3) are covariant under $O(2)$ transformations $\boldsymbol{x} \mapsto \boldsymbol{x}' = R\boldsymbol{x}$, L is covariant under 2-dimensional rotations, $L \to -L$ under x_1-inversion and the same applies under x_2-inversion; this implies that the spectra $\Sigma_{x_i}(\Lambda)$ of all coordinate operators x_i are equal, and for this reason it is reasonable to focus the attention only to x_1. The spectrum Σ_{x_1} for $\Lambda = 1, 2$ is presented in formulae (8.2-8.3) of the appendix.

More generally, on the basis $\mathcal{B}$ of $\mathcal{H}_\Lambda$ the operator x_1 is represented by the $(2\Lambda{+}1) \times (2\Lambda{+}1)$ symmetric tri-diagonal matrix [cf. (4.1)]

$$X^\Lambda = \frac{1}{2}\begin{pmatrix} 0 & b_\Lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ b_\Lambda & 0 & b_{\Lambda-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{\Lambda-1} & 0 & b_{\Lambda-2} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{2-\Lambda} & 0 & b_{1-\Lambda} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{1-\Lambda} & 0 \end{pmatrix} = X_0^\Lambda + O\left(\frac{1}{\Lambda^2}\right),$$

where $X_0^\Lambda := \frac{1}{2}P(0,1,1)$, and it is obvious that all the eigenvalues of X^Λ are real.

Let $\Sigma_0^\Lambda := \{\widetilde{\alpha}_h(\Lambda)\}_{h=1}^{2\Lambda+1}$ be the set of the eigenvalues of X_0^Λ arranged in descending order; according to (4.2) one has

$$\widetilde{\alpha}_h(\Lambda) = \cos\left(\frac{h\pi}{2\Lambda+2}\right), \quad h = 1, 2, \cdots, 2\Lambda+1. \tag{4.5}$$

It is easy to see that $\alpha \in \Sigma_0^\Lambda \Rightarrow -\alpha \in \Sigma_0^\Lambda$, all the eigenvalues of Σ_0^Λ are simple, $\widetilde{\alpha}_1(\Lambda+1) > \widetilde{\alpha}_1(\Lambda)$ and Σ_0^Λ becomes uniformly dense in $[-1,1]$ as $\Lambda \to \infty$.

In section 8.2 it is shown that the same holds true also for the spectrum Σ^Λ of x_Λ, in particular one has

Theorem 4.2.1 *(A) If α is an eigenvalue of X^Λ, then also $-\alpha$ is.*

(B) For all Λ, all eigenvalues of X^Λ are simple; they are denoted as $\alpha_1(\Lambda), \alpha_2(\Lambda), \ldots, \alpha_{2\Lambda+1}(\Lambda)$, in decreasing order.

(C) Let $k_D(\Lambda) \geq \Lambda(\Lambda-1)(2\Lambda+3)^2(2\Lambda+4)^4/4\pi^4$, then

$$\alpha_1(\Lambda+1) > \alpha_1(\Lambda) \quad \forall \Lambda \in \mathbb{N}. \tag{4.6}$$

(D) Σ^Λ becomes uniformly dense in $[-1,1]$ as $\Lambda \to \infty$, in particular

$$\lim_{\Lambda\to+\infty} \alpha_1(\Lambda) = 1 \quad \text{and} \quad \alpha_1(\Lambda) \geq 1 - \frac{\pi^2}{8(\Lambda+1)^2} \quad \forall \Lambda \in \mathbb{N}. \tag{4.7}$$

Let $\boldsymbol{\chi} := \sum_{n=-\Lambda}^{\Lambda} \chi_n \psi_n$, the eigenvalue equation $x_1 \boldsymbol{\chi} = \alpha \boldsymbol{\chi}$ amounts to

$$\frac{b_\Lambda}{2} \chi_{\pm(\Lambda-1)} = \alpha \chi_{\pm\Lambda}, \qquad \frac{b_n \chi_{n-1} + b_{n+1} \chi_{n+1}}{2} = \alpha \chi_n \quad \text{if } |n| < \Lambda; \tag{4.8}$$

on the other hand, $b_n \to 1$ in the commutative limit and in section 8.2.4 it is shown that $\alpha_h(\Lambda) \simeq \cos\left(\frac{h\pi}{2\Lambda+2}\right)$ in the limit $\Lambda \to +\infty$, so (4.3) and (4.4) imply

$$x_1 \boldsymbol{\chi}_h(\Lambda) = \alpha_h(\Lambda) \boldsymbol{\chi}_h(\Lambda) \Longrightarrow \chi_{h,n}(\Lambda) \simeq \sqrt{\frac{2}{2\Lambda+2}} \sin\left(\frac{hn\pi}{2\Lambda+2}\right).$$

4.3 Spectrum of x_i in the $O(3)$-equivariant fuzzy sphere

The spectrum of x_0 is studied in the following lines, this is not a restriction since the covariance of the algebra under $O(3)$ transformations $\boldsymbol{x} \mapsto \boldsymbol{x}' = R\boldsymbol{x}$, $\boldsymbol{L} \mapsto \boldsymbol{L}' = R\boldsymbol{L}$ implies that the spectra $\Sigma_{x_i}(\Lambda)$ of all coordinate operators x_i of the new fuzzy space are equal; on the other hand, because of $[x_0, L_0] = 0$, it is possible to simultaneously diagonalize x_0 and L_0.

Eq. $(3.6)_1$ and

$$\begin{cases} L_0 \boldsymbol{\chi}_\alpha^\beta = \beta \boldsymbol{\chi}_\alpha^\beta \\ x_0 \boldsymbol{\chi}_\alpha^\beta = \alpha \boldsymbol{\chi}_\alpha^\beta \end{cases}, \tag{4.9}$$

implies

$$\beta = m \in \{-\Lambda, -\Lambda+1, \cdots, \Lambda-1, \Lambda\} \quad \text{and} \quad \boldsymbol{\chi}_\alpha^m = \sum_{l=|m|}^{\Lambda} \chi_{\alpha,l}^m \boldsymbol{\psi}_l^m; \tag{4.10}$$

so $x_0 \boldsymbol{\chi}_\alpha^m = \alpha \boldsymbol{\chi}_\alpha^m$ can be re-written as

$$\begin{cases} \chi_{\alpha,|m|+1}^m c_{|m|+1} G(|m|+1, m, 2) = \alpha \chi_{\alpha,|m|}^m \\ \chi_{\alpha,|m|}^m c_{|m|+1} F(|m|, m, 2) + \chi_{\alpha,|m|+2}^m c_{|m|+2} G(|m|+2, m, 2) = \alpha \chi_{\alpha,|m|+1}^m \\ \chi_{\alpha,|m|+1}^m c_{|m|+2} F(|m|+1, m, 2) + \chi_{\alpha,|m|+3}^m c_{|m|+3} G(|m|+3, m, 2) = \alpha \chi_{\alpha,|m|+2}^m \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ \chi_{\alpha,\Lambda-2}^m c_{\Lambda-1,3} F(\Lambda-2, m, 2) + \chi_{\alpha,\Lambda}^m c_{\Lambda,3} G(\Lambda, m, 2) = \alpha \chi_{\alpha,\Lambda-1}^m \\ c_{\Lambda,3} F(\Lambda-1, m, 2) \chi_{\alpha,\Lambda-1}^m = \alpha \chi_{\alpha,\Lambda}^m \end{cases} \tag{4.11}$$

which in turn can be rewritten in the matrix form $B_m(\Lambda)\chi = \alpha\chi$, where $\chi = \left(\chi_{\alpha,|m|}^m, \chi_{\alpha,|m|+1}^m, \ldots, \chi_{\alpha,\Lambda}^m\right)^T$ and $B_m(\Lambda)$ is the following $n(\Lambda; m) \times n(\Lambda; m)$ symmetric tridiagonal matrix

$$B_m(\Lambda) = \begin{pmatrix} 0 & c_{|m|+1}G(|m|+1,m,2) & \vdots & 0 \\ c_{|m|+1}G(|m|+1,m,2) & 0 & \vdots & 0 \\ 0 & c_{|m|+2}G(|m|+2,m,2) & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \vdots & c_{\Lambda,3}G(\Lambda,m,2) \\ 0 & 0 & \vdots & 0 \end{pmatrix},$$

or equivalently $M_m(\Lambda;\alpha)\,\chi = 0$, where 0 here is the null vector, with

$$n = n(\Lambda;m) := \Lambda - |m| + 1, \qquad M_m(\Lambda;\alpha) := B_m(\Lambda) - \alpha I_{n(\Lambda;m)}.$$

It is well known that the problem of determining analytically the eigenvalues of a square matrix of large rank is absolutely not trivial, but the $B_m(\Lambda)$ have several good properties (for example they are symmetric and tri-diagonal) which will help in studying their spectra. First of all,

Remark 1 *All the eigenvalues of $B_m(\Lambda)$ are real, and $B_m(\Lambda) \equiv B_{-m}(\Lambda)$ implies that it is reasonable to focus the attention to the cases $\beta = m \in \{0, 1, \cdots, \Lambda\}$.*

As for the fuzzy circle,

Theorem 4.3.1 *(A) If α is an eigenvalue of $B_m(\Lambda)$, then also $-\alpha$ is.*

(B) For all Λ, m, all eigenvalues of $B_m(\Lambda)$ are simple; they are denoted with $\alpha_1(\Lambda;m), \alpha_2(\Lambda;m), \ldots, \alpha_{n(\Lambda;m)}(\Lambda;m)$, in decreasing order.

(C) Let $\alpha_1(\Lambda;m)$ be the highest eigenvalue of $B_m(\Lambda)$, then

$$\alpha_1(\Lambda;0) > \alpha_1(\Lambda;1) > \cdots > \alpha_1(\Lambda;\Lambda), \tag{4.12}$$

and

$$\alpha_1(\Lambda+1;0) > \alpha_1(\Lambda;0) \quad \textit{definitively, if } k_D(\Lambda) \geq \Lambda^6. \tag{4.13}$$

(D) $\Sigma_{B_0(\Lambda)}$ becomes uniformly dense in $[-1,1]$ as $\Lambda \to \infty$, in particular

$$\lim_{\Lambda\to+\infty} \alpha_1(\Lambda;0) = 1 \quad \textit{and} \quad \alpha_1(\Lambda;0) \geq 1 - \frac{\pi^2}{2(\Lambda+2)^2} \quad \forall \Lambda \geq 2. \tag{4.14}$$

Item *(C)* of last theorem allows also to make a connection between these new localized states and the classical ones because the $\alpha_1(\Lambda;0)$-eigenstate approximates a quantum particle on S^2 concentrated (because of the above equivalence between the $\alpha_1(\Lambda;0)$-eigenstate and the most localized state of the new fuzzy space (see chapter 5) on the North pole and rotating around the x_3-axis; on the other hand, taking a classical particle forced to stay on S^2 and in the position $(0,0,1)$ then it must be

$$L_3 = (\underline{\boldsymbol{L}})_3 = \left(\underline{\boldsymbol{r}} \times \underline{\boldsymbol{p}}\right)_3 = 0,$$

as for this new case.

Note that, the spectrum $\Sigma_{B_0(\Lambda)}$ contains exactly $\Lambda + 1$ eigenvalues and the highest one fulfills (4.12), for this reason the attention is focused only on that matrix.

It is important to point out that the proof of item *(D)* can be trivially rearranged in order to prove that it holds for $\Sigma_{B_m(\Lambda)}$ and $\alpha_1(\Lambda; m)$ also if $m > 0$ is any other fixed integer.

Let $m \in \mathbb{N}_0$ and assume that $\boldsymbol{\chi}_\alpha^m := \sum_{l=m}^{\Lambda} \chi_{\alpha,l}^m \boldsymbol{\psi}_l^m$ is a common eigenstate of x_0 and L_0; let $\{\widetilde{\alpha}_h(\Lambda; m)\}_{h=1}^{\Lambda-m+1}$ be the set of the eigenvalues of $P_{\Lambda-m+1}\left(0, \frac{1}{2}, \frac{1}{2}\right)$ arranged in descending order; according to (4.2) one has

$$\widetilde{\alpha}_h(\Lambda, m) = \cos\left(\frac{h\pi}{\Lambda - m + 2}\right), \qquad h = 1, 2, \cdots, \Lambda - m + 1.$$

One can prove (as for section 8.2.4) that $\alpha_h(\Lambda; m) \simeq \cos\left(\frac{h\pi}{\Lambda-m+2}\right)$ in the limit $\Lambda \to +\infty$, although in this case $c_{l,3}G(l,m,2) \nrightarrow \frac{1}{2}$. On the other hand, when $|m| \ll l$, it is possible to approximate well $c_{l,3}G(l,m,2) \simeq \frac{1}{2}$ in the commutative limit, for this reason it is expected that

$$\chi_{\alpha_h(\Lambda;m),l}^m \simeq \sqrt{\frac{2}{\Lambda - m + 2}} \sin\left(\frac{hl\pi}{\Lambda - m + 2}\right),$$

as for the $D = 2$ case.

4.4 Spectrum of x_i in the $O(D)$-equivariant fuzzy hypersphere when $D > 3$

In this section (which is based on an unpublished article) we do the analysis of the spectrum of x_D, this is not a restriction since the covariance of the algebra under $O(D)$ transformations $\boldsymbol{x} \mapsto \boldsymbol{x}' = R\boldsymbol{x}$, $\boldsymbol{L} \mapsto \boldsymbol{L}' = R\boldsymbol{L}$ implies that the spectra $\Sigma_{x_i}(\Lambda)$ of all coordinate operators x_i of the new fuzzy space are equal; on the other hand, because of

$$[x_D, L_{1,2}] = [x_D, C_3] = \cdots = [x_D, C_d] = 0,$$

it is possible to simultaneously diagonalize x_D and $L_{1,2}, \cdots, C_d$.

Eq. (2.13) and

$$\begin{cases} x_D\boldsymbol{\chi_\alpha} = \alpha_1\boldsymbol{\chi_\alpha} \\ L_{1,2}\boldsymbol{\chi_\alpha} = \alpha_2\boldsymbol{\chi_\alpha} \\ C_3\boldsymbol{\chi_\alpha} = \alpha_3\boldsymbol{\chi_\alpha} \\ \dots \\ C_d\boldsymbol{\chi_\alpha} = \alpha_d\boldsymbol{\chi_\alpha} \end{cases}, \tag{4.15}$$

imply

$$\begin{aligned} &\alpha_d = l_{d-1} \in \{0, 1, \cdots, \Lambda-1, \Lambda\} \\ &\quad \dots \\ &\alpha_3 = l_2 \in \{0, 1, \cdots, l_3-1, l_3\} \\ &\alpha_2 = l_1 \in \{-l_2, -l_2+1, \cdots, l_2-1, l_2\} \\ \text{and} \quad &\boldsymbol{\chi}_{\alpha,{}_d\boldsymbol{l}} = \sum_{l=l_{d-1}}^{\Lambda} \chi_{\alpha,l,{}_d\boldsymbol{l}}\boldsymbol{\psi}_{l,{}_d\boldsymbol{l}}; \end{aligned} \tag{4.16}$$

so $x_D\boldsymbol{\chi}_{\alpha,{}_d\boldsymbol{l}} = \alpha\boldsymbol{\chi}_{\alpha,{}_d\boldsymbol{l}}$ can be re-written as

$$\begin{cases} \chi_{\alpha,l_{d-1}+1,{}_d\boldsymbol{l}}c_{l_{d-1}+1,D}G(l_{d-1}+1, l_{d-1}, d) = \alpha\chi_{\alpha,l_{d-1},{}_d\boldsymbol{l}} \\ \chi_{\alpha,l_{d-1},{}_d\boldsymbol{l}}c_{l_{d-1}+1,D}F(l_{d-1}, l_{d-1}, d) + \chi_{\alpha,l_{d-1}+2,{}_d\boldsymbol{l}}c_{l_{d-1}+2,D}G(l_{d-1}+2, l_{d-1}, d) = \alpha\chi_{\alpha,l_{d-1}+1,{}_d\boldsymbol{l}} \\ \chi_{\alpha,l_{d-1}+1,{}_d\boldsymbol{l}}c_{l_{d-1}+2,D}F(l_{d-1}+1, l_{d-1}, d) + \chi_{\alpha,l_{d-1}+3,{}_d\boldsymbol{l}}c_{l_{d-1}+3,D}G(l_{d-1}+3, l_{d-1}, d) = \alpha\chi_{\alpha,l_{d-1}+2,{}_d\boldsymbol{l}} \\ \quad\vdots \quad\vdots \quad\vdots \quad\vdots \quad\vdots \quad\vdots \quad\vdots \quad\vdots \\ \chi_{\alpha,\Lambda-2,{}_d\boldsymbol{l}}c_{\Lambda-1,D}F(\Lambda-2, l_{d-1}, d) + \chi_{\alpha,\Lambda,{}_d\boldsymbol{l}}c_{\Lambda,D}G(\Lambda, l_{d-1}, d) = \alpha\chi_{\alpha,\Lambda-1,{}_d\boldsymbol{l}} \\ c_{\Lambda,D}F(\Lambda-1, l_{d-1}, d)\chi_{\alpha,\Lambda-1,{}_d\boldsymbol{l}} = \alpha\chi_{\alpha,\Lambda,{}_d\boldsymbol{l}} \end{cases} \tag{4.17}$$

which in turn can be rewritten in the matrix form $\Theta_{l_{d-1}}(\Lambda)\chi = \alpha\chi$, where $\chi = \left(\chi_{\alpha,l_{d-1},{}_d\boldsymbol{l}}, \chi_{\alpha,l_{d-1}+1,{}_d\boldsymbol{l}}, \dots, \chi_{\alpha,\Lambda,{}_d\boldsymbol{l}}\right)^T$ and $\Theta_{l_{d-1}}(\Lambda)$ is the following $N(\Lambda; l_{d-1}) \times N(\Lambda; l_{d-1})$ symmetric tridiagonal matrix

$$\Theta_{l_{d-1}}(\Lambda) = \begin{pmatrix} 0 & v_{l_{d-1}+1,l_{d-1},D} & 0 & 0 \\ v_{l_{d-1}+1,l_{d-1},D} & 0 & v_{l_{d-1}+2,l_{d-1},D} & 0 \\ 0 & v_{l_{d-1}+2,l_{d-1},D} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & v_{\Lambda,l_{d-1},D} \\ 0 & 0 & v_{\Lambda,l_{d-1},D} & 0 \end{pmatrix},$$

or equivalently $\Xi_{l_{d-1}}(\Lambda;\alpha)\,\chi = 0$, where 0 here is the null vector, with

$$n = n(\Lambda; l_{d-1}) := \Lambda - l_{d-1} + 1, \quad \Xi_{l_{d-1}}(\Lambda;\alpha) := \Theta_{l_{d-1}}(\Lambda) - \alpha I_{n(\Lambda;l_{d-1})},$$

$$\text{and } v_{l,l_{d-1},D} := c_{l,D}G(l, l_{d-1}, d).$$

It is well known that the problem of determining analytically the eigenvalues of a square matrix of large rank is absolutely not trivial, but the $B_{l_{d-1}}(\Lambda)$ have several good properties (for example they are symmetric and tri-diagonal) which will help in studying their spectra. First of all,

Remark 2 *All the eigenvalues of $B_{l_{d-1}}(\Lambda)$ are real.*

Then, as for the dimensions $D=2,3$, one has

Theorem 4.4.1 *(A) If α is an eigenvalue of $B_{l_{d-1}}(\Lambda)$, then also $-\alpha$ is.*

(B) For all Λ, l_{d-1}, all eigenvalues of $B_{l_{d-1}}(\Lambda)$ are simple; they are denoted with $\alpha_1(\Lambda;l_{d-1}), \alpha_2(\Lambda;l_{d-1}),\ ...,\alpha_{n(\Lambda;l_{d-1})}(\Lambda;l_{d-1})$, in decreasing order.

(C) Let $\alpha_1(\Lambda;m)$ be the highest eigenvalue of $B_m(\Lambda)$, then

$$\alpha_1(\Lambda;0)>\alpha_1(\Lambda;1)>\cdots>\alpha_1(\Lambda;\Lambda). \tag{4.18}$$

(D) $\Sigma_{B_0(\Lambda)}$ becomes uniformly dense in $[-1,1]$ as $\Lambda\to\infty$, in particular

$$\lim_{\Lambda\to+\infty}\alpha_1(\Lambda;0)=1. \tag{4.19}$$

Chapter 5

Coherent states

5.1 Preliminaries

5.1.1 Basics about Coherent States

Coherent states (CS) were originally introduced in quantum mechanics on $\mathbb{R}^3$ as states [21, 22, 23] saturating the Heisenberg uncertainty relations (HUR) $\Delta x_i \Delta p_i \geq \hbar/2$ and mapped into each other by the Heisenberg-Weyl group; they make up an overcomplete set yielding a nice resolution of the identity. The latter properties are usually taken as minimal requirements [27] for defining CS in general: a set of CS $\{\phi_l\}_{l\in\Omega}$ is a particular set of vectors of a Hilbert space $\mathcal{H}$, where l is an element of an appropriate (topological) label space Ω, such that the following properties hold:

1. **Continuity**: the vector ϕ_l is a strongly continuous function of the label l.
2. **Resolution of the identity**: there exists on Ω an integration measure such that
$$I = \int_\Omega P_l \, dl, \qquad P_l := \phi_l \langle \phi_l, \cdot \rangle \equiv |\phi_l\rangle\langle\phi_l|; \tag{5.1}$$
3. or, at least, **Completeness**: $\overline{\mathrm{Span}\,\{\phi_l : l \in \Omega\}} = \mathcal{H}$;

the first two properties characterize a strong SCS, while the first and third a weak SCS.

A. M. Perelomov and R. Gilmore develop [24, 45] the concept of CS when Ω is a Lie group G acting on a Hilbert space $\mathcal{H}$ via an unitary irreducible representation T (see e.g. Perelomov's book [31]). Actually, most arguments hold also if the group G is not Lie. Fixed $\boldsymbol{\phi}_0 \in \mathcal{H}$ Perelomov defines $\boldsymbol{\phi}_g := T(g)\boldsymbol{\phi}_0$ and the *coherent-state system* $\{T, \boldsymbol{\phi}_0\}$ as

$$\{T, \boldsymbol{\phi}_0\} := \{\boldsymbol{\phi}_g := T(g)\boldsymbol{\phi}_0 \mid g \in G\}. \tag{5.2}$$

Clearly $\{T, \phi_0\} = \{T, \phi_g\}$ for all $g \in G$. The maximal subgroup H of G formed by elements h fulfilling

$$\phi_h = \exp[i\alpha(h)]\phi_0,$$

with some function $\alpha : H \to \mathbb{R}$, is called the isotropy subgroup for ϕ_0. Clearly, $g' = gh$ implies

$$\phi_{g'} = T(g)T(h)\phi_0 = T(g)\exp[i\alpha(h)]\phi_0 = \exp[i\alpha(h)]\phi_g,$$

i.e. $\phi_{g'}, \phi_g$ belong to the same ray. Therefore equivalence classes $x(g) := \{g' = gh \mid h \in H\}$, i.e. elements of the coset space $X := G/H$, are in one-to-one correspondence with coherent rays, or equivalently with coherent 1-dimensional projections (states): hence one shall denote $P_g := \phi_g\langle\phi_g, \cdot\rangle = P_{g'}$ also as P_x. A left-invariant measure $d\mu(g)$ on G induces an invariant measure dx on X. T is said square-integrable if $I_T \equiv \int_X |\langle\phi_0, T[g(x)]\phi_0\rangle|^2\, dx < \infty$ (this is automatically true if G, or at least X, is compact, because then the volume of X is finite); here $g(x)$ is any (smooth) map from X to G such that $g(x) \in x$ [the result does not depend on the representative element in x because it is invariant under the replacement $g \mapsto gh$; $g(x)$ can be seen as a section of a $U(1)$-fiber bundle on X]. If T is square-integrable then the integral defining the operator $B := \int_X P_x\, dx$ is automatically convergent. From the identities $T(g')P_xT(g'^{-1}) = P_{x'}$ (with $x' := g'x$) and the invariance of dx it follows that $T(g')B\,T(g'^{-1}) = B$, and therefore B is central; then by Schur lemma there is $b \in \mathbb{R}^+$ such that $B = bI$. One can determine b taking the mean value of both sides on ϕ_0; one easily finds $b\langle\phi_0, \phi_0\rangle = I_T$. In general the set $\{\phi_{g(x)}\}_{x\in X}$ is overcomplete (this is certainly the case if X is a *continuum*); one can extract a basis out of it in many different ways. Introducing the normalized integration measure $d\nu(x) := dx/b$ one finds the first resolution of the identity in

$$I = \int_X P_x\, d\nu(x), \qquad I = \int_G P_g\, d\mu'(g); \tag{5.3}$$

the second holds if H has a finite volume h, with $d\mu'(g) := d\mu(g)/bh$, so $\{T, \phi_0\}$ is a strong SCS. In particular, Perelomov applies (chpt. 4 in [31]) these notions to the irreducible representation (π_l, V_l) of $G = SU(2)$ selecting a vector ϕ_0 that minimizes the square dispersion $(\Delta\boldsymbol{L})^2$. As explained in the introduction, one possible such ϕ_0 is the highest weight vector $|l, l\rangle \in V_l$, i.e. the eigenvector of L_3 with the highest eigenvalue l ($L_3|l, m\rangle = m|l, m\rangle$ with $|m| \le l$, in standard ket notation), whereby $\langle L_1\rangle = \langle L_2\rangle = 0$, $(\Delta\boldsymbol{L})^2 = (\Delta\boldsymbol{L})^2_{min} = l$. Therefore these CS coincide with the socalled *coherent spin* [46] or *Bloch* states. By the $SU(2)$ invariance of $(\Delta\boldsymbol{L})^2$, all elements $\phi_g \in \{\pi_l, \phi_0 = |l, l\rangle\}$ - including $|l, -l\rangle \sim T(e^{i\pi L_1})|l, l\rangle$ - have the same minimal dispersion. As the isotropy subgroup H is that $SO(2)$ of rotations $e^{i\varphi L_3}$ around the $\vec{z}$-axis, the states associated with this system are in one-to-one correspondence with the points of $SO(3)/SO(2) = S^2$. The latter sphere can be considered as the phase manifold for spin (angular momentum); these coherent states are the closest to the classical ones on such a sphere. Applying the rescaling (2) one immediately finds that also in the Madore FS the space uncertainty is minimal on the $|\boldsymbol{\phi}_g\rangle$'s and equal to (13).

Out of the ϕ_g's only the vectors proportional to $|l, \pm l\rangle$ saturate (i.e. satisfy as equalities) for all i, j the uncertainty relations $\Delta L_i \, \Delta L_j \geq |\varepsilon^{ijk}\langle L_k\rangle|/2$, which follow from the commutation relation $[L_i, L_j] = i\varepsilon^{ijk} L_k$ (on them one has in addition $\langle L_1\rangle = \langle L_2\rangle = 0 = \Delta L_3$, $|\langle L_3\rangle| = l$, $\Delta L_1 = \Delta L_2 = \sqrt{l/2}$). Incidentally, the authors in Ref. [47] consider also two alternative definitions of sets of optimally localized states: the set of 'intelligent states', that saturate the uncertainty relation $\Delta L_1 \, \Delta L_2 \geq |\langle L_3\rangle|/2$, and the set of 'minimum uncertainty states', for which $\Delta L_1 \, \Delta L_2$ has a local minimum (note that then in general $\Delta L_1 \, \Delta L_3$, $\Delta L_2 \, \Delta L_3$ are not minimized). But neither one is invariant under arbitrary rotation, in contrast with the definition of Perelomov and of the present thesis; one can easily show (see e.g. [22] pp. 27-28) that these states are 'fewer' than the points of S^2, i.e cannot be put in one-to-one correspondence with the points of S^2, but just of a finite number of lines on S^2.

5.1.2 Uncertainty relations and coherent states on commutative S^1

Let x_1, x_2 be Cartesian coordinates on $\mathbb{R}^2$, $\partial_i \equiv \partial/\partial x_i$, $L = -i(x_1\partial_2 - x_2\partial_1)$ be the angular momentum operator up to $\hbar$. From $[L, x_1] = ix_2$, $[L, x_2] = -ix_1$ one derives in the standard way the uncertainty relations (UR)

$$(\Delta L)^2(\Delta x_1)^2 \geq \frac{1}{4}\langle x_2\rangle^2, \qquad (\Delta L)^2(\Delta x_2)^2 \geq \frac{1}{4}\langle x_1\rangle^2, \qquad (\Delta L)^2(\Delta \boldsymbol{x})^2 \geq \frac{1}{4}\langle \boldsymbol{x}\rangle^2 \tag{5.4}$$

the third inequality is obtained summing the first two. These commutation relations and UR hold not only for the operators on $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2)$, but also for those on $\mathcal{H} = \mathcal{L}^2(S^1)$. In the latter case the x_i fulfill the constraint $\boldsymbol{x}^2 \equiv x_1^2 + x_2^2 = 1$, or equivalently $x_+x_- = 1$, where $x_\pm := x_1 \pm ix_2$, whence $(x_+)^{-n} = (x_-)^n$, and the third inequality represents a lower bound for the dispersion $\Delta L \, |\Delta \boldsymbol{x}|$ in phase space; L is the momentum along the circle. The inequalities (5.4) are therefore the analog [48] on the circle of the Heisenberg UR (it is important to underline that adopting the azimuthal angle φ as the observable canonically conjugate to L, $[\varphi, L] = i$, would be inconsistent). The orthonormal basis $\mathcal{B} := \{\boldsymbol{\psi}_n\}_{n\in\mathbb{Z}}$ of $\mathcal{L}^2(S^1)$, $\sqrt{2\pi}\boldsymbol{\psi}_n := e^{in\varphi} = (x_+)^n$ consists of eigenvectors of L, $L\psi_n = n\psi_n$, while $x_\pm$ acta as ladder operators: $x_\pm\psi_n = \boldsymbol{\psi}_{n\pm1}$. These relations characterize the basic[1] unitary irreducible representation T of the $*$-algebra $\mathcal{A}$ of observables generated by $L, x_\pm$ fulfilling $[L, x_\pm] = \pm x_\pm$, $x_+x_- = x_-x_+ = 1$, $L^\dagger = L$, $x_+^\dagger = x_-$. The $\boldsymbol{\psi}_n$ saturate the inequalities (5.4), because on them $(\Delta L)^2 = \langle x_1\rangle = \langle x_2\rangle = 0$,

[1]The inequivalent unitary irreducible representation of $\mathcal{A}$ are parametrized by $\alpha \in [0, 2\pi[$, entering $L\psi_n = (n + \alpha)\boldsymbol{\psi}_n$.

while $(\Delta x_i)^2 = 1/2$; in appendix 9.3 it is shown that in fact these are the only states saturating (5.4). The decomposition of the identity associated to $\mathcal{B}$ (first equality)

$$I = \sum_n P_n = \int_{G/H} P_x \, d\mu(x), \qquad P_n = \boldsymbol{\psi}_n \langle \boldsymbol{\psi}_n, \cdot \rangle \tag{5.5}$$

thus involves all and only the states saturating (5.4), i.e. is of the type (5.1) with labels $n \in \Omega \equiv \mathbb{Z}$; the second equality is explained once noted that $\mathcal{H} = \mathcal{L}^2(S^1)$ carries a unitary irreducible representation of the group

$$G := \{(x_+)^n e^{i(aL+b)} \,|\, (a,b,n) \in \mathbb{R}^2 \times \mathbb{Z}\} \simeq U(1) \times U(1) \ltimes \mathbb{Z} \tag{5.6}$$

(consisting of $*$-automorphisms of the algebra of observables) with product rule

$$(x_+)^n e^{i(aL+b)} \, (x_+)^{n'} e^{i(a'L+b')} = (x_+)^{n+n'} e^{i[(a+a')L+(b+b'+an')]};$$

$e^{iaL} \boldsymbol{\psi}(\varphi) = \boldsymbol{\psi}(\varphi + a)$, i.e. e^{iaL} is the translation operator along the circle (it rotates φ by an angle a), while $x_\pm \psi_m = \boldsymbol{\psi}_{m\pm1}$, i.e. $x_\pm$ act as discretized boost operators in the (anti)clockwise direction. G acts transitively on the set of states saturating the HUR (5.4), i.e. the eigenvectors of L. $H = \{e^{i(aL+b)}\} \simeq [U(1)]^2$ is the isotropy subgroup of $\boldsymbol{\psi}_0$ (and of all other $\boldsymbol{\psi}_n$), and $G/H = \{(x_+)^n \,|\, n \in \mathbb{Z}\}$, hence integrating over G/H amounts to summing over $n \in \mathbb{Z}$. In this broader sense $\{T, \boldsymbol{\psi}_0\}$ is a strong SCS.

5.1.3 Uncertainty relations and coherent states on commutative S^2

From the commutation relation $[L_i, L_j] = i\varepsilon^{ijk} L_k$ (for all i,j), valid on $\mathcal{L}^2(\mathbb{R}^3)$ and $\mathcal{L}^2(S^2)$, one derives in the standard way the UR

$$\Delta L_1 \, \Delta L_2 \geq \frac{1}{2} |\langle L_3 \rangle|, \qquad \Delta L_2 \, \Delta L_3 \geq \frac{1}{2} |\langle L_1 \rangle|, \qquad \Delta L_3 \, \Delta L_1 \geq \frac{1}{2} |\langle L_2 \rangle|. \tag{5.7}$$

As already said, the set of coherent spin states within $\mathcal{H} = V_l$ is the subset of states minimizing $(\Delta \boldsymbol{L})^2$. Among them only $|l,l\rangle$, $|l,-l\rangle$ saturate (5.7). Is there some UR which is saturated by all coherent spin states? In appendix 9.1 it is shown not only that the answer is affirmative, but that such a UR is actually l-independent and valid on all of $\mathcal{L}^2(S^2)$:

Theorem 5.1.1 *The following uncertainty relation holds on* $\mathcal{L}^2(S^2) = \bigoplus_{l=0}^{\infty} V_l$

$$(\Delta \boldsymbol{L})^2 \geq |\langle \boldsymbol{L} \rangle| \qquad \Leftrightarrow \qquad \langle \boldsymbol{L}^2 \rangle \geq |\langle \boldsymbol{L} \rangle| \, (|\langle \boldsymbol{L} \rangle| + 1) \,, \tag{5.8}$$

and is saturated by the spin coherent states $\boldsymbol{\phi}_{l,g} = \pi_l(g)|l,l\rangle \in V_l \subset \mathcal{L}^2(S^2)$, $g \in SO(3)$, $l \in \mathbb{N}_0$.

Remarks:

1. The theorem seems to be new, albeit the proof is very simple. One cannot obtain inequality (5.19) directly from (5.7) or the Robertson inqualities[2].

2. Summing Perelomov's resolutions of the identities for all V_l, the result is the resolution of the identity for $\mathcal{L}^2(S^2)$

$$I=\sum_{l=0}^{\infty}C_l\int_{SO(3)}d\mu(g)\,P_{l,g},\qquad P_{l,g}=\boldsymbol{\phi}_{l,g}\langle\boldsymbol{\phi}_{l,g},\cdot\rangle,\quad C_l=\frac{2l+1}{8\pi^2},\quad \boldsymbol{\phi}_{l,g}:=T(g)Y_l^l; \tag{5.9}$$

this holds also integrating over S^2 [instead of $SO(3)$] and replacing $C_l\mapsto 2\pi C_l$.

From the commutation relation $[L_i,x_j]=i\varepsilon^{ijk}x_k$ (for all i,j), valid on $\mathcal{L}^2(\mathbb{R}^3)$, and $\mathcal{L}^2(S^2)$, one derives in the standard way the UR

$$\begin{aligned}
\Delta L_1\,\Delta x_2\geq\frac{1}{2}|\langle x_3\rangle|,\qquad \Delta L_1\,\Delta x_3\geq\frac{1}{2}|\langle x_2\rangle|,\\
\Delta L_2\,\Delta x_1\geq\frac{1}{2}|\langle x_3\rangle|,\qquad \Delta L_2\,\Delta x_3\geq\frac{1}{2}|\langle x_1\rangle|,\\
\Delta L_3\,\Delta x_1\geq\frac{1}{2}|\langle x_2\rangle|,\qquad \Delta L_3\,\Delta x_2\geq\frac{1}{2}|\langle x_1\rangle|.
\end{aligned} \tag{5.10}$$

Relations (5.10) are analogs of the Heisenberg UR (HUR), as the L_i are the 'momentum' components along the sphere. Alternative ones can be found e.g. in [49]. In the literature it is not easy to find works investigating whether they can be saturated.

5.2 Coherent and localized states on the fuzzy circle S^1_Λ

5.2.1 $O(2)$-invariant UR and CS systems on S^1_Λ

First of all, since relations (3.3)-(3.1) are as in the commutative case, the 'Heisenberg' UR (5.4) hold, the eigenvectors $\boldsymbol{\psi}_n$ of L make up again a set of states saturating (5.4), because on them $(\Delta L)^2=\langle x_1\rangle=\langle x_2\rangle=0$, while

$$(\Delta x_i)^2=\begin{cases}\frac{1}{2}\left(1+\frac{n^2}{k_D}\right),\\ \frac{1}{4}\left[1+\frac{\Lambda(\Lambda-1)}{k_D}\right],\end{cases}\qquad (\Delta\boldsymbol{x})^2=\begin{cases}1+\frac{n^2}{k_D} & \text{if } |n|<\Lambda,\\ \frac{1}{2}\left[1+\frac{\Lambda(\Lambda-1)}{k_D}\right] & \text{if } |n|=\Lambda.\end{cases}$$

[2] Using (5.7) one can obtain the weaker inequality $(\Delta\boldsymbol{L})^2\geq|\langle\boldsymbol{L}\rangle|\sqrt{3/4}$: (5.7) implies the inequalities $2\Delta L_1^2\,\Delta L_2^2\geq\langle L_3\rangle^2/2$, $(\Delta L_1^4+\Delta L_2^4)/2\geq\langle L_3\rangle^2/4$ and the ones obtained permuting $1,2,3$ cyclically; summing all of them, one has $(\Delta\boldsymbol{L})^4\geq\langle\boldsymbol{L}\rangle^2|3/4$.

The first resolution of the identity in (5.5) still holds,

$$I=\sum_n P_n=\int_{G/H} P_x\, d\mu(x), \qquad P_n=\boldsymbol{\psi}_n\langle\boldsymbol{\psi}_n,\cdot\rangle, \tag{5.11}$$

provided n runs over $\Omega\equiv\{-\Lambda,1-\Lambda,...,\Lambda\}$ instead of $\mathbb{Z}$. For the second one to be valid one should replace $\mathbb{Z}$ by $\mathbb{Z}_{2\Lambda+1}$ in the definition (5.6) of G, more precisely replace $(x_+)^n$ by u^n, where the unitary operator u is defined by $u\psi_\Lambda=\boldsymbol{\psi}_{-\Lambda}$, $u\psi_n=\boldsymbol{\psi}_{n+1}$ otherwise. Such a G is a subgroup of the group of $*$-automorphisms of $\mathcal{A}_\Lambda$. In appendix 9.3 it is shown that in $\mathcal{H}_\Lambda$ again only the $\boldsymbol{\psi}_n$ saturate all of the inequalities of (5.4). Nevertheless, there is a whole family (parametrized by $\mu\in\mathbb{R}$) of complete sets of states saturating $(5.4)_1$ alone. These states are eigenvectors of $a_1^\mu:=L-i\mu x_1$ (They are explicitly determined for $\Lambda=1$), and the family interpolates between the set of eigenvectors of L and the set of eigenvectors of x_1.

In the commutative case the spacial uncertainties $\Delta x_1,\Delta x_2$ can be simultaneously as small as one wishes. In the fuzzy case even the Robertson UR

$$4\left(\Delta x_1\right)^2\left(\Delta x_2\right)^2\geq\langle L'\rangle^2+\langle x_1x_2+x_2x_1\rangle^2, \qquad L':=-\frac{L}{k}{}_D+\left[1+\frac{\Lambda(\Lambda+1)}{k}{}_D\right]\frac{\widetilde{P}_\Lambda-\widetilde{P}_{-\Lambda}}{2},$$

which follow from $(3.1)_1$ and is slightly stronger than the Schrödinger UR, is not particularly stringent, in that the right-hand side vanishes on a large class of states[3], hence does not exclude that either Δx_1 or Δx_2 vanish. However, the latter cannot vanish simultaneously, because $(\Delta\boldsymbol{x})^2$ is bounded from below (see section 5.2.2).

In the following lines (5.2) is applied adopting $T=\pi_\Lambda$ and as a G not $SO(3)$ (the largest Λ-independent subgroup of the group of $*$-automorphism of $\mathcal{A}_\Lambda$), but its subgroup $G=SO(2)$; hence $\mathcal{H}_\Lambda$ carries a *reducible* representation of G, so that completeness and resolution of the identity are not automatic. Consider a generic unit vector $\boldsymbol{\omega}=\sum_{m=-\Lambda}^{\Lambda}\omega_m\psi_m$ and let

$$\boldsymbol{\omega}_\alpha:=e^{i\alpha L}\boldsymbol{\omega}=\sum_{m=-\Lambda}^{\Lambda}e^{i\alpha m}\omega_m\psi_m, \qquad P_\alpha:=\boldsymbol{\omega}_\alpha\langle\boldsymbol{\omega}_\alpha,\cdot\rangle,$$

$(\boldsymbol{\omega}_0\equiv\boldsymbol{\omega})$. The system $A:=\{\boldsymbol{\omega}_\alpha\}_{\alpha\in[0,2\pi[}$ is complete provided $\omega_m\neq 0$ for all m (then it is also overcomplete). Defining $B:=\int_0^{2\pi}d\alpha\, P_\alpha$ one finds

$$B\psi_n=\overline{\omega_n}\int_0^{2\pi}\boldsymbol{\omega}_\alpha e^{-i\alpha n}\,d\alpha=\overline{\omega_n}\sum_{m=-\Lambda}^{\Lambda}\omega_m\psi_m\int_0^{2\pi}e^{i\alpha(m-n)}\,d\alpha=2\pi|\omega_n|^2\psi_n,$$

[3]In fact, on the generic vector $\boldsymbol{\chi}=\sum_{m=-\Lambda}^{\Lambda}\chi_m\psi_m$ one finds $\langle L'\rangle_{\boldsymbol{\chi}}=\sum_{m=1}^{\Lambda-1}[|\chi_{-m}|^2-|\chi_m|^2]m/k_D+[|\chi_\Lambda|^2-|\chi_{-\Lambda}|^2][1/2+\Lambda(\Lambda-1)/2k_D]$, which vanishes e.g. if $|\chi_{-m}|=|\chi_m|$ for all m, and $\langle x_1x_2+x_2x_1\rangle=\langle x_+^2-x_-^2\rangle/2i$. which vanishes if e.g. all $\chi_m\in\mathbb{R}$, so that $\langle x_+^2\rangle$ is real.

implying $B = \sum_{n=-\Lambda}^{\Lambda} 2\pi|\omega_n|^2 \widetilde{P}_n$; this is proportional to the identity only if $|\omega_n|^2$ is independent of n and therefore (since $\boldsymbol{\omega}$ is normalized) if $|\omega_n|^2 = 1/(2\Lambda+1)$. Setting $\omega_n = e^{i\beta_n}/\sqrt{2\Lambda+1}$ one finds the following resolutions of the identity, parametrized by $\beta \in (\mathbb{R}/2\pi\mathbb{Z})^{2\Lambda+1}$:

$$I = \frac{2\Lambda+1}{2\pi}\int_0^{2\pi} d\alpha\, P_\alpha^\beta, \qquad P_\alpha^\beta := \boldsymbol{\omega}_\alpha^\beta \langle \boldsymbol{\omega}_\alpha^\beta, \cdot\rangle, \qquad \boldsymbol{\omega}_\alpha^\beta := \sum_{m=-\Lambda}^{\Lambda} \frac{e^{i(\alpha m+\beta_m)}}{\sqrt{2\Lambda+1}} \boldsymbol{\psi}_m. \tag{5.12}$$

By choosing $\beta_{-m} = \beta_m$ the strong SCS $\{\boldsymbol{\omega}_\alpha^\beta\}$ is fully $O(2)$-equivariant, because is mapped into itself also by the unitary transformation $\boldsymbol{\psi}_m \mapsto \boldsymbol{\psi}_{-m}$ that corresponds to the transformation of the coordinates (with determinant -1) $(x_1, x_2) \mapsto (x_1, -x_2)$. What is the β minimizing $(\Delta \boldsymbol{x})^2$? In appendix 9.2 it is shown that on the states $\boldsymbol{\omega}_\alpha^\beta$

$$\langle L\rangle = 0, \qquad (\Delta L)^2 = \langle L^2\rangle = \frac{\Lambda(\Lambda+1)}{3} \quad \text{for all } \alpha, \beta, \tag{5.13}$$

$$\langle \boldsymbol{x}^2\rangle \le \frac{2\Lambda}{2\Lambda+1} + \frac{2(\Lambda-1)\Lambda(\Lambda+1)}{3(2\Lambda+1)k_D}, \qquad \langle x_+\rangle = \frac{e^{-i\alpha}}{2\Lambda+1}\sum_{m=1-\Lambda}^{\Lambda} e^{i(\beta_{m-1}-\beta_m)} b_m. \tag{5.14}$$

Therefore $\langle \boldsymbol{x}\rangle^2 = |\langle x_+\rangle|^2$ is maximal, and $(\Delta \boldsymbol{x})^2 = \langle \boldsymbol{x}^2\rangle - \langle \boldsymbol{x}\rangle^2$ is minimal, if $\beta = 0$; then

$$\langle x_+\rangle_{\boldsymbol{\phi}_\alpha} = \frac{2\, e^{-i\alpha}}{2\Lambda+1}\sum_{m=1}^{\Lambda} b_m, \qquad (\Delta \boldsymbol{x})^2 < \frac{1}{\Lambda+1}\left(\frac{1}{2}+\frac{1}{3\Lambda}\right) \overset{\Lambda\ge 2}{\le} \frac{2}{3(\Lambda+1)} \tag{5.15}$$

where $\boldsymbol{\phi}_\alpha := \boldsymbol{\omega}_\alpha^0$; in particular $\langle x_2\rangle_{\boldsymbol{\phi}} = 0$, $\langle x_1\rangle_{\boldsymbol{\phi}} = \langle x_+\rangle_{\boldsymbol{\phi}} \in \mathbb{R}$, where $\boldsymbol{\phi} := \boldsymbol{\phi}_0 = \boldsymbol{\omega}_0^0$. The corresponding strong SCS is denoted with $\mathcal{S}^1 := \{\boldsymbol{\phi}_\alpha\}_{\alpha\in[0,2\pi[}$.

The $\boldsymbol{\omega}_\alpha^\beta$ have no limit in $\mathcal{L}^2(S^1)$ as $\Lambda \to \infty$, since all their components in the canonical basis $\{\boldsymbol{\psi}_n\}_{n\in\mathbb{Z}}$ go to zero; the renormalized $\sqrt{2\Lambda+1}\boldsymbol{\phi}_\alpha/2\pi$ have at least a limit in the space of distributions, more precisely go to δ_α, where δ_α is the Dirac δ on the circle centered at angle $\varphi = \alpha$.

5.2.2 $O(2)$-invariant overcomplete set of states minimizing $(\Delta \boldsymbol{x})^2$

As $(\Delta \boldsymbol{x})^2$ is $O(2)$-invariant, so is the set $\mathcal{W}^1$ of states on S^1_Λ minimizing $(\Delta \boldsymbol{x})^2$. Therefore one can first look for a state $\underline{\boldsymbol{\chi}} \in \mathcal{W}^1$ such that $\langle x_2\rangle = 0$, and then recover the whole $\mathcal{W}^1$ as $\mathcal{W}^1 = \{\underline{\boldsymbol{\chi}}_\alpha := e^{i\alpha L}\widehat{\boldsymbol{\chi}} \,|\, \alpha \in [0, 2\pi[\}$. This is an $O(2)$-invariant, overcomplete set of states (i.e. a weak SCS) in one-to-one correspondence with the points of the circle. The determination in closed form of $\underline{\boldsymbol{\chi}}, \mathcal{W}^1$ for general Λ is presumably not possible. Since it is $\boldsymbol{x}^2 = 1 + O(1/\Lambda^2)$ (except on $\boldsymbol{\psi}_{\pm\Lambda}$), it is reasonable to think that the eigenstate $\widehat{\boldsymbol{\chi}}$ of x_1 with highest eigenvalue (or the eigenstate with opposite eigenvalue) approximates $\underline{\boldsymbol{\chi}}$ at order $O(1/\Lambda^2)$. But also the determination in closed form of such an eigenvector is presumably not possible. Here $\underline{\boldsymbol{\chi}}, \widehat{\boldsymbol{\chi}}$ are explicitely determined for $\Lambda = 1$, while for general Λ it is calculated a set of states having a smaller $(\Delta\boldsymbol{x})^2$ than that of the $\boldsymbol{\phi}_\alpha$ of the previous subsection, more precisely going to zero as $1/\Lambda^2$; this is done with the help of the results of chapter 4, where a detailed study of the x_i-eigenvalue problem is carried out.

When $\Lambda = 1$ normalized eigenvectors and eigenvalues of x_1 are given by

$$\boldsymbol{\chi}_0 = \frac{\boldsymbol{\psi}_{-1} - \boldsymbol{\psi}_1}{\sqrt{2}}, \quad x_1\boldsymbol{\chi}_0 = 0, \qquad \boldsymbol{\chi}_\pm = \frac{\boldsymbol{\psi}_{-1} \pm \sqrt{2}\boldsymbol{\psi}_0 + \boldsymbol{\psi}_1}{2}, \quad x_1\boldsymbol{\chi}_\pm = \pm\frac{\sqrt{2}}{2}\boldsymbol{\chi}_\pm \tag{5.16}$$

One easily checks that on $\widehat{\boldsymbol{\chi}} \equiv \boldsymbol{\chi}_+$ it is $\langle \boldsymbol{x}^2 \rangle = 3/4$, $\langle x_+ \rangle = \sqrt{2}/2$, and therefore $(\Delta\boldsymbol{x})^2 = 1/4$. On the other hand in section 9.2 it is shown that $(\Delta\boldsymbol{x})^2$ is slightly smaller on $\underline{\boldsymbol{\chi}}$:

$$\underline{\boldsymbol{\chi}} = \frac{\sqrt{5}}{4}\left[\boldsymbol{\psi}_{-1} + \boldsymbol{\psi}_1\right] + \frac{\sqrt{3}}{\sqrt{8}}\boldsymbol{\psi}_0 \qquad \Rightarrow \qquad (\Delta\boldsymbol{x})^2 = (\Delta\boldsymbol{x})^2_{min} = \frac{7}{32}. \tag{5.17}$$

For general Λ, on the basis $\mathcal{B}_\Lambda$ of $\mathcal{H}_\Lambda$ the operator x_1 is represented by the $(2\Lambda+1) \times (2\Lambda+1)$ matrix

$$x_\Lambda = \frac{1}{2}\begin{pmatrix} 0 & b_\Lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ b_\Lambda & 0 & b_{\Lambda-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{\Lambda-1} & 0 & b_{\Lambda-2} & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_{2-\Lambda} & 0 & b_{1-\Lambda} \\ 0 & 0 & 0 & 0 & \cdots & 0 & b_{1-\Lambda} & 0 \end{pmatrix} = X_0^\Lambda + O\!\left(\frac{1}{\Lambda^2}\right),$$

where $X_0^\Lambda := \frac{1}{2}P_{2\Lambda+1}(0,1,1)$ [see (4.1)]. The spectrum Σ_0^Λ of X_0^Λ is $\{\cos[\pi n/(2\Lambda+2)]\}_{n=1,2,\ldots,2\Lambda+1}$ (see section 4.1); $\Sigma_0^{\Lambda+1}$, Σ_0^Λ interlace, i.e. between any two subsequent eigenvalues in $\Sigma_0^{\Lambda+1}$ there is exactly one in Σ_0^Λ, and Σ_0^Λ becomes uniformly dense in $[-1,1]$ as $\Lambda \to \infty$. In chapter 4 it is shown that the same properties hold true also for $x_\Lambda \simeq x_1$, by studying its spectrum. Here as a first good estimate of $\widehat{\boldsymbol{\chi}}$ the eigenvector $\boldsymbol{\chi}$ of the Toeplitz matrix X_0^Λ with the maximal eigenvalue $\lambda_M = \cos[\pi/(2\Lambda+2)]$ is taken. The associated $(\Delta\boldsymbol{x})^2_{\boldsymbol{\chi}}$, which is a first good estimate of $(\Delta\boldsymbol{x})^2_{min}$ and goes to zero as $1/\Lambda^2$, fulfills (see appendix 9.2)

$$(\Delta\boldsymbol{x})^2_{\boldsymbol{\chi}} < \frac{3.5}{(\Lambda+1)^2}. \tag{5.18}$$

5.3 Coherent and localized states on the fuzzy sphere S^2_Λ

5.3.1 $O(3)$-invariant UR and CS systems on S^2_Λ

First of all, since the commutation relations $[L_i, L_j] = i\varepsilon^{ijk} L_k$ are as on S^2, then not only the UR (5.7), but also Theorem 5.3.1 and the resolution of the identity (5.9) hold, provided l runs over $\{0, 1, ..., \Lambda\}$ instead of $\mathbb{N}_0$:

Theorem 5.3.1 *The uncertainty relation*

$$(\Delta \boldsymbol{L})^2 \geq |\langle \boldsymbol{L} \rangle| \qquad \Leftrightarrow \qquad \langle \boldsymbol{L}^2 \rangle \geq |\langle \boldsymbol{L} \rangle| \left(|\langle \boldsymbol{L} \rangle| + 1 \right) \tag{5.19}$$

holds on $\mathcal{H}_\Lambda = \oplus_{l=0}^{\Lambda} V_l$ *and is saturated by the spin coherent states* $\boldsymbol{\phi}_{l,g} := \boldsymbol{\pi}_\Lambda(g) \boldsymbol{\psi}_l^l \in V_l$, $l \in \{0, 1, ..., \Lambda\}$, $g \in SO(3)$. *Moreover on* $\mathcal{H}_\Lambda$ *the following resolution of identity holds:*

$$I = \sum_{l=0}^{\Lambda} C_l \int_{SO(3)} d\mu(g) P_{l,g}, \qquad C_l = \frac{2l+1}{8\pi^2}, \qquad P_{l,g} = \boldsymbol{\phi}_{l,g} \langle \boldsymbol{\phi}_{l,g}, \cdot \rangle. \tag{5.20}$$

It is possible to parametrize $g \in SO(3)$, the invariant measure and the integral over $SO(3)$ through the Euler angles $\varphi, \theta, \boldsymbol{\psi}$:

$$g = e^{\varphi I_3} e^{\theta I_2} e^{\psi I_3} \quad \text{where } I_3 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_2 := \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{5.21}$$

$$\boldsymbol{\pi}_\Lambda(g) = e^{i\varphi L_3} e^{i\theta L_2} e^{i\psi L_3}, \qquad \int\limits_{SO(3)} d\mu(g) = \int\limits_0^{2\pi} d\varphi \int\limits_0^{\pi} d\theta \sin\theta \int\limits_0^{2\pi} d\psi = 8\pi^2. \tag{5.22}$$

Since the commutation relations $[L_i, x_j] = i\varepsilon^{ijk} x_k$ hold also on S^2_Λ, so do the UR (5.10). However here it is not investigatd whether they (or some alternative ones) can be saturated, because seems to be this is not known even for the commutative S^2.

In the commutative case the spacial uncertainties $\Delta x_1, \Delta x_2, \Delta x_3$ can be simultaneously as small as one wishes, because $[x_i, x_j] = 0$. In the fuzzy case even the Robertson UR

$$4 \left(\Delta x_1 \right)^2 \left(\Delta x_2 \right)^2 \geq \langle L_3' \rangle^2 + \langle x_1 x_2 + x_2 x_1 \rangle^2, \qquad L_3' := \left(\frac{I}{k_D} - k_D \widetilde{P}_\Lambda \right) L_3,$$

and its permutations, which follow from (3.8) and are slightly stronger than the Schrödinger UR, are not particularly stringent, in that the right-hand side vanishes on a large class of states[4], hence does not exclude that either $\Delta x_1, \Delta x_2$ or Δx_3 vanish. However, in the next lines it is shown that they cannot vanish simultaneously, because $(\Delta \boldsymbol{x})^2$ is bounded from below (see section 5.3.2). Summing the Schrödinger UR

$$\begin{aligned} & \frac{(\Delta x_1)^4 + (\Delta x_2)^4}{2} \geq (\Delta x_1)^2 (\Delta x_2)^2 \geq \frac{\langle L_3' \rangle^2}{4} \\ \Rightarrow \quad & \frac{(\Delta x_1)^4 + (\Delta x_2)^4}{2} + 2 (\Delta x_1)^2 (\Delta x_2)^2 \geq \frac{3}{4} \langle L_3' \rangle^2, \end{aligned}$$

and the ones with permuted indices $1, 2, 3$ one finds the $O(3)$-invariant UR

$$(\Delta \boldsymbol{x})^4 \geq \frac{3}{4} \langle \boldsymbol{L}' \rangle^2. \tag{5.23}$$

Note that on the eigenstates of $x_0, L_0 \equiv L_3$, with $L_0 = m$ it is $\langle L'_\pm \rangle = 0$ and $|\langle \boldsymbol{L}' \rangle| = |\langle L_3' \rangle| = |m| \left(1/k_D - k_D \langle \widetilde{P}_\Lambda \rangle\right)$; in particular for $m = 0$ the right-hand side of (5.23) is zero. It is left for possible future investigation to determine the states, if any, saturating the UR (5.23); clearly there can be no saturation on a state such that $\langle L_3' \rangle = 0$, because as said $(\Delta \boldsymbol{x})^2$ has a positive minimum.

In the nest lines (5.2) is applied adopting as a G not $SO(4)$ (the largest Λ-independent subgroup of the group of $*$-automorphism of $\mathcal{A}_\Lambda$), but its subgroup $G = SO(3)$ with Lie algebra spanned by the L_i, and $T = \boldsymbol{\pi}_\Lambda$. By (2.34), $(\mathcal{H}_\Lambda, \boldsymbol{\pi}_\Lambda)$ is a *reducible* representation of G, more precisely the direct sum of the irreducible representations (V_l, π_l), $l = 0, ..., \Lambda$; therefore completeness and resolution of the identity are not automatic. Fixed a normalized vector $\boldsymbol{\omega} \in \mathcal{H}_\Lambda$, for $g \in G$ let

$$\boldsymbol{\omega}_g := \boldsymbol{\pi}_\Lambda(g) \boldsymbol{\omega}, \qquad\qquad P_g := \boldsymbol{\omega}_g \langle \boldsymbol{\omega}_g, \cdot \rangle. \tag{5.24}$$

The system $A := \{\boldsymbol{\omega}_g\}_{g \in G}$ is complete provided that for all l there exists at least one h such that $\omega_l^h \neq 0$ (then it is also overcomplete). In appendix 9.4 we do the proof of the following

Theorem 5.3.2 *If* $\boldsymbol{\omega} = \sum\limits_{l=0}^{\Lambda} \sum\limits_{h=-l}^{l} \omega_l^h \boldsymbol{\psi}_l^h$ *fulfills*

$$\sum_{h=-l}^{l} |\omega_l^h|^2 = \frac{2l+1}{(\Lambda+1)^2}, \qquad l = 0, 1, ..., \Lambda, \tag{5.25}$$

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[4]In fact, on the generic vector $\boldsymbol{\chi} = \sum\limits_{l=0}^{\Lambda} \sum\limits_{m=-l}^{l} \chi_l^m \psi_l^m$ one finds $\langle L_3' \rangle_{\boldsymbol{\chi}} = \sum\limits_{l=0}^{\Lambda-1} \sum\limits_{m=1}^{l} [|\chi_l^{-m}|^2 - |\chi_l^m|^2] \frac{m}{k} + \frac{1+\frac{\Lambda^2}{k_D}}{2\Lambda+1} \sum\limits_{m=1}^{\Lambda} [|\chi_\Lambda^{-m}|^2 - |\chi_\Lambda^m|^2]$, which vanishes e.g. if $|\chi_l^{-m}| = |\chi_l^m|$ for all l, m, and $\langle x_1 x_2 + x_2 x_1 \rangle = \langle x_+^2 - x_-^2 \rangle / 2i$, which vanishes if e.g. all $\chi_l^m \in \mathbb{R}$, so that $\langle x_+^2 \rangle$ is real.

then the following resolution of the identity on $\mathcal{H}_\Lambda$ holds:

$$I=\frac{(\Lambda+1)^2}{8\pi^2}\int_{SO(3)}d\mu(g)\,P_g,\qquad P_g:=\boldsymbol{\omega}_g\langle\boldsymbol{\omega}_g,\cdot\rangle,\qquad \boldsymbol{\omega}_g:=\boldsymbol{\pi}_\Lambda(g)\boldsymbol{\omega}. \tag{5.26}$$

If $\omega_l^h=\omega_l^{-h}$ the strong SCS $\{\boldsymbol{\omega}_g\}_{g\in SO(3)}$ is fully $O(3)$-equivariant.

In particular, choosing $\boldsymbol{\omega}=\boldsymbol{\omega}^\beta:=\sum_{l=0}^\Lambda\boldsymbol{\psi}_l^l e^{i\beta_l}\sqrt{2l{+}1}/(\Lambda+1)$ one finds a family of strong SCS $\{\boldsymbol{\omega}_g^\beta\}_{g\in SO(3)}$ and associated resolutions of the identity parametrized by $\beta\equiv(\beta_0,...,\beta_\Lambda)\in(\mathbb{R}/2\pi\mathbb{Z})^{\Lambda+1}$. In appendix 9.6 the uncertainties $(\Delta\boldsymbol{L})^2,(\Delta\boldsymbol{x})^2$ are calculated on this strong SCS; the first is independent of β,g, the second is minimal if $\beta=0$. Then they are given by

$$(\Delta\boldsymbol{L})^2=\frac{\Lambda(2\Lambda^3{+}32\Lambda^2{+}65\Lambda{+}36)}{36(\Lambda{+}1)^2},\qquad\qquad (\Delta\boldsymbol{x})^2<\frac{3}{\Lambda+1}. \tag{5.27}$$

It is possible to construct a strong SCS with a larger $(\Delta\boldsymbol{L})^2$ and a smaller $(\Delta\boldsymbol{x})^2$. Choosing $\boldsymbol{\omega}=\boldsymbol{\phi}^\beta=\sum_{l=0}^\Lambda\boldsymbol{\psi}_l^0e^{i\beta_l}\sqrt{2l{+}1}/(\Lambda{+}1)$ [this is suggested by the arguments following (5.23) and the ones of next subsection] one again finds a family of strong SCS and associated resolutions of the identity parametrized by $\beta\equiv(\beta_0,...,\beta_\Lambda)\in(\mathbb{R}/2\pi\mathbb{Z})^{\Lambda+1}$. This SCS is fully $O(3)$-equivariant. Since $\boldsymbol{\phi}^\beta$ are eigenvectors of L_3 (actually with zero eigenvalue), the isotropy group $H=\{e^{i\psi L_3}\,|\,\psi\in\mathbb{R}\}\simeq SO(2)$ is nontrivial, and the resolution of the identity holds also with the integral extended over just the coset space $S^2\simeq SO(3)/SO(2)\ni g=e^{\varphi I_3}e^{i\theta I_2}$:

$$\begin{aligned}I=\frac{(\Lambda+1)^2}{4\pi}\int_0^{2\pi}\!d\varphi\int_0^{\pi}\!d\theta\,\sin\theta\,P_g^\beta,\qquad & P_g^\beta=\boldsymbol{\phi}_g^\beta\langle\boldsymbol{\phi}_g^\beta,\cdot\rangle,\\ \boldsymbol{\phi}_g^\beta=\sum_{l=0}^{\Lambda}\frac{e^{i\beta_l}\sqrt{2l{+}1}}{\Lambda{+}1}&\boldsymbol{\pi}_\Lambda(g)\boldsymbol{\psi}_l^0.\end{aligned} \tag{5.28}$$

In the appendix the uncertainties $(\Delta\boldsymbol{L})^2,(\Delta\boldsymbol{x})^2$ are calculated on the SCS $\{\boldsymbol{\phi}_g^\beta\}_{g\in G}$; this is the analog of the SCS (5.12-5.15). Again $(\Delta\boldsymbol{x})^2$ is smallest if $\beta=0$. Correspondingly, one finds

$$(\Delta\boldsymbol{L})^2=\frac{\Lambda(\Lambda{+}2)}{2},\qquad\qquad (\Delta\boldsymbol{x})^2<\frac{1}{\Lambda+1}. \tag{5.29}$$

5.3.2 $O(3)$-invariant overcomplete set of states minimizing $(\Delta\boldsymbol{x})^2$

As $(\Delta\boldsymbol{x})^2$ is $O(3)$-invariant, so is the set $\mathcal{W}^2$ of states on S^2_Λ minimizing $(\Delta\boldsymbol{x})^2$. Arguing as in the introduction, one can first look for the states $\underline{\boldsymbol{\chi}}\in\mathcal{W}^2$ on which $\langle x_0\rangle=|\langle\boldsymbol{x}\rangle|$ [whence $\langle x_\pm\rangle=0$, $(\Delta\boldsymbol{x})^2=\langle\boldsymbol{x}^2\rangle-\langle x_0\rangle^2$], and then recover the whole $\mathcal{W}^2$ as $\mathcal{W}^2=\{\underline{\boldsymbol{\chi}}_g:=\boldsymbol{\pi}_\Lambda(g)\underline{\boldsymbol{\chi}}\,|\,g\in SO(3)\}$. Presumably it is not possible to determine the most localized state $\underline{\boldsymbol{\chi}}^2$ in closed form for general Λ. Since eq. (3.9) implies that $\boldsymbol{x}^2 \geq \frac{1}{2}$ on the $\boldsymbol{L}^2 = \Lambda(\Lambda+1)$ eigenspace and $\boldsymbol{x}^2 = 1+O(1/\Lambda^2)$ on the orthogonal complement, $(\Delta\boldsymbol{x})^2 = \langle\boldsymbol{x}^2\rangle - \langle x_0\rangle^2$ on the eigenvector $\widehat{\boldsymbol{\chi}}$ of x_0 with highest eigenvalue exceeds $(\Delta\boldsymbol{x})^2_{min}$ at most by a term $O(1/\Lambda^2)$. Presumably it is not possible to determine $\widehat{\boldsymbol{\chi}}$ in closed form for general Λ either; determining analytically the eigenvalues and eigenvectors of a square matrix of large rank is an absolutely nontrivial problem. Nevertheless in chapter 4 we do a detailed study of their properties. In particular, since $[x_0, L_0] = 0$, it is possible to simultaneously diagonalize x_0 and L_0. By $(3.6)_1$ the eigenvalues of L_0 are $m \in \{-\Lambda, 1{-}\Lambda, ..., \Lambda\}$; let $\mathcal{H}^m_\Lambda$ be the corresponding eigenspaces. One can look for eigenvectors of both x_0, L_0 in the form (4.9).

Note that $L_0\boldsymbol{\chi} = m\boldsymbol{\chi}$ (with any m) implies $\langle x_\pm\rangle_{\boldsymbol{\chi}} = 0$, $|\langle\boldsymbol{x}\rangle_{\boldsymbol{\chi}}| = |\langle x_0\rangle_{\boldsymbol{\chi}}|$. The second equation in (4.9) turns out to be an eigenvalue equation for a real, symmetric and tri-diagonal square matrix $B_m(\Lambda)$ having dimension $\Lambda - |m| + 1$. It is easy to see that it is possible to focus the attention only to the cases $m \in \{0, 1, \cdots, \Lambda\}$; by (4.12), the eigenvector $\widehat{\boldsymbol{\chi}}$ of x_0 with the highest eigenvalue $\alpha_1(\Lambda; 0)$ belongs to $\mathcal{H}^0_\Lambda$. The matrix representing x_0 in the basis $\{\boldsymbol{\psi}^0_l\}_{l=0,...,\Lambda}$ of $\mathcal{H}^0_\Lambda$ is

$$B_0 = B_0(\Lambda) = \begin{pmatrix} 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & a_{\Lambda-1} & 0 & a_\Lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & a_\Lambda & 0 \end{pmatrix}, \tag{5.30}$$

where

$$a_l := c_{l,3}G(l,0,2) = \sqrt{1+\frac{l^2}{k_D}}\sqrt{\frac{l^2}{4l^2-1}} > \frac{1}{2} \qquad \forall l \leq \Lambda, \quad \forall \Lambda \in \mathbb{N},$$

and this implies (see proposition 8.1.2)

$$\|B_0\chi\|_2 > \left\|\frac{1}{2}P_{\Lambda+1}(0,1,1)\chi\right\|_2 \qquad \forall \chi \in \mathbb{R}^{\Lambda+1}_+. \tag{5.31}$$

The normalized vector $\widetilde{\chi} \equiv (\widetilde{\chi}_0, ..., \widetilde{\chi}_l) \in \mathbb{R}^{\Lambda+1}_+$ maximizing the right-hand side is the eigenvector of $\frac{1}{2}P_{\Lambda+1}(0,1,1)$ with highest eigenvalue $\lambda_1 = \cos[\pi/(\Lambda{+}2)]$:

$$\widetilde{\chi}_l = \sqrt{\frac{2}{\Lambda+2}}\sin\left[\frac{(l+1)\pi}{\Lambda+2}\right], \qquad 0 \leq l \leq \Lambda;$$

Hence as the highest lower bound for $\left|\langle\boldsymbol{x}\rangle_{\widehat{\chi}}\right| = \langle x_0\rangle_{\widehat{\chi}} = \alpha_1(\Lambda;0) = \|B_0\widehat{\chi}\|_2 \,/\, \|\widehat{\chi}\|$ one finds

$$\alpha_1(\Lambda;0) \geq \langle x_0\rangle_{\widetilde{\chi}} = \|B_0\widetilde{\chi}\|_2 > \left\|P_{\Lambda+1}\left(0,\frac{1}{2},\frac{1}{2}\right)\widetilde{\chi}\right\|_2 = \cos\left(\frac{\pi}{\Lambda{+}2}\right). \tag{5.32}$$

This finally suggests that a quite stringent upper bound for $(\Delta\boldsymbol{x})^2_{min}$ should be $(\Delta\boldsymbol{x})^2$ on $\widetilde{\boldsymbol{\chi}} = \sum_{l=0}^{\Lambda} \widetilde{\chi}_l \boldsymbol{\psi}_l^0 \in \mathcal{H}_\Lambda^0$. In fact, in the appendix it is shown that

$$(\Delta\boldsymbol{x})^2_{\widetilde{\boldsymbol{\chi}}} \overset{\Lambda\geq 3}{<} \frac{\pi^2}{(\Lambda+2)^2} + \frac{1}{(\Lambda+1)^2} < \frac{11}{(\Lambda+1)^2}. \tag{5.33}$$

This leads to the important result mentioned in the introduction: the smallest space dispersion on the new fuzzy sphere is smaller than the one (13) on the Madore's FS when $l = \Lambda$, i.e. the cutoffs of the two fuzzy spaces are the same; in formulas,

$$(\Delta\boldsymbol{x})^2_{min} \leq (\Delta\boldsymbol{x})^2_{\widetilde{\boldsymbol{\chi}}} < (\Delta\boldsymbol{x})^2_{minMadore} \equiv \frac{1}{\Lambda+1}. \tag{5.34}$$

Replacing $\underline{\boldsymbol{\chi}}$ by $\widehat{\boldsymbol{\chi}}, \widetilde{\boldsymbol{\chi}}$ in the definition of $\mathcal{W}^2$ one respectively obtains fully $O(3)$-invariant weak SCS $\widehat{\mathcal{W}}^2, \widetilde{\mathcal{W}}^2$ approximating $\mathcal{W}^2$. Since $\widehat{\boldsymbol{\chi}}, \widetilde{\boldsymbol{\chi}}$ are eigenvectors of L_0, the corresponding isotropy subgroup of $SO(3)$ is isomorphic to $SO(2)$, and the rays of the elements of $\widehat{\boldsymbol{\chi}}, \widetilde{\boldsymbol{\chi}}$ are in one-to-one correspondence with the points of the sphere $S^2 \simeq SO(3)/SO(2)$. The fact that the eigenvalue is zero is in agreement with the classical picture of the particle: the angular momentum $\boldsymbol{L} = \boldsymbol{r} \wedge \boldsymbol{p}$ is orthogonal to the position vector $\boldsymbol{r}$, hence if $\boldsymbol{r} \simeq \boldsymbol{k_D}$ (i.e. the particle is located concentrated around the north pole) then $\boldsymbol{L}$ is approximately orthogonal to the $x_3 \equiv x_0$-axis, and $L_3 \equiv L_0 \simeq 0$.

Chapter 6

Conclusions, outlook and comparison with literature

The construction of the $O(D)$-equivariant fuzzy sphere in the second section has been done through the imposition of a sufficiently low (and Λ-dependent, with $\Lambda \in \mathbb{N}$) energy cutoff $\overline{E} := \Lambda\,(\Lambda + D - 2)$ on the quantum mechanics of a particle subject to a rotation-invariant potential $V(r)$ having a very deep minimum in $r = 1$, and regulated by a confining parameter $k(\Lambda) \geq [\Lambda(\Lambda + D - 2)]^2$, which expresses the sharpness of that minimum.

The output is a sequence $\{\mathcal{A}_{\Lambda,D}\}_{\Lambda\in\mathbb{N}}$ of finite-dimensional algebras. Every operator $A \in \mathcal{A}_{\Lambda,D}$ acts on the corresponding Hilbert space of admitted states $\mathcal{H}_{\Lambda,D}$, which is also finite-dimensional and can be realized using an irreducible representation of $U\boldsymbol{so}(D+1)$ (the one having $l_D \equiv \Lambda$), but also a reducible representation of $U\boldsymbol{so}(D)$; in fact it can be decomposed through the irreps of $U\boldsymbol{so}(D)$ having $0 \leq l \leq \Lambda$.

The algebraic relations involving $\overline{L}_{h,j}, \overline{x}_p$ are invariant under parity, as well as under any $O(D)$-transformation of the coordinates, and this was expected because of the application of a rotation-invariant energy-cutoff to a theory having the same covariance; then, as shown, the projected theory has inherited that symmetry. It is also important to underline that these relations are nothing but the generalizations, to the D-dimensional case, of the ones calculated for $D = 2$ and $D = 3$.

The focal point is the definition 2.3.1. It is inspired by the action of a generic coordinate t_h on a spherical harmonic, and it allowed to repeat (in the generic D-dimensional case) what was done in [13, 14]; in fact, in almost all the proof it was fundamental that the action of $L_{h,D}$ on $Y_{\boldsymbol{l}}$ coincides, more or less, with the one of the coordinate t_h on $Y_{l_{d-1},\cdots,l_1}$ this is also in agreement with the Wigner-Eckart theorem, and this is also in agreement with the Wigner-Eckart theorem, because both t_h and $L_{h,D}$ transform in the same way under $SO(d)$.

Another crucial point of this section is the research of all the eigenfunctions of $\boldsymbol{L}^2$ (section 7.0.3) on S^d, for this reason the goal was the determination of an orthonormal basis of eigenfunctions $\{Y_{\boldsymbol{l}}\}_{\boldsymbol{l}}$ for $\boldsymbol{L}^2$ in $\mathcal{L}^2\left(S^d\right)$; this returned an orthonormal basis of $\mathcal{H}_{\Lambda,D}$ and then the subsequent possibility of calculating explicitely the action of $\overline{x}_p$ and $\overline{L}_{h,j}$ on every state ψ (section 2.3.2).

On the other hand, every space (here $\bar{l}$ is a fixed number of $\mathbb{N}_0$)

$$span\left\{Y_{\boldsymbol{l}}(\theta_d,\cdots,\theta_1) : \bar{l} \equiv l_d \geq \cdots \geq l_2 \geq |l_1|, l_i \in \mathbb{Z}\forall i\right\}$$

is the representation space of an irrep of $\boldsymbol{so}(D)$, the one corresponding to $\boldsymbol{L}^2 \equiv \bar{l}\left(\bar{l}+D-2\right)I$; it is important to underline that the Cartan subalgebra is too small to be a CSCO, which means that the sets of their eigenvalues do not univocally identify all spherical harmonics; then, in this case, one is not able to write down explicitely an orthonormal basis of $\boldsymbol{L}^2$-eigenfunctions in $\mathcal{L}^2\left(S^d\right)$ [as for (2.15)] and, consequently, to calculate the action of $\overline{x}_h$ and $\overline{L}_{h,j}$ on every quantum state ψ.

The aforementioned definition of the components $L_{h,j}$ of the D-dimensional angular momentum operator was also fundamental to realize the algebra of observables $\mathcal{A}_{\Lambda,D}$ with a suitable irreducible representation of $U\boldsymbol{so}(D+1)$. In fact, in that realization the 'projected' coordinate operator $\overline{x}_h$ is identified with $L_{h,D+1}$ up to some scalar left and right factors.

Finally, we do the proof of the convergence of this new fuzzy hypersphere to quantum mechanics on S^d in the commutative limit $\Lambda \to +\infty$, this was also expected because in that limit the potential $V(r)$ forces the particle to stay on the unit sphere, which (from the mathematical point of view) is represented by $c_{l,D} \to 1$, and then that every operator $\overline{x}_h$ converges to the corresponding $t_h\cdot$.

We now compare our fuzzy spheres with with other ones appeared in the literature; in [50] the authors build their two fuzzy versions of S^3:

- In the first case, from $\mathbb{C}P^3_F$ they firstly obtain a fuzzy S^4_F using the fact that $\mathbb{C}P^3$ is a S^2 bundle over S^4 and that there is a well defined matrix approximation of $\mathbb{C}P^3 \simeq \frac{SU(4)}{U(3)}$, then they construct S^3_F from this S^4_F.

- In the second case, they obtain S^3_F starting from the orthogonal Grassmanian $\frac{SO(5)}{SO(3)\times SO(3)}$ and then using the existence of a well defined matrix approximation of the algebra of functions on this Grassmanian, in other words they consider fuzzy orthogonal Grassmanians.

A well-known fuzzy 4-sphere is built in [34], and it essentially coincides with [51]; there the author considers the Dirac Γ matrices, which form the 4-dimensional spin representation of $\boldsymbol{so}(5)$, and are used in the n-fold symmetric tensor representation of Γ (here Sym means the restriction to the completely symmetrized tensor product space)

$$G_i^{(n)} := \left(\Gamma_i \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma_i \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \Gamma_i\right)_{Sym},$$

for $i = 1, \cdots, 5$. The $G_i^{(n)}$ defined above are $N \times N$ matrices, with

$$N = \frac{(n+1)(n+2)(n+3)}{6},$$

and they fulfill

$$\sum_i \left[G_i^{(n)}\right]^2 = n(n+4)\mathbb{I}_N.$$

Then, from

$$X_i := \frac{r}{n} G_i^{(n)} \quad \text{it follows} \quad \sum_i X_i^2 = r^2 \mathbb{I}_N + O\left(\frac{1}{n}\right).$$

The representations of $Spin(5)$ [or equivalently $Sp(2)$] are considered in [52] in order to build another fuzzy S^4; in particular, the irrep $\left(\frac{L}{2}, \frac{L}{2}\right)$ contains the 5 Dirac matrices J_a, $a = 1, \cdots, 5$, which can be realized as the symmetrization of L copies of the Γ matrices in the $Spin(5)$ fundamental representation:

$$J_a := \left(\Gamma_a \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} + \mathbb{I} \otimes \Gamma_a \otimes \cdots \otimes \mathbb{I} + \cdots + \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \Gamma_a\right)_{Sym},$$

where Sym means the projection in the totally symmetrized irreducible representation.

The J_a fulfill $J_a J_a = L(L+4)\mathbb{I}$, then from $X_a := \frac{R}{\sqrt{L(L+4)}} J_a$, it follows $X_a X_A = R^2 \mathbb{I}$ and that in the limit $L \to +\infty$ the algebra becomes commutative.

In [35] the authors approximate the sphere $S^N \cong \frac{SO(N+1)}{SO(N)}$ starting from the cartesian co-coordinates X^a, the angular momentum components $L_{a,b}$ in $\mathbb{R}^{N+1}$, with $a, b \in \{1, 2, \cdots, N+1\}$, and then also the $L_{A,B}$ in $\mathbb{R}^{N+2}$, with $A, B \in \{1, 2, \cdots, N+2\}$. The definition $X_a := \mu L_{a,N+2}$, with $\mu \in \mathbb{R}$, returns Snyder-type commutation relations

$$\left[X^a, X^b\right] = -i\mu^2 L_{a,b},$$

and also that (here $C_2^{N'}$ is the square angular momentum in $\mathbb{R}^{N'}$)

$$X_a X_a = \mu^2 \left[C_2^{N+2} - C_2^{N+1}\right],$$

which is central in the fundamental spinor representation of $Spin(N+2)$

$$X_a X_a = \frac{\mu^2(N+1)}{2} \mathbb{I}.$$

In agreement with the above construction, Sperling and Steinacker [36, 53] build their approximation S_N^4 of S^4 with a reducible representation of $U\boldsymbol{so}(5)$ (as for the above S_Λ^4) on a Hilbert space V obtained decomposing an irreducible representation π of $U\boldsymbol{so}(6)$ characterized by a triple of highest weights (n_1, n_2, N); so

$End(V) \simeq \pi[U\boldsymbol{so}(6)]$, in analogy with our scheme. The elements $X^a := r\mathcal{M}^{a6}$ play the role of noncommutative cartesian coordinates and they fulfill *Snyder*-type commutation relations (as for the above S^4_Λ). As a consequence the $O(5)$-scalar $\mathcal{R}^2 = X^a X^a$ is no longer central, but its spectrum is still very close to 1 if $N \gg n_1, n_2$ [because then the decomposition of V contains few irreducible representations under $SO(5)$].

On the other hand, if $n_1 = n_2 = 0$, the representation of $U\boldsymbol{so}(5)$ turns out to be irreducible (unlike the above S^4_Λ) [the highest weight is $(0, 0, N)$], and one obtain the basic fuzzy 4-sphere S^4_N, which is essentially the same of [34, 35], but in the case $N \equiv 4$:

$$X^a X_a = \mathcal{R}^2 = \frac{1}{4} N(N + 4)\mathbb{I},$$

so the coordinates can be trivially 'normalized'; furthermore, from $\boldsymbol{su}(4) \simeq \boldsymbol{so}(6)$ it follows

$$\mathcal{H}_\Lambda = (0, 0, N)_{\boldsymbol{su}(4)} = (0, N)_{\boldsymbol{so}(5)}.$$

The authors fuzzy approximate the quantum mechanics on the 4-sphere with the algebra $End\,(\mathcal{H}_N)$, and it fulfills

$$End\,(\mathcal{H}_N) = (0, 0, N) \otimes (N, 0, 0) = \bigoplus_{n=0}^{N} (n, 0, n),$$

which is its decomposition in the $\boldsymbol{su}(4)$ harmonics.

In turn, every $(n, 0, n)$ decomposes in this way in the $\boldsymbol{so}(5)$ harmonics:

$$(n, 0, n) = \bigoplus_{m=0}^{n} (n - m, 2m).$$

So, in $End\,(\mathcal{H}_N)$, there are

$$\bigoplus_{n=0}^{N} (n, 0),$$

which corresponds to the algebra $\mathcal{A}_{N,D}$ when $D = 5$, but there are also 'further modes', i.e. the representations $(n, 2s)$ with $s \geq 1$, that can be seen as higher spin algebras in the Vasiliev theory.

Their physical interpretation of $End(V)$ is that it represents a fuzzy approximation of some fiber bundle on a sphere S^4 (rather than of the algebra of observables of a quantum particle on a S^4).

In addition, in the analysis of the spectra $\Sigma_{x_i}(\Lambda)$ of the new fuzzy spaces (chapter 4) it has been shown the following:

1. $O(D)$-equivariance: the spectrum Σ_{x_i} of each x_i, for all choices of the orthogonal axes, is the same.

2. Parity property:
$$\alpha \in \Sigma_{x_i}(\Lambda) \Rightarrow -\alpha \in \Sigma_{x_i}(\Lambda)\,.$$

3. Monotonicity of the maximal eigenvalue with respect to Λ:
$$\max \Sigma_{x_i}(\Lambda) < \max \Sigma_{x_i}(\Lambda+1) \quad \text{and} \quad \lim_{\Lambda\to+\infty} [\max \Sigma_{x_i}(\Lambda)] = 1.$$

4. Density property
$$\Sigma_{x_i}(\Lambda) \text{ becomes uniformly dense in } [-1,1] \text{ when } \Lambda \to +\infty.$$

5. On the new fuzzy sphere S^2_Λ the state $\boldsymbol{\chi}$ most localized around the North pole fulfills the property $L_3\boldsymbol{\chi} = 0$ (item *(C)* of theorem 4.3.1), as the generalized quantum state (distribution) $2\delta(\theta)/\sin\theta \simeq \delta(x_1)\delta(x_2)$ on S^2 concentrated on the North pole (here θ is the colatitude); the classical counterpart of this property is that the classical particle on S^2 in the position $(0,0,1)$ has zero L_3 (z-component of the angular momentum).

It is important to underline that these are welcome properties for a x_i-operator which is required to approximate well, in the commutative limit, the x_i-coordinate of a quantum particle forced to stay on the unit sphere S^2.

Moreover, the spectrum of L_i is $\Sigma_{L_i}(\Lambda) = \{-\Lambda, 1-\Lambda, ..., \Lambda\}$ for all $i = 1,2,3$, by the $SO(3)$-covariance, and fulfills properties 1,2 (the multiplicity of the eigenvalue m is $\Lambda - |m| + 1$).

In the Madore fuzzy sphere, since the x_i are obtained by the rescaling (2) of angular momentum operators acting in an irreducible representation, then all x_i have again the same spectrum as x_3, by $SO(3)$-covariance, and this is obtained by the rescaling of the spectrum of L_3; this leads to the eigenvalues (all simple) and eigenvectors
$$x_3\varphi_m = \frac{m}{\sqrt{\Lambda^2+\Lambda}}\varphi_m \qquad \text{with } m \in \{-\Lambda, \cdots, \Lambda\}\,,$$
where $\Lambda := (n-1)/2$. Hence also in this case properties 1-4 are fulfilled. However, for this reason there is no longer room for independent observables playing the role of angular momentum operators on the carrier Hilbert space V_Λ, and property 5 is lost.

For this reason, and the other ones mentioned in the introduction, it is more natural to interpret the L_i in the irreducible representation (π_Λ, V_Λ) still as the inthrinsic angular momentum components of a particle of spin Λ, and the states (rays) in V_Λ as states on the corresponding spin phase manifold. Then, since the spin degrees of freedom have no classical limit, it is not possible to define also position observables or see any state $\boldsymbol{\varphi} \in V_\Lambda$ as an approximation of a classical point in S^2-configuration space; the algebra $\mathcal{A}_n$ should be seen simply as the spin phase space algebra, not as a fuzzyfication of the algebra of configuration space observables on S^2.

In chapter 5 various strong and weak systems of coherent states (SCS)[1] have been introduced on the fuzzy spheres S^1_Λ, S^2_Λ, and we do also a study of their localizations in configuration as well as (angular) momentum space. As on the commutative spheres S^d ($d=1,2$), these localizations can be respectively expressed in terms of the uncertainties $\Delta x_i, \Delta L_{ij}$, or in terms of their $O(D)$-invariant ($D \equiv d+1$) quadratic polynomials $(\Delta \boldsymbol{x})^2, (\Delta \boldsymbol{L})^2$ (sums of the variances of the x_i and L_{ij}, respectively); as a consequence, the localizations expressed through $(\Delta \boldsymbol{x})^2, (\Delta \boldsymbol{L})^2$ are preferable because reference-frame independent. General bounds (e.g. uncertainty relations following from commutation relations) for $\Delta x_i, \Delta L_{ij}, (\Delta \boldsymbol{x})^2, (\Delta \boldsymbol{L})^2$ are determined, it is estimated the latter on these SCS, and then it is partly investigated which SCS may saturate these bounds. Preliminarly we do a discussion about these issues for the commutative circle S^1 and sphere S^2, because the literature for the latter seems incomplete.

In particular, after the derivation of the $O(3)$-invariant uncertainty relation (5.19) (both on S^2 and on S^2_Λ), we do a discussion about its virtues compared to the $\Delta L_i \Delta L_j$ uncertainty relations (5.7), and then it is shown that the system of spin coherent states saturates it (see theorems 5.3.1 and 5.3.1); also for the commutative S^2 this result is new. Moreover, we do a discussion about the Heisenberg (i.e. $\Delta x \Delta L$) type uncertainty relations (HUR) (5.4), which hold both on S^1 and on S^1_Λ, and the states saturating them: it has been shown that only the eigenvectors[2] ψ_n of L saturate both $(5.4)_{1-2}$, or equivalently the $O(2)$-invariant inequality $(5.4)_3$, while there is a complete family (parametrized by $\mu \in \mathbb{R}$) of states saturating $(5.4)_1$ alone (these states are eigenvectors of $a_1^\mu := L - i\mu x_1$); the family interpolates between the set of eigenvectors of L and the set of eigenvectors of x_1.

Moreover, for $d=1,2$ a large class of strong SCSs is built on S^d_Λ applying $SO(D)$-transformations on suitable initial states $\boldsymbol{\omega} \in \mathcal{H}_\Lambda$, see eq. (5.12) and Theorem 5.3.2; in particular, the SCS have been chosen so that they minimize (within the class) either $(\Delta \boldsymbol{L})^2$, or $(\Delta \boldsymbol{x})^2$; the SCS $\mathcal{S}^d$ minimizing $(\Delta \boldsymbol{x})^2$ is fully $O(D)$-equivariant, its states (rays) are actually in one-to-one correspondence with points of $S^d \simeq SO(D)/SO(d)$, and their $(\Delta \boldsymbol{x})^2$ is smaller than the uncertainty (13) in Madore FS, i.e. satisfies $(\Delta \boldsymbol{x})^2 < 1/(\Lambda+1)$ - see (5.15), (5.29) [more careful computations will lead to lower upper bounds for $(\Delta \boldsymbol{x})^2$].

For both $d=1,2$ a fully $O(D)$-equivariant, weak SCS $\mathcal{W}^d = \{\underline{\boldsymbol{\chi}}_g := \pi_\Lambda(g)\underline{\boldsymbol{\chi}} \mid g \in SO(D)\}$ have been introduced; it consists of states minimizing $(\Delta \boldsymbol{x})^2$ within the whole Hilbert space $\mathcal{H}_\Lambda$; the states (rays) of $\mathcal{W}^d$ are actually again in one-to-one

[1] A strong SCS yields a resolution of the identity; a weak SCS is just (over)complete.

[2] The ψ_n make an orthonormal basis of the Hilbert space; in a broad (but rather unconventional) sense this basis can be considered the system of coherent states associated to the group (5.6), semidirect product of a Lie group times a discrete one.

correspondence with the points of $S^d \simeq SO(D)/SO(d)$. They are determined up to order $O(1/\Lambda^2)$, with the help of the results of chapter 4 the vector $\underline{\boldsymbol{\chi}}$ is approximated as the eigenvector $\widetilde{\boldsymbol{\chi}}$ with maximal eigenvalue of a suitable Toeplitz tridiagonal matrix, and denoted as $\widetilde{\mathcal{W}}^d$ the corresponding SCS; this eigenvector is in turn very close to the eigenvector with maximal eigenvalue of x_1 (resp. $x_0 \equiv x_3$), because numerical computations suggest that $\left\|X^\Lambda\right\|_2$ and $\|B_0(\Lambda)\|_2$ both converge with order 2 to 1.

For these reasons the strong SCS $\mathcal{S}^d$ (or alternatively the weak one $\mathcal{W}^d$, if a resolution of the identity is not needed) can be considered the system of quantum states that is the 'closest' approximation to S^d.

It is important to underline that the states of the strong SCS $\mathcal{S}^2$ (resp. of the weak SCS $\mathcal{W}^2$, $\widetilde{\mathcal{W}}^2$) are better localized than the most localized states of the Madore fuzzy sphere with the same cutoff ($l = \Lambda$) by a factor smaller than 1, see (5.29) [resp. by a power of $1/\Lambda$, see (5.34)]. On S^2_Λ the state $\boldsymbol{\chi} \in \mathcal{S}^2$ centered around the North pole (i.e. with $\langle x_1 \rangle = \langle x_2 \rangle = 0$, $\langle x_3 \rangle > 0$) fulfills the property $L_3 \boldsymbol{\chi} = 0$; the classical counterpart of this property is that a classical particle at the North pole of S^2 has zero L_3 (z-component of the angular momentum), see section 5.3.2. As noted in chapter 4, such a property is impossible to realize on the Madore-Hoppe FS. For these reasons, and the other ones mentioned in the introduction, it is reasonable to see the fuzzy sphere S^2_Λ as a much more realistic fuzzy approximation of a classical S^2 configuration space.

Finally, the construction of various systems of coherent states on the new fuzzy circle and fuzzy sphere will be very useful to study quantum mechanics and above all quantum field theory on these fuzzy spaces.

Chapter 7

Appendix A

7.0.1 The action of $C_{\widetilde{D}}$ in $\mathbb{R}^D$ and in $\mathbb{R}^{\widetilde{D}}$, when $2 \leq \widetilde{D} < D$

Let $(x_1, \cdots, x_D)$ be the rectangular coordinates in $\mathbb{R}^D$ and $(r, \theta_d, \cdots, \theta_1)$ the spherical ones:

$$\begin{aligned} x_1 &= r \sin\theta_d \sin\theta_{d-1} \cdots \sin\theta_2 \cos\theta_1, \\ x_2 &= r \sin\theta_d \sin\theta_{d-1} \cdots \sin\theta_2 \sin\theta_1, \\ x_3 &= r \sin\theta_d \sin\theta_{d-1} \cdots \cos\theta_2, \\ &\vdots \\ x_d &= r \sin\theta_d \cos\theta_{d-1}, \\ x_D &= r \cos\theta_d, \end{aligned} \tag{7.1}$$

with $r \geq 0$, $\theta_1 \in [0, 2\pi[$ and $\theta_2, \cdots, \theta_d \in [0, \pi]$.

First of all, in both $\mathbb{R}^{\widetilde{D}}$ and $\mathbb{R}^D$ the equality

$$C_{\widetilde{D}} \overset{(6)}{=} \sum_{1 \leq j < h \leq \widetilde{D}} L^2_{j,h}$$

holds, but the crucial difference is that the expression of x_p in polar coordinates (7.1) changes when one passes from $\mathbb{R}^{\widetilde{D}}$

$$\widehat{x}_p := r' \sin\theta_{\widetilde{D}-1} \cdots \sin\theta_p \cos\theta_{p-1},$$

to $\mathbb{R}^D$

$$x_p := r \sin\theta_d \cdots \sin\theta_{\widetilde{D}} \sin\theta_{\widetilde{D}-1} \cdots \sin\theta_p \cos\theta_{p-1} = \frac{r}{r'} \sin\theta_d \cdots \sin\theta_{\widetilde{D}} \widehat{x}_p,$$

where

$$r' := \sqrt{\sum_{p=1}^{\widetilde{D}} \widehat{x}_p^2} \quad \text{and} \quad r := \sqrt{\sum_{p=1}^{D} x_p^2}.$$

This means that, in order to understand the difference between the action of the operator $C_{\widetilde{D}}$ in the two ambient spaces, one can focus the attention only on the differences between the action (♣) of $L_{j,h}$ in $\mathbb{R}^{\widetilde{D}}$ and the one (♠) in $\mathbb{R}^D$.

According to this, if $f(x_D, \cdots, x_1)$ is a differentiable function on $\mathbb{R}^D$ and $r' = r$, then

$$\widehat{x}_j \frac{\partial}{\partial \widehat{x}_h} f(x_D, \cdots, x_1) = \widehat{x}_j \sin\theta_d \cdots \sin\theta_{\widetilde{D}} \frac{\partial f}{\partial x_h}(x_D, \cdots, x_1) = x_j \frac{\partial f}{\partial x_h}(x_D, \cdots, x_1);$$

which implies that the action ♣ on the sphere $S_r^{\widetilde{D}-1}$ coincides with the one ♠ on S_r^d, in particular they coincide on the corresponding unit spheres.

7.0.2 About the regularity of $f(r)$ in (2.2)

In the case of a second order linear ODE

$$y'(z) + P(z)y'(z) + Q(z)y(z) = 0, \tag{7.2}$$

a point $z_0 \in \mathbb{C}$ is singular for the equation if $P(z)$ and $Q(z)$ have an isolated singularity at $z = z_0$; z_0 is a fuchsian point if $P(z)$ has a pole of order at most 1 in $z = z_0$ and $Q(z)$ has a pole of order at most 2 in $z = z_0$.

Fuchs theorem states that in the neighborhood of a fuchsian point every solution of (7.2) is a combination of the two independent ones having the following behavior:

$$y_1(z) = (z - z_0)^{\alpha_1} w_1(z) \quad \text{and} \quad y_2(z) = (z - z_0)^{\alpha_2} w_2(z),$$

where α_i are the solutions of the algebraic equation

$$x^2 + (p_0 - 1)x + q_0 = 0,$$

$w_i(z)$ are holomorphic functions which do not vanish in $z = z_0$,

$$p_0 = \lim_{z\to z_0} (z - z_0)P(z) \quad \text{and} \quad q_0 = \lim_{z\to z_0} (z - z_0)^2 Q(z).$$

From this last theorem, applied to (2.2) under the hypothesis

$$\lim_{r\to 0^+} r^2 V(r) = T \in \mathbb{R}^+, \tag{7.3}$$

it follows

$$p_0 = D - 1 \quad \text{and} \quad q_0 = -\left[l(l + D - 2) + T\right],$$

then

$$\alpha_1 = \frac{2 - D + \sqrt{(D-2)^2 + 4\left[l(l+D-2)+T\right]}}{2} \overset{T>0}{\geq} \frac{2 - D + \sqrt{(D-2)^2}}{2} = 0,$$

$$\alpha_2 = \frac{2 - D - \sqrt{(D-2)^2 + 4\left[l(l+D-2)+T\right]}}{2} \overset{T>0}{<} 0.$$

Hence

$$f(r) = \gamma r^{\alpha_1} w_1(r) + \delta r^{\alpha_2} w_2(r) \quad \text{when} \quad r \to 0;$$

in addition, according to the self-adjointness of H, it must be

$$\boldsymbol{\psi} \in D(H) \equiv D(H^*) = \left\{ \boldsymbol{\psi} \in \mathcal{L}^2\left(\mathbb{R}^D\right) : \boldsymbol{\psi} \text{ is twice differentiable and } H\psi \in \mathcal{L}^2\left(\mathbb{R}^D\right) \right\},$$

which implies $\delta \equiv 0$ and then $f(0) = 0$.

7.0.3 The D-dimensional spherical harmonics

In this section it is explained how to determine an orthonormal basis of $\mathcal{L}^2\left(S^d\right)$ made up of eigenfunctions Y of $\boldsymbol{L}^2$ in $\mathbb{R}^D$.

7.0.3.1 The resolution of $\boldsymbol{L}^2 Y = l(l+D-2)Y$ by separation of variables

First of all, from (9) it follows that $L_{1,2}$ and all these C_p operators can be simultaneously diagonalized; in addition, in section 7.0.1 we do the proof that C_p coincides with the opposite of the Laplace-Beltrami operator $\Delta_{S^{p-1}}$ on the sphere S^{p-1} in every dimension D, then from [54] p. 21, it follows

$$\begin{aligned} \Delta =& \frac{\partial^2}{\partial r^2} + (D-1)\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\boldsymbol{L^2}, \\ \boldsymbol{L}^2 =& -(1-t_2)\frac{\partial^2}{\partial t_2} + (D-1)t\frac{\partial}{\partial t} + \frac{1}{1-t_2}C_d, \end{aligned} \tag{7.4}$$

where $t = \cos\theta_d$.

Furthermore, when $\theta \in [0, \pi]$,

$$\frac{\partial}{\partial \cos\theta} = \frac{\partial \theta}{\partial \cos\theta}\frac{\partial}{\partial\theta} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial \cos^2\theta} &= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) = \frac{1}{\sin\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta^2}\right) \\ &= -\frac{\cos\theta}{\sin^3\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\theta^2}. \end{aligned}$$

According to this,

$$\begin{aligned} C_D = \boldsymbol{L}^2 &= -(1-t_2)\frac{\partial^2}{\partial t_2} + (D-1)t\frac{\partial}{\partial t} + \frac{1}{1-t_2}C_d \\ &= -\frac{\partial^2}{\partial\theta_d^2} + \frac{\cos\theta_d}{\sin\theta_d}\frac{\partial}{\partial\theta_d} - (D-1)\frac{\cos\theta_d}{\sin\theta_d}\frac{\partial}{\partial\theta_d} + \frac{1}{\sin^2\theta_d}C_d \\ &= -\frac{\partial^2}{\partial\theta_d^2} - (D-2)\frac{\cos\theta_d}{\sin\theta_d}\frac{\partial}{\partial\theta_d} + \frac{1}{\sin^2\theta_d}C_d \\ &= -\frac{1}{\sin^{d-1}\theta_d}\frac{\partial}{\partial\theta_d}\left(\sin^{d-1}\theta_d\frac{\partial}{\partial\theta_d}\right) + \frac{1}{\sin^2\theta_d}C_d. \end{aligned} \tag{7.5}$$

The aforementioned proof of $(7.4)_2$ and also (7.5) can be trivially generalized to every dimension, which means that, when $n \in \{3, \cdots, D\}$,

$$\begin{aligned} C_n = -\frac{1}{\sin^{n-2}\theta_{n-1}}\frac{\partial}{\partial\theta_{n-1}}\left(\sin^{n-2}\theta_{n-1}\frac{\partial}{\partial\theta_{n-1}}\right) + \frac{1}{\sin^2\theta_{n-1}}C_{n-1}, \\ \text{while} \quad L_{1,2} = \frac{1}{i}\frac{\partial}{\partial\theta_1} \Rightarrow C_2 = -\frac{\partial^2}{\partial\theta_1^2}. \end{aligned} \tag{7.6}$$

Section 7.0.1 and (7.6) suggest to apply a separation of variables in the resolution of $C_pY = l_{p-1}(l_{p-1}+p-2)Y$ for $p = 2, \cdots D$; then $Y = Y_1(\theta_d, \cdots, \theta_2)g_1(\theta_1)$, (7), $(7.6)_2$ and $C_2Y = L_{1,2}^2Y = l_1^2Y$ with $l_1 \in \mathbb{Z}$ imply $g_1(\theta_1) = Ce^{il_1\theta_1}$, with $l_1 \in \mathbb{Z}$.

The constant C can be fixed by requiring that

$$\int_0^{2\pi} g_1g_1^*d\theta_1 = 1,$$

which implies $C = \frac{1}{\sqrt{2\pi}}$.

Furthermore

$$C_3 = -\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right) + \frac{1}{\sin^2\theta_2}C_2 \quad \text{and} \quad C_3Y \overset{(7)}{=} l_2(l_2+1)Y,$$

while $L_{h,j}^\dagger = L_{h,j}$ and the fact that every operator $B := A^\dagger A$ has positive spectrum imply

$$\langle Y, C_3Y\rangle \geq \langle Y, C_2Y\rangle \Longleftrightarrow l_2^2 + l_2 - l_1^2 \geq 0 \quad \text{with } l_1, l_2 \in \mathbb{Z},$$

and this is possible if and only if $l_2 \geq |l_1|$.

The separation of variables

$$Y_1(\theta_{D-1}, \cdots, \theta_2) = Y_2(\theta_{D-1}, \cdots, \theta_3)g_2(\theta_2),$$

returns

$$l_2(l_2+1)g_2 = -\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial g_2}{\partial\theta_2}\right) + \frac{1}{\sin^2\theta_2}l_1^2g_2,$$

and setting $z = \cos\theta_2$, then

$$\frac{\partial}{\partial\theta_2} = \frac{\partial z}{\partial\theta_2}\frac{\partial}{\partial z} = -\sqrt{1-z^2}\frac{\partial}{\partial z},$$

so one has to solve

$$l_2(l_2+1)g_2 = \frac{\partial}{\partial z}\left(-(1-z^2)\frac{\partial g_2}{\partial z}\right) + \frac{l_1^2}{1-z^2}g_2,$$

which is equivalent to

$$\left[(1-z^2)\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} + l_2(l_2+1) - \frac{l_1^2}{1-z^2}\right]g_2 = 0.$$

This last equation is the general Legendre differential equation (see [55] formula 8.1.1 p. 332) and the solution is the associated Legendre function of first kind:

$$g_2(z) = CP_{l_2}^{l_1}(z) \Longrightarrow g_2(\theta_2) = CP_{l_2}^{l_1}(\cos\theta_2).$$

The constant C can be determined by requiring that

$$|C|^2 \int_0^\pi P_{l_2}^{l_1}(\cos\theta_2)\left[P_{l_2'}^{l_1'}(\cos\theta_2)\right]^* \sin\theta_2 d\theta_2 = \delta_{l_2}^{l_2'}\delta_{l_1}^{l_1'},$$

and after the replacement $z = \cos\theta_2$ it becomes

$$|C|^2 \int_{-1}^1 P_{l_2}^{l_1}(z)\left[P_{l_2'}^{l_1'}(z)\right]^* dz = \delta_{l_2}^{l_2'}\delta_{l_1}^{l_1'}.$$

The equalities

$$\int_{-1}^1 P_n^m(x)P_l^m(x)dx = 0 \quad (l \neq n) \quad \text{and} \quad \int_{-1}^1 \left[P_n^m(x)\right]^2 dx = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \tag{7.7}$$

from [55] formulas 8.14.11, 8.14.13 p. 338 and $P_l^m(x) \in \mathbb{R}\ \forall x \in \mathbb{R}$ imply

$$|C| = \sqrt{\frac{2l_2+1}{2}}\sqrt{\frac{(l_2-l_1)!}{(l_2+l_1)!}}, \tag{7.8}$$

then

$$g_2(\cos\theta_2) = \sqrt{\frac{2l_2+1}{2}}\sqrt{\frac{(l_2-l_1)!}{(l_2+l_1)!}}P_{l_2}^{l_1}(\cos\theta_2).$$

On the other hand, $l_2 \geq |l_1|$ and $l_1, l_2 \in \mathbb{Z}$ imply that the formula 8.2.5 in [55] (here Q_ν^μ is the associated Legendre function of second kind)

$$P_\nu^{-\mu}(z) = \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[P_\nu^\mu(z) - \frac{2}{\pi}e^{-i\mu\pi}\sin(\mu\pi)Q_\nu^\mu\right]$$

becomes

$$P_\nu^\mu(z) = \frac{(\nu+\mu)!}{(\nu-\mu)!}P_\nu^{-\mu}(z); \tag{7.9}$$

then

$$g_2(\cos\theta_2) = \sqrt{\frac{2l_2+1}{2}}\sqrt{\frac{(l_2+l_1)!}{(l_2-l_1)!}}P_{l_2}^{-l_1}(\cos\theta_2).$$

This last procedure can be repeated for the angular variables $\theta_3, \cdots, \theta_d$, because $(7.6)_1$ links every C_n with C_{n-1}, for this reason one can now work with a generic C_n.

From $(7.6)_1$,

$$C_{n-1}Y \overset{(7)}{=} l_{n-2}(l_{n-2}+n-3)Y \quad \text{and} \quad C_nY \overset{(7)}{=} l_{n-1}(l_{n-1}+n-2)Y$$

it follows $l_{n-1} \geq l_{n-2}$ and

$$l_{n-1}(l_{n-1}+n-2)g_{n-1} = \left[-\frac{\partial^2}{\partial\theta_{n-1}^2} - (n-2)\frac{\cos\theta_{n-1}}{\sin\theta_{n-1}}\frac{\partial}{\partial\theta_{n-1}} + \frac{l_{n-2}(l_{n-2}+n-3)}{\sin^2\theta_{n-1}}\right] g_{n-1}.$$

The replacement $z = \cos\theta_{n-1}$ implies

$$\frac{\partial}{\partial\theta_{n-1}} = \frac{\partial z}{\partial\theta_{n-1}}\frac{\partial}{\partial z} = -\sqrt{1-z^2}\frac{\partial}{\partial z},$$

and then the last ODE becomes

$$l_{n-1}(l_{n-1}+n-2)g_{n-1} = \frac{\partial}{\partial z}\left[-(1-z^2)\frac{\partial g_{n-1}}{\partial z}\right] + \frac{l_{n-2}(l_{n-2}+n-3)}{1-z^2}g_{n-1},$$

which is equivalent to

$$\left[(1-z^2)\frac{\partial^2}{\partial z^2} - (n-1)z\frac{\partial}{\partial z} + l_{n-1}(l_{n-1}+n-2) - \frac{l_{n-2}(l_{n-2}+n-3)}{1-z^2}\right] g_{n-1} = 0.$$

Assume that

$$g_{n-1}(z) = \left(1-z^2\right)^{\frac{3-n}{4}} f_{n-1}(z) \tag{7.10}$$

and that the function $f_{n-1}(z)$ (which is determined in the following lines) has a zero in $z=1$ of order higher than $\frac{n-3}{4}$ (see section 7.0.4 for the proof of this); then

$$g'_{n-1}(z) = \frac{z(n-3)}{2}\left(1-z^2\right)^{-\frac{1+n}{4}} f_{n-1}(z) + \left(1-z^2\right)^{\frac{3-n}{4}} f'_{n-1}(z)$$

and

$$\begin{aligned} g'_{n-1}(z) =& \frac{n-3}{2}\left(1-z^2\right)^{-\frac{1+n}{4}} f_{n-1}(z) + \frac{z(n-3)}{2}\frac{z(n+1)}{2}\left(1-z^2\right)^{-\frac{5+n}{4}} f_{n-1}(z) \\ &+ z(n-3)\left(1-z^2\right)^{-\frac{1+n}{4}} f'_{n-1}(z) + \left(1-z^2\right)^{\frac{3-n}{4}} f'_{n-1}(z), \end{aligned}$$

which implies

$$-(n-1)zg'_{n-1}(z) = -(n-1)z\left[\frac{z(n-3)}{2}\left(1-z^2\right)^{-\frac{1+n}{4}} f_{n-1}(z) + \left(1-z^2\right)^{\frac{3-n}{4}} f'_{n-1}(z)\right]$$

$$= \left(1-z^2\right)^{\frac{3-n}{4}}\left[-\frac{z(n^2-4n+3)}{2(1-z^2)}f_{n-1}(z) - (n-1)zf'_{n-1}(z)\right]$$

and, similarly,

$$\begin{aligned}(1-z^2)g'_{n-1} = \left(1-z^2\right)^{\frac{3-n}{4}} \Bigg[&\frac{n-3}{2}f_{n-1}(z) + \frac{z^2(n^2-2n-3)}{4(1-z^2)}f_{n-1}(z) \\ &+z(n-3)f'_{n-1}(z) + (1-z^2)f'_{n-1}(z)\Big].\end{aligned}$$

Furthermore,

$$\begin{aligned}&-\frac{l_{n-2}(l_{n-2}+n-3)}{1-z^2} + \frac{z^2(n^2-2n+3)}{4(1-z^2)} - \frac{z^2(n^2-4n+3)}{2(1-z^2)} \\ &= -\frac{l_{n-2}(l_{n-2}+n-3)}{1-z^2} + \frac{z^2(n^2-2n-3-2n^2+8n-6)}{4(1-z^2)} \\ &= -\frac{l_{n-2}(l_{n-2}+n-3)}{1-z^2} + \frac{z^2(-n^2+6n-9)}{4(1-z^2)} \\ &= \frac{1}{4}(n^2-6n+9) - \frac{l_{n-2}(l_{n-2}+n-3)+\frac{1}{4}(n^2-6n+9)}{1-z^2}.\end{aligned}$$

At this point, the first term of the ODE for f_{n-1} [after deleting the common factor $(1-z^2)^{\frac{3-n}{4}}$] is

$$(1-z^2)f'_{n-1}(z),$$

the second term is

$$z(n-3)f'_{n-1}(z) - z(n-1)f'_{n-1}(z) = -2zf'_{n-1}(z),$$

the third term is

$$\begin{aligned}&l_{n-1}(l_{n-1}+n-2)f_{n-1}(z) + \frac{n-3}{2}f_{n-1}(z) + \frac{n^2-6n+9}{4}f_{n-1}(z) \\ =&\left(l_{n-1}^2 + l_{n-1}n - 2l_{n-1} + \frac{n^2-4n+3}{4}\right)f_{n-1}(z) \\ =&\left(l_{n-1}+\frac{n-3}{2}\right)\left(l_{n-1}+\frac{n-3}{2}+1\right)f_{n-1}(z) \\ =&l'(l'+1)f_{n-1}(z),\end{aligned}$$

with $l' := l_{n-1} + \frac{n-3}{2}$, and the last term is

$$\frac{l_{n-2}^2 + l_{n-2}n - 3l_{n-2} + n^2 - 6n + 9}{1-z^2}f_{n-1}(z) = \frac{\left(l_{n-2}+\frac{n-3}{2}\right)^2}{1-z^2}f_{n-1}(z) = \frac{(m')^2}{1-z^2}f_{n-1}(z),$$

with $m' := l_{n-2} + \frac{n-3}{2}$.

This means that there is another associated Legendre equation:

$$\left[(1-z^2)\frac{\partial^2}{\partial z^2} - 2z\frac{\partial}{\partial z} + l'(l'+1) - \frac{(m')^2}{1-z^2}\right]f_{n-1}(z) = 0,$$

and then the solution is [here the constant C is fixed as done in (7.8)]

$$
\begin{aligned}
f_{n-1}(\cos\theta_{n-1}) &= \sqrt{\frac{2l'+1}{2}}\sqrt{\frac{(l'-m')!}{(l'+m')!}}P_{l'}^{m'}(\cos\theta_{n-1}) \\
&= \sqrt{\frac{2l_{n-1}+n-2}{2}}\sqrt{\frac{(l_{n-1}-l_{n-2})!}{(l_{n-1}+l_{n-2}+n-3)!}}P_{l_{n-1}+\frac{n-3}{2}}^{l_{n-2}+\frac{n-3}{2}}(\cos\theta_{n-1}),
\end{aligned}
$$

which implies

$$
g_{n-1}(\cos\theta_{n-1}) = \sqrt{\frac{2l_{n-1}+n-2}{2}}\sqrt{\frac{(l_{n-1}-l_{n-2})!}{(l_{n-1}+l_{n-2}+n-3)!}}\,[\sin\theta_{n-1}]^{\frac{3-n}{2}}\,P_{l_{n-1}+\frac{n-3}{2}}^{l_{n-2}+\frac{n-3}{2}}(\cos\theta_{n-1}).
$$

It is obvious that $l'+m', l'-m' \in \mathbb{N}_0$, so

$$
g_{n-1}(\cos\theta_{n-1}) \overset{(7.9)}{=} \sqrt{\frac{2l_{n-1}+n-2}{2}}\sqrt{\frac{(l_{n-1}+l_{n-2}+n-3)!}{(l_{n-1}-l_{n-2})!}}\,[\sin\theta_{n-1}]^{\frac{3-n}{2}}\,P_{l_{n-1}+\frac{n-3}{2}}^{-\left(l_{n-2}+\frac{n-3}{2}\right)}(\cos\theta_{n-1}).
$$

Summarizing,

$$
Y_{\boldsymbol{l}}(\theta_d,\cdots,\theta_2,\theta_1) = \frac{e^{il_1\theta_1}}{\sqrt{2\pi}}\left[\prod_{n=2}^{D-1} {}_n\overline{P}_{l_n}^{l_{n-1}}(\theta_n)\right], \tag{7.11}
$$

where

$$
{}_j\overline{P}_L^M(\theta) := \sqrt{\frac{2L+j-1}{2}}\sqrt{\frac{(L+M+j-2)!}{(L-M)!}}\,[\sin\theta]^{\frac{2-j}{2}}\,P_{L+\frac{j-2}{2}}^{-\left(M+\frac{j-2}{2}\right)}(\cos\theta). \tag{7.12}
$$

7.0.3.2 The orthonormality of $Y_{\boldsymbol{l}}$

The $Y_{\boldsymbol{l}}$ built above are eigenvectors of self-adjoint operators, so

$$
\int_{S^d} Y_{\boldsymbol{l}} Y_{\boldsymbol{l}'}^* d\alpha = 0 \quad \text{if} \quad \boldsymbol{l} \neq \boldsymbol{l}';
$$

in order to prove $(2.13)_3$ it remains to show that

$$
\int_{S^d} Y_{\boldsymbol{l}} Y_{\boldsymbol{l}}^* d\alpha = 1,
$$

and this is done recursively.

First of all, it is important to underline that from (7.11-7.12) it follows

$$
\int_{S^2} Y_{\boldsymbol{l}} Y_{\boldsymbol{l}'}^* d\alpha = \int_0^{2\pi} \frac{e^{i(l_1-l_1')\theta_1}}{\sqrt{2\pi}} d\theta_1 \cdot \left[\prod_{n=2}^{d} \int_0^{\pi} {}_n\overline{P}_{l_n}^{l_{n-1}}(\theta_n)\,{}_n\overline{P}_{l_n'}^{l_{n-1}'}(\theta_n)\sin^{n-1}\theta_n d\theta_n\right].
$$

While from

$$\int_0^{2\pi} \frac{e^{i(l_1-l_1')\theta_1}}{2\pi} d\theta_1 = \delta_{l_1}^{l_1'},$$

it follows that, if $l_1 \neq l_1'$, then $(2.13)_3$ vanishes; otherwise, if $l_1 = l_1' \geq 0$ (the case $l_1 < 0$ is essentially the same), then

$$\begin{aligned}
\int_0^{\pi} {}_2\overline{P}_{l_2}^{l_1}(\theta_2) {}_2\overline{P}_{l_2'}^{l_1}(\theta_2) \sin\theta_2 d\theta_2 &\overset{(7.12)}{=} \sqrt{\frac{2l_2+1}{2}\frac{(l_2+l_1)!}{(l_2-l_1)!}}\sqrt{\frac{2l_2'+1}{2}\frac{(l_2'+l_1)!}{(l_2'-l_1)!}} \int_0^{\pi} P_{l_2}^{-l_1}(\theta_2) P_{l_2'}^{-l_1}(\theta_2) \sin\theta_2 d\theta_2 \\
&\overset{(7.9)}{=} \sqrt{\frac{2l_2+1}{2}\frac{(l_2-l_1)!}{(l_2+l_1)!}}\sqrt{\frac{2l_2'+1}{2}\frac{(l_2'-l_1)!}{(l_2'+l_1)!}} \int_0^{\pi} P_{l_2}^{l_1}(\theta_2) P_{l_2'}^{l_1}(\theta_2) \sin\theta_2 d\theta_2 \\
&\overset{x=\cos\theta_2}{=} \sqrt{\frac{2l_2+1}{2}\frac{(l_2-l_1)!}{(l_2+l_1)!}}\sqrt{\frac{2l_2'+1}{2}\frac{(l_2'-l_1)!}{(l_2'+l_1)!}} \int_{-1}^{1} P_{l_2}^{l_1}(x) P_{l_2'}^{l_1}(x) dx \\
&\overset{(7.7)}{=} \delta_{l_2}^{l_2'}
\end{aligned}$$

and if $l_2 \neq l_2'$, then $(2.13)_3$ vanishes.

In general, if $l_i = l_i'$ for $i \in \{1, \cdots, n-1\}$,

$$\begin{aligned}
\int_0^{\pi} {}_2\overline{P}_{l_n}^{l_{n-1}}(\theta_n) {}_2\overline{P}_{l_n'}^{l_{n-1}}(\theta_n) \sin\theta_n d\theta_n &\overset{(7.12)}{=} \sqrt{\frac{2l_n+n-1}{2}\frac{(l_n+l_{n-1}+n-2)!}{(l_n-l_{n-1})!}}\sqrt{\frac{2l_n'+1}{2}\frac{(l_n'+l_{n-1}+n-2)!}{(l_n'-l_{n-1})!}} \\
&\quad \cdot \int_0^{\pi} P_{l_n+\frac{n-2}{2}}^{-\left(l_{n-1}+\frac{n-2}{2}\right)}(\theta_n) P_{l_n'+\frac{n-2}{2}}^{-\left(l_{n-1}+\frac{n-2}{2}\right)}(\theta_n) \sin\theta_n d\theta_n \\
&\overset{(7.9)}{=} \sqrt{\frac{2l_n+n-1}{2}\frac{(l_n-l_{n-1})!}{(l_n+l_{n-1}+n-2)!}}\sqrt{\frac{2l_n'+1}{2}\frac{(l_n'-l_{n-1})!}{(l_n'+l_{n-1}+n-2)!}} \\
&\quad \cdot \int_0^{\pi} P_{l_n+\frac{n-2}{2}}^{-\left(l_{n-1}+\frac{n-2}{2}\right)}(\theta_n) P_{l_n'+\frac{n-2}{2}}^{-\left(l_{n-1}+\frac{n-2}{2}\right)}(\theta_n) \sin\theta_n d\theta_n \\
&\overset{x=\cos\theta_n}{=} \sqrt{\frac{2l_n+n-1}{2}\frac{(l_n-l_{n-1})!}{(l_n+l_{n-1}+n-2)!}}\sqrt{\frac{2l_n'+1}{2}\frac{(l_n'-l_{n-1})!}{(l_n'+l_{n-1}+n-2)!}} \\
&\quad \cdot \int_{-1}^{1} P_{l_n+\frac{n-2}{2}}^{-\left(l_{n-1}+\frac{n-2}{2}\right)}(x) P_{l_n'+\frac{n-2}{2}}^{-\left(l_{n-1}+\frac{n-2}{2}\right)}(x) dx \\
&\overset{(7.7)}{=} \delta_{l_n}^{l_n'};
\end{aligned}$$

and this proves $(2.13)_3$.

7.0.3.3 The Y_l seen as homogeneous polynomials

The next proposition uses the equality (see section 7.0.4 for its proof)

$${}_h\overline{P}_l^m(\theta) = (\sin\theta)^m \widetilde{P}_{l+\frac{h-2}{2}}^{-\left(m+\frac{h-2}{2}\right)}(\cos\theta) \equiv (\sin\theta)^m \left\{[\cos\theta]^{l-m} + [\cos\theta]^{l-m-2} + \cdots\right\}, \tag{7.13}$$

which is true up to any multiplicative constant before every power of $\cos\theta$ and $\sin\theta$.

Proposition 7.0.1 *Every D-dimensional spherical harmonic Y_l can be written as a homogeneous polynomial of degree l in the t_h variables.*

Proof. This proof is given by induction over D; if $D = 3$ and $m \geq 0$ [the assumption $m < 0$ is essentially equivalent, because of (7.9)], then (7.13) implies

$$ {}_2\overline{P}_l^m(\theta_2) = (\sin\theta_2)^m \left[(\cos\theta_2)^{l-m} + (\cos\theta_2)^{l-m-2} + (\cos\theta_2)^{l-m-4} + \cdots \right], $$

so

$$ \begin{aligned} &Y_l^m(\theta_2, \theta_1) = {}_2\overline{P}_l^m(\theta_2)\, e^{im\theta_1} = (t_1 + it_2)^m \\ &\cdot \left[(t_3)^{l-m} + (t_3)^{l-m-2}(t_1t_1 + t_2t_2 + t_3t_3) + (t_3)^{l-m-4}(t_1t_1 + t_2t_2 + t_3t_3)^2 + \cdots \right]; \end{aligned} $$

which means that the claim is true for $D = 3$.

Let $D > 3$ and assume that the claim is true for $D - 1$, then there exists $\widehat{P}_{l_{d-1},\cdots,l_1}$, a suitable homogeneous polynomial of degree l_{d-1} in the $t_1, \cdots, t_d$ variables, such that

$$ Y_{l_{d-1},\cdots,l_1} = \prod_{h=2}^{d-1} {}_h\overline{P}_{l_h}^{l_{h-1}}(\theta_h) \cdot e^{il_1\theta_1} = \widehat{P}_{l_{d-1},\cdots,l_1}(t_1, \cdots, t_d). $$

On the other hand, the polar system of coordinates (7.1) depends on the dimension of the carrier space, and then, in $\mathbb{R}^D$,

$$ \prod_{h=2}^{d-1} {}_h\overline{P}_{l_h}^{l_{h-1}}(\theta_h) \cdot e^{il_1\theta_1} = (\sin\theta_d)^{l_{d-1}}\, \widehat{P}_{l_{d-1},\cdots,l_1}(t_1, \cdots, t_d), $$

for the same $\widehat{P}$.

This,

$$ \begin{aligned} {}_d\overline{P}_{l_d}^{l_{d-1}}(\theta_d) &= (\sin\theta_d)^{l_{d-1}} \left[(\cos\theta_d)^{l-l_{d-1}} + (\cos\theta_d)^{l-l_{d-1}-2} + (\cos\theta_d)^{l-l_{d-1}-4} + \cdots \right] \\ &= (\sin\theta_d)^{l_{d-1}} \Big[(t_D)^{l-l_{d-1}} + (t_D)^{l-l_{D-1}-2}(t_1t_1 + \cdots + t_Dt_D) \\ &\qquad + (t_D)^{l-l_{d-1}-4}(t_1t_1 + \cdots + t_Dt_D)^2 + \cdots \Big] \end{aligned} $$

and (2.11) imply the claim. □

7.0.3.4 The Y_l are a basis of $\mathcal{L}^2\left(S^d\right)$

Let

- $\mathcal{P}_l^D$ be the vector space of polynomials in the $x_1, \cdots, x_D$ variables of degree at most l;
- $\mathcal{Q}_l^D$ be the vector space of homogeneous polynomials in the $x_1, \cdots, x_D$ variables of degree l;
- $\mathcal{T}_l^D$ be the vector space of homogeneous harmonic polynomials in the $x_1, \cdots, x_D$ variables of degree l (the $q \in \mathcal{Q}_l^D$ fulfilling $\Delta q = 0$);
- $\widetilde{\mathcal{P}}_l^D$, $\widetilde{\mathcal{Q}}_l^D$ and $\widetilde{\mathcal{T}}_l^D$ be the restriction to the sphere S^d of $\mathcal{P}_l^D$, $\mathcal{Q}_l^D$ and $\mathcal{T}_l^D$, respectively;
- $$\Omega_{l,D} := \bigoplus_{m=0}^{l} \widetilde{\mathcal{T}}_m^D, \quad \widehat{\Omega}_{l,D} := \bigoplus_{m=0}^{l} V_{m,D}. \tag{7.14}$$

The goal is to show that

$$\forall f \in \mathcal{L}^2\left(S^d\right), \forall \varepsilon > 0 \exists l \in \mathbb{N}_0, \exists g \in \widehat{\Omega}_{l,D} \quad \text{such that} \quad \|f-g\|_2 < \varepsilon. \tag{7.15}$$

The density of $C^0\left(S^d\right)$ in $\mathcal{L}^2\left(S^d\right)$ implies that it is sufficient to show (7.15) for a generic continuous function on the unit sphere; on the other hand, from the Stone-Weierstrass theorem it follows that the function f can be replaced with a polynomial, without loss of generality. According to this and

$$\mathcal{P}_l^D = \bigoplus_{m=0}^{l} \mathcal{Q}_m^D$$

it remains to show that every homogeneous polynomial can be approximated by the harmonic homogeneous ones, and then that $\Omega_{l,D} \equiv \widehat{\Omega}_{l,D}\ \forall l, D$.

In order to do this, let

$$L : p(x_1, \cdots, x_D) \in \mathcal{P}_l^D \longrightarrow \left(x_1^2 + \cdots + x_D^2\right) p(x_1, \cdots, x_D) \mathcal{P}_{l+2}^D,$$

and define in this way an internal product in $\mathcal{P}_l^D$:

$$\langle x_1^{n_1} \cdots x_D^{n_D}, x_1^{m_1} \cdots x_D^{m_D} \rangle_l := (n_1)! \cdots (n_D)! \quad \text{if} \quad n_1 = m_1, \cdots, n_D = m_D,$$

0 otherwise.

L is linear and

$$\langle L[p], q \rangle_{l+2} = \langle p, \Delta q \rangle_l \quad \forall p \in \mathcal{P}_l^D \quad \text{and} \quad q \in \mathcal{P}_{l+2}^D,$$

which means that $L^* = \Delta$; then

$$\mathcal{Q}^D_{l+2} = L\left(\mathcal{Q}^D_l\right) \oplus Ker\left(L^*\right) = r^2\mathcal{Q}^D_l \oplus \mathcal{T}^D_{l+2}.$$

This implies (the dimension of $\mathcal{Q}^D_l$ is the the number of ways to sample l elements from a set of D elements allowing for duplicates, but disregarding different orderings)

$$\begin{aligned}\dim\left(\mathcal{T}^D_l\right) &= \dim\left(\mathcal{Q}^D_l\right) - \dim\left(\mathcal{Q}^D_{l-2}\right)\\ &= \binom{D+l-1}{l} - \binom{D+l-3}{l-2} = \binom{D+l-3}{l-2}\frac{(D+2l-2)(D-1)}{l(l-1)}\\ &= \binom{D+l-3}{l-1}\frac{D+2l-2}{l},\end{aligned} \tag{7.16}$$

and also that

$$\mathcal{P}^D_l = \mathcal{T}^D_l \oplus r^2\mathcal{T}^D_{l-2} \oplus r^4\mathcal{T}^D_{l-4} \oplus \cdots \quad \Longrightarrow \quad \widetilde{\mathcal{Q}}^D_l = \widetilde{\mathcal{T}}^D_l \oplus \widetilde{\mathcal{T}}^D_{l-2} \oplus \widetilde{\mathcal{T}}^D_{l-4} \oplus \cdots,$$

in other words, every homogeneous polynomial on the sphere is a linear combination of homogeneous harmonic polynomials.

Furthermore,

$$h \in \mathcal{T}^D_l \quad \Longrightarrow \quad p = r^l q, \quad \Delta p = 0 \quad \text{and with} \quad q \in \widetilde{\mathcal{T}}^D_l;$$

this and $(7.4)_1$ imply

$$\boldsymbol{L}^2 q = l(l+D-2)q.$$

Then, both $q \in \widetilde{\mathcal{T}}^D_l$ and $Y_{\boldsymbol{l}}$ are eigenfunctions of $\boldsymbol{L}^2$ with eigenvalue $l(l+D-2)$ and homogeneous polynomials in the t_h variables of degree l; this and (7.14) imply that $\Omega_{l,D} \equiv \widehat{\Omega}_{l,D}\ \forall l, D$ is equivalent to the proof of the following

Theorem 7.0.1

$$\widetilde{\mathcal{T}}^D_{\bar{l}} = V_{\bar{l},D} \quad \forall \bar{l} \in \mathbb{N}_0, \forall D \in \mathbb{N}. \tag{7.17}$$

Proof. This proof is by induction on the dimension D of the carrier space $\mathbb{R}^D$. When $D = 3$,

$$\dim\left(\widetilde{\mathcal{T}}^3_{\bar{l}}\right) = \dim\left(\mathcal{T}^3_{\bar{l}}\right) \overset{(7.16)}{=} \binom{\bar{l}}{\bar{l}-1}\frac{2\bar{l}+1}{\bar{l}} = 2\bar{l}+1,$$

and

$$\begin{aligned}V_{\bar{l},D} &\overset{(2.14)}{=} span\left\{Y_{\boldsymbol{l}} : \bar{l} \equiv l \geq l_{d-1} \geq \cdots \geq l_2 \geq |l_1|, l_i \in \mathbb{Z}\forall i\right\}\\ &= span\left\{Y^m_{\bar{l}} : |m| \leq \bar{l}, m \in \mathbb{Z}\right\},\end{aligned}$$

so (7.17) is true when $D = 3$.

Assume that it is true for d, this means that

$$\dim V_{l_{d-1},d} = \binom{D + l_{d-1} - 4}{l_{d-1} - 1} \frac{D + 2l_{d-1} - 3}{l_{d-1}}; \tag{7.18}$$

this, the hockey stick identity (see [56] formula (2))

$$\begin{aligned}
\binom{n+1}{r+1} &= \sum_{i=r}^{n} \binom{i}{r} = \sum_{i=r}^{n} \frac{i!}{(i-r)!r!} \overset{m=i-r}{=} \sum_{m=0}^{n-r} \frac{(m+r)!}{m!r!} = \sum_{m=0}^{n-r} \binom{m+r}{m} \\
&\overset{n=a+r}{\Longrightarrow} \sum_{m=0}^{a} \binom{m+r}{m} = \binom{a+r+1}{r+1}
\end{aligned} \tag{7.19}$$

and (7.11) imply

$$\begin{aligned}
\dim V_{\bar{l},D} &= \sum_{m=0}^{\bar{l}} \dim V_{m,d} \overset{(7.18)}{=} \sum_{m=0}^{\bar{l}} \binom{D+m-4}{m-1} \frac{D+2m-3}{m} \\
&= \sum_{m=0}^{\bar{l}} \frac{(D+m-4)!}{(D-3)!(m-1)!} \frac{D+2m-3}{m} \\
&= \sum_{m=0}^{\bar{l}} \frac{(D+m-4)!}{(D-4)!m!} + 2 \sum_{m=0}^{\bar{l}} m \frac{(D+m-4)!}{(D-3)!m!} \\
&= \sum_{m=0}^{\bar{l}} \binom{D-4+m}{m} + 2 \sum_{m=1}^{\bar{l}} \binom{D-4+m}{m-1} \\
&= \sum_{m=0}^{\bar{l}} \binom{D-4+m}{m} + 2 \sum_{n=0}^{\bar{l}-1} \binom{D-3+n}{n} \\
&\overset{(7.19)}{=} \binom{\bar{l}+D-4+1}{D-4+1} + 2 \binom{\bar{l}-1+D-3+1}{D-3+1} \\
&= \binom{D+\bar{l}-3}{D-3} + 2 \binom{D+\bar{l}-3}{D-2} \\
&= \frac{(D+\bar{l}-3)!}{(D-3)!\bar{l}!} + 2 \frac{(D+\bar{l}-3)!}{(D-2)!(\bar{l}-1)!} \\
&= \binom{D+\bar{l}-3}{\bar{l}-1} \frac{2\bar{l}+D-2}{\bar{l}} = \dim \widetilde{\mathcal{T}}_{\bar{l}}^{D},
\end{aligned}$$

so the proof is finished. □

According to this last proof

- The spherical harmonics Y_l are the harmonic homogeneous polynomials on the unit sphere S^d;
- The spherical harmonics are an orhonormal basis of $\mathcal{L}^2\left(S^d\right)$;
- The collection of operators $\{L_{1,2}, C_2, \cdots, C_D\}$ is a CSCO for the $\boldsymbol{L}^2$-eigenfunctions in $\mathcal{L}^2\left(S^d\right)$;
- The dimension of $\mathcal{H}_{\Lambda,D}$ coincides with the one of $\mathcal{T}_\Lambda^{D+1}$ (and then also with the one of $V_{\Lambda,D+1}$), so
$$\dim \mathcal{H}_{\Lambda,D} = \binom{D+\Lambda-2}{\Lambda-1} \frac{D+2\Lambda-1}{\Lambda}. \tag{7.20}$$
- Every $V_{l,D}$ is the representation space of a $SO(D)$-irrep, the one with $\boldsymbol{L}^2 \equiv l(l+D-2)I$, and
$$\begin{aligned} V_{l,D} \quad &\text{is isomorphic to} \quad \bigoplus_{m=0}^{l} V_{m,d} \quad \text{if } D>3, \\ \text{while } V_{l,3} \quad &\text{is isomorphic to} \quad \bigoplus_{m=-l}^{l} V_{m,2}; \end{aligned}$$
once one defines $V_{m,2}$ as the representation space of the $SO(2)$-irrep, the one with $L_{1,2} \equiv mI$.

7.0.4 The associated Legendre function of first kind

In this section $L, l, h \in \mathbb{N}_0$ and the behavior of ${}_h\overline{P}_L^l(\theta)$ is investigated, in order to prove the regularity of $g_{n-1}(z)$ in (7.10).

The equations (7.9) and (7.12) imply that ${}_h\overline{P}_L^l(\theta)$ basically coincides (up to a multiplicative constant, depending on L, l and h) with
$$[\sin\theta]^{\frac{2-h}{2}} P_{L+\frac{h-2}{2}}^{l+\frac{h-2}{2}}(\cos\theta)$$
where P_r^s is the associated Legendre function of first kind, $L+\frac{h-2}{2}$ (and also $l+\frac{h-2}{2}$) is integer if and only if h is even, while it is half-integer if and only if h is odd; according to this, one has to analyze the following two cases.

7.0.4.1 The case h even

When h is even, then from eq. (6) pag. 148 and eq. (17) pag. 151 in [43]
$$P_l^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m P_l(x)}{dx^m}, \quad P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l \tag{7.21}$$

it follows (in the next equations there is not any multiplicative constant, depending on the indices of P, because they are not relevant in this case, except when that constant is 0)

$$P_{L+\frac{h-2}{2}}^{-(l+\frac{h-2}{2})}(\cos\theta) = (\sin\theta)^{l+\frac{h-2}{2}}\, \widetilde{P}_{L+\frac{h-2}{2}}^{l+\frac{h-2}{2}}(\cos\theta)\,,$$

where $\widetilde{P}_{L+\frac{h-2}{2}}^{l+\frac{h-2}{2}}(\cos\theta)$ is a polynomial of degree $L-l$ in $\cos\theta$ which does not contain any terms of degree $L-l-(2n+1)$, with $n \in \mathbb{N}_0$; so,

$${}_h\overline{P}_L^l(\theta) = (\sin\theta)^l\, \widetilde{P}_{L+\frac{h-2}{2}}^{l+\frac{h-2}{2}}(\cos\theta)\,.$$

In addition, from (7.9) and (7.21) it follows that the highest coefficient multiplying a power of $\cos\theta$ in $P_L^l(\cos\theta)$, when $L \geq |l|$ and $L, l \in \mathbb{Z}$ is

$$\frac{(2L)!}{2^L L!} \overset{L\leq\Lambda}{\leq} \frac{(2\Lambda)!}{2^\Lambda \Lambda!} < 2^\Lambda\left[(2\Lambda+1)!!\right]^2.$$

7.0.4.2 The case h odd

In [43] eq. (7) pag. 122 there is another explicit expression of the associated Legendre function of first kind (pay attention to the different fonts P and P, while here $\boldsymbol{F}$ is the Gauss hypergeometric function)

$$\begin{aligned} P_L^l(z) &= \frac{2^l}{\Gamma(1-l)}\left(\frac{z+1}{z-1}\right)^{\frac{l}{2}} \boldsymbol{F}\left(1-l+L, -l-L, 1-l, \frac{1-z}{2}\right) \\ &= \frac{2^l}{\Gamma(1-l)} \frac{1}{(z^2-1)^{\frac{l}{2}}} \boldsymbol{F}\left(1-l+L, -l-L, 1-l, \frac{1-z}{2}\right), \end{aligned} \tag{7.22}$$

while in [43] eq. (5) pag. 143 there is the following definition

$$\mathrm{P}_L^l(x) := e^{\frac{1}{2}il\pi} P_L^l(x+i0). \tag{7.23}$$

So, putting together (7.22) and (7.23),

$$\begin{aligned} \mathrm{P}_L^l(x) &= e^{\frac{1}{2}il\pi} \frac{2^l}{\Gamma(1-l)} \frac{1}{(x^2-1)^{\frac{l}{2}}} \boldsymbol{F}\left(1-l+L, -l-L, 1-l, \frac{1-x}{2}\right) \\ &= \frac{2^l}{\Gamma(1-l)} \frac{1}{(1-x^2)^{\frac{l}{2}}} \boldsymbol{F}\left(1-l+L, -l-L, 1-l, \frac{1-x}{2}\right); \end{aligned} \tag{7.24}$$

in this thesis there the P font is always used, but this is only a stylistic choice, in fact it is always referring to this ‘real’ function P of (7.24).

In addition, from [43] p. 161 (12)-(14) it follows

$$\begin{aligned}
\sqrt{1-x^2}P_L^l(x) &= \frac{1}{2L+1}\left[(L-l+1)(L-l+2)P_{L+1}^{l-1}(x)-(L+l-1)(L+l)P_{L-1}^{l-1}(x)\right],\\
\sqrt{1-x^2}P_L^l(x) &= \frac{1}{2L+1}\left[-P_{L+1}^{l+1}(x)+P_{L-1}^{l+1}(x)\right],\\
xP_L^l(x) &= \frac{1}{2L+1}\left[(L-l+1)P_{L+1}^{l}(x)+(L+l)P_{L-1}^{l}(x)\right];
\end{aligned} \tag{7.25}$$

so, if $L=l=\frac{1}{2}$, eq. (11) p. 101 in [43]

$$\cos az = \boldsymbol{F}\left(\frac{1}{2}a,-\frac{1}{2}a,\frac{1}{2},(\sin z)^2\right),$$

implies

$$\begin{aligned}
P_{\frac{1}{2}}^{\frac{1}{2}}(\cos\theta) &= \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\frac{1}{\sqrt{\sin\theta}}\boldsymbol{F}\left(1,-1,\frac{1}{2},\frac{1-\cos\theta}{2}\right)\\
&= \sqrt{\frac{2}{\pi}}\frac{1}{\sqrt{\sin\theta}}\boldsymbol{F}\left(1,-1,\frac{1}{2},\sin^2\frac{\theta}{2}\right) = \sqrt{\frac{2}{\pi}}\frac{\cos\theta}{\sqrt{\sin\theta}}.
\end{aligned} \tag{7.26}$$

If $L=-l=\frac{1}{2}$, then eq. (4) p. 101, eq. (18) p. 102 and eq. (3) p.105 in [43]

$$\begin{aligned}
\boldsymbol{F}(-a,b,b,z) = (1+z)^a \quad , \quad \boldsymbol{F}(a,b,c,z) = \boldsymbol{F}(b,a,c,z),\\
\boldsymbol{F}(a,b,c,z) = (1-z)^{-a}\boldsymbol{F}\left(a,c-b,c,\frac{z}{z-1}\right),
\end{aligned}$$

imply

$$\begin{aligned}
\boldsymbol{F}\left(2,0,\frac{3}{2},\frac{1-x}{2}\right) &= \left(1-\frac{1-x}{2}\right)^{-2}\boldsymbol{F}\left(2,\frac{3}{2},\frac{3}{2},\frac{\frac{1-x}{2}}{\frac{1-x}{2}-1}\right)\\
&= \left(\frac{1+x}{2}\right)^{-2}F\left[-(-2),\frac{3}{2},\frac{3}{2},-\left(-\frac{\frac{1-x}{2}}{\frac{-1-x}{2}}\right)\right]\\
&= \left(\frac{1+x}{2}\right)^{-2}\left(1-\frac{\frac{1-x}{2}}{\frac{-1-x}{2}}\right)^{-2}\\
&= \left(\frac{1+x}{2}\right)^{-2}\left(\frac{1}{\frac{1+x}{2}}\right)^{-2} = 1;
\end{aligned}$$

and then

$$P_{\frac{1}{2}}^{-\frac{1}{2}}(\cos\theta) = \frac{2^{-\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)}\sqrt{\sin\theta}\boldsymbol{F}\left(2,0,\frac{3}{2},\frac{1-\cos\theta}{2}\right) = \sqrt{\frac{2}{\pi}}\sqrt{\sin\theta}. \tag{7.27}$$

Once calculated these $P_{\frac{1}{2}}^{\pm\frac{1}{2}}$, (7.25) leads to

$$\cos\theta P_{\frac{1}{2}}^{-\frac{1}{2}}(\cos\theta) = P_{\frac{3}{2}}^{-\frac{1}{2}}(\cos\theta) = \sqrt{\frac{2}{\pi}}\cos\theta\sqrt{\sin\theta} \tag{7.28}$$

and

$$\sin\theta P_{\frac{1}{2}}^{-\frac{1}{2}}(\cos\theta) = \frac{1}{2}[2\cdot 3]P_{\frac{3}{2}}^{-\frac{3}{2}}(\cos\theta) = \sqrt{\frac{2}{\pi}}\sin\theta\sqrt{\sin\theta}; \tag{7.29}$$

for completeness, according to (7.9),

$$P_{\frac{3}{2}}^{\frac{3}{2}}(\cos\theta) = 2\sqrt{\frac{2}{\pi}}\sin\theta\sqrt{\sin\theta} \quad , \quad P_{\frac{3}{2}}^{\frac{1}{2}}(\cos\theta) = 2\sqrt{\frac{2}{\pi}}\cos\theta\sqrt{\sin\theta}. \tag{7.30}$$

At this point, in order to conclude the proof of the regularity of $g_{n-1}(z)$ in (7.10), it is necessary the following

Proposition 7.0.2 *Let L and l be half-integer and positive, $0 \le l \le L$, then*

$$P_L^{-l}(\cos\theta) = (\sin\theta)^l\, \widetilde{P}_L^{-l}(\cos\theta),$$

where $\widetilde{P}_L^{-l}(\cos\theta)$ is a polynomial of degree $L-l$ in $\cos\theta$ which does not contain any term of degree $L-l-(2h+1)$, with $h \in \mathbb{N}_0$.

Proof. This is proved by induction over L. When $L=\frac{1}{2}$ and $L=\frac{3}{2}$, (7.26)-(7.29) imply that the claim is true in these two cases. Let $L=\frac{1}{2}+n$, with $2 \le n \in \mathbb{N}$ and assume that the claim is true for $n-1$, then $(7.25)_2$ implies, if $n > n'+1 \in \mathbb{N}$ and $n' \in \mathbb{N}_0$,

$$\sin\theta P_{\frac{1}{2}+(n-1)}^{-\left[\frac{1}{2}+(n'-1)\right]} = P_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+n'\right]} + P_{\frac{1}{2}+(n-2)}^{-\left[\frac{1}{2}+n'\right]},$$

then, from

$$\sin\theta P_{\frac{1}{2}+(n-1)}^{-\left[\frac{1}{2}+(n'-1)\right]}(\cos\theta) = (\sin\theta)^{\frac{1}{2}+n'}\, \widetilde{P}_{\frac{1}{2}+(n-1)}^{-\left[\frac{1}{2}+(n'-1)\right]}(\cos\theta)$$

and

$$P_{\frac{1}{2}+(n-2)}^{-\left[\frac{1}{2}+n'\right]}(\cos\theta) = (\sin\theta)^{\frac{1}{2}+n'}\, \widetilde{P}_{\frac{1}{2}+(n-2)}^{-\left[\frac{1}{2}+n'\right]}(\cos\theta),$$

it follows that also

$$P_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+n'\right]}(\cos\theta) = (\sin\theta)^{\frac{1}{2}+n'}\, \widetilde{P}_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+n'\right]}(\cos\theta),$$

where $\widetilde{P}_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+n'\right]}$ is a polynomial of degree $n-n'$ in $\cos\theta$ which does not contain any term of degree $n-n'-(2h+1)$, with $h \in \mathbb{N}_0$.

On the other hand, $(7.25)_3$ implies

$$\cos\theta P_{\frac{1}{2}+(n-1)}^{-\left[\frac{1}{2}+(n-1)\right]}(\cos\theta) = P_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+(n-1)\right]}(\cos\theta) + 0 = P_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+(n-1)\right]}(\cos\theta)\,;$$

so

$$P_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+(n-1)\right]}(\cos\theta) = (\sin\theta)^{\frac{1}{2}+(n-1)}\,\widetilde{P}_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+(n-1)\right]}(\cos\theta)\,,$$

where $\widetilde{P}_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+(n-1)\right]}(\cos\theta)$ is a polynomial in $\cos\theta$ of degree 1.

Furthermore, $(7.25)_1$ implies

$$\sin\theta P_{\frac{1}{2}+(n-1)}^{-\left[\frac{1}{2}+(n-1)\right]}(\cos\theta) = P_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+n\right]}(\cos\theta) = (\sin\theta)^{\frac{1}{2}+n}\,\widetilde{P}_{\frac{1}{2}+(n-1)}^{-\left[\frac{1}{2}+(n-1)\right]}(\cos\theta)\,,$$

so the claim is true also for n, because this last equality means that

$$P_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+\widetilde{n}\right]}(\cos\theta) = [\sin\theta]^{\frac{1}{2}+\widetilde{n}}\,\widetilde{P}_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+\widetilde{n}\right]}(\cos\theta)\,,$$

where $\widetilde{P}_{\frac{1}{2}+n}^{-\left[\frac{1}{2}+\widetilde{n}\right]}(\cos\theta)$ is a polynomial of degree $n-\widetilde{n}$ in $\cos\theta$ which does not contain any term of degree $n-\widetilde{n}-(2h+1)$, with $h\in\mathbb{N}_0$. $\square$

It is important to underline that the hypothesis $l\geq 0$ in the last proof is not stringent, in fact the same result can be proved also when l is negative, because of (7.9).

In addition, from (7.26)-(7.30) it turns out that the highest coefficient multiplying a power of $\cos\theta$ in $P_L^l(\cos\theta)$, when $\frac{3}{2}\geq L\geq |l|$ and $L,l\in\frac{\mathbb{Z}}{2}$ is always less or equal that $2L+1$; on the other hand, from (7.25) it follows

$$\begin{aligned}
P_{L+1}^{l-1}(x) &= \frac{2L+1}{(L-l+1)(L-l+2)}\sqrt{1-x^2}P_L^l(x) + \frac{(L+l-1)(L+l)}{(L-l+1)(L-l+2)}P_{L-1}^{l-1}(x),\\
P_{L+1}^{l+1}(x) &= -\sqrt{1-x^2}(2L+1)P_L^l(x) + P_{L-1}^{l+1}(x),\\
P_{L+1}^{l}(x) &= \frac{2L+1}{L-l+1}xP_L^l(x) - \frac{L+l}{L-l+1}P_{L-1}^{l}(x);
\end{aligned}$$

and then that the highest coefficient multiplying a power of $\cos\theta$ in $P_{L+1}^{l'}(\cos\theta)$ is less then $[2(L+1)+1]^2$ times the sum of the highest coefficient multiplying a power of $\cos\theta$ in $P_L^{l'}(\cos\theta)$.

According to this, by recursion one has that the highest coefficient multiplying a power of $\cos\theta$ in $P_L^l(\cos\theta)$ is

$$c2^L\left[(2L+1)!!\right]^2 \overset{L\leq\Lambda}{\leq} 2^\Lambda\left[(2\Lambda+1)!!\right]^2.$$

7.0.5 The square-integrability of $\psi_{\boldsymbol{l},D}$

In this section we do the proof that that every $\psi_{\boldsymbol{l},D}$ is square-integrable and also the explicit calculation of $M_{l,D}$.

The integral

$$\int_{\mathbb{R}^D} |\psi_{\boldsymbol{l},D}|^2 \, dx$$

can be factorized in this way:

$$\begin{aligned}\int_{\mathbb{R}^D} |\psi_{\boldsymbol{l},D}|^2 \, dx &= |M_{l,D}|^2 \left(\int_0^{+\infty} \frac{[g_{0,l,D}(r)]^2}{r^d} r^d dr \right) \\ &\quad \cdot \left[\int_{S^d} |Y_{\boldsymbol{l}}|^2 \left(\sin^{d-1}\theta_d \sin^{d-2}\theta_{d-1} \cdots \sin\theta_2 \right) d\theta_1 d\theta_2 \cdots d\theta_d \right] \\ &\overset{(2.13)_2}{=} |M_{l,D}|^2 \int_0^{+\infty} [g_{0,l,D}(r)]^2 \, dr.\end{aligned}$$

So, proceeding as in section 6.5 of [13],

$$\int_{\mathbb{R}^D} |\psi_{\boldsymbol{l},D}|^2 \, dx = 1 \Longleftrightarrow M_{l,D} = \frac{\sqrt[8]{k_{l,D}}}{\sqrt[4]{\pi}}. \tag{7.31}$$

7.0.6 The action of t_h on $Y_{\boldsymbol{l}}$

First of all

Definition 7.0.1 *Let $L \geq |l|$ and $2 \leq j \in \mathbb{N}$, then*

$$\begin{aligned} A(L,l,j) &:= \sqrt{\frac{(L+l+j-1)(L+l+j)}{(2L+j-1)(2L+j+1)}}, \\ B(L,l,j) &:= -\sqrt{\frac{(L-l-1)(L-l)}{(2L+j-1)(2L+j-3)}}, \\ C(L,l,j) &:= -\sqrt{\frac{(L-l+2)(L-l+1)}{(2L+j-1)(2L+j+1)}}, \\ D(L,l,j) &:= \sqrt{\frac{(L+l+j-2)(L+l+j-3)}{(2L+j-1)(2L+j-3)}}, \\ F(L,l,j) &:= \sqrt{\frac{(L+l+j-1)(L-l+1)}{(2L+j-1)(2L+j+1)}}, \\ G(L,l,j) &:= \sqrt{\frac{(L-l)(L+l+j-2)}{(2L+j-1)(2L+j-3)}}. \end{aligned} \tag{7.32}$$

They fulfill

$$
\begin{aligned}
&A(L,l,j) = D(L+1,l+1,j), \quad B(L,l,j) = C(L-1,l+1,j),\\
&F(L,l,j) = G(L+1,l,j), \quad F(L,l,j)A(L+1,l,j) = A(L,l,j)F(L+1,l+1,j),\\
&G(L,l,j)B(L-1,l,j) = B(L,l,j)G(L-1,l+1,j),
\end{aligned}
\tag{7.33}
$$

but it is also important to point out something about the generalized associated Legendre functions P_r^s.

From (7.25) and (7.32), it follows

$$
\begin{aligned}
[\sin\theta]\,{}_j\overline{P}_L^l(\theta) &= \sqrt{\frac{2L+j-1}{2}}\sqrt{\frac{(L+l+j-2)!}{(L-l)!}}\,[\sin\theta]^{\frac{2-j}{2}}\,[\sin\theta]\,P_{L+\frac{j-2}{2}}^{-(l+\frac{j-2}{2})}(\cos\theta)\\
&= \sqrt{\frac{2L+j-1}{2}}\sqrt{\frac{(L+l+j-2)!}{(L-l)!}}\,[\sin\theta]^{\frac{2-j}{2}}\,\frac{1}{2L+j-1}\\
&\cdot\Big\{(L+l+j-1)(L+l+j)P_{[(L+1)+\frac{j-2}{2}]}^{-[(l+1)+\frac{j-2}{2}]}(\cos\theta)\\
&\quad -(L-l)(L-l-1)P_{[(L-1)+\frac{j-2}{2}]}^{-[(l+1)+\frac{j-2}{2}]}(\cos\theta)\Big\}\\
&= \sqrt{\frac{2L+j+1}{2}}\sqrt{\frac{(L+l+j)!}{(L-l)!}}\,[\sin\theta]^{\frac{2-j}{2}}\\
&\cdot P_{[(L+1)+\frac{j-2}{2}]}^{-[(l+1)+\frac{j-2}{2}]}(\cos\theta)\sqrt{\frac{(L+l+j-1)(L+l+j)}{(2L+j-1)(2L+j+1)}}\\
&- \sqrt{\frac{2L+j-3}{2}}\sqrt{\frac{(L+l+j-2)!}{(L-l-2)!}}\,[\sin\theta]^{\frac{2-j}{2}}\\
&\cdot P_{[(L-1)+\frac{j-2}{2}]}^{-[(l+1)+\frac{j-2}{2}]}(\cos\theta)\sqrt{\frac{(L-l-1)(L-l)}{(2L+j-1)(2L+j-3)}}\\
&= \sqrt{\frac{(L+l+j-1)(L+l+j)}{(2L+j-1)(2L+j+1)}}\,{}_j\overline{P}_{L+1}^{l+1}(\theta)\\
&+ \left[-\sqrt{\frac{(L-l-1)(L-l)}{(2L+j-1)(2L+j-3)}}\right]{}_j\overline{P}_{L-1}^{l+1}(\theta)\\
&= A(L,l,j)\,{}_j\overline{P}_{L+1}^{l+1}(\theta) + B(L,l,j)\,{}_j\overline{P}_{L-1}^{l+1}(\theta);
\end{aligned}
\tag{7.34}
$$

$$
\begin{aligned}
\left[\sin\theta\right]{}_j\overline{P}^l_L(\theta) &= \sqrt{\frac{2L+j-1}{2}}\sqrt{\frac{(L+l+j-2)!}{(L-l)!}}\left[\sin\theta\right]^{\frac{2-j}{2}}\left[\sin\theta\right]P^{-(l+\frac{j-2}{2})}_{L+\frac{j-2}{2}}(\cos\theta)\\
&= \sqrt{\frac{2L+j-1}{2}}\sqrt{\frac{(L+l+j-2)!}{(L-l)!}}\left[\sin\theta\right]^{\frac{2-j}{2}}\frac{1}{2L+j-1}\\
&\cdot\left\{-P^{-[(l-1)+\frac{j-2}{2}]}_{[(L+1)+\frac{j-2}{2}]}(\cos\theta)+P^{-[(l-1)+\frac{j-2}{2}]}_{[(L-1)+\frac{j-2}{2}]}(\cos\theta)\right\}\\
&= -\sqrt{\frac{2L+j+1}{2}}\sqrt{\frac{(L+l+j-2)!}{(L-l+2)!}}\left[\sin\theta\right]^{\frac{2-j}{2}}\\
&\cdot P^{-[(l-1)+\frac{j-2}{2}]}_{[(L+1)+\frac{j-2}{2}]}(\cos\theta)\sqrt{\frac{(L-l+2)(L-l+1)}{(2L+j-1)(2L+j+1)}}\\
&+\sqrt{\frac{2L+j-3}{2}}\sqrt{\frac{(L+l+j-4)!}{(L-l)!}}\left[\sin\theta\right]^{\frac{2-j}{2}}\\
&\cdot P^{-[(l-1)+\frac{j-2}{2}]}_{[(L-1)+\frac{j-2}{2}]}(\cos\theta)\sqrt{\frac{(L+l+j-2)(L+l+j-3)}{(2L+j-1)(2L+j-3)}}\\
&= \left[-\sqrt{\frac{(L-l+2)(L-l+1)}{(2L+j-1)(2L+j+1)}}\right]{}_j\overline{P}^{l-1}_{L+1}(\theta)\\
&+\sqrt{\frac{(L+l+j-2)(L+l+j-3)}{(2L+j-1)(2L+j-3)}}{}_j\overline{P}^{l-1}_{L-1}(\theta)\\
&= C(L,l,j){}_j\overline{P}^{l-1}_{L+1}(\theta)+D(L,l,j){}_j\overline{P}^{l-1}_{L-1}(\theta);
\end{aligned}
\tag{7.35}
$$

$$
\begin{aligned}
\left[\cos\theta\right]{}_j\overline{P}^l_L(\theta) &= \sqrt{\frac{2L+j-1}{2}}\sqrt{\frac{(L+l+j-2)!}{(L-l)!}}\left[\sin\theta\right]^{\frac{2-j}{2}}\left[\cos\theta\right]P^{-(l+\frac{j-2}{2})}_{L+\frac{j-2}{2}}(\cos\theta)\\
&= \sqrt{\frac{2L+j-1}{2}}\sqrt{\frac{(L+l+j-2)!}{(L-l)!}}\left[\sin\theta\right]^{\frac{2-j}{2}}\frac{1}{2L+j-1}\\
&\left\{(L+l+j-1)P^{-[(l)+\frac{j-2}{2}]}_{[(L+1)+\frac{j-2}{2}]}(\cos\theta)+(L-l)P^{-[(l)+\frac{j-2}{2}]}_{[(L-1)+\frac{j-2}{2}]}(\cos\theta)\right\}\\
&= \left[\sqrt{\frac{(L+l+j-1)(L-l+1)}{(2L+j-1)(2L+j+1)}}\right]{}_j\overline{P}^{l}_{L+1}(\theta)\\
&+\left[\sqrt{\frac{(L-l)(L+l+j-2)}{(2L+j-1)(2L+j-3)}}\right]{}_j\overline{P}^{l}_{L-1}(\theta)\\
&= F(L,l,j){}_j\overline{P}^{l}_{L+1}(\theta)+G(L,l,j){}_j\overline{P}^{l}_{L-1}(\theta).
\end{aligned}
\tag{7.36}
$$

These last relations are fundamental, in fact they are used in order to understand the action of a coordinate t_h (seen as a multiplication operator) on a D-dimensional spherical harmonic $Y_{\boldsymbol{l}}$.

Remark 3 *Let* $t_\pm := \frac{x_1 \pm i x_2}{\sqrt{2} r}$ *and* $t_\nu := \frac{x_\nu}{r}$*, when* $\nu \in \{1, 2, \cdots, D\}$*; obviously* $t_+ t_- + t_- t_+ = (t_1)^2 + (t_2)^2$*, so (7.34)-(7.36) can be used to write* $t_h Y_{\boldsymbol{l}}$ *in terms of other D-dimensional spherical harmonics, for instance*

$$t_+ Y_{\boldsymbol{l}} = \sin\theta_d \sin\theta_{d-1} \cdots \sin\theta_2 \left[\prod_{n=2}^{d} {}_n\overline{P}_{l_n}^{l_{n-1}}(\theta_n) \right] \frac{1}{\sqrt{2\pi}} e^{i(l_1+1)\theta_1};$$

then in the product $\sin(\theta_2) \cdot {}_2P_{l_2}^{l_1}(\theta_2)$ *it is necessary to use (7.34), because* t_+ *changes* $e^{i l_1 \theta_1}$ *to* $e^{i(l_1+1)\theta_1}$*, so*

$$\begin{aligned} t_+ Y_{\boldsymbol{l}} = & \sin\theta_d \sin\theta_{d-1} \cdots \sin\theta_3 \left[A(l_2, l_1, 2)\, {}_2P_{l_2+1}^{l_1+1}(\theta_1) + B(l_2, l_1, 2)\, {}_2P_{l_2-1}^{l_1+1}(\theta_1) \right] \\ & \cdot \left[\prod_{n=3}^{d} {}_n\overline{P}_{l_n}^{l_{n-1}}(\theta_n) \right] \frac{1}{\sqrt{2\pi}} e^{i(l_1+1)\theta_1} \end{aligned}$$

and so on with the remaining factors $\sin\theta_j \cdot {}_j\overline{P}_{l'_j}^{l'_{j-1}}(\theta_j)$*.*

Of course, this last procedure can be repeated also for t_- and then for every t_ν with $\nu \in \{3, \cdots, D\}$, while the actions of t_1 and t_2 can be recovered from the ones of t_+ and t_-. According to this, let

$$R_{h,D}(\boldsymbol{l}; \boldsymbol{l'}) := \langle Y_{\boldsymbol{l'}}, t_h Y_{\boldsymbol{l}} \rangle \tag{7.37}$$

and this definition implies that, in general

$$t_\nu Y_{\boldsymbol{l}} = \sum_{\substack{l'_j : |l_j - l'_j| = 1 \\ \text{for } j = \nu-1, \cdots, d}} R_{\nu,D}(\boldsymbol{l}; \boldsymbol{l'_\nu}) \cdot Y_{\boldsymbol{l'_\nu}}, \quad \text{where} \quad \boldsymbol{l'_\nu} := (l', l'_{d-1}, \cdots, l'_{\nu-1}, l_{\nu-2}, \cdots, l_1). \tag{7.38}$$

Remark 3 and (7.37) suggest that every $R_{\nu,D}$ can be written as a sum of elements, where every addend is a product of several A, B, C, D, F, G; it is important to note that there are some simple rules, reported in the the next lines, which help to calculate every $R_{\nu,D}$.

The first rule is that the generic term of a $R_{\nu,D}$ is always written in an 'ordered' way, in fact the factors appear in this 'order':

$$R_{\nu,D}(\cdots; \cdots) = \cdots + \cdots D(l_{j+2}, l_{j+1}, j+2) B(l_{j+1}, l_j, j+1) A(l_j, l_{j-1}, j) \cdots + \cdots$$

in other words a factor having third argument $j+1$ is always right-multiplied by a factor having third argument j and always left-multiplied by a factor having thirs argument $j+2$.

Remark 4 *The other rules are these ones:*

- *Every* $A(l_j, l_{j-1}, j)$ *is always left-multiplied by an* $A(l_{j+1}, l_j, j{+}1)$ *or* $B(l_{j+1}, l_j, j+1)$*;*
- *Every* $B(l_j, l_{j-1}, j)$ *is always left-multiplied by an* $C(l_{j+1}, l_j, j{+}1)$ *or* $D(l_{j+1}, l_j, j+1)$*;*
- *Every* $C(l_j, l_{j-1}, j)$ *is always left-multiplied by an* $A(l_{j+1}, l_j, j{+}1)$ *or* $B(l_{j+1}, l_j, j+1)$*;*
- *Every* $D(l_j, l_{j-1}, j)$ *is always left-multiplied by an* $C(l_{j+1}, l_j, j{+}1)$ *or* $D(l_{j+1}, l_j, j+1)$*;*
- *In* $R_{1,D}$ *the first factor (from right to left) is* $A(l_2, l_1, 2)$*, or* $B(l_2, l_1, 2)$*, or* $C(l_2, l_1, 2)$*, or* $D(l_2, l_1, 2)$*;*
- *In* $R_{2,D}$ *the first factor (from right to left) is* $A(l_2, l_1, 2)$*, or* $B(l_2, l_1, 2)$*, or* $C(l_2, l_1, 2)$*, or* $D(l_2, l_1, 2)$*;*
- *If* $\nu \geq 3$*, in order to calculate* $R_{\nu,D}$*, it is better to start by using* (7.36) *with* $\theta = \theta_{\nu-1}$*, and then go 'backward'.*

7.0.7 Proof of $(2.19)_1$

The definition 2.3.1 (which uses the R coefficients) allows to take the relations among the coordinates t_h (seen as multiplication operators) and obtain from them some relations among the components $L_{h,j}$ of the D-dimensional angular momentum operator.

In particular,

$$(t_1)^2 + (t_2)^2 + \cdots + (t_D)^2 = t_+t_- + t_-t_+ + (t_3)^2 + \cdots + (t_D)^2 = 1$$

implies

$$\left[(t_1)^2 + (t_2)^2 + \cdots + (t_D)^2\right] Y_{\boldsymbol{l}} = Y_{\boldsymbol{l}}; \tag{7.39}$$

but (7.38) implies also that [here $Z\left(\boldsymbol{l}, \boldsymbol{l'}\right)$ are suitable coefficients, which can be obtained from the Rs]

$$\left[(t_1)^2 + (t_2)^2 + \cdots + (t_D)^2\right] Y_{\boldsymbol{l}} = \sum_{\substack{j=1,\cdots,d \\ |l_j - l'j| \leq 2}} Z\left(\boldsymbol{l}, \boldsymbol{l'}\right) Y_{\boldsymbol{l'}}; \tag{7.40}$$

in addition, (7.39) and (7.40) imply $Z\left(\boldsymbol{l}, \boldsymbol{l'}\right) = 0$ if there is at least one j such that $l_j \neq l'_j$. On the other hand, it is obvious that

$$Z\left(\boldsymbol{l}, \boldsymbol{l}\right) =: Z\left(\boldsymbol{l}\right) = 1; \tag{7.41}$$

so only the $Z\left(\boldsymbol{l}\right)$ are relevant.

Remark 5 *Equations (7.34)-(7.36) imply that*

- *if* $R_{h,D}\left(\cdots, l_j, l_{j-1}, \cdots; \cdots, l'_j, l'_{j-1}, \cdots\right)$ *contains a factor* $A(l_j, l_{j-1}, j)$, *then* $l'_j = l_j + 1$ *and* $l'_{j-1} = l_{j-1} + 1$;
- *if* $R_{h,D}\left(\cdots, l_j, l_{j-1}, \cdots; \cdots, l'_j, l'_{j-1}, \cdots\right)$ *contains a factor* $B(l_j, l_{j-1}, j)$, *then* $l'_j = l_j - 1$ *and* $l'_{j-1} = l_{j-1} + 1$;
- *if* $R_{h,D}\left(\cdots, l_j, l_{j-1}, \cdots; \cdots, l'_j, l'_{j-1}, \cdots\right)$ *contains a factor* $C(l_j, l_{j-1}, j)$, *then* $l'_j = l_j + 1$ *and* $l'_{j-1} = l_{j-1} - 1$;
- *if* $R_{h,D}\left(\cdots, l_j, l_{j-1}, \cdots; \cdots, l'_j, l'_{j-1}, \cdots\right)$ *contains a factor* $D(l_j, l_{j-1}, j)$, *then* $l'_j = l_j - 1$ *and* $l'_{j-1} = l_{j-1} - 1$;
- *if* $R_{h,D}\left(\cdots, l_j, l_{j-1}, \cdots; \cdots, l'_j, l'_{j-1}, \cdots\right)$ *contains a factor* $F(l_j, l_{j-1}, j)$, *then* $l'_j = l_j + 1$ *and* $l'_{j-1} = l_{j-1}$;
- *if* $R_{h,D}\left(\cdots, l_j, l_{j-1}, \cdots; \cdots, l'_j, l'_{j-1}, \cdots\right)$ *contains a factor* $G(l_j, l_{j-1}, j)$, *then* $l'_j = l_j - 1$ *and* $l'_{j-1} = l_{j-1}$;

in other words, these A, B, C, D, F, G *express that an index is raising or lowering, as in* remark 3.

Furthermore

$$\begin{aligned} 1 =& t_+ t_- + t_- t_+ + (t_3)^2 + \cdots + (t_D)^2 \\ =& [\cos\theta_d]^2 + [\sin\theta_d]^2 \left\{ [\cos\theta_{d-1}]^2 + [\sin\theta_{d-1}]^2 \left\{ \cdots \left\{ [\cos\theta_2]^2 + [\sin\theta_2]^2 \right\} \cdots \right\} \right\} \end{aligned} \tag{7.42}$$

implies

$$\begin{aligned} \left\{ [\cos\theta_2]^2 + [\sin\theta_2]^2 \right\} Y_{\boldsymbol{l}} =& \left\{ [F(l_2, l_1, 2)]^2 + [A(l_2, l_1, 2)]^2 + [C(l_2, l_1, 2)]^2 \right. \\ & \left. + [G(l_2, l_1, 2)]^2 + [B(l_2, l_1, 2)]^2 + [D(l_2, l_1, 2)]^2 \right\} \\ & \cdot Y_{\boldsymbol{l}} \\ =:& \left\{ Z_{1,2}(l_2) + Z_{2,2}(l_2) \right\} Y_{\boldsymbol{l}}; \end{aligned} \tag{7.43}$$

while remark 4 implies

$$\begin{aligned} & \left\{ [\cos\theta_3]^2 + [\sin\theta_3]^2 \left\{ [\cos\theta_2]^2 + [\sin\theta_2]^2 \right\} \right\} Y_{\boldsymbol{l}} = \\ & \left\{ [F(l_3, l_2, 3)]^2 + [A(l_3, l_2, 3)]^2 Z_{1,2}(l_2) + [C(l_3, l_2, 3)]^2 Z_{2,2}(l_2) \right. \\ & \left. + [G(l_3, l_2, 3)]^2 + [B(l_3, l_2, 3)]^2 Z_{1,2}(l_2) + [D(l_3, l_2, 3)]^2 Z_{2,2}(l_2) \right\} \\ & \cdot Y_{\boldsymbol{l}} \\ =:& \left\{ Z_{1,3}(l_3, l_2) + Z_{2,3}(l_3, l_2) \right\} Y_{\boldsymbol{l}}; \end{aligned} \tag{7.44}$$

and so on with the other elements of (7.42), *so*

$$\begin{aligned}
&\left\{[\cos\theta_j]^2 + [\sin\theta_j]^2\left\{[\cos\theta_{j-1}]^2 + [\sin\theta_{j-1}]^2\left\{\cdots\left\{[\cos\theta_2]^2 + [\sin\theta_2]^2\right\}\right\}\right\}\right\} Y_{\boldsymbol{l}} = \\
&\left\{[F(l_j, l_{j-1}, j)]^2 + [A(l_j, l_{j-1}, j)]^2 Z_{1,j-1}\left(l_{j-1}, l_{j-2}, \cdots, l_2\right)\right. \\
&+[C(l_3, l_2, 3)]^2 Z_{2,j-1}\left(l_{j-1}, l_{j-2}, \cdots, l_2\right) + [G(l_j, l_{j-1}, j)]^2 \\
&\left.+[B(l_j, l_{j-1}, j)]^2 Z_{1,j-1}\left(l_{j-1}, l_{j-2}, \cdots, l_2\right) + [D(l_3, l_2, 3)]^2 Z_{2,j-1}\left(l_{j-1}, l_{j-2}, \cdots, l_2\right)\right\} \\
&\cdot Y_{\boldsymbol{l}} \\
=:& \left\{Z_{1,j}\left(l_j, l_{j-1}, \cdots, l_2\right) + Z_{2,j}\left(l_j, l_{j-1}, \cdots, l_2\right)\right\} Y_{\boldsymbol{l}}.
\end{aligned} \tag{7.45}$$

It is important to underline that every $Z_{h,j}$ defined above does not depend on the dimension D of the ambient space, and this is a direct consequence of the factorization in (7.11).

A crucial result of this section is the following

Proposition 7.0.3

$$Z_{1,d}(\boldsymbol{l}) = \frac{l+d-1}{2l+d-1} \quad , \quad Z_{2,d}(\boldsymbol{l}) = \frac{l}{2l+d-1}. \tag{7.46}$$

Proof. The proof is by induction on the dimension D of the carrier space $\mathbb{R}^D$. If $D = 3$, then

$$\begin{aligned}
t_+ Y_{l_2,l_1} &= A(l_2, l_1, 2) Y_{l_2+1,l_1+1} + B(l_2, l_1, 2) Y_{l_2-1,l_1+1}, \\
t_- Y_{l_2,l_1} &= C(l_2, l_1, 2) Y_{l_2+1,l_1-1} + D(l_2, l_1, 2) Y_{l_2-1,l_1-1}, \\
t_3 Y_{l_2,l_1} &= F(l_2, l_1, 2) Y_{l_2+1,l_1} + G(l_2, l_1, 2) Y_{l_2-1,l_1};
\end{aligned}$$

and

$$\begin{aligned}
\left[t_+t_- + t_-t_+ + (t_3)^2\right] Y_{l_2,l_1} = &\left\{\frac{1}{2}[A(l_2, l_1, 2)]^2 + \frac{1}{2}[B(l_2, l_1, 2)]^2 + \frac{1}{2}[C(l_2, l_1, 2)]^2 + \right. \\
&\left.\frac{1}{2}[D(l_2, l_1, 2)]^2 + [F(l_2, l_1, 2)]^2 + [G(l_2, l_1, 2)]^2\right\} Y_{l_2,l_1};
\end{aligned}$$

so

$$Z_{1,2}(l_2) = \frac{l_2+3-3}{2l_2+3-3} = \frac{1}{2}, \quad Z_{2,2}(l_2) = \frac{l_2}{2l_2+3-3} = \frac{1}{2},$$

then (7.46) is true when $D = 3$. Let $D > 3$ and assume that (7.46) is true for

$D-1$, from (7.45) it follows

$$\begin{aligned}
Z_{1,d}\left(l, l_{d-1}, \cdots, l_2\right) = & \left[F(l, l_{d-1}, d)\right]^2 \\
& + \left[A(l, l_{d-1}, d)\right]^2 Z_{1,d-1}\left(l_{d-1}, l_{d-2}, \cdots, l_2\right) \\
& + \left[C(l, l_{d-1}, d)\right]^2 Z_{2,d-1}\left(l_{d-1}, l_{d-2}, \cdots, l_2\right) \\
= & \frac{(l+l_{d-1}+d-1)(l-l_{d-1}+1)}{(2l+d-1)(2l+D)} \\
& + \frac{(l+l_{d-1}+d-1)(l+l_{d-1}+d)}{(2l+d-1)(2l+D)} \frac{l_{d-1}+d-2}{2l_{d-1}+d-2} \\
& + \frac{(l-l_{d-1}+2)(l-l_{d-1}+1)}{(2l+d-1)(2l+D)} \frac{l_{d-1}}{2l_{d-1}+d-2} \\
= & \frac{l+d-1}{2l+d-1},
\end{aligned} \tag{7.47}$$

and

$$\begin{aligned}
Z_{2,d}\left(l, l_{d-1}, \cdots, l_2\right) = & \left[G(l, l_{d-1}, d)\right]^2 \\
& + \left[B(l, l_{d-1}, d)\right]^2 Z_{1,d-1}\left(l_{d-1}, l_{d-2}, \cdots, l_2\right) \\
& + \left[D(l, l_{d-1}, d)\right]^2 Z_{2,d-1}\left(l_{d-1}, l_{d-2}, \cdots, l_2\right) \\
= & \frac{(l-l_{d-1})(l+l_{d-1}+d-2)}{(2l+d-1)(2l+D-4)} \\
& + \frac{(l-l_{d-1}-1)(l-l_{d-1})}{(2l+d-1)(2l+D-4)} \frac{l_{d-1}+d-2}{2l_{d-1}+d-2} \\
& + \frac{(l+l_{d-1}+d-2)(l+l_{d-1}+D-4)}{(2l+d-1)(2l+D-4)} \frac{l_{d-1}}{2l_{d-1}+d-2} \\
= & \frac{l}{2l+d-1};
\end{aligned} \tag{7.48}$$

so the proof is finished. □

It is interesting to note that, because of this last proposition,

$$Z_{1,d}\left(\boldsymbol{l}\right) + Z_{2,d}\left(\boldsymbol{l}\right) = 1, \tag{7.49}$$

which agrees with

$$Y_{\boldsymbol{l}} = \left[t_+t_- + t_-t_+ + (t_3)^2 + \cdots + (t_D)^2\right] Y_{\boldsymbol{l}} = \left\{Z_{1,d}\left(\boldsymbol{l}\right) + Z_{2,d}\left(\boldsymbol{l}\right)\right\} Y_{\boldsymbol{l}}. \tag{7.50}$$

Here comes the proof of $(2.19)_1$.

Theorem 7.0.2 *The definition 2.3.1 implies*

$$\boldsymbol{L}^2 Y_{\boldsymbol{l}} := \sum_{1\le i<j\le D} L^2_{h,j} Y_{\boldsymbol{l}} = l\,(l+D-2)\, Y_{\boldsymbol{l}}. \tag{7.51}$$

Proof. This proof is by induction on the dimension D of the carrier space; if $D=2$, then $\boldsymbol{L}^2 Y_{l_1} = L^2_{1,2} Y_{l_1} = (l_1)^2 Y_{l_1}$; so (7.51) is true for $D=2$. Let $D>2$ and assume that (7.51) is true for $D-1$, which means that

$$\sum_{1\le i<j\le d} L^2_{h,j} Y_{\boldsymbol{l}} = l_{d-1}\,(l_{d-1}+D-3)\, Y_{\boldsymbol{l}}. \tag{7.52}$$

From remark 5, proposition 7.0.3 and definition 2.3.1 it follows

$$\begin{aligned}
&\sum_{i=1}^{d} L^2_{h,D} Y_{\boldsymbol{l}} = \left(d_{l,l_{d-1},D}\right)^2 Z_{2,d-1}\,(l_{d-1},\cdots,l_1)\, Y_{\boldsymbol{l}} \\
&+ \left(d_{l,l_{d-1}+1,D}\right)^2 Z_{1,d-1}\,(l_{d-1},\cdots,l_1)\, Y_{\boldsymbol{l}} \\
&= \left\{ [(l+1)(l+D-3) - l_{d-1}(l_{d-1}+D-4)]\, \frac{l_{d-1}}{2l_{d-1}+D-3} \right\} Y_{\boldsymbol{l}} \\
&+ \left\{ [(l+1)(l+D-3) - (l_{d-1}+1)(l_{d-1}+D-3)]\, \frac{l_{d-1}+D-3}{2l_{d-1}+D-3} \right\} Y_{\boldsymbol{l}} \\
&= [l\,(l+D-2) - l_{d-1}\,(l_{d-1}+D-3)]\, Y_{\boldsymbol{l}}.
\end{aligned} \tag{7.53}$$

The proof can be now completed because

$$\boldsymbol{L}^2 Y_{\boldsymbol{l}} = \sum_{1\le h<j\le D} L^2_{h,j} Y_{\boldsymbol{l}} = \sum_{1\le h<j\le d} L^2_{h,j} Y_{\boldsymbol{l}} + \sum_{j=1}^{d} L^2_{j,D} Y_{\boldsymbol{l}} \overset{(7.52)\&(7.53)}{=} l\,(l+D-2)\, Y_{\boldsymbol{l}}. \tag{7.54}$$

□

7.0.8 Proof of $(2.19)_2$

The definition 2.3.1 is given by induction on the dimension D of the carrier space $\mathbb{R}^D$, this means that, in order to prove $(2.19)_2$, it is sufficient to show that

$$\begin{aligned}
&[L_{h,D}, L_{j,D}] = i L_{h,j} \quad , \quad [L_{h,j}, L_{j,D}] = \frac{1}{i} L_{h,D}, \\
&[L_{h,j}, L_{p,D}] = 0 \text{ if } D \neq h,j \text{ and } p \neq h,j.
\end{aligned} \tag{7.55}$$

7.0.8.1 Proof of $(7.55)_1$

Let $h<j$, of course $[t_h, t_j] Y_{\boldsymbol{l}} = 0\ \forall h,j$, but this and (7.38) can be used to obtain some informations about the action of $[L_{h,D}, L_{j,D}]$ on a spherical harmonic $Y_{\boldsymbol{l}}$.

It is important to point out that

Remark 6 *Let* $1 \leq h < j \leq d$, *then* $t_h t_j Y_{l_{d-1},l_{d-2},\cdots,l_1}$ *can be written as a linear combination of* $(D-1)$*-dimensional spherical harmonics* $Y_{l'_{d-1},\cdots,l'_1}$ *with, in principle,* $|l_h - l'_h| \leq 2$, $\forall h \leq d-1$.

More precisely, $t_h \cdot$ *on* $Y_{l_{d-1},l_{d-2},\cdots,l_1}$, *'modifies' only the integers* $l_{d-1}, \cdots, l_{h-1}$, *while* $t_j \cdot$ *'modifies' only* $l_{d-1}, \cdots, l_{j-1}$, *then the 'modified' integers from the action of* $t_h t_j \cdot$, *as the ones from the action of* $t_j t_h \cdot$ *are* $l_{d-1}, \cdots, l_{h-1}$ *and, in particular,* $|l_p - l'_p| \leq 2$ *if* $p \in \{d-1, \cdots, j-1\}$, *while* $|l_p - l'_p| = 1$ *if* $p \in \{j-2, \cdots, h-1\}$.

Then

$$0 = [t_h, t_j] \, Y_{{}_d\boldsymbol{l}} = \sum_{\substack{|l'_{p_1} - l_{p_1}| \leq 2 \\ p_1 = d-1,\cdots,j-1 \\ |l'_{p_2} - l_{p_2}| = 1 \\ p_2 = j-2,\cdots,h-1}} Q_{D,h,j} \left({}_d\boldsymbol{l}, {}_d\boldsymbol{l}'_h \right) Y_{{}_d\boldsymbol{l}'_h}, \tag{7.56}$$

where

$${}_d\boldsymbol{l} := (l_{d-1}, \cdots, l_1) \quad \textit{and} \quad {}_d\boldsymbol{l}'_h := (l'_{d-1}, \cdots, l'_{h-1}, l_{h-2}, \cdots, l_1) \, .$$

Furthermore, the definition 2.3.1 implies that the action of $L_{h,D}$ on $Y_{\boldsymbol{l}}$ is similar to the action of the coordinate t_h on $Y_{{}_d\boldsymbol{l}}$, the only difference is given by the presence of the d coefficients; so

$$[L_{h,D}, L_{j,D}] \, Y_{\boldsymbol{l}} = \sum_{\substack{|l'_{p_1} - l_{p_1}| \leq 2 \\ p_1 = d-1,\cdots,j-1 \\ |l'_{p_2} - l_{p_2}| = 1 \\ p_2 = j-2,\cdots,h-1}} \widetilde{Q}_{D,h,j} \left(\boldsymbol{l}, \boldsymbol{l}'_h \right) Y_{\boldsymbol{l}'_h}, \tag{7.57}$$

where

$$\boldsymbol{l}'_h := (l, l'_{d-1}, \cdots, l'_{h-1}, l_{h-2}, \cdots, l_1) \, .$$

It is necessary to prove the following

Proposition 7.0.4

$$\widetilde{Q}_{D,h,j} \left(\boldsymbol{l}, \boldsymbol{l}'_h \right) = 0$$

if there exists at least one $p \in \{d-1, \cdots, j-1\}$ *such that* $|l_p - l'_p| = 2$.

Proof. First of all, if $l_{d-1} \neq l'_{d-1}$, for example $l'_{d-1} = l_{d-1} + 2$ (the case $l'_{d-1} = l_{d-1} - 2$ is similar), then

$$\widetilde{Q}_{D,h,j} \left(\boldsymbol{l}, \boldsymbol{l}'_h \right) = -d_{l,l_{d-1}+1,D} d_{l,l_{d-1}+2,D} Q_{D,h,j} \left(\boldsymbol{l}, \boldsymbol{l}'_h \right),$$

but $Q_{D,h,j} \left(\boldsymbol{l}, \boldsymbol{l}'_h \right) = 0$ because of (7.56), and this implies $\widetilde{Q}_{D,h,j} \left(\boldsymbol{l}, \boldsymbol{l}'_h \right) = 0$.

Furthermore, if $j \leq d-1$ and $l_{d-1} = l'_{d-1}$, while $l'_{d-2} = l_{d-2} + 2$ (the case $l'_{d-2} = l_{d-2} - 2$ is similar), then it must be

$$\begin{aligned} \left\langle Y_{\boldsymbol{l}'_h}, L_{h,D} L_{j,D} Y_{\boldsymbol{l}} \right\rangle = & \left[d_{l,l_{d-1},D} B \left(l_{d-1}, l_{d-2}, d-1 \right) d_{l,l_{d-1},D} A \left(l_{d-1} - 1, l_{d-2} + 1, d-1 \right) \right. \\ & \left. + d_{l,l_{d-1}+1,D} A \left(l_{d-1}, l_{d-2}, d-1 \right) d_{l,l_{d-1}+1,D} B \left(l_{d-1} + 1, l_{d-2} + 1, d-1 \right) \right] g \left(\boldsymbol{l}, \boldsymbol{l}'_h \right) \\ =: & \widetilde{g} \left(l, l_{d-1}, l_{d-2} \right) g \left(\boldsymbol{l}, \boldsymbol{l}'_h \right), \end{aligned}$$

for a certain function g and, similarly,

$$\begin{aligned}\left\langle Y_{\boldsymbol{l}'_h}, L_{j,D} L_{h,D} Y_{\boldsymbol{l}}\right\rangle = &\left[d_{l,l_{d-1},D} B\left(l_{d-1}, l_{d-2}, d-1\right) d_{l,l_{d-1},D} A\left(l_{d-1}-1, l_{d-2}+1, d-1\right)\right. \\ &\left.+d_{l,l_{d-1}+1,D} A\left(l_{d-1}, l_{d-2}, d-1\right) d_{l,l_{d-1}+1,D} B\left(l_{d-1}+1, l_{d-2}+1, d-1\right)\right] g\left(\boldsymbol{l}, \boldsymbol{l}'_h\right) \\ =:&\widetilde{g}\left(l, l_{d-1}, l_{d-2}\right) g\left(\boldsymbol{l}, \boldsymbol{l}'_h\right),\end{aligned}$$

for the same function g, because $Q_{D,h,j}\left(\boldsymbol{l}, \boldsymbol{l}'_h\right)=0$; so, also in this case $\widetilde{Q}_{D,h,j}\left(\boldsymbol{l}, \boldsymbol{l}'_h\right)=0$.

In general, if there is a $p \in \{d-1, \cdots, j+1\}$ such that $l'_{p-1}=l_{p-1}+2$ (the case $l'_{p-1}=l_{p-1}-2$ is similar), while $l_q=l'_q$ $\forall q \geq p$, then

$$\begin{aligned}\left\langle Y_{\boldsymbol{l}'_h}, L_{h,D} L_{j,D} Y_{\boldsymbol{l}}\right\rangle =&g_1\left(l, l_{d-1}, \cdots, l_p\right) \\ &\cdot\left[A\left(l_p, l_{p-1}, p\right) B\left(l_p+1, l_{p-1}+1, p\right)+B\left(l_p, l_{p-1}, p\right) A\left(l_p-1, l_{p-1}+1, p\right)\right] \\ &\cdot g_2\left(\boldsymbol{l}, \boldsymbol{l}'_h\right),\end{aligned}$$

and

$$\begin{aligned}\left\langle Y_{\boldsymbol{l}'_h}, L_{j,D} L_{h,D} Y_{\boldsymbol{l}}\right\rangle =&g_1\left(l, l_{d-1}, \cdots, l_p\right) \\ &\cdot\left[A\left(l_p, l_{p-1}, p\right) B\left(l_p+1, l_{p-1}+1, p\right)+B\left(l_p, l_{p-1}, p\right) A\left(l_p-1, l_{p-1}+1, p\right)\right] \\ &\cdot g_2\left(\boldsymbol{l}, \boldsymbol{l}'_h\right),\end{aligned}$$

for the same functions g_1 (because $l_q=l'_q$ $\forall q \geq p$) and g_2 [because $Q_{D,h,j}\left(\boldsymbol{l}, \boldsymbol{l}'_h\right)=0$]; so, also in this case, $\widetilde{Q}_{D,h,j}\left(\boldsymbol{l}, \boldsymbol{l}'_h\right)=0$.

Finally, if $l'_{j-1}=l_{j-1}+2$ (the case $l'_{j-1}=l_{j-1}-2$ is similar), $l'_{j-2}=l_{j-2}+1$ (also here, the case $l'_{j-2}=l_{j-2}-1$ is similar), while $l_q=l'_q$ $\forall q \geq j$, then

$$\begin{aligned}\left\langle Y_{\boldsymbol{l}'_h}, L_{h,D} L_{j,D} Y_{\boldsymbol{l}}\right\rangle =&g_3\left(l, l_{d-1}, \cdots, l_j\right) \\ &\cdot\left[F\left(l_{j-1}, l_{j-2}, j-1\right) A\left(l_{j-1}+1, l_{j-2}, j-1\right)\right] \\ &\cdot g_4\left(\boldsymbol{l}, \boldsymbol{l}'_h\right),\end{aligned}$$

$$\begin{aligned}\left\langle Y_{\boldsymbol{l}'_h}, L_{j,D} L_{h,D} Y_{\boldsymbol{l}}\right\rangle =&g_3\left(l, l_{d-1}, \cdots, l_j\right) \\ &\cdot\left[A\left(l_{j-1}, l_{j-2}, j-1\right) F\left(l_{j-1}+1, l_{j-2}+1, j-1\right)\right] \\ &\cdot g_4\left(\boldsymbol{l}, \boldsymbol{l}'_h\right),\end{aligned}$$

for the same functions g_3 (because $l_q=l'_q$ $\forall q \geq j$) and g_4 [because $Q_{D,h,j}\left(\boldsymbol{l}, \boldsymbol{l}'_h\right)=0$]; so, because of (7.33), $\widetilde{Q}_{D,h,j}\left(\boldsymbol{l}, \boldsymbol{l}'_h\right)=0$. □

The last proof implies

$$\left[L_{h,D}, L_{j,D}\right] Y_{\boldsymbol{l}}=\sum_{\substack{|l'_p-l_p|=1 \\ p=j-2, \cdots, h-1}} \widetilde{Q}_{D,h,j}\left(\boldsymbol{l}, \boldsymbol{l}'_h\right) Y_{\boldsymbol{l}'_h}, \tag{7.58}$$

and from now on assume that $l'_p=l_p$ $\forall p \geq j-1$, otherwise $\widetilde{Q}_{D,h,j}\left(\boldsymbol{l}, \boldsymbol{l}'_h\right)=0$.

It is necessary to further investigate about these

$$\widetilde{Q}_{D,h,j}\left(\boldsymbol{l},\boldsymbol{l}'_h\right) \quad \text{when } |l'_p - l_p| = 1, p \in \{j-2,\cdots,h-1\}.$$

In order to do this, let

$$\begin{aligned}
T_1^1(l_p,l_{p-1},p) &:= A(l_p,l_{p-1},p)G(l_p+1,l_{p-1}+1,p) - F(l_p,l_{p-1},p)B(l_p+1,l_{p-1},p),\\
T_2^1(l_p,l_{p-1},p) &:= B(l_p,l_{p-1},p)F(l_p-1,l_{p-1}+1,p) - G(l_p,l_{p-1},p)A(l_p-1,l_{p-1},p),\\
T_3^1(l_p,l_{p-1},p) &:= C(l_p,l_{p-1},p)G(l_p+1,l_{p-1}-1,p) - F(l_p,l_{p-1},p)D(l_p+1,l_{p-1},p),\\
T_4^1(l_p,l_{p-1},p) &:= D(l_p,l_{p-1},p)F(l_p-1,l_{p-1}-1,p) - G(l_p,l_{p-1},p)C(l_p-1,l_{p-1},p);
\end{aligned}$$

they fulfill

$$\begin{aligned}
T_1^1(l_p,l_{p-1},p) &= \frac{\sqrt{(l_p+l_{p-1}+p-1)(l_p-l_{p-1})}}{2l_p+p-1} = \frac{d_{l_p,l_{p-1}+1,p+1}}{2l_p+p-1},\\
T_2^1(l_p,l_{p-1},p) &= -\frac{\sqrt{(l_p+l_{p-1}+p-1)(l_p-l_{p-1})}}{2l_p+p-1} = -\frac{d_{l_p,l_{p-1}+1,p+1}}{2l_p+p-1},\\
T_3^1(l_p,l_{p-1},p) &= -\frac{\sqrt{(l_p+l_{p-1}+p-2)(l_p-l_{p-1}+1)}}{2l_p+p-1} = -\frac{d_{l_p,l_{p-1},p+1}}{2l_p+p-1},\\
T_4^1(l_p,l_{p-1},p) &= \frac{\sqrt{(l_p+l_{p-1}+p-2)(l_p-l_{p-1}+1)}}{2l_p+p-1} = \frac{d_{l_p,l_{p-1},p+1}}{2l_p+p-1}.
\end{aligned} \tag{7.59}$$

Similarly, for $n \geq 2$, let

$$\begin{aligned}
T_1^n(l_{p+n-1},l_p,l_{p-1},p) :=& \left[A(l_{p+n-1},l_{p+n-2},p+n-1)\right]^2 T_1^{n-1}(l_{p+n-2},l_p,l_{p-1},p)\\
&+ \left[C(l_{p+n-1},l_{p+n-2},p+n-1)\right]^2 T_2^{n-1}(l_{p+n-2},l_p,l_{p-1},p)\\
=&\frac{d_{l_p,l_{p-1}+1,p+1}}{2l_{p+n-1}+p+n-2},\\
T_2^n(l_{p+n-1},l_p,l_{p-1},p) :=& \left[B(l_{p+n-1},l_{p+n-2},p+n-1)\right]^2 T_1^{n-1}(l_{p+n-2},l_p,l_{p-1},p)\\
&+ \left[D(l_{p+n-1},l_{p+n-2},p+n-1)\right]^2 T_2^{n-1}(l_{p+n-2},l_{p-1},p)\\
=&-\frac{d_{l_p,l_{p-1}+1,p+1}}{2l_{p+n-1}+p+n-2},\\
T_3^n(l_{p+n-1},l_p,l_{p-1},p) :=& \left[A(l_{p+n-1},l_{p+n-2},p+n-1)\right]^2 T_3^{n-1}(l_{p+n-2},l_p,l_{p-1},p)\\
&+ \left[C(l_{p+n-1},l_{p+n-2},p+n-1)\right]^2 T_4^{n-1}(l_{p+n-2},l_p,l_{p-1},p)\\
=&-\frac{d_{l_p,l_{p-1},p+1}}{2l_{p+n-1}+p+n-2},\\
T_4^n(l_{p+n-1},l_p,l_{p-1},p) :=& \left[B(l_{p+n-1},l_{p+n-2},p+n-1)\right]^2 T_3^{n-1}(l_{p+n-2},l_p,l_{p-1},p)\\
&+ \left[D(l_{p+n-1},l_{p+n-2},p+n-1)\right]^2 T_4^{n-1}(l_{p+n-2},l_p,l_{p-1},p)\\
=&\frac{d_{l_p,l_{p-1},p+1}}{2l_{p+n-1}+p+n-2}.
\end{aligned} \tag{7.60}$$

Assume (witout loss of generality) that $l'_{j-2} = l_{j-2} - 1$, then

$$\begin{aligned}
\widetilde{Q}_{D,h,j}\left(\boldsymbol{l},\boldsymbol{l'_h}\right) = \Big\{ \left(d_{l,l_{d-1}+1,D}\right)^2 & \left\{[A(l_{d-1},l_{d-2},d-1)]^2 \ \left\{[A(l_{d-2},l_{d-3},D-3)]^2 \{\cdots\} \right.\right. \\
& \qquad\qquad \left. + [C(l_{d-2},l_{d-3},D-3)]^2 \{\cdots\}\right\} \\
& + [C(l_{d-1},l_{d-2},d-1)]^2 \left\{[B(l_{d-2},l_{d-3},D-3)]^2 \{\cdots\} \right. \\
& \qquad\qquad \left.\left. + [D(l_{d-2},l_{d-3},D-3)]^2 \{\cdots\}\right\}\right\} \\
+ \left(d_{l,l_{d-1},D}\right)^2 & \left\{[B(l_{d-1},l_{d-2},d-1)]^2 \ \left\{[A(l_{d-2},l_{d-3},D-3)]^2 \{\cdots\} \right.\right. \\
& \qquad\qquad \left. + [C(l_{d-2},l_{d-3},D-3)]^2 \{\cdots\}\right\} \\
& + [D(l_{d-1},l_{d-2},d-1)]^2 \left\{[B(l_{d-2},l_{d-3},D-3)]^2 \{\cdots\} \right. \\
& \qquad\qquad \left.\left.\left. + [D(l_{d-2},l_{d-3},D-3)]^2 \{\cdots\}\right\}\right\}\right\} \\
& \cdot R_{h,j-1}\left(\boldsymbol{l}|^{j,h}, \widetilde{\boldsymbol{l'}}|^{j,h}\right),
\end{aligned} \tag{7.61}$$

where

$$\boldsymbol{l}|^{j,h} := \left(l_{j-2},\cdots,l_{h-1},l_{h-2}\right) \quad , \quad \widetilde{\boldsymbol{l'}}|^{j,h} := \left(l_{j-2}-1,\cdots,l'_{h-1},l_{h-2}\right).$$

Remark 7 *The $\{\cdots\}$ in (7.61) is such that*

- *every $[A(l_h,l_{h-1},h)]^2$ is always left-multiplied by $[A(l_{h+1},l_h,h+1)]^2$ or $[B(l_{h+1},l_h,h+1)]^2$;*
- *every $[B(l_h,l_{h-1},h)]^2$ is always left-multiplied by $[C(l_{h+1},l_h,h+1)]^2$ or $[D(l_{h+1},l_h,h+1)]^2$;*
- *every $[C(l_h,l_{h-1},h)]^2$ is always left-multiplied by $[A(l_{h+1},l_h,h+1)]^2$ or $[B(l_{h+1},l_h,h+1)]^2$;*
- *every $[D(l_h,l_{h-1},h)]^2$ is always left-multiplied by $[C(l_{h+1},l_h,h+1)]^2$ or $[D(l_{h+1},l_h,h+1)]^2$;*
- *the most 'internal' term of $\{\cdots\}$ is $T_p\left(l_{j-1},l_{j-2},j\right)$, with $p\in\{1,2,3,4\}$;*
- *every $T_1^1\left(l_{j-1},l_{j-2},j\right)$ and $T_2^1\left(l_{j-1},l_{j-2},j\right)$ are always left-multiplied by $[A(l_j,l_{j-1},j)]^2$ or $[B(l_j,l_{j-1},j)]^2$;*
- *every $T_3^1\left(l_{j-1},l_{j-2},j\right)$ and $T_4^1\left(l_{j-1},l_{j-2},j\right)$ are always left-multiplied by $[C(l_j,l_{j-1},b)]^2$ or $[D(l_j,l_{j-1},j)]^2$.*

This and (7.60) imply

$$\begin{aligned}\widetilde{Q}_{D,h,j}\left(\boldsymbol{l},\boldsymbol{l'}_h\right) = \Big\{ & \left(d_{l,l_{d-1}+1,D}\right)^2 \{\, [A(l_{d-1},l_{D-3},D-2)]^2\, T_3^{D-j-1}\left(l_{D-3},l_{j-1},l_{j-2},j-1\right) \\ & \qquad + [C(l_{d-1},l_{D-3},D-2)]^2\, T_4^{D-j-1}\left(l_{D-3},l_{j-1},l_{j-2},j-1\right)\Big\} \\ & + \left(d_{l,l_{d-1},D}\right)^2 \{\, [B(l_{d-1},l_{D-3},D-2)]^2\, T_3^{D-j-1}\left(l_{D-3},l_{j-1},l_{j-2},j-1\right) \\ & \qquad + [D(l_{d-1},l_{D-3},D-2)]^2\, T_4^{D-j-1}\left(l_{D-3},l_{j-1},l_{j-2},j-1\right)\Big\}\Big\} \\ & \qquad \cdot R_{h,j-1}\left(\boldsymbol{l}|^{j,h},\widetilde{\boldsymbol{l'}}|^{j,h}\right) \\ \overset{(7.60)}{=} & \left[-\left(d_{l,l_{d-1}+1,D}\right)^2+\left(d_{l,l_{d-1},D}\right)^2\right]\frac{d_{l_{j-1},l_{j-2},j}}{2l_{d-1}+D-3}R_{h,j-1}\left(\boldsymbol{l}|^{j,h},\widetilde{\boldsymbol{l'}}|^{j,h}\right) \\ = & \; d_{l_{j-1},l_{j-2},j}R_{h,j-1}\left(\boldsymbol{l}|^{j,h},\widetilde{\boldsymbol{l'}}|^{j,h}\right);\end{aligned} \tag{7.62}$$

and this proves the following

Theorem 7.0.3 *The equation* (7.61)*, using the rules of remark 7, becomes*

$$\widetilde{Q}_{D,h,j}\left(\boldsymbol{l},\boldsymbol{l'}_h\right) = d_{l_{j-1},l_{j-2},j}R_{h,j-1}\left(\boldsymbol{l}|^{j,h},\widetilde{\boldsymbol{l'}}|^{j,h}\right). \tag{7.63}$$

The same job can be done with the assumption $l'_{j-2} = l_{j-2}+1$, in this case the result is an equation like (7.61), but with T_1 and T_2 instead of T_3 and T_4, respectively; and in this case it turns out that

$$\widetilde{Q}_{D,h,j}\left(\boldsymbol{l},\boldsymbol{l'}_h\right) = -d_{l_{j-1},l_{j-2}+1,j}\cdot R_{h,j-1}\left(\boldsymbol{l}|^{j,h},\widehat{\boldsymbol{l'}}|^{j,h}\right), \tag{7.64}$$

where $\widehat{\boldsymbol{l'}} := (l_{j-2}+1,\cdots,l'_{h-1},l_{h-2})$.

Finally, definition 2.3.1 and (7.63-7.64) imply

$$[L_{h,D},L_{j,D}] = iL_{h,j}. \tag{7.65}$$

7.0.8.2 Proof of $(7.55)_2$ and $(7.55)_3$

Let $1 \le h < j \le d$, the definition 2.3.1 implies that the action of $L_{j,D}$ in $\mathbb{R}^D$ is the same of t_j in $\mathbb{R}^d$, the only difference is given by the $\frac{1}{i}d_{l,l'_{d-1},D}$ coefficients and their signs (here $l'_{d-1} = l_{d-1}$ or $l'_{d-1} = l_{d-1}+1$), but the action of $L_{h,j}$ on a $Y_{\boldsymbol{l}}$ does not change the indices l and l_{d-1}.

According to this and (2.25), from

$$[L_{h,j},x_j] = \frac{1}{i}x_h \quad \text{it follows} \quad [L_{h,j},L_{j,D}] = \frac{1}{i}L_{h,D},$$

and from

$$[L_{h,j},x_p] = 0 \quad \text{if} \quad p \ne h,j \quad \text{it follows} \quad [L_{h,j},L_{p,D}] = 0 \text{ if } D \ne h,j \text{ and } p \ne h,j.$$

7.0.9 On the action of 'projected' coordinate operators $\overline{x}_h$

The behavior (2.15)-(2.16) of a generic $\boldsymbol{\psi_{l,D}}$ and the expression of the integration measure $d\boldsymbol{x}$ of $\mathbb{R}^D$ in spherical coordinates

$$d\boldsymbol{x} = r^{D-1}\sin^{d-1}(\theta_d)\sin^{d-2}(\theta_{d-1})\cdots\sin(\theta_2)drd\theta_1 d\theta_2\cdots d\theta_d$$

allow to factorize the scalar product $\langle\psi_{\boldsymbol{l'},D}, \overline{x}_h\boldsymbol{\psi_{l,D}}\rangle_{\mathbb{R}^D}$ in this way:

$$\langle\psi_{\boldsymbol{l'},D}, \overline{x}_h\boldsymbol{\psi_{l,D}}\rangle = \langle f_{0,l',D}, rf_{0,l,D}\rangle_{R^+}\cdot\langle Y_{\boldsymbol{l'}}, t_hY_{\boldsymbol{l}}\rangle_{S^d},$$

where

$$\langle f_{0,l',D}, rf_{0,l,D}\rangle_{\mathbb{R}^+} := M_{l,D}M_{l',D}\int_0^{+\infty} re^{-\sqrt{k_{l,D}}\left(r-\widetilde{r}_{l,D}\right)^2}e^{-\sqrt{k_{l',D}}\left(r-\widetilde{r}_{l',D}\right)^2}dr, \quad (7.66)$$

while the value of the 'angular' scalar product

$$\langle Y_{\boldsymbol{l'}}, t_hY_{\boldsymbol{l}}\rangle_{S^d} = \int_{S^d} Y^*_{\boldsymbol{l'}}t_hY_{\boldsymbol{l}}\left[\sin^{d-1}(\theta_d)\sin^{d-2}(\theta_{d-1})\cdots\sin(\theta_2)\right]d\theta_1 d\theta_2\cdots d\theta_d$$

is

$$\langle Y_{\boldsymbol{l'}}, t_hY_{\boldsymbol{l}}\rangle_{S^d} \equiv R_{h,D}\left(\boldsymbol{l},\boldsymbol{l'}\right),$$

according to sections 7.0.3 and 7.0.6.

On the other hand, as for section 6.6 in [13],

$$\langle f_{0,l\pm1,D}, rf_{0,l,D}\rangle_{\mathbb{R}^+} = M_{l,D}M_{l\pm1,D}\; e^{-\frac{\sqrt{k_{l,D}k_{l\pm1,D}}\left(\widetilde{r}_{l,D}-\widetilde{r}_{l\pm1,D}\right)^2}{2\left(\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}\right)}}\sqrt{\frac{2\pi}{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}}}\;\widehat{r}_{l,l\pm1,D};$$

with

$$\widehat{r}_{l,l\pm1,D} = \frac{\sqrt{k_{l,D}}\widetilde{r}_{l,D}+\sqrt{k_{l\pm1,D}}\widetilde{r}_{l\pm1,D}}{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}}. \quad (7.67)$$

Then, in order to calculate $\langle f_{0,l\pm1,D}, rf_{0,l,D}\rangle_{\mathbb{R}^+}$ at leading orders in $1/\sqrt{k_D}$, the following steps are needed.

First of all

$$\widetilde{r}_{l,D} = 1+\frac{b(l,D)}{2k_D}-3\frac{\left[b(l,D)\right]^2}{4k_D^2}+9\frac{\left[b(l,D)\right]^3}{8k^3}-\frac{27\left[b(l,D)\right]^4}{16k^4}+O\left(k_D^{-5}\right); \quad (7.68)$$

while

$$\begin{aligned}\sqrt{k_{l,D}} =& \sqrt{2k_D}+\frac{3}{2\sqrt{2k_D}}b(l,D)-\frac{9}{8}\frac{\left[b(l,D)\right]^2}{2k_D\sqrt{2k_D}}+\frac{27}{16}\frac{\left[b(l,D)\right]^3}{4k_D^2\sqrt{2k_D}}\\ &-\frac{405}{128}\frac{\left[b(l,D)\right]^4}{8k^3\sqrt{2k_D}}+O\left(k_D^{-4}\right),\end{aligned} \quad (7.69)$$

implies

$$\begin{aligned}\sqrt{k_{l,D}k_{l\pm1,D}} =& 2k_D + \frac{3}{2}\left[b(l,D)+b(l\pm1,D)\right] - \frac{9}{8}\frac{\left[b(l,D)-b(l\pm1,D)\right]^2}{2k_D}\\ &+ \frac{27}{16}\frac{\left[b(l,D)\right]^3+\left[b(l\pm1,D)\right]^3-\left[b(l,D)\right]^2\left[b(l\pm1,D)\right]}{4k_D^2}\\ &- \frac{\left[b(l\pm1,D)\right]^2\left[b(l,D)\right]}{4k_D^2} + O\left(k_D^{-3}\right),\end{aligned} \tag{7.70}$$

$$\begin{aligned}\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}} =& 2\sqrt{2k_D} + \frac{3}{2}\frac{b(l,D)+b(l\pm1,D)}{\sqrt{2k_D}} - \frac{9}{8}\frac{\left\{\left[b(l,D)\right]^2+\left[b(l\pm1,D)\right]^2\right\}}{2k_D\sqrt{2k_D}}\\ &+ \frac{27}{16}\frac{\left\{\left[b(l,D)\right]^3+\left[b(l\pm1,D)\right]^3\right\}}{4k_D^2\sqrt{2k_D}} - \frac{405}{128}\frac{\left\{\left[b(l,D)\right]^4+\left[b(l\pm1,D)\right]^4\right\}}{8k^3\sqrt{2k_D}}\\ &+ O\left(k_D^{-3}\right),\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}} =& \frac{1}{2\sqrt{2k_D}} - \frac{3}{8}\frac{\left[b(l,D)+b(l\pm1,D)\right]}{(2k_D)^{\frac{3}{2}}}\\ &+ \frac{9}{16}\frac{\left[b(l,D)\right]^2+\left[b(l\pm1,D)\right]^2+b(l,D)b(l\pm1,D)}{(2k_D)^{\frac{5}{2}}} + O\left(k_D^{-3}\right),\end{aligned} \tag{7.71}$$

$$\begin{aligned}\sqrt{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}} =& \sqrt{2}\sqrt[4]{2k_D} + \frac{3\sqrt{2}}{8}\frac{b(l,D)+b(l\pm1,D)}{(2k_D)^{\frac{3}{4}}}\\ &- \frac{45\sqrt{2}}{128}\frac{\left[b(l,D)\right]^2+\left[b(l\pm1,D)\right]^2+\frac{2}{5}b(l,D)b(l\pm1,D)}{(2k_D)^{\frac{7}{4}}}\\ &- \frac{567\sqrt{2}}{1024}\frac{\left[b(l,D)\right]^3+\left[b(l+1,D)\right]^2+\frac{1}{3}\left[b(l,D)\right]^2b(l\pm1,D)}{(2k_D)^{\frac{11}{4}}}\\ &+ \frac{\frac{1}{3}b(l,D)\left[b(l\pm1,D)\right]^2}{(2k_D)^{\frac{11}{4}}} + O\left(k_D^{-3}\right),\end{aligned}$$

$$\begin{aligned}\frac{1}{\sqrt{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}}} =& \frac{1}{\sqrt{2}\sqrt[4]{2k_D}} - \frac{3\sqrt{2}}{16}\frac{b(l,D)+b(l\pm1,D)}{(2k_D)^{\frac{5}{4}}}\\ &+ \frac{63\sqrt{2}}{256}\frac{\left[b(l,D)\right]^2+\left[b(l\pm1,D)\right]^2+\frac{6}{7}b(l,D)b(l\pm1,D)}{(2k_D)^{\frac{9}{4}}} + O\left(k_D^{-3}\right).\end{aligned} \tag{7.72}$$

So, from (7.31) and (7.70), it follows

$$\begin{aligned}\sqrt{\pi}M_{l,D}M_{l\pm1,D} =&\sqrt[8]{k_{l,D}k_{l\pm1,D}} = \sqrt[4]{2k_D} + \frac{3}{8}\frac{[b(l,D)+b(l\pm1,D)]}{(2k_D)^{\frac{3}{4}}}\\ &-\frac{63}{128}\frac{[b(l,D)]^2+[b(l\pm1,D)]^2-\frac{2}{7}b(l,D)b(l\pm1,D)}{(2k_D)^{\frac{7}{4}}}\\ &+\frac{945}{1024}\frac{[b(l,D)]^3+[b(l\pm1,D)]^3-\frac{1}{5}[b(l,D)]^2b(l\pm1,D)}{(2k_D)^{\frac{11}{4}}}\\ &-\frac{\frac{1}{5}[b(l\pm1,D)]^2b(l,D)}{(2k_D)^{\frac{11}{4}}}+O\left(k_D^{-3}\right),\end{aligned}$$

and then

$$M_{l,D}M_{l\pm1,D}\sqrt{\frac{2\pi}{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}}} \overset{(7.72)}{=} 1-\frac{9}{64}\frac{[b(l,D)-b(l\pm1,D)]^2}{4k_D^2}+O\left(k_D^{-3}\right). \tag{7.73}$$

Furthermore, from

$$\sqrt{k_{l,D}}\widetilde{r}_{l,D} \overset{(7.68)\&(7.69)}{=} \sqrt{2k_D}+\frac{5b(l,D)}{2\sqrt{2k_D}}-\frac{21}{8}\frac{[b(l,D)]^2}{2k_D\sqrt{2k_D}}+\frac{81}{16}\frac{[b(l,D)]^3}{4k_D^2\sqrt{2k_D}}+O\left(k_D^{-3}\right),$$

it follows

$$\begin{aligned}\sqrt{k_{l,D}}\widetilde{r}_{l,D}+\sqrt{k_{l+1,D}}\widetilde{r}_{l+1,D} =&2\sqrt{2k_D}+\frac{5[b(l,D)+b(l\pm1,D)]}{2\sqrt{2k_D}}-\frac{21}{8}\frac{\left\{[b(l,D)]^2+[b(l\pm1,D)]^2\right\}}{2k_D\sqrt{2k_D}}\\ &+\frac{81}{16}\frac{\left\{[b(l,D)]^3+[b(l\pm1,D)]^3\right\}}{4k_D^2\sqrt{2k_D}}+O\left(k_D^{-3}\right);\end{aligned}$$

then the last equalities and (7.67) imply

$$\begin{aligned}\widehat{r}_{l,l\pm1,D} =&1+\frac{1}{2}\frac{b(l,D)+b(l\pm1,D)}{2k_D}-\frac{9}{8}\frac{[b(l,D)]^2+[b(l+1,D)]^2+\frac{2}{3}b(l,D)b(l\pm1,D)}{4k_D^2}\\ &+O\left(k_D^{-3}\right);\end{aligned}$$

Similarly,

$$\begin{aligned}\left(\widetilde{r}_{l,D}-\widetilde{r}_{l\pm1,D}\right)^2 \overset{(7.68)}{=}&\frac{[b(l,D)-b(l\pm1,D)]^2}{4k_D^2}\\ &-\frac{6\left\{[b(l,D)]^3+[b(l\pm1,D)]^3-[b(l,D)]^2b(l\pm1,D)\,[b(l\pm1,D)]^2b(l,D)\right\}}{8k^3}\\ &+O\left(k_D^{-4}\right),\end{aligned} \tag{7.74}$$

$$\frac{\sqrt{k_{l,D}k_{l\pm1,D}}\left(\widetilde{r}_{l,D}-\widetilde{r}_{l\pm1,D}\right)^2}{2\left(\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}\right)} \overset{(7.70),(7.71)\&(7.74)}{=} \frac{1}{4}\frac{\left[b(l,D)-b(l\pm1,D)\right]^2}{(2k_D)^{\frac{3}{2}}}$$
$$-\frac{21}{16}\frac{\left[b(l,D)\right]^3+\left[b(l\pm1,D)\right]^3-\left[b(l,D)\right]^2 b(l\pm1,D)-\left[b(l\pm1,D)\right]^2 b(l,D)}{(2k_D)^{\frac{5}{2}}}+O\left(k_D^{-3}\right),$$

which implies

$$\begin{aligned} e^{-\frac{\sqrt{k_{l,D}k_{l\pm1,D}}\left(\widetilde{r}_{l,D}-\widetilde{r}_{l\pm1,D}\right)^2}{2\left(\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}\right)}} =& 1-\frac{1}{4}\frac{\left[b(l,D)-b(l\pm1,D)\right]^2}{(2k_D)^{\frac{3}{2}}} \\ &+\frac{21}{16}\frac{\left[b(l,D)\right]^3+\left[b(l\pm1,D)\right]^3}{(2k_D)^{\frac{5}{2}}} \\ &-\frac{\left[b(l,D)\right]^2 b(l\pm1,D)+\left[b(l\pm1,D)\right]^2 b(l,D)}{(2k_D)^{\frac{5}{2}}} \\ &+O\left(k_D^{-3}\right). \end{aligned} \tag{7.75}$$

So, according to the above equalities,

$$\begin{aligned} \langle f_{0,l\pm1,D}, r f_{0,l,D}\rangle_{\mathbb{R}^+} =& 1+\frac{1}{2}\frac{\left[b(l,D)+b(l\pm1,D)\right]}{2k_D}-\frac{1}{4}\frac{\left[b(l,D)-b(l\pm1,D)\right]^2}{(2k_D)^{\frac{3}{2}}} \\ &-\frac{81}{64}\frac{\left[b(l,D)\right]^2+\left[b(l\pm1,D)\right]^2+\frac{54}{5}b(l,D)b(l\pm1,D)}{4k_D^2}+O\left(k_D^{-\frac{5}{2}}\right). \end{aligned} \tag{7.76}$$

Furthermore, (2.22) and this last equality imply

$$\left(c_{l+1,D}\right)^2-\left(c_{l,D}\right)^2=\frac{b(l+1,D)-b(l-1,D)}{2k_D}+O\left(\frac{1}{k_D^2}\right)=\frac{2l+D-2}{k_D}+O\left(\frac{1}{k_D^2}\right). \tag{7.77}$$

Similarly, the scalar product $\langle\psi_{\boldsymbol{l'},D}, t_h\boldsymbol{\psi}_{\boldsymbol{l},D}\rangle_{\mathbb{R}^D}$ can be factorized, obtaining

$$\langle\psi_{\boldsymbol{l'},D}, t_h\boldsymbol{\psi}_{\boldsymbol{l},D}\rangle=\langle f_{0,l',D}, f_{0,l,D}\rangle_{R^+}\cdot\langle Y_{\boldsymbol{l'}}, t_h Y_{\boldsymbol{l}}\rangle_{S^d},$$

and also in this case

$$\langle Y_{\boldsymbol{l'}}, t_h Y_{\boldsymbol{l}}\rangle_{S^d}\equiv R_{h,D}\left(\boldsymbol{l},\boldsymbol{l'}\right)$$

does not vanish if $l'=\pm1$. On the other hand, as for the previous ‘radial’ scalar product,

$$\begin{aligned} \langle f_{0,l\pm1,D}, f_{0,l,D}\rangle_{R^+} =& M_{l,D}M_{l\pm1,D}\int_0^{+\infty} e^{-\frac{\sqrt{k_{l,D}}}{2}\left(r-\widetilde{r}_{l,D}\right)^2} e^{-\frac{\sqrt{k_{l\pm1,D}}}{2}\left(r-\widetilde{r}_{l\pm1,D}\right)^2}dr \\ \simeq& M_{l,D}M_{l\pm1,D}\int_{-\infty}^{+\infty} e^{-\frac{\sqrt{k_{l,D}}}{2}\left(r-\widetilde{r}_{l,D}\right)^2} e^{-\frac{\sqrt{k_{l\pm1,D}}}{2}\left(r-\widetilde{r}_{l\pm1,D}\right)^2}dr, \end{aligned}$$

with

$$\begin{aligned}
&\int_{-\infty}^{+\infty} e^{-\frac{\sqrt{k_{l,D}}}{2}\left(r-\widetilde{r}_{l,D}\right)^2} e^{-\frac{\sqrt{k_{l\pm1,D}}}{2}\left(r-\widetilde{r}_{l\pm1,D}\right)^2} dr \\
=&e^{-\frac{\sqrt{k_{l,D}}\widetilde{r}^2_{l,D}+\sqrt{k_{l\pm1,D}}\widetilde{r}^2_{l\pm1,D}}{2}} \int_{-\infty}^{+\infty} e^{-r^2\frac{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}}{2}+2r\frac{\sqrt{k_{l,D}}\widetilde{r}_{l,D}+\sqrt{k_{l\pm1,D}}\widetilde{r}_{l\pm1,D}}{2}} dr \\
=&e^{-\frac{\sqrt{k_{l,D}}\widetilde{r}^2_{l,D}+\sqrt{k_{l\pm1,D}}\widetilde{r}^2_{l\pm1,D}}{2}+\frac{\left(\sqrt{k_{l,D}}\widetilde{r}_{l,D}+\sqrt{k_{l\pm1,D}}\widetilde{r}_{l\pm1,D}\right)^2}{2\left(\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}\right)}} \int_{-\infty}^{+\infty} e^{-\frac{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}}{2}\left(r-\widehat{r}_{l,l\pm1,D}\right)^2} dr \\
=&e^{-\frac{\sqrt{k_{l,D}k_{l\pm1,D}}}{2\left(\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}\right)}\left(\widetilde{r}_{l,D}-\widetilde{r}_{l\pm1,D}\right)^2} \int_{-\infty}^{+\infty} e^{-\frac{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}}{2}\left(r-\widehat{r}_{l,l\pm1,D}\right)^2} dr \\
=&e^{-\frac{\sqrt{k_{l,D}k_{l\pm1,D}}}{2\left(\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}\right)}\left(\widetilde{r}_{l,D}-\widetilde{r}_{l\pm1,D}\right)^2} \sqrt{\frac{2\pi}{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}}},
\end{aligned}$$

then

$$\begin{aligned}
\left\langle f_{0,l\pm1,D}, f_{0,l,D}\right\rangle_{R^+} =&M_{l,D}M_{l\pm1,D}e^{-\frac{\sqrt{k_{l,D}k_{l\pm1,D}}}{2\left(\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}\right)}\left(\widetilde{r}_{l,D}-\widetilde{r}_{l\pm1,D}\right)^2} \sqrt{\frac{2\pi}{\sqrt{k_{l,D}}+\sqrt{k_{l\pm1,D}}}} \\
&\overset{(7.73)\&(7.75)}{=} 1+O\left(\frac{1}{k_D^{\frac{3}{2}}}\right).
\end{aligned} \tag{7.78}$$

7.0.10 The algebraic relations fulfilled by $\overline{L}_{h,j}$ and $\overline{x}_s$

7.0.10.1 Proof of (2.24)

The proofs of section 7.0.8.1 can be used here to calculate $\left[\overline{x}_h, \overline{x}_j\right]\boldsymbol{\psi}_{\boldsymbol{l},D}$ when $h < j$ and $l < \Lambda$; this is possible because definition 2.3.1 implies that the action of $L_{h,D+1}$ in $\mathbb{R}^{D+1}$ is very similar to the one of t_h (and also of $\overline{x}_h$) in $\mathbb{R}^D$. In fact, the only difference is the replacement of $-\frac{1}{i}d_{l_D,l+1,D+1}$ and $\frac{1}{i}d_{l_D,l,D+1}$ with $c_{l+1,D}$ and $c_{l,D}$, respectively.

Then it must be $l'_p = l_p$ $\forall p \geq j-1$ and

$$\left[\overline{x}_h; \overline{x}_j\right]\boldsymbol{\psi}_{\boldsymbol{l},D} = \sum_{\substack{|l'_h - l_h|=1 \\ h=j-2,\cdots,h-1}} \widehat{Q}_{D,h,j}\left(\boldsymbol{l}, \boldsymbol{l}'_h\right)\boldsymbol{\psi}_{\boldsymbol{l}'_h,D}. \tag{7.79}$$

If $l'_{j-2} = l_{j-2} - 1$, then

$$\widehat{Q}_{D,h,j}\left(\boldsymbol{l},\boldsymbol{l}'_h\right) = \Big\{\left(c_{l+1,D}\right)^2 \big\{ \left[A(l,l_{d-1},D-1)\right]^2 T_3^{D-j}\left(l_{d-1},l_{j-1},l_{j-2},j-1\right) + \left[C(l,l_{d-1},D-1)\right]^2 T_4^{D-j}\left(l_{d-1},l_{j-1},l_{j-2},j-1\right)\big\} + \left(c_{l,D}\right)^2 \big\{ \left[B(l,l_{d-1},D-1)\right]^2 T_3^{D-j}\left(l_{d-1},l_{j-1},l_{j-2},j-1\right) + \left[D(l,l_{d-1},D-1)\right]^2 T_4^{D-j}\left(l_{d-1},l_{j-1},l_{j-2},j-1\right)\big\}\Big\} \cdot R_{h,j-1}\left(\boldsymbol{l}|^{j,h},\widetilde{\boldsymbol{l}'}|^{j,h}\right). \tag{7.80}$$

The equations (7.60) and (7.77) imply

$$\begin{aligned}\widehat{Q}_{D,h,j}\left(\boldsymbol{l},\boldsymbol{l}'_h\right) &= \left[-\left(c_{l+1,D}\right)^2 + \left(c_{l,D}\right)^2\right]\frac{d_{l_{j-1},l_{j-2},j}}{2l+D-2}R_{h,j-1}\left(\boldsymbol{l}|^{j,h},\widetilde{\boldsymbol{l}'}|^{j,h}\right)\\ &= -\frac{d_{l_{j-1},l_{j-2},j}}{k_D(\Lambda)}R_{h,j-1}\left(\boldsymbol{l}|^{j,h},\widetilde{\boldsymbol{l}'}|^{j,h}\right);\end{aligned} \tag{7.81}$$

similarly, if $l'_{j-2} = l_{j-2} + 1$, then

$$\widehat{Q}_{D,h,j}\left(\boldsymbol{l},\boldsymbol{l}'_h\right) = \frac{d_{l_{j-1},l_{j-2}+1,j}}{k_D(\Lambda)}R_{h,j-1}\left(\boldsymbol{l}|^{j,h},\widehat{\boldsymbol{l}'}|^{j,h}\right); \tag{7.82}$$

and then, when $l < \Lambda$,

$$\left[\overline{x}_h,\overline{x}_j\right]\boldsymbol{\psi}_{\boldsymbol{l},D} = -i\frac{\overline{L}_{h,j}}{k_D(\Lambda)}\boldsymbol{\psi}_{\boldsymbol{l},D}.$$

On the other hand, if $l = \Lambda$, the only difference is that $c_{\Lambda+1,D} = 0$ and then (of course, the calculations of section 7.0.8.1 are used also here)

$$\left[\overline{x}_h,\overline{x}_j\right]\boldsymbol{\psi}_{\Lambda,l_{d-1},\cdots,l_1,D} = i\frac{\left(c_{\Lambda,D}\right)^2}{2\Lambda+D-2}\overline{L}_{h,j}\boldsymbol{\psi}_{\Lambda,l_{d-1},\cdots,l_1,D}.$$

According to this,

$$\left[\overline{x}_h,\overline{x}_j\right] = i\left[-\frac{I}{k_D(\Lambda)} + \left(\frac{1}{k_D(\Lambda)} + \frac{\left(c_{\Lambda,D}\right)^2}{2\Lambda+D-2}\right)\widehat{P}_{\Lambda,D}\right]\overline{L}_{h,j}.$$

7.0.10.2 Proof of (2.26)

The proofs of section 7.0.7 can be used here to calculate the value of $\boldsymbol{x}^2\psi_{\boldsymbol{l},D}$; in fact it is easy to see that, when $l < \Lambda$,

$$\begin{aligned}\boldsymbol{x}^2\boldsymbol{\psi}_{\boldsymbol{l},D} &= \left[\left(c_{l+1,D}\right)^2 Z_{1,d}\left(\boldsymbol{l}\right) + \left(c_{l,D}\right)^2 Z_{2,d}\left(\boldsymbol{l}\right)\right]\boldsymbol{\psi}_{\boldsymbol{l},D}\\ &\overset{(7.49)}{=} \left\{1 + \frac{[b(l,D)+b(l+1,D)]Z_{1,d}\left(\boldsymbol{l}\right)}{2k_D(\Lambda)} + \frac{[b(l,D)+b(l-1,D)]Z_{2,d}\left(\boldsymbol{l}\right)}{2k_D(\Lambda)} + O\left(\frac{1}{k_D^2}\right)\right\}\boldsymbol{\psi}_{\boldsymbol{l},D}\\ &= \left\{1 + \frac{b(l,D) + [b(l+1,D)]\frac{l+D-2}{2l+D-2} + [b(l-1,D)]\frac{l}{2l+D-2}}{2k_D(\Lambda)} + O\left(\frac{1}{k_D^2}\right)\right\}\boldsymbol{\psi}_{\boldsymbol{l},D}.\end{aligned}$$

On the other hand, if $l=\Lambda$, $c_{\Lambda+1,D}=0$; so

$$\boldsymbol{x}^2\boldsymbol{\psi}_{\Lambda,l_{d-1},\cdots,l_1,D}=\left[(c_{\Lambda,D})^2\frac{\Lambda}{2\Lambda+D-2}\right]\boldsymbol{\psi}_{\Lambda,l_{d-1},\cdots,l_1,D}.$$

And then, [up to $O\left(\frac{1}{k_D^2}\right)$]

$$\begin{aligned}\boldsymbol{x}^2\boldsymbol{\psi}_{\boldsymbol{l},D}=&\left\{1+\frac{b(l,D)+[b(l+1,D)]\frac{l+D-2}{2l+D-2}+[b(l-1,D)]\frac{l}{2l+D-2}}{2k_D(\Lambda)}\right.\\&\left.-\left[\left(1+\frac{b(\Lambda,D)+b(\Lambda+1,D)}{2k_D(\Lambda)}\right)\frac{\Lambda+D-2}{2\Lambda+D-2}\right]\widehat{P}_{\Lambda,D}\right\}\boldsymbol{\psi}_{\boldsymbol{l},D}.\end{aligned}$$

7.0.11 The product of two D-dimensional spherical harmonics

First of all, it is important to summarize that in section 7.0.4 it has been shown that (in the following equations there is not any multiplicative constant, depending on the indices of P, because they are not relevant also in this case, except when that constant is 0)

$$P_l^{-m}(\cos\theta)=(\sin\theta)^m\,\widetilde{P}_l^{-m}(\cos\theta)\,,$$

where $0\le m\le l$, $\widetilde{P}_l^{-m}(\cos\theta)$ is a polynomial of degree $l-m$ in $\cos\theta$ which does not contain any term of degree $l-m-(2n+1)$, with $n\in\mathbb{N}_0$; so, coming back to ${}_jP_L^l(\theta)$,

$${}_h\overline{P}_l^m(\theta)=(\sin\theta)^m\,\widetilde{P}_{l+\frac{h-2}{2}}^{-\left(m+\frac{h-2}{2}\right)}(\cos\theta)=(\sin\theta)^m\left\{[\cos\theta]^{l-m}+[\cos\theta]^{l-m-2}+\cdots\right\}. \tag{7.83}$$

It is now possible to calculate the product of two spherical harmonics $Y_{\boldsymbol{l'}}$ and $Y_{\boldsymbol{l}}$; first of all, $e^{il'_1\theta_1}e^{il_1\theta_1}=e^{i(l_1+l'_1)\theta_1}$, then

$$\begin{aligned}&{}_2\overline{P}_{l'_2}^{l'_1}(\theta_2)\,{}_2\overline{P}_{l_2}^{l_1}(\theta_2)\,e^{i(l_1+l'_1)\theta_1}\\&\overset{(7.83)}{=}(\sin\theta_2)^{l'_1}\left[(\cos\theta)^{l'_2-l'_1}+(\cos\theta)^{l'_2-l'_1-2}+(\cos\theta)^{l'_2-l'_1-4}+\cdots\right]{}_2\overline{P}_{l_2}^{l_1}(\theta_2)\,e^{i(l_1+l'_1)\theta_1}\\&\overset{(7.25)}{=}\left[{}_2\overline{P}_{l_2+l'_2}^{l_1+l'_1}(\theta_2)+{}_2\overline{P}_{l_2+l'_2-2}^{l_1+l'_1}(\theta_2)+{}_2\overline{P}_{l_2+l'_2-4}^{l_1+l'_1}(\theta_2)+\cdots\right]e^{i(l_1+l'_1)\theta_1}.\end{aligned} \tag{7.84}$$

Similarly,

$$\begin{aligned}&\left[{}_3\overline{P}_{l'_3}^{l'_2}(\theta_3)\,{}_3\overline{P}_{l_3}^{l_2}(\theta_3)\right]{}_2\overline{P}_{l_2+l'_2}^{l_1+l'_1}(\theta_2)\\&\overset{(7.83)}{=}(\sin\theta_3)^{l'_2}\left[(\cos\theta_3)^{l'_3-l'_2}+(\cos\theta_3)^{l'_3-l'_2-2}+(\cos\theta_3)^{l'_3-l'_2-4}+\cdots\right]{}_2\overline{P}_{l_2+l'_2}^{l_1+l'_1}(\theta_2)\\&\overset{(7.25)}{=}\left[{}_3\overline{P}_{l_3+l'_3}^{l_2+l'_2}(\theta_3)+{}_3\overline{P}_{l_3+l'_3-2}^{l_2+l'_2}(\theta_3)+{}_3\overline{P}_{l_3+l'_3-2}^{l_2+l'_2}(\theta_3)+\cdots\right]{}_2\overline{P}_{l_2+l'_2}^{l_1+l'_1}(\theta_2)\,.\end{aligned} \tag{7.85}$$

Furthermore, in order to calculate

$$ {}_3\overline{P}^{l'_2}_{l'_3}(\theta_3)\, {}_3\overline{P}^{l_2}_{l_3}(\theta_3)\, {}_2\overline{P}^{l_1+l'_1}_{l_2+l'_2-2}(\theta_2)\,, $$

the formula $(7.25)_2$ must be used $l'_2 - 1$ times and then 1 time the formula $(7.25)_1$ in correspondence of $\sin\theta_3\cdot$, while the formula $(7.25)_3$ must be used in correspondence of $\cos\theta_3\cdot$; then

$$ \begin{aligned} &{}_3\overline{P}^{l'_2}_{l'_3}(\theta_3)\, {}_3\overline{P}^{l_2}_{l_3}(\theta_3)\, {}_2\overline{P}^{l_1+l'_1}_{l_2+l'_2-2}(\theta_2) \\ &\overset{(7.83)}{=} (\sin\theta_3)^{l'_2}\left[(\cos\theta_3)^{l'_3-l'_2} + (\cos\theta_3)^{l'_3-l'_2-2} + (\cos\theta_3)^{l'_3-l'_2-4} + \cdots\right] {}_3\overline{P}^{l_2}_{l_3}(\theta_3)\, {}_2\overline{P}^{l_1+l'_1}_{l_2+l'_2-2}(\theta_2) \\ &\overset{(7.25)}{=} \left[{}_3\overline{P}^{l_2+l'_2-2}_{l_3+l'_3}(\theta_3) + {}_3\overline{P}^{l_2+l'_2-2}_{l_3+l'_3-2}(\theta_3) + {}_3\overline{P}^{l_2+l'_2-2}_{l_3+l'_3-4}(\theta_3) + \cdots\right] {}_2\overline{P}^{l_1+l'_1}_{l_2+l'_2-2}(\theta_2)\,, \end{aligned} \tag{7.86} $$

and so on with the other angles and factors.

According to this,

$$ Y_{\boldsymbol{l'}} Y_{\boldsymbol{l}} = \sum_{l'=0}^{l+l'} \sum_{l'_{d-1}=0}^{l_{d-1}+l'_{d-1}} \cdots \sum_{l'_2=0}^{l_2+l'_2} \gamma_{\boldsymbol{l'}} Y_{\boldsymbol{l'}}, \quad \text{where} \quad \boldsymbol{l'} := \left(l', l'_{d-1}, \cdots, l'_2, l_1 + l'_1\right); \tag{7.87} $$

so, this last equation describes the action of the generic multiplication operator $Y_{\boldsymbol{l'}}\cdot$ on the Hilbert space of D-dimensional spherical harmonics.

Furthermore, from section 7.0.3.3 and the fact that the t_h commute it follows

$$ \begin{aligned} Y_{\boldsymbol{l}} &= \sum_{\substack{\boldsymbol{\alpha}\in(\mathbb{N}_0)^D \\ \|\boldsymbol{\alpha}\|_1 = l}} c^{\boldsymbol{\alpha}}_{\boldsymbol{l}} (t_1)^{\alpha_1} (t_2)^{\alpha_2} \cdots (t_D)^{\alpha_D} \\ &= \sum_{\substack{\boldsymbol{\alpha}\in(\mathbb{N}_0)^D \\ \|\boldsymbol{\alpha}\|_1 = l}} c^{\boldsymbol{\alpha}}_{\boldsymbol{l}} \frac{(\alpha_1)!(\alpha_2)!\cdots(\alpha_D)!}{l!} \sum_h N\left(h, \boldsymbol{\alpha}, t_1, t_2, \cdots, t_D\right), \end{aligned} $$

where $c^{\boldsymbol{\alpha}}_{\boldsymbol{l}}$ is a suitable constant and $N\left(h, \boldsymbol{\alpha}, t_1, t_2, \cdots, t_D\right)$ is the ordered monomial obtained applying π_h (the permutation with ripetition of l objects with α_1 identical objects of type 1, α_2 identical objects of type 2,..., α_D identical objects of type D) to the monomial $(t_1)^{\alpha_1} (t_2)^{\alpha_2} \cdots (t_D)^{\alpha_D}$.

Inspired by this, define the fuzzy approximations $\widehat{Y}_{\boldsymbol{l}}$ of the spherical harmonics $Y_{\boldsymbol{l}}$ as

$$ \widehat{Y}_{\boldsymbol{l}} := \sum_{\substack{\boldsymbol{\alpha}\in(\mathbb{N}_0)^D \\ \|\boldsymbol{\alpha}\|_1 = l}} c^{\boldsymbol{\alpha}}_{\boldsymbol{l}} \frac{(\alpha_1)!(\alpha_2)!\cdots(\alpha_D)!}{l!} \sum_h N\left(h, \boldsymbol{\alpha}, \overline{x}_1, \overline{x}_2, \cdots, \overline{x}_D\right). \tag{7.88} $$

It is also important to underline that

Remark 8 *From* $|A|, |B|, |C|, |D|, |F|, |G| \leq 1$ *it follows*

$$
\begin{aligned}
\left|\sin\theta_j \overline{P}^l_L(\theta)\right| &\leq \left|_j\overline{P}^{l+1}_{L+1}(\theta)\right| + \left|_j\overline{P}^{l+1}_{L-1}(\theta)\right|, \\
\left|\sin\theta_j \overline{P}^l_L(\theta)\right| &\leq \left|_j\overline{P}^{l-1}_{L+1}(\theta)\right| + \left|_j\overline{P}^{l-1}_{L-1}(\theta)\right|, \\
\left|\cos\theta_j \overline{P}^l_L(\theta)\right| &\leq \left|_j\overline{P}^{l}_{L+1}(\theta)\right| + \left|_j\overline{P}^{l}_{L-1}(\theta)\right|.
\end{aligned}
\tag{7.89}
$$

This, the recursive procedures of section 7.0.4 and the calculations of this section, imply that the product $_j\overline{P}^{l'}_{L'}(\theta)\, _j\overline{P}^{l}_{L}(\theta)$ *when* $\Lambda \geq L' \geq l' \geq 0$ *and* $\Lambda \geq L \geq l \geq 0$ *is the sum of (at most)* $\Lambda 2^\Lambda$ *terms* $_j\overline{P}^{l'}_{L'}(\theta)$*, and some of them may have the same indices. Furthermore, (7.89) implies that the product of* $_j\overline{P}^{l}_{L}$ *by* $\sin\theta$ *and* $\cos\theta$ *returns coefficients that are bounded by* 1*, while in (2.13) every* $_j\overline{P}^{l}_{L}(\theta)$ *contains a normalization constant which is less or equal than* $(2\Lambda)!$*.*

7.0.12 Some proofs about convergence

Let $\varphi \in \mathcal{H}_{\Lambda,D}$, with $\|\varphi\|_2 = 1$, and

$$
\varphi = \sum_{\substack{0\leq l\leq\Lambda \\ l_{h-1}\leq l_h \text{ for } h=d,\cdots,3 \\ |l_1|\leq l_2}} \varphi_{\boldsymbol{l}} \boldsymbol{\psi}_{\boldsymbol{l},D}
$$

be the decomposition of φ in that orthonormal basis of $\mathcal{H}_{\Lambda,D}$; of course, $\|\varphi\|_2 = 1$ implies $|\varphi_{\boldsymbol{l}}| \leq 1$.

According to these statements,

$$
\begin{aligned}
\|(\overline{x}_h - t_h)\,\varphi\|_2 \leq & \sum_{\substack{0\leq l\leq\Lambda \\ l_{h-1}\leq l_h \text{ for } h=d,\cdots,3 \\ |l_1|\leq l_2}} |\varphi_{\boldsymbol{l}}|\, \|(\overline{x}_h - t_h)\,\boldsymbol{\psi}_{\boldsymbol{l},D}\|_2 \\
\overset{|\varphi_{\boldsymbol{l}}|\leq 1}{\leq} & \sum_{\substack{0\leq l\leq\Lambda \\ l_{h-1}\leq l_h \text{ for } h=d,\cdots,3 \\ |l_1|\leq l_2}} \|(\overline{x}_h - t_h)\,\boldsymbol{\psi}_{\boldsymbol{l},D}\|_2 \\
\overset{*}{\leq} & [\dim\mathcal{H}_{\Lambda,D}]^2 \left(\frac{b(l+1,D) + 2b(l,D) + b(l-1,D)}{4k_D}\right) \\
\overset{\#}{\leq} & [\dim\mathcal{H}_{\Lambda,D}]^2 \frac{b(\Lambda,D)}{k_D};
\end{aligned}
$$

where the $*$ inequality follows from the fact that the sum is of $\dim\mathcal{H}_{\Lambda,D}$ elements, that both $\overline{x}_h\psi_{\boldsymbol{l},D}$ and $t_h\psi_{\boldsymbol{l},D}$ can be written as the linear combination of (at most) $\dim\mathcal{H}_{\Lambda,D}$ elements, that $|A|, |B|, |C|, |D|, |F|, |G| \leq 1$, that $\langle\boldsymbol{\psi}_{\boldsymbol{l}',D}, \overline{x}_h\boldsymbol{\psi}_{\boldsymbol{l},D}\rangle_{\mathbb{R}^D}$ and

$\langle \boldsymbol{\psi}_{\boldsymbol{l}',D}, t_h \boldsymbol{\psi}_{\boldsymbol{l},D} \rangle_{\mathbb{R}^D}$ do not vanish if $l' = l \pm 1$, and the values of the corresponding 'radial' scalal product are

$$\langle f_{0,l\pm 1,D}, r f_{0,l,D} \rangle_{\mathbb{R}^+} \overset{(7.76)}{=} 1 + \frac{1}{2} \frac{[b(l,D) + b(l \pm 1, D)]}{2k_D} + O\left(\frac{1}{k_D^{\frac{3}{2}}}\right)$$

$$\text{and} \quad \langle f_{0,l\pm,D}, f_{0,l,D} \rangle_{\mathbb{R}^+} \overset{(7.78)}{=} 1 + O\left(\frac{1}{k_D^{\frac{3}{2}}}\right);$$

while the # inequality follows from $(2.5)_1$.

So, if

$$k_D(\Lambda) \geq \Lambda \left[\dim \mathcal{H}_{\Lambda,D}\right]^2 b(\Lambda, D), \text{ then } \|(\overline{x}_h - t_h)\varphi\|_2 \overset{\Lambda \to +\infty}{\longrightarrow} 0.$$

In section 7.0.11 we do the product between two generic D-dimensional spherical harmonics and the in section 7.0.3.3 it is shown that every D-dimensional spherical harmonic is a homogeneous polynomial in the t_h variables, this suggested the definition (7.88).

Those $\widehat{Y}_{\boldsymbol{l}}$ are the fuzzy spherical harmonics, they are elements of $B[\mathcal{L}^2(S^d)]$ and, in particular,

Remark 9 *The action of $\widehat{Y}_{\boldsymbol{l}}$ on $Y_{\boldsymbol{l}'}$ can be obtained through the following replacements to $Y_{\boldsymbol{l}} \cdot Y_{\boldsymbol{l}'}$:*

- *replace every $A(l, l_{d-1}, D-1)$ with $c_{l+1,D} A(l, l_{d-1}, D-1)$;*
- *replace every $B(l, l_{d-1}, D-1)$ with $c_{l,D} B(l, l_{d-1}, D-1)$;*
- *replace every $C(l, l_{d-1}, D-1)$ with $c_{l+1,D} C(l, l_{d-1}, D-1)$;*
- *replace every $D(l, l_{d-1}, D-1)$ with $c_{l,D} D(l, l_{d-1}, D-1)$;*
- *replace every $F(l, l_{d-1}, D-1)$ with $c_{l+1,D} F(l, l_{d-1}, D-1)$;*
- *replace every $G(l, l_{d-1}, D-1)$ with $c_{l,D} G(l, l_{d-1}, D-1)$.*

7.0.12.1 Proof of Proposition 2.5.1

Let $\phi, f \in B\left(S^d\right)$, then

$$(f - \hat{f}_\Lambda)\phi = \sum_{l=0}^{\Lambda} \sum_{\substack{l_{d-1} \leq l \\ l_{h-1} \leq l_h \text{ for } h=d-1,\cdots,3 \\ |l_1| \leq l_2}} Y_{\boldsymbol{l}} \chi_{\boldsymbol{l}} + \sum_{l > \Lambda} \sum_{\substack{l_{d-1} \leq l \\ l_{h-1} \leq l_h \text{ for } h=d-1,\cdots,3 \\ |l_1| \leq l_2}} Y_{\boldsymbol{l}} (f\phi)_{\boldsymbol{l}}, \tag{7.90}$$

where

$$\chi_{\boldsymbol{l}} := (f\phi)_{\boldsymbol{l}} - (\hat{f}_\Lambda\phi)_{\boldsymbol{l}} \quad , \quad (f\phi)_{\boldsymbol{l}} = \langle Y_{\boldsymbol{l}}, f\phi\rangle \quad \text{and} \quad \left(\hat{f}_\Lambda\phi\right)_{\boldsymbol{l}} = \langle Y_{\boldsymbol{l}}, \hat{f}_\Lambda\phi\rangle;$$

in particular

$$\begin{aligned}\chi_{\boldsymbol{l}} &= \left\langle Y_{\boldsymbol{l}}, \left(f - \hat{f}_\Lambda\right)\phi\right\rangle = \left\langle Y_{\boldsymbol{l}}, \sum_{l'=0}^{2\Lambda} \sum_{\substack{l'_{d-1}\le l' \\ l'_{h-1}\le l'_h \text{ for } h=d-1,\cdots,3 \\ |l'_1|\le l'_2}} f_{\boldsymbol{l'}}\left(Y_{\boldsymbol{l'}} - \widehat{Y}_{\boldsymbol{l'}}\right)\phi\right\rangle \\ &= \sum_{l'=0}^{2\Lambda} \sum_{\substack{l'_{d-1}\le l' \\ l'_{h-1}\le l'_h \text{ for } h=d-1,\cdots,3 \\ |l'_1|\le l'_2}} \sum_{l'=0}^{+\infty} \sum_{\substack{l'_{d-1}\le l' \\ l'_{h-1}\le l'_h \text{ for } h=d-1,\cdots,3 \\ |l'_1|\le l'_2}} f_{\boldsymbol{l'}}\phi_{\boldsymbol{l'}} \ \left\langle Y_{\boldsymbol{l}}, \left(Y_{\boldsymbol{l'}} - \widehat{Y}_{\boldsymbol{l'}}\right)Y_{\boldsymbol{l'}}\right\rangle.\end{aligned} \tag{7.91}$$

On the other hand,

$$\|(f-\hat{f}_\Lambda)\phi\|_2 = \sum_{l=0}^{\Lambda} \sum_{\substack{l_{d-1}\le l \\ l_{h-1}\le l_h \text{ for } h=d-1,\cdots,3 \\ |l_1|\le l_2}} |\chi_{\boldsymbol{l}}|^2 + \sum_{l>\Lambda} \sum_{\substack{l_{d-1}\le l \\ l_{h-1}\le l_h \text{ for } h=d-1,\cdots,3 \\ |l_1|\le l_2}} |(f\phi)_{\boldsymbol{l}}|^2, \tag{7.92}$$

the second sum goes to zero as $\Lambda \to \infty$, it remains to show that the first sum does as well.

The sum over l' in (7.91) consists of at most $\dim \mathcal{H}_{2\Lambda,D}$ elements, as for the one over l' (because $0 \le l \le \Lambda$),; the equality (7.87) can be applied in this case, and it implies that both $Y_{\boldsymbol{l'}}Y_{\boldsymbol{l'}}$ and $\widehat{Y}_{\boldsymbol{l'}}Y_{\boldsymbol{l'}}$ can be written as a linear combination of, at most, $\dim \mathcal{H}_{2\Lambda,D}$ basis elements, then the sum in (7.91) is made up by at most $[\dim \mathcal{H}_{2\Lambda,D}]^3$ non-vanishing addends, while the one over l in (7.92) is of at most $\dim \mathcal{H}_{\Lambda,D}$ elements.

In addition, the fact that in (2.13) every ${}_j\overline{P}^{l}_{L}(\theta)$ contains a normalization constant which is less or equal than $(2\Lambda)!$, that the highest coefficient multiplying a power of $\cos\theta$ in $P^l_L(\cos\theta)$ is less or equal than

$$2^\Lambda\left[(2\Lambda+1)!!\right]^2,$$

that $|A|, |B|, |C|, |D|, |F|, |G| \le 1$, that $\langle \boldsymbol{\psi}_{\boldsymbol{l'},D}, \overline{x}_h\boldsymbol{\psi}_{\boldsymbol{l},D}\rangle_{\mathbb{R}^D}$ and $\langle \boldsymbol{\psi}_{\boldsymbol{l'},D}, t_h\boldsymbol{\psi}_{\boldsymbol{l},D}\rangle_{\mathbb{R}^D}$ do not vanish if $l' = l \pm 1$, and the values of the corresponding 'radial' scalal product are

$$\langle f_{0,l\pm1,D}, r f_{0,l,D}\rangle_{\mathbb{R}+} \overset{(7.76)}{=} 1 + \frac{1}{2}\frac{[b(l,D) + b(l\pm1,D)]}{2k_D} + O\left(\frac{1}{k_D^{\frac{3}{2}}}\right)$$

$$\text{and} \quad \langle f_{0,l\pm,D}, f_{0,l,D}\rangle_{\mathbb{R}+} \overset{(7.78)}{=} 1 + O\left(\frac{1}{k_D^{\frac{3}{2}}}\right)$$

imply

$$\sum_{l=0}^{\Lambda} \sum_{\substack{l_{d-1}\le l \\ l_{h-1}\le l_h \text{ for } h=d-1,\cdots,3 \\ |l_1|\le l_2}} |\chi_{\boldsymbol{l}}|^2 \le [\dim \mathcal{H}_{\Lambda,D}] \left\{ [\dim \mathcal{H}_{2\Lambda,D}]^3 \left[(2\Lambda)!\right]^D 2^{\Lambda D} \left[(2\Lambda+1)!!\right]^{2D} \frac{\Lambda b(\Lambda,D)}{k_D(\Lambda)} \right\}^2$$

So, if

$$k_D\,(\Lambda) \ge \Lambda^2 [\dim \mathcal{H}_{2\Lambda,D}]^3 \left[(2\Lambda)!\right]^D 2^{\Lambda D} \left[(2\Lambda+1)!!\right]^{2D} b(\Lambda,D) \sqrt{\dim \mathcal{H}_{\Lambda,D}},$$

then

$$\|(f-\hat{f}_\Lambda)\phi\|_2 \le \|f\|^2 \|\phi\|^2 \frac{1}{\Lambda^2} + \sum_{l>\Lambda} \sum_{\substack{l_{d-1}\le l \\ l_{h-1}\le l_h \text{ for } h=d-1,\cdots,3 \\ |l_1|\le l_2}} |(f\phi)_{\boldsymbol{l}}|^2 \overset{\Lambda\to\infty}{\longrightarrow} 0, \quad (7.93)$$

i.e. $\widehat{f}_\Lambda \to f\cdot$ strongly for all $f \in B\big(S^d\big)$, as claimed.

The replacement $f \mapsto fg$, implies that $\widehat{(fg)}_\Lambda \to (fg)\cdot$ strongly for all $f, g \in B\big(S^d\big)$, while from (7.93) it follows

$$\|(f-\hat{f}_\Lambda)\phi\|_2 \le \|f\|^2 \|\phi\|^2 \frac{1}{\Lambda^2} + \|f\phi\|^2 \le \left(\frac{\|f\|^2}{\Lambda^2} + \|f\|_\infty^2\right) \|\phi\|^2, \qquad (7.94)$$

with

$$\sqrt{\frac{\|f\|^2}{\Lambda^2} + \|f\|_\infty^2} \le \sqrt{\|f\|^2 + \|f\|_\infty^2} \le \|f\| + \|f\|_\infty, \qquad (7.95)$$

and then

$$\|\hat{f}_\Lambda\phi\| \le \|(\hat{f}_\Lambda - f)\phi\| + \|f\phi\| \le \|(\hat{f}_\Lambda - f)\phi\| + \|f\|_\infty \|\phi\| \overset{(7.94)\&(7.95)}{\le} (\|f\| + 2\|f\|_\infty)\, \|\phi\|, \qquad (7.96)$$

i.e. the operator norms $\|\hat{f}_\Lambda\|_{op}$ of the $\hat{f}_\Lambda$ are bounded uniformly in Λ: $\|\hat{f}_\Lambda\|_{op} \le \|f\| + 2\|f\|_\infty$. Therefore, as claimed, (7.93) implies

$$\begin{aligned} \|(fg - \hat{f}_\Lambda \hat{g}_\Lambda)\phi\| &\le \|(f-\hat{f}_\Lambda)g\phi\| + \|\hat{f}_\Lambda (g-\hat{g}_\Lambda)\phi\| \\ &\le \|(f-\hat{f}_\Lambda)(g\phi)\| + \|\hat{f}_\Lambda\|_{op}\; \|(g-\hat{g}_\Lambda)\phi\| \overset{\Lambda\to\infty}{\longrightarrow} 0. \quad (7.97) \end{aligned}$$

Chapter 8

Appendix B

8.1 A very useful proposition

The following proposition is very useful

Proposition 8.1.1 *Let* $A = (a_{i,j})_{i,j=1}^n$ *be a square matrix such that* $a_{i,j} \geq 0 \; \forall i, j$, *then there exist a vector* $\widehat{\chi} \in \mathbb{R}_+^n$ *fulfilling*

$$\|\widehat{\chi}\|_2 = 1 \quad \textit{and} \quad \|A\left[\widehat{\chi}\right]\|_2 = \|A\|_2.$$

Proof. By definition

$$\|A\|_2 = \sup_{\|\chi\|_2=1} \|A[\chi]\|_2,$$

the Weierstrass theorem implies that

$$\sup_{\|\chi\|_2=1} \|A[\chi]\|_2 = \max_{\|\chi\|_2=1} \|A[\chi]\|_2, \tag{8.1}$$

so it is possible to consider a vector $\widetilde{\chi} \in \mathbb{R}^n$ fulfilling (8.1) and $\|\widetilde{\chi}\|_2 = 1$. One needs to prove that $\widetilde{\chi}_i \geq 0$ for all i. If $\widetilde{\chi}_j < 0$ for some j in $\{1, 2, \cdots, n\}$, then the vector $\widehat{\chi} := (|\widetilde{\chi}_1|, |\widetilde{\chi}_2|, \cdots, |\widetilde{\chi}_n|)^T$. It is such that $\|\widehat{\chi}\|_2 = \|\widetilde{\chi}\|_2 = 1$ and

$$\begin{aligned}
\|A\left[\widetilde{\chi}\right]\|_2 = \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n a_{i,j}\widetilde{\chi}_j\right)^2} &\overset{a_{i,j}\geq 0}{\leq} \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n a_{i,j}|\widetilde{\chi}_j|\right)^2} \\
&= \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n a_{i,j}\widehat{\chi}_j\right)^2} = \|A\left[\widehat{\chi}\right]\|_2 .
\end{aligned}$$

This last inequality proves that one can consider in the realization of the maximum the ‘positive’ vector $\widehat{\chi}$, instead of $\widetilde{\chi}$, so the proof is finished. □

Proposition 8.1.2 *Let $A = (a_{i,j})_{i,j=1}^n$ and $B = (b_{i,j})_{i,j=1}^n$ be square matrices fulfilling $0 \leq a_{i,j} \leq b_{i,j}$ $\forall i, j$, then*

$$\|A\|_2 \leq \|B\|_2.$$

Proof. According to proposition 8.1.1 it is possible to consider a vector $\widehat{\chi} \in \mathbb{R}^n_+$ with $\|\widehat{\chi}\|_2 = 1$ fulfilling $\|A\|_2 = \|A[\widehat{\chi}]\|_2$; so

$$\|A\|_2 = \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n a_{i,j}\widehat{\chi}_j\right)^2} \stackrel{a_{i,j}\leq b_{i,j}}{\leq} \sqrt{\sum_{i=1}^n \left(\sum_{j=1}^n b_{i,j}\widehat{\chi}_j\right)^2} \leq \|B\|_2.$$

□

8.2 The proofs about the x_i spectrum in S^1_Λ

8.2.1 Proof of item (A) in theorem 4.2.1

Consider the unitary and involutive operator $U_1 = U_1^\dagger = U_1^{-1}$ corresponding to the inversion operator of the x_1-axis (this exists by the $O(2)$-covariance of the new model[1]: $U_1\, x_1\, U_1 = -x_1$, $U_1 x_2 U_1 = x_2$. Then $x_1\boldsymbol{\chi} = \alpha\boldsymbol{\chi}$ implies $x_1(U_1\boldsymbol{\chi}) = -\alpha(U_1\boldsymbol{\chi})$, i.e. $U_1\boldsymbol{\chi}$ is an eigenvector of x_1 with the opposite eigenvalue.

8.2.2 Proof of item (B) in theorem 4.2.1

According to the last proof, if I_n is the $n\times n$ identity matrix and $M_\Lambda(\alpha) := x_\Lambda + \alpha I_{2\Lambda+1}$, then the eigenvalue problem for x_Λ is equivalent to solve $\det[M_\Lambda(\alpha)] = 0$. In order to do this, let M^n_Λ be the $n \times n$ submatrix of M_Λ formed by the first n rows and columns, then

$$p_\Lambda(\alpha) := \det[M_\Lambda(\alpha)] \quad \text{and} \quad p^n_\Lambda(\alpha) := \det\{M^n_\Lambda(\alpha)\}.$$

It is not difficult to see that

- when $\Lambda = 1$, then

$$\begin{vmatrix} \alpha & \frac{b_1}{2} & 0 \\ \frac{b_1}{2} & \alpha & \frac{b_0}{2} \\ 0 & \frac{b_0}{2} & \alpha \end{vmatrix} = \alpha\left[\alpha^2 - \frac{(b_0)^2}{4} - \frac{(b_1)^2}{4}\right] =: p_1(\alpha) \Longrightarrow \begin{cases} \alpha_1(1) = \frac{\sqrt{(b_0)^2+(b_1)^2}}{2} = \frac{\sqrt{2}}{2}, \\ \alpha_2(1) = 0, \\ \alpha_3(1) = -\frac{\sqrt{(b_0)^2+(b_1)^2}}{2} - \frac{\sqrt{2}}{2}; \end{cases} \tag{8.2}$$

[1]U_1 is obtained by projection on $\mathcal{H}_\Lambda$ of the original unitary operator $\tilde{U}_1$ acting on $\mathcal{L}^2(\mathbb{R}^2)$ as follows: $\tilde{U} : \boldsymbol{\psi}(x_1, x_2) \to \boldsymbol{\psi}(-x_1, x_2)$.

- when $\Lambda = 2$, then

$$p_2(\alpha) := \begin{vmatrix} \alpha & \frac{b_2}{2} & 0 & 0 & 0 \\ \frac{b_2}{2} & \alpha & \frac{b_1}{2} & 0 & 0 \\ 0 & \frac{b_1}{2} & \alpha & \frac{b_0}{2} & 0 \\ 0 & 0 & \frac{b_0}{2} & \alpha & \frac{b_{-1}}{2} \\ 0 & 0 & 0 & \frac{b_{-1}}{2} & \alpha \end{vmatrix}$$

$$= \alpha \left\{ \alpha^4 - \alpha^2 \frac{\left[(b_2)^2 + (b_1)^2 + (b_0)^2 + (b_{-1})^2 \right]}{4} + \frac{(b_1 b_{-1})^2 + (b_2 b_0)^2 + (b_2 b_{-1})^2}{16} \right\}$$

$$\Longrightarrow \quad \begin{cases} \alpha_1(2) = \sqrt{\frac{1}{8}} \sqrt{A_2 + \sqrt{B_2}} = \frac{1}{2}\sqrt{3 + \frac{2}{k_D}}, \\ \alpha_2(2) = \sqrt{\frac{1}{8}} \sqrt{A_2 - \sqrt{B_2}} = \frac{1}{2}\sqrt{1 + \frac{2}{k_D}}, \\ \alpha_3(2) = 0, \\ \alpha_4(2) = -\sqrt{\frac{1}{8}} \sqrt{A_2 - \sqrt{B_2}} = -\frac{1}{2}\sqrt{1 + \frac{2}{k_D}}, \\ \alpha_5(2) = -\sqrt{\frac{1}{8}} \sqrt{A_2 + \sqrt{B_2}} = -\frac{1}{2}\sqrt{3 + \frac{2}{k_D}}, \end{cases} \tag{8.3}$$

because $A_2 := (b_2)^2 + (b_1)^2 + (b_0)^2 + (b_{-1})^2 = 4\left(1 + \frac{1}{k_D}\right)$ and

$$\begin{aligned} B_2 :=& 2\left[(b_1 b_0)^2 - (b_2 b_0)^2 + (b_{-1} b_0)^2 + (b_2 b_1)^2 - (b_{-1} b_1)^2 - (b_2 b_{-1})^2 \right] \\ &+ (b_2)^4 + (b_1)^4 + (b_0)^4 + (b_{-1})^4 = 4. \end{aligned}$$

- in general, when $\Lambda > 2$, one can calculate $p_\Lambda(\alpha)$ through the use of this recursion formula:

$$\begin{aligned} p_\Lambda^2(\alpha) &:= \det\{M_\Lambda^2(\alpha)\} = \alpha^2 - \left(\tfrac{b_\Lambda}{2}\right)^2, \\ p_\Lambda^3(\alpha) &:= \det\{M_\Lambda^3(\alpha)\} = \alpha\left[\alpha^2 - \tfrac{(b_\Lambda)^2 + (b_{\Lambda-1})^2}{4}\right], \\ p_\Lambda^4(\alpha) &:= \alpha\left[p_\Lambda^3(\alpha)\right] - \left(\tfrac{b_{\Lambda-2}}{2}\right)^2 p_\Lambda^2(\alpha), \\ p_\Lambda^5(\alpha) &:= \alpha\left[p_\Lambda^4(\alpha)\right] - \left(\tfrac{b_{\Lambda-3}}{2}\right)^2 p_\Lambda^3(\alpha), \\ &\vdots \\ p_\Lambda(\alpha) &= \alpha\left[p_\Lambda^{2\Lambda}(\alpha)\right] - \left(\tfrac{b_{1-\Lambda}}{2}\right)^2 p_\Lambda^{2\Lambda-1}(\alpha). \end{aligned} \tag{8.4}$$

So the claim is true because of (8.4) and the following

Theorem 8.2.1 The Favard theorem, [57] (p. 60)
Let $\{p_n(x) = x_n + \cdots\}$ $(n = 0, 1, \cdots)$ be a sequence of polynomials with real coefficients, satisfying a recursion formula

$$p_n(x) = (x - \beta_n)\, p_{n-1}(x) - \Sigma_n p_{n-2}(x) \tag{8.5}$$

with positive Σ_n and real β_n; then there exists a distribution $d\alpha$ such that

$$\int_{-\infty}^{+\infty} p_n(x)p_m(x)d\alpha(x) = 0 \quad (m \neq n).$$

Theorem 8.2.2 [58] (p. 44)
The zeros of the orthogonal polynomials $p_n(x)$, associated with distribution $d\alpha(x)$ on the interval $[a, b]$ are distinct and are located in the interior of the interval $[a, b]$.

8.2.3 Proof of item (C) in theorem 4.2.1

First of all, $\rho(A) = \|A\|_2$ for every symmetric matrix A, where $\rho(A)$ is the spectral radius, i.e.

$$\rho(A) := max\left\{|\lambda_j| : \lambda_j \in \Sigma_A\right\}.$$

From $1 \leq b_n \leq \sqrt{1 + \frac{\Lambda(\Lambda-1)}{k_D(\Lambda)}}$ and proposition 8.1.2, one has

$$\alpha_1(\Lambda) = \left\|X^\Lambda\right\|_2 \leq \sqrt{1 + \frac{\Lambda(\Lambda-1)}{k_D(\Lambda)}} \left\|P_{2\Lambda+1}\left(0, \frac{1}{2}, \frac{1}{2}\right)\right\|_2 = \sqrt{1 + \frac{\Lambda(\Lambda-1)}{k_D(\Lambda)}} \cos\left(\frac{\pi}{2\Lambda+2}\right)$$

and

$$\alpha_1(\Lambda+1) = \|x_{\Lambda+1}\|_2 \geq \left\|P_{2\Lambda+3}\left(0, \frac{1}{2}, \frac{1}{2}\right)\right\|_2 = \cos\left(\frac{\pi}{2\Lambda+4}\right).$$

On the other hand, by algebraic calculations, one can easily see that

$$\sqrt{1 + \frac{\Lambda(\Lambda-1)}{k_D(\Lambda)}} \cos\left(\frac{\pi}{2\Lambda+2}\right) \leq \cos\left(\frac{\pi}{2\Lambda+4}\right)$$

is equivalent to

$$\begin{aligned} k_D(\Lambda) &\geq \frac{\Lambda(\Lambda-1)\cos^2\left(\frac{\pi}{2\Lambda+2}\right)}{\cos^2\left(\frac{\pi}{2\Lambda+4}\right) - \cos^2\left(\frac{\pi}{2\Lambda+2}\right)} \\ &= \frac{\Lambda(\Lambda-1)\cos^2\left(\frac{\pi}{2\Lambda+2}\right)}{2\sin\left(\frac{\pi(2\Lambda+3)}{(2\Lambda+2)(2\Lambda+4)}\right)\sin\left(\frac{\pi}{(2\Lambda+2)(2\Lambda+4)}\right)\left[\cos\left(\frac{\pi}{2\Lambda+4}\right) + \cos\left(\frac{\pi}{2\Lambda+2}\right)\right]}. \end{aligned}$$

And from

$$\frac{a+1}{a(a+2)} > \frac{1}{1+a}, \quad \frac{\cos^2\left(\frac{\pi}{2\Lambda+2}\right)}{\cos\left(\frac{\pi}{2\Lambda+4}\right) + \cos\left(\frac{\pi}{2\Lambda+2}\right)} \leq \frac{1}{2} \quad \forall \Lambda \in \mathbb{N} \text{ and } \sin x \geq x^2 \quad \forall x \in \left[0, \frac{1}{2}\right],$$

it follows

$$\frac{\Lambda(\Lambda-1)\cos^2\left(\frac{\pi}{2\Lambda+2}\right)}{2\sin\left(\frac{\pi(2\Lambda+3)}{(2\Lambda+2)(2\Lambda+4)}\right)\sin\left(\frac{\pi}{(2\Lambda+2)(2\Lambda+4)}\right)\left[\cos\left(\frac{\pi}{2\Lambda+4}\right)+\cos\left(\frac{\pi}{2\Lambda+2}\right)\right]} < \frac{\Lambda(\Lambda-1)}{4\left(\frac{\pi}{2\Lambda+3}\frac{\pi}{(2\Lambda+2)(2\Lambda+4)}\right)^2}$$
$$< \frac{1}{4\pi^4}\Lambda(\Lambda-1)(2\Lambda+2)^2(2\Lambda+3)^2(2\Lambda+4)^2.$$

According to this,

$$k_D(\Lambda) \geq \frac{1}{4\pi^4}\Lambda(\Lambda-1)(2\Lambda+2)^2(2\Lambda+3)^2(2\Lambda+2)^4 \quad \Rightarrow \quad \alpha_1(\Lambda) < \alpha_1(\Lambda+1) \quad \forall\Lambda \in \mathbb{N}.$$

8.2.4 Proof of item (D) in theorem 4.2.1

The scheme of the proof is the following:

- First of all,
$$\lim_{\Lambda\to+\infty} \alpha_1(\Lambda) = 1. \tag{8.6}$$
- Then it is shown that, in the limit $\Lambda \to +\infty$, X^Λ can be approximated by $P_\Lambda\left(0, \frac{1}{2}, \frac{1}{2}\right)$; so one can consider the spectra of both matrices.
- For every $\Lambda \in \mathbb{N}$ it is possible to define a continuous, odd and increasing (with respect to x) function $G_\Lambda(x)$ mapping one spectrum into the other.
- From lemma 8.2.1 and lemma 8.2.2 it follows theorem 8.2.3, which tells that $\lim_{\Lambda\to+\infty} G_\Lambda(x) = x$ $\forall x \in [-1, 1]$.
- Finally, in theorem 8.2.4, it is shown that $G_\Lambda \to I$ uniformly, and this trivially implies the claim of (D).

As for the previous proof, from

$$\frac{1}{2} \leq \frac{b_n}{2} \leq \frac{\sqrt{1+\frac{\Lambda(\Lambda-1)}{k_D}}}{2} \quad \forall n \in \{\Lambda, \Lambda-1, \cdots, 2-\Lambda, 1-\Lambda\}$$

and proposition 8.1.2 one obtains

$$\left\| P_{2\Lambda+1}\left(0, \frac{1}{2}, \frac{1}{2}\right)\right\|_2 \leq \left\|X^\Lambda\right\|_2 \leq \left\| P_{2\Lambda+1}\left(0, \frac{\sqrt{1+\frac{\Lambda(\Lambda-1)}{k_D}}}{2}, \frac{\sqrt{1+\frac{\Lambda(\Lambda-1)}{k_D}}}{2}\right)\right\|_2,$$

which is equivalent to

$$\cos\left(\frac{\pi}{2\Lambda+2}\right) \leq \alpha_1(\Lambda) \leq \sqrt{1+\frac{\Lambda(\Lambda-1)}{k_D}}\cos\left(\frac{\pi}{2\Lambda+2}\right), \tag{8.7}$$

this and $k_D = k_D(\Lambda) \geq \Lambda^2(\Lambda+1)^2$ concludes the proof of (8.6).

The inequality $(4.7)_2$ follows trivially from (8.7), $\cos x \geq 1 - \frac{x^2}{2}$ $\forall x \in [0,1]$ and $\frac{\pi}{2\Lambda+2} \leq 1$ $\forall \Lambda \in \mathbb{N}$.

Corollary 6.3.8 in [59] p. 370 states that (here M_n is the space of $n \times n$ complex matrices)

Let $A, E \in M_n$, assume that A is Hermitian and that $A+E$ is normal, let $\{\lambda_1, \cdots, \lambda_n\}$ be the eigenvalues of A arranged in increasing order ($\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$) and let $\left\{\widehat{\lambda}_1, \cdots, \widehat{\lambda}_n\right\}$ be the eigenvalues of $A+E$, ordered so that $Re\left(\widehat{\lambda}_1\right) \leq Re\left(\widehat{\lambda}_2\right) \leq \cdots \leq Re\left(\widehat{\lambda}_n\right)$. Then

$$\left[\sum_{i=1}^{n} \left|\widehat{\lambda}_i - \lambda_i\right|^2\right]^{\frac{1}{2}} \leq \|E\|_2 . \tag{8.8}$$

According to this, setting $A := P_{2\Lambda+1}\left(0, \frac{1}{2}, \frac{1}{2}\right)$, $E := X^\Lambda - P_{2\Lambda+1}\left(0, \frac{1}{2}, \frac{1}{2}\right)$, then A and $A+E$ are both symmetric, so (8.8) becomes

$$\left[\sum_{i=1}^{2\Lambda+1} |\alpha_i(\Lambda) - \widetilde{\alpha}_i(\Lambda)|^2\right]^{\frac{1}{2}} \leq \|E\|_2 .$$

From $\sqrt{1+x} \leq 1 + \frac{x}{2}$, $k_D = k_D(\Lambda) \geq \Lambda^2(\Lambda+1)^2$ and $|n| \leq \Lambda$ one obtains

$$\frac{1}{2}\left[\sqrt{1+\frac{n(n-1)}{k_D}} - 1\right] \leq \frac{n(n-1)}{4k_D} \leq \frac{1}{4(\Lambda+1)^2},$$

so proposition 8.1.2 implies

$$\|E\|_2 \leq \left\| P_{2\Lambda+1}\left(0, \frac{1}{4(\Lambda+1)^2}, \frac{1}{4(\Lambda+1)^2}\right)\right\|_2 = \frac{1}{2(\Lambda+1)^2}\cos\left(\frac{\pi}{2\Lambda+2}\right) < \frac{1}{2(\Lambda+1)^2}$$

and then

$$\left[\sum_{i=1}^{2\Lambda+1} |\alpha_i(\Lambda) - \widetilde{\alpha}_i(\Lambda)|^2\right]^{\frac{1}{2}} < \frac{1}{2(\Lambda+1)^2} \quad \forall \Lambda. \tag{8.9}$$

For every $\Lambda \in \mathbb{N}$ it is possible to define a continuous function $G_\Lambda : [-1,1] \to [-\alpha_1(\Lambda), \alpha_1(\Lambda)]$ such that $G_\Lambda[\widetilde{\alpha}_n(\Lambda)] = \alpha_n(\Lambda)$, $G_\Lambda(-x) = -G_\Lambda(x)$, $G_\Lambda(x) = \alpha_1(\Lambda)\, \forall x \in [\widetilde{\alpha}_1(\Lambda), 1]$, for instance one can join two 'consecutive' points $(\widetilde{\alpha}_i(\Lambda), \alpha_i(\Lambda))$ and $(\widetilde{\alpha}_{i+1}(\Lambda), \alpha_{i+1}(\Lambda))$ by a straight line; furthermore, because of

$$G_\Lambda[\widetilde{\alpha}_n(\Lambda)] = \alpha_n(\Lambda) < G_\Lambda[\widetilde{\alpha}_{n-1}(\Lambda)] = \alpha_{n-1}(\Lambda),$$

one can assume that every function $G_\Lambda(x)$ is also increasing with respect to x.

The $G_\Lambda(x)$ are all odd functions so one can restict the attention to the $x \in [0,1]$, but it is also true that the continuity and the monotonicity of every G_Λ implies that

$$\forall \varepsilon > 0, \forall x \in [0,1] \exists \delta = \delta(\varepsilon, \Lambda, x) \text{ s.t. } y \in [0,1], \begin{cases} |x-y| < \delta \Rightarrow |G_\Lambda(x) - G_\Lambda(y)| < \varepsilon, \\ |x-y| > \delta \Rightarrow |G_\Lambda(x) - G_\Lambda(y)| > \varepsilon. \end{cases}$$

At this point the following lemma is needed

Lemma 8.2.1 *Let $\varepsilon > 0$ and $\overline{x} \in [0,1]$ such that*

$$\limsup_{\Lambda \to +\infty} |\overline{x} - G_\Lambda(\overline{x})| = 0,$$

then

$$\liminf_{\Lambda \to +\infty} \delta(\varepsilon, \Lambda, \overline{x}) = \widetilde{\delta}(\varepsilon, \overline{x}) > 0. \tag{8.10}$$

Proof. Let $\varepsilon > 0$ and assume, per absurdum, that

$$\liminf_{\Lambda \to +\infty} \delta(\varepsilon, \Lambda, \overline{x}) = 0,$$

then one can find a sequence $\left\{\widetilde{\Lambda}_n\right\}_{n \in \mathbb{N}}$ such that

$$\lim_n \delta(\varepsilon, \widetilde{\Lambda}_n, \overline{x}) = 0 \tag{8.11}$$

and, correspondingly, because of (8.11) one can assume that n is sufficiently large so that there exists $x \in [0,1]$ with $\frac{\varepsilon}{4} > |\overline{x} - x| > \delta(\varepsilon, \widetilde{\Lambda}_n, \overline{x})$, $|\overline{x} - G_{\widetilde{\Lambda}_n}(\overline{x})| < \frac{\varepsilon}{4}$ and $|G_{\widetilde{\Lambda}_n}(\overline{x}) - G_{\widetilde{\Lambda}_n}(x)| > \varepsilon$; then

$$\begin{aligned} |x - G_{\widetilde{\Lambda}_n}(x)| &= |x - \overline{x} + \overline{x} - G_{\widetilde{\Lambda}_n}(\overline{x}) + G_{\widetilde{\Lambda}_n}(\overline{x}) - G_{\widetilde{\Lambda}_n}(x)| \\ &\geq |G_{\widetilde{\Lambda}_n}(\overline{x}) - G_{\widetilde{\Lambda}_n}(x)| - |\overline{x} - x| - |\overline{x} - G_{\widetilde{\Lambda}_n}(\overline{x})| \\ &\geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}. \end{aligned}$$

This last inequality and (4.5) implies that there exist a finite set of indices I with $|I| = m(n)$ such that the correspondings eigenvalues of $P_{2\widetilde{\Lambda}_n+1}\left(0, \frac{1}{2}, \frac{1}{2}\right)$, in symbols $\left\{\widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right)\right\}_{i \in I}$, fulfill

$$\frac{\varepsilon}{4} > \left|\overline{x} - \widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right)\right| > \delta(\varepsilon, \widetilde{\Lambda}_n, \overline{x}) \quad \forall i \in I \Longrightarrow \left|\widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right) - G_{\widetilde{\Lambda}_n}\left[\widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right)\right]\right| > \frac{\varepsilon}{2} \quad \forall i \in I$$

and of course (8.11) implies that $m(n) \overset{n \to +\infty}{\longrightarrow} +\infty$, so

$$\lim_n \left[\sum_{i \in I} \left|\widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right) - G_{\widetilde{\Lambda}_n}\left[\widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right)\right]\right|^2\right] = +\infty,$$

which disagrees with (8.9), so the proof is finished. □

Let

$$A := \left\{x \in [0,1] : \limsup_{\Lambda\to+\infty} |x - G_\Lambda(x)| = 0\right\},$$

then $0 \in A$ and also $1 \in A$ because

$$\lim_{\Lambda\to+\infty} \alpha_1(\Lambda) = \lim_{\Lambda\to+\infty} \widetilde{\alpha}_1(\Lambda) = \lim_{\Lambda\to+\infty} G_\Lambda[\widetilde{\alpha}_1(\Lambda)] = 1.$$

In order to prove item (D) in theorem 4.2.1 one needs the following

Lemma 8.2.2 *If* $0 \le \overline{x} \le 1, \overline{x} \in A$, *then* $\exists \sigma > 0$ *such that*

$$x \in \left]\max\{\overline{x}-\sigma, 0\}, \min\{\overline{x}+\sigma, 1\}\right[\Longrightarrow x \in A.$$

Proof. Let $\varepsilon > 0$, then lemma 8.2.1 implies

$$\liminf_{\Lambda\to+\infty} \delta(\varepsilon, \Lambda, \overline{x}) = \widetilde{\delta}(\varepsilon, \overline{x}) > 0;$$

so, if $\sigma := \min\left\{\frac{\delta(\varepsilon,\overline{x})}{2}, \varepsilon\right\}$ and $x \in \left]\max\{\overline{x}-\sigma, 0\}, \min\{\overline{x}+\sigma, 1\}\right[$, then

$$\begin{aligned}\limsup_{\Lambda\to+\infty} |x - G_\Lambda(x)| &= \limsup_{\Lambda\to+\infty} |x - G_\Lambda(x) - \overline{x} + \overline{x} - G_\Lambda(\overline{x}) + G_\Lambda(\overline{x})| \\ &\le \limsup_{\Lambda\to+\infty} |x - \overline{x}| + |\overline{x} - G_\Lambda(\overline{x})| + |G_\Lambda(x) - G_\Lambda(\overline{x})| \le 2\varepsilon,\end{aligned}$$

of course ε can be chosen arbitrary small, so the proof is finished. □

According to this, one has

Corollary 8.2.1

$$A = [0,1]$$

or

$$A = [0, x_1[\,\dot\cup\,]x_2, x_3[\,\dot\cup \cdots \dot\cup\,]x_s, 1] \quad \textit{and} \quad B := [0,1] \setminus A = [x_1, x_2] \,\dot\cup\, [x_3, x_4] \,\dot\cup \cdots,$$

where $x_1 < x_2 < x_3 < x_4 \cdots$ *are suitable points of* $]0,1[$.

It is now possible to prove the following

Theorem 8.2.3

$$A = [0,1]$$

Proof. Assume, per absurdum, that $A \ne [0,1]$, then corollary 8.2.1 implies

$$B := [0,1] \setminus A = [x_1, x_2] \,\dot\cup\, [x_3, x_4] \,\dot\cup \cdots, \tag{8.12}$$

so if $x \in A$, $\delta > 0$, $x_1 - \delta < x < x_1$ and $\limsup_{\Lambda \to +\infty} |x_1 - G_\Lambda(x_1)| = k_D > 0$, then

$$\begin{aligned}\limsup_{\Lambda \to +\infty} |G_\Lambda(x) - G_\Lambda(x_1)| &= \limsup_{\Lambda \to +\infty} |G_\Lambda(x) - x_1 + x_1 - G_\Lambda(x_1)| \\ &\leq \limsup_{\Lambda \to +\infty} |G_\Lambda(x) - x_1| + |x_1 - G_\Lambda(x_1)| \\ &\leq \delta + k_D,\end{aligned} \tag{8.13}$$

because $x \in A$.

On the other hand

$$\begin{aligned}\limsup_{\Lambda \to +\infty} |G_\Lambda(x) - G_\Lambda(x_1)| &= \limsup_{\Lambda \to +\infty} |G_\Lambda(x) - x_1 + x_1 - G_\Lambda(x_1)| \\ &\geq \limsup_{\Lambda \to +\infty} |x_1 - G_\Lambda(x_1)| - |G_\Lambda(x) - x_1| \\ &\geq k_D - \delta.\end{aligned} \tag{8.14}$$

According to this, one has

$$\lim_{\delta \to 0} \left[\limsup_{\Lambda \to +\infty} |G_\Lambda(x) - G_\Lambda(x_1)| - k_D \right] = 0$$

so $\limsup_{\Lambda \to +\infty} G_\Lambda(x_1) = k_D + x_1$ and then there exists a sequence $\left\{ \widetilde{\Lambda}_n \right\}_{n \in \mathbb{N}}$ such that

$$\lim_{n \to +\infty} G_{\widetilde{\Lambda}_n}(x_1) = k_D + x_1,$$

but $G_\Lambda(x)$ is increasing with respect to x, so

$$\liminf_{n \to +\infty} G_{\widetilde{\Lambda}_n}(x) \geq k_D + x_1 \quad \forall x \in \left[x_1, x_1 + \frac{k_D}{2} \right].$$

This implies

$$\liminf_{n \to +\infty} \left| x - G_{\widetilde{\Lambda}_n}(x) \right| \geq k_D + x_1 - \left(x_1 + \frac{k_D}{2} \right) = \frac{k_D}{2} \quad \forall x \in \left[x_1, x_1 + \frac{k_D}{2} \right].$$

This last inequality and (4.5) implies that there exist a finite set of indices I with $|I| = m(n)$ such that the correspondings eigenvalues of $P_{2\widetilde{\Lambda}_n+1}\left(0, \frac{1}{2}, \frac{1}{2}\right)$, in symbols $\left\{ \widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right) \right\}_{i \in I}$, fulfill

$$\widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right) \in \left[x_1, x_1 + \frac{k_D}{2} \right] \quad \forall i \in I \quad \Longrightarrow \left| \widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right) - G_{\widetilde{\Lambda}_n}\left[\widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right) \right] \right| > \frac{k_D}{4} \quad \forall i \in I$$

and of course $m(n) \overset{n \to +\infty}{\longrightarrow} +\infty$, so

$$\lim_{n \to +\infty} \left[\sum_{i \in I} \left| \widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right) - G_{\widetilde{\Lambda}_n}\left[\widetilde{\alpha}_i\left(\widetilde{\Lambda}_n\right) \right] \right|^2 \right] = +\infty,$$

which disagrees with (8.9), so the proof is finished. □

According to this, one has

$$\lim_{\Lambda\to+\infty} G_\Lambda(x) = x \quad \forall x \in [0,1],$$

in the next theorem the sequence $\{\Lambda\}_{\Lambda\in\mathbb{N}}$ and its subsequences are always denoted with the same notation.

Theorem 8.2.4

$$\limsup_{\Lambda\to+\infty}\left[\max_{x\in[0,1]}\{|x-G_\Lambda(x)|\}\right]=0.$$

Proof. Assume, per absurdum, that

$$\limsup_{\Lambda\to+\infty}\left[\max_{x\in[0,1]}\{|x-G_\Lambda(x)|\}\right]=M>0,$$

and set

$$x_\Lambda := \max_{x\in[0,1]}\{|x-G_\Lambda(x)|\};$$

one has (up to a suitable subsequence)

$$\lim_{\Lambda\to+\infty}|x_\Lambda - G_\Lambda(x_\Lambda)| = M.$$

The sequence $\{x_\Lambda\}_{\Lambda\in\mathbb{N}}$ is bounded, so (up to a further suitable subsequence)

$$\lim_{\Lambda\to+\infty} x_\Lambda = \overline{x} \in [0,1] = A,$$

at this point, choose ε, x so that

$$0<\varepsilon<\frac{M}{8} \quad , \quad \sigma := \min\left\{\frac{\widetilde{\delta}(\varepsilon,\overline{x})}{2},\frac{M}{8}\right\}>0 \quad , \quad x\in\left[\overline{x}-\frac{\sigma}{2},\overline{x}+\frac{\sigma}{2}\right]$$

and Λ such that

$$|x_\Lambda - G_\Lambda(x_\Lambda)| > \frac{M}{2} \quad , \quad |x-x_\Lambda|<\sigma \quad , \quad |\overline{x}-x_\Lambda|<\sigma,$$

then (if Λ is sufficiently large)

$$\begin{aligned}|x-G_\Lambda(x)| &\geq |x_\Lambda - G_\Lambda(x_\Lambda)| - |x-x_\Lambda| - |G_\Lambda(\overline{x})-G_\Lambda(x_\Lambda)| - |G_\Lambda(\overline{x})-G_\Lambda(x)|\\ &> \frac{M}{2}-\sigma-\varepsilon-\varepsilon>\frac{M}{8}.\end{aligned}$$

This last inequality implies that there exist a finite set of indices I with $|I| = m(\Lambda)$ such that the correspondings eigenvalues of $P_{2\widetilde{\Lambda}_n+1}\left(0,\frac{1}{2},\frac{1}{2}\right)$, in symbols $\{\widetilde{\alpha}_i(\Lambda)\}_{i\in I}$, fulfill

$$\widetilde{\alpha}_i(\Lambda)\in\left[\overline{x}-\frac{\sigma}{2},\overline{x}+\frac{\sigma}{2}\right] \quad \forall i\in I \quad \Longrightarrow |\widetilde{\alpha}_i(\Lambda)-G_\Lambda[\widetilde{\alpha}_i(\Lambda)]|>\frac{M}{8} \quad \forall i\in I$$

and of course $m(\Lambda) \stackrel{\Lambda\to+\infty}{\longrightarrow} +\infty$, so

$$\lim_{\Lambda\to+\infty}\left[\sum_{i\in I}\left|\widetilde{\alpha}_i\left(\Lambda\right) - G_\Lambda\left[\widetilde{\alpha}_i\left(\Lambda\right)\right]\right|^2\right] = +\infty,$$

which disagrees with (8.9), so the proof is finished. $\square$

The proof of item (D) of theorem 4.2.1 can be completed, because if $\varepsilon > 0$ the last theorem implies that there exists a $\widetilde{\Lambda} = \widetilde{\Lambda}\left(\varepsilon\right)$ such that $\left|x - G_\Lambda\left(x\right)\right| < \varepsilon$ $\forall\Lambda > \widetilde{\Lambda}$ and $\forall x \in [0,1]$, while (4.5) implies

$$\begin{aligned}\left|\widetilde{\alpha}_{n+1}\left(\Lambda\right) - \widetilde{\alpha}_n\left(\Lambda\right)\right| &= \left|\cos\left[\frac{(n+1)\pi}{\Lambda+1}\right] - \cos\left(\frac{n\pi}{\Lambda+1}\right)\right| \\ &= \left|2\sin\left[\frac{(2n+1)\pi}{\Lambda+1}\right]\sin\left(\frac{\pi}{\Lambda+1}\right)\right| \leq 2\sin\left(\frac{\pi}{\Lambda+1}\right),\end{aligned}$$

this means that there exists a $\widehat{\Lambda} = \widehat{\Lambda}\left(\varepsilon\right)$ such that $\left|\widetilde{\alpha}_i\left(\Lambda\right) - \widetilde{\alpha}_{i+1}\left(\Lambda\right)\right| < \varepsilon\ \forall\Lambda > \widetilde{\Lambda}$, $\forall i$.

Finally, if $\overline{\Lambda}\left(\varepsilon\right) = \max\left\{\widehat{\Lambda}\left(\varepsilon\right), \widetilde{\Lambda}\left(\varepsilon\right)\right\}$, then $\forall\Lambda > \overline{\Lambda}$ one has

$$\begin{aligned}\left|\alpha_i\left(\Lambda\right) - \alpha_{i+1}\left(\Lambda\right)\right| &\leq \left|\alpha_i\left(\Lambda\right) - \widetilde{\alpha}_i\left(\Lambda\right)\right| + \left|\alpha_{i+1}\left(\Lambda\right) - \widetilde{\alpha}_{i+1}\left(\Lambda\right)\right| + \left|\widetilde{\alpha}_i - \widetilde{\alpha}_{i+1}\right| \\ &= \left|G_\Lambda\left[\widetilde{\alpha}_i\left(\Lambda\right)\right] - \widetilde{\alpha}_i\left(\Lambda\right)\right| + \left|G_\Lambda\left[\widetilde{\alpha}_{i+1}\left(\Lambda\right)\right] - \widetilde{\alpha}_{i+1}\left(\Lambda\right)\right| + \left|\widetilde{\alpha}_i - \widetilde{\alpha}_{i+1}\right| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,\end{aligned}$$

so the proof is completed.

8.3 The proofs about the x_i spectrum in S^2_Λ

8.3.1 Proof of item (A) in theorem 4.3.1

Consider the unitary and involutive operator $U_0 = U_0^\dagger = U_0^{-1}$ corresponding to the inversion operator of the x_3-axis (this exists by the $O(3)$-covariance of this new model): $U_0\,x_0\,U_0 = -x_0$, $U_0 x_\pm U_0 = x_\pm$. Then $x_0\boldsymbol{\chi} = \alpha\boldsymbol{\chi}$ implies $x_0(U_0\boldsymbol{\chi}) = -\alpha(U_0\boldsymbol{\chi})$, i.e. $U_0\boldsymbol{\chi}$ is an eigenvector of x_0 with the opposite eigenvalue.

8.3.2 Proof of item (B) in theorem 4.3.1

According to the last proof, one can equivalently set $M_m\left(\Lambda;\alpha\right) := B_m\left(\Lambda\right) + \alpha I_{\Lambda-m+1}$, then the eigenvalue problem for $B_m\left(\Lambda\right)$ is equivalent to solve $\det\left[M_m(\Lambda;\alpha)\right] = 0$; in order to do this, let M_m^h be the $h\times h$ submatrix of M_m formed by the first h rows and columns, then

$$p_{n(\Lambda;m)}(\alpha) := \det\left[M_m(\Lambda;\alpha)\right] \quad \text{and} \quad p^h_{n(\Lambda;m_1)}(\alpha) := \det\left\{M^h_{m_1}\left(\Lambda;\alpha\right)\right\},$$

where $n(\Lambda; m') := \Lambda - |m'| + 1$ is the degree of the polynomial $p_{n(\Lambda,m')}$.

It is not difficult to see that

- when $n = 1 \Leftrightarrow |m| = \Lambda$, then $\alpha = 0$;
- when $n = 2$, then
$$\begin{vmatrix} \alpha & c_{\Lambda,3}G(\Lambda, \Lambda-1, 2) \\ c_{\Lambda,3}G(\Lambda, \Lambda-1, 2) & \alpha \end{vmatrix} = \alpha^2 - (c_{\Lambda,3}G(\Lambda, \Lambda-1, 2))^2$$
$$=: p_2(\alpha) \Rightarrow \alpha_{1,2} = \pm c_{\Lambda,3}G(\Lambda, \Lambda-1, 2);$$
- when $n = 3$, then
$$\begin{vmatrix} \alpha & c_{\Lambda-1,3}G(\Lambda-1, \Lambda-2, 2) & 0 \\ c_{\Lambda-1,3}G(\Lambda-1, \Lambda-2, 2) & \alpha & c_{\Lambda,3}G(\Lambda, \Lambda-2, 2) \\ 0 & c_{\Lambda,3}G(\Lambda, \Lambda-2, 2) & \alpha \end{vmatrix}$$
$$= \alpha\left[\alpha^2 - (c_{\Lambda,3}G(\Lambda, \Lambda-2, 2))^2\right] - \alpha\left(c_{\Lambda-1,3}G(\Lambda-1, \Lambda-2, 2)\right)^2 =: p_3(\alpha);$$
- in general, let $\widetilde{n} = n(\Lambda; \widetilde{m})$, then one can calculate $p_{\widetilde{n}}(\alpha)$ through the use of this recursion formula
$$\begin{aligned} p^2_{\widetilde{n}}(\alpha) &:= \det\{M^2_{\widetilde{m}}(\Lambda; \alpha)\}, \\ p^3_{\widetilde{n}}(\alpha) &:= \det\{M^3_{\widetilde{m}}(\Lambda; \alpha)\}, \\ p^4_{\widetilde{n}}(\alpha) &:= \alpha\left[p^3_{\widetilde{n}}(\alpha)\right] - \left(c_{\widetilde{m}+3,3}G(\widetilde{m}+3, \widetilde{m}, 2)\right)^2 p^2_{\widetilde{n}}(\alpha), \\ p^5_{\widetilde{n}}(\alpha) &:= \alpha\left[p^4_{\widetilde{n}}(\alpha)\right] - \left(c_{\widetilde{m}+4,3}G(\widetilde{m}+4, \widetilde{m}, 2)\right)^2 p^3_{\widetilde{n}}(\alpha), \\ &\vdots \\ p_{\widetilde{n}}(\alpha) &= \alpha\left[p^{\widetilde{n}-1}_{\widetilde{n}}(\alpha)\right] - \left(c_{\Lambda,3}G(\Lambda, \widetilde{m}, 2)\right)^2 p^{\widetilde{n}-2}_{\widetilde{n}}(\alpha). \end{aligned} \tag{8.15}$$

Then the proof of item (B) follows trivially from (8.15), theorem 8.2.1 and theorem 8.2.2, as for section 8.2.2.

8.3.3 Proof of (4.12) in theorem 4.3.1

In this proof the following theorem (here $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of orthogonal polynomials) is used:

Theorem 8.3.1 [58] (p. 46)
Let $x_1 < x_2 < \cdots < x_2$ be the zeros of $p_n(x)$. Then each interval $[x_\nu, x_{\nu+1}]$ contains exactly one zero of $p_{n+1}(x)$.

This is the scheme of the proof:

- First of all, theorem 8.2.1 is used to prove that there exist a $\mathbb{R}$-measure such that the polynomials $\left\{p^h_{n(\Lambda;m)}\right\}^{n(\Lambda;m)}_{h=1}$ are orthogonal with respect to that measure; this implies that one can apply theorem 8.2.2 getting that all the roots of every polynomial $p^h_{n(\Lambda;m)}$ are real and simple.

- Then lemma 8.3.2 and theorem 8.3.1 can be used to prove also that

$$\rho\left(B_m\right) = \|B_m\|_2 < \|B_{m-1}^{n(\Lambda;m)}\|_2 = \rho\left(B_{m-1}^{n(\Lambda;m)}\right),$$

  where ρ is the spectral radius.

- This last inequality involving the spectral radii trivially implies (4.12).

According to this, let's start with the first point of this scheme.

Lemma 8.3.1 *The roots of* $p^h_{n(\Lambda;m)}$ *are real and simple, and if* $\alpha_1^\nu(\Lambda;m) > \alpha_2^\nu(\Lambda;m) > \cdots > \alpha_\nu^\nu(\Lambda;m)$ *are the zeros of* $p^\nu_{n(\Lambda;m)}(\alpha)$*, then every interval* $\left[\alpha_{i+1}^{\nu+1}(\Lambda;m), \alpha_i^{\nu+1}(\Lambda;m)\right]$ *contains exactly one zero of* $p^\nu_{n(\Lambda;m)}(\alpha)$*.*

Proof. The matrices $B_m^h(\Lambda)$ are all symmetric, so the roots of $p^h_{n(\Lambda;m)}(\alpha)$ are real; while the sequence of polynomials $\left\{p^h_{n(\Lambda;m)}\right\}_{h=1}^{n(\Lambda;m)}$ fulfill the recurrence relation (8.15) and because of theorem 8.2.1 one has that there exists a distribution $d\Theta(\alpha)$ such that

$$\int_{-\infty}^{+\infty} p^j_{n(\Lambda;m)}(\alpha) p^h_{n(\Lambda;m)}(\alpha) d\Theta(\alpha) = 0 \quad (j \neq h).$$

Finally, theorem 8.2.2 and theorem 8.3.1 can be applied to the set $\{p^h_{n(\Lambda;m)}(\alpha)\}_{h=1}^{n(\Lambda;m)}$ of polynomials, so the proof is finished. $\square$

First of all, an inequality involving the B_m-matrix elements is proved, which implies the aforementioned inequality between the spectral radii.

Lemma 8.3.2 *Let*

$$1 \leq m \leq \Lambda, \quad j \in \mathbb{N}_0, \quad 1 \leq l := m + j \leq \Lambda; \tag{8.16}$$

then

$$c_{l,3} G(l, m-1, 2) > c_{l+1,3} G(l+1, m, 2). \tag{8.17}$$

Proof. Because of (7.32) and (7.76), one has

$$c_{l,3} G(l, m-1, 2) = \sqrt{1 + \frac{l^2}{k_D}} \sqrt{\frac{(l+m-1)(l-m+1)}{4l^2-1}}$$

and

$$c_{l+1,3} G(l+1, m, 2) = \sqrt{1 + \frac{(l+1)^2}{k_D}} \sqrt{\frac{(l+m+1)(l-m+1)}{4(l+1)^2-1}},$$

then (8.17) becomes

$$\left(1+\frac{l^2}{k_D}\right)\left(\frac{l+m-1}{4l^2-1}\right)-\left(1+\frac{(l+1)^2}{k_D}\right)\left(\frac{l+m+1}{4(l+1)^2-1}\right)>0$$

$\forall\ 1 \le m \le \Lambda$ and $1 \le l \le \Lambda$; by algebraic calculations, one can prove that the last inequality is equivalent to the following one:

$$\underbrace{\left[k_D+l^2\right](l+m-1)(2l+3)}_{A}-\underbrace{\left[k_D+(l+1)^2\right](l+m+1)(2l-1)}_{B}>0 \quad (8.18)$$

$\forall\ 1 \le m \le \Lambda$ and $1 \le l \le \Lambda$.

Furthermore, one has

$$\begin{aligned} A = &\quad 2k_Dl^2+2k_Dlm+kl+3k_Dm-3k_D+2l^4+2l^3m+l^3+3l^2m-3l^2,\\ B = &\quad 2k_Dl^2+2k_Dlm+kl-km-k_D+2l^4+2l^3m+5l^3+3l^2m+3l^2-l-m-1; \end{aligned}$$

finally, (8.18) becomes

$$A-B=4k_Dm-2k_D-4l^3-6l^2+l+m+1>0$$

$\forall\ 1 \le m \le \Lambda$ and $1 \le l \le \Lambda$.

From $k_D(\Lambda) \ge \Lambda^2(\Lambda+1)^2$ it follows

$$4k_Dm-2k_D-4l^3-6l^2+l+m+1 \ge 2\Lambda^2(\Lambda+1)^2-4\Lambda^3-6\Lambda^2=2\Lambda^2\left(\Lambda^2-2\right)>0 \quad \forall\Lambda\ge 2,$$

while when $\Lambda=1$

$$4k_Dm-2k_D-4l^3-6l^2+l+m+1 \ge 2[1^2(2)^2]-4-6+3=1,$$

so the proof is finished. □

Lemma 8.3.3 *Let $m \ge 1$, then*

$$\|B_m\|_2 < \|B^{n(\Lambda;m)}_{m-1}\|_2.$$

Proof. The matrices B_m and $B^{n(\Lambda;m)}_{m-1}$ have the same dimensions, they are

$$B_m=\begin{pmatrix} 0 & c_{m+1,3}G(m+1,m,2) & \vdots & 0\\ c_{m+1,3}G(m+1,m,2) & 0 & \vdots & 0\\ 0 & c_{m+2,3}G(m+1,m,2) & \vdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \vdots & c_{\Lambda,3}G(\Lambda,m,2)\\ 0 & 0 & \vdots & 0 \end{pmatrix} \quad (8.19)$$

and

$$
B_{m-1}^{n(\Lambda;m)} = \begin{pmatrix}
0 & c_{m,3}G(m,m-1,2) & \vdots & 0 \\
c_{m,3}G(m,m-1,2) & 0 & \vdots & 0 \\
0 & c_{m+1,3}G(m+1,m-1,2) & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \vdots & c_{\Lambda-1,3}G(\Lambda-1,m-1,2) \\
0 & 0 & \vdots & 0
\end{pmatrix}. \tag{8.20}
$$

Lemma 8.3.2, together with proposition 8.1.2, (8.19) and (8.20), imply

$$
\|B_m\|_2 < \|B_{m-1}^{n(\Lambda;m)}\|_2,
$$

so the proof is finished. □

At this point, let $\alpha_1(\Lambda) := \max\{\alpha_1(\Lambda;0);\alpha_1(\Lambda;1);\cdots;\alpha_1(\Lambda;\Lambda)\}$ and assume, per absurdum, that $\alpha_1(\Lambda) = \alpha_1(\Lambda;m)$ with $m > 0$. One can take the matrix B_{m-1} and its elements; from lemma 8.3.3 it follows

$$
\|B_m\|_2 < \|B_{m-1}^{n(\Lambda;m)}\|_2; \tag{8.21}
$$

and from lemma 8.3.1 one has that the eigenvalues of $B_{m-1}^{n(\Lambda;m)}$ 'separate' the ones of B_{m-1}, then

$$
\rho\left(B_{m-1}^{n(\Lambda;m)}\right) < \rho\left(B_{m-1}\right). \tag{8.22}
$$

The inequalities (8.21) and (8.22) lead to $\alpha_1(\Lambda) < \alpha_1(\Lambda;m-1)$, but this is not possible. It is possible to conclude that $\alpha_1(\Lambda) = \alpha_1(\Lambda;0)$ and with the same procedure one can prove the other inequalities in (4.12).

8.3.4 Proof of (4.13) in theorem 4.3.1

Let

$$
\widehat{B}_0(\Lambda) := \begin{pmatrix}
0 & G(1,0,2) & \vdots & 0 \\
G(1,0,2) & 0 & \vdots & 0 \\
0 & G(2,0,2) & \vdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \vdots & G(\Lambda,0,2) \\
0 & 0 & \vdots & 0
\end{pmatrix}
$$

and its spectrum $\{\widehat{\alpha}_i(\Lambda;0)\}_{i=1}^{\Lambda+1}$, where the eigenvalues are arranged in descending order.

First of all, from $1 \le c_{l,3} \le \sqrt{1+\frac{\Lambda^2}{k_D(\Lambda)}}$ $\forall 1 \le l \le \Lambda$ and proposition 8.1.2, it follows

$$\alpha_1(\Lambda;0) = \|B_0(\Lambda)\|_2 \le \sqrt{1+\frac{\Lambda^2}{k_D(\Lambda)}}\left\|\widehat{B}_0(\Lambda)\right\|_2 = \sqrt{1+\frac{\Lambda^2}{k_D(\Lambda)}}\widehat{\alpha}_1(\Lambda;0)$$

and $\alpha_1(\Lambda+1;0) = \|B_0(\Lambda+1)\|_2 \ge \left\|\widehat{B}_0(\Lambda+1)\right\|_2 = \widehat{\alpha}_1(\Lambda+1;0)$; then, by algebraic calculations, one has

$$\sqrt{1+\frac{\Lambda^2}{k_D(\Lambda)}}\widehat{\alpha}_1(\Lambda;0) \le \widehat{\alpha}_1(\Lambda+1;0) \Leftrightarrow k_D(\Lambda) \ge \frac{\Lambda^2\left[\widehat{\alpha}_1(\Lambda;0)\right]^2}{\left[\widehat{\alpha}_1(\Lambda+1;0)\right]^2-\left[\widehat{\alpha}_1(\Lambda;0)\right]^2}. \tag{8.23}$$

As done for section 8.2.2, one can use theorem 8.2.1 and theorem 8.2.2 to prove that $\widehat{\alpha}_1(\Lambda+1;0) > \widehat{\alpha}_1(\Lambda;0)$ $\forall \Lambda \in \mathbb{N}$, while it is obvious that

$$\sqrt{\frac{l^2}{4l^2-1}} > \frac{1}{2} \quad \forall l \in \mathbb{N} \quad \Longrightarrow \quad \left\|\widehat{B}_0(\Lambda)\right\|_2 = \widehat{\alpha}_1(\Lambda;0) > \cos\left(\frac{\pi}{\Lambda+2}\right) \quad \forall \Lambda \in \mathbb{N};$$

finally, in section 8.3.5 it is shown that $\alpha_1(\Lambda;0) \to 1$ when $\Lambda \to +\infty$.

According to this, one has $\widehat{\alpha}_1(\Lambda;0) \uparrow 1$, $\widehat{\alpha}_1(\Lambda;0) = \cos\left(\frac{\pi}{\Lambda+2}\right) + \varepsilon(\Lambda)$ with $\varepsilon(\Lambda) \ge 0$ and $\varepsilon(\Lambda) \to 0$.

It is well known that $\cos x = 1 - \frac{x^2}{2} + o(x_3)$, then it is obvious that $\varepsilon(\Lambda) = \frac{1}{\Lambda} + o\left(\frac{1}{\Lambda}\right)$ when $\Lambda \to +\infty$ is not possible, because it is in constrast with $\widehat{\alpha}_1(\Lambda;0) = \cos\left(\frac{\pi}{\Lambda+2}\right) + \varepsilon(\Lambda) \le 1$ $\forall \Lambda$; for the same reason, it must be

$$\varepsilon(\Lambda) < \frac{\pi^2}{2(\Lambda+2)^2} \quad \text{when } \Lambda \to +\infty.$$

Finally, this and

$$\cos\left(\frac{\pi}{\Lambda+3}\right) - \cos\left(\frac{\pi}{\Lambda+2}\right) = \frac{\pi}{\Lambda^3} + o\left(\frac{1}{\Lambda^3}\right)$$

implies

$$\widehat{\alpha}_1(\Lambda+1;0) - \widehat{\alpha}_1(\Lambda;0) = \frac{\widetilde{C}}{\Lambda^3} + o\left(\frac{1}{\Lambda^3}\right) \quad \text{when } \Lambda \to +\infty,$$

for a suitable constant $\widetilde{C} > 0$.

Coming back to (8.23), from $\sqrt{\frac{1}{3}} \le \widehat{\alpha}_1(\Lambda;0) < 1$ $\forall \Lambda \in \mathbb{N}$ one has

$$\frac{\Lambda^2\left[\widehat{\alpha}_1(\Lambda;0)\right]^2}{\left[\widehat{\alpha}_1(\Lambda+1;0)\right]^2-\left[\widehat{\alpha}_1(\Lambda;0)\right]^2} \le \frac{1}{2\sqrt{\frac{1}{3}}}\frac{\Lambda^2}{\widehat{\alpha}_1(\Lambda+1;0)-\widehat{\alpha}_1(\Lambda;0)} = \frac{1}{2\widetilde{C}\sqrt{\frac{1}{3}}}\Lambda^5 + O\left(\Lambda^6\right)$$

when $\Lambda \to +\infty$.

Then

$$k_D(\Lambda) \ge \Lambda^6 \Rightarrow \alpha_1(\Lambda+1;0) > \alpha_1(\Lambda;0) \quad \text{definitively.}$$

8.3.5 Proof of item (D) in theorem 4.3.1

First of all, from $\sqrt{\frac{l^2}{4l^2-1}} > \frac{1}{2}$ $\forall l \in \mathbb{N}$ and proposition 8.1.2, it follows

$$\left\| P_{\Lambda+1}\left(0, \frac{1}{2}, \frac{1}{2}\right) \right\|_2 = \cos\left(\frac{\pi}{\Lambda+2}\right) < \alpha_1(\Lambda; 0) = \|B_0(\Lambda)\|_2, \tag{8.24}$$

then the inequality $(4.14)_2$ follows trivially from (8.24), $\cos x \geq 1 - \frac{x^2}{2}$ $\forall x \in [0, 1]$ and $\frac{\pi}{\Lambda+2} \leq 1$ $\forall \Lambda \geq 2$.

On the other hand, if $\boldsymbol{\chi}_1$ is the x_0-eigenvector having $\alpha_1(\Lambda, 0)$ eigenvalue, then $L_0 \boldsymbol{\chi}_1 = 0$, which implies

$$\langle \boldsymbol{\chi}_1, x_+ \boldsymbol{\chi}_1 \rangle = 0, \quad \langle \boldsymbol{\chi}_1, x_- \boldsymbol{\chi}_1 \rangle = 0 \Rightarrow \langle \boldsymbol{\chi}_1, x_1 \boldsymbol{\chi}_1 \rangle = 0, \quad \langle \boldsymbol{\chi}_1, x_2 \boldsymbol{\chi}_1 \rangle = 0;$$

so, from

$$(\Delta \boldsymbol{x})^2_{\boldsymbol{\chi}_1} := \left\langle \boldsymbol{\chi}_1, \boldsymbol{x}^2 \boldsymbol{\chi}_1 \right\rangle - \sum_{i=1}^{3} \langle \boldsymbol{\chi}_1, x_i \boldsymbol{\chi}_1 \rangle^2 \geq 0,$$

it follows

$$[\alpha_1(\Lambda, 0)]^2 = \langle \boldsymbol{\chi}_1, x_0 \boldsymbol{\chi}_1 \rangle^2 \leq \left\langle \boldsymbol{\chi}_1, \boldsymbol{x}^2 \boldsymbol{\chi}_1 \right\rangle \overset{(3.9)}{\leq} 1 + \frac{\Lambda(\Lambda+1)+1}{k_D(\Lambda)}. \tag{8.25}$$

It is obvious that (8.24) and (8.25) trivially implies

$$\lim_{\Lambda \to +\infty} \alpha_1(\Lambda, 0) = 1.$$

Once proved this, then the proof of (D) is essentially the same of section 8.2.4, the only difference is that here $A = P_{\Lambda+1}\left(0, \frac{1}{2}, \frac{1}{2}\right)$, $A + E = B_0(\Lambda)$ and $\|E\|_2 \leq 2\left\{\sqrt{1 + \frac{1}{(\Lambda+1)^2}}\left[\frac{1}{2} + \frac{1}{12}\right] - \frac{1}{2}\right\}$, which follows from proposition 8.1.2, (4.2) and

$$\begin{aligned}
c_{l,3} G(l, m, 2) &= \sqrt{1 + \frac{l^2}{k_D(\Lambda)}} \sqrt{\frac{l^2 - m^2}{4l^2 - 1}} \leq \sqrt{1 + \frac{1}{(\Lambda+1)^2}} \sqrt{\frac{l^2}{4l^2-1}} \\
&\leq \sqrt{1 + \frac{1}{(\Lambda+1)^2}} \left[\frac{1}{2} + \left(\sqrt{\frac{l^2}{4l^2-1}} - \frac{1}{2}\right)\right] \\
&= \sqrt{1 + \frac{1}{(\Lambda+1)^2}} \left[\frac{1}{2} + \left(\frac{\frac{1}{4(4l^2-1)}}{\sqrt{\frac{l^2}{4l^2-1}} + \frac{1}{2}}\right)\right] \\
&\leq \sqrt{1 + \frac{1}{(\Lambda+1)^2}} \left[\frac{1}{2} + \frac{1}{12}\right].
\end{aligned}$$

8.4 The proofs about the x_i spectrum in S_Λ^d when $d>2$

8.4.1 Proof of item (A) in theorem 4.4.1

This proof is essentially the same of section 8.3.1, one has only to replace x_0 with x_D and $x_\pm$ with x_h, $h \neq D$.

8.4.2 Proof of item (B) in theorem 4.4.1

This proof is essentially the same of section 8.3.2, one has only to use in this case the $\Xi_{l_{d-1}}(\Lambda;\alpha)$ matrix.

8.4.3 Proof of (4.18) in theorem 4.4.1

This proof is essentially the same of section 8.3.3, one has only to prove that

Lemma 8.4.1 *Let*

$$1 \leq m \leq \Lambda, \quad j \in \mathbb{N}, \quad 1 \leq l := m+j \leq \Lambda; \tag{8.26}$$

then

$$c_{l,D}G(l,m-1,d) \geq c_{l,D}G(l,m,d). \tag{8.27}$$

Proof. Because of (7.32) and (7.76), one has

$$c_{l,D}G(l,m-1,d) = \sqrt{1+\frac{b(l,D)+b(l-1,D)}{2k_D}}\sqrt{\frac{(l-m+1)(l+m+d-3)}{(2l+d-1)(2l+d-3)}}$$

and

$$c_{l,D}G(l,m,d) = \sqrt{1+\frac{b(l,D)+b(l-1,D)}{2k_D}}\sqrt{\frac{(l-m)(l+m+d-2)}{(2l+d-1)(2l+d-3)}},$$

then (8.17) becomes

$$(l-m+1)(l+m+d-3)-(l-m)(l+m+d-2) = d+2m-3 > 0$$

and this is true because of (8.26) and $d \geq 3$. □

Similarly to section 8.3.3, from this it follows that

$$\rho\left(\Theta_{l_{d-1}-1}(\Lambda)\right) > \rho\left(\Theta_{l_{d-1}}(\Lambda)\right), \tag{8.28}$$

and then the inequality (4.18).

8.4.4 Proof of item (D) in theorem 4.4.1

In order to clarify the notation, let $\chi = \left(\chi^0, \chi^1, \cdots, \chi^\Lambda\right)^T \in \mathbb{R}^{\Lambda+1}$, so applying the matrix Θ_0 to this vector, and calculating the norm $\|\|_2$, one obtains (here and later on $v_l := v_{l,0,D}$)

$$\begin{aligned}\|\Theta_0[\chi]\|_2^2 = &\left(v_1\chi^1\right)^2 + \left(v_1\chi^0 + v_2\chi^2\right)^2 + \left(v_2\chi^1 + v_3\chi^3\right)^2 + \left(v_3\chi^2 + v_4\chi^4\right)^2 + \cdots \\ &\cdots + \left(v_{\Lambda-2}\chi^{\Lambda-3} + v_{\Lambda-1}\chi^{\Lambda-1}\right)^2 + \left(v_{\Lambda-1}\chi^{\Lambda-2} + v_\Lambda\chi^\Lambda\right)^2 + \left(v_\Lambda\chi^{\Lambda-1}\right)^2;\end{aligned} \tag{8.29}$$

then one can try to find some informations about $\alpha_0(\Lambda)$ by calculating (8.29) on particular algebraic vectors χ. In particular, if

$$\chi \equiv \widetilde{\chi} = \begin{pmatrix} \frac{1}{\sqrt{\Lambda+1}} \\ \frac{1}{\sqrt{\Lambda+1}} \\ \frac{1}{\sqrt{\Lambda+1}} \\ \vdots \\ \frac{1}{\sqrt{\Lambda+1}} \end{pmatrix} ,$$

then (8.29) becomes

$$\|\Theta_0[\chi]\|_2^2 = \frac{1}{\Lambda+1}\left[2\sum_{l=1}^{\Lambda}(v_l)^2 + 2\sum_{l=2}^{\Lambda} v_l v_{l-1}\right]; \tag{8.30}$$

with

$$\begin{aligned}2\sum_{l=1}^{\Lambda}(v_l)^2 = 2\sum_{l=1}^{\Lambda}[c_{l,D}G(l,0,d)]^2 &\overset{c_{l,D}\geq 1}{\geq} 2\sum_{l=1}^{\Lambda}\frac{l(l+d-2)}{(2l+d-1)(2l+d-3)} \\ &\geq 2\sum_{l=1}^{\Lambda}\frac{l(l+d-2)}{(2l+d-1)^2} \\ &= 2\left[\sum_{l=1}^{\Lambda}\frac{l^2}{(2l+d-1)^2} + (d-2)\sum_{l=1}^{\Lambda}\frac{l}{(2l+d-1)^2}\right]\end{aligned} \tag{8.31}$$

and

$$\begin{aligned}2\sum_{l=2}^{\Lambda} v_l v_{l-1} =& 2\sum_{l=2}^{\Lambda} c_{l,D}G(l,0,d)c_{l-1,D}G(l-1,0,d) \\ &\overset{c_{m,D}\geq 1}{\geq} 2\sum_{l=2}^{\Lambda} G(l,0,d)G(l-1,0,d) \overset{\$}{\geq} 2\sum_{l=2}^{\Lambda}[G(l,0,d)]^2 \\ &\overset{(8.31)}{\geq} 2\left[\sum_{l=1}^{\Lambda}\frac{l^2}{(2l+d-1)^2} + (d-2)\sum_{l=1}^{\Lambda}\frac{l}{(2l+d-1)^2}\right],\end{aligned} \tag{8.32}$$

where the inequality \$ follows from

$$\begin{aligned} G(l,0,d) \geq G(l-1,0,d) &\Leftrightarrow \frac{l(l+d-2)}{(2l+d-1)(2l+d-3)} \geq \frac{(l-1)(l+d-3)}{(2l+d-3)(2l+d-5)} \\ &\Leftrightarrow l(l+d-2)(2l+d-5) \geq (l-1)(l+d-3)(2l+d-1) \\ &\Leftrightarrow d^2-4d+3 \geq 0, \end{aligned}$$

which is true because $d \geq 3$. According to this,

$$\begin{aligned} \|\Theta_0[\chi]\|_2^2 &\geq \frac{2}{\Lambda+1}\left[\sum_{l=1}^{\Lambda}\frac{l^2}{(2l+d-1)^2} + (d-2)\sum_{l=1}^{\Lambda}\frac{l}{(2l+d-1)^2}\right] \\ &+ \frac{2}{\Lambda+1}\left[\sum_{l=2}^{\Lambda}\frac{l^2}{(2l+d-1)^2} + (d-2)\sum_{l=2}^{\Lambda}\frac{l}{(2l+d-1)^2}\right] \\ &\geq \frac{4}{\Lambda+1}\left[\sum_{l=1}^{\Lambda}\frac{l^2}{(2l+d-1)^2}\right] - \frac{2}{\Lambda+1}\left[\frac{1}{d+1}+\frac{d-2}{d+1}\right] \\ &= \frac{1}{\Lambda+1}\left[\sum_{l=1}^{\Lambda}\frac{l^2}{\left(l+\frac{d-1}{2}\right)^2}\right] - \frac{2(d-1)}{(\Lambda+1)(d+1)} \\ &= \frac{1}{\Lambda+1}\left[\sum_{l=1}^{\Lambda}\frac{1}{\left(1+\frac{d-1}{2l}\right)^2}\right] - \frac{2(d-1)}{(\Lambda+1)(d+1)}. \end{aligned} \tag{8.33}$$

So, from

$$\frac{1}{(1+x)^2} \geq 1-2x \Leftrightarrow 1 \geq 1-2x^3-3x^2 \tag{8.34}$$

which is true when $x > 0$, and from [60]

$$\ln\left(n+\frac{1}{2}\right)+\gamma+\frac{1}{24(n+a)^2} \leq \sum_{k=1}^{n}\frac{1}{k} \leq \ln\left(n+\frac{1}{2}\right)+\gamma+\frac{1}{24(n+b)^2}, \tag{8.35}$$

with

$$a := \frac{1}{\sqrt{24\left[1-\gamma-\ln\left(\frac{3}{2}\right)\right]}} \simeq 0.55, \quad b := \frac{1}{2}$$

and γ is the Euler-Mascheroni constant; it follows

$$\begin{aligned} \sum_{l=1}^{\Lambda}\frac{1}{\left(1+\frac{d-1}{2l}\right)^2} &\overset{(8.34)}{\geq} \sum_{l=1}^{\Lambda}(1-2\frac{d-1}{2l}) = \Lambda-(d-1)\sum_{l=1}^{\Lambda}\frac{1}{l} \\ &\overset{(8.35)}{\geq} \Lambda-(d-1)\left[\ln\left(\Lambda+\frac{1}{2}\right)+\gamma+\frac{1}{24\left(\Lambda+\frac{1}{2}\right)^2}\right]. \end{aligned} \tag{8.36}$$

Then

$$\begin{aligned}\alpha_0(\Lambda) &:= \rho\left(\Theta_0\right) \geq \|\Theta_0[\chi]\|_2^2 \\ &\geq \sqrt{\frac{1}{\Lambda+1}\left\{\Lambda-(d-1)\left[\ln\left(\Lambda+\frac{1}{2}\right)+\gamma+\frac{2}{d+1}+\frac{1}{24\left(\Lambda+\frac{1}{2}\right)^2}\right]\right\}}\end{aligned} \quad (8.37)$$

On the other hand, one can easily prove a bound from above for $\alpha_0(\Lambda)$, but it is important to point out that

$$\boldsymbol{\chi} \in \mathcal{H}_\Lambda \text{ and } \|\boldsymbol{\chi}\| = 1 \Rightarrow \left\langle \boldsymbol{\chi}, \boldsymbol{L}^2 \boldsymbol{\chi}\right\rangle \leq \Lambda(\Lambda+D-2). \quad (8.38)$$

Theorem 8.4.1 *The maximal eigenvalue α_0 of Θ_0 and the corresponding eigenvector $\boldsymbol{\chi}_0$ fulfill*

$$[\alpha_0(\Lambda)]^2 \leq 1 + \frac{D^2-2D+1+\Lambda(\Lambda+D-2)}{4k_D} \quad \forall \Lambda \in \mathbb{N}. \quad (8.39)$$

Proof. The equalities

$$C_d\boldsymbol{\chi}_0 = \cdots = C_3\boldsymbol{\chi}_0 = L_{1,2}\boldsymbol{\chi}_0 = 0$$

imply

$$\langle \boldsymbol{\chi}_0 | \overline{x}_+ | \boldsymbol{\chi}_0 \rangle = \langle \boldsymbol{\chi}_0 | \overline{x}_- | \boldsymbol{\chi}_0 \rangle = \langle \boldsymbol{\chi}_0 | \overline{x}_h | \boldsymbol{\chi}_0 \rangle = 0$$

for all $h \neq D$.

Consequently, the inequality $\Delta\mathcal{R}^2_{\boldsymbol{\chi}_0} \geq 0$, or more explicitely

$$0 \leq \left[\left\langle \boldsymbol{\chi}_0 \right| \left(\overline{x}_D\right)^2 \left|\boldsymbol{\chi}_0\right\rangle - \left(\langle \boldsymbol{\chi}_0 | \overline{x}_D | \boldsymbol{\chi}_0 \rangle\right)^2\right] + \sum_{h=1}^{d} \left\langle \boldsymbol{\chi}_0 \right| \left(\overline{x}_h\right)^2 \left|\boldsymbol{\chi}_0\right\rangle,$$

becomes

$$\left[\alpha_0\left(\Lambda\right)\right]^2 \leq \sum_{h=1}^{D} \left\langle \boldsymbol{\chi}_0 \right| \left(\overline{x}_h\right)^2 \left|\boldsymbol{\chi}_0\right\rangle = \left\langle \boldsymbol{\chi}_0 | \mathcal{R}^2 | \boldsymbol{\chi}_0 \right\rangle,$$

with

$$\begin{aligned}\boldsymbol{x}^2\boldsymbol{\psi}_{\boldsymbol{l},D} &\overset{(2.26)}{=} \left\{1+\frac{b(l,D)+[b(l+1,D)]\frac{l+D-2}{2l+D-2}+[b(l-1,D)]\frac{l}{2l+D-2}}{2k_D(\Lambda)}\right. \\ &\quad \left.-\left[\left(1+\frac{b(\Lambda,D)+b(\Lambda+1,D)}{2k_D(\Lambda)}\right)\frac{\Lambda+D-2}{2\Lambda+D-2}\right]\widehat{P}_{\Lambda,D}\right\}\boldsymbol{\psi}_{\boldsymbol{l},D} \\ &= \left\{1+\frac{D^2-2D+1+4\boldsymbol{L}^2}{4k_D(\Lambda)}\right. \\ &\quad \left.-\left[\left(1+\frac{b(\Lambda,D)+b(\Lambda+1,D)}{2k_D(\Lambda)}\right)\frac{\Lambda+D-2}{2\Lambda+D-2}\right]\widehat{P}_{\Lambda,D}\right\}\boldsymbol{\psi}_{\boldsymbol{l},D}\end{aligned} \quad (8.40)$$

and these two last equations, together with (8.38), imply obviously (8.39); so the proof is finished. □

The inequalities (8.37) and (8.39) trivially imply the following

Corollary 8.4.1 *The maximal eigenvalue $\alpha_0 = \alpha_0(\Lambda)$ of $\overline{x}_D$ fulfills*

$$\lim_{\Lambda\to+\infty} \alpha_0(\Lambda) = 1. \tag{8.41}$$

Once proved this, then the proof of (D) is essentially the same of section 8.2.4, the only difference is that here $A = P_{\Lambda+1}\left(0, \frac{1}{2}, \frac{1}{2}\right)$, $A + E = \Theta_0(\Lambda)$ and

$$\|E\|_2 \leq 2\left\{\sqrt{1+\frac{b(\Lambda,D)+b(\Lambda-1,D)}{2k_D(\Lambda)}}-\frac{1}{2}\right\},$$

which follows from proposition 8.1.2, (4.2) and

$$\begin{aligned} c_{l,D}G(l,0,d) &= \sqrt{1+\frac{b(l,D)+b(l-1,D)}{2k_D(\Lambda)}}\sqrt{\frac{l(l+d-2)}{(2l+d-1)(2l+d-3))}} \\ &\leq \sqrt{1+\frac{b(\Lambda,D)+b(\Lambda-1,D)}{2k_D(\Lambda)}}. \end{aligned}$$

8.5 Some useful summations

From $h(h{+}1)(h{+}2)...(h{+}j{+}1) - (h{-}1)h(h{+}1)...(h{+}j) = (j{+}2)h(h{+}1)...(h{+}j)$ (with $j \in \mathbb{N}_0$) it follows

$$\sum_{h=1}^{n} h(h{+}1)...(h{+}j) = \frac{1}{j{+}2}n(n{+}1)(n{+}2)...(n{+}j{+}1); \tag{8.42}$$

this implies, in particular,

$$\sum_{h=1}^{n} h^2 = \sum_{h=1}^{n}[h(h{+}1) - h] = \frac{n(n{+}1)(n{+}2)}{3} - \frac{n(n{+}1)}{2} = \frac{n(n{+}1)(2n{+}1)}{6}, \tag{8.43}$$

There are also other equalities that are useful:

$$\sum_{h=1}^{n} h^3 = \frac{n^2(n+1)^2}{4}, \qquad \sum_{h=1}^{n} h(2h+1) = \frac{4n^3+9n^2+5n}{6}, \tag{8.44}$$

$$\sum_{h=1}^{n} h(h+1)(2h+1) = \frac{1}{2}n(n+1)^2(n+2), \tag{8.45}$$

$$\sum_{h=1}^{n} [h(h+1)+1](2h+1) = \frac{(n+1)^2(n^2+2n+2)}{2}, \qquad \sum_{h=1}^{n} h\left(1-\frac{1}{2h}\right) = \frac{n^2}{2}. \tag{8.46}$$

Using the inequalities $1+x/2 \geq \sqrt{1+x} \geq 1+x/2-x^2/8$ (the first one is valid for $x \geq -1$, the second for $x \leq 8$) one has

$$1+\frac{m(m-1)}{2k_D} \geq b_m \geq 1+\frac{m(m-1)}{2k_D}-\frac{m^2(m-1)^2}{(2k_D)^2} \qquad (8.47)$$

$$\Rightarrow \qquad n+\frac{(n-1)n(n+1)}{6k} \geq \sum_{m=1}^{n} b_m \geq n+\frac{(n-1)n(n+1)}{6k}-\frac{(n-1)n(n+1)(3n^2-2)}{60k^2} \qquad (8.48)$$

Using trigonometric formulae it is straightforward to show that

$$\sum_{m=2}^{n} \cos\left[\frac{\pi(2m-1)}{2n+2}\right] = 0 \qquad (8.49)$$

(the terms cancel pairwise: the terms with $m=2,n$ cancel each other, the terms with $m=3,n-1$ cancel each other, etc.), and

$$\begin{aligned} 2\sin\left[\frac{\pi(n+1+m)}{2n+2}\right]\sin\left[\frac{\pi(n+m)}{2n+2}\right] &= \cos\left[\frac{\pi}{2n+2}\right]-\cos\left[\frac{\pi(2n+1+2m)}{2n+2}\right] \\ &= \cos\left[\frac{\pi}{2n+2}\right]+\cos\left[\frac{\pi(2m-1)}{2n+2}\right]. \qquad (8.50) \end{aligned}$$

Chapter 9

Appendix C

9.1 Proof of Theorems 5.3.1 and 5.3.1

Consider a generic Hilbert space $\mathcal{H}$ carrying a unitary representation of $O(3)$. For any vector $\boldsymbol{\psi} \in \mathcal{H}$, let $g \in O(3)$ be a 3×3 matrix such that the expectation values of L_j on $\boldsymbol{\psi}$ fulfill

$$g_{ij}\langle L_j\rangle = \delta_i^3 |\langle \boldsymbol{L}\rangle|. \tag{9.1}$$

The expectation values of the $L_j, \boldsymbol{L}^2$ on the states $\boldsymbol{\psi}$, $\boldsymbol{\psi}' := U(g)\boldsymbol{\psi}$ fulfill $\langle L_1\rangle' = \langle L_2\rangle' = 0$, $\langle L_3\rangle' = |\langle \boldsymbol{L}\rangle'| = |\langle \boldsymbol{L}\rangle| \geq 0$, $\langle \boldsymbol{L}^2\rangle' = \langle \boldsymbol{L}^2\rangle$ (the second equalities hold because $U(g)$ is unitary). Hence $\boldsymbol{\psi}$ fulfills/saturates (5.19) iff $\boldsymbol{\psi}'$ respectively fulfills/saturates

$$\langle \boldsymbol{L}^2\rangle' - \langle L_3\rangle' \left(\langle L_3\rangle' + 1\right) \geq 0. \tag{9.2}$$

If $\mathcal{H} = V_l$ the first term equals $l(l+1)$, the inequality (9.2) is fulfilled, and it is saturated by $\boldsymbol{\psi}' = |l, l\rangle$, because $\mathrm{Spec}(L_3) = \{-l, 1-l, ..., l\}$.

Now assume that $\mathcal{H}$ can be decomposed as the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of orthogonal subspaces $\mathcal{H}_1, \mathcal{H}_2$ carrying subrepresentations of $O(3)$ and on which (5.19) is fulfilled; moreover, let $\Gamma_i \subset \mathcal{H}_i$ be the subsets of vectors saturating (5.19). Decomposing $\boldsymbol{\psi}' = a_1\psi_1 + a_2\psi_2$ and setting $\alpha := |a_1|^2$, one finds $0 \leq \alpha \leq 1$, $|a_2|^2 = 1-\alpha$, and

$$\begin{aligned} &\langle \boldsymbol{L}^2\rangle' - \langle L_3\rangle' \left(\langle L_3\rangle' + 1\right) \\ &= \alpha\langle \boldsymbol{L}^2\rangle_1 + (1-\alpha)\langle \boldsymbol{L}^2\rangle_2 - \left[\alpha\langle L_3\rangle_1 + (1-\alpha)\langle L_3\rangle_2\right]^2 - \left[\alpha\langle L_3\rangle_1 + (1-\alpha)\langle L_3\rangle_2\right] =: f(\alpha), \end{aligned} \tag{9.3}$$

where $\langle A\rangle_i \equiv \langle A\rangle_{\boldsymbol{\psi}_i}$. The polynomial $f'(\alpha)$ vanishes only at one point $\alpha' \in \mathbb{R}$, which however is of maximum for $f(\alpha)$, because $f'(\alpha) = -\left[\langle L_3\rangle_1 - \langle L_3\rangle_2\right]^2 \leq 0$. Hence the minimum point of $f(\alpha)$ in the interval $[0, 1]$ is either 0 or 1. But, by the above assumptions,

$$f(1) = \langle \boldsymbol{L}^2\rangle_1 - |\langle \boldsymbol{L}\rangle_1| \left(|\langle \boldsymbol{L}\rangle_1| + 1\right) \geq 0,$$

$$f(0) = \langle \boldsymbol{L}^2\rangle_2 - |\langle \boldsymbol{L}\rangle_2| \left(|\langle \boldsymbol{L}\rangle_2| + 1\right) \geq 0,$$

proving that (5.19) is fulfilled on $\mathcal{H}$. Moreover, the set of states of $\mathcal{H}$ saturating the inequality is clearly $\Gamma = \Gamma_1 \cup \Gamma_2$.

Choosing first $\mathcal{H}_1 = V_0$ and $\mathcal{H}_2 = V_1$, then $\mathcal{H}_1 = V_0 \oplus V_1$ and $\mathcal{H}_2 = V_2$, and so on, one thus iteratively proves the statements of Theorems 5.3.1 and 5.3.1 for pure states.

Similarly, also mixed states (i.e. density operators) ρ fulfill (5.19), but cannot saturate it: abbreviating $\langle A\rangle \equiv \langle A\rangle_\rho := \mathrm{tr}(\rho A)$, let $g \in O(3)$ be a 3×3 matrix such that the expectation values of L_j on ρ fulfill (9.1). Then the expectation values of $L_j, \boldsymbol{L}^2$ on the state $\rho' = U(g)\rho U^{-1}(g)$ fulfill $\langle L_1\rangle' = \langle L_2\rangle' = 0$, $\langle L_3\rangle' = |\langle \boldsymbol{L}\rangle'| = |\langle \boldsymbol{L}\rangle| \geq 0$, $\langle \boldsymbol{L}^2\rangle' = \langle \boldsymbol{L}^2\rangle$, and ρ fulfills/saturates (5.19) iff ρ' fulfills/saturates (9.2). If $\rho' = \alpha\rho_1 + (1-\alpha)\rho_2$, the left-hand side of (9.2) again takes the form (9.3). Hence, reasoning as before, one finds that ρ fulfills (5.19), and that there are no mixed states saturating this inequality.

9.2 Proofs of some results regarding S^1_Λ

On a vector $\boldsymbol{\chi} = \sum_{m=-\Lambda}^{\Lambda} \chi_m \psi_m$ one has $x_+\boldsymbol{\chi} = \sum_{m=-\Lambda}^{\Lambda-1} \chi_m b_{m+1} \boldsymbol{\psi}_{m+1}$, and

$$\langle x_+\rangle_{\boldsymbol{\chi}} = \sum_{m=1-\Lambda}^{\Lambda} \overline{\chi_m}\chi_{m-1} b_m; \tag{9.4}$$

$$\begin{aligned}\langle \boldsymbol{x}^2\rangle_{\boldsymbol{\chi}} &= \sum_{m=1-\Lambda}^{\Lambda-1} \left(1+\frac{m^2}{k_D}\right)|\chi_m|^2 + \frac{1}{2}\left[1+\frac{\Lambda(\Lambda-1)}{k_D}\right](|\chi_\Lambda|^2 + |\chi_{-\Lambda}|^2)\\ &= \langle \boldsymbol{\chi},\boldsymbol{\chi}\rangle + \sum_{m=1-\Lambda}^{\Lambda-1} \frac{m^2}{k_D}|\chi_m|^2 + \frac{1}{2}\left[\frac{\Lambda(\Lambda-1)}{k_D}-1\right](|\chi_\Lambda|^2 + |\chi_{-\Lambda}|^2) \end{aligned}\tag{9.5}$$

In the next lines we do firstly the proof (5.13),

$$\langle L\rangle_{\phi^\beta_\alpha} = \frac{1}{2\Lambda+1}\sum_{m=-\Lambda}^{\Lambda} m = 0, \quad \langle L^2\rangle_{\phi^\beta_\alpha} = \frac{1}{2\Lambda+1}\sum_{m=-\Lambda}^{\Lambda} m^2 = \frac{2}{2\Lambda+1}\sum_{m=1}^{\Lambda} m^2 \overset{(8.43)}{=} \frac{\Lambda(\Lambda+1)}{3}.$$

Then, the proof of (5.14). By (3.9), (8.43), (9.4-9.5)

$$\begin{aligned}\langle \boldsymbol{x}^2\rangle_{\phi^\beta_\alpha} &= \langle \boldsymbol{\phi}^\beta_\alpha, \boldsymbol{x}^2 \boldsymbol{\phi}^\beta_\alpha\rangle = 1 + \frac{2}{2\Lambda+1}\sum_{m=1}^{\Lambda}\frac{m^2}{k_D} - \frac{1}{2\Lambda+1}\left[\frac{\Lambda(\Lambda+1)}{k_D}+1\right]\\ &= 1+\frac{\Lambda(\Lambda+1)}{3k_D} - \frac{1}{2\Lambda+1}\left[\frac{\Lambda(\Lambda+1)}{k_D}+1\right] = \frac{2\Lambda}{2\Lambda+1} + \frac{2(\Lambda-1)\Lambda(\Lambda+1)}{3(2\Lambda+1)k_D},\end{aligned}$$

as claimed. Now it is possible to prove (5.15):

$$
\begin{aligned}
(\Delta\boldsymbol{x})^2_{\boldsymbol{\phi}^\beta_\alpha} &= \langle\boldsymbol{x}^2\rangle_{\boldsymbol{\phi}^\beta_\alpha} - |\langle x_+\rangle_{\boldsymbol{\phi}^\beta_\alpha}|^2 = \frac{2\Lambda}{2\Lambda+1} + \frac{2(\Lambda^2-1)}{3(2\Lambda+1)k_D} - \frac{4}{(2\Lambda+1)^2}\left[\sum_{m=1}^{\Lambda} b_m\right]^2 \\
&\overset{1\le b_m}{\le} \frac{2\Lambda}{2\Lambda+1} + \frac{2(\Lambda^2-1)\Lambda}{3(2\Lambda+1)k_D} - \frac{4\Lambda^2}{(2\Lambda+1)^2} \\
&\overset{(3.5)}{\le} \frac{2\Lambda}{2\Lambda+1} - \frac{4\Lambda^2}{(2\Lambda+1)^2} + \frac{2(\Lambda-1)}{3(2\Lambda+1)\Lambda(\Lambda+1)} < \frac{2\Lambda}{(2\Lambda+1)^2} + \frac{1}{3\Lambda(\Lambda+1)} \\
&< \frac{2\Lambda}{4\Lambda(\Lambda+1)} + \frac{1}{3\Lambda(\Lambda+1)} = \frac{1}{\Lambda+1}\left(\frac{1}{2} + \frac{1}{3\Lambda}\right) \overset{\Lambda\ge 2}{\le} \frac{2}{3(\Lambda+1)}.
\end{aligned}
$$

At this point comes the proof of (5.17). On a generic normalized $\boldsymbol{\chi}$ (9.4-9.5) with $\Lambda = 1$ gives

$$
\begin{aligned}
\langle\boldsymbol{x}^2\rangle_{\boldsymbol{\chi}} = \frac{1}{2}\left[1+|\chi_0|^2\right] = \frac{1}{2}\left[1+s\right], \qquad \langle x_+\rangle_{\boldsymbol{\chi}} = \overline{\chi_0}\chi_{-1} + \overline{\chi_1}\chi_0, \\
|\langle x_+\rangle_{\boldsymbol{\chi}}|^2 = |\chi_0|^2\left(|\chi_1|^2+|\chi_{-1}|^2\right) + \left(\chi_0^2\overline{\chi_1\chi_{-1}} + \overline{\chi_0}^2\chi_1\chi_{-1}\right) = s(1-s) + 2st\cos\alpha, \\
(\Delta\boldsymbol{x})^2_{\boldsymbol{\chi}} = \langle\boldsymbol{x}^2\rangle_{\boldsymbol{\chi}} - |\langle x_+\rangle_{\boldsymbol{\chi}}|^2 = \frac{1}{2}\left[1-s\right] + s^2 - 2st\cos\alpha \qquad (9.6)
\end{aligned}
$$

where $s := |\chi_0|^2 \le 1$, $t := |\chi_1\chi_{-1}|$, and α is the phase of $\chi_0^2\overline{\chi_1\chi_{-1}}$; by the Cauchy-Schwarz inequality $t \le \left(|\chi_1|^2+|\chi_{-1}|^2\right)/2 = (1-s)/2$. For fixed s, (9.6) is minimized by $\alpha = 0$ and $t = (1-s)/2$ (namely $|\chi_1| = |\chi_{-1}| = \sqrt{t} = \sqrt{(1-s)/2}$), what then yields

$$
(\Delta\boldsymbol{x})^2_{\boldsymbol{\chi}} = \frac{1}{2}(1-s) + s^2 - s(1-s) = 2s^2 - \frac{3}{2}s + \frac{1}{2}.
$$

This is minimized by $s = 3/8$, and the minimum value is $(\Delta\boldsymbol{x})^2_{min} = 7/32$, as claimed. The corresponding minimizing vectors are $\underline{\boldsymbol{\chi}} = \frac{\sqrt{5}}{4}\left[e^{i\beta}\boldsymbol{\psi}_{-1} + e^{i\gamma}\boldsymbol{\psi}_1\right] + \frac{\sqrt{3}}{\sqrt{8}}e^{i(\beta+\gamma)/2}\boldsymbol{\psi}_0$; the one in (5.17) is chosen so that $\langle x_+\rangle \in \mathbb{R}$.

Next, the proof of (5.18). Up to normalization, the components of the eigenvector $\boldsymbol{\chi}$ of the Toeplitz matrix X_0 with the maximal eigenvalue ($\lambda_M = \cos\left[\pi/(2\Lambda+2)\right]$) are [see (4.1)]

$$
\chi_m = \sin\left[\frac{\pi(\Lambda+1+m)}{2\Lambda+2}\right] = \cos\left[\frac{\pi m}{2\Lambda+2}\right]; \qquad (9.7)
$$

then $\langle \boldsymbol{\chi}, \boldsymbol{\chi} \rangle = \Lambda{+}1$,

$$
\begin{aligned}
\langle \boldsymbol{\chi}, \boldsymbol{x}^2\boldsymbol{\chi} \rangle &\overset{(9.5)}{=} \langle \boldsymbol{\chi}, \boldsymbol{\chi} \rangle + 2\sum_{m=1}^{\Lambda-1} \frac{m^2}{k_D}\chi_m^2 + \left[\frac{\Lambda(\Lambda-1)}{k_D} - 1\right]\chi_\Lambda^2 \\
&= \Lambda + 1 + 2\sum_{m=1}^{\Lambda-1} \frac{m^2}{k_D}\chi_m^2 + \left[\frac{\Lambda(\Lambda-1)}{k_D} - 1\right]\chi_\Lambda^2 \\
&\overset{\chi_m^2\le 1}{\le} \Lambda + 1 + 2\sum_{m=1}^{\Lambda-1} \frac{m^2}{k_D} + \left[\frac{\Lambda(\Lambda-1)}{k_D} - 1\right]\chi_\Lambda^2 \\
&\overset{(3.5)}{\le} \Lambda + 1 + 2\sum_{m=1}^{\Lambda-1} \frac{m^2}{k_D} \overset{(8.43)}{=} \Lambda + 1 + \frac{\Lambda(\Lambda-1)(2\Lambda-1)}{3k_D},
\end{aligned}
$$

which implies

$$
\langle \boldsymbol{x}^2 \rangle_{\boldsymbol{\chi}} = \frac{\langle \boldsymbol{\chi}, \boldsymbol{x}^2\boldsymbol{\chi} \rangle}{\langle \boldsymbol{\chi}, \boldsymbol{\chi} \rangle} \le 1 + \frac{\Lambda(\Lambda-1)(2\Lambda-1)}{3k_D(\Lambda+1)} \overset{(3.5)}{\le} 1 + \frac{\Lambda(\Lambda-1)(2\Lambda-1)}{3\Lambda^2(\Lambda+1)^3} \le 1 + \frac{1}{(\Lambda+1)^2}. \tag{9.8}
$$

Moreover, due to (8.47), (8.48), $\chi_{-m} = \chi_m \in \mathbb{R}$, it is $\langle x_1 \rangle_{\boldsymbol{\chi}} = \langle x_+ \rangle_{\boldsymbol{\chi}}$ because the latter is real, whence

$$
\begin{aligned}
\langle \boldsymbol{\chi}, x_1\boldsymbol{\chi} \rangle &\overset{(9.4)}{=} 2\sum_{m=1}^{\Lambda} b_m \sin\left[\frac{\pi(\Lambda{+}1{+}m)}{2\Lambda{+}2}\right] \sin\left[\frac{\pi(\Lambda{+}m)}{2\Lambda{+}2}\right] \\
&\overset{b_m\ge 1}{\ge} 2\sum_{m=1}^{\Lambda} \sin\left[\frac{\pi(\Lambda{+}1{+}m)}{2\Lambda{+}2}\right] \sin\left[\frac{\pi(\Lambda{+}m)}{2\Lambda{+}2}\right] \\
&\overset{(8.50)}{=} \sum_{m=1}^{\Lambda} \left\{\cos\left[\frac{\pi}{2\Lambda{+}2}\right] + \cos\left[\frac{\pi(2m{-}1)}{2\Lambda{+}2}\right]\right\} \\
&\overset{(8.49)}{=} (\Lambda+1)\cos\left[\frac{\pi}{2\Lambda{+}2}\right] \quad \Longrightarrow
\end{aligned}
$$

$$
\langle x_1 \rangle_{\boldsymbol{\chi}}^2 = \left(\frac{\langle \boldsymbol{\chi}, x_1\boldsymbol{\chi} \rangle}{\langle \boldsymbol{\chi}, \boldsymbol{\chi} \rangle}\right)^2 \ge \cos^2\left[\frac{\pi}{2\Lambda{+}2}\right] = 1 - \sin^2\left[\frac{\pi}{2\Lambda{+}2}\right] \ge 1 - \left(\frac{\pi}{2\Lambda{+}2}\right)^2, \tag{9.9}
$$

$$
\begin{aligned}
(\Delta \boldsymbol{x})_{\boldsymbol{\chi}}^2 &= \langle \boldsymbol{x}^2 \rangle_{\boldsymbol{\chi}} - \langle x_1 \rangle_{\boldsymbol{\chi}}^2 \\
&\overset{(9.8)\&(9.9)}{\le} 1 + \frac{1}{(\Lambda+1)^2} - 1 + \left(\frac{\pi}{2\Lambda+2}\right)^2 \\
&= \frac{1 + \frac{\pi^2}{4}}{(\Lambda+1)^2} < \frac{3.5}{(\Lambda+1)^2}.
\end{aligned} \tag{9.10}
$$

9.3 States saturating the Heisenberg UR (5.4) on S^1, S^1_Λ

For any $\mu \in \mathbb{R}$, $i = 1,2$ let $a_i^\mu := L - i\mu x_i$, $z_i := \langle L\rangle - i\mu\langle x_i\rangle$, $A_i^\mu := a_i^\mu - z_i$. The inequality $0 \le \langle A_i^{\mu\dagger} A_i^\mu\rangle = (\Delta L)^2 + \mu^2(\Delta x_i)^2 + \mu\epsilon^{ij}\langle x_j\rangle$ (here $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\epsilon^{21} = 1$, and a sum over $j = 1,2$ is understood) is saturated on the states annihilated by A_i^μ, which are the eigenvectors $\boldsymbol{\chi} = \sum_n \chi_n\psi_n$ of a_i^μ; here the sum runs over $n \in \mathbb{Z}$ for S^1 [where $\boldsymbol{\psi}_n$ is $(x_+)^n = e^{in\varphi}$], over $n \in I_\Lambda := \{-\Lambda, 1–\Lambda, ..., \Lambda\}$ for S^1_Λ. One can just stick to $i = 1$; the UR will be thus saturated on the eigenvectors of a_1^μ. The results for a_2^μ can be obtained by a rotation of $\pi/2$, by the $O(2)$-equivariance.

One easily checks that $a_1^\mu\boldsymbol{\chi} = z\boldsymbol{\chi}$ in $\mathcal{H} = \mathcal{L}^2(S^1)$ amounts to the equations

$$2\chi_n(n - z) - i\mu(\chi_{n+1} + \chi_{n-1}) = 0, \qquad n \in \mathbb{Z}. \tag{9.11}$$

One way to fulfill them (with a non trivial $\boldsymbol{\chi}$) is with $\mu = 0$; this implies $\chi_n = 0$ for all n but one, i.e. $\boldsymbol{\chi} \propto \psi_m$ for some $m \in \mathbb{Z}$, and $z = \langle L\rangle = m$. This is actually the only way: if $\mu \neq 0$ then the equations can be used as recurrence relations to determine all the χ_n as combinations of two, e.g. χ_0, χ_1; if the latter vanish so do all χ_n, otherwise the resulting sequence does not lead to a $\boldsymbol{\chi} \in \mathcal{H}$ because $\sum_n |\chi_n|^2 = \infty$. In fact, rewriting (9.11) in the form $\chi_{n+1} = -\chi_{n-1} + C_n\chi_n$, with $C_n := \frac{2}{i\mu}(n - z)$ it is easy to iteratively prove the relation

$$\chi_{n+1} = \chi_n Q_n - \frac{\chi_0}{Q_1 Q_2 \ldots Q_{n-1}}, \qquad Q_1 := C_1, \quad Q_n := C_n - \frac{1}{Q_{n-1}}.$$

This implies that as $n \to \infty$ $|C_n| \to \infty$, $|Q_n| \simeq |C_n| \to \infty$, $|\chi_{n+1}/\chi_n|^2 \simeq |Q_n|^2 \to \infty$, whence by the D'Alembert criterion the series $\sum_{n=0}^\infty |\chi_n|^2$ diverges. The $\boldsymbol{\psi}_m$ are also eigenvectors of $a^2_{\mu=0}$ and therefore saturate not only $(5.4)_1$, but also $(5.4)_2$, and therefore all of (5.4).

One easily checks that the eigenvalue equation $a_1^\mu\boldsymbol{\chi} = z\boldsymbol{\chi}$ in $\mathcal{H}_\Lambda$ (i.e. on S^1_Λ) amounts to the equations

$$\begin{aligned}
&2\chi_{-\Lambda}(\Lambda + z) + i\mu\, b_{1-\Lambda}\chi_{1-\Lambda} = 0,\\
&2\chi_n(n - z) - i\mu(b_{n+1}\chi_{n+1} + b_n\chi_{n-1}) = 0, \qquad n = 1-\Lambda, 2-\Lambda, ..., \Lambda-1,\\
&2\chi_\Lambda(\Lambda - z) - i\mu\, b_\Lambda\chi_{\Lambda-1} = 0
\end{aligned} \tag{9.12}$$

(actually the second equations include also the first, third, because for $n = \pm\Lambda$, $b_{-\Lambda} = b_{\Lambda+1} = 0$). One way to fulfill (9.12) is with $\mu = 0$; this implies $\chi_n = 0$ for all n but one, i.e. $\boldsymbol{\chi} \propto \psi_m$ for some $m \in I_\Lambda$, and $z = \langle L\rangle = m$. But nontrivial solutions exist also with nonzero $\mu \neq 0$. In fact, equations (9.12) can be used as recurrence relations to determine all the χ_n in terms of one. It is possible to use them in the order to express first $\chi_{1-\Lambda}$ as $\chi_{-\Lambda}$ times a factor, then $\chi_{2-\Lambda}$ as $\chi_{-\Lambda}$ times another factor, etc., then the last equation amounts to the eigenvalue equation, a polynomial equation in z of degree $(2\Lambda+1)$. Note that if z is an eigenvalue and $\boldsymbol{\chi}$ the corresponding eigenvector then also $z' = -z$ is an eigenvalue with corresponding eigenvector characterized by components $\chi'_n = (-1)^n \chi_{-n}$. Since $a_2^\mu = e^{-i\pi L/2} a_1^\mu e^{i\pi L/2}$, to each eigenvector $\boldsymbol{\chi}$ of a_1^μ there corresponds the one $\boldsymbol{\chi}' = e^{-i\pi L/2}\boldsymbol{\chi}$ of a_2^μ with the same eigenvalue z and components related by $\chi'_n = \chi_n(-i)^n$. Hence $\boldsymbol{\chi}$ cannot be a simultaneous eigenvector of a_1^μ, a_2^μ and therefore again cannot saturate all of (5.4), but only one of the first two inequalities, unless $\mu = 0$, namely unless it is an eigenvector of L; hence again the $\boldsymbol{\psi}_m$ are the only states saturating all of (5.4).

Here the eigenvectors of a_1^μ are determined for $\Lambda = 1$. The eigenvalue equation amounts to $z(z^2 - 1 + \mu^2/2) = 0$. One can easily find that (9.12) admits the following solutions:

$$z = 0,\ \pm\sqrt{1-\frac{\mu^2}{2}},\ \boldsymbol{\chi} = \chi_{-1}\left\{\boldsymbol{\psi}_{-1} + \frac{2i}{\mu}(1+z)\boldsymbol{\psi}_0 - \left[1+\frac{4z}{\mu^2}(1+z)\right]\boldsymbol{\psi}_1\right\}. \quad (9.13)$$

$\|\boldsymbol{\chi}\|_2 = 1$ amounts to $\frac{|\chi_{-1}|^2}{\mu^4}\left\{\mu^4 + 4\mu^2|1+z|^2 + |\mu^2+4z(1+z)|^2\right\} = 1$. This leads to

$$\begin{aligned} &z = 0 \ \Rightarrow\ |\chi_{-1}|^2 = \frac{\mu^2}{2\mu^2+4} \\ &z = \pm\sqrt{1-\frac{\mu^2}{2}} \ \Rightarrow\ z \in \begin{cases} \mathbb{R}, \\ i\mathbb{R}, \end{cases} \quad |\chi_{-1}|^2 = \begin{cases} \frac{\mu^4}{32(1+z)-8\mu^2} & \text{if } \mu^2 \le 2, \\ \frac{1}{4} & \text{if } \mu^2 \ge 2. \end{cases} \end{aligned}$$

In the $\mu \to 0$ limit one recovers the eigenvectors $\boldsymbol{\psi}_1, \boldsymbol{\psi}_0, \boldsymbol{\psi}_{-1}$ of L with eigenvalues $-1, 0, 1$, whereas in the $\mu \to \infty$ limit the eigenvectors $\varphi_-, \varphi_0, \varphi_+$ of x_1 with eigenvalues $-\sqrt{2}/2, 0, \sqrt{2}/2$ (they are obtained in the reverse order $\varphi_+, \varphi_0, \varphi_-$ in the limit $\mu \to -\infty$). On the other hand if $\mu^2 = 2$ then all eigenvalues coincide with the zero eigenvalue, which remains with geometric multiplicity 1; in other words, in this case (only) there is no basis of $\mathcal{H}_\Lambda$ consisting of eigenvectors of a_1^μ. Moreover, recalling that $z = \langle L\rangle - i\mu\langle x_1\rangle$ one finds that if $\mu^2 \le 2$ then $\langle x_1\rangle = 0$ on all eigenvectors (because z is real), whereas if $\mu^2 \ge 2$ then $\langle L\rangle = 0$ on all eigenvectors (because z is purely imaginary). One easily checks that

$$\langle x_1\rangle + i\langle x_2\rangle = \langle x_+\rangle = \frac{2i}{\mu}|\chi_{-1}|^2\left[2 + z + \bar{z} + \frac{4z}{\mu^2}|1+z|^2\right],$$

leading to

$$z=0,\ \mu^2\le 2\ \Rightarrow\ \langle x_1\rangle=0,\ \langle x_2\rangle=\frac{2\mu}{\mu^2+2},\ \langle\boldsymbol{x}\rangle^2=\frac{4\mu^2}{(\mu^2+2)^2},\tag{9.14}$$

$$z=0,\ \mu^2\ge 2\ \Rightarrow\ \langle x_1\rangle=0,\ \langle x_2\rangle=\frac{1}{\mu},\ \langle\boldsymbol{x}\rangle^2=\frac{1}{\mu^2},\tag{9.15}$$

$$z=\pm\sqrt{1-\frac{\mu^2}{2}},\ \mu^2\le 2\ \Rightarrow\ \langle x_1\rangle=0,\ \langle x_2\rangle=\frac{\mu}{2},\ \langle\boldsymbol{x}\rangle^2=\frac{\mu^2}{4},\tag{9.16}$$

$$z=\pm i\sqrt{\frac{\mu^2}{2}-1},\ \mu^2\ge 2\ \Rightarrow\ \langle x_1\rangle=\frac{\mp 1}{\mu}\sqrt{\frac{\mu^2}{2}-1},\ \langle x_2\rangle=\frac{1}{\mu},\ \langle\boldsymbol{x}\rangle^2=\frac{1}{2}.\tag{9.17}$$

As on $\mathcal{H}_1$ it is $\boldsymbol{x}^2=1-(\tilde{P}_1+\tilde{P}_{-1})/2$, one finds

$$\langle\boldsymbol{x}^2\rangle=1-\frac{|\chi_{-1}|^2}{2}\left\{1+\left|1+\frac{4z}{\mu^2}(1+z)\right|^2\right\}$$

leading to

$$z=0,\ \mu^2\le 2\ \Rightarrow\ \langle\boldsymbol{x}^2\rangle=\frac{\mu^2+4}{2(\mu^2+2)},\ (\Delta\boldsymbol{x})^2=\frac{1}{2}+\frac{2-3\mu^2}{(\mu^2+2)^2},\tag{9.18}$$

$$z=0,\ \mu^2\ge 2\quad\Rightarrow\quad\langle\boldsymbol{x}^2\rangle=\frac{\mu^2+4}{2(\mu^2+2)},\ (\Delta\boldsymbol{x})^2=\frac{\mu^4+2\mu^2-4}{2\mu^2(\mu^2+2)},\tag{9.19}$$

$$z=\pm\sqrt{1-\frac{\mu^2}{2}},\ \mu^2\le 2\quad\Rightarrow\quad\langle\boldsymbol{x}^2\rangle=\frac{1}{2}+\frac{\mu^2}{8},\ (\Delta\boldsymbol{x})^2=\frac{1}{2}-\frac{\mu^2}{8},\tag{9.20}$$

$$z=\pm i\sqrt{\frac{\mu^2}{2}-1},\ \mu^2\ge 2\quad\Rightarrow\quad\langle\boldsymbol{x}^2\rangle=\frac{3}{4},\ (\Delta\boldsymbol{x})^2=\frac{1}{4}.\tag{9.21}$$

One also finds

$$z=0,\quad\Rightarrow\quad(\Delta L)^2=\langle L^2\rangle=\frac{\mu^2}{\mu^2+2},\tag{9.22}$$

$$z=\pm\sqrt{1-\frac{\mu^2}{2}},\ \mu^2\le 2\quad\Rightarrow\quad(\Delta L)^2=1-\frac{\mu^2}{4}-\left[1-\frac{\mu^2(1+z)}{4(1+z)-\mu^2}\right]^2,\tag{9.23}$$

$$z=\pm i\sqrt{\frac{\mu^2}{2}-1},\ \mu^2\ge 2\quad\Rightarrow\quad(\Delta L)^2=\langle L^2\rangle=\frac{1}{2}.\tag{9.24}$$

For all μ $\boldsymbol{\chi}_\alpha:=e^{i\alpha L}\boldsymbol{\chi}$ is characterized by the same $(\Delta\boldsymbol{L})^2,(\Delta\boldsymbol{x})^2$ as $\boldsymbol{\chi}$. For all $\mu\neq 0$ and any of the eigenvectors $\boldsymbol{\chi}$ of a_1^μ the system $X:=\{\boldsymbol{\chi}_\alpha\}_{\alpha\in[0,2\pi[}$ is complete (actually overcomplete), but the resolution of the identity $\int_0^{2\pi}d\alpha\,\boldsymbol{\chi}_\alpha\langle\boldsymbol{\chi}_\alpha,\cdot\rangle=cI$ does not hold.

9.4 Proof of Theorem 5.3.2

This is based on the following two lemmas:

Lemma 9.4.1 *Let* $P^h = \sum_{l=|h|}^{\Lambda} \boldsymbol{\psi}_l^h \langle \boldsymbol{\psi}_l^h, \cdot \rangle$ *be the projector on the* $L_3 = h$ *eigenspace. Then*

$$\int_0^{2\pi} d\alpha\, e^{i\alpha(L_3-h)} = 2\pi P^h. \tag{9.25}$$

This can be proved applying both sides to the basis vectors $\boldsymbol{\psi}_l^m$. In subsection 9.5 we do the proof of

Lemma 9.4.2 (Lemma *I*) *If* $|h|, |n| \le l, j$ *then*

$$\int_0^{\pi} d\theta \sin\theta\, \langle \boldsymbol{\psi}_j^n, e^{i\theta L_2} \boldsymbol{\psi}_j^h \rangle \, \langle e^{i\theta L_2} \boldsymbol{\psi}_l^h, \boldsymbol{\psi}_l^n \rangle = \frac{2}{2l+1} \delta_{lj}. \tag{9.26}$$

Now let $B := \int_{SO(3)} d\mu(g)\, P_g^{\beta}$, with a generic $\boldsymbol{\omega} = \sum\limits_{l=0}^{\Lambda} \sum\limits_{h=-l}^{l} \omega_l^h \boldsymbol{\psi}_l^h$; here comes the computation of $B\boldsymbol{\psi}_l^n$ ($|n| \le l$):

$$\begin{aligned}
B\boldsymbol{\psi}_l^n &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\alpha\, e^{i\varphi L_3} e^{i\theta L_2} e^{i\alpha L_3} \boldsymbol{\omega}\, \langle e^{i\theta L_2} e^{i\alpha L_3} \boldsymbol{\omega}, e^{-i\varphi L_3} \boldsymbol{\psi}_l^n \rangle \\
&\overset{(3.6)}{=} \int_0^{2\pi} d\varphi\, e^{i\varphi(L_3-n)} \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\alpha\, e^{i\theta L_2} e^{i\alpha L_3} \boldsymbol{\omega}\, \langle e^{i\theta L_2} e^{i\alpha L_3} \boldsymbol{\omega}, \boldsymbol{\psi}_l^n \rangle \\
&\overset{(9.25)}{=} 2\pi \sum_{j=|n|}^{\Lambda} \boldsymbol{\psi}_j^n \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\alpha\, \langle \boldsymbol{\psi}_j^n, e^{i\theta L_2} e^{i\alpha L_3} \boldsymbol{\omega} \rangle \, \langle e^{i\theta L_2} e^{i\alpha L_3} \boldsymbol{\omega}, \boldsymbol{\psi}_l^n \rangle
\end{aligned}$$

$$= 2\pi \sum_{j=|n|}^{\Lambda} \boldsymbol{\psi}_j^n \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\alpha \sum_{h=-l}^{l} \overline{\omega_l^h} \sum_{m=-j}^{j} \omega_j^m \langle \boldsymbol{\psi}_j^n, e^{i\theta L_2} e^{i\alpha L_3} \boldsymbol{\psi}_j^m \rangle \, \langle e^{i\theta L_2} e^{i\alpha L_3} \boldsymbol{\psi}_l^h, \boldsymbol{\psi}_l^n \rangle$$

$$\begin{aligned}
&= 2\pi \sum_{j=|n|}^{\Lambda} \boldsymbol{\psi}_j^n \sum_{h=-l}^{l} \overline{\omega_l^h} \sum_{m=-j}^{j} \omega_j^m \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\alpha\, e^{i\alpha(m-h)} \langle \boldsymbol{\psi}_j^n, e^{i\theta L_2} \boldsymbol{\psi}_j^m \rangle \, \langle e^{i\theta L_2} \boldsymbol{\psi}_l^h, \boldsymbol{\psi}_l^n \rangle \\
&\overset{(9.25)}{=} (2\pi)^2 \sum_{j=|n|}^{\Lambda} \boldsymbol{\psi}_j^n \sum_{h=-m_{jl}}^{m_{jl}} \overline{\omega_l^h} \omega_j^h \int_0^{\pi} d\theta \sin\theta\, \langle \boldsymbol{\psi}_j^n, e^{i\theta L_2} \boldsymbol{\psi}_j^h \rangle \, \langle e^{i\theta L_2} \boldsymbol{\psi}_l^h, \boldsymbol{\psi}_l^n \rangle
\end{aligned}$$

where $m_{jl} := \min\{j, l\}$. By (9.26) this becomes $B\boldsymbol{\psi}_l^n = \boldsymbol{\psi}_l^n \sum\limits_{h=-l}^{l} |\omega_l^h|^2 8\pi^2/(2l+1)$. In order that this equals $C\boldsymbol{\psi}_l^n$, i.e. that $B = CI$ with some constant $C > 0$, it must be $\sum\limits_{h=-l}^{l} |\omega_l^h|^2 = C(2l+1)/8\pi^2$ for all $l = 0, ..., \Lambda$. Summing over l and imposing that $\boldsymbol{\omega}$ be normalized one finds

$$1 = \|\boldsymbol{\omega}\|_2 = \sum_{l=0}^{\Lambda}\sum_{h=-l}^{l}|\omega_l^h|^2 = \sum_{l=0}^{\Lambda}\frac{2l+1}{8\pi^2}C = \frac{(\Lambda+1)^2}{8\pi^2}C \quad \Rightarrow \quad C = \frac{8\pi^2}{(\Lambda+1)^2}, \tag{9.27}$$

as claimed. The strong SCS $\{\boldsymbol{\omega}_g\}_{g\in SO(3)}$ is fully $O(3)$-equivariant if $\omega_l^h = \omega_l^{-h}$, because then it is mapped into itself also by the unitary transformation $\boldsymbol{\psi}_l^h \mapsto \boldsymbol{\psi}_l^{-h}$ that corresponds to the transformation of the coordinates (with determinant -1) $(x_1, x_2, x_3) \mapsto (x_1, -x_2, x_3)$.

9.5 Proof of Lemma *I*

First of all, denoting as $F(a, b; c; z)$ the Gauss hypergeometric function and as $(z)_n$ the Pochhammer's symbol, then, by definition,

$$(z)_n := \frac{\Gamma(z+n)}{\Gamma(z)} \quad \text{and} \quad \boldsymbol{F}(-n, b; c; z) := \sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m z^m (b)_m}{(c)_m}. \tag{9.28}$$

According to [55] p. 561 eq 15.4.6, one has

$$\boldsymbol{F}(-n, \alpha+1+\beta+n; \alpha+1; x) = \frac{n!}{(\alpha+1)_n}P_n^{(\alpha,\beta)}(1-2x), \tag{9.29}$$

where $P_n^{(\alpha,\beta)}$ is the Jacobi polynomial. From p. 556 eq. 15.1.1 one has

$$\boldsymbol{F}(a, b; c; z) = \boldsymbol{F}(b, a; c; z), \tag{9.30}$$

p. 559 eq. 15.3.3

$$\boldsymbol{F}(a, b; c; z) = (1-z)^{c-a-b}\boldsymbol{F}(c-a, c-b; c; z) \tag{9.31}$$

and from p. 774

$$\begin{aligned}&\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx \\ &= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}\delta_{nm}.\end{aligned} \tag{9.32}$$

In addition, one needs the following

Proposition 9.5.1 *Let $l \geq s \geq h \geq -l$ and*

$$f(l, h, s) := \begin{cases} \prod_{j=h}^{s-1}[l(l+1) - j(j+1)] & \text{if} \quad h < s, \\ 1 & \text{if} \quad h = s; \end{cases} \tag{9.33}$$

then

$$f(l, h, s) = \frac{(l-h)!(l+s)!}{(l+h)!(l-s)!}. \tag{9.34}$$

Proof. When $h = s$,

$$f(l,h,h) = 1 = \frac{(l-h)!(l+h)!}{(l+h)!(l-h)!};$$

assume that $h < s$ and (induction hypothesis)

$$f(l,h,s-1) = \frac{(l-h)!(l+s-1)!}{(l+h)!(l-s+1)!},$$

so

$$\begin{aligned} f(l,h,s) = f(l,h,s-1)\,[l(l+1)-(s-1)s] &= \frac{(l-h)!(l+s-1)!}{(l+h)!(l-s+1)!}(l+s)(l-s+1) \\ &= \frac{(l-h)!(l+s)!}{(l+h)!(l-s)!}. \end{aligned}$$

$\square$

In the same way one can prove that, when $l \geq s \geq h \geq -l$, and setting

$$g(l,h,s) := \begin{cases} \prod_{j=h+1}^{s} [l(l+1) - j(j-1)] & \text{if } \; h < s, \\ 1 & \text{if } \; h = s; \end{cases} \tag{9.35}$$

then

$$g(l,h,s) = \frac{(l-h)!(l+s)!}{(l+h)!(l-s)!}; \tag{9.36}$$

so, when $l \geq s \geq h \geq -l$,

$$\begin{gathered} f(l,h,h) = 1 = g(l,-h,-h) \quad \text{and} \\ f(l,h,s) = \prod_{j=h}^{s-1} [l(l+1) - j(j+1)] = \prod_{j=-s+1}^{-h} [l(l+1) - j(j-1)] = g(l,-s,-h). \end{gathered} \tag{9.37}$$

It is important to point out that, when $0 \le n \le h \le l$,

$$
\begin{aligned}
&\left\langle e^{2\log\left(\cos\frac{\theta}{2}\right)L_0} e^{\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_l^h, e^{-\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_l^n\right\rangle \\
\overset{(9.33)}{=}& \left\langle \sum_{s=h}^{l}\left(\cos\frac{\theta}{2}\right)^{2s}\frac{\left(\tan\frac{\theta}{2}\right)^{s-h}}{(s-h)!}\sqrt{f(l,h,s)}\boldsymbol{\psi}_l^s, \sum_{r=n}^{l}\frac{(-1)^{r-n}\left(\tan\frac{\theta}{2}\right)^{r-n}}{(r-n)!}\sqrt{f(l,n,r)}\boldsymbol{\psi}_l^r\right\rangle \\
\overset{n\le h}{=}& (-1)^{h-n}\sum_{s=h}^{l}(-1)^{s-h}\frac{\left[\tan\frac{\theta}{2}\right]^{s-h}}{(s-h)!}\sqrt{f(l,h,s)}\left(\cos\frac{\theta}{2}\right)^{2s}\frac{\left[\tan\frac{\theta}{2}\right]^{s-n}}{(s-n)!}\sqrt{f(l,n,s)} \\
=& (-1)^{h-n}\sum_{s=h}^{l}(-1)^{s-h}\frac{1}{(s-h)!}\sqrt{f(l,h,s)}\left(\cos\frac{\theta}{2}\right)^{n+h}\left(\sin\frac{\theta}{2}\right)^{2s-n-h}\frac{1}{(s-n)!}\sqrt{f(l,n,s)} \\
\overset{(9.34)}{=}& (-1)^{h-n}\left(\cos\frac{\theta}{2}\right)^{n+h}\left(\sin\frac{\theta}{2}\right)^{h-n}\sqrt{\frac{(l-n)!(l-h)!}{(l+n)!(l+h)!}} \\
&\cdot\sum_{s=h}^{l}(-1)^{s-h}\frac{(l+s)!}{(l-s)!(s-n)!(s-h)!}\left(\sin\frac{\theta}{2}\right)^{2(s-h)} \\
\overset{j=s-h}{=}& (-1)^{h-n}\left(\cos\frac{\theta}{2}\right)^{n+h}\left(\sin\frac{\theta}{2}\right)^{h-n}\sqrt{\frac{(l-n)!(l-h)!}{(l+n)!(l+h)!}} \\
&\cdot\sum_{j=0}^{l-h}(-1)^{j}\frac{(l+h+j)!}{(l-h-j)!(h-n+j)!(j)!}\left(\sin\frac{\theta}{2}\right)^{2j}, \\
\overset{(9.28)}{=}& (-1)^{h-n}\left(\cos\frac{\theta}{2}\right)^{n+h}\left(\sin\frac{\theta}{2}\right)^{h-n}\sqrt{\frac{(l-n)!(l-h)!}{(l+n)!(l+h)!}} \\
&\cdot\frac{(l+h)!}{(l-h)!(h-n)!}F\left(-(l-h), l+h+1; h-n+1; \left(\sin\frac{\theta}{2}\right)^2\right) \\
\overset{(9.29)}{=}& (-1)^{h-n}\left(\cos\frac{\theta}{2}\right)^{n+h}\left(\sin\frac{\theta}{2}\right)^{h-n}\sqrt{\frac{(l-n)!(l+h)!}{(l+n)!(l-h)!}} \\
&\cdot\frac{1}{(h-n)!}\frac{(l-h)!(h-n)!}{(l-n)!}P_{l-h}^{(h-n,h+n)}\left(1-2\sin^2\frac{\theta}{2}\right) \\
=& (-1)^{h-n}\left(\cos\frac{\theta}{2}\right)^{n+h}\left(\sin\frac{\theta}{2}\right)^{h-n}\sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}P_{l-h}^{(h-n,h+n)}\left(1-2\sin^2\frac{\theta}{2}\right),
\end{aligned}
\tag{9.38}
$$

$$
\begin{aligned}
A :=& \left\langle e^{-2\log\left(\cos\frac{\theta}{2}\right)L_0} e^{-\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_l^{-h}, e^{\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_l^{-n}\right\rangle \\
\overset{(9.35)}{=}& \left\langle \sum_{s=-h}^{-l}\left(\cos\frac{\theta}{2}\right)^{-2s}\frac{(-1)^{-s-h}\left(\tan\frac{\theta}{2}\right)^{-s-h}}{(-s-h)!}\sqrt{g(l,s,-h)}\boldsymbol{\psi}_l^s, \sum_{r=-n}^{-l}\frac{\left(\tan\frac{\theta}{2}\right)^{-r-n}}{(-r-n)!}\sqrt{g(l,r,-n)}\boldsymbol{\psi}_l^r\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
A &\overset{(9.37)}{=} \left\langle \sum_{s=-h}^{-l} \left(\cos\frac{\theta}{2}\right)^{-2s} \frac{(-1)^{-s-h}\left(\tan\frac{\theta}{2}\right)^{-s-h}}{(-s-h)!} \sqrt{f(l,h,-s)}\boldsymbol{\psi}_l^s, \sum_{r=-n}^{-l} \frac{\left(\tan\frac{\theta}{2}\right)^{-r-n}}{(-r-n)!} \sqrt{f(l,n,-r)}\boldsymbol{\psi}_l^r \right\rangle \\
&\overset{r,s\to -r,-s}{=} \left\langle \sum_{s=h}^{l} \left(\cos\frac{\theta}{2}\right)^{2s} \frac{(-1)^{s-h}\left(\tan\frac{\theta}{2}\right)^{s-h}}{(s-h)!} \sqrt{f(l,h,s)}\boldsymbol{\psi}_l^{-s}, \sum_{r=n}^{l} \frac{\left(\tan\frac{\theta}{2}\right)^{r-n}}{(r-n)!} \sqrt{f(l,n,r)}\boldsymbol{\psi}_l^{-r} \right\rangle \\
&\overset{n\leq h}{=} \sum_{s=h}^{l} (-1)^{s-h} \frac{\left[\tan\frac{\theta}{2}\right]^{s-h}}{(s-h)!} \sqrt{f(l,h,s)} \left(\cos\frac{\theta}{2}\right)^{2s} \frac{\left[\tan\frac{\theta}{2}\right]^{s-n}}{(s-n)!} \sqrt{f(l,n,s)} \\
&= \sum_{s=h}^{l} (-1)^{s-h} \frac{\left[\tan\frac{\theta}{2}\right]^{s-h}}{(s-h)!} \sqrt{f(l,h,s)} \left(\cos\frac{\theta}{2}\right)^{2s} \frac{\left[\tan\frac{\theta}{2}\right]^{s-n}}{(s-n)!} \sqrt{f(l,n,s)} \\
&\overset{(9.38)}{=} \left(\cos\frac{\theta}{2}\right)^{n+h} \left(\sin\frac{\theta}{2}\right)^{h-n} \sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}} P_{l-h}^{(h-n,h+n)} \left(1-2\sin^2\frac{\theta}{2}\right),
\end{aligned}
\tag{9.39}
$$

$$
\begin{aligned}
B &:= \left\langle e^{2\log\left(\cos\frac{\theta}{2}\right)L_0} e^{\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_l^{-h}, e^{-\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_l^n \right\rangle \\
&\overset{(9.33)}{=} \left\langle \sum_{s=-h}^{l} \left(\cos\frac{\theta}{2}\right)^{2s} \frac{\left(\tan\frac{\theta}{2}\right)^{s+h}}{(s+h)!} \sqrt{f(l,-h,s)}\boldsymbol{\psi}_l^s, \sum_{r=n}^{l} \frac{(-1)^{r-n}\left(\tan\frac{\theta}{2}\right)^{r-n}}{(r-n)!} \sqrt{f(l,n,r)}\boldsymbol{\psi}_l^r \right\rangle \\
&\overset{-h\leq n}{=} \sum_{s=n}^{l} (-1)^{s-n} \frac{\left[\tan\frac{\theta}{2}\right]^{s+h}}{(s+h)!} \sqrt{f(l,-h,s)} \left(\cos\frac{\theta}{2}\right)^{2s} \frac{\left[\tan\frac{\theta}{2}\right]^{s-n}}{(s-n)!} \sqrt{f(l,n,s)} \\
&= \sum_{s=n}^{l} (-1)^{s-n} \frac{1}{(s+h)!} \sqrt{f(l,-h,s)} \left(\cos\frac{\theta}{2}\right)^{n-h} \left(\sin\frac{\theta}{2}\right)^{2s-n+h} \frac{1}{(s-n)!} \sqrt{f(l,n,s)} \\
&\overset{(9.34)}{=} \left(\cos\frac{\theta}{2}\right)^{n-h} \left(\sin\frac{\theta}{2}\right)^{h+n} \sqrt{\frac{(l-n)!(l+h)!}{(l+n)!(l-h)!}} \\
&\quad \cdot \sum_{s=n}^{l} (-1)^{s-n} \frac{(l+s)!}{(l-s)!(s-n)!(s+h)!} \left(\sin\frac{\theta}{2}\right)^{2(s-n)} \\
&\overset{j=s-n}{=} \left(\cos\frac{\theta}{2}\right)^{n-h} \left(\sin\frac{\theta}{2}\right)^{h+n} \sqrt{\frac{(l-n)!(l+h)!}{(l+n)!(l-h)!}} \\
&\quad \cdot \sum_{j=0}^{l-n} (-1)^{j} \frac{(l+n+j)!}{(l-n-j)!(j)!(h+n+j)!} \left(\sin\frac{\theta}{2}\right)^{2j} \\
&\overset{(9.28)}{=} \left(\cos\frac{\theta}{2}\right)^{n-h} \left(\sin\frac{\theta}{2}\right)^{h+n} \sqrt{\frac{(l-n)!(l+h)!}{(l+n)!(l-h)!}} \\
&\quad \cdot \frac{(l+n)!}{(l-n)!(h+n)!} F\left(-(l-n), l+n+1; h+n+1; \left(\sin\frac{\theta}{2}\right)^2\right),
\end{aligned}
$$

$$
\begin{aligned}
B &\overset{(9.31)}{=} \left(\cos\frac{\theta}{2}\right)^{n-h}\left(\sin\frac{\theta}{2}\right)^{h+n}\sqrt{\frac{(l+n)!(l+h)!}{(l-n)!(l-h)!}}\frac{1}{(h+n)!} \\
&\quad\cdot\left[1-\left(\sin\frac{\theta}{2}\right)^2\right]^{h-n} F\left(l+h+1,-(l-h);h+n+1;\left(\sin\frac{\theta}{2}\right)^2\right) \\
&\overset{(9.29)}{=} \left(\cos\frac{\theta}{2}\right)^{h-n}\left(\sin\frac{\theta}{2}\right)^{h+n}\sqrt{\frac{(l+n)!(l+h)!}{(l-n)!(l-h)!}}\frac{1}{(h+n)!} \\
&\quad\cdot\frac{(l-h)!(h+n)!}{(l+n)!}P_{l-h}^{(h+n,h-n)}\left(1-2\sin^2\frac{\theta}{2}\right) \\
&= \left(\cos\frac{\theta}{2}\right)^{h-n}\left(\sin\frac{\theta}{2}\right)^{h+n}\sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}P_{l-h}^{(h+n,h-n)}\left(1-2\sin^2\frac{\theta}{2}\right)
\end{aligned}
\tag{9.40}
$$

and

$$
\begin{aligned}
&\left\langle e^{-2\log\left(\cos\frac{\theta}{2}\right)L_0}e^{-\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_l^h, e^{\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_l^{-n}\right\rangle \\
&\overset{(9.35)}{=}\left\langle\sum_{s=h}^{-l}\left(\cos\frac{\theta}{2}\right)^{-2s}\frac{(-1)^{h-s}\left(\tan\frac{\theta}{2}\right)^{h-s}}{(h-s)!}\sqrt{g(l,s,h)}\boldsymbol{\psi}_l^s,\sum_{s=-n}^{-l}\frac{\left(\tan\frac{\theta}{2}\right)^{-s-n}}{(-s-n)!}\sqrt{g(l,s,-n)}\boldsymbol{\psi}_l^s\right\rangle \\
&\overset{(9.37)}{=}\left\langle\sum_{s=h}^{-l}\left(\cos\frac{\theta}{2}\right)^{-2s}\frac{(-1)^{h-s}\left(\tan\frac{\theta}{2}\right)^{h-s}}{(h-s)!}\sqrt{f(l,-h,-s)}\boldsymbol{\psi}_l^s,\sum_{s=-n}^{-l}\frac{\left(\tan\frac{\theta}{2}\right)^{-s-n}}{(-s-n)!}\sqrt{f(l,n,-s)}\boldsymbol{\psi}_l^s\right\rangle \\
&\overset{s\to-s}{=}\left\langle\sum_{s=-h}^{l}\left(\cos\frac{\theta}{2}\right)^{2s}\frac{(-1)^{s+h}\left(\tan\frac{\theta}{2}\right)^{s+h}}{(s+h)!}\sqrt{f(l,-h,s)}\boldsymbol{\psi}_l^{-s},\sum_{s=n}^{l}\frac{\left(\tan\frac{\theta}{2}\right)^{s-n}}{(s-n)!}\sqrt{f(l,n,s)}\boldsymbol{\psi}_l^{-s}\right\rangle \\
&\overset{-h\leq n}{=}\sum_{s=n}^{l}(-1)^{s+h}\frac{\left[\tan\frac{\theta}{2}\right]^{s+h}}{(s+h)!}\sqrt{f(l,-h,s)}\left(\cos\frac{\theta}{2}\right)^{2s}\frac{\left[\tan\frac{\theta}{2}\right]^{s-n}}{(s-n)!}\sqrt{f(l,n,s)}, \\
&=(-1)^{h+n}\sum_{s=n}^{l}(-1)^{s-n}\frac{\left[\tan\frac{\theta}{2}\right]^{s+h}}{(s+h)!}\sqrt{f(l,-h,s)}\left(\cos\frac{\theta}{2}\right)^{2s}\frac{\left[\tan\frac{\theta}{2}\right]^{s-n}}{(s-n)!}\sqrt{f(l,n,s)} \\
&\overset{(9.40)}{=}(-1)^{h+n}\left(\cos\frac{\theta}{2}\right)^{h-n}\left(\sin\frac{\theta}{2}\right)^{h+n}\sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}P_{l-h}^{(h+n,h-n)}\left(1-2\sin^2\frac{\theta}{2}\right).
\end{aligned}
\tag{9.41}
$$

Finally, when $l \geq h \geq n \geq 0$, one has

$$
\begin{aligned}
&\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}\frac{1}{2^{2h}}\cdot\frac{2^{(h+n)+(h-n)+1}}{2(l-h)+(h+n)+(h-n)+1} \\
&\quad\cdot\frac{\Gamma((l-h)+(h+n)+1)\Gamma((l-h)+(h-n)+1)}{(l-h)!\Gamma((l-h)+(h+n)+(h-n)+1)}=\frac{2}{2l+1},
\end{aligned}
\tag{9.42}
$$

$$\begin{aligned}&\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}\frac{1}{2^{2h}}\cdot\frac{2^{(h-n)+(h+n)+1}}{2(l-h)+(h-n)+(h+n)+1}\\&\quad\cdot\frac{\Gamma((l-h)+(h-n)+1)\Gamma((l-h)+(h+n)+1)}{(l-h)!\Gamma((l-h)+(h-n)+(h+n)+1)}=\frac{2}{2l+1}.\end{aligned}\tag{9.43}$$

It is now possible to prove the aforementioned lemma.

Assume that $0\le n\le h\le l$; by means of the Gauss decomposition, $e^{i\theta L_2}$ can be written in the 'antinormal form' (see e.g. eq. (4.3.14) in [31])

$$e^{i\theta L_2}=e^{-\tan\frac{\theta}{2}L_-}e^{2\log\left(\cos\frac{\theta}{2}\right)L_0}e^{\tan\frac{\theta}{2}L_+};\tag{9.44}$$

hence

$$\begin{aligned}&\int_0^\pi d\theta\sin\theta\,\langle\boldsymbol{\psi}_j^n,e^{i\theta L_2}\boldsymbol{\psi}_j^h\rangle\,\langle e^{i\theta L_2}\boldsymbol{\psi}_l^h,\boldsymbol{\psi}_l^n\rangle\\&=\int_0^\pi d\theta\sin\theta\,\overline{\left\langle e^{2\log\left(\cos\frac{\theta}{2}\right)L_0}e^{\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_j^h,e^{-\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_j^n\right\rangle}\left\langle e^{2\log\left(\cos\frac{\theta}{2}\right)L_0}e^{\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_l^h,e^{-\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_l^n\right\rangle\\&\overset{(9.38)}{=}2(-1)^{2(h-n)}\int_0^\pi d\theta\left(\cos\frac{\theta}{2}\right)^{2(n+h)+1}\left(\sin\frac{\theta}{2}\right)^{2(h-n)+1}\\&\quad\cdot\sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}\sqrt{\frac{(j-h)!(j+h)!}{(j+n)!(j-n)!}}\\&\quad\cdot P_{l-h}^{(h-n,h+n)}\left(1-2\sin^2\frac{\theta}{2}\right)P_{j-h}^{(h-n,h+n)}\left(1-2\sin^2\frac{\theta}{2}\right)\\&\overset{x=1-2\sin^2\frac{\theta}{2}}{=}\sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}\sqrt{\frac{(j-h)!(j+h)!}{(j+n)!(j-n)!}}\int_{-1}^{1}\frac{dx}{2^{2h}}\,(1-x)^{h-n}(1+x)^{h+n}P_{l-h}^{(h-n,h+n)}(x)\,P_{j-h}^{(h-n,h+n)}(x)\\&\overset{(9.32)\&(9.42)}{=}\frac{2}{2l+1}\delta_{lj}.\end{aligned}$$

On the other hand, in order to calculate $\int_0^\pi d\theta\sin\theta\langle\boldsymbol{\psi}_j^{-n},e^{i\theta L_2}\boldsymbol{\psi}_j^{-h}\rangle\langle e^{i\theta L_2}\boldsymbol{\psi}_l^{-h},\boldsymbol{\psi}_l^{-n}\rangle$, one can use now the 'normal form' of the Gauss decomposition (see e.g. eq. (4.3.12) in [31])

$$e^{i\theta L_2}=e^{\tan\frac{\theta}{2}L_+}e^{-2\log\left(\cos\frac{\theta}{2}\right)L_0}e^{-\tan\frac{\theta}{2}L_-},\tag{9.45}$$

and then

$$
\begin{aligned}
&\int_0^\pi d\theta \sin\theta \, \langle \boldsymbol{\psi}_j^{-n}, e^{i\theta L_2}\boldsymbol{\psi}_j^{-h}\rangle \, \langle e^{i\theta L_2}\boldsymbol{\psi}_l^{-h}, \boldsymbol{\psi}_l^{-n}\rangle \\
&= \int_0^\pi d\theta \sin\theta \, \overline{\left\langle e^{-2\log\left(\cos\frac{\theta}{2}\right)L_0} e^{-\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_j^{-h}, e^{\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_j^{-n}\right\rangle} \\
&\quad \cdot \left\langle e^{-2\log\left(\cos\frac{\theta}{2}\right)L_0} e^{-\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_l^{-h}, e^{\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_l^{-n}\right\rangle \\
&\overset{(9.39)}{=} 2\int_0^\pi d\theta \left(\cos\frac{\theta}{2}\right)^{2(n+h)+1}\left(\sin\frac{\theta}{2}\right)^{2(h-n)+1}\sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}\sqrt{\frac{(j-h)!(j+h)!}{(j+n)!(j-n)!}} \\
&\quad \cdot P_{l-h}^{(h-n,h+n)}\left(1-2\sin^2\frac{\theta}{2}\right) P_{j-h}^{(h-n,h+n)}\left(1-2\sin^2\frac{\theta}{2}\right) \\
&\overset{x=1-2\sin^2\frac{\theta}{2}}{=} \sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}\sqrt{\frac{(j-h)!(j+h)!}{(j+n)!(j-n)!}}\frac{1}{2^{2h}} \\
&\quad \cdot \int_{-1}^{1} dx\,(1-x)^{h-n}(1+x)^{h+n} P_{l-h}^{(h-n,h+n)}(x)\, P_{j-h}^{(h-n,h+n)}(x) \\
&\overset{(9.32)\&(9.42)}{=} \frac{2}{2l+1}\delta_{lj}.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
&\int_0^\pi d\theta \sin\theta \, \langle \boldsymbol{\psi}_j^{n}, e^{i\theta L_2}\boldsymbol{\psi}_j^{-h}\rangle \, \langle e^{i\theta L_2}\boldsymbol{\psi}_l^{-h}, \boldsymbol{\psi}_l^{n}\rangle \\
&\overset{(9.44)}{=} \int_0^\pi d\theta \sin\theta \, \overline{\left\langle e^{2\log\left(\cos\frac{\theta}{2}\right)L_0} e^{\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_j^{-h}, e^{-\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_j^{n}\right\rangle} \left\langle e^{2\log\left(\cos\frac{\theta}{2}\right)L_0} e^{\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_l^{-h}, e^{-\tan\frac{\theta}{2}L_+}\boldsymbol{\psi}_l^{n}\right\rangle \\
&\overset{(9.40)}{=} 2\int_0^\pi d\theta \left(\cos\frac{\theta}{2}\right)^{2(h-n)+1}\left(\sin\frac{\theta}{2}\right)^{2(h+n)+1}\sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}\sqrt{\frac{(j-h)!(j+h)!}{(j+n)!(j-n)!}} \\
&\quad \cdot P_{l-h}^{(h+n,h-n)}\left(1-2\sin^2\frac{\theta}{2}\right) P_{j-h}^{(h+n,h-n)}\left(1-2\sin^2\frac{\theta}{2}\right) \\
&\overset{x=1-2\sin^2\frac{\theta}{2}}{=} \sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}\sqrt{\frac{(j-h)!(j+h)!}{(j+n)!(j-n)!}}\frac{1}{2^{2h}} \\
&\quad \cdot \int_{-1}^{1} dx\,(1-x)^{h+n}(1+x)^{h-n} P_{l-h}^{(h+n,h-n)}(x)\, P_{j-h}^{(h+n,h-n)}(x) \\
&\overset{(9.32)\&(9.43)}{=} \frac{2}{2l+1}\delta_{lj}
\end{aligned}
$$

and, finally, as claimed,

$$
\begin{aligned}
E &:= \int_0^\pi d\theta \sin\theta \,\langle \boldsymbol{\psi}_j^{-n}, e^{i\theta L_2}\boldsymbol{\psi}_j^h\rangle \,\langle e^{i\theta L_2}\boldsymbol{\psi}_l^h, \boldsymbol{\psi}_l^{-n}\rangle \\
&\overset{(9.45)}{=} \int_0^\pi d\theta \sin\theta \,\overline{\left\langle e^{-2\log\left(\cos\frac{\theta}{2}\right)L_0} e^{-\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_j^h, e^{\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_j^{-n}\right\rangle} \left\langle e^{-2\log\left(\cos\frac{\theta}{2}\right)L_0} e^{-\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_l^h, e^{\tan\frac{\theta}{2}L_-}\boldsymbol{\psi}_l^{-n}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(9.41)}{=} 2(-1)^{2(h+n)} \int_0^\pi d\theta \left(\cos\frac{\theta}{2}\right)^{2(h-n)+1} \left(\sin\frac{\theta}{2}\right)^{2(h+n)+1} \sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}\sqrt{\frac{(j-h)!(j+h)!}{(j+n)!(j-n)!}} \\
&\quad \cdot P_{l-h}^{(h+n,h-n)}\left(1-2\sin^2\frac{\theta}{2}\right) P_{j-h}^{(h+n,h-n)}\left(1-2\sin^2\frac{\theta}{2}\right),
\end{aligned}
$$

$$
\begin{aligned}
E &\overset{x=1-2\sin^2\frac{\theta}{2}}{=} \sqrt{\frac{(l-h)!(l+h)!}{(l+n)!(l-n)!}}\sqrt{\frac{(j-h)!(j+h)!}{(j+n)!(j-n)!}}\frac{1}{2^{2h}} \\
&\qquad \cdot \int_{-1}^{1} dx\,(1-x)^{h+n}(1+x)^{h-n} P_{l-h}^{(h+n,h-n)}(x)\, P_{j-h}^{(h+n,h-n)}(x) \\
&\overset{(9.32)\&(9.43)}{=} \frac{2}{2l+1}\delta_{lj}.
\end{aligned}
$$

9.6 Proofs of some results regarding S^2_Λ

Proof of (5.27). $L_+\boldsymbol{\omega}^\beta = 0$, $L_-\boldsymbol{\omega}^\beta$ is a combination of $\boldsymbol{\psi}_l^{l-1}$, therefore is orthogonal to $\boldsymbol{\omega}^\beta$. Hence

$$
\langle L_\pm\rangle_{\boldsymbol{\omega}^\beta} = 0, \quad \Rightarrow \quad |\langle \boldsymbol{L}\rangle_{\boldsymbol{\omega}^\beta}| = \langle L_0\rangle_{\boldsymbol{\omega}^\beta} = \sum_{l=0}^{\Lambda} \frac{l(2l+1)}{(\Lambda+1)^2} \overset{(8.44)_2}{=} \frac{\Lambda(4\Lambda+5)}{6(\Lambda+1)};
$$

while

$$
\left\langle \boldsymbol{L}^2\right\rangle_{\boldsymbol{\omega}^\beta} = \sum_{l=0}^{\Lambda} \frac{l(l+1)(2l+1)}{(\Lambda+1)^2} \overset{(8.45)}{=} \frac{\frac{1}{2}\Lambda(\Lambda+1)^2(\Lambda+2)}{(\Lambda+1)^2} = \frac{\Lambda(\Lambda+2)}{2}.
$$

Replacing these results in $(\Delta\boldsymbol{L})^2_{\boldsymbol{\omega}^\beta} = \langle \boldsymbol{L}^2\rangle_{\boldsymbol{\omega}^\beta} - \langle \boldsymbol{L}\rangle^2_{\boldsymbol{\omega}^\beta}$, one finds

$$
(\Delta\boldsymbol{L})^2_{\boldsymbol{\omega}^\beta} = \frac{\Lambda(\Lambda+2)}{2} - \left(\frac{\Lambda(4\Lambda+5)}{6(\Lambda+1)}\right)^2 = \frac{\Lambda(2\Lambda^3+32\Lambda^2+65\Lambda+36)}{36(\Lambda+1)^2}.
$$

On the other hand, $x_0\boldsymbol{\omega}^\beta$ is a combination of $\boldsymbol{\psi}_{l-1}^l, \boldsymbol{\psi}_{l+1}^l$, therefore is orthogonal to $\boldsymbol{\omega}^\beta$, and $\langle x_0\rangle = 0$. Hence

$$
\langle \boldsymbol{x}\rangle^2 = \langle x_1\rangle^2 + \langle x_2\rangle^2 + \langle x_3\rangle^2 = \frac{\langle x_+ + x_-\rangle^2}{4} - \frac{\langle x_+ - x_-\rangle^2}{4} + \langle x_0\rangle^2 = \langle x_+\rangle\langle x_-\rangle = |\langle x_+\rangle|^2,
$$

$$(\Delta\boldsymbol{x})^2 = \left\langle \boldsymbol{x}^2 \right\rangle - |\langle x_+ \rangle|^2 .$$

But

$$\begin{aligned}
\left\langle \boldsymbol{x}^2 \right\rangle_{\boldsymbol{\omega}^\beta} &\overset{(3.9)_2}{=} 1 + \sum_{l=0}^{\Lambda} \frac{l(l+1)+1}{k_D} \frac{2l+1}{(\Lambda+1)^2} - \left[1 + \frac{(\Lambda{+}1)^2}{k_D}\right] \frac{1}{\Lambda{+}1} \\
&= \frac{\Lambda}{\Lambda{+}1} - \frac{\Lambda{+}1}{k_D} + \frac{1}{k_D(\Lambda+1)^2}\left[2\sum_{l=0}^{\Lambda} l(l+1)(l+2) - 3\sum_{l=0}^{\Lambda} l(l+1) + \sum_{l=0}^{\Lambda}(2l+1)\right] \\
&\overset{(8.46)_1}{=} \frac{\Lambda}{\Lambda{+}1} - \frac{\Lambda{+}1}{k_D} + \frac{1}{k_D(\Lambda{+}1)^2}\left[\frac{\Lambda}{2}(\Lambda{+}1)(\Lambda{+}2)(\Lambda{+}3) - \Lambda(\Lambda{+}1)(\Lambda{+}2) + (\Lambda+1)^2\right] \\
&= \frac{\Lambda}{\Lambda{+}1} - \frac{\Lambda{+}1}{k_D} + \frac{1+(\Lambda+1)^2}{2k_D} = \frac{\Lambda}{\Lambda{+}1} + \frac{\Lambda^2}{2k_D} \overset{(3.5)}{\leq} \frac{\Lambda}{\Lambda{+}1} + \frac{1}{2(\Lambda+1)^2}, \qquad (9.46)
\end{aligned}$$

while

$$\begin{aligned}
x_+\boldsymbol{\omega}^\beta &\overset{(3.7)}{=} \sum_{l=0}^{\Lambda-1} e^{i\beta_l} \frac{\sqrt{2l+1}}{\Lambda+1} c_{l+1,3} B_l^{+,l} \boldsymbol{\psi}_{l+1}^{l+1} \overset{(7.32)}{=} \sum_{l=0}^{\Lambda-1} e^{i\beta_l} \frac{\sqrt{2l+1}}{\Lambda+1} c_{l+1,3} \left(-\sqrt{\frac{2l+2}{2l+3}}\right) \boldsymbol{\psi}_{l+1}^{l+1} \\
&= -\sum_{l=1}^{\Lambda} e^{i\beta_{l-1}} \sqrt{\frac{(2l)(2l-1)}{2l+1}} \frac{c_{l,3}}{\Lambda+1} \boldsymbol{\psi}_l^l \quad \Longrightarrow \\
\langle x_+ \rangle_{\boldsymbol{\omega}^\beta} &= -\sum_{l=1}^{\Lambda} e^{i(\beta_{l-1}-\beta_l)} \frac{c_{l,3}\sqrt{(2l)(2l-1)}}{(\Lambda+1)^2},
\end{aligned}$$

so

$$\langle \boldsymbol{x} \rangle^2_{\boldsymbol{\omega}^\beta} = |\langle x_+ \rangle_{\boldsymbol{\omega}^\beta}|^2 = \left| \sum_{l=1}^{\Lambda} \frac{c_{l,3} 2l \sqrt{1-\frac{1}{2l}}}{(\Lambda+1)^2} e^{i(\beta_{l-1}-\beta_l)} \right|^2 .$$

Since all $\frac{c_{l,3}\sqrt{(2l)(2l-1)}}{(\Lambda+1)^2} > 0$, to maximize $|\langle x_+ \rangle_{\boldsymbol{\omega}^\beta}|$, and thus minimize $(\Delta\boldsymbol{x})^2_{\boldsymbol{\omega}^\beta}$, one needs to take all the β_l equal (mod. 2π), in particular $\beta_l = 0$.

In this case, from $\sqrt{1-\frac{1}{2l}} \geq 1 - \frac{1}{2l}$ $\forall l \in \mathbb{N}$ and $c_{l,3} \geq 1$, it follows (here and below $\boldsymbol{\omega} \equiv \boldsymbol{\omega}^0$)

$$\langle \boldsymbol{x} \rangle^2_{\boldsymbol{\omega}} \geq \left[\frac{2}{(\Lambda+1)^2} \sum_{l=1}^{\Lambda} l\left(1 - \frac{1}{2l}\right)\right]^2 \overset{(8.46)_2}{=} \left\{\frac{2}{(\Lambda+1)^2}\left[\frac{\Lambda^2}{2}\right]\right\}^2 = \frac{\Lambda^4}{(\Lambda+1)^4}.$$

Finally,

$$\begin{aligned}
(\Delta\boldsymbol{x})^2_{\boldsymbol{\omega}} &= \left\langle \boldsymbol{x}^2 \right\rangle_{\boldsymbol{\omega}} - \langle \boldsymbol{x} \rangle^2_{\boldsymbol{\omega}} \leq \frac{\Lambda}{\Lambda{+}1} + \frac{1}{2(\Lambda+1)^2} - \frac{\Lambda^4}{(\Lambda+1)^4} = \frac{2\Lambda(\Lambda+1)^3 + (\Lambda+1)^2 - 2\Lambda^4}{2(\Lambda+1)^4} \\
&= \frac{6\Lambda^3 + 7\Lambda^2 + 4\Lambda + 1}{2(\Lambda+1)^4} < \frac{3\Lambda^3 + 9\Lambda^2 + 9\Lambda + 3}{(\Lambda+1)^4} = \frac{3}{\Lambda+1}.
\end{aligned}$$

Proof of (5.29). $L_0\boldsymbol{\phi}^\beta = 0$, while $L_\pm\boldsymbol{\phi}^\beta$ are combinations of $\boldsymbol{\psi}_l^{\pm 1}$, therefore are orthogonal to $\boldsymbol{\phi}^\beta$; similarly, $x_\pm\boldsymbol{\phi}^\beta$ are combinations of $\boldsymbol{\psi}_{l-1}^{\pm 1}, \boldsymbol{\psi}_{l+1}^{\pm 1}$, therefore are orthogonal to $\boldsymbol{\phi}^\beta$. Hence

$$\langle L_a\rangle_{\boldsymbol{\phi}^\beta} = 0, \quad \langle x_\pm\rangle_{\boldsymbol{\phi}^\beta} = 0 \qquad \Rightarrow \qquad \langle \boldsymbol{L}\rangle_{\boldsymbol{\phi}^\beta} = 0, \quad |\langle \boldsymbol{x}\rangle_{\boldsymbol{\phi}^\beta}| = |\langle x_0\rangle_{\boldsymbol{\phi}^\beta}|.$$

Replacing these results in $(\Delta\boldsymbol{L})^2 = \langle \boldsymbol{L}^2\rangle_{\boldsymbol{\phi}^\beta} - \langle \boldsymbol{L}\rangle_{\boldsymbol{\phi}^\beta}^2$ and using (5.28), one finds, as claimed

$$(\Delta\boldsymbol{L})^2 = \langle \boldsymbol{L}^2\rangle_{\boldsymbol{\phi}^\beta} = \langle \boldsymbol{\phi}^\beta, \boldsymbol{L}^2\boldsymbol{\phi}^\beta\rangle = \sum_{l=1}^{\Lambda} \frac{l(l+1)(2l+1)}{(\Lambda+1)^2} \overset{(8.45)}{=} \frac{\Lambda(\Lambda+2)}{2}.$$

On the other hand,

$$\begin{aligned}
x_0\boldsymbol{\phi}^\beta &\overset{(3.7)}{=} \sum_{l=0}^{\Lambda} e^{i\beta_l}\frac{\sqrt{2l+1}}{\Lambda+1}\big(c_{l,3}G(l,0,2)\boldsymbol{\psi}_{l-1}^0 + c_{l+1,3}F(l,0,2)\boldsymbol{\psi}_{l+1}^0\big)\\
&\overset{(7.32)}{=} \sum_{l=0}^{\Lambda} \frac{e^{i\beta_l}}{\Lambda+1}\left(c_{l,3}\frac{l}{\sqrt{2l-1}}\boldsymbol{\psi}_{l-1}^0 + c_{l+1,3}\frac{l+1}{\sqrt{2l+3}}\boldsymbol{\psi}_{l+1}^0\right)\\
&= \sum_{l=1}^{\Lambda} \frac{e^{i\beta_l}c_{l,3}}{\Lambda+1}\frac{l}{\sqrt{2l-1}}\boldsymbol{\psi}_{l-1}^0 + \sum_{l=0}^{\Lambda-1} \frac{e^{i\beta_l}c_{l+1,3}}{\Lambda+1}\frac{l+1}{\sqrt{2l+3}}\boldsymbol{\psi}_{l+1}^0\\
&= \sum_{l=0}^{\Lambda-1} \frac{e^{i\beta_{l+1}}c_{l+1,3}}{\Lambda+1}\frac{l+1}{\sqrt{2l+1}}\boldsymbol{\psi}_{l}^0 + \sum_{l=1}^{\Lambda} \frac{e^{i\beta_{l-1}}c_{l,3}}{\Lambda+1}\frac{l}{\sqrt{2l+1}}\boldsymbol{\psi}_{l}^0, \qquad \Longrightarrow\\
\langle x_0\rangle_{\boldsymbol{\phi}^\beta} &= \sum_{l=0}^{\Lambda-1} e^{i(\beta_{l+1}-\beta_l)}c_{l+1,3}\frac{l+1}{(\Lambda+1)^2} + \sum_{l=1}^{\Lambda} e^{i(\beta_{l-1}-\beta_l)}c_{l,3}\frac{l}{(\Lambda+1)^2}\\
&= \sum_{l=1}^{\Lambda} \frac{2lc_{l,3}}{(\Lambda+1)^2}\cos(\beta_{l-1}-\beta_l),
\end{aligned}$$

this means that $\langle x_0\rangle_{\boldsymbol{\phi}^\beta}^2 \equiv \langle \boldsymbol{x}\rangle_{\boldsymbol{\phi}^\beta}^2$ is maximal when $\beta \equiv 0$, and in this case one has (here and later on $\boldsymbol{\phi} \equiv \boldsymbol{\phi}^0$)

$$\langle \boldsymbol{x}\rangle_{\boldsymbol{\phi}}^2 \overset{c_{l,3}\geq 1}{\geq} \left[\sum_{l=1}^{\Lambda} \frac{2l}{(\Lambda+1)^2}\right]^2 \overset{(8.42)}{=} \frac{\Lambda^2}{(\Lambda+1)^2}.$$

One easily checks that $\langle \boldsymbol{x}^2\rangle_{\boldsymbol{\phi}^\beta} = \langle \boldsymbol{x}^2\rangle_{\boldsymbol{\omega}^\beta}$; hence, using (9.46), on $\boldsymbol{\phi}$ it follows, as claimed

$$\begin{aligned}
(\Delta\boldsymbol{x})_{\boldsymbol{\phi}}^2 = \langle \boldsymbol{x}^2\rangle_{\boldsymbol{\phi}} - \langle \boldsymbol{x}\rangle_{\boldsymbol{\phi}}^2 &\leq \frac{\Lambda}{\Lambda+1} + \frac{1}{2(\Lambda+1)^2} - \frac{\Lambda^2}{(\Lambda+1)^2}\\
&= \frac{2\Lambda(\Lambda+1)+1-2\Lambda^2}{2(\Lambda+1)^2} = \frac{2\Lambda+1}{2(\Lambda+1)^2} < \frac{1}{\Lambda+1}.
\end{aligned}$$

Proof of (5.33).

$$
\begin{aligned}
\left\langle \boldsymbol{x}^2 \right\rangle_{\widetilde{\boldsymbol{\chi}}} &\overset{(3.9)_2}{=} \sum_{l=0}^{\Lambda} \left|\widetilde{\chi}^l\right|^2 + \frac{\left[\sum_{l=0}^{\Lambda} l(l+1)\left|\widetilde{\chi}^l\right|^2\right]+1}{k_D(\Lambda)} - \left[1+\frac{(\Lambda+1)^2}{k_D(\Lambda)}\right]\frac{\Lambda+1}{2\Lambda+1}\left|\widetilde{\chi}^\Lambda\right|^2 \\
&\overset{\|\widetilde{\boldsymbol{\chi}}\|_2=1}{=} 1 + \frac{\left[\sum_{l=0}^{\Lambda} l(l+1)\left|\widetilde{\chi}^l\right|^2\right]+1}{k_D(\Lambda)} - \left[1+\frac{(\Lambda+1)^2}{k_D(\Lambda)}\right]\frac{\Lambda+1}{2\Lambda+1}\left|\widetilde{\chi}^\Lambda\right|^2 \\
&\le 1 + \frac{\frac{2}{\Lambda+2}\left[\sum_{l=1}^{\Lambda} l(l+1)\right]+1}{k_D(\Lambda)} \overset{(8.42)}{\le} 1 + \frac{\frac{2}{3}\Lambda(\Lambda+1)+1}{k_D(\Lambda)} \overset{(3.5)}{\le} 1 + \frac{\frac{2}{3}\Lambda(\Lambda+1)+1}{\Lambda^2(\Lambda+1)^2},
\end{aligned}
\tag{9.47}
$$

so, putting together (5.32) and (9.47), one obtains, as claimed,

$$
\begin{aligned}
(\Delta \boldsymbol{x})^2_{\widetilde{\boldsymbol{\chi}}} &= \left\langle \boldsymbol{x}^2 \right\rangle_{\widetilde{\boldsymbol{\chi}}} - \langle \boldsymbol{x}\rangle^2_{\widetilde{\boldsymbol{\chi}}} = \left\langle \boldsymbol{x}^2 \right\rangle_{\widetilde{\boldsymbol{\chi}}} - \langle x_0\rangle^2_{\widetilde{\boldsymbol{\chi}}} < 1 - \cos^2\left(\frac{\pi}{\Lambda+2}\right) + \frac{\frac{2}{3}\Lambda(\Lambda+1)+1}{\Lambda^2(\Lambda+1)^2} \\
&= \sin^2\left(\frac{\pi}{\Lambda+2}\right) + \frac{\frac{2}{3}+\frac{2}{3\Lambda}+\frac{1}{\Lambda^2}}{(\Lambda+1)^2} \overset{\Lambda\ge 3}{<} \frac{\pi^2}{(\Lambda+2)^2} + \frac{1}{(\Lambda+1)^2} < \frac{11}{(\Lambda+1)^2}.
\end{aligned}
$$

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