On the derivability, with respect to initial data, of the solution of a stochastic differential equation with lipschitz coeffcients

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ON THE DERIVABILITY, WITH RESPECT TO THE INITIAL DATA, OF THE SOLUTION OF A STOCHASTIC DIFFERENTIAL EQUATION WITH LIPSCHITZ COEFFICIENTS

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Abstract : We consider a stochastic differential equation, driven by a Brownian motion, with Lipschitz coefficients. We prove that the corresponding flow is, almost surely, almost everywhere derivable with respect to the initial data for any time, and the process defined by the Jacobian matrices is a $GL_n(\mathbb{R})$-valued continuous solution of a linear stochastic differential equation. In dimension one, this process is given by an explicit formula. These results partially extend those which are known when the coefficients are $C^{1,\alpha}$-Hölder continuous. Dirichlet forms are involved in the proofs.

Key words : Stochastic differential equations, Stochastic flows, Brownian motion, Dirichlet forms, Opérateur carré du champ.

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1- Introduction

For $d \geq 1$, we denote by $(\Omega , P , \mathcal{F} , (\mathcal{F}_t)_{t\geq 0} , (B_t)_{t\geq 0})$ the standard Brownian motion in $\mathbb{R}^d$ starting from 0 :

Ω is the Wiener space,

$$\Omega = \{ \omega : \mathbb{R}_+ \longrightarrow \mathbb{R}^d ; \omega \text{ is continuous and } \omega(0)=0 \}$$

equipped with the metric of the uniform convergence on compacts. (Ω is a Frechet space).

$\forall t\geq 0 \quad \forall \omega \in \Omega \quad B_t(\omega) = \omega(t)$.

$\mathcal{F}$ is the Borel σ-field of Ω and P is the Wiener measure on $(\Omega , \mathcal{F})$.

$\mathcal{F}_t$ is the least σ-field containing the P-negligible sets of $\mathcal{F}$ for which all B_s ,$0\leq s\leq t$, are measurable.

We consider two Borel functions

$$\sigma : \mathbb{R}_+\times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$$
$$b : \mathbb{R}_+\times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

where n is a fixed natural number and $\mathbb{R}^{n\times d}$ denotes the space of real (n,d)-matrices.

In the following, we are interested in the stochastic differential equation

$$(*) \qquad \begin{cases} dX_t^x = \sigma(t,X_t^x).dB_t + b(t,X_t^x)\, dt \\ X_0^x = x \end{cases}$$

Throughout the sequel, we will suppose that the following Lipschitz conditions are fulfilled :

There exists a function $C : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that,

$$\forall T \geq 0 \quad \forall t \in [0,T] \quad \forall x,y \in \mathbb{R}^n$$
$$|\sigma(t,x)| \vee |b(t,x)| \leq C(T)\,(1 + |x|)$$
$$|\sigma(t,x) - \sigma(t,y)| \vee |b(t,x) - b(t,y)| \leq C(T)\,|x - y|$$

where $|\ |$ stands for a norm in the suitable space.

The following fundamental result is wellknown :

<u>**Theorem**</u> ([6]) : *There exists a solution* $(X_t^x)_{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n}$ *to (*) which is a* $(\mathcal{F}_t)$*-adapted and continuous process* ("continuous" meaning as usual : $\forall\omega\in\Omega \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^n \longrightarrow X_t^x(\omega)\in\mathbb{R}^n$ is a continuous function). *This solution is unique up to* P-*indistinguishability and, for* P-*almost all* ω *, for any* $t\geq 0$ *the map*

$$X_t^{\cdot}(\omega) : x \in \mathbb{R}^n \longrightarrow X_t^x(\omega) \in \mathbb{R}^n$$

is an onto homeomorphism.

In the sequel, we will denote by (X_t^x) "this" solution.

For the study of the smoothness with respect to the initial data x , the coefficients σ and b are usually supposed to fulfil smoothness assumptions, namely $C^{1,\alpha}$-Hölder conditions with respect to x.

Under such additional conditions, $X_t^{\cdot}(\omega)$ is shown to be an onto C^1-diffeomorphism (for P-almost all ω , for all $t\geq 0$) and, for P-almost all ω ,

$$(t,x) \in \mathbb{R}_+\times\mathbb{R}^n \longrightarrow \frac{\partial}{\partial x}\Big(X_t^x(\omega)\Big) \in GL_n(\mathbb{R})$$

is continuous (c.f.[6]).

This paper is devoted to the following problem :

What remains of the above conclusions if we suppose only the forementioned Lipschitz conditions ?

We obtain actually (see Theorem 1 in §3) a weakened form of these conclusions preserving some properties of continuity, with respect to t, and of invertibility of the Jacobian matrix $\frac{\partial}{\partial x}\Big(X_t^x(\omega)\Big)$ (the derivative being understood in the weak sense).

These results were announced in [3], part I.

As a corollary of our main result, we obtain that, under Lipschitz conditions, the solution of

$$X_t = X_0 + \int_0^t \sigma(s,X_s)\, dB_s + \int_0^t b(s,X_s)\, ds$$

with a random initial data X_0 admitting a density and independent of $(B_t)_{t\geq 0}$, is such that X_t admits a density for each t.

This result simplifies hypotheses which are often used. For example, the results of [7] about the reversibility of diffusion processes are valid without $C^{1,\alpha}$ assumptions on the coefficients (c.f. [7] Part 4).

Thanks to our result of invertibility of the Jacobian matrix of the flow, we can also prove (Theorem 2 in §4) that the process thus defined is a solution to a linear stochastic differential equation on the product space $\mathbb{R}^n \times \Omega$, and, in dimension one, we obtain an explicit formula.

Dirichlet spaces play an important role in this work. They are introduced in the following paragraph.

2- Preliminaries and notations

a) Let h be a fixed continuous positive function on $\mathbb{R}^n$ such that

$$\int h(x)\, dx = 1 \quad \text{and} \quad \int |x|^2\, h(x)\, dx < +\infty .$$

Obviously, $L^2(h\, dx) \subset L^1_{loc}(\mathbb{R}^n)$.

We define the space $\mathbf{d}$ by

$$\mathbf{d} = \{\, f \in L^2(h\, dx) \;;\; \forall 1\leq j\leq n \quad \frac{\partial}{\partial x_j} f \in L^2(h\, dx) \,\}$$

where $\frac{\partial}{\partial x_j}$ denotes the derivative in the sense of distributions.

$\mathbf{d}$ is a subspace of the Sobolev space $H^1_{loc}(\mathbb{R}^n)$.

$\mathbf{d}$ equipped with the norm

$$\|f\|_{\mathbf{d}} = \left[\int f^2\, h\, dx + \sum_{j=1}^{n} \int \left(\frac{\partial f}{\partial x_j}\right)^2 h\, dx\right]^{1/2}$$

is a Hilbert space which is a classic Dirichlet space ([4],[5]).

b) Let $\tilde{\Omega}$ be the product space

$$\tilde{\Omega} = \mathbb{R}^n \times \Omega$$

which is also a Frechet space.
We endow $\tilde{\Omega}$ with the measure $(\tilde{\mathcal{F}}, \tilde{P})$ where $\tilde{\mathcal{F}}$ denotes the Borel σ-field of $\tilde{\Omega}$ and $\tilde{P}$ denotes the product measure $h\,dx \times P$.
Let, for $t \geq 0$

$$\tilde{B}_t(x,\omega) = B_t(\omega)$$

and $\tilde{\mathcal{F}}_t$ be the least σ-field containing the $\tilde{P}$-negligible sets of $\tilde{\mathcal{F}}$ for which all $\tilde{B}_s$,$0 \leq s \leq t$, are measurable.
$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, (\tilde{\mathcal{F}}_t), (\tilde{B}_t))$ is a Brownian motion in $\mathbb{R}^d$ starting form 0 .

c) Let $e_1,\ldots,e_n$ be the canonical basis in $\mathbb{R}^n$.
We define, for each $1 \leq i \leq n$, D_i by

$D_i = \{u : \tilde{\Omega} \longrightarrow \mathbb{R}\ ;\ \exists \tilde{u} : \tilde{\Omega} \longrightarrow \mathbb{R}$ Borel function such that $u = \tilde{u}$ $\tilde{P}$-a.e. and $\forall (x,\omega) \in \tilde{\Omega}$ $t \in \mathbb{R} \longrightarrow \tilde{u}(x+te_i,\omega)$ is locally absolutely continuous$\}$.

D_i is considered a set of classes (with respect to the $\tilde{P}$-a.e. equality).
If u is in D_i and if $\tilde{u}$ is associated with it according to the above definition, we can write

$$\nabla_i u(x,\omega) = \lim_{t \to 0} \frac{\tilde{u}(x+te_i,\omega) - \tilde{u}(x,\omega)}{t} .$$

It was proved in [2], in a more general context, that $\nabla_i u$ is well defined $\tilde{P}$-a.e. and this definition does not depend on the representative $\tilde{u}$, up to $\tilde{P}$-a.e. equality.
We denote by $\tilde{d}$ the space

$$\tilde{d} = \{ u \in L^2(\tilde{P}) \cap \left(\bigcap_{i=1}^{n} D_i \right) ;\ \forall 1 \leq i \leq n \quad \nabla_i u \in L^2(\tilde{P}) \} .$$

It was also proved in [2] that $\tilde{d}$ equipped with the norm

$$\|u\|_{\tilde{d}} = \left[\int u^2 \, d\tilde{P} + \sum_{i=1}^{n} \int (\nabla_i u)^2 \, d\tilde{P} \right]^{1/2}$$

is a Hilbert space which is a general Dirichlet space ([1],[2]).

We define the "carré du champ" by

$$\forall u,v \in \tilde{\mathbf{d}} \qquad \gamma(u,v) = \sum_{i=1}^{n} (\nabla_i u)\,(\nabla_i v) \quad .$$

The following proposition connects the space $\tilde{\mathbf{d}}$ with the space $\mathbf{d}$:

Proposition 1 : *If* $u \in \tilde{\mathbf{d}}$, *for P-almost all* ω *in* Ω ,

$$u(\cdot,\omega) \in \mathbf{d} \quad \textit{and} \quad \forall 1 \leqslant i \leqslant n \quad \frac{\partial}{\partial x_i} u(x,\omega) = \nabla_i u(x,\omega) \quad dx\text{-a.e.}$$

Proof : By definition, if $u \in \tilde{\mathbf{d}}$, there exist a $\tilde{P}$-negligible set F and Borel functions $\tilde{u}_1,\ldots,\tilde{u}_n$ such that

$$\forall 1 \leqslant i \leqslant n \qquad \tilde{u}_i = u \quad \text{on} \quad \complement F \quad ,$$

and, $\forall (x,\omega)$ $\quad t \longrightarrow \tilde{u}_i(x+te_i,\omega)$ is locally absolutely continuous.

We can also assume

$$\forall 1 \leqslant i \leqslant n \quad \forall (x,\omega) \quad \nabla_i u(x,\omega) = \lim_{t \to 0} \frac{\tilde{u}_i(x+te_i,\omega) - \tilde{u}_i(x,\omega)}{t} \quad .$$

Therefore, there exists a P-negligible set N such that, for any $\omega \notin N$,
there exists a dx-negligible set $\mathfrak{X}_\omega$ with

$$\forall 1 \leqslant i \leqslant n \quad \forall x \notin \mathfrak{X}_\omega \quad \tilde{u}_i(x,\omega) = u(x,\omega) \quad ,$$

and

$$u(\cdot,\omega) \in L^2(h\,dx)$$
$$\forall 1 \leqslant i \leqslant n \quad \nabla_i u(\cdot,\omega) \in L^2(h\,dx) \quad .$$

It is then easy to see that, for any $\omega \notin N$, for $1 \leqslant i \leqslant n$,we have

$$\frac{\partial}{\partial x_i} u(\cdot,\omega) = \nabla_i u(\cdot,\omega)$$

in the sense of distributions.

■

If U and V belong to $\tilde{\mathbf{d}}^n$, we will denote by $\gamma(U,V)$ the matrix $(\gamma(U_i,V_j))_{1 \leqslant i,j \leqslant n}$.

d) We denote by $(\tilde{X}_t)_{t\geq 0}$ the "unique" $(\tilde{\mathcal{F}}_t)$-adapted continuous solution to

$$(\tilde{*}) \qquad \begin{cases} d\tilde{X}_t = \sigma(t,\tilde{X}_t).d\tilde{B}_t + b(t,\tilde{X}_t)\, dt \\ \tilde{X}_0 = x \end{cases}$$

Proposition 2 : *There exists a* $\tilde{P}$*-negligible set* $\tilde{N}$ *such that*

$$\forall (x,\omega) \notin \tilde{N} \qquad \forall t\geq 0 \qquad \tilde{X}_t(x,\omega) = X_t^x(\omega) \quad .$$

Proof : It is easy to see that

$$\forall t \geq 0 \qquad (x,\omega) \longrightarrow X_t^x(\omega) \in L^2(\tilde{P})$$

$$\text{and} \qquad \forall T \geq 0 \qquad \left\| \sup_{0\leq t\leq T} |X_t^x(\omega)| \right\|_{L^2(\tilde{P})} < +\infty \quad .$$

Therefore, the stochastic integral

$$\int_0^t \sigma(s,X_s^x(\omega)).d\tilde{B}_s(x,\omega)$$

is well defined $\tilde{P}$-a.e. , and $\tilde{P}$-a.e. equal to

$$\int_0^t \sigma(s,X_s^x(\omega)).dB_s(\omega) \quad .$$

Then, the result follows from the uniqueness of the solution of $(\tilde{*})$.

■

Remark : Equations $(*)$ and $(\tilde{*})$ are almost identical, but the fact that the uniqueness is slightly weaker for $(\tilde{*})$ than for $(*)$, because the $\tilde{P}$-evanescent sets affect ω and x , will allow us to perform operations on $\tilde{X}$ which are not defined for X .

e) Let h be the set of all measurable $(\tilde{\mathcal{F}}_t)$-adapted processes, (α_t) ,such that the map

$$t \longrightarrow \alpha_t$$

belongs to $\mathcal{L}^2_{loc}(dt;\tilde{d})$.

Proposition 3 : *Let* $(\alpha_t) \in h^{n\times d}$, $(\beta_t) \in h^n$, $u \in d^n$, *and*

$$Z_t = u + \int_0^t \alpha_s . d\tilde{B}_s + \int_0^t \beta_s \, ds \quad .$$

Then, for any $t \geq 0$,

$$Z_t \in \tilde{\mathfrak{d}}^n \quad ,$$

$$\left(\| Z_t \|_{\tilde{\mathfrak{d}}^n}\right)^2 \leq 3\left(\left(\| u \|_{\mathfrak{d}^n}\right)^2 + \int_0^t \left(\| \alpha_s \|_{\tilde{\mathfrak{d}}^n \times d}\right)^2 ds + t\int_0^t \left(\| \beta_s \|_{\tilde{\mathfrak{d}}^n}\right)^2 ds\right) \quad ,$$

$$\forall j,k,\ell \qquad \int_0^t [\gamma(Z_j(s),\alpha_{k,\ell}(s))]^2 \, ds < +\infty \qquad \tilde{P}\text{-a.e.}$$

and

$$\gamma(Z_t, Z_t) = \left(\frac{\partial u}{\partial x}\right) \left(\frac{\partial u}{\partial x}\right)^* + \sum_{\ell=1}^{d} \int_0^t [\gamma(Z_s, \alpha_{.,\ell}(s)) + \gamma(\alpha_{.,\ell}(s), Z_s)] \, d\tilde{B}_s^\ell$$

$$+ \int_0^t \left[\gamma(Z_s, \beta_s) + \gamma(\beta_s, Z_s) + \sum_{\ell=1}^{d} \gamma(\alpha_{.,\ell}(s), \alpha_{.,\ell}(s))\right] ds$$

(where $*$ denotes the transposition).

Proof :

- We notice first that the subset of $\mathbf{h}$ consisting of those (α_t)'s such that

$$\forall t \geq 0 \qquad \alpha_t = \sum_{k=0}^{\infty} \alpha_k \, 1_{[k/r,(k+1)/r[}(t)$$

with $r \in \mathbb{N}^*$, $\forall k$ $\alpha_k \in \tilde{\mathfrak{d}}$ and α_k $\tilde{\mathcal{F}}_{k/r}$-measurable, is dense in the space $\mathbf{h}$.

- If

$$\alpha_t = \sum_{k=0}^{\infty} \alpha_k \, 1_{[k/r,(k+1)/r[}(t)$$

with $\forall k$ $\alpha_k \in \tilde{\mathfrak{d}}^{n \times d}$ and α_k $\tilde{\mathcal{F}}_{k/r}$-measurable, then

$$Z_t := \int_0^t \alpha_s . d\tilde{B}_s = \sum_{k=0}^{\infty} \alpha_k . \left(\tilde{B}_{(k+1)/r \wedge t} - \tilde{B}_{k/r \wedge t}\right)$$

belongs to $\tilde{\mathfrak{d}}^n$ and

$$\left(\|Z_t\|_{[L^2(\tilde{P})]^n}\right)^2 = \int_0^t \left(\|\alpha_s\|_{[L^2(\tilde{P})]^{n\times d}}\right)^2 ds \ , \ \nabla_i Z_t = \int_0^t (\nabla_i \alpha_s).d\tilde{B}_s$$

and, therefore

$$\left(\|Z_t\|_{\tilde{d}^n}\right)^2 = \int_0^t \left(\|\alpha_s\|_{\tilde{d}^n\times d}\right)^2 ds \quad .$$

Likewise, if

$$\beta_t = \sum_{k=0}^{\infty} \beta_k \ 1_{[k/r,(k+1)/r[}(t)$$

with $\forall k \ \ \beta_k \in \tilde{d}^n$ and β_k $\tilde{\mathcal{F}}_{k/r}$-measurable, then

$$Z_t := \int_0^t \beta_s \ ds$$

belongs to $\tilde{d}^n$, $\nabla_i Z_t = \int_0^t (\nabla_i \beta_s) \ ds$ and

$$\left(\|Z_t\|_{\tilde{d}^n}\right)^2 \leq t \int_0^t \left(\|\beta_s\|_{\tilde{d}^n}\right)^2 ds \quad .$$

- Let now (α_t) and (β_t) be like above, $u \in d^n$ and

$$Z_t = u + \int_0^t \alpha_s.d\tilde{B}_s + \int_0^t \beta_s \ ds \quad ,$$

then

$$\nabla_i(Z_j(t)) = \frac{\partial u_j}{\partial x_i} + \int_0^t \nabla_i(\alpha_{j,.}(s)).d\tilde{B}_s + \int_0^t \nabla_i(\beta_j(s)) \ ds$$

and therefore

$$\nabla_i(Z_j(t)) \ \nabla_i(Z_k(t)) = \frac{\partial u_j}{\partial x_i}\frac{\partial u_k}{\partial x_i}$$

$$+ \int_0^t [\nabla_i(Z_j(s)) \ \nabla_i(\alpha_{k,.}(s)) + \nabla_i(Z_k(s)) \ \nabla_i(\alpha_{j,.}(s))].d\tilde{B}_s$$

$$+ \int_0^t [\nabla_i(Z_j(s)) \ \nabla_i(\beta_k(s)) + \nabla_i(Z_k(s)) \ \nabla_i(\beta_j(s))] \ ds$$

$$+ \int_0^t \left[\sum_{\ell=1}^{d} \nabla_i(\alpha_{j,\ell}(s)) \ \nabla_i(\alpha_{k,\ell}(s))\right] ds \quad ,$$

which proves the proposition in this case.

• In the general case, we approximate α and β by means of processes of the previous type, and we use the same arguments as in the proof of Theorem 5.2. in [1].

■

Proposition 4 : *Let* $t \in \mathbb{R}_+ \longrightarrow Y_t \in [L^2(\tilde{P})]^n$ *be a continuous function such that*

$$\forall t>0 \quad Y_t \in \tilde{d}^n \quad \textit{and} \quad \forall T>0 \quad \sup_{t\in[0,T]} \|Y_t\|_{\tilde{d}^n} < +\infty \quad .$$

Let us consider a Borel function $s : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *such that there exists* $C : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *satisfying*

$\forall T>0 \quad \forall t \in [0,T] \quad \forall x,y \in \mathbb{R}^n$
$|s(t,x)| \leq C(T)\ (1 + |x|)$ *and* $|s(t,x) - s(t,y)| \leq C(T)\ |x - y|$

Then $t \longrightarrow s(t,Y_t)$ *belongs to* $\mathcal{L}^2_{loc}(dt;\tilde{d})$.

Proof : It is easy to see that

$$\forall t\geq 0 \quad s(t,Y_t) \in \tilde{d} \quad ,$$

and $\left(\|s(t,Y_t)\|_{\tilde{d}}\right)^2 \leq 2\ C(T)^2 \left[1 + \left(\|Y_t\|_{\tilde{d}}\right)^2\right]$ for $t \in [0,T]$.

It is therefore enough to prove the weak measurability of

$$t \in \mathbb{R}_+ \longrightarrow s(t,Y_t) \in \tilde{d} \quad .$$

If s is continuous, $t \longrightarrow s(t,Y_t)$ is weakly continuous.

Now, there exists a sequence (s_p) such that

$\forall p$ s_p is continuous and fulfils the same assumptions as s , and, for almost every t , $\forall x$ $s_p(t,x) \longrightarrow s(t,x)$.

Then, for almost every t ,

$$s_p(t,Y_t) \longrightarrow s(t,Y_t) \quad \text{weakly in } \tilde{d}$$

and therefore the weak measurability is proved.

■

3- Main result

<u>**Theorem 1**</u> : *For P-almost every* ω ,

$$\forall t \geq 0 \quad X_t^{\cdot}(\omega) \in \mathbf{d}^n \subset \left[H^1_{loc}(\mathbb{R}^n)\right]^n$$

and there exists a $(\tilde{\mathcal{F}}_t)$*-adapted* $GL_n(\mathbb{R})$*-valued continuous process* $(M_t)_{t \geq 0}$ *such that, for P-almost every* ω,

$$\forall t \geq 0 \quad \left[\frac{\partial}{\partial x}\left(X_t^x(\omega)\right) = M_t(x,\omega) \quad dx\text{-a.e.} \right]$$

(where $\frac{\partial}{\partial x}$ denotes the derivative in the sense of distributions).

Remarks :

1) For P-almost all ω , for any $0<\beta<1$, $X_t^{\cdot}(\omega)$ is β-Hölder continuous locally uniformly with respect to t (c.f. [6]), but, of course, this result does not imply the first part of the theorem.

2) In the deterministic case ($\sigma = 0$) , the first part of the theorem is obvious since, for any t , $X_t^{\cdot}$ is actually Lipschitz continuous, but, even in this case, the second part does not seem as obvious.

3) It is clear that the process (M_t) is unique up to $\tilde{P}$-indistinguishability.

Proof :

<u>Lemma 1</u> : $\forall t \geq 0 \quad \tilde{X}_t \in \tilde{\mathbf{d}}^n$,

$t \in \mathbb{R}_+ \longrightarrow \tilde{X}_t \in \tilde{\mathbf{d}}^n$ *is continuous and, for* $t \geq 0$,

$$\begin{aligned}\gamma(\tilde{X}_t,\tilde{X}_t) = I &+ \sum_{\ell=1}^{d} \int_0^t \left[\gamma\left(\tilde{X}_s,\sigma_{\cdot\cdot\ell}(s,\tilde{X}_s)\right) + \gamma\left(\sigma_{\cdot\cdot\ell}(s,\tilde{X}_s),\tilde{X}_s\right)\right] d\tilde{B}_s^\ell \\ &+ \int_0^t \left[\gamma\left(\tilde{X}_s,b(s,\tilde{X}_s)\right) + \gamma\left(b(s,\tilde{X}_s),\tilde{X}_s\right)\right] ds \\ &+ \int_0^t \sum_{\ell=1}^{d} \gamma\left(\sigma_{\cdot\cdot\ell}(s,\tilde{X}_s),\sigma_{\cdot\cdot\ell}(s,\tilde{X}_s)\right) ds\end{aligned}$$

(where I denotes the (n,n)-identity matrix).

Proof : We define

$$\tilde{X}^0_t = x$$

and, $\forall r \geq 0$,

$$\tilde{X}^{r+1}_t = x + \int_0^t \sigma(s,\tilde{X}^r_s)\cdot d\tilde{B}_s + \int_0^t b(s,\tilde{X}^r_s)\, ds \quad .$$

According to Propositions 3 and 4,

$\forall r \geq 0 \quad t \longrightarrow \tilde{X}^r_t$ is a continuous map form $\mathbb{R}_+$ into $\tilde{d}^n$.

It follows also from Proposition 3 that

$$\forall T \geq 0 \qquad \sup_{\substack{t\in[0,T] \\ r\geq 0}} \left\| \tilde{X}^r_t \right\|_{\tilde{d}^n} < +\infty \quad .$$

Since $\tilde{X}^r_t \longrightarrow \tilde{X}_t$ in $[L^2(\tilde{P})]^n$ when $r \longrightarrow +\infty$,

$$\tilde{X}_t \in \tilde{d}^n \quad \text{and} \quad \tilde{X}^r_t \longrightarrow \tilde{X}_t \quad \text{weakly in} \quad \tilde{d}^n \quad .$$

Using again Propositions 3 and 4, since

$$\tilde{X}_t = x + \int_0^t \sigma(s,\tilde{X}_s).d\tilde{B}_s + \int_0^t b(s,\tilde{X}_s)\, ds \quad ,$$

we get the properties stated in the lemma.

■

<u>Lemma 2</u> : *There exist predictable* $\mathbb{R}^{n\times n}$*-valued processes,* V *and* $(U_\ell)_{1\leq \ell\leq d}$, *bounded on each interval* $[0,T]$, *such that*

$$\gamma\left(\tilde{X}_t,\tilde{X}_t\right) = I + \sum_{\ell=1}^{d} \int_0^t \{U_\ell(s),\gamma\left(\tilde{X}_s,\tilde{X}_s\right)\}\, d\tilde{B}^\ell_s$$

$$+ \int_0^t \left[\{V_s,\gamma\left(\tilde{X}_s,\tilde{X}_s\right)\} + \sum_{\ell=1}^{d} U_\ell(s)\ \gamma\left(\tilde{X}_s,\tilde{X}_s\right)\ U^*_\ell(s) \right] ds$$

(where $\{A,B\}$ is set for $A\,B^* + B\,A^*$).

Proof : This lemma proceeds from the previous one by using the following

<u>Factorisation lemma</u> : *Let* $u \in h^n$ *and* F *be one of the functions* b *or* $\sigma_{\cdot\cdot\ell}$ $(1\leq \ell\leq n)$. *There exists a predictable*

$\mathbb{R}^{n\times n}$-valued locally bounded process, (U_t) *, such that, for almost all* t *,*

$$\gamma(F(t,u(t)),u(t)) = U_t\ \gamma(u(t),u(t)) \qquad \tilde{P}\text{-a.e.}$$

$$\gamma(F(t,u(t)),F(t,u(t))) = U_t\ \gamma(u(t),u(t))\ U_t^* \qquad \tilde{P}\text{-a.e.}$$

This factorisation lemma can be proved quite in the same way as the similar "Lemme 14 de factorisation" in [2], by approximating F by C^1 functions.

■

<u>Lemma 3</u> : *For any* $T>0$,

$$\left\| \sup_{t\in[0,T]} |\tilde{X}_t| \right\|_{L^2(\tilde{P})} < +\infty$$

$$\left\| \sup_{t\in[0,T]} |\gamma(\tilde{X}_t,\tilde{X}_t)| \right\|_{L^1(\tilde{P})} < +\infty$$

$\forall 1\le i,j\le n \quad \exists A_{i,j} \quad \forall 0\le s\le t\le T$

$$\int \left(\nabla_i\left(\tilde{X}_j(t)\right) - \nabla_i\left(\tilde{X}_j(s)\right)\right)^4 d\tilde{P} \le A_{i,j}\,|t-s|^2 \quad .$$

Proof : The first property is clear (and wellknown).
Likewise, according to Lemma 2,

$$\left\| \sup_{t\in[0,T]} |\gamma(\tilde{X}_t,\tilde{X}_t)| \right\|_{L^2(\tilde{P})} < +\infty \quad ,$$

wich implies the second property.
According to Proposition 3, if $0\le s\le t\le T$,

$$\gamma(\tilde{X}_t-\tilde{X}_s,\tilde{X}_t-\tilde{X}_s) =$$

$$\sum_{\ell=1}^{d} \int_s^t \left[\gamma\left(\tilde{X}_u-\tilde{X}_s,\sigma_{\cdot\cdot\ell}(u,\tilde{X}_u)\right) + \gamma\left(\sigma_{\cdot\cdot\ell}(u,\tilde{X}_u),\tilde{X}_u-\tilde{X}_s\right)\right] d\tilde{B}_u^\ell$$

$$+ \int_s^t \left[\gamma\left(\tilde{X}_u-\tilde{X}_s,b(u,\tilde{X}_u)\right) + \gamma\left(b(u,\tilde{X}_u),\tilde{X}_u-\tilde{X}_s\right)\right] du$$

$$+ \int_s^t \sum_{\ell=1}^{d} \gamma\left(\sigma_{\cdot\cdot\ell}(u,\tilde{X}_u),\sigma_{\cdot\cdot\ell}(u,\tilde{X}_u)\right) du \quad .$$

Denoting, for $s \le u \le t$, $\left\| \gamma\left(\tilde{X}_u - \tilde{X}_s, \tilde{X}_u - \tilde{X}_s\right) \right\|_{L^2(\tilde{P})}$ by $\varphi(u)$,we have

$$\varphi(t) \le C \left[\sqrt{\int_s^t \varphi(u)\, du} + \int_s^t \sqrt{\varphi(u)}\, du + (t-s) \right] .$$

If $A(t) := \sup_{s \le u \le t} \varphi(u)$,

$$A(t) \le C \left(\sqrt{t-s}\,\sqrt{A(t)} + (t-s)\sqrt{A(t)} + (t-s)\right) ,$$

$$A(t) - C' \sqrt{t-s}\,\sqrt{A(t)} - C\,(t-s) \le 0$$

and therefore

$$\sqrt{A(t)} \le \frac{\sqrt{t-s}}{2} \left(C' + \sqrt{C'^2 + 4\,C}\right) .$$

We get then

$$\forall 0 \le s \le t \le T \qquad \left\| \gamma\left(\tilde{X}_t - \tilde{X}_s, \tilde{X}_t - \tilde{X}_s\right) \right\|_{L^2(\tilde{P})} \le C_T\ (t-s) ,$$

wich implies the third property.

■

Lemma 3 yields, thanks to Kolmogorov's criterion, the existence of a $(\tilde{\mathcal{F}}_t)$-adapted $\mathbb{R}^{n\times n}$-valued and continuous process, $(M_t)_{t\ge 0}$, such that, for any $t \ge 0$ and for any $1 \le i, j \le n$,

$$(M_t)_{j,i} = \nabla_i \left(\tilde{X}_j(t)\right) \quad \tilde{P}\text{-a.e.}$$

By definition

$$\forall t \ge 0 \quad M_t M_t^* = \gamma(\tilde{X}_t, \tilde{X}_t) \quad \tilde{P}\text{-a.e.}$$

However, if one considers the stochastic differential equation in $\mathbb{R}^{n\times n}$

$$\begin{cases} dN_t = \sum_{\ell=1}^{d} U_\ell(t)\, N_t\, d\tilde{B}_t^\ell + V_t\, N_t\, dt \\ N_0 = I \end{cases}$$

it is known (see e.g. [8] Lemma 5.3) that the continuous solution (N_t) is $GL_n(\mathbb{R})$-valued, and $(N_t N_t^*)$ satisfies to the same equation as $(\gamma(\tilde{X}_t, \tilde{X}_t))$ in Lemma 2.

By uniqueness, for $\tilde{P}$-almost all (x,ω),

$$\forall t \qquad M_t M_t^*(x,\omega) = N_t N_t^*(x,\omega)$$

and therefore, it can be assumed

$$\forall (x,\omega) \qquad \forall t \qquad M_t \in GL_n(\mathbb{R}) \quad .$$

By Propositions 1 and 2 and Lemma 1, there exists a P-negligible subset N of Ω such that, for all $\omega \notin N$,

$$\exists \mathfrak{X}_\omega \text{ , dx-negligible, } \forall t \geq 0 \quad \forall x \notin \mathfrak{X}_\omega \qquad \tilde{X}_t(x,\omega) = X_t^x(\omega)$$

and

$$\forall t \in \mathbb{Q}_+ \qquad \tilde{X}_t(\cdot,\omega) \in d^n \quad \text{and} \quad \frac{\partial}{\partial x}\tilde{X}_t(x,\omega) = M_t(x,\omega) \quad \text{dx-a.e.}$$

If $\omega \notin N$, $t_p \in \mathbb{Q}_+$, $t \geq 0$ and $t_p \longrightarrow t$, we have

$$\forall x \qquad X_{t_p}^x(\omega) \longrightarrow X_t^x(\omega)$$

$$\text{and} \qquad \frac{\partial}{\partial x}\left(X_{t_p}^x(\omega)\right) = M_{t_p}(x,\omega) \longrightarrow M_t(x,\omega) \qquad \text{dx-a.e.}$$

By Lemma 3, we can assume

$$\forall \omega \notin N \qquad \forall T > 0 \qquad \sup_{t\in[0,T]} \left| X_t^\cdot(\omega) \right| \in L^2(h\,dx) \quad ,$$

$$\sup_{t\in[0,T]} \left| \gamma(\tilde{X}_t, \tilde{X}_t) \right| (\cdot,\omega) \in L^1(h\,dx) \quad .$$

In particular, if $\omega \notin N$,

$$\sup_p \left| X_{t_p}^\cdot(\omega) \right| \quad \text{and} \quad \sup_p \left| \frac{\partial}{\partial x}\left(X_{t_p}^x(\omega)\right) \right| \in L^2(h\,dx) \quad .$$

Therefore, if $\omega \notin N$,

$$X_t^\cdot(\omega) \in d^n \qquad \text{and} \qquad \frac{\partial}{\partial x}\left(X_t^x(\omega)\right) = M_t(x,\omega) \qquad \text{dx-a.e}$$

The proof of Theorem 1 is complete.

■

<u>Corollary</u> : *Let* X_0 *be an* $\mathbb{R}^n$*-valued random variable admitting a density and independent of the Brownian motion* $(B_t)_{t\geq 0}$. *The solution of*

$$X_t = X_0 + \int_0^t \sigma(s,X_s).dB_s + \int_0^t b(s,X_s)\,ds$$

is such that X_t *possesses a density for each* t .

(Obviously, in the above statement as in the following proof, the basic probability space (Ω,P) is supposed to be replaced by a product $(\Omega_0\times\Omega, P_0\times P)$ and X_0 depends only on the first variable. For the sake of simplicity, we keep the notation (Ω,P) in place of $(\Omega_0\times\Omega, P_0\times P)$).

Proof : Let p be the density of X_0 . For a Borelian subset B of $\mathbb{R}^n$ we can write

$$P\{X_t\in B\} = \int_{\mathbb{R}^n} P\{X_t^x\in B\}\; p(x)\; dx = \int_\Omega\left[\int_{\mathbb{R}^n} 1_{\{x;X_t^x(\omega)\in B\}} p(x)\; dx\right] dP \quad .$$

By Theorem 1, for P-almost all ω , the map $x \longrightarrow X_t^x(\omega)$ belongs to $\left[H^1_{loc}(\mathbb{R}^n)\right]^n$ and admits an invertible Jacobian matrix dx-almost everywhere, and therefore (cf [2] part I) the image of the Lebesgue measure by this map is absolutely continuous. Consequently, if B is negligible we have $P\{X_t\in B\} = 0$.

■

4- Description of the process $(M_t)_{t\geq 0}$ as a solution of a S.D.E.

We denote by σ' (resp. b') any Borel function such that

$$\forall t\geq 0 \qquad \left(\sigma'(t,x) = \frac{\partial}{\partial x}\sigma(t,x) \quad dx\text{-a.e.}\right)$$

$$\left[\text{resp.}\quad b'(t,x) = \frac{\partial}{\partial x} b(t,x) \quad dx\text{-a.e.}\right] \quad .$$

<u>**Theorem 2**</u> : $(M_t)_{t\geq 0}$ *is the* $\mathbb{R}^{n\times n}$*-valued* $(\tilde{\mathcal{F}}_t)$*-adapted continuous solution of*

$$\begin{cases} dM_t = \left[\sigma'(t,\tilde{X}_t)\cdot M_t\right]\cdot d\tilde{B}_t + \left[b'(t,\tilde{X}_t)\cdot M_t\right] dt \\ M_0 = I \end{cases}$$

Proof : We saw, in the previous paragraph, that

$$\forall t \qquad \Upsilon\left(\tilde{X}_t, \tilde{X}_t\right)(x,\omega) \in GL_n(\mathbb{R}) \qquad \tilde{P}\text{-a.e.}$$

Therefore, according to Théorème 3 in [2], for any $t\geq 0$, the

image measure of $\tilde{P}$ by $\tilde{X}_t$ is absolutely continuous with respect to the Lebesgue measure dx on $\mathbb{R}^n$.
In other words

$$\forall t>0 \quad \forall A \subset \mathbb{R}^n$$
$$(A \text{ dx-negligible}) \Longrightarrow \left(\{(x,\omega)\ ;\ \tilde{X}_t(x,\omega) \in A\} \text{ is } \tilde{P}\text{-negligible}\right) .$$

In particular, for any $t>0$, $\sigma'(t,\tilde{X}_t)$ (resp. $b'(t,\tilde{X}_t)$) does not depend on versions of σ' (resp. b') up to $\tilde{P}$-almost everywhere equality.

Let us define now, for $k \in \mathbb{N}$,

$$\sigma^{(k)}(t,x) = k^n \int \sigma(t,x-y)\ \varphi(ky)\ dy$$

$$b^{(k)}(t,x) = k^n \int b(t,x-y)\ \varphi(ky)\ dy ,$$

where φ denotes a non negative C^1 function on $\mathbb{R}^n$ with compact support and such that

$$\int \varphi(y)\ dy = 1 .$$

Consequently, $\sigma^{(k)}$ (resp. $b^{(k)}$) fulfils the same Lipschitz condition as σ (resp. b) and

$$\forall t\geq 0 \quad \sigma^{(k)}(t,\cdot) \text{ and } b^{(k)}(t,\cdot) \text{ are } C^1 \text{ functions} .$$

Moreover, for $T\geq 0$, there exists $C\geq 0$ such that

$$\forall k>0 \quad \forall t \in [0,T] \quad \forall x \in \mathbb{R}^n$$
$$|\sigma^{(k)}(t,x) - \sigma(t,x)| \vee |b^{(k)}(t,x) - b(t,x)| \leq \frac{C}{k}$$

and $\forall t\geq 0$
$$\left(\lim_{k\to\infty} \frac{\partial}{\partial x}\sigma^{(k)}(t,x)=\sigma'(t,x) \text{ and } \lim_{k\to\infty} \frac{\partial}{\partial x}b^{(k)}(t,x)=b'(t,x) \text{ dx-a.e.}\right) .$$

Let us define, for any $k \in \mathbb{N}$,

$$\tilde{X}_t^{(k)} = x + \int_0^t \sigma^{(k)}(s,\tilde{X}_s)\cdot d\tilde{B}_s + \int_0^t b^{(k)}(s,\tilde{X}_s)\ ds .$$

According to the fact that, for any t, $\sigma^{(k)}(t,\cdot)$ and $b^{(k)}(t,\cdot)$ are C^1 functions, one can easily see that

$$\forall t\geq 0 \qquad \tilde{X}_t^{(k)} \in \tilde{d}^n$$

and, if we denote by $M_t^{(k)}$ the matrix $\left(\nabla_i\left(\tilde{X}_j^{(k)}(t)\right)\right)_{1\leqslant i,j\leqslant n}$,

$$M_t^{(k)} = I + \int_0^t \left[\frac{\partial}{\partial x}\sigma^{(k)}(s,\tilde{X}_s)\cdot M_s\right]\cdot d\tilde{B}_s + \int_0^t \left[\frac{\partial}{\partial x}b^{(k)}(s,\tilde{X}_s)\cdot M_s\right] ds \quad .$$

By dominated convergences and in view of the property stated at the beginning of the proof, we get, for any $t\geqslant 0$,

$$\lim_{k\to\infty} \tilde{X}_t^{(k)} = \tilde{X}_t \quad \text{in} \quad [L^2(\tilde{P})]^n \quad \text{and}$$

$$\lim_{k\to\infty} M_t^{(k)} = \tilde{M}_t \quad \text{in} \quad [L^2(\tilde{P})]^{n\times n} \quad ,$$

where $\tilde{M}_t$ is defined by

$$\tilde{M}_t = I + \int_0^t \left[\sigma'(s,\tilde{X}_s)\cdot M_s\right]\cdot d\tilde{B}_s + \int_0^t \left[b'(s,\tilde{X}_s)\cdot M_s\right] ds \quad .$$

Consequently

$$\forall t\geqslant 0 \qquad \left(\tilde{M}_t = M_t \quad \tilde{P}\text{-a.e.}\right)$$

and, by continuity, $(M_t)_{t\geqslant 0}$ is the solution to the equation given in the statement of the theorem.

■

Let us consider now the case $n=1$. In this case we can obtain an explicit formula.
According to the one-dimensional Lipschitzian functional calculus for local Dirichlet spaces (c.f. [1]), we can take $U_t = F'(t,u(t))$ in the factorisation lemma, where F' is any Borel function such that

$$\forall t\geqslant 0 \qquad \left(F'(t,x) = \frac{\partial}{\partial x}F(t,x) \quad dx\text{-a.e.}\right) \quad ,$$

and Lemma 2 in part 3 can now be written

$$\gamma\left(\tilde{X}_t,\tilde{X}_t\right) = 1 + 2\sum_{\ell=1}^{d}\int_0^t \sigma_\ell'\left(s,\tilde{X}_s\right)\,\gamma\left(\tilde{X}_s,\tilde{X}_s\right)\,d\tilde{B}_s^\ell$$

$$+ \int_0^t \left[\sum_{\ell=1}^{d}\left[\sigma_\ell'\left(s,\tilde{X}_s\right)\right]^2 + 2b'\left(s,\tilde{X}_s\right)\right]\gamma\left(\tilde{X}_s,\tilde{X}_s\right)\,ds$$

where σ_ℓ' and b' are, as before, fixed Borelian versions of the derivatives of σ_ℓ and b with respect to the second variable.
Then, if we consider on $(\Omega\ ,\ (\mathcal{F}_t)\ ,\ P)$ the process

$$Y_t^x = \exp\left[\sum_{\ell=1}^{d}\left(\int_0^t \sigma_\ell'(s,X_s^x)\ dB_s^\ell - \frac{1}{2}\int_0^t\left[\sigma_\ell'(s,X_s^x)\right]^2 ds\right) + \int_0^t b'(s,X_s^x)\ ds\right]$$

it is easy to see that $\left(Y_t^x\right)^2$ satisfies the equation

$$Z_t = 1 + \sum_{\ell=1}^{d}\int_0^t 2\sigma_\ell'(s,X_s^x)\ Z_s\,dB_s^\ell + \int_0^t\left[\sum_{\ell=1}^{d}\left(\sigma_\ell'(s,X_s^x)\right)^2 + 2b'(s,X_s^x)\right] Z_s\,ds.$$

From this we deduce that for $\tilde{P}$-almost all (x,ω)

$$\left(Y_.^x(\omega)\right)^2 = \gamma\left(\tilde{X}_.,\tilde{X}_.\right)(x,\omega)$$

and then

$$Y_.^x(\omega) = M_.(x,\omega)\quad .$$

Finally our main result gives the following property :

$$P\text{-almost surely}\quad \left(X_t^\alpha(\omega) - X_t^\beta(\omega) = \int_\alpha^\beta Y_t^x(\omega)\ dx \qquad \forall t\geq 0\quad \forall \alpha,\beta\in\mathbb{R}\right)$$

and we see that two processes such as Y , defined with different versions of σ' and b' , are $\tilde{P}$-indistinguishable.

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